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Title: Rigidity of Vertex-Regular Actions on Fuchsian Buildings

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We present a method by which torsion-free groups of automorphisms of a 2-dimensional hyperbolic building which act simply transitively on the vertex set can be constructed, and prove that any such group can be obtained by this construction. The method produces groups defined by finite presentations with strong small cancellation properties, and we prove that when the building is Fuchsian with a regular fundamental chamber, two such groups are isomorphic if and only if there is an isomorphism taking generators to generators and relators to relators. Using these results, we find and classify all the torsion-free vertex-regular groups of automorphisms of Bourdon's building $I_{5,5}$.

Rigidity of Vertex-Regular Actions on Fuchsian Buildings

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Chair of the Department of Mathematics

Dean of the Graduate School

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Samantha T. Smith, Author

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1. INTRODUCTION

1.1. An Informal Overview

A 2-dimensional hyperbolic building is a metric space X constructed by gluing together copies of a hyperbolic polygon P, in such a way that X can be expressed as a union of copies of the hyperbolic plane $\mathbb{H}^2$ satisfying certain axioms (see Definition 2.4.0.1). Two-dimensional hyperbolic buildings share axioms similar to those of the simplicial complexes known as Tits buildings, but cannot be considered either a generalization or a special case, since 2-dimensional hyperbolic buildings need not have a simplicial structure, and Tits buildings need not be negatively curved. One can view 2-dimensional hyperbolic buildings as a higher-dimensional analogue of trees. Indeed, as trees arise as the universal cover of 1-dimensional cellular metric spaces, 2-dimensional hyperbolic buildings often arise as the universal cover of 2-dimensional cellular metric spaces ([13]).

The first example of a 2-dimensional hyperbolic building in the literature was produced by M. Bourdon in [4]. In his paper, Bourdon explored the geometric and combinatorial properties of the buildings he constructed, which later became known as Bourdon's buildings, and proved, in particular, that his buildings possess a property known as Mostow rigidity (see Theorem 2.4.2.2). Mostow rigidity provides a connection between the cellular metric structure of the building and the algebraic structure of its *cocompact lattices*, groups of cellular isometries from the building to itself with finite cell stabilizers and compact orbit space. Later ([30]), Mostow rigidity was shown to hold for a much more general class of 2-dimensional hyperbolic buildings called Fuchsian buildings (see Definition 2.4.0.1).

In [18], R. Kangaslampi and A. Vdovina produced and studied a collection of torsion-free cocompact lattices of a simplicial Fuchsian building acting freely and transitively on the building's vertices. We will call such a torsion-free cocompact lattice *vertex-regular*. Kangaslampi and Vdovina used Mostow rigidity to produce a group invariant of those lattices, which they called the *dual graph* of the lattice, and used this dual graph to distinguish some of their groups. Still, the classification is not yet complete; in Section 2.5, we will produce examples of two non-isomorphic vertex-regular lattices with isomorphic dual graphs.

In this thesis, we consider the problem of constructing and completely classifying up to group isomorphism the vertex-regular cocompact lattices of any Fuchsian building. First, we will prove that each torsion-free vertex-regular cocompact lattice of a 2-dimensional hyperbolic building corresponds to a finite presentation $\langle S \mid R \rangle$ which can easily be determined from the orbit space, possessing properties described in the following theorem.

Main Theorem 1. *Suppose there exists a torsion-free vertex-regular cocompact lattice G of the 2-dimensional hyperbolic building X. Then*

- *the link at each vertex of X is graph-isomorphic to the same graph Γ, and*
- *Γ contains an even number of vertices.*

Moreover, there is a presentation $\langle S \mid R \rangle$ of G satisfying the following properties:

(i) The size of S is half the number of vertices of Γ,

(ii) each element of R has length equal to the number of sides of the hyperbolic polygon used to tile X, and

(iii) the Whitehead graph of $\langle S \mid R\rangle$ *is graph-isomorphic to* Γ*.*

We then prove that if two such lattices are isomorphic as groups, and the building is Fuchsian with regular 2-cells (i.e. all open 1-cells are pairwise isometric), Mostow rigidity implies that we can find an isomorphism taking the generators of the presentation of one lattice to the generators (and their inverses) of the presentation of the other lattice.

Main Theorem 2. *Let* X *be a Fuchsian building tiled by a regular hyperbolic polygon. Let* $G_1 = \langle S_1 \mid R_1\rangle$ *and* $G_2 = \langle S_2 \mid R_2\rangle$ *be any two torsion-free vertex-regular cocompact lattices of* X*, where* $\langle S_1 \mid R_1\rangle$ *and* $\langle S_2 \mid R_2\rangle$ *are the presentations obtained in Main Theorem 1. If* G_1 *and* G_2 *are isomorphic, then there is an isomorphism from* G_1 *to* G_2 *taking* $S_1^{\pm}$ *bijectively to* $S_2^{\pm}$*.*

We also show that the presentations $\langle S \mid R\rangle$ of Main Theorem 1 possess certain small cancellation properties, which depend on the local structure of the building and on the number of sides in the hyperbolic polygon P used to construct the building. Historically, small cancellation has been used to deduce "spelling theorems," results that place restrictions on the kinds of words in the generators of a finitely presented group which can possibly be trivial. Such theorems have been instrumental in the solutions to the word problem and isomorphism problem for certain classes of groups, including surface groups. We will use two spelling theorems in combination with Mostow rigidity to show that if two torsion-free vertex-regular cocompact lattices of a Fuchsian building are isomorphic, we can find an isomorphism taking generators to generators and relators to relators.

Main Theorem 3. *Let* X *be a Fuchsian building tiled by a regular hyperbolic polygon. Let* $G_1 = \langle S_1 \mid R_1\rangle$ *and* $G_2 = \langle S_2 \mid R_2\rangle$ *be any two torsion-free vertex-regular cocompact lattices of* X*, where* $\langle S_1 \mid R_1\rangle$ *and* $\langle S_2 \mid R_2\rangle$ *are the presentations obtained in Main Theorem 1. Let* $\psi : G_1 \to G_2$ *be a group isomorphism taking* $S_1^{\pm}$ *bijectively to* $S_2^{\pm}$*. Then for every relator* $r \in R_1$*, the image under* ψ *is a symmetrized relator,* $\psi(r) \in R_2^*$*.*

All together, these results have the following consequence. In finite time, using a simple computer algorithm, we can construct a collection of groups containing, up to isomorphism, all the torsion-free vertex-regular cocompact lattices of any Fuchsian building and completely classify the groups in this collection.

In general, the collection we construct may contain cocompact lattices of multiple distinct Fuchsian buildings, and an example of such a collection was exhibited in [18]. However, in the special case of Bourdon's buildings, we obtain a converse to Main Theorem 1 producing conditions on a finite presentation $\langle S \mid R \rangle$ which in fact characterize its torsion-free vertex-regular cocompact lattices. This allows us to produce, up to isomorphism, the exact set of torsion-free vertex-regular cocompact lattices of any of Bourdon's buildings. We then apply these results to obtain and classify all the torsion-free vertex-regular cocompact lattices of the simplest of Bourdon's buildings, denoted $I_{5,5}$.

Main Theorem 4. *Up to isomorphism, there are exactly 8,882 torsion-free vertex-regular cocompact lattices of* $I_{5,5}$.

We point out that every torsion-free vertex-regular cocompact lattice of one of Bourdon's buildings is word-hyperbolic (Proposition 2.4.0.1; [6], Proposition 8.19) and does not split essentially as a free product with amalgamation or an HNN-extension over a finite or cyclic subgroup ([5], [12]). The isomorphism problem was shown to be solvable for such groups by Z. Sela ([25]). Main Theorems 2 and 3 reaffirm and illustrate this result by providing a solution to the isomorphism problem for these lattices which can be practically implemented.

Our results have relevance to a well-known open question first posed in the 1980s by M. Gromov ([17]), who asked whether every word-hyperbolic group is residually finite, i.e. if every nontrivial element in the group remains nontrivial in some finite quotient. Although D. Wise has shown ([29]) that every torsion-free cocompact lattice of a Fuchsian building is residually finite if the polygon used to tile the building has at least 6 sides, the groups acting on Bourdon's buildings that we have produced have the potential to be non-residually finite word-hyperbolic groups. The groups produced in [18] also have this potential; however, the groups we obtain possess different properties. While the groups produced in [18] were all positively presented with finite abelianization, we have obtained examples which are positively presented and others that are not positively presented; some have finite abelianization and others have infinite abelianization; some are even cyclically presented. Our presentations possess small cancellation properties that differ from those found in [18]. Finally, while the groups of [18] all have Kazhdan's property (T), none of our groups have Kazhdan's property (T).

1.2. A Statement of the Problem

In this thesis, we consider the problem of constructing all possible torsion-free groups acting by cellularly and isometrically on a Fuchsian building and freely and transitively on the set of vertices. We also seek an easy combinatorial technique by which these groups can be classified completely up to group isomorphism.

1.3. An Outline of the Thesis

We begin in Chapter 2 by presenting the terminology and known results which we will need throughout this thesis. We will also review the work of Cartwright et al. in [9] and Kangaslampi and Vdovina in [18], on which this thesis builds.

We then move in Chapter 3 to our main results. We will begin by showing that any torsion-free vertex-regular cocompact lattice of a Fuchsian building admits a straightforward finite presentation, and find conditions on a finite presentation which guarantee that the corresponding group is a torsion-free vertex-regular lattice of a Fuchsian building.

Next, we leverage Mostow rigidity to show that any group isomorphism between two such torsion-free lattices determines a group isomorphism taking generators to generators. We then investigate the small cancellation properties possessed by the presentations of these lattices, and use these to prove that any group isomorphism between two such lattices determines a group isomorphism taking generators to generators and relators to relators.

In Chapter 4, we apply the results of Chapter 3 to obtain, up to isomorphism, all the torsion-free vertex-regular lattices of one of Bourdon's buildings. (A complete list of the groups is presented in Appendix A.) We summarize the properties of these groups.

We will conclude in Chapter 5 with a discussion and ideas for future work.

