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The problem of acceleration or speed-up of a convergent complex series $\Sigma a_n$, i.e., finding a series $\Sigma b_n$ which converges more rapidly than a given series $\Sigma a_n$, and which has the same sum, has occupied the interest of various mathematicians, dating back at least to E.E. Kummer in 1837. In many cases, only real series have been considered; in particular, series of positive terms or alternating series.

To the author's knowledge, there is no basic treatment of this subject in the literature to date, and it is hoped that this paper will serve, at least as a beginning, to fill this gap. Such an exposition should present some of the methods in some type of unified setting and, at
the same time, bring new information to light. The author believes that both of these objectives have been "partially" fulfilled, while presenting a more or less self-contained introduction to some of the aspects of speed-up.
ERROR ANALYSIS, CONVERGENCE, DIVERGENCE, AND THE ACCELERATION OF CONVERGENCE

by

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CHAPTER I

INTRODUCTION

Given a complex series \( \sum_{n=0}^{\infty} a_n \), we shall write \( \sum_{n}^{\infty} a_n \) for \( \sum_{n=0}^{\infty} a_n \), and, if \( \sum_{n}^{\infty} a_n \) converges, \( S = \sum_{n}^{\infty} a_n \).

Similarly, if \( \sum_{n}^{\infty} a_n \) converges, then \( S' = \sum_{n}^{\infty} a_n \). Given two convergent series \( \sum_{n}^{\infty} a_n \) and \( \sum_{n}^{\infty} a_n \), the latter is said to converge more rapidly than the former iff

\[
\frac{(S' - S')}{(S - S')} \to 0 \quad \text{as} \quad n \to \infty.
\]

If \( \sum_{n}^{\infty} a_n \) converges,

"MR(\( \sum_{n}^{\infty} a_n \))" will denote the class of all series \( \sum_{n}^{\infty} b_n \) which converge more rapidly to \( S \) than \( \sum_{n}^{\infty} a_n \), i.e.,

\( \sum_{n}^{\infty} b_n \in \text{MR}(\sum_{n}^{\infty} a_n) \) iff \( \sum_{n}^{\infty} b_n \) converges more rapidly to \( S \) than \( \sum_{n}^{\infty} a_n \). The concept of "acceleration" or "speed-up" can now be defined as the problem of finding a series \( \sum_{n}^{\infty} b_n \) such that \( \sum_{n}^{\infty} b_n \in \text{MR}(\sum_{n}^{\infty} a_n) \). We will say that \( \sum_{n}^{\infty} b_n \) converges with the same rapidity as \( \sum_{n}^{\infty} a_n \) iff there are numbers \( A \) and \( B \) such that \( 0 < A < \frac{|S' - S'|}{|S - S'|} < B \).

The notation "<." means that \( < \) holds for all sufficiently large \( n \). If "*" denotes any relation, "*."
will be used in the same manner, while "*:" means that *
holds for infinitely many positive integers n. Simi-
larly, \( f(x) \leq g(x) \) iff \( f(x) \leq g(x) \) for all suffi-
ciently large values of the real variable x.

Various methods, found in the literature, for ob-
taining a series \( \Sigma a_n \in \text{MR}(\Sigma a_n) \) may be summarized as fol-
lows. A sequence \( \{b_n\} \) is proposed, and then the partial
sums \( S_n' \) are specified by the equation \( S_n' = S_n + b_{n+1} \)
for \( n \geq 0 \). It is immediate that \( a_n' = a_0 + b_1 \), and

\[
a'_n = a_n + b_{n+1} - b_n \quad \text{for} \quad n \geq 1.
\]

It seems somewhat advantageous to set \( b_n = a_n a_n \)
for \( n \geq 1 \), and specify the "transform sequence" \( \{a_n\} \).
In doing so, we set \( S_{a_n} = S_n + a_{n+1} a_{n+1} \) for \( n \geq 0 \),
\( a_{a_0} = S_{a_0} = a_0 + a_1 a_1 \), and \( a_{a_n} = S_{a_n} - S_{a(n-1)} \)

\[
= a_n + a_{n+1} a_{n+1} - a_n a_n \quad \text{for} \quad n \geq 1.
\]

It follows that if \( \Sigma a_n \) converges, and \( a_n = 0 \) or \( a_n = 0 \), then
\( S_{a_n} = S_n \), and thus \( \Sigma a_{a_n} \notin \text{MR}(\Sigma a_n) \). Consequently, we
shall usually consider only series \( \Sigma a_n \) for which
\( a_n \neq 0 \). If \( \Sigma a_{a_n} \) converges, its sum will be denoted by
\( S_a \).

Suppose that \( \Sigma a_n \) converges and \( a_n \neq 0 \) for \( n \geq 0 \).
The optimal choice of \( \{a_n\} \) for acceleration should yield \( S_{a_n} = S \) for \( n \geq 0 \). Thus \( S_n + a_{n+1}a_{n+1} = S \) and we must have \( a_{n+1} = (S-S_n)/a_{n+1} \) for \( n \geq 0 \). We easily verify that
\[
S_{an} = S_n + a_{n+1}a_{n+1} = S_n + a_{n+1}(S-S_n)/a_{n+1} = S
\]
for \( n \geq 0 \), with \( a_n = (S-S_{n-1})/a_n \) for \( n \geq 1 \). Hence this transform sequence is the "exact" solution to our problem of speed-up. In general we must satisfy ourselves with an approximation to this solution. We now turn to some of these "approximations".

For each \( n \) such that \( a_{n-1} \neq 0 \) we write \( r_n = a_n/a_{n-1} \). The notation \( Q_n = n(1-r_n) \), \( Q = \lim Q_n \), and \( r = \lim r_n \) of Lubkin (17, p. 228-229) will be used (Lubkin uses "\( R^n \) in place of our "\( r^n \)").

Aitken's \( \delta^2 \)-process will be treated in detail in this paper and can be obtained by defining its transform sequence \( \{\delta_n\} \) as follows:

1.1 \( \delta_n = 1/(1-r_n) \) if \( r_n \neq 1 \), \( \delta_n = 0 \) otherwise.

The notation in 1.1 will be adhered to throughout this paper. Various other processes considered in this paper can be described by defining their corresponding transform sequence. We enumerate some of them as follows:
1.2 \( a_n = 1/(1-r) \).

1.3 \( a_n = (1-r_{n-1})/(1-2r_n + r_{n-1}r_n) \) for \( n \geq 2, \ a_1 = -1/r_1 \).

1.4 \( a_n = n/(Q-1) \).

1.5 \( a_n = Q/(Q-1)(1-r_n) = nQ/(Q-1)Q_n = Q\delta_n/(Q-1) \).

1.6 \( a_n = s/(s-1)(1-r_n), \ s = \lim a_n/a_{\delta n} \).

Among publications in which 1.1 is found are the following: Aitken (1,p.301), Forsythe (11, p. 310), Hartree (12, p. 233), Householder (13, p. 117), Isakson (14, p. 443), Lubkin (17, p. 228), Pflanz (18, p. 27), Samuelson (20, p. 131), Schmidt (21, p. 376), Shanks (23, p. 233), Todd (28, p. 5, 86, 115, 187, 197, 260). We find 1.2 in Lubkin (17, p. 232), Shanks (22, p. 39) and (23, p. 25-26); 1.3 in Lubkin (17, p. 229); 1.4 in Szász (26, p. 274); 1.5 in Lubkin (17, p. 232), Pflanz (18, p. 25); 1.6 in Shanks (23, p. 39).

Lubkin calls \( \Sigma a_{\delta n} \) the T transformation, \( \Sigma a_{\alpha n} \) of 1.2 the Ratio transformation, and \( \Sigma a_{\alpha n} \) of 1.3 the W transformation. The transformation defined by 1.5 is found in Lubkin's Theorem 8 (17, p. 232). Daniel Shanks calls \( \Sigma a_{\alpha n} \) of 1.6 the \( e_{1}^{(s)} \) transformation.

The author suggests the use of the following transform sequences for acceleration.
1.7 \( \alpha_n = \frac{n+a}{Q-1} \), \( n \) some complex number.

1.8 \( \alpha_n = \frac{n+a}{Q_n-1} \), \( n \) some complex number.

The sequence 1.7 reduces to 1.4, if \( a = 0 \). A method for determining the most appropriate value for \( a \) in 1.7 will be indicated by an example at the end of Chapter V. The sequence 1.8, with \( a = 0 \), is suggested for application to power series \( \sum a_n \) where

\[ a_n = b_n z^n \] \( n \geq 0 \).

Given any sequence \( \{x_n\} \) we define, for every \( n \),

\[ \Delta x_n = x_{n+1} - x_n \quad \text{and} \quad \Delta^2 x_n = \Delta(\Delta x_n) = \Delta x_{n+1} - \Delta x_n \]

\[ = x_{n+2} - 2x_{n+1} + x_n. \] No use will be made of the higher order differences \( \Delta^k x_n, \ k \geq 3 \).

Aitken's \( \delta^2 \)-process can be formulated in various ways. In particular, assuming that division by zero is excluded, we have:

1.9 \( S_{\delta n} = S_n + \alpha_{n+1} \delta_{n+1} = S_n + \alpha_{n+1}/(1 - r_{n+1}), \ n \geq 0 \).

1.10 \( S_{\delta n} = (S_{n-1} S_{n+1} - S_{n}^2)/(S_{n-1} - 2S_n + S_{n+1}), \ n \geq 1 \).

1.11 \( S_{\delta n} = \frac{S_{n-1} S_{n}}{\Delta S_{n-1} \Delta S_{n}} \cdot \frac{1}{\Delta S_{n-1} \Delta S_{n}} \), \( n \geq 1 \).

1.12 \( S_{\delta n} = S_{n-1} - (\Delta S_{n-1})^2/\Delta^2 S_{n-1}, \ n \geq 1 \).

1.13 \( S_{\delta n} = S_n - (\Delta S_{n-1} \Delta S_{n})/\Delta^2 S_{n-1}, \ n \geq 1 \).
1.14 \[ S_{6n} = S_{n+1} - \frac{(\Delta S_n)^2}{\Delta^2 S_{n-1}}, \quad n \geq 1. \]

Moreover, if we define \[ F(x, y, z) = \frac{xz-y^2}{x-2y+z}, \]
and \[ x-2y+z \neq 0, \]
we have \[ F(x+a, y+a, z+a) = a + F(x, y, z), \]
for every \( a \), and 1.10 becomes,

1.15 \[ S_{6n} = F(S_{n-1}, S_n, S_{n+1}), \quad n \geq 1. \]

The function \( F \) also satisfies \[ F(c, x, cy, cz) = cF(x, y, z). \]

We see that these two properties of \( F \) may be of some use in actual numerical calculations. For example, suppose that \( S_1 = 15.001418373, \ S_2 = 15.000304169, \)
and \( S_3 = 15.000065221. \)

Then, \( S_{62} = F(S_1, S_2, S_3) = 15.000065221 \]
\[ + 10^{-9} F(1353152, 238948, 0) = 15.000065221 \]
\[ + (10^{-9})[-(238948)^2]/[1353152-2(238948)-0] = etc. \]

The \( \delta^2 \)-process has the following geometrical interpretation. Suppose that \( S_n \to S, \) so that

\( (S_n, S_{n+1}) \to (S, S). \)

The points \( (S, S) \) and \( (S_n, S_{n+1}), \)
\( n \geq 0, \) are graphed. The straight line through two successive points \( (S_{n-1}, S_n) \) and \( (S_n, S_{n+1}) \) is intersected with the line \( y = x. \)

Denoting this point of intersection by \( (S_{\delta n}, S_{\delta n}) \) yields Aitken's \( \delta^2 \)-process. This interpretation is found in Todd (28, p. 260), but no mention is made of the \( \delta^2 \)-process there. Also, Todd (28, p. 5) credits the \( \delta^2 \)-process to Kummer (16, p. 206-214).
Returning to the exact solution for speed-up

\[ a_n = \frac{(S-S_{n-1})}{a_n}, \quad n \geq 1, \]
we have

\[ a_n = \frac{(a_n + (S-S_n))}{a_n} = 1 + \frac{(S-S_n)}{a_n} = 1 + T_{n+1}, \]
if we set \( T_{n+1} = \frac{(S-S_n)}{a_n} \)
for \( n \geq 1 \). Hence \( 1 + T_{n+1}, \ n \geq 1, \) is the exact solution.