2. MATHEMATICAL BACKGROUND

2.1. Polygonal Complexes

Much of what we will describe in this section follows the presentation of [6]. The reader is directed to this book for a more general treatment of polygonal complexes and CAT(-1) spaces. Below, we will present only what is needed for our work in this thesis.

Definition 2.1.0.1. *Let (X, d) be a metric space. A* ***geodesic*** *from a point $x \in X$ to a point $y \in X$ is a map $f : [0, \ell] \to X$ such that $f(0) = x$, $f(\ell) = y$, and $|t-s| = d(f(t), f(s))$ for all $0 \leq t, s \leq 1$. In particular, $d(x, y) = \ell$, and the image of f can be thought of as a path whose length realizes the distance between x and y. In a common abuse of language, we will call the image of f a geodesic if the map itself does not need to be specified.*

The space X is called a ***geodesic metric space*** *if for any choice of points x and y in X, there is a geodesic from x to y.*

If (X, d) is a geodesic metric space, the ***convex hull*** *of a subset $Y \subseteq X$ is the smallest subset Z of X containing Y satisfying the following property: If $x, y \in Z$ and $f : [0, \ell] \to X$ is a geodesic from x to y, then the image of f is contained in Z.*

The hyperbolic plane $\mathbb{H}^2$ with the standard metric is a geodesic metric space. A ***convex polygon*** *in $\mathbb{H}^2$ is a subspace of $\mathbb{H}^2$ with nonempty interior and which is equal to the convex hull of a finite set of points. The boundary of such a polygon consists of a union of geodesic segments, any two of which intersect in at most one point. If these geodesic segments have equal length, the polygon is said to be* ***regular****.*

Definition 2.1.0.2. *Let $\{\sigma_\alpha\}_{\alpha \in I}$ be a collection of convex polygons in the hyperbolic plane $\mathbb{H}^2$. A* ***$M_{(-1)}$-polygonal complex*** *is a quotient X of the disjoint union $\sqcup_\alpha \sigma_\alpha$ such that the quotient map $p : \sqcup_\alpha \sigma_\alpha \to X$ satisfies the following two properties:*

(1) For every $\alpha \in I$, the restriction of p to the interior of each face of σ_α is injective.

(2) *For all* $\alpha_1, \alpha_2 \in I$ *and* $x_1 \in \sigma_{\alpha_1}, x_2 \in \sigma_{\alpha_2}$*, if* $p(x_1) = p(x_2)$*, then there is an isometry* h *from the minimal closed cell* τ *containing* x_1 *to the minimal closed cell containing* x_2 *such that* $p = p \circ h$ *on* τ*.*

A group G acts on a $M_{(-1)}$-polygonal complex X by **cellular isometries** if, for every $g \in G$, the corresponding homeomorphism $X \to X$ is distance-preserving and maps open cells homeomorphically to open cells. We will call the action **vertex-regular** if, in addition, the homeomorphism acts freely and transitively on the set of vertices of X.

If a connected $M_{(-1)}$-polygonal complex X has only finitely many isometry types of cells, it follows from the work of M. Bridson ([6], Part I, Theorem 7.50) that X is a complete geodesic metric space under the metric defined as follows: Let $f : [a, b] \to X$ be a path in X, and suppose there exists a partition $a = t_0 < t_1 < \cdots < t_{p+1} = b$ such that, for all $i = 0, \ldots, p$, the image $f((t_i, t_{i+1}))$ is contained in an open cell σ_i° of X and $f|_{(t_i,t_{i+1})}$ is a geodesic in the metric on this open cell induced by $\mathbb{H}^2$. We then define the length of this path as

$$\ell(f) = \sum_{i=0}^{p} \ell_{\sigma_i^\circ}(f|_{(t_i,t_{i+1})}),$$

where $\ell_{\sigma_i^\circ}$ is the function giving the length of geodesic segments in $\sigma_i^\circ \subset \mathbb{H}^2$. The distance between two points is then defined as the minimum of the lengths of such paths between them.

Throughout the remainder of this thesis, the $M_{(-1)}$-polygonal complex X will always be connected with finitely many isometry types of cells. Moreover, we will only treat $M_{(-1)}$-polygonal complexes which are **locally finite**, meaning that every closed cell is disjoint from all but finitely many other closed cells.

Definition 2.1.0.3. *Let* X *be a* $M_{(-1)}$*-polygonal complex. For every* $x \in X$*, we define a metric space* $Lk(x, X)$ *called the* ***link*** *of* x *in* X*, as follows. The underlying set is a sphere* $S(x, \epsilon)$ *in* X *of radius* ϵ *centered at* x*, where* ϵ *is taken small enough that if* $0 < \epsilon' < \epsilon$*, then* $S(x, \epsilon)$ *and* $S(x, \epsilon')$ *are homeomorphic.*

This set has a natural graph structure in which vertices arise from the intersection of $S(x,\epsilon)$ with 1-cells, and edges arise from the intersection of $S(x,\epsilon)$ with 2-cells. This graph is finite since, by assuption, X is locally finite. If the graph is also connected, we can therefore define a metric on this graph by specifying that the length of each edge is equal to the angle between the edges in X corresponding to the endpoints.

For example, the link of any vertex in the hyperbolic plane tiled by right-angled pentagons is a cycle graph on four edges of length $\frac{\pi}{2}$, as depicted in Figure 2.1.

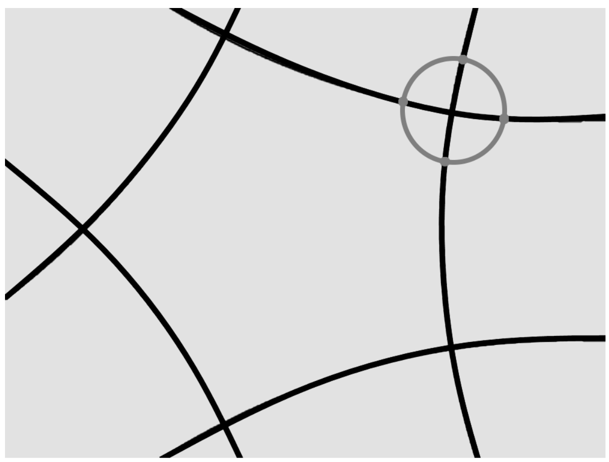

FIGURE 2.1: The link of a vertex in the hyperbolic plane.

2.1.1 The CAT(-1) Condition

Definition 2.1.1.1. *Let (X, d_X) be a geodesic metric space.*

Let $x, y, z \in X$ be arbitrary, and let $[x,y], [y,z], [x,z]$ be any geodesic segments connecting x to y, y to z, and z to x respectively. Denote $\Delta = [x,y] \cup [y,z] \cup [z,x]$.

Let p, q, r be any points in $\mathbb{H}^2$ satisfying $d(p,q) = d_X(x,y)$, $d(q,r) = d_X(y,z)$, and $d(p,r) = d_X(x,z)$, where d denotes the standard metric on $\mathbb{H}^2$. There exists a unique triangle $\overline{\Delta}$ in $\mathbb{H}^2$ whose vertices are p, q, r and whose edges are geodesic segments.

*For $c \in [x,y]$, we call $\overline{c} \in \mathbb{H}^2$ a **comparison point** for c if $\overline{c}$ is contained in the geodesic segment from p to q and $d(x,c) = d(p,\overline{c})$.*

We say that X is a ***CAT(−1) space*** *if for any choice of $x, y, z, [x, y], [y, z], and [z, x]$ in X, any choice of $p, q,$ and r in $\mathbb{H}^2$, and for any $a, b \in \Delta$, there exist comparison points $\overline{a}, \overline{b} \in \overline{\Delta}$ such that*

$$d(a, b) \leq d(\overline{a}, \overline{b}).$$

Loosely, CAT(−1) spaces have geodesic triangles which are thinner than triangles in $\mathbb{H}^2$ of comparable size. This condition is global and highly restrictive. For example, all CAT(−1) spaces are contractible (see [6], Part II, Proposition 1.4).

Thus, there is the following local alternative. We say that X is **locally CAT(−1)** if for any $x \in X$, there is a small ball $B(x, \delta)$ so that $B(x, \delta)$ is CAT(−1) in the subspace metric.

The next theorem, often known as *Gromov's Link Condition*, provides an easy way to check if a locally finite $M_{(-1)}$-polygonal complex is locally CAT(−1).

Theorem 2.1.1.1 (Gromov's Link Condition, [6], Part II, Theorem 5.2 and Lemma 5.6)**.** *Let X be an $M_{(-1)}$-polygonal complex. Then X is locally CAT(−1) if and only if for each vertex $v \in X$, every injective loop in $Lk(v, X)$ has length at least 2π.*

Remarkably, in the presence of simple connectivity, the local CAT(−1) condition and global CAT(−1) condition are equivalent. This is a consequence of Cartan-Hadamard Theorem. To avoid introducing additional terminology which will not be needed in this thesis, we present the theorem as applied only to $M_{(-1)}$-polygonal complexes. For the most general version of the result, see [6], Part II, Theorem 4.1.

Theorem 2.1.1.2. *Let K be a locally CAT(−1) $M_{(-1)}$-polygonal complex. Then the universal cover $\tilde{K}$ inherits a $M_{(-1)}$-polygonal complex structure from K: its cells are obtained by taking all possible lifts to $\tilde{K}$ of cells of K. Moreover, $\tilde{K}$, in the length metric induced by this $M_{(-1)}$-polygonal structure, is globally CAT(−1).*

2.2. Group Presentations

Let S be a nonempty set. In this thesis, we will always assume that S is finite. Denote the elements of S by $x_0, x_1, \ldots, x_n$, and let S^{-1} denote the set of symbols $x_0^{-1}, x_1^{-1}, \ldots, x_n^{-1}$. Denote by $S^{\pm}$ the disjoint union $S \sqcup S^{-1}$.

The **free group** generated by S, denoted $F(S)$, is the set of strings formed by elements of $S^{\pm}$. The group operation is concatenation, the identity element is the empty string, and the inverse of x_i is x_i^{-1} for all $i = 0, \ldots, n$.

Every word in the free group $F(S)$ can be expressed in the form $x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$, where $x_{i_1}, \ldots, x_{i_k} \in S$ and $\epsilon_1, \ldots, \epsilon_k \in \{1, -1\}$. This word will be called **positive** if $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_k = 1$. It will be called **reduced** if $x_{i_j}^{\epsilon_j} \neq x_{i_{j+1}}^{-\epsilon_{j+1}}$ for all $j = 1, \ldots, k-1$. It will be **cyclically reduced** if in addition to being reduced, we have $x_{i_k}^{\epsilon_k} \neq x_{i_1}^{-\epsilon_1}$.

Let R be a set of elements of $F(S)$, possibly empty. In this thesis, we will always assume that R is finite. The **normal closure** of R in $F(S)$, denoted $N(R)$, consists of all products of conjugates $w^{-1}r^{\epsilon}w$, where $w \in F(S)$, $\epsilon \in \{1, -1\}$, and $r \in R$. Then $N(R)$ forms a normal subgroup of $F(S)$. The quotient $F(S)/N(R)$ is called **the group defined by the presentation** $\langle S \mid R \rangle$. In a slight abuse of notation, we will sometimes write $G = \langle S \mid R \rangle$ to mean that G is the group defined by the presentation $\langle S \mid R \rangle$. The elements of the set R are called the **relators** of the presentation. A presentation is said to be **positive** if each of its relators is positive.

If an element $r \in R$ is written $x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$, where $\epsilon_1, \ldots, \epsilon_k \in \{1, -1\}$ and $x_{i_1}, \ldots, x_{i_k} \in S$, we will often speak of a special conjugate of r called a **cyclic permutation**, which is obtained by conjugating r by some initial subword $x_{i_1}^{\epsilon_1} \cdots x_{i_j}^{\epsilon_j}$, $1 \leq j \leq k$. Informally, a cyclic permutation of r can be obtained by "rotating" the first letter of r to the end, or the last letter of r to the front, as many times as desired. By a **cyclic subword** of r, we mean a subword of some cyclic permutation of r.

Let g be an element of $F(S)$ written in the form $x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$, where $\epsilon_1, \ldots, \epsilon_k \in \{1, -1\}$, $x_{i_1}, \ldots, x_{i_k} \in S$, and $x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$ is reduced. The integer k is called the **length** of g (as measured in $F(S)$) and in this case we will write $|g| = k$.

To every presentation $\langle S \mid R \rangle$, we can associate a 2-complex in the following way: To a single 0-cell, attach an oriented 1-cell for every element of S. For each element r of R of length n, take an n-gon, and assign its edges labellings and orientations so that reading around the boundary spells out r. Then attach this 2-cell according to the labelling and orientation on the boundary. The result is called the **presentation 2-complex** of $\langle S \mid R \rangle$. It follows from the Seifert van Kampen theorem that the fundamental group of the presentation 2-complex of $\langle S \mid R \rangle$ is the group defined by $\langle S \mid R \rangle$.

For an example, Figure 2.2 depicts a schematic of the presentation 2-complex of the presentation

$$\langle a, b, c, d, e \mid ab^2ad, bc^2be, cd^2ca, de^2db, ea^2ec \rangle.$$

To assemble the presentation 2-complex from the schematic, identify every vertex of every pentagon to a single point, and identify edges with the same labelling, respecting orientation.

To every presentation $\langle S \mid R \rangle$, we can also associate a graph in the following way. For every generator $x \in S$, create two vertices x^- and x^+. If $x^\epsilon y^\delta$ is a cyclic subword of some relator R, draw an edge

- from x^- to y^+ if $\epsilon = \delta = 1$,
- from x^+ to y^- if $\epsilon = \delta = -1$,
- from x^+ to y^+ if $\epsilon = -1$ and $\delta = 1$,
- from x^- to y^- if $\epsilon = 1$ and $\delta = -1$.

The resulting graph is called the **Whitehead graph** of $\langle S \mid R \rangle$.

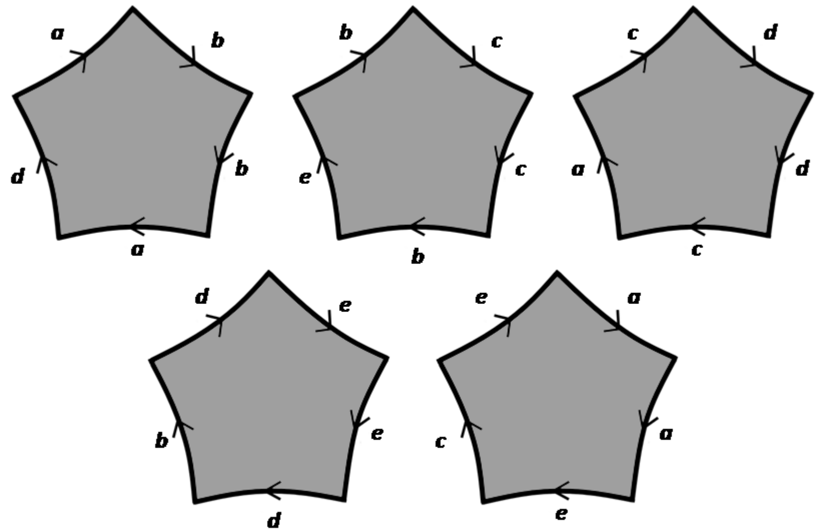

FIGURE 2.2: A depiction of a presentation 2-complex.

Note that the Whitehead graph of a presentation is naturally graph-isomorphic to the link of the vertex in the presentation 2-complex, if the latter is given the structure of an $M_{(-1)}$-polygonal complex. This is because a small sphere centered at this vertex will intersect each edge corresponding to a generator x in two points, which we label x^+ and x^-. Adopt the convention that x^+ is the first point and x^- is the second point encountered as we traverse the edge labelled x in the forward direction. Then any 2-cell around whose boundary we can read the cyclic subword xy determines an edge in the link between x^- and y^+, as shown in Figure 2.3.

2.3. Small Cancellation

The material in this section follows the presentation of [21]. The reader is directed to this book for a more general treatment of small cancellation theory.

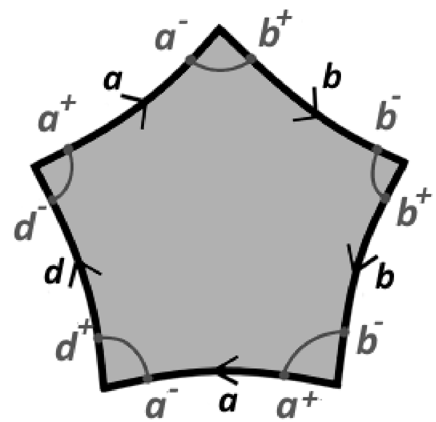

FIGURE 2.3: The relator ab^2ad gives five edges in the Whitehead graph.

The group defined by $\langle S \mid R\rangle$ remains unaltered if R is changed in any of the following ways:

- An element of R is replaced by a conjugate or an inverse.
- A conjugate or an inverse of an element of R is added to R.

Hence, the group defined by $\langle S \mid R\rangle$ remains unchanged if R is replaced by its **symmetrization** R^*. This set is the smallest subset of $F(S)$ which contains R and is closed under taking cyclic permutations and inverses.

Definition 2.3.0.1 (Small Cancellation Conditions)**.** *Let $\langle S \mid R\rangle$ be a group presentation.*

Suppose that $r_1, r_2 \in R^*$, where r_1 and r_2 are distinct. If we can write $r_1 = bc_1$ and $r_2 = bc_2$, then b is a ***piece*** *(of $\langle S \mid R\rangle$).*

Let p be a positive integer. The presentation $\langle S \mid R\rangle$ satisfies the $C(p)$ ***condition*** *if no element of R^* is a product of fewer than p pieces.*

Let $\lambda > 0$. The presentation $\langle S \mid R\rangle$ satisfies the $C'(\lambda)$ ***condition*** *if whenever $r \in R^*$ and $r = bc$ where b is a piece, then $|b| < \lambda|r|$.*

Let q be a positive integer. The presentation $\langle S \mid R\rangle$ satisfies the $T(q)$ ***condition*** *if, for every integer h satisfying $3 \leq h < q$, the Whitehead graph of $\langle S \mid R\rangle$ has no reduced* *cycles on h edges.*

We emphasize that the small cancellation conditions are possessed by the presentation, and not by the group the presentation defines. Informally, these properties express that if two elements of the set of symmetrized relators are multiplied together, only a small amount of cancellation will occur, unless one element is the inverse of the other.

Groups defined by presentations satisfying $C'(\lambda)$-$T(q)$ for small λ and large q, or $C(p)$-$T(q)$ for large p and q, often possess additional special properties. The following properties, in particular, will be used in the proof of Theorem 3.2.0.2 to determine circumstances under which a map between two finitely presented groups determines a group isomorphism.

Theorem 2.3.0.1 ([21], Theorem 4.4)**.** *Let $\langle S \mid R \rangle$ have the $C(3)$ and $T(6)$ properties. Let w be a nontrivial, cyclically reduced word with $w \in N(R)$, i.e. w is trivial in the group defined by $\langle S \mid R \rangle$. Then either $w \in R^*$, or some cyclically reduced conjugate w^* of w can be written*

$$w^* = u_1 s_1 \cdots u_j s_j$$

where for every $k = 1, \ldots, j$, there exists $r_k \in R^*$ such that

$$r_k = s_k b_1 \cdots b_{i(s_k)}$$

where $b_1, \ldots, b_{i(s_k)}$ are pieces, and

$$\sum_{k=1}^{j} (2.5 - i(s_k)) \geq 3.$$

Theorem 2.3.0.2 ([21], Theorem 4.6)**.** *Let $\langle S \mid R \rangle$ have the $C'(\frac{1}{4})$ and $T(4)$ properties. Let w be a nontrivial, cyclically reduced word with $w \in N(R)$, i.e. w is trivial in the group defined by $\langle S \mid R \rangle$. Then either $w \in R^*$, or some cyclically reduced conjugate of w contains either*

1. *two disjoint subwords* t_1, t_2*, each satisfying* $r_i = t_i c_i$ *for some* $r_i \in R^*$ *and* $|t_i| > \frac{3}{4}|r_i|, i = 1, 2,$ *or*

2. *four disjoint subwords* $t_1, \ldots, t_4$*, each satisfying* $r_i = t_i c_i$ *for some* $r_i \in R^*$ *and* $|t_i| > \frac{1}{2}|r_i|, i = 1, \ldots, 4.$

2.4. Hyperbolic Buildings

Fix an m-sided convex polygon P of the hyperbolic plane $\mathbb{H}^2$ whose edges are labelled, reading around the boundary, by $1, \ldots, m$, and whose vertices are labelled by one of the 2-sets $\{1, 2\}$, ..., $\{m-1, m\}, \{m, 1\}$ according to the edges to which the vertex is incident. Suppose every dihedral angle of P is of the form π/k, where k is an integer. Then the images of P under the group generated by the reflections through the edges of P form a tessellation of $\mathbb{H}^2$, and induce on $\mathbb{H}^2$ the structure of a labelled $M_{(-1)}$-polygonal complex ([22], Theorem IV.H.11). Denote this labelled complex by A_P.