Suppose that \( \sum a_n \) converges and \( n \) is any integer \( \geq 1 \) such that \( a_{n-1} \neq 0 \). We then formally define

\[ T_n = \frac{(S-S_{n-1})}{a_{n-1}}. \]

Various relations are satisfied by the quantities \( T_n \), some of which we now state and prove:

1.17 \( T_n = r_n(1+T_{n+1}), \) if \( a_{n-1}a_n \neq 0 \).

1.18 \((1-r_n)(1+T_{n+1}) = 1 + T_{n+1} - T_n, \) if \( a_{n-1}a_n \neq 0 \).

1.19 \[ [(1-r_n)/a_n](S-S_{n-1}) = 1 + T_{n+1} - T_n, \] if \( a_{n-1}a_n \neq 0 \).

1.20 \( T_{n+1} = \frac{r_n}{1-r_n} + \frac{(T_{n+1} - T_n)/(1-r_n)}{1-r_n}, \) if \( r_n \neq 0 \) or \( 1 \).

1.21 \( T_n = r_n + r_n r_{n+1} + \cdots + (r_n r_{n+1} \cdots r_{n+k}) + \cdots, \) if \( a_m \neq 0 \) for \( m \geq n-1 \).

For 1.17, \( T_n = \frac{(S-S_{n-1})}{a_{n-1}} = \frac{(a_n + S-S_n)}{a_{n-1}} = \frac{a_n}{a_{n-1}} + \frac{(a_n/a_{n-1})[(S-S_n)/a_n]}{a_{n-1}} = r_n + \frac{r_n T_{n+1}}{1-r_n} = r_n (1+T_{n+1}). \) Thus, \( (1-r_n)(1+T_{n+1}) = 1 + T_{n+1} - r_n (1+T_{n+1}) = 1 + T_{n+1} - T_n, \) i.e., 1.18
holds. Consequently, \[ ((1-r_n)/a_n)(S - S_{n-1}) \]
\[ = (1-r_n)[(S - S_{n-1})/a_n] = (1-r_n)(T_n/r_n) = (1-r_n)(1+T_{n+1}) \]
\[ = 1/T_{n+1} - T_n, \] and thus 1.19 holds. From 1.18, \( 1+T_{n+1} \)
\[ = 1/(1-r_n) + (T_{n+1} - T_n)/(1-r_n), \] so that \( T_{n+1} \)
\[ = 1/(1-r_n) - 1 + (T_{n+1} - T_n)/(1-r_n) = r_n/(1-r_n) \]
\[ + (T_{n+1} - T_n)/(1-r_n), \] i.e., 1.20 holds. Finally,
\[ T_n = (S - S_{n-1})/a_{n-1} = (a_n + a_{n+1} + \cdots + a_{n+k} + \cdots)/a_{n-1} \]
\[ = a_n/a_{n-1} + a_{n+1}/a_{n-1} + \cdots + a_{n+k}/a_{n-1} + \cdots = a_n/a_{n-1} \]
\[ + (a_n a_{n+1})/(a_{n-1} a_n) + \cdots + (a_n a_{n+1} \cdots a_{n+k})/(a_{n-1} a_{n-2} \cdots a_{n+k-1}) \]
\[ + \cdots = r_n r_{n+1} + \cdots + (r_n r_{n+1} \cdots r_{n+k}) + \cdots, \] i.e., 1.21 holds.

Given a series \( \Sigma a_n \), not necessarily convergent,
we define
\[ 1.22 \quad T_{n,k} = (S_{n+k} - S_{n-1})/a_{n-1}, \] for \( k \geq -1 \) and \( a_{n-1} \neq 0 \).

We note that \( T_{n,-1} = 0 \). Also, if \( k \) is any integer \( \geq 0 \),
and \( n \) is any integer such that \( a_m \neq 0 \) for \( n - 1 \leq m \leq n + k \), then
\[ 1.23 \quad T_{n,k} = r_n + r_n r_{n+1} + \cdots + (r_n r_{n+1} \cdots r_{n+k}). \]

We also define \( a_n \sim \beta_n \) iff \( a_n/\beta_n \to 1 \) as \( n \to \infty \).
The abbreviation "n.a.s.c." is used both for "necessary and sufficient condition" and "necessary and sufficient conditions."

Instead of a convergent series \( \Sigma a_n \), one may desire
to accelerate the convergence of a sequence of complex numbers $S_n$. We then set $S_{an} = S_n + a_n a_{n+1}$, where $a_n = \Delta S_{n-1} = S_n - S_{n-1}$, $r_n = a_n / a_{n-1}$, and $\{a_n\}$ is a prescribed transform sequence. If $s = \lim S_n$, we require that $(S - S_{an})/(S - S_n) \to 0$ in order that $\{S_{an}\}$ converge more rapidly to $S$ than $\{S_n\}$. Thus we may view acceleration from either the series or sequential viewpoint. They are clearly one and the same thing.
CHAPTER II

ACCELERATION, RAPIDITY OF CONVERGENCE, AITKEN'S δ²-PROCESS, AND DIVERGENCE

All series in this chapter are assumed complex unless explicitly stated to the contrary.

Theorem 2.1. The conditions (1) \( r_n \to 0 \), (2) \( T_n \to 0 \), and (3) \( T_n/r_n \to 1 \) are equivalent.

Proof: If \( T_n \to 0 \), then \( a_n \neq 0 \) so that
\[
r_n = T_n/(1+T_{n+1}) \to 0.
\]
Conversely, assume that \( r_n \to 0 \).

Let \( 0 < \varepsilon < 1 \). Then \( |r_n| \leq \varepsilon \), so that
\[
|T_n| = |r_n + r_n r_{n+1} + \cdots| \leq |r_n| + |r_n||r_{n+1}| + \cdots \leq \varepsilon/(1-\varepsilon)
\]
and thus \( T_n \to 0 \).

If \( T_n \to 0 \), then \( T_n/r_n = 1+T_{n+1} \to 1 \). Conversely, if \( T_n/r_n \to 1 \), then \( T_{n+1} = T_n/r_n - 1 \to 0 \).

Q.E.D.

Theorem 2.2. If \( T_n \to t \) for some complex number \( t \), then:

1. \( r = t/(1+t) \), \( |r| \leq 1 \), and \( r \neq 1 \).
2. \( t = r/(1-r) \) and \(-\frac{1}{2} \leq \text{Re}t\).

If, in addition, \( \{a_n\} \) is a sequence of complex numbers
such that \( \alpha_n \to \alpha_0 \) for some complex number \( \alpha_0 \), then:

1. \( S_\alpha = S \).
2. \( \sum a_n \in \text{MR}(\sum a_n) \) if and only if \( \alpha_0 = 1/(1-r) \).
3. \( \sum a_n \) converges with the same rapidity as \( \sum a_n \)
   if and only if \( \alpha_0 \neq 1/(1-r) \).

Proof: Since \( \{T_n\} \) converges and \( T_n = r_n(1+T_{n+1}) \),
\( T_n \neq 0 \) and \( T_n \neq -1 \). Consequently \( t \neq -1 \), since
otherwise \( |r_n| = |T_n/(1+T_{n+1})| \to +\infty \), which is impossible
since \( a_n \to 0 \). Thus, \( r_n = T_n/(1+T_{n+1}) \to t/(1+t) \), i.e.,
\( r = t/(1+t) \neq 1 \). Clearly, \( |r| \leq 1 \) so that (1) holds.
From (1), \( t = r/(1-r) \) and \( |t|/|(-1)-t| = |t/(1+t)| \)
\( = |r| \leq 1 \). Thus, \( |t| \leq |(-1)-t| \), which is equivalent
to \(-\frac{1}{2} \leq \text{Re } t \), so that (2) holds. (3) holds since
\( S_\alpha = S + a_n + a_{n+1} \to S + 0 \alpha_0 = S \). Since \( T_n \neq 0 \), we
have \( (S-S_{n-1}) \neq 0 \). If \( t = 0 \), then \( r_n/T_n \to 1 = 1-r \),
according to (1), (2) and Theorem 2.1. If \( t \neq 0 \), then
\( r_n/T_n \to r/t = (1-r) \) from (1) and (2). In either case,
\[(S-S_\alpha)/(S-S_n) = [S-(S_n+a_n+a_{n+1})]/(S-S_n) \]
\( = 1-a_{n+1} a_{n+1}/(S-S_n) = 1-a_{n+1} r_{n+1}/T_{n+1} \to 1-\alpha_0(1-r) \).
Hence, (4) and (5) hold, since \( 1-\alpha_0(1-r) = 0 \) is equi-
valent to \( \alpha_0 = 1/(1-r) \). Q.E.D.
Corollary 2.3. If \( \{T_n\} \) converges, then \( \Sigma a_n \in \text{MR}(\Sigma a_n) \).

Proof: Suppose \( T_n \to t \). From (1) of Theorem 2.2, 
\( r_n \to r \) where \( r \neq 1 \). Thus \( \delta_n = \frac{1}{1-r_n} \to \frac{1}{1-r} \), 
so that \( \Sigma a_n \in \text{MR}(\Sigma a_n) \) according to (4) of Theorem 2.2. 
Q.E.D.

We inquire if the convergence of \( \{T_n\} \) is also 
necessary for \( \Sigma a_n \in \text{MR}(\Sigma a_n) \). In the following chapter, 
we shall see that the answer is in the negative. There it 
will be proven that \( \Sigma a_n \in \text{MR}(\Sigma a_n) \) if and only if 
\( T_{n+1} - T_n \to 0 \).

Theorem 2.4. If \( \Sigma a_n \) and \( \Sigma a_\delta n \) are convergent real 
series, then \( S = S_\delta \).

Proof: Assume that \( S \neq S_\delta \). Since \( a_n \delta_n = S_\delta (n-1) - S(n-1) \) 
\to \( S_\delta - S \neq 0 \), \( \delta_n \neq 0 \) and \( a_n/(1-r_n) = a_n \delta_n \to S_\delta - S \neq 0 \). 
Thus \( a_n \to 0 \) implies that \( 1-r_n \to 0 \), i.e., \( r_n \to r = 1 \) so 
that \( 0 < r_n \) and \( 0 < T_n \). From \( 1+T_{n+1} - T_n \)

\[ \frac{[1-r_n]}{a_n} (S - S_{n-1}) \to 0, \]  
we have \( 1+T_{n+1} - T_n < \frac{1}{2} \) and 
\( 0 < T_{n+1} < T_n \), which implies that \( \{T_n\} \) converges. 
From (1) of Theorem 2.2, \( r \neq 1 \), which contradicts \( r = 1 \).
Thus our assumption is false, and \( S = S_0 \). Q.E.D.

Lubkin (17, p. 230) gave the first published proof of Theorem 2.4 for real series. The proof of this theorem for the complex case is given in Theorem 2.6, and to the author's knowledge is the first such proof.

**Theorem 2.5.** If \( (1-r_n)/a_n \to L \neq 0 \), then \( \Sigma a_n \) diverges.

**Proof:** Assume that \( \Sigma a_n \) converges. We may suppose that

\( L = 1-i \); since otherwise \( \Sigma a'_n \) converges where

\[
    a'_n = a_n L/(1-i) \quad \text{and} \quad (1-r'/a'_n) = (1-r_n)/(a_n L/(1-i)) \to 1-i.
\]

Accordingly, \( (1-r_n)/a_n = ((Re a_n)/|a_n|^2 - (Re a_{n-1})/|a_{n-1}|^2) + i[(Im a_{n-1})/|a_{n-1}|^2 - (Im a_n)/|a_n|^2] \to 1-i.\) Consequently, \( (Re a_{n-1})/|a_{n-1}|^2 < (Re a_n)/|a_n|^2 \) so that \( (Re a_n)/|a_n|^2 \to L_1 \) for some \( L_1 \leq +\infty \). If \( L_1 < +\infty \), then \( Re [(1-r_n)/a_n] \to L_1 - L_1 = 0 \), which is impossible since \( Re [(1-r_n)/a_n] \to 1 \). Thus \( L_1 = +\infty \) and \( 0 < Re a_n \).