Definition 2.4.0.1. *Let* X *be a connected* $M_{(-1)}$*-polygonal complex whose edges are labelled with integers* $1, \ldots, m$ *and whose vertices are labelled with 2-sets of integers* $\{1, 2\}$, ..., $\{m-1, m\}, \{m, 1\}$*. Let* P *be the labelled convex polygon described in the previous paragraph, and suppose that for every 2-cell* c *of* X*, which we call a* ***chamber****, there is a cellular isometry (an* ***isomorphism****)* $c \to P$ *which also preserves the labels of the edges and vertices.*

We say that X *is a* ***2-dimensional hyperbolic building*** *(of* ***type*** P*) if it has a family of subcomplexes, called* ***apartments****, isomorphic to* A_P *by a label-preserving isomorphism, with the following properties:*

1. *Given any two chambers, there is an apartment containing both.*

2. *For any two apartments* A_1, A_2 *that share a chamber, there is a label-preserving*

isomorphism from A_1 to A_2 fixing $A_1 \cap A_2$ pointwise.

The polygon P is called the ***fundamental chamber*** *of X.*

We say that X is a ***Fuchsian building*** *if in addition, there are integers $q_i \geq 3$, $i = 1, \ldots, k$, such that each edge of X labelled by i is contained in exactly q_i chambers.*

The $M_{(-1)}$-polygonal complexes which can occur as 2-dimensional hyperbolic buildings are severely limited, as shown by the following proposition.

Definition 2.4.0.2. *For two vertices u and v in a connected graph, define $D(u, v)$ to be the minimum integer k such that there exists a sequence of sequentially adjacent vertices $u = v_1, \ldots, v_{k-1} = v$. The* ***diameter*** *of a graph is $\max\{D(u, v)\}$, where the maximum is taken over all the vertices u and v of the graph. In other words, the diameter of a graph is the minimum number of edges needed to connect the vertices in the graph which are furthest apart.*

The ***girth*** *of a graph is the length of the shortest nontrivial cycle in the graph.*

A ***generalized n-gon*** *is a graph with diameter n and girth $2n$.*

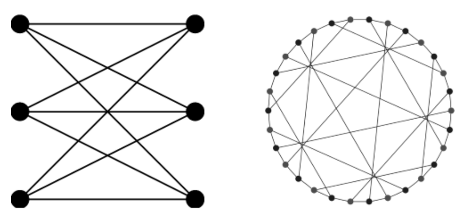

FIGURE 2.4: A complete bipartite graph (left) is a generalized 2-gon. The Tutte-Coxeter graph (right) is a generalized 4-gon.

Proposition 2.4.0.1 ([13], Proposition 1.5)**.** *If X is a 2-dimensional hyperbolic building, then X is CAT(-1) and, hence, contractible. Moreover, for every vertex v of X, there exists an integer $n \geq 2$ such that the link $Lk(v, X)$ is a generalized n-gon.*

Despite these restrictions, 2-dimensional hyperbolic buildings arise frequently as universal covers of $M_{(-1)}$-polygonal complexes, provided that their links are suitably nice.

Theorem 2.4.0.1 ([13], Theorémè 2.1'). *Let X be a CAT(-1) $M_{(-1)}$-polygonal complex, where each open 2-cell is isometric to the interior of fixed polygon P in $\mathbb{H}^2$, and each open 1-cell is isometric to the interior of some side of P. If, for every vertex v of X, there exists an integer $n \geq 2$ such that the link $Lk(v, X)$ is a generalized n-gon, then X is a 2-dimensional hyperbolic building.*

The following corollary is now a consequence of Theorem 2.1.1.2.

Corollary 2.4.0.1 ([13], Corollaire 2.4). *Let Y be a connected, locally CAT(-1) $M_{(-1)}$-polygonal complex, where each open 2-cell is isometric to the interior of a fixed polygon P in $\mathbb{H}^2$, and each open 1-cell is isometric to the interior of some side of P. If, for every vertex v of Y, there exists an integer $n \geq 2$ such that the link $Lk(v, Y)$ is a generalized n-gon, then the universal cover of Y is a 2-dimensional hyperbolic building.*

2.4.1 Bourdon's Buildings

Some of the most notable examples of Fuchsian buildings are those introduced by M. Bourdon ([4]). These buildings stand out among the Fuchsian buildings because they can be defined uniquely up to cellular isometry given information only about their fundamental chamber and vertex links.

Theorem 2.4.1.1 ([4], Proposition 2.2.1). *For every integer $p \geq 5$ and $q \geq 2$, there exists a 2-dimensional hyperbolic building $I_{p,q}$ whose chambers are regular hyperbolic p-gons with dihedral angles $\pi/2$ and edge lengths 1, and whose link at each vertex is the complete bipartite graph on $q + q$ vertices, $K(q, q)$.*

Moreover, suppose X is any simply connected polygonal 2-complex satisfying the following properties:

1. *Its 2-cells are isometric to a regular hyperbolic p-gon of angles $\pi/2$ and edge lengths 1, attached by their edges and vertices so that any two 2-cells have at most one edge or one vertex in common, and*

2. *The link at each of its vertices is the graph* $K(q,q)$.

Then X *is cellularly isometric to* $I_{p,q}$.

Definition 2.4.1.1. *Let* $p \geq 5$ *and* $q \geq 2$ *be integers. Define* ***Bourdon's building*** $I_{p,q}$ *to be the 2-dimensional hyperbolic building whose chambers are regular hyperbolic* p*-gons with dihedral angles* $\pi/2$ *and edge lengths 1, and whose link at each vertex is the complete bipartite graph on* $q+q$ *vertices,* $K(q,q)$.

Note that every edge of $I_{p,q}$ is contained in exactly q chambers, since every vertex in $K(q,q)$ is incident to exactly q edges. Hence, $I_{p,q}$ is Fuchsian if $q \geq 3$.

2.4.2 Lattices of Hyperbolic Buildings

For a 2-dimensional hyperbolic building X, we will call a cellular isometry $X \to X$ an **automorphism** of X. The cellular isometry need not be label-preserving. A **cocompact lattice** of X is a group G of automorphisms of X such that the orbit space X/G is compact and the stabilizer of each cell is finite.

Bourdon's buildings and, more generally, all Fuchsian buildings enjoy a special property known as *Mostow rigidity*, which restricts the potential group isomorphisms between cocompact lattices. This property is similar to the properties known as Mostow rigidity in the field of Riemannian geometry, but distinguishes itself by taking into account not only the building's geometric structure, but also its cellular structure.

Theorem 2.4.2.1 ([30], Corollary 1.3)**.** *Let* X_1, X_2 *be two Fuchsian buildings with regular chambers, and suppose* G *is a cocompact lattice of both* X_1 *and* X_2*. Then there is a cellular isometry* $\phi : X_1 \to X_2$ *which is* G*-equivariant, that is,* $\phi g = g\phi$ *for all* $g \in G$.

Now suppose that G_1 is a cocompact lattice of X_1 and G_2 is a cocompact lattice of X_2, where X_1 and X_2 are Fuchsian buildings with regular fundamental chambers. If $\Phi : G_1 \to G_2$ is an isomorphism, then $G_2 = \Phi(G_1)$ can be identified with G_1, and G_1 can be thought of as a cocompact lattice of both X_1 and X_2. So there is a cellular isometry $\phi : X_1 \to X_2$ such that $\phi g = \Phi(g)\phi$ for all $g \in G$. This implies $\phi g \phi^{-1} = \Phi(g)$, and therefore, $\phi G_1 \phi^{-1} = G_2$. These observations are summarized below.

Theorem 2.4.2.2. *Let X_1, X_2 be two Fuchsian buildings with regular chambers, and suppose that G_i is a cocompact lattice of X_i for $i = 1, 2$. If G_1 and G_2 are isomorphic as groups, then there is a cellular isometry $\phi : X_1 \to X_2$ so that $\phi G_1 \phi^{-1} = G_2$ as subsets of the group of automorphisms of X_2.*

This theorem was first discovered in the case $X_1 = X_2 = I_{p,q}$ by M. Bourdon in [4].

2.5. Vertex-Regular Groups Acting on Buildings: A Brief Literature Review

To our knowledge and at the time of writing, there exist only two papers in which torsion-free vertex-regular groups of automorphisms of a building have been studied.

The first was written by Cartwright et al. in 1993 ([9]). At this time, the notion of a 2-dimensional hyperbolic building as presented in Definition 2.4.0.1 had not yet been developed, and instead, Cartwright et al. viewed buildings as an abstract simplicial complex without a metric structure. They proved that any group which acted by simplicial isomorphisms on the building, simply transitively on the vertex set, admitted a straightforward presentation, whose generators could be obtained by considering the vertices adjacent to a fixed vertex v in the building, and whose relators could be read off the boundaries of the 2-dimensional simplices containing v as a vertex ([9], Section 2). Because 2-dimensional simplices have three edges, the relators obtained by Cartwright et al. all had length 3, and thus they were referred to as triangle presentations. In Section 3 of [9], Cartwright et al. determined circumstances under which triangle presentations determined a group acting by simplicial isomorphisms on buildings satisfying certain additional properties.

Cartwright et al. later communicated their methods to Kangaslampi and Vdovina ([18]), who applied the technique to 2-dimensional hyperbolic buildings whose fundamental chamber was also simplicial. They obtained conditions under which a triangle presentation defined a torsion-free lattice of a building whose link at each vertex was the Tutte-Coxeter graph (a generalized 4-gon on 30 vertices and 45 edges), and constructed all such positive presentations using a computer algorithm.

Kangaslampi and Vdovina observed that most of the presentations they had obtained were redundant. In many cases, a permutation of the generators would take one presentation to another. Using this observation, they reduced their list of over 7100 triangle presentations to merely 45. Noting that the hyperbolic structure of their buildings allowed them to make use of Mostow rigidity, Kangaslampi and Vdovina associated to each presentation a group invariant called the *dual graph*, constructed in the following way. Define one vertex for each generator, and one vertex for each relator. Draw an edge between two vertices if they correspond to a generator and a relator containing that generator. Note that the resulting graph is bipartite: the vertices corresponding to generators can be colored black, and the vertices corresponding to relators can be colored white.

To show the dual graph is a group invariant, take two groups G_1 and G_2 defined by two triangle presentations and suppose they are isomorphic. Mostow rigidity produces a cellular isometry conjugating one group to another, $\phi G_1 \phi^{-1} = G_2$. We claim that ϕ determines a cellular isometry $\overline{\phi} : X/G_1 \to X/G_2$ between orbit spaces. So let σ be any cell of X/G_1. Fix one lift $\tilde{\sigma}$ of σ to X, and let $\overline{\phi}(\sigma)$ be the cell which is the orbit of $\phi(\tilde{\sigma})$ under the action of G_2. To show this is well-defined, note that any other lift of σ can be written $g_1(\tilde{\sigma})$ for some $g_1 \in G_1$. Then $\phi(g_1(\tilde{\sigma})) = (g_2\phi)(\tilde{\sigma})$ for some $g_2 \in G_2$. Since $\phi(\tilde{\sigma})$ and $(g_2\phi)(\tilde{\sigma})$ have the same orbit, $\overline{\phi}$ is well-defined. Now the cellular isometry $\overline{\phi}$, in turn, defines a color-preserving graph isomorphism of dual graphs.

Unfortunately, the dual graph is not sufficient to completely classify lattices of

Fuchsian buildings up to group isomorphism. For example, in this thesis we will obtain groups defined by presentations

$$\langle x_0, \ldots, x_4 \mid x_0^2x_1x_0x_2, x_0x_3x_1^2x_4, x_0x_4x_1x_3^2, x_1x_2^2x_3x_2, x_2x_4^2x_3x_4\rangle$$

and

$$\langle x_0, \ldots, x_4 \mid x_0^2x_1x_0x_3, x_0x_3x_1^2x_4, x_0x_4^2x_1x_3, x_1x_2^2x_4x_2, x_2x_3^2x_4x_3\rangle$$

which are torsion-free vertex-regular lattices of Bourdon's building $I_{5,5}$. The presentations have isomorphic dual graphs but define non-isomorphic groups. The latter claim is most easily checked by considering the derived series of the groups, since one can see via the software GAP (Groups, Algorithms, and Programming, [14]) that the abelianization of the derived subgroup of the first group is trivial, while for the second group this abelianization is of order 4, $Z_2 \times Z_2$.

Thus, Kangaslampi and Vdovina could only use the dual graph to produce 23 presentations defining groups which were necessarily pairwise non-isomorphic. There remains the possibility that, for example, all 45 of the presentations found by these authors could define non-isomorphic groups.

3. METHODS AND CONSTRUCTIONS

3.1. Constructing Torsion-Free Vertex-Regular Cocompact Lattices of a Fuchsian Building

We begin with a short lemma.

Lemma 3.1.0.1. *Let G be a torsion-free group acting cellularly on a polygonal complex X. If the action of G on X is vertex-regular, then the stabilizer of every cell of X is trivial and the action of G on X is free.*

Proof. If $g \in G$ stabilizes a cell c of X, then g permutes the vertices of c. Hence, some power of g fixes the vertices of c and is trivial, so g is trivial.

Now suppose that $g \in G$ is such that $gx = x$ for some $x \in X$. Let c be any cell of X containing x. Then g stabilizes c, so $g = 1$. □

Our first theorem shows that every torsion-free vertex-regular cocompact lattice of a 2-dimensional hyperbolic building admits a presentation of a simple type.

Theorem 3.1.0.1. *Suppose there exists a torsion-free vertex-regular cocompact lattice G of a 2-dimensional hyperbolic building X. Then*

- *the link at each vertex of X is graph-isomorphic to the same graph Γ, and*
- *Γ contains an even number of vertices.*

Moreover, there is a presentation $\langle S \mid R \rangle$ of G satisfying the following properties:

(i) The size of S is half the number of vertices of Γ,

(ii) each element of R has length equal to the number of sides of the fundamental chamber of X, and

(iii) the Whitehead graph of $\langle S \mid R \rangle$ *is graph-isomorphic to* Γ*.*

Note that (ii) and (iii) imply that the number of edges of the fundamental chamber of X *divides the number of edges of* Γ*.*

Proof. Let G be a torsion-free vertex-regular cocompact lattice of X. Since the action of G is transitive on vertices, and is cellular, it preserves the link at each vertex. It follows immediately that the link at each vertex is graph-isomorphic to the same graph; call it Γ.

We claim that the action of G on X is properly discontinuous, meaning that for any compact subset K of X, the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite. The set K is contained in the union of a finite collection $\{c_1, \ldots, c_\ell\}$ of cells of X. If $g \in G$ is such that $gK \cap K \neq \emptyset$, then g maps some cell c_i onto some cell c_j, $1 \leq i, j \leq \ell$. Because the stabilizer of each cell is trivial, the group element taking c_i to c_j is unique. We conclude that $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite, its size bounded above by $\ell(\ell - 1)$.

Since the action of G on X is also free, the orbit map $X \to Y := X/G$ is a covering projection, and

- $\pi_1(Y) \cong G$, since X is simply connected;
- Y has one vertex, since the action of G on the vertices of X is transitive;
- Y inherits a cellular structure from X, since the action of G on X is cellular; and
- the link at the single vertex of Y is isomorphic to Γ, since the action of G on X preserves the link at each vertex.

The first and second items imply that G has a presentation $\langle S \mid R \rangle$ whose presentation 2-complex is Y, and the fourth implies the Whitehead graph of $\langle S \mid R \rangle$ is Γ. In particular, the number of vertices in Γ is $2|S|$.

The third item implies that all the elements of R have length equal to the number of sides of any of the 2-cells in Y, which in turn is equal to the number of sides in the fundamental chamber. $\square$

Note that Theorem 3.1.0.1 implies that, up to isomorphism, the number of vertex-regular cocompact lattices of any 2-dimensional hyperbolic building X is finite, bounded above by $(2k)^p$, where $2k$ is the number of vertices in the link of any vertex of X and p is the number of edges in the link.

Definition 3.1.0.1. *Let G be a torsion-free vertex-regular cocompact lattice of a 2-dimensional hyperbolic building X. The presentation $\langle S \mid R\rangle$ determined by the orbit space X/G, satisfying properties (i)-(iii) of Theorem 3.1.0.1, will be called the* ***scaffolded presentation*** *of G.*

Conversely to Theorem 3.1.0.1, if a group is defined by a finite presentation whose presentation 2-complex satisfies certain metric and combinatorial properties, that group will be a torsion-free vertex-regular cocompact lattice of a 2-dimensional hyperbolic building. The following result was first observed without proof in [18]; we present our proof below.

Proposition 3.1.0.1 ([18])**.** *Let $G = \langle S \mid R\rangle$, and suppose that $\langle S \mid R\rangle$ satisfies the following properties:*

(i) The sets S and R are finite,

(ii) The length of each element of R is equal to a fixed integer m, $m \geq 3$,

(iii) the Whitehead graph of $\langle S \mid R\rangle$ is a generalized n-gon, and

(iv) $n > \frac{m}{m-2}$.

Then G is a torsion-free vertex-regular cocompact lattice of a 2-dimensional hyperbolic building X.

Proof. First, we show that the presentation 2-complex Y of G admits a metric turning it into a locally CAT(-1) $M_{(-1)}$-polygonal complex. Metrize the open 2-cells of Y so that they are isometric to the interior of an m-sided regular polygon P whose dihedral angles are $\frac{\pi}{n}$ and whose edge lengths are 1. Since the girth of the Whitehead graph of $\langle S \mid R\rangle$ is a generalized n-gon, the length of the shortest embedded loop in Y is $2n\left(\frac{\pi}{n}\right) = 2\pi$. Since $n > \frac{m}{m-2}$, rearrangement produces $\pi(m-2) > m\left(\frac{\pi}{n}\right)$, and therefore P is hyperbolic. Metrize the open 1-cells of Y so that they are isometric to the interior of any side of P.

It is now immediate from Corollary 2.4.0.1 that the universal cover of Y is a 2-dimensional hyperbolic building X. The fundamental group of Y is the group G, and it acts on X by cellular isometries. Since X is contractible, Y is a finite-dimensional $K(G,1)$, and therefore G is torsion-free (see [7], Corollary 2.5). The action of G is vertex-regular, since Y has a single 0-cell. Finally, that G is a cocompact lattice of X follows from compactness of Y and Lemma 3.1.0.1. □

By Bourdon's uniqueness result (Theorem 2.4.1.1), any group defined by a presentation satisfying properties (i) - (iv) of Proposition 3.1.0.1 with Whitehead graph $K(q,q)$ is necessarily a cocompact lattice of one of Bourdon's buildings. Thus, we get the following equivalence.

Proposition 3.1.0.2. *Let G be a group, and let $p \geq 5, n \geq 1$ be integers. The following are equivalent:*

(1) G is a torsion-free vertex-regular cocompact lattice of Bourdon's building $I_{p,np}$.