Similarly, \( (Im a_{n-1})/|a_{n-1}|^2 < (Im a_n)/|a_n|^2 \) and \( 0 < Im a_n \). Hence setting \( a_n = |a_n|e^{i\theta_n} \) we may chose \( \theta_n \) such that \( 0 < \theta_n < \pi/2 \). From

\[
    T_n = a_n/a_{n-1} + a_{n+1}/a_{n-1} + \cdots + a_{n+k}/a_{n-1} + \cdots
\]

\[
    = |a_n/a_{n-1}|e^{i(\theta_n - \theta_{n-1})} + |a_{n+1}/a_{n-1}|e^{i(\theta_n - \theta_{n-1})} + \cdots
\]
\[= \left| a_n \right| \cos (\theta_n - \theta_{n-1}) + \ldots + \left| a_{n+k} \right| \cos (\theta_{n+k} - \theta_{n-1}) \]
\[+ \ldots \right| a_{n-1} \right| + (\text{Im} T_n)i\]

and \(0 < \theta_n < \pi/2\), we have \(0 < \text{Re} T_n\). Since
\[1 + T_{n+1} - T_n = \left[(1-r_n)/a_n\right](S - S_{n-1}) \to 0,\]
we have
\[1 + \text{Re} T_{n+1} - \text{Re} T_n = \text{Re} (1 + T_{n+1} - T_n) \to 0.\]
Thus \(\text{Re} T_{n+1} - \text{Re} T_n < -\frac{1}{2}\) for \(n \geq N\), where \(N\) is some positive integer. Consequently,
\[\text{Re} T_{N+n} = \text{Re} T_N + \sum_{i=1}^{n} \text{Re}[T_{N+i} - T_{N+i-1}] < \text{Re} T_N - \frac{n}{2} \to -\infty\]
as \(n \to \infty\). Hence, \(\text{Re} T_n < 0\) which contradicts \(0 < \text{Re} T_n\). Consequently our initial assumption cannot hold, i.e., \(\Sigma a_n\) must diverge. Q.E.D.

**Theorem 2.6.** If \(\Sigma a_n\) and \(\Sigma b_n\) both converge, then \(S = S_\delta\).

**Proof:** Assume that \(S \neq S_\delta\). Then \(a_n b_n = S_\delta (n-1) S_{n-1} \to S_\delta - S \neq 0\) so that \(b_n \neq 0\) and \(a_n/(1-r_n)\)
\[= a_n b_n - S_\delta - S \neq 0.\]
Thus \((1-r_n)/a_n \to 1/(S_\delta - S) \neq 0\), which implies, in view of Theorem 2.5, that \(\Sigma a_n\) diverges, a contradiction. Therefore our assumption cannot hold, i.e., \(S = S_\delta\). Q.E.D.
It should be kept in mind throughout the remainder of this paper that, according to the preceding theorem, the statements "\(\sum_{\delta} a_n \in MR(\Sigma a_n)\)" and "\(\Sigma a_{\delta n}\) converges more rapidly than \(\Sigma a_n\)" are equivalent.

Lemma 2.7. Suppose that \(\Sigma a_n\) is a convergent series, \(a_n \neq 0\), and \(c_n = c + S_n - S\) for \(n \geq 0\) where \(c\) is some complex number. Then,

\[
1 + c \left(\frac{1-r^n}{a_n} + \frac{c_{n-1}}{a_{n-1}} - \frac{c_n}{a_n}\right) = \frac{1-r^n}{a_n} (S - S_{n-1}).
\]

Proof: We have

\[
1 + c \left(\frac{1-r^n}{a_n} + \frac{c_{n-1}}{a_{n-1}} - \frac{c_n}{a_n}\right) = 1 + c \left(\frac{1}{a_n}\right) \frac{c + S_{n-1} - S}{a_{n-1}}
\]

\[
- \frac{c + S_{n-1} - S}{a_n} = 1 + \frac{S - S_n}{a_n} - \frac{S - S_{n-1}}{a_{n-1}} = \frac{S - S_{n-1}}{a_n} - \frac{S - S_{n-1}}{a_{n-1}}
\]

\[
= (\frac{1}{a_n} - \frac{1}{a_{n-1}}) (S - S_{n-1}) = (\frac{1-r^n}{a_n}) (S - S_{n-1}). \quad Q.E.D.
\]

Theorem 2.8. If \(\{(1-r^n)/a_n\}\) is bounded, then the complex series \(\Sigma a_n\) diverges.

Proof: Assume that \(\Sigma a_n\) converges. Since \(\{(1-r^n)/a_n\}\) is bounded, there is an \(\epsilon > 0\) such that \(\mid \epsilon (1-r^n)/a_n \mid < 1/4\). Let \(c\) be any complex number satisfying \(\mid c \mid = \epsilon\) so that
(1) \(-\Re c(1-r_n)/a_n < \frac{1}{4}\).

Setting \(c_n = c + S_n - S\), for \(n \geq 0\), we have \(c_n \to c\).

From Lemma 2.7,

\[
\Re \left[ 1 + c \left( \frac{1-r_n}{a_n} \right) + \frac{c_{n-1}}{a_{n-1}} - \frac{c_n}{a_n} \right] = \Re \frac{1-r_n}{a_n}(S-S_{n-1}) \to 0
\]

and thus,

(2) \(1 + \Re c \left( \frac{1-r_n}{a_n} \right) + \Re \frac{c_{n-1}}{a_{n-1}} - \Re \frac{c_n}{a_n} < \frac{1}{4}\).

Using (1) and (2),

\[
\frac{1}{2} + \Re \frac{c_{n-1}}{a_{n-1}} < \Re \frac{c_n}{a_n} - \Re c \left( \frac{1-r_n}{a_n} \right) - \frac{1}{4} < \Re \frac{c_n}{a_n},
\]

from which it is easily seen that \(\Re c_n/a_n \to +\infty\) and \(\Re c_n/a_n > 0\). Since \(\Re c_n/a_n > 0\) and \(c_n \to c\), we conclude that

(3) \(a_n \notin \{z: \arg c + 3\pi/4 \leq \arg z \leq \arg c + 5\pi/4\}\).

Choosing \(\arg c\) successively in (3) as \(0, \pi/2, \pi,\) and \(3\pi/2\), we conclude that \(a_n\) is not in the complex plane for large \(n\), which is absurd. Hence, our initial assumption cannot hold, i.e., \(\sum a_n\) must diverge. Q.E.D.

For the series \(\sum a_n\) where \(a_n = 1/\ln n\) for \(n \geq 2\), we have \((1-r_n)/a_n = 1/a_n - 1/a_{n-1} = \ln n - \ln(n-1) \to 0\) so that, from Theorem 2.8, \(\sum a_n\) diverges. Similarly, with \(a_n = 1/(n+1)\) for \(n \geq 0\), we
have \( \frac{1}{a_n} - \frac{1}{a_{n-1}} = (n+1) - n = 1 \) for \( n \geq 1 \), and thus \( \sum a_n \) diverges. For the divergent series \( \sum a_n \) where \( a_n = \frac{1}{n \ln n} \), we have

\[
\frac{1}{a_n} - \frac{1}{a_{n-1}} = n \ln n - (n-1) \ln (n-1) = (n-1)[\ln n - \ln(n-1)] + \ln n \to \infty,
\]

so that Theorem 2.8 is not applicable, and thus appears to be a very limited criterion for divergence.

**Theorem 2.9.** If \( \Sigma a_n \) is a convergent series, then some subsequence of \( \{S_{\delta n}\} \) converges to \( S \).

**Proof:** Suppose \( \Sigma a_n \) is convergent and assume that no subsequence of \( \{S_{\delta n}\} \) converges to \( S \). Since \( S_{\delta n} - S_n = a_{n+1} \delta_{n+1} \), our assumption holds if and only if no subsequence of \( \{a_{\delta n}\} \) converges to zero, and this is equivalent to \( |a_{\delta n}| > B \) for some \( B > 0 \). Thus

\[
|(1-r_n)/a_n| = 1/|a_{\delta n}| < 1/B.
\]

From Theorem 2.8, \( \Sigma a_n \) diverges, a contradiction. Therefore our assumption cannot be true, i.e., some subsequence of \( \{S_{\delta n}\} \) converges to \( S \). Q.E.D.

Theorem 2.9 clearly yields a second proof of Theorem 2.6.

**Example 2.10.** It is not necessarily true that if \( \Sigma a_n \) converges, \( \Sigma a_{\delta n} \) will also converge. In particular,
Lubkin (1, p. 240) considers the series \( \sum a_n = 1 + 1/2 - 1/3 - 1/4 + 1/5 + 1/6 - 1/7 - 1/8 + 1/9 + \cdots \) which converges while \( \sum a_{\delta n} \) diverges. However, according to Theorem 2.9 some subsequence of \( \{S_{\delta n}\} \) must converge to \( S \). Hence, of course, this is evident since \( r_n < 0 \) and
\[
S_{\delta n} = S_n + a_{n+1}/(1-r_{n+1}).
\]
This particular series shows that the \( \delta^2 \)-process is not regular.

**Example 2.11.** Lubkin (17, p. 240) also shows that the series
\[
\sum a_n = 1 + 1/(1+1) + 1/2^2 + 2^2/(2^4+1) + 1/3^2 + 3^2/(3^4+1) + \cdots
\]
converges while \( \sum a_{\delta n} \) diverges. Again, according to Theorem 2.9, some subsequence of \( \{S_{\delta n}\} \) must converge to \( S \). This is not so obvious by inspection as was the case in Example 2.10.

**Theorem 2.12.** If \( \sum a_n \) is a series such that \( \sum a_{\delta n} \) is properly divergent, i.e., \( |S_{\delta n}| \to \infty \), as \( n \to \infty \), then \( \sum a_n \) diverges.

**Proof:** Assume that \( \sum a_n \) is convergent. From Theorem 2.9 some subsequence of \( \{S_{\delta n}\} \) converges to \( S \), so that
\[
|S_{\delta n}| \not\to \infty \text{ as } n \to \infty, \text{ i.e., } \sum a_{\delta n} \text{ is not properly divergent.} \quad \text{Q.E.D.}
\]
Theorem 2.13. A n.a.s.c. that \( \{T_n\} \) converge is that
\[ r_n \to r \neq 1 \text{ and } T_{n+1} - T_n \to 0. \]

Proof: The necessity follows from (1) of Theorem 2.2 and the fact that \( \{T_n\} \) converges implies that \( T_{n+1} - T_n \to 0. \)

For the sufficiency, \( r \neq 1 \) implies that
\[ r_n (1-r_n) \neq 0. \]
From 1.20, \( T_{n+1} = r_n/(1-r_n) \)
+ \( (T_{n+1} - T_n)/(1-r_n) \to r/(1-r)\). Q.E.D.

Theorem 2.14. If \( r_n \to r \) where \( |r| < 1 \), then
\[ T_n \to r/(1-r). \]

Proof: Since \( |r| < 1, r \neq 1 \) and \( \sum a_n \) converges, so that \( T_n \) exists for large \( n \). Let \( \epsilon > 0 \) and \( \rho \) be any number such that \( |r| < \rho < 1 \). There exists an integer \( N \) such that for \( n \geq N \) and \( m > N \) we have
\[ |r_n| < \rho \text{ and } |r_n - r_m| < \epsilon(1-\rho). \]
Thus, for each \( n \geq N \) we have
\[ |T_{n+1} - T_n| = |[r_{n+1} - r_n] + [r_{n+1} r_{n+2} - r_n r_{n+1}] + \cdots + [(r_{n+1} \cdots r_{n+k+1}) - (r_n \cdots r_{n+k})] + \cdots| \]
\[ \leq |r_{n+1} - r_n| + |r_{n+1} - r_{n+2} - r_n| + \cdots + |r_{n+1} \cdots r_{n+k}| + |r_{n+k+1} - r_n| + \cdots \]
\[ < \epsilon(1-\rho) + \rho \epsilon(1-\rho) + \cdots + \rho^k \epsilon(1-\rho) + \cdots = \epsilon. \]
Hence, \( |T_{n+1} - T_n| \to 0 \), i.e., \( T_{n+1} - T_n \to 0. \) From
Theorem 2.13. \( \{T_n\} \) converges. Consequently, 
\[ T_n \to \frac{r}{1-r} \] according to (2) of Theorem 2.2. Q.E.D.

**Theorem 2.15.** Suppose that \( r_n \to r \) where \( |r| < 1 \), and let \( \{a_n\} \) be a complex sequence converging to some complex number \( a_0 \). Then \( T_n \to t \) for some complex number \( t \), and conditions (1) through (5) of Theorem 2.2 hold.

**Proof:** From Theorem 2.14, \( \{T_n\} \) converges. Now apply Theorem 2.2. Q.E.D.

According to Theorem 2.15, \( \Sigma a_n \in \text{MR}(\Sigma a_n) \) if \( r = 0 \). Nevertheless, the reader should be forewarned in case \( r = 0 \). In particular, let \( \Sigma a_n = \frac{\infty}{n} \). We have \( r_n \to -1/n \) for \( n \geq 1 \), and \( \delta_n = 1/(1-r_n) \)
\[ = \frac{1}{1+(1/n)} = n/(n+1) = 1-1/(n+1) = 1+r_n+1 \text{ for } n \geq 2. \]
Consequently, \( S_{\delta_n} = S_n + a_{n+1} \delta_{n+1} = S_n + a_{n+1} (1+r_{n+2}) = S_{n+2} \)
for \( n \geq 1 \). Hence \( \{\delta_n\} \) appears to be a poor selection for accelerating the convergence of \( \Sigma a_n \).

**Lemma 2.16.** If \( |r| < 1 \), then \( T_n/r_n \to 1/(1-r) \).

**Proof:** If \( r = 0 \), then \( T_n/r_n \to 1 = 1/(1-r) \) according to Theorem 2.1. If \( r \neq 0 \), then \( T_n/r_n \to \frac{|r|/(1-r)}{r} = 1/(1-r) \) according to Theorem 2.14. Q.E.D.
Theorem 2.17. Suppose that \( \Sigma a_n \) and \( \Sigma a'_n \) are series such that \(|r| < 1\) and \(|r'| < 1\). Then:

1. \( \Sigma a'_n \) converges more rapidly than \( \Sigma a_n \) if and only if \( a'_n/a_n \to 0 \).
2. \( \Sigma a'_n \) converges with the same rapidity as \( \Sigma a_n \) if and only if there are numbers \( a \) and \( b \) such that \( 0 < a < a'_n/a_n < b \).

Proof: From Lemma 2.16, \( T_n/r_n \to 1/(1-r) \) and \( T'_n/r'_n \to 1/(1-r') \).

If \( a'_n/a_n \to 0 \),

\[
\frac{S' - S'_{n-1}}{S - S_{n-1}} = \frac{a'_n}{a_n} \cdot \frac{T'_n/r'_n}{T_n/r_n} \to 0 \cdot \frac{1/(1-r')}{1/(1-r)} = 0.
\]

Conversely, if \( \Sigma a'_n \) converges more rapidly than \( \Sigma a_n \),

\[
\frac{a'_n}{a_n} = \frac{T'_n/r'_n}{T_n/r_n} \cdot \frac{S' - S'_{n-1}}{S - S_{n-1}} \to \frac{1/(1-r)}{1/(1-r')} \cdot 0 = 0.
\]

This proves (1).

Assume that \( a \) and \( b \) are numbers such that \( 0 < a < a'_n/a_n < b \). Since \( \left| T'_n/r'_n \right| / \left| T_n/r_n \right| \to \left| (1-r)/(1-r') \right| \neq 0 \), there are numbers \( c \) and \( d \) such that \( 0 < c < \left| (T'_n/r'_n)/(T_n/r_n) \right| < d \). Thus,

\[
0 < ac < \left| \frac{S' - S'_{n-1}}{S - S_{n-1}} \right| = \left| \frac{a'_n}{a_n} \right| \cdot \left| \frac{T'_n/r'_n}{T_n/r_n} \right| < bd.
\]
Assume that $A$ and $B$ are numbers such that $0 < A < |(S'-S_{n-1})/(S-S_{n-1})| < B$. As above, there are numbers $c$ and $d$ such that $0 < c < |(T_n/r_n)/(T'_n/r'_n)| < d$. Thus,

$$0 < Ac < \left| \frac{a'_n}{a_n} \right| = \left| \frac{T_n/r_n}{T'_n/r'_n} \right| \left| \frac{S'-S_{n-1}}{S-S_{n-1}} \right| < B.$$

Lemma 2.18. If $|r_n| \leq \rho < 1/2$ for some number $\rho$, then $0 < (1-2\rho)/(1-\rho) \leq |T_n/r_n| \leq 1/(1-\rho)$.

Proof: We have $|T_n| \leq |r_n| + |r_n r_{n+1}| + \cdots + |r_n \cdots r_{n+k}| + \cdots \leq |r_n|/(1-\rho) \leq \rho/(1-\rho) < 1$. Thus, $|T_n/r_n| \leq 1/(1-\rho)$ and $|T_n/r_n| = |1+T_{n+1}| \geq ||1| - |T_{n+1}|| = 1 - |T_{n+1}| \geq 1 - \rho/(1-\rho) = (1-2\rho)/(1-\rho) > 0$. Q.E.D.

Theorem 2.19. Suppose that $\Sigma a_n$, $\Sigma a'_n$ are series such that $a'_n/a_n \rightarrow 0$, and $|r_n| \leq \rho_1 < 1/2$, $|r'_n| \leq \rho_2 < 1$ for some numbers $\rho_1, \rho_2$. Then $\Sigma a'_n$ converges more rapidly than $\Sigma a_n$.

Proof: From Lemma 2.18, $0 < (1-2\rho_1)/(1-\rho_1) \leq |T_n/r_n|$. Also, $|T'_n/r'_n| = |1+r'_{n+1}+r'_{n+1} r'_{n+2}+\cdots| \leq 1/(1-\rho_2)$. Thus,
According to the following counterexample, Theorem 2.19 fails to hold if we replace \( p_1 < \frac{1}{2} \) by \( p_1 \leq 1 \) and \( p_2 < 1 \) by \( p_2 \leq 1 \).

**Counterexample 2.20.** For \( n \geq 0 \), define \( a_n = (-1)^n / (n+1) \) and \( a'_n = 1 / (n+1)(n+2) \). Then \( a_n' / a_n \to 0 \), \( r_n' \to r' = 1 \), and \( r_n + r = -1 \). Since \( S' - S'_n = 1 / (n+2) \) and \( |S - S_n| \leq |a_{n+1}| = 1 / (n+2) \), we have \( |S' - S'_n| / |S - S_n| \geq 1 \), and thus \( \Sigma a'_n \) does not converge more rapidly than \( \Sigma a_n \).
CHAPTER III

BASIC THEOREMS FOR ACCELERATION, AITKEN'S $\delta^2$-PROCESS, AND LUBKIN'S $W$ TRANSFORMATION

All series in this chapter are assumed to be complex. The first two theorems of this chapter, the second theorem in particular, are basic for a study of acceleration.

Theorem 3.1. Suppose that $\Sigma a_n$ is a complex series

\[ \{b_n\} \] is a complex sequence, and $\Sigma a'_n$ is a series with partial sums $S'_n = S_n + b_{n+1}$. Then $\Sigma a'_n \in \text{MR}(\Sigma a_n)$ if and only if $b_{n+1} \sim S-S_n \rightarrow 0$.

Proof: If either condition holds, then $S-S_n = S-S'_n + b_{n+1}$ \neq 0, so that $b_{n+1}/(S-S_n) + (S-S'_n)/(S-S_n) = 1$. Thus

\[ (S-S'_n)/(S-S_n) \rightarrow 0 \] and $S-S_n \rightarrow 0$ if, and only if,

\[ b_{n+1}/(S-S_n) \rightarrow 1 \] and $S-S_n \rightarrow 0$; but this is equivalent to $b_{n+1} \sim S-S_n \rightarrow 0$. Q.E.D.

From Theorem 3.1, we see that the class of all sequences $\{c_n\}$ such that $\Sigma a'_n \in \text{MR}(\Sigma a_n)$, where $S'_n = S_n + c_{n+1}$, is completely determined by one such sequence $\{b_n\}$; the required condition being that $c_n \sim b_n$. 
Similarly, we now show that if $\sum a_\alpha n \in \text{MR}(\Sigma a_n)$, then $\sum a_\beta n \in \text{MR}(\Sigma a_n)$, if and only if $\beta_n \sim \alpha_n$.

**Theorem 3.2.** Suppose that $\sum a_\alpha n \in \text{MR}(\Sigma a_n)$. Then $\sum a_\beta n \in \text{MR}(\Sigma a_n)$ if and only if $\beta_n \sim \alpha_n$.

**Proof:** From Theorem 3.1, $a_n+1 \alpha a_n+1 \sim S-S_n \rightarrow 0$. Hence, from Theorem 3.1, $\sum a_\beta n \in \text{MR}(\Sigma a_n)$ if and only if $a_n+1 \beta n+1 \sim S-S_n$, and this is equivalent to $a_n+1 \beta n+1 \sim a_n+1 \beta n+1$, that is, $\beta_{n+1} \sim \alpha_{n+1}$. Q.E.D.

**Lemma 3.3.** If $(1-r_n)(1-r_{n+1}) \neq 0$, then $a_\delta n/a_n \neq 0$.

$$a_\delta n/a_n = 1/(1-r_{n+1}) - 1/(1-r_n) = r_{n+1}/(1-r_{n+1}) - r_n/(1-r_n)$$

$$= (r_{n+1} - r_n)/(1-r_n)(1-r_{n+1})$$

**Proof:** Since $r_n \neq 1$ and $r_{n+1} \neq 1$, we have $\delta_n$ as $1/(1-r_n)$ and $\delta_{n+1} = 1/(1-r_{n+1})$. Thus, $a_\delta n/a_n$

$$= (a_n + a_{n+1} \delta_{n+1} - a_n \delta_n)/a_n = 1+r_{n+1} \delta_{n+1} - \delta_n = r_{n+1}/(1-r_{n+1})$$

$$+ 1 - 1/(1-r_n) = r_{n+1}/(1-r_{n+1}) - r_n/(1-r_n) = [r_{n+1}(1-r_{n+1}) - r_n(1-r_{n+1})]/(1-r_n)(1-r_{n+1}) = (r_{n+1} - r_n)/(1-r_n)(1-r_{n+1})$$

$$= 1/(1-r_{n+1}) - 1/(1-r_n). \text{ Q.E.D.}$$

**Theorem 3.4.** Suppose that $a_\delta n/a_n \rightarrow 0$. Then
Theorem 3.5. Suppose that \( \frac{a_0}{a_n} \rightarrow 0 \). Then

\[ \Sigma a_0 \in \text{MR}(\Sigma a_n) \] if and only if \( \Sigma a_n \in \text{MR}(\Sigma a_n) \), where

\[ a_n = \frac{(1-r_{n-1})}{(1-2r_n + r_{n-1}r_n)}. \]

**Proof:** Suppose that \( \Sigma a_0 \in \text{MR}(\Sigma a_n) \). As in the proof of Theorem 3.4, \( 1-2r_n + r_{n-1}r_n \neq 0 \). Hence, \( a_n/\delta_n = (1-r_{n-1})(1-r_n)/(1-2r_n + r_{n-1}r_n) \neq 0 \). Therefore, \( \frac{a_0}{a_n} \rightarrow 0 \) and, from Theorem 3.2,

\[ \Sigma a_0 \in \text{MR}(\Sigma a_n). \]
Suppose that $\Sigma a_n \in MR(\Sigma a_n)$. Then $r_n \neq 1$, and thus $\alpha_n/\delta_n = 1/(1-a_\delta(n-1)/a_{n-1}) \to 1$. From Theorem 3.2, $\Sigma a_\delta n \in MR(\Sigma a_n)$.

**Theorem 3.6.** $\Sigma a_n \in MR(\Sigma a_n)$, $\alpha_n \sim T_n/r_n$, and $\alpha_n \sim 1+T_{n+1}$ are equivalent.

**Proof:** From Theorem 3.1, $\Sigma a_n \in MR(\Sigma a_n)$ if and only if $a_{n+1}/a_{n+1} \sim S-S_n \to 0$; and this is equivalent to $\alpha_{n+1} \sim (S-S_n)/a_{n+1} = T_{n+1}/r_{n+1}$. Moreover, $\alpha_n \sim T_n/r_n$ is equivalent to $\alpha_n \sim 1+T_{n+1}$, since $T_n/r_n = 1+T_{n+1}$.

Q.E.D.