(2) G admits a presentation $\langle S \mid R\rangle$ where the set S contains np elements, the set R contains n^2p elements, every element of R is a word of length p, and the Whitehead graph of $\langle S \mid R\rangle$ is graph-isomorphic to $K(np,np)$.

We remark that the restriction to the case $q = np$ is due to the observation in Theorem 3.1.0.1 that the number of edges of the fundamental chamber divides the number of edges in the link of any 2-dimensional hyperbolic building admitting a torsion-free vertex-regular cocompact lattice.

Proof. The forward direction is immediate from Theorem 3.1.0.1, since the fundamental chamber of $I_{p,np}$ is p-sided and the link at each vertex contains $2np$ vertices and n^2p^2 edges.

By Proposition 3.1.0.1, if $\langle S \mid R \rangle$ satisfies the properties of (2), it defines a torsion-free vertex-regular cocompact lattice of some 2-dimensional hyperbolic building X whose link at each vertex is $K(np, np)$ and whose fundamental chamber is isometric to the regular hyperbolic polygon of edge lengths 1 and dihedral angles $\frac{\pi}{2}$. By Theorem 2.4.1.1, X is cellularly isometric to $I_{p,np}$, and thus the group defined by $\langle S \mid R \rangle$ is a torsion-free vertex-regular cocompact lattice of $I_{p,np}$. □

3.2. Classifying Torsion-Free Vertex-Regular Cocompact Lattices of Fuchsian Buildings

In this section, we turn to the problem of classifying the torsion-free vertex-regular cocompact lattices of a Fuchsian building. We show that in the presence of a free and transitive action on the vertex set, the Mostow rigidity of Theorem 2.4.2.2 translates to a group-theoretic rigidity according to the following result.

Theorem 3.2.0.1. *Let X be a Fuchsian building with regular fundamental chamber. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ be any two torsion-free vertex-regular cocompact lattices of X, where $\langle S_1 \mid R_1 \rangle$ and $\langle S_2 \mid R_2 \rangle$ are scaffolded presentations. If G_1 and G_2 are isomorphic, then there is an isomorphism from G_1 to G_2 taking $S_1^{\pm}$ bijectively to $S_2^{\pm}$.*

Proof. Denote by $2k$ the number of vertices in the link of any vertex of X. Denote the metric on X by d, and let c denote the length of any edge of P.

Let $i = 1$ or 2. Although we already know that G_i is isomorphic to $F(S_i)/N(R_i)$, we will require a systematic way to produce specific group isomorphisms $F(S_i)/N(R_i) \to G_i$ by which elements of S_i (viewed as a subset of $F(S_i)/N(R_i)$) can be identified with automorphisms in $G_i \subset \mathrm{Aut}(X)$. We will do this using the following method: Let $Y_i := X/G_i$ denote the orbit space. Fix an orientation of each edge of Y_i, and fix an identification of the loops of Y_i with the elements of S_i. Now for each vertex w in X, we define a group isomorphism $\tau_w^i : F(S_i)/N(R_i) \to G_i$ by defining $\tau_w^i(x)$ for all $x \in S_i$. The oriented loop in Y_i corresponding to x lifts to an oriented edge in X beginning at w. The terminal endpoint of this edge is a vertex of X adjacent to w. There is a unique automorphism in G_i mapping w to this vertex. Define $\tau_w^i(x)$ to be this automorphism. It follows from covering space theory that τ_w^i is a group isomorphism for every vertex w.

Next, we make the following observation. Again, let $i = 1$ or 2. For any vertex w and for any $x \in S_i$, the vertices $\tau_w^i(x)(w)$ are adjacent to x by construction of τ_w^i. Therefore $d(w, \tau_w^i(x)(w)) = c$. Since G_i acts on X by isometries and τ_w^i is a group isomorphism, it follows that $d(\tau_w^i(x^{-1})(w), w) = c$; therefore $\tau_w^i(x^{-1})(w)$ is also adjacent to w. There are exactly $2k$ vertices adjacent to w, and exactly $2k$ elements of the form $\tau_w^i(x)(w)$, $x \in S_i^{\pm}$, so every vertex adjacent to w can be written $\tau_w^i(x)(w)$, $x \in S_i^{\pm}$. Since the action of G_i on X is free, we conclude that if $g \in G_i$ is such that $d(gw, w) = c$, then $g \in \{\tau_w^i(x)(w) : x \in S_i^{\pm}\}$.

Finally, fix a vertex v of X. By Mostow rigidity of X, there exists a cellular isometry ϕ of X such that $\phi G_1 \phi^{-1} = G_2$ as subgroups of $\mathrm{Aut}(X)$. Let $x \in S_1^{\pm}$ be arbitrary. Then $\tau_v^1(x)(v)$ is adjacent to v.

The isometry $\phi\tau_v^1(x)\phi^{-1}$ is an automorphism of X contained in G_2. Since ϕ acts on X by isometries, we get

$$\begin{aligned} d((\phi\tau_v^1(x)\phi^{-1})(\phi(v)), \phi(v)) &= d((\phi\tau_v^1(x))(v), \phi(v)) \\ &= d(\tau_v^1(x)(v), v) \\ &= c. \end{aligned}$$

We conclude that $\phi\tau_v^1(x)\phi^{-1} = \tau_{\phi(v)}^2(y)$ for some $y \in S_2^{\pm}$.

Thus, the desired isomorphism is conjugation by ϕ. $\square$

Thus, to check whether two groups defined by presentations $\langle S_1 \mid R_1 \rangle$ and $\langle S_2 \mid R_2 \rangle$ are isomorphic, one must only check $(k!)2^k$ possible assignments of elements of S_1 to elements of $S_2^{\pm}$.

Nevertheless, the problem of group isomorphism between vertex-regular cocompact lattices of a Fuchsian building remains difficult, as it is not easy, in general, to tell if a given generator assignment even defines a group homomorphism.

However, we show that scaffolded presentations $\langle S \mid R \rangle$ also satisfy small cancellation properties which further simplify the isomorphism problem considerably.

Theorem 3.2.0.2. *Let X be a Fuchsian building with regular fundamental chamber. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ be any two torsion-free vertex-regular cocompact lattices X, where $\langle S_1 \mid R_1 \rangle$ and $\langle S_2 \mid R_2 \rangle$ are scaffolded presentations. Let $\psi : G_1 \to G_2$ be a group isomorphism taking $S_1^{\pm}$ bijectively to $S_2^{\pm}$. Then for every relator $r \in R_1$, the image under ψ is a symmetrized relator, $\psi(r) \in R_2^*$.*

Proof. Let m denote the number of sides of the fundamental chamber of X. The link of any vertex in X is a generalized n-gon, $n \geq 2$ (Proposition 2.4.0.1).

We begin by proving the following claim: If $n = 2$, then $m \geq 5$. So suppose for eventual contradiction that $n = 2$ and $m = 3$ or 4. Let α denote the dihedral angle of the fundamental chamber. Since X is CAT(-1), by Gromov's Link Condition, the length of the shortest embedded loop in the link at every vertex is $\geq 2\pi$. At the same time, since $n = 2$, the length of the shortest embedded loop in the link at every vertex is 4α. Thus $\alpha \geq \frac{\pi}{2}$, which is impossible for a hyperbolic 3-gon or 4-gon.

Thus, the following two cases are exhaustive:

- Case 1: $m \geq 5$.

- Case 2: $n \geq 3$.

Proof of Case 1: We start with the following lemma.

Lemma 3.2.0.1. *Let $\langle S \mid R \rangle$ be a scaffolded presentation, where the fundamental chamber of the building X has $m \geq 5$ sides. Then $\langle S \mid R \rangle$ satisfies the $C'(1/4)$ and $T(4)$ small cancellation conditions.*

Proof. Since $m \geq 5$, all the elements of R have length ≥ 5. At the same time, the Whitehead graph of $\langle S \mid R \rangle$ is a generalized n-gon, $n \geq 2$ (Proposition 2.4.0.1), and thus has no double edges. It follows that the longest piece of $\langle S \mid R \rangle$ has length 1, for if there were a piece of the form xy, there would be two edges with endpoints x^- and y^+. Since $1 < \frac{1}{4}(5) \leq \frac{1}{4}(m)$, we can say that $\langle S \mid R \rangle$ is $C'(1/4)$.

To show $\langle S \mid R \rangle$ is $T(4)$, it suffices to observe that the shortest cycle in a generalized n-gon, $n \geq 2$, is of length ≥ 4. □

Now suppose that $G_1 = \langle S_1 \mid R_1 \rangle, G_2 = \langle S_2 \mid R_2 \rangle$, and $\psi : G_1 \to G_2$ are as described in the theorem's hypothesis, and let $r \in R_1$ be arbitrary. Write $r = y_1^{\epsilon_1} y_2^{\epsilon_2} \cdots y_m^{\epsilon_m}$ where $y_1, \ldots, y_m \in S_1$ and $\epsilon_1, \ldots, \epsilon_m \in \{1, -1\}$. Then $\psi(r)$ can be written

$$\psi(r) = z_1^{\delta_1} \cdots z_m^{\delta_m}$$

where $z_1, \ldots, z_m \in S_2$ and $\delta_1, \ldots, \delta_m \in \{1, -1\}$.

By Theorem 2.3.0.2, either $\psi(r) \in R_2^*$, or some cyclically reduced conjugate of $\psi(r)$ contains either

1. two disjoint subwords, each of length $> \frac{3}{4}m$, or

2. four disjoint subwords, each of length $> \frac{1}{2}m$.

These last two cases are impossible, since they both imply that the length of $\psi(r)$ is greater than m.

We conclude that $\psi(r) \in R_2^*$.

Proof of Case 2: We again start with a lemma.

Lemma 3.2.0.2. *Let $\langle S \mid R\rangle$ be a scaffolded presentation, where the link of any vertex of X is a generalized n-gon, $n \geq 3$. Then $\langle S \mid R\rangle$ satisfies the $C(3)$ and $T(6)$ small cancellation conditions.*

Proof. Once again, because the Whitehead graph of $\langle S \mid R\rangle$ has no double edges, every piece of $\langle S \mid R\rangle$ has length 1. Since every element of R has length $m \geq 3$, it follows that $\langle S \mid R\rangle$ satisfies $C(3)$.