**Lemma 3.7.** If $\Sigma a_n$ is a convergent series and $n$ is a positive integer such that $T_{n+1}-T_n \neq -1$, then

$$\frac{(S-S_\delta(n-1))/(S-S_{n-1})}{(S-S_\delta(n-1))/(S-S_{n-1})} = \frac{(T_{n+1}-T_n)/(1+T_{n+1}-T_n)}{a_n(1+T_{n+1})}.$$  

**Proof:** From $(1-r_n)(1+T_{n+1}) = l+T_{n+1}-T_n \neq 0$, $T_{n+1} \neq -1$ and $r_n \neq 1$. Thus $S-S_{n-1} = a_n(1+T_{n+1}) \neq 0$. We then have

$$\frac{(S-S_\delta(n-1))/(S-S_{n-1})}{(S-S_\delta(n-1))/(S-S_{n-1})} = \frac{(S-S_{n-1}-a_n \delta_n)/(S-S_{n-1})}{S-S_{n-1}}$$

$$= 1 - \frac{a_n}{S-S_{n-1}} \frac{1}{1-r_n} = 1 - \frac{1}{T_n} \frac{T_n/(1+T_{n+1})}{1-T_n/(1+T_{n+1})}.$$
Theorem 3.8. \( \Sigma a_\delta n \in MR(\Sigma a_n) \) if and only if \( T_{n+1} - T_n \to 0 \).

1st Proof: From Theorem 3.6, \( \Sigma a_\delta n \in MR(\Sigma a_n) \) if and only if \( \delta_n \sim 1 + T_{n+1} \), and this is equivalent to

\[
(1 + T_{n+1})(1-r_n) \to 1, \text{ since } \delta_n = \frac{1}{1-r_n}. \quad \text{Finally,}

(1 + T_{n+1})(1-r_n) \to 1 \text{ if and only if } T_{n+1} - T_n \to 0, \text{ since}

T_{n+1} - T_n = (1 + T_{n+1})(1-r_n) - 1. \quad \text{Q.E.D.}

2nd Proof: If \( T_{n+1} - T_n \to 0 \), then \( T_{n+1} - T_n \neq -1 \). Thus, from Lemma 3.7,

\[
\frac{(S-S_\delta(n-1))}{(S-S_{n-1})} = \frac{(T_{n+1} - T_n)}{(1 + T_{n+1} - T_n)} \to 0. \quad \text{Conversely, suppose that}

\frac{(S-S_\delta(n-1))}{(S-S_{n-1})} \to 0. \quad \text{Then } a_n \neq 0 \text{ and } r_n \neq 1, \text{ since } \delta_n \neq 0. \quad \text{We must have}

1 + T_{n+1} - T_n \neq 0, \text{ since}

otherwise \( (1-r_n)(T_n/r_n) = 1 + T_{n+1} - T_n =: 0 \), \( T_n =: 0 \), \( \text{and } S-S_{n-1} =: 0; \quad \text{a contradiction. From Lemma 3.7,}

\( \frac{(T_{n+1} - T_n)}{(1 + T_{n+1} - T_n)} = \frac{(S-S_\delta(n-1))}{(S-S_{n-1})} \to 0, \quad \text{and}

thus } \frac{T_{n+1} - T_n}{0}. \quad \text{Q.E.D.}

The preceding theorem immediately yields the corollary, also proven in the previous chapter, that the
convergence of \( \{T_n\} \) implies \( \Sigma a_\delta n \in MR(\Sigma a_n) \).

**Lemma 3.9.** If \( \Sigma a_n \) is a convergent series and \( n \) is a positive integer such that \( a_{n-1}a_n a_{n+1} \neq 0 \), then

\[
r_{n+1} - r_n = (T_{n+2} - T_{n+1})(1-r_n)(1-r_{n+1}) - (T_{n+2} - T_{n+1})(1-r_n) \\
+ (T_{n+1} - T_n)(1-r_{n+1}).
\]

**Proof:** We have \( (1-r_n)(1+T_{n+1}) = 1-r_n + T_{n+1} - r_n T_{n+1} \)
\[
= 1+T_{n+1} - r_n(1+T_{n+1}) = 1+T_{n+1} - T_n, \quad \text{so that} \quad T_{n+1} - T_n \\
= (1-r_n)(1+T_{n+1}) - 1. \quad \text{Similarly,} \quad T_{n+2} - T_{n+1} \\
= (1-r_{n+1})(1+T_{n+2}) - 1. \quad \text{Thus,} \quad (T_{n+2} - T_{n+1})(1-r_n)(1-r_{n+1}) \\
- (T_{n+2} - T_{n+1})(1-r_n) + (T_{n+1} - T_n)(1-r_{n+1}) \\
= (T_{n+2} - T_{n+1})(1-r_n)(1-r_{n+1}) - (1-r_n)[(1-r_{n+1})(1+T_{n+2}) - 1] \\
+ (1-r_{n+1})[(1-r_n)(1+T_{n+1}) - 1] = (T_{n+2} - T_{n+1})(1-r_n)(1-r_{n+1}) \\
+ (1-r_n)(1-r_{n+1})(1+T_{n+2}) - (1-r_{n+1}) \\
+ (1-r_n)(1-r_{n+1})(1+T_{n+1}) = (1-r_n)(1-r_{n+1})[(T_{n+2} - T_{n+1}) \\
- (1+T_{n+2}) + (1+T_{n+1})] + r_{n+1} - r_n = r_{n+1} - r_n. \quad \text{Q.E.D.}
\]

**Lemma 3.10.** If \( \Sigma a_n \) is a convergent series and \( n \) is a positive integer such that \( (1-r_n)(1-r_{n+1})a_{n+1} \neq 0 \), then

\[
a^\delta n / a_n = (T_{n+2} - T_{n+1}) - (T_{n+2} - T_{n+1})/(1-r_{n+1}) \\
+ (T_{n+1} - T_n)/(1-r_n).
\]
Proof: We have $a_{n-1}a_n a_{n+1} \neq 0$, and $a_{\delta n}/a_n$
$= (r_{n+1} - r_n)/(1-r_n)(1-r_{n+1})$ according to Lemma 3.3. Now
apply Lemma 3.9. Q.E.D.

Lemma 3.11. If $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$ and $0 < B \leq |1-r_n|$ for
some number $B$, then $a_{\delta n}/a_n \to 0$.

Proof: From Theorem 3.8, $T_{n+1} - T_n \to 0$. Using Lemma 3.10
and $0 < B \leq |1-r_n|$, it is obvious that $a_{\delta n}/a_n \to 0$.
Q.E.D.

Theorem 3.12. Suppose that $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$ and
$0 < B \leq |1-r_n|$. Then $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$, where $a_n$
$= (1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$ or $a_n$
$= (1-r_{n-1})/(1-2r_n+r_{n-1} r_n)$.

Proof: From Lemma 3.11, $a_{\delta n}/a_n \to 0$. We now apply
Theorem 3.4, if $a_n = (1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$; or
Theorem 3.5, if $a_n = (1-r_{n-1})/(1-2r_n+r_{n-1} r_n)$. Q.E.D.

Theorem 3.13. If $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$ and $|r_n| \leq B$ for
some number $B$, then $r_{n+1} - r_n \to 0$.

Proof: From Theorem 3.8, Lemma 3.9, and $|r_n| \leq B$, it
is obvious that \( r_{n+1}-r_n \to 0 \). Q.E.D.

**Theorem 3.14.** Suppose that \( |r_n| \leq \rho < 1 \) for some number \( \rho \). Then a n.a.s.c. that \( \Sigma a_{\delta n} \in MR(\Sigma a_n) \) is that \( r_{n+1}-r_n \to 0 \).

**Proof:** Since \( |r_n| \leq \rho < 1 \), \( \Sigma a_n \) converges.

The necessity follows from Theorem 3.13.

For the sufficiency, let \( \epsilon' > 0 \). Since \( r_{n+1}-r_n \to 0 \), \( |r_{n+1}-r_n| \leq \epsilon'/(1-\rho)^2 \). With \( \epsilon = \epsilon'/(1-\rho)^2 \),

\[
|T_{n+1}-T_n| = |(r_{n+1}-r_n)+r_{n+1}(r_{n+2}-r_n)+r_{n+1}r_{n+2}(r_{n+3}-r_n)+\cdots+\sum(r_{n+1}\cdots r_{n+k-1})(r_{n+k}-r_n)+\cdots|
\]

\[
\leq |r_{n+1}-r_n| + |r_{n+1}| + |r_{n+2}| + \cdots + |r_{n+1}\cdots r_{n+k-1}| + |r_{n+k}| + \cdots
\]

\[
\leq \epsilon + 2\epsilon |r_{n+1}| + \cdots + k\epsilon |r_{n+1}\cdots r_{n+k-1}| + \cdots
\]

\[
\leq \epsilon + [1+2\rho+3\rho^2+\cdots+k\rho^{k-1}+\cdots] = \epsilon/(1-\rho^2) = \epsilon'.
\]

Hence \( T_{n+1}-T_n \to 0 \), and thus, from Theorem 3.8, \( \Sigma a_{\delta n} \in MR(\Sigma a_n) \). Q.E.D.

**Corollary 3.15.** Suppose that \( |r_n| \leq \rho < 1 \) for some number \( \rho \), and \( \Sigma a_{\delta n} \in MR(\Sigma a_n) \). Suppose, in addition, that \( q \) is an integer and \( a'_n = a_n z^{n+q} \) for every \( n \).

Then \( \Sigma a'_{\delta n} \in MR(\Sigma a'_n) \), for each complex number \( z \).
satisfying $0 < |z| < 1/\rho$.

**Proof:** From Theorem 3.14, $r_{n+1} - r_n \to 0$. Let $z$ be any complex number such that $0 < |z| < 1/\rho$. Then

$$|r_n'| = |r_n z| \leq |r_n| |z| < 1 \quad \text{and} \quad r_{n+1}' - r_n' = r_{n+1} z - r_n z = z(r_{n+1} - r_n) \to 0.$$  

Thus $\Sigma a_n' \in \text{MR}(\Sigma a_n')$, according to Theorem 3.14. Q.E.D.

**Corollary 3.16.** Suppose that $|r_n| \leq \rho < 1$ for some number $\rho$, and $r_{n+1} - r_n \to 0$. Suppose, in addition, that $q$ is an integer and $a_n' = a_n z^{n+q}$ for every $n$. Then $\Sigma a_n' \in \text{MR}(\Sigma a_n')$, for each complex number $z$ satisfying $0 < |z| < 1/\rho$.

**Proof:** From Theorem 3.14, $\Sigma a_n \in \text{MR}(\Sigma a_n)$. We now apply Corollary 3.15. Q.E.D.

**Lemma 3.17.** If $0 < A \leq |1 - r_n| \leq B$, then $a_{n+1}/a_n$ is an integer and $a_{n+1}/a_n \to 0$ if and only if $r_{n+1} - r_n \to 0$.

**Proof:** Since $0 < A \leq |1 - r_n| \leq B$, $0 < A^2 \leq |(1 - r_n)(1 - r_{n+1})| \leq B^2$. Hence from Lemma 3.3, $a_{n+1}/a_n = (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1})$. Thus, from
Theorem 3.20. If $|r_n| \leq \rho < 1$, then $a_{\delta n}/a_n \rightarrow 0$ if and only if $r_{n+1} - r_n \rightarrow 0$. Q.E.D.

Lemma 3.18. If $|r_n| \leq \rho < 1$, then $a_{\delta n}/a_n = (r_{n+1} - r_n)/(1-r_n)(1-r_{n+1})$, and $a_{\delta n}/a_n \rightarrow 0$ if and only if $r_{n+1} - r_n \rightarrow 0$.

Proof: From $|r_n| \leq \rho < 1$, $0 < 1-\rho \leq |1-r_n| \leq 2$. We now apply Lemma 3.17. Q.E.D.

Theorem 3.19. Suppose that $|r_n| \leq \rho < 1$. Then $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$ if and only if $a_{\delta n}/a_n \rightarrow 0$.

Proof: Lemma 3.18, $a_{\delta n}/a_n \rightarrow 0$ if and only if $r_{n+1} - r_n \rightarrow 0$. From Theorem 3.14, $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$ if and only if $r_{n+1} - r_n \rightarrow 0$. Consequently, $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$ if and only if $a_{\delta n}/a_n \rightarrow 0$. Q.E.D.

Theorem 3.20. If $|r_n| \leq \rho < 1$ and $a_{\delta n}/a_n \rightarrow 0$, then $\Sigma a_n \in \text{MR}(\Sigma a_n)$, where $a_n = (1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$ or $a_n = (1-r_{n+1})/(1-2r_n+r_{n-1}r_n)$.

Proof: From Theorem 3.19, $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$. From Theorem 3.4, $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$ if $a_n = (1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$. 0 < A^2 \leq |(1-r_n)(1-r_{n+1})| \leq B^2$, $a_{\delta n}/a_n \rightarrow 0$ if and only if $r_{n+1} - r_n \rightarrow 0$. Q.E.D.
If $\alpha_n = (1-r_{n-1})/(1-2r_n + r_{n-1} r_n)$, we may apply Theorem 3.5 to obtain $\Sigma \alpha_n \in \text{MR}(\Sigma \alpha_n)$. Q.E.D.

Theorem 3.21. If $|r_n| < \rho < 1$ and $r_{n+1} - r_n \to 0$, then $\Sigma \alpha_n \in \text{MR}(\Sigma \alpha_n)$, where $\alpha_n = (1-r_{n+1})/(1-2r_{n+1} + r_n r_{n+1})$

or $\alpha_n = (1-r_{n-1})/(1-2r_n + r_{n-1} r_n)$.

Proof: From Lemma 3.18, $a_{\infty}/a_n \to 0$. We now apply Theorem 3.20. Q.E.D.
CHAPTER IV

RAPIDITY OF CONVERGENCE AND VARIOUS METHODS
FOR ACCELERATING CONVERGENCE. A VACUOUS THEOREM

In this chapter, both real and complex series will be considered. Various methods for accelerating convergence will be treated. That part of Lubkin's Theorem 6 (17, p. 231) concerning acceleration will be shown to have no application if $r_n \to 1$. That part of his Theorem 7 (17, p. 232) concerning acceleration will be proven to be vacuous.

If $\alpha, \beta$ are real numbers and $0 \leq \beta < \pi/2$, the notation $\langle \alpha, \beta \rangle$ will be used to denote the set of complex numbers $z$ such that $|\arg z - \alpha| \leq \beta$ for some $\arg z$. Thus $\langle \alpha, \beta \rangle$ is the infinite sector in the complex plane, subtending the angle $2\beta$ and bisected by the ray $\theta = \alpha$. If $\beta = 0$, $\langle \alpha, \beta \rangle$ degenerates to the ray $\theta = \alpha$.

The following theorem appears to be the only one of general character, concerning rapidity of convergence, which is found in Knopp (15, p. 279-280).

Theorem 4.1. Suppose that $\Sigma a_n$ and $\Sigma b_n$ are convergent series of positive terms. Then $\Sigma a_n$ converges more rapidly than $\Sigma b_n$ if $a_n/b_n \to 0$. 
According to Counterexample 2.20, Theorem 4.1 fails to hold for arbitrary convergent complex series \( \sum a_n, \sum b_n \).

The converse of Theorem 4.1 is false. That is, if \( \sum a_n \) and \( \sum b_n \) are series of positive terms, and \( \sum a_n \) converges more rapidly than \( \sum b_n \), then it is not necessarily true that \( \frac{a_n}{b_n} \to 0 \). This is made obvious by the following theorem.

**Theorem 4.2.** Suppose that \( \sum a_n \) and \( \sum b_n \) are series of positive terms, and that \( \sum a_n \) converges more rapidly than \( \sum b_n \). Then \( a_0 + a_1 + a_1 + \cdots + a_n + \cdots \) converges more rapidly than \( a_0 + b_0 + a_1 + b_1 + \cdots + a_n + b_n + \cdots \).

**Proof:** We have
\[
\frac{a_n}{b_n} \to 0
\]
as \( n \to \infty \), and
\[
\frac{a_n}{b_n} \to 0
\]
as \( n \to \infty \). Q.E.D.

As previously noted, Theorem 4.2 shows that the converse of Theorem 4.1 is false; however, we do have the
following theorem.

**Theorem 4.3.** Suppose that $\Sigma a_n$ and $\Sigma b_n$ are convergent series of positive terms. Then $a_n/b_n \to 0$ if, and only if $\Sigma a_n$, converges more rapidly than $\Sigma b_n$, for each subsequence $\{n'\}$ of $\{n\}$.

**Proof:** If $a_n/b_n \to 0$ and $\{n'\}$ is any subsequence of $\{n\}$, then $a_{n'}/b_{n'} \to 0$ and, according to Theorem 4.1, $\Sigma a_{n'}$ converges more rapidly than $\Sigma b_{n'}$.

Assume that $a_n/b_n \not\to 0$. Then there is an $\epsilon > 0$ and a subsequence $\{n'\}$ of $\{n\}$ such that $a_{n'}/b_{n'} \geq \epsilon$. Consequently, $\sum_{k=n}^{\infty} a_k \geq \epsilon \sum_{k=n}^{\infty} b_k$, and thus $\Sigma a_{n'}$ does not converge more rapidly than $\Sigma b_{n'}$. Q.E.D.

**Lemma 4.4.** If $\Sigma a_n$ is a convergent complex series such that $a_n \in \langle \alpha, \beta \rangle$ for some set $\langle \alpha, \beta \rangle$, then $\sum_{k=n}^{\infty} |a_k|$ is

$$\leq |\sum_{k=n}^{\infty} a_k|/\cos \beta.$$ 

**Proof:** We may assume that $\alpha = 0$, since with $b_n = a_ne^{-ia}$ for $n \geq 0$, we have $b_n \in \langle 0, \beta \rangle$, $|\sum_{k=n}^{\infty} a_k| = |\sum_{k=n}^{\infty} b_k|$, and $\sum_{k=n}^{\infty} |a_k| = \sum_{k=n}^{\infty} |b_k|$. Since $a_n \in \langle 0, \beta \rangle$, we may
set \( a_n = |a_n| e^{i\theta_n} \) where \( |\theta_n| \leq \beta < \pi/2 \). Thus,

\[
\cos \theta_n \geq \cos \beta \quad \text{and} \quad |\sum_{k=1}^{\infty} a_k| = |\sum_{k=1}^{\infty} a_k \cos \theta_k |
\]

\[
\geq \sum_{k=1}^{\infty} |a_k| \cos \beta = (\cos \beta) \sum_{k=1}^{\infty} |a_k|.
\]

Q.E.D.

**Theorem 4.5.** Suppose that \( \Sigma a_n, \Sigma b_n \) are complex series such that \( \Sigma a_n \) converges and \( a_n \in \langle \alpha, \beta \rangle \) for some set \( \langle \alpha, \beta \rangle \). Then \( b_n/a_n \to 0 \) if and only if \( \Sigma b_n \), converges more rapidly than \( \Sigma a_n \), for every subsequence \( \langle n' \rangle \) of \( \{n\} \).

**Proof:** If \( a_n = 0 \), then \( a_{n'} = 0 \) for some subsequence \( \langle n' \rangle \) of \( \{n\} \), and both conditions in the conclusion of our theorem fail to hold. Thus we may assume that \( a_n \neq 0 \).

Suppose that \( b_n/a_n \to 0 \), \( \epsilon > 0 \), and \( \langle n' \rangle \) is any subsequence of \( \{n\} \). Then \( \|b_n\| \leq \epsilon \|a_n\| \cos \beta \), and \( \Sigma |b_{n'}|, \Sigma |a_{n'}| \) both converge, since \( \Sigma |a_n| \) converges according to Lemma 4.4. Hence,

\[
\sum_{k=1}^{\infty} |b_{n'}| \leq \epsilon \sum_{k=1}^{\infty} |a_{n'}| \leq \epsilon \sum_{k=1}^{\infty} |a_k|, \quad \text{the last inequality following from Lemma 4.4. Thus} \quad \Sigma b_{n'} \text{ converges}.
\]
more rapidly than $\Sigma a_n$.

Suppose that $b_n/a_n \not\to 0$. Then there is an $\varepsilon > 0$ and a subsequence $\{n'\}$ of $\{n\}$ such that $|b_{n'}| > \varepsilon |a_{n'}|$.

Since $b_n, \varepsilon : \langle \alpha', \pi/4 \rangle$ for some real $\alpha'$, there is a subsequence $\{n^*\}$ of $\{n'\}$ such that $b_{n^*} \varepsilon : \langle \alpha', \pi/4 \rangle$ and $|b_{n^*}| > \varepsilon |a_{n^*}|$. If $\Sigma b_{n^*}$ does not converge, there is nothing to prove. Hence, assume that $\Sigma b_{n^*}$ converges. From $|b_{n^*}| > \varepsilon |a_{n^*}|$ and Lemma 4.4,

$\sum_{k=n}^{\infty} b_{k^*} \geq (\cos \pi/4) \sum_{k=n}^{\infty} |b_{k^*}| \geq (\varepsilon \cos \pi/4) \sum_{k=n}^{\infty} |a_{k^*}|$,

$> (\varepsilon \cos \pi/4) \sum_{k=n}^{\infty} |a_{k^*}|$, and thus $\Sigma b_{n^*}$ does not converge more rapidly than $\Sigma a_{n^*}$. Q.E.D.

Corollary 4.6. Suppose that $\Sigma a_n$ is a convergent series such that $a_n \varepsilon : \langle \alpha, \beta \rangle$ for some set $\langle \alpha, \beta \rangle$. Then a n.a.s.c. that $\Sigma a_{\delta n}$ converge more rapidly than $\Sigma a_n$, for each subsequence $\{n'\}$ of $\{n\}$, is that $a_{\delta n}/a_n \to 0$.

Proof: Set $a_{\delta n} = b_n$ and apply Theorem 4.5. Q.E.D.
Theorem 4.7. Suppose that $\Sigma a_n$ is a convergent real series such that $r_n \leq r_{n+1}$ and $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. Suppose, in addition, that $q$ is an integer and $a_n = a_n z^{n+q}$ for every $n$. Then $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$ for each complex number $z$ satisfying $0 < |z| \leq 1$.

Proof: Let $0 < |z| \leq 1$. Since $\Sigma a_n$ converges and $r_n \leq r_{n+1}$, $r_n \to r$ where $-1 < r \leq 1$. If $r < 1$, then $|r_n| \leq \rho$ for some number $\rho$, and $0 < |z| < 1/\rho$. Since $r_{n+1} - r_n \to 0$, Corollary 3.16 implies $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$. Suppose that $r = 1$. We note that $0 < r_n$, so that $0 < a_n$ or $a_n < 0$. In either case, $\Sigma |a_n|$ converges. Also, $|r'_n| = |r_n z| \leq |r_n|$, and thus $\Sigma a'_n$ converges absolutely. In view of Theorem 3.8, $T_{n+1} - T_n \to 0$. Since $r'_n = r_n z$, $T'_n = r'_n + r'_{n-1} + \cdots + (r'_n \cdots r'_{n+k})z^k + \cdots$ $+ (r'_n \cdots r'_{n+k})z^k \cdots + \cdots = r_n z + r_n r_{n+1} z^2 + \cdots + (r_n \cdots r_{n+k})z^k + \cdots$. Thus, $|T'_{n+1} - T'_n| = |(r_{n+1} - r_n)z + r_{n+1}(r_{n+2} - r_n)z^2 + \cdots + (r_{n+1} \cdots r_{n+k-1})(r_{n+k} - r_n)z^k + \cdots |$ $\leq |r_{n+1} - r_n| + |r_{n+1}(r_{n+2} - r_n)| + \cdots + |(r_{n+1} \cdots r_{n+k-1})(r_{n+k} - r_n)|$ $+ \cdots = (r_{n+1} - r_n) + r_{n+1}(r_{n+2} - r_n) + \cdots + (r_{n+1} \cdots r_{n+k-1})(r_{n+k} - r_n) + \cdots$
as $n \to \infty$. Hence $T_{n+1}' - T_n' \to 0$, and thus $\Sigma a_{\delta n}' \in MR(\Sigma a_n')$
according to Theorem 3.8. Q.E.D.

**Theorem 4.8.** If $\Sigma a_n$ is a real series, $0 < r_n$, and $\Sigma a_{\delta n} \in MR(\Sigma a_n)$, then $r_n < 1$ and $0 < Q_n$.

**Proof:** Since $0 < r_n$, $T_n > 0$. From Theorem 3.6,

$$
\delta_n = \frac{1}{1-r_n} \sim \frac{T_n}{r_n} > 0, \text{ so that } 1 - r_n > 0. \text{ Thus, }
$$

$r_n < 1$ and $0 < n(1-r_n) = Q_n$. Q.E.D.