Since the shortest cycle in the Whitehead graph of $\langle S \mid R\rangle$ has length $2n \geq 6$, $\langle S \mid R\rangle$ is $T(6)$. □

Suppose that $G_1 = \langle S_1 \mid R_1\rangle, G_2 = \langle S_2 \mid R_2\rangle$, and $\psi : G_1 \to G_2$ are as described in the theorem's hypothesis, and let $r \in R_1$ be arbitrary. Write $r = y_1^{\epsilon_1} y_2^{\epsilon_2} \cdots y_m^{\epsilon_m}$ where $y_1, \dots, y_m \in S_1$ and $\epsilon_1, \dots, \epsilon_m \in \{1, -1\}$. Then $\psi(r)$ can be written

$$\psi(r) = z_1^{\delta_1} \cdots z_m^{\delta_m}$$

where $z_1, \dots, z_m \in S_2$ and $\delta_1, \dots, \delta_m \in \{1, -1\}$.

Denote $w = \psi(r)$. By Theorem 2.3.0.1, either $w \in R_2^*$, or some cyclically reduced conjugate w^* can be written

$$w^* = u_1 s_1 \cdots u_j s_j$$

where for every $k = 1, \dots, j$, there exists $r_k \in R_2^*$ such that

$$r_k = s_k b_1 \cdots b_{i(s_k)},$$

the $b_1 \dots, b_{i(s_k)}$ are pieces, and

$$\sum_{k=1}^{j} (2.5 - i(s_k)) \geq 3.$$

We claim that $w \in R_2^*$. Otherwise, since every element of R_2^* has length m, and all the pieces have length 1, we get

$$m = |r_k| = |s_k| + i(s_k)$$

for every $k = 1, \ldots, j$.

Since w^* has length m, we get

$$
\begin{aligned}
m &= |w^*| \\
&\geq \sum_{k=1}^{j} |s_k| \\
&= \sum_{k=1}^{j} (m - i(s_k)) \\
&\geq \sum_{k=1}^{j} (m - 2.5) + 3 \\
&= j(m - 2.5) + 3 \\
&\geq m + 0.5,
\end{aligned}
$$

since $j \geq 1$ and $m \geq 3$. This is clearly impossible.

We conclude that $w \in R_2^*$ in this case also. □

4. VERTEX-REGULAR COCOMPACT LATTICES OF $I_{5,5}$

4.1. A Description of the Method

In this section, we will apply the results of Chapter 3 to the simplest possible choice of Bourdon's building $I_{p,np}$, namely $I_{5,5}$. With this choice of p and n, Proposition 3.1.0.2 takes the following form:

Corollary 4.1.0.1. *Let G be a group. Then G is a torsion-free vertex-regular cocompact lattice of $I_{5,5}$ if and only if it admits a presentation $\langle S \mid R\rangle$ satisfying the following four properties:*

(i) S contains 5 elements,

(ii) R contains 5 elements,

(iii) every element of R is a word of length 5, and

(iv) the Whitehead graph of $\langle S \mid R\rangle$ is $K(5,5)$.

To ease notation and increase legibility, we will refer to one of these presentations by a 25-digit serial number xxxxx-xxxxx-xxxxx-xxxxx-xxxxx, composed of five subsequences of length five, each written using the digits 0 through 9. For $i = 0, \ldots, 4$, the digit i represents the generator x_i in the 5-element generating set S. For $i = 5, \ldots, 9$, the digit i represents the inverse generator x_{i-5}^{-1} in S^{-1}. Each length five subsequence represents a relator of the group.

For example, the serial number 00102-03484-08669-12189-22374 corresponds to the group generated by $x_0, \ldots, x_4$ with relators

$$x_0^2x_1x_0x_2,\ x_0x_3x_4x_3^{-1}x_4,\ x_0x_3^{-1}x_1^{-2}x_4^{-1},\ x_1x_2x_1x_3^{-1}x_4^{-1},\ x_2^2x_3x_2^{-1}x_4.$$

To simplify our search for presentations satisfying conditions (i) - (iv) of Corollary 4.1.0.1, we introduce the following proposition.

Proposition 4.1.0.1. *Let G be a group defined by a presentation $\langle S \mid R \rangle$, where S satisfies the properties (i) - (iv) of Corollary 4.1.0.1. Suppose that for every generator $x \in S$, the vertices x^+ and x^- in the Whitehead graph are oppositely colored. Then G admits a presentation $\langle S \mid R' \rangle$, where S and R' satisfy properties (i) - (iv) as well as the additional property*

(v) all the words in R' are positive.

Proof. First, note that replacing the generator x with x^{-1} will swap the coloring of x^- and x^+. By doing this repeatedly, we can guarantee that all the vertices of the form x^- are black and the vertices of the form x^+ are white. Thus, there are no cyclic subwords of the form xy^{-1} or $x^{-1}y$; there can only be those of the form $x^{-1}y^{-1}$ or xy. By replacing a relator with its inverse if necessary, we can then guarantee that every relator is positive. □

With this proposition in mind, we use the following algorithm to find presentations satisfying properties (i) - (iv) of Proposition 4.1.0.1.

(1) Generate all possible length 5 sequences from the digits 0 through 4.

(2) From this list, eliminate any cyclic redundances. For example, given sequences 00102 and 01020, we remove 01020.

(3) From this list, eliminate any sequences which will create double edges in the associated Whitehead graph. For example, we remove the sequence 00011 and the sequence 12012.

(4) From the remaining set of sequences, form all possible subsets of size 5. These subsets will form our serial numbers.

(5) Generate the corresponding Whitehead graph, and check for double edges.

(6) If the Whitehead graph has no double edges, it is automatically $K(5,5)$, and we keep the serial number.

We enact this algorithm using a program we have written with the computer software GAP (Groups, Algorithms, and Programming, [14]). As a result, we obtain the following.

Proposition 4.1.0.2. *There are 71,424 positive presentations $\langle S \mid R \rangle$ satisfying properties (i) - (v) of Proposition 4.1.0.1.*

Now we find the vertex-regular lattices of $I_{5,5}$ which do not admit positive presentations using the following algorithm.

(1) Generate all possible length 5 sequences from the digits 0 through 9.

(2) From this list, eliminate cyclic redundances.

(3) From this list, eliminate inverses. For example, given sequences 01234 and 98765, we remove 98765.

(4) From this list, eliminate sequences which represent relators that are not cyclically reduced, as these will create loops in the Whitehead graph. For example, we remove 01602.

(5) From this list, eliminate sequences which will result in double edges in the Whitehead graph. For example, we remove 00011 and 12012, as well as 01365.

(6) From the remaining set of sequences, form all possible subsets of length 5. These subsets will form our serial numbers.

(7) Generate the corresponding Whitehead graph, and check if it is isomorphic to $K(5,5)$.

(8) If it is isomorphic to $K(5,5)$, check if there is at least one generator x such that x^- and x^+ are equally colored.

(9) If there is at least one generator x such that x^- and x^+ are equally colored, we keep the serial number.

We also enact this algorithm in GAP, obtaining the following;

Proposition 4.1.0.3. *There are 21,156,862 presentations $\langle S \mid R\rangle$ satisfying properties (i) - (iv) of Corollary 4.1.0.1, as well as the additional property*

(v') there exists an element $x \in S$ such that x^+ and x^- have equal colors.

We remark that the algorithms we have presented up to this point can be adapted to find the positive and non-positive scaffolded presentations of the torsion-free vertex-regular cocompact lattices of $I_{p,q}$ for any choice of integers $p \geq 5$ and $q \geq 2$. If p does not divide q^2, no such lattice exists. If p divides q^2, the scaffolded presentations of these lattices can be represented by serial numbers with q^2 digits in base $(2q)$, consisting of $\frac{q^2}{p}$ subsequences of length p.

We now apply Theorem 3.2.0.2 to classify the groups defined by these 71,424 + 21,156,862 presentations up to isomorphism. To reduce computation time, we make use of the following two corollaries. Note that although Corollary 4.1.0.2 can be generalized to any $I_{p,q}$, Corollary 4.1.0.3 can only be generalized to $I_{5,5n}$.

Corollary 4.1.0.2. *Suppose $G_1 = \langle S_1 \mid R_1\rangle$ satisfies the properties (i) - (v) of Proposition 4.1.0.1, and $G_2 = \langle S_2 \mid R_2\rangle$ satisfies the properties (i) - (v') of Proposition 4.1.0.3. Then G_1 and G_2 are not isomorphic.*

Proof. Suppose by way of contradiction that G_1 and G_2 are isomorphic. By Theorem 3.2.0.2, there exists an isomorphism $\psi : G_1 \to G_2$ taking $S_1^{\pm}$ bijectively to $S_2^{\pm}$ and such that $\psi(r) \in R_2^*$ for all $r \in R_1^*$. Let $z \in S_2$ be such that z^+ and z^- are equally colored in the Whitehead graph of $\langle S_2 \mid R_2 \rangle$. Then there exists $y \in S_1^{\pm}$ such that $\psi(y) = z$. If $y \in S_1$, there exists a relator in R_1 containing the cyclic subword y^2; in this case, denote by r this relator. If $y \in S_1^-$, there exists a relator in R_1 containing the cyclic subword y^{-2}. In this case, denote by r the inverse of this relator.

Then $\psi(r) \in R_2^*$, and hence, an element of R_2 contains a cyclic subword equal to either z^2 or z^{-2}. In either case, there is an edge between z^+ and z^- in the Whitehead graph, contradicting assumption. □

Corollary 4.1.0.3. *Suppose $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ satisfy the properties (i) - (v) of Proposition 4.1.0.1. If G_1 and G_2 are isomorphic, then there is an isomorphism from G_1 to G_2 taking S_1 bijectively either to S_2 or to S_2^{-1}.*

Proof. Suppose that G_1 and G_2 are isomorphic. By Theorem 3.2.0.1, there is an isomorphism $\psi : G_1 \to G_2$ taking S_1 injectively into $S_2^{\pm}$. Suppose that for some generators $x, y \in S_1$ we have $\psi(x) \in S_2$ and $\psi(y) \in S_2^{-1}$.