**Lemma 4.9.** Suppose that $\Sigma a_n$ is a real convergent series

such that $a_{\delta n}/a_n \to 0$ and $0 \leq r_n$. Then $r_n < 1$,

$r_{n+1} - r_n \to 0$, and $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

**Proof:** Since $0 \leq r_n$, $a_n \in \langle 0,0 \rangle$ or $a_n \in \langle \pi,0 \rangle$.

From Corollary 4.6 and Theorem 2.6, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ since

$a_{\delta n}/a_n \to 0$. Thus, according to Theorem 4.8, $r_n < 1$, so that $|r_n| \leq 1$. Hence $r_{n+1} - r_n \to 0$ in view of Theorem

3.13. Q.E.D.

**Theorem 4.10.** Suppose that $\Sigma a_n$ is a convergent

real series such that $r_n \leq r_{n+1}$ and $a_{\delta n}/a_n \to 0$. Suppose,

in addition, that $q$ is an integer and $a'_n = a_n z^{n+q}$ for
every \( n \). Then \( \sum a'_n \in \text{MR}(\sum a'_n) \) for every complex number \( z \) such that \( 0 < |z| \leq 1 \).

**Proof:** Since \( \sum a_n \) converges, \( r_n \to r \) where \(-1 < r \leq 1\).

If \( r < 1 \), we may complete the proof in the same manner as in the proof of Theorem 4.7. If \( r = 1 \), then

\[
0 \leq r_n, \quad \text{and} \quad \sum a_{\delta n} \in \text{MR}(\sum a_n) \quad \text{according to Lemma 4.9.}
\]

We may now apply Theorem 4.7 to complete the proof. Q.E.D.

**Theorem 4.11.** Suppose that \( \sum a_n \) is a convergent series such that \( a_n \in \langle \alpha, \beta \rangle \) for some set \( \langle \alpha, \beta \rangle \). Then a n.a.s.c. that \( \sum a_{\delta n} \), converge more rapidly than \( \sum a_n \), for each subsequence \( \{n'\} \) of \( \{n\} \), is that

\[
(r_{n+1} - r_n)/(1-r_n)(1-r_{n+1}) \to 0.
\]

**Proof:** For the sufficiency, \( \delta_n = 1/(1-r_n) \) since

\[
(r_{n+1} - r_n)/(1-r_n)(1-r_{n+1}) \quad \text{exists for large \( n \).}
\]

Thus

\[
a_{\delta n}/a_n = (r_{n+1} - r_n)/(1-r_n)(1-r_{n+1}) \to 0.
\]

From Corollary 4.6, \( \sum a_{\delta n} \), converges more rapidly than \( \sum a_n \), for each subsequence \( \{n'\} \) of \( \{n\} \).

For the necessity, \( \delta_n \neq 0 \); since if \( \delta_n = 0 \), then \( S_{\delta n} = S_n \), and thus, \( \sum a_{\delta n} \) does not converge more rapidly than \( \sum a_n \), a contradiction. Hence,
\[ \delta_n = \frac{1}{1-r_n} \] and, from Corollary 4.6,
\[ \frac{(r_{n+1}-r_n)}{(1-r_n)(1-r_{n+1})} = \frac{a_{\delta_n}}{a_n} \to 0. \] Q.E.D.

**Theorem 4.12.** If \( \Sigma a_n \) is a real series such that \( r = 1 \) and \( |n(n+1)(r_{n+1}-r_n)| \leq 1 \), then \( \Sigma a_n \) diverges.

**Proof:** By hypothesis, \( 1-r_n \to 0 \) and \( |r_{n+1}-r_n| \leq 1/n(n+1) \). Thus,
\[
1-r_n = \sum_{k=n}^{\infty} [(1-r_k)-(1-r_{k+1})] \leq \sum_{k=n}^{\infty} |r_{k+1}-r_k| \\
\leq \sum_{k=n}^{\infty} \frac{1}{k(k+1)} = \frac{1}{n},
\]
from which \( 1-1/n \leq r_n \). Since \( \Sigma a_n', a_n' = 1/n, \) diverges and \( r_n' = (n-1)/n = 1-1/n \leq r_n \), \( \Sigma a_n \) must diverge. Q.E.D.

**Corollary 4.13.** If \( \Sigma a_n \) is a real series such that \( r = 1 \) and \( n^2(r_{n+1}-r_n) \to 0 \), then \( \Sigma a_n \) diverges.

**Proof:** Since \( n^2(r_{n+1}-r_n) \to 0 \), \( n(n+1)(r_{n+1}-r_n) \to 0 \) so that \( |n(n+1)(r_{n+1}-r_n)| \leq 1 \). We now apply Theorem 4.12. Q.E.D.

Lubkin (17, p. 231-232) has proven the following two theorems.
Theorem 6. If $\sum a_n$ is a convergent real series, $r_n > 0$, $Q_n > K > 0$, and $n^2(r_{n+1} - r_n) \to 0$, as $n \to \infty$, then $\sum a_{\delta n} \in \text{MR}(\sum a_n)$.

Theorem 7. If $\sum a_n$ is a convergent real series, $Q$ exists (as a finite limit), and $n^2(r_{n+1} - r_n) \to 0$, then $\sum a_{\delta n} \in \text{MR}(\sum a_n)$.

If $\sum a_n$ is a real series such that $\{n^2(r_{n+1} - r_n)\}$ is bounded, then $\sum |r_{n+1} - r_n|$ converges since $|r_{n+1} - r_n| \leq B/n^2$ for some number $B$. Thus $\sum (r_{n+1} - r_n)$ converges, from which $r_n \to r$ for some number $r$. In view of Corollary 4.13, it is now evident that $0 \leq r < 1$, if the hypothesis of Theorem 6 is satisfied. Consequently if $r = 1$, the hypothesis of Theorem 6 cannot be satisfied. On the other hand, $r = 1$ if $Q$ exists. Hence, according to Corollary 4.13, the hypothesis of Theorem 7 can never be fulfilled.

(1) If $\text{Re } Q_n \to Q'$ and $\text{Re } n^2(r_{n+1} - r_n) \to P'$, then $P' = Q'$.
(2) If $\text{Im } Q_n \to Q''$ and $\text{Im } n^2(r_{n+1} - r_n) \to P''$, then $P'' = Q''$.
(3) If $Q_n \to Q$ and $n^2(r_{n+1} - r_n) \to P$, then $P = Q$.
Proof: We first note that \( Q_{n+1} - Q_n = (n+1)(l - r_{n+1}) - n(l - r_n) \)
\[ = n(l - r_{n+1}) + (l - r_{n+1}) - n(l - r_n) = (l - r_{n+1}) - n(r_{n+1} - r_n) \]
and
\[ n(Q_{n+1} - Q_n) = n(l - r_{n+1}) - n^2(r_{n+1} - r_n) = (n+1)(l - r_{n+1}) - (l - r_{n+1}) - n^2(r_{n+1} - r_n). \]

Assume that \( P' \neq Q' \). Set \( Q_n' = \text{Re } Q_n' \). Since
\[ \text{Re } n(l - r_n) \to Q', \text{Re } (l - r_n) \to 0. \]
Thus, \( n(Q_n' - Q_n') \)
\[ = Q_{n+1}' - \text{Re } (l - r_{n+1}) - \text{Re } n^2(r_{n+1} - r_n) \to Q' - 0 - P' = Q' - P' \neq 0. \]
Let \( L = (Q' - P')/2 \). If \( L > 0 \), then \( n \Delta Q_n' \geq L \). Hence there is a positive integer \( m \) such that \( Q_m' + \Delta Q_m' + \Delta Q_{m+1}' + \cdots + \Delta Q_{m+n-1}' \to +\infty \), so that \( Q_n' \to +\infty \), a contradiction. If \( L < 0 \), then \( n \Delta Q_n' \leq L \). Hence there is a positive integer \( m \) such that \( Q_m' + \Delta Q_m' + \cdots + \Delta Q_{m+n-1}' \to -\infty \), so that \( Q_n' \to -\infty \), a contradiction. Thus we must have \( P' = Q' \). This proves (1). The proof of (2) follows in a similar manner, and (3) is an immediate consequence of (1) and (2). Q.E.D.

Theorem 4.14 again shows that the hypothesis of Lubkin's Theorem 7, previously mentioned, can never be fulfilled, since we would have \( Q = 0 \) and \( \Sigma a_n \) would diverge.

Theorem 4.15. If \( 0 < K \leq \text{Re } Q_n' \) and \( \text{Re } [n^2(r_{n+1} - r_n)] \to 0 \),
then \( \text{Re } Q_n < \text{Re } Q_{n+1} \) and \( \text{Re } Q_n \to +\infty \).
Proof: Since $\text{Re } n^2(r_{n+1} - r_n) \to 0$, $\text{Re } n(n+1)(r_{n+1} - r_n) \to 0$.

Also, $(n+1)(Q_n - Q_{n+1}) = -Q_{n+1} + (n+1) Q_n - nQ_{n+1} = -Q_{n+1} + n(n+1)(r_{n+1} - r_n)$. Thus, with $Q'_n = \text{Re } Q_n,$

$$(n+1)(Q'_n - Q'_{n+1}) = -Q'_n + \text{Re } n(n+1)(r_{n+1} - r_n) \leq -K + \text{Re } n(n+1)(r_{n+1} - r_n) < 0$$

from which $Q'_n < Q'_{n+1}$. Hence, $Q'_n \to Q'$ where $K < Q' \leq +\infty$. If $Q' < +\infty$, $Q' = 0$ according to (1) of Theorem 4.14; this is a contradiction.

Thus, $Q' = +\infty$. Q.E.D.

Theorem 4.16. Suppose that $\Sigma a_n$ is a convergent series such that (1) $a_n \in \langle a, \beta \rangle$ for some set $\langle a, \beta \rangle$ and (2) $Q_n \to \infty$. Suppose further that \{P_n\} is a sequence such that (3) $P_n/Q_{n+1} \to 0$ and (4) $n |Q_{n+1} - Q_n| \leq |P_n Q_n|$. Then $a_{\delta n}/a_n \to 0$ and $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$.

Proof: From (2), $\delta_n = 1/(1-r_n)$ and $a_{\delta n}/a_n$

$= n(Q_n - Q_{n+1})/Q_n Q_{n+1} + 1/Q_{n+1}$. From (2), $1/Q_n \to 0$. From (3) and (4), $|n(Q_n - Q_{n+1})/Q_n Q_{n+1}| \leq |P_n Q_n/Q_n Q_{n+1}|$

$= |P_n/Q_{n+1}| \to 0$. Thus $a_{\delta n}/a_n \to 0$. Hence $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$ according to Corollary 4.6 and Theorem 2.6. Q.E.D.
Theorem 4.17. Suppose that $\Sigma a_n$ is a real series such that $-1 < r_n \leq r_{n+1}$, $Q_n \leq Q_{n+1}$, and $Q_n \to +\infty$. Then $a_{\delta n}/a_n \to 0$ and $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$.

Proof: Since $Q_n = n(1-r_n) \to +\infty$, $r_n < 1$. Hence $-1 < r_n \leq r_{n+1} < 1$ and thus $r_n \to r$ where $-1 < r \leq 1$.

If $r < 1$, it is obvious that $|r_n| \leq \rho < 1$ for some number $\rho$. Also $r_{n+1} - r_n \to 0$. Thus from Theorem 3.14, $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$. Suppose that $r = 1$. Then

$0 < r_n \leq r_{n+1} < 1$, and $a_n \in <0,0>$ or $a_n \in \langle \pi,0\rangle$.

Also, $a_{\delta n}/a_n = 1/(1-r_{n+1}) - 1/(1-r_n) \geq 0$ and $0 \leq a_{\delta n}/a_n = n(Q_n - Q_{n+1})/Q_n Q_{n+1} + 1/Q_{n+1}$. Hence, with $P_n = 1$, we have $0 \leq n(Q_{n+1} - Q_n)/Q_n Q_{n+1} \leq 1/Q_{n+1}$, $n|Q_{n+1} - Q_n| \leq |P_n Q_n|$, and $P_n/Q_{n+1} \to 0$. Since $Q_n \to +\infty$, $\Sigma a_n$ converges. Thus, from Theorem 4.16, $a_{\delta n}/a_n \to 0$ and $\Sigma a_{\delta n} \in \text{MR}(\Sigma a_n)$. Q.E.D.