Since the Whitehead graph of $\langle S_1 \mid R_1 \rangle$ is complete bipartite, there exists a relator containing xy as a cyclic subword. Without loss of generality, suppose this relator is written $xyz_1z_2z_3$ with $z_1, z_2, z_3 \in S_1$. Then $\psi(x)\psi(y)\psi(z_1)\psi(z_2)\psi(z_3)$ is product of 5 letters in $S_2^{\pm}$ which is trivial in G_2.

Note that since G_2 admits a presentation in which every relator is positive of length 5, there exists a group homomorphism $\lambda : G_2 \to Z_5$ from G_2 to the cyclic group of order 5 which takes each generator of G_2 to the sole generator of Z_5. Then $\lambda(\psi(x)\psi(y)) = 0$ in Z_5. At the same time, $\lambda(\psi(x)\psi(y)\psi(z_1)\psi(z_2)\psi(z_3)) = 0$ in Z_5. This implies $\lambda(\psi(z_1)\psi(z_2)\psi(z_3)) = 0$ in Z_5. But this is impossible, since no sum of three integers in $\{1, -1\}$ is congruent to 0 modulo 5. □

In light of Corollaries 4.1.0.2 and 4.1.0.3, we use the following algorithm to determine which of the presentations found define isomorphic groups:

For the presentations satisfying properties (i) - (v) of Proposition 4.1.0.1:

(1) For each of the 71,424 presentations found, consider all 240 possible injections from the set $\{0, 1, \ldots, 4\}$ into either the set $\{0, 1, \ldots, 4\}$ or the set $\{5, 6, \ldots, 9\}$.

(2) This map induces a map on serial numbers. If the image of one serial number under this map can be matched to another serial number after reordering, cyclically permuting, and/or inverting its subsequences, we identify the groups they define to the same isomorphism class.

For the presentations satisfying properties (i) - (v) of Proposition 4.1.0.3:

(1) For each of the 21,156,862 presentations found, consider all 3840 possible functions from the set $\{0, 1, \ldots, 4\}$ into the set $\{0, 1, \ldots, 9\}$ such that if $x, y \in \{0, 1, \ldots, 4\}$ are distinct, their images in $\{0, 1, \ldots, 9\}$ are not congruent modulo 5.

(2) This map induces a map on serial numbers. If the image of one serial number under this map can be matched to another serial number after reordering, cyclically permuting, and/or inverting its subsequences, we identify the groups they define to the same isomorphism class.

After applying this algorithm, we obtain the following.

Theorem 4.1.0.1. *Of the 71,424 presentations satisfying properties (i) - (v) of Proposition 4.1.0.1, exactly 308 define groups that are distinct up to isomorphism.*

Of the 21,156,802 presentations satisfying properties (i) - (v') of Proposition 4.1.0.3, exactly 8,574 define groups that are distinct up to isomorphism.

Hence, there are exactly 8,882 torsion-free vertex-regular cocompact lattices of $I_{5,5}$.

The complete list of torsion-free vertex-regular cocompact lattices we have obtained, together with the source code used to construct and classify those lattices, can be found in the data set [28]. We direct the reader to the data set's README file for more details.

4.2. Properties of the Groups Found

In this section, we briefly list and explain some properties of the 8,882 groups found in Theorem 4.1.0.1. To avoid introducing a large amount of terminology which is beyond the scope of this thesis, we direct the reader to the given citations for definitions of the terms below.

We first consider the properties possessed not only by the torsion-free vertex-regular cocompact lattices of $I_{5,5}$, but by any cocompact lattice of any 2-dimensional hyperbolic building. All such lattices are word-hyperbolic, since they act properly discontinuously, cocompactly, by isometries on a 2-dimensional hyperbolic building, which is a CAT(-1) space. For an introduction to word-hyperbolicity of groups, see [6], Part III. As a consequence, the groups have no Baumslag-Solitar subgroups, and the groups are not simple. The groups have trivial center and trivial Frattini subgroup ([20]). Moreover, the groups do not split non-trivially as a graph product of groups, since they are torsion-free with no $\mathbb{Z}^2$ subgroups ([16]).

To any word-hyperbolic group, we can associate a compact metric space called the *hyperbolic boundary.* The topological properties of the boundary inform the algebraic properties of the group. For an introduction to the hyperbolic boundary and its connections to the associated group, see the 2002 survey paper written by N. Benakli and I. Kapovich ([2]). The boundary of any cocompact lattice of a 2-dimensional hyperbolic building is 1-dimensional and connected ([12], Corollary A.9). As a consequence, the group is non-elementary, and therefore SQ-universal ([11], [23]). The group is not virtually free, but it contains the free group on two generators as a subgroup, and hence is not virtually solvable and has exponential growth. If the group is torsion-free, it has geometric and cohomological dimension 2 ([3]).

For any torsion-free vertex-regular cocompact lattice of a 2-dimensional hyperbolic building, the presentation 2-complex of the associated scaffolded presentation is an Eilenberg-Maclane space for the lattice, because the universal cover of this complex is contractible. It follows that the group is aspherical. If in particular the building is $I_{p,np}$, the group has Euler characteristic $1 - np + n^2p \geq 1$, since the Eilenberg-Maclane space has one 0-cell, np 1-cells, and n^2p 2-cells. As a consequence, the group is not amenable, and it is not a virtual 3-manifold group ([24]).

Next, we consider the properties possessed by cocompact lattices of one of Bourdon's buildings $I_{p,q}$, not necessarily torsion-free or vertex-regular. The hyperbolic boundary of any such group is homeomorphic to the Menger curve, a 1-dimensional, connected, locally connected, compact topological space with no local cut points ([12], Main Theorem). Therefore, the group is not a virtual surface group, since the boundary is not a circle. The group does not split as a free product with amalgamation over a finite or cyclic subgroup of infinite index in both factors ([5]). The group does not split as an HNN-extension over a finite subgroup with infinite index in the base group ([10], [15], [17]). If the group is torsion-free, it is hence both Hopf and co-Hopf ([26], [27]).

In the special case of the torsion-free vertex-regular cocompact lattices of $I_{5,5}$, none of the groups has Kazhdan's property (T), since at least one of the groups has infinite abelianization. For an introduction to Kazhdan's property (T), see the 2008 survey paper written by B. Bekka, P. de la Harpe, and A. Valette ([1]).

4.3. Cyclically Presented Groups

Our technique produces the four serial numbers

(1) 00121-02332-03044-11343-14224

(2) 00121-02203-04114-13443-23324

(3) 00102-03133-04434-11214-22423

(4) 00123-02114-03422-04431-13324

corresponding to groups which are isomorphic, respectively, to the groups with serial numbers:

(1) 00131-11242-22303-33414-44020

(2) 00141-11202-22313-33424-44030

(3) 00102-11213-22324-33430-44041

(4) 00143-11204-22310-33421-44032

via the permutations $2 \leftrightarrow 3$, $2 \leftrightarrow 4$, $3 \leftrightarrow 4$, and $2 \leftrightarrow 4$.

These latter serial numbers correspond to positive *cyclic presentations*. These are presentations of the form

$$\langle x_0, \ldots, x_n \mid r, \theta(r), \theta^2(r), \ldots, \theta^n(r) \rangle$$

where $\theta : F(\{x_0, \ldots, x_n\}) \to F(\{x_0, \ldots, x_n\})$ is the group homomorphism defined by $\theta(x_i) = x_{i+1}$ for all $i = 0, \ldots, n$, indices modulo $n + 1$.

Thus, we have obtained four length five positive cyclic presentations defining pairwise non-isomorphic aspherical groups.

5. CONCLUSIONS

5.1. Discussion of the Results

We begin by remarking that the main results of this paper can be used to complete the classification of the groups produced in [18]. Each of the positive triangle presentations presented in [18] satisfies the small cancellation conditions $C(3)$-$T(6)$.

The most critical aspect of the proofs of Theorems 3.2.0.1 and, by extension, 3.2.0.2 was the Mostow rigidity of Fuchsian buildings, the most general class of 2-dimensional hyperbolic buildings for which Mostow rigidity is known to hold. If the reader encounters a building which is not Fuchsian, but is Mostow rigid and in which every edge of the building has equal length, the conclusions of these theorems will still be true.

Indeed, Theorem 3.2.0.1 will remain true even for Mostow rigid polygonal complexes which are not buildings and which are not hyperbolic. If the reader encounters this case, we encourage them to study the number of sides of the 2-cells and the girth of the link at each vertex to see if their groups possess presentations satisfying the $C(3)$-$T(6)$ or $C'(1/4)$-$T(4)$ properties, or other small cancellation properties yielding similar spelling theorems. If so, the reader will be able to apply Theorem 3.2.0.2 as well. However, we do not know if any such polygonal complexes exist.

5.2. Directions for Future Work

The authors Kangaslampi and Vdovina, together with L. Carbone, went on to produce groups with torsion acting simply transitively on the vertex set of simplicial 2-dimensional hyperbolic buildings ([8]). We are interested to see if this method can be adapted to produce groups with torsion acting simply transitively on the vertex set of other hyperbolic buildings and, in particular, $I_{5,5}$. We are also interested to see if there is any generalization of Theorems 3.2.0.1 and 3.2.0.2 to the torsion-case, since our current proofs will not work in this situation, as the action on the building is no longer guaranteed to be free.

Since the groups we have found are hyperbolic, we are interested in methods which can be applied to these groups to study their residual finiteness. To this point, we have not found any technique or theorem which can be successfully adapted to our case.

Our groups also provide examples of hyperbolic groups whose boundary is nonempty and connected, i.e. they are one-ended. Thus, it may be worthwhile to investigate these groups for surface subgroups, in response to M. Gromov's famous open question: Does every one-ended hyperbolic group have a surface subgroup? A preliminary investigation was conducted by Kangaslampi and Vdovina in the case of a simplicial 2-dimensional hyperbolic building in [19].

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