As previously noted, Lubkin's Theorem 6 is not applicable if $r_n \to r = 1$, and his Theorem 7, in which $r_n \to 1$, is vacuous. This is not the case with Theorem 4.17. In particular, if $Q_n = a_n^p$ where $a > 0$ and $0 < p < 1$, it can be verified that $r_n \to 1$ and Theorem
4.17 is applicable. The same is true with $Q_n = a_n/(\ln n)^p$ where $a > 0$ and $p > 0$. Moreover, the proof of Theorem 4.17 shows that the theorem itself is a special case of Theorem 4.16. Consequently, Theorem 4.16 is also applicable with $r_n \to 1$.

**Theorem 4.18.** If $\Sigma a_n$ is a complex series such that $\Sigma a_{\alpha n}$, $\alpha_n = n/(Q_n - 1)$, and $\Sigma a_{\delta n}$ both converge more rapidly to $S$ than $\Sigma a_n$, then $Q_n \to \infty$.

**Proof:** From Theorem 3.2, $\alpha_n \sim \delta_n$, i.e., $n/(Q_n - 1) \sim n/Q_n$. Hence, $(Q_n - 1)/Q_n = 1 - 1/Q_n \to 1$, and thus $Q_n \to \infty$. Q.E.D.

**Theorem 4.19.** Suppose that $\Sigma a_n$ is a complex series such that $Q_n \to \infty$. Then $\Sigma a_{\alpha n}$, $\alpha_n = n/(Q_n - 1)$, converges more rapidly to $S$ than $\Sigma a_n$ if and only if $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

**Proof:** Since $Q_n \to \infty$, $\delta_n/\alpha_n = [n/Q_n][(Q_n - 1)/n] = 1 - 1/Q_n \to 1$, i.e., $\delta_n \sim \alpha_n$. Thus, from Theorem 3.2, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ if and only if $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. Q.E.D.

**Theorem 4.20.** Suppose that $\Sigma a_n$ is a real series such that $-1 < r_n \leq r_{n+1}$, $Q_n < Q_{n+1}$, and $Q_n \to +\infty$. Suppose, in addition, that $q$ is an integer and $a_n' = a_n z^{n+q}$ for
every n. Then for each complex number z satisfying
0 < |z| ≤ 1, Σa'n ε MR(Σa'n) and Σa'n ε MR(Σa'n), where
a'n = (1-r'n)/(1-2r'n+r'n, r') or a'n = n/(Q'n-1).

Proof: From Theorem 4.17, a'n/a'n → 0 and Σa'n ε MR(Σa'n).
Let z be any complex number such that 0 < |z| ≤ 1. From
Theorem 4.7, Σa'n ε MR(Σa'n).

Suppose a'n = (1-r'n)/(1-2r'n-r'n-1, r'). If z = 1,
a'n/a'n = a'n/a'n → 0. If z ≠ 1, a'n/a'n
= (r'n+1-r')/(1-r')(1-r'n+1) = (zr'n+1-zr')/(1-zr')(1-zr'n+1)
→ 0/(1-zr)(1-zr) = 0, since r'n → r where -1 < r ≤ 1.
In either case, Theorem 3.5 implies Σa'n ε MR(Σa'n).

Suppose that a'n = n/(Q'n-1). Then Q'n = n(1-r'n)
= n(1-zr'n) → ∞. From Theorem 4.19 and Σa'n ε MR(Σa'n),
Σa'n ε MR(Σa'n). Q.E.D.

Lemma 4.21. If Σa'n is a complex series such that Q'n → Q
where Re Q > 1, then n(l-|r'n|)→ Re Q, Σa'n converges
absolutely, na'n → 0, and Σa'n = S where a'n = n/(Q-1).

Proof: Let a, b be any numbers satisfying 1 < a < Re Q < b.
Geometrically, it can be seen that |n-b| ≤ |n-Q'n| ≤ |n-a|
so that 1-b/n ≤ |1-Q'n/n| = |r'n| ≤ 1-a/n, and thus
a \leq n(1-|r_n|) \leq b \quad \text{and} \quad |\text{Re } Q - n(1-|r_n|)| \leq |b-a|.

With |b-a| > 0 taken arbitrarily small, we thus conclude that 
\lim_{n \to \infty} n(1-|r_n|) = \text{Re } Q. \quad \text{Since} \quad |r_n| \leq 1 - a/n, \quad \Sigma a_n \text{ converges absolutely. Since} \quad |r_n| \leq 1 \quad \text{and} \quad \Sigma|a_n| \text{ converges,}

n|a_n| \to 0, \quad \text{i.e.,} \quad n a_n \to 0 \quad (15, \text{p. 124}). \quad \text{Consequently,}

\Sigma a_n = S_n - a_{n+1} a_{n+1} = S_n - a_{n+1} (n+1)/(Q-1) \to S, \quad \text{i.e.,}

\Sigma a_{a_n} = S. \quad \text{Q.E.D.}

Theorem 4.22. If \Sigma a_n is a complex series such that

a_n \in \langle a', 0 \rangle \quad \text{for some set} \quad \langle a', 0 \rangle \quad \text{and} \quad Q_n \to Q \quad \text{where}

\text{Re } Q > 1, \quad \text{then} \quad T_n/n \to 1/(Q-1) \quad \text{and} \quad \Sigma a_n \in \text{MR}(\Sigma a_n) \quad \text{where}

a_n = n/(Q-1).

\text{Proof:} \quad \text{From Lemma 4.21,} \quad \Sigma a_{a_n} = S. \quad \text{Also,} \quad a_{a_n}/a_n

= 1 + r_{n+1} a_{n+1} - a_n = 1 + (1-Q_{n+1}/(n+1))(n+1)/(Q-1) - n/(Q-1)

= 1 + (n+1)/(Q-1) - Q_{n+1}/(Q-1) - n/(Q-1) = 1 + 1/(Q-1) - Q_{n+1}/(Q-1)

= (Q-Q_{n+1})/(Q-1) \to 0. \quad \text{Thus, from Theorem 4.5,} \quad \Sigma a_{a_n}

\quad \text{converges more rapidly than} \quad \Sigma a_n. \quad \text{From Theorem 3.6,} \quad n/(Q-1)

\quad = a_n \sim T_n/r_n, \quad \text{so that} \quad T_n/n \sim r_n/(Q-1) \to 1/(Q-1) \quad \text{and}

T_n/n \to 1/(Q-1). \quad \text{Q.E.D.}

Szász (26, p. 274) has proven Theorem 4.22 in the following form for real series: If \quad u_n > 0, \quad a > 1, \quad and
$u_n/u_{n-1} = 1-a/n + \gamma_{n-1}/n$ where $\gamma_n \to 0$, then the transform $t_n = s_n + (n+1)u_{n+1}/(a-1)$ converges more rapidly than $s_n = u_0 + u_1 + u_2 + \ldots + u_n$, and $|s-t_n| < \gamma_{n+1}(s-s_n)/(a-1)$ where $\gamma_n = \max_{k \geq n} |\gamma_k|$. A slight error is evident here, since strict equality cannot hold if $\gamma_n = 0$. We now generalize Theorem 4.22 by removing the condition $a_n \in \langle \alpha', \beta \rangle$.

**Theorem 4.23.** If $Q_n \to Q$ where $\text{Re } Q > 1$, then $T_n/n \to 1/(Q-1)$, and $\Sigma a_{an} \in \text{MR}(\Sigma a_n)$ where $a_n = n/(Q-1)$.

**Proof:** We have $r_n = 1-Q/n = 1-Q/n-(Q_n-Q)/n$. Setting $\gamma_{n-1} = Q_n-Q$, $r_n = 1-Q/n-\gamma_{n-1}/n$ where $\gamma_n \to 0$. Hence, $na_n = na_{n-1}-Qa_{n-1}-\gamma_{n-1}a_{n-1} = (n-1)a_{n-1}+(1-Q)a_{n-1}-\gamma_{n-1}a_{n-1}$ and, replacing $n$ by $n+1$, $(n+1)a_{n+1} = na_n+(1-Q)a_n-\gamma n a_n$.

Consequently $na_n - (n+1)a_{n+1} = (Q-1)a_n + \gamma_n a_n$. From Lemma 4.21, $na_n \to 0$ and $\Sigma a_n$ converges. Thus $na_n$

$$\sum_{k=n}^{\infty} [ka_k - (k+1)a_{k+1}] = (Q-1) \sum_{k=n}^{\infty} a_k + \sum_{k=n}^{\infty} \gamma_k a_k.$$ From Lemma 4.21, $\Sigma |a_n|$ converges, so that $|na_n-(Q-1)\sum_{k=n}^{\infty} a_k|$

$$= |\sum_{k=n}^{\infty} \gamma_k a_k| \leq \sum_{k=n}^{\infty} |\gamma_k a_k| \leq \gamma_n \sum_{k=n}^{\infty} |a_k|$$ where $\gamma_n$

$$= \max_{k \geq n} |\gamma_k| \to 0. \text{ Dividing by } |na_{n-1}|, \ |r_n-(Q-1)T_n/n|$$

$$\leq \gamma_n \sum_{k=n}^{\infty} |a_k|/|na_{n-1}|. \text{ Setting } a'_n = |a_n|,$$
\[ r'_n = \frac{a'_n}{a'_{n-1}} = |r_n|, \quad Q'_n = n(1-r'_n) = n(1-|r_n|), \quad \text{and} \]
\[ T'_n = \sum_{k=n}^{\infty} |a_k|/|a_{n-1}|, \quad \text{we have} \quad Q'_n \to Q' = \text{Re} Q \quad \text{from Lemma 4.21}, \quad \text{and} \]
\[ \sum_{k=n}^{\infty} |a_k|/|n a_{n-1}| = T'_n/n \to 1/(Q'-1) \quad \text{from Theorem 4.22}. \]
Thus, \[ |r_n - (Q-1)T_n/n| \leq \gamma_n T'_n/n \to 0, \]
so that \((Q-1)T_n/n \to 1\) since \(r_n \to 1\). Hence \(T_n/n \to 1/(Q-1)\), and \(n/(Q-1) \sim T_n \sim T_n/r_n\), i.e., \(a_n \sim T_n/r_n\). From Theorem 3.6, \(\sum a_n \in \text{MR}(\sum a_n)\). Q.E.D.

**Corollary 4.24.** If \(Q_n \to Q\) where \(\text{Re} Q > 1\), then
\[ T'_{n+1} - T_n \to 1/(Q-1). \]

**Proof:** Using Theorem 4.23,
\[ T'_{n+1} - T_n = T'_{n+1} - r_n (1+T'_{n+1}) \]
\[ = (1-r_n) T'_{n+1} - r_n = Q_n T'_{n+1}/n - r_n \to Q/(Q-1) - 1 = 1/(Q-1). \]
Q.E.D.

Suppose that \(Q_n \to Q\) where \(\text{Re} Q > 1\). Recalling that \(a_n = 1+T'_{n+1}, n \geq 2\), yields the best transform for accelerating convergence, we are led quite naturally to the transform sequence 1.5 in the Introduction by Corollary 4.24 and the following estimate:
\[ 1+T'_{n+1} = 1/(1-r_n) \]
\[ + (T'_{n+1} - T_n)/(1-r_n) \approx 1/(1-r_n) + [1/(Q-1)]/(1-r_n) \]
\[ = Q/(Q-1)(1-r_n). \]
Theorem 4.25. Suppose that \( Q_n \to Q \) where \( \text{Re} \, Q > 1 \). Then \( \sum a_n \in \text{MR}(\sum a_n) \) if and only if \( a_n/n \to 1/(Q-1) \).

Proof: From Theorem 4.23, \( \sum a_n \in \text{MR}(\sum a_n) \) where \( \beta_n = n/(Q-1) \). Thus, from Theorem 3.2, \( \sum a_n \in \text{MR}(\sum a_n) \) if and only if \( a_n \sim \beta_n \), i.e., \( a_n \sim n/(Q-1) \). But this is equivalent to \( a_n/n \to 1/(Q-1) \). Q.E.D.

Corollary 4.26. Suppose that \( Q_n \to Q \) where \( \text{Re} \, Q > 1 \), and that \( a_n = n/(Q_n-1) \). Then \( \sum a_n \in \text{MR}(\sum a_n) \).

Proof: We have \( a_n/n = 1/(Q_n-1) \to 1/(Q-1) \). Thus, from Theorem 4.25, \( \sum a_n \in \text{MR}(\sum a_n) \). Q.E.D.

Theorem 4.27. Suppose that \( Q_n \to Q \) where \( \text{Re} \, Q > 1 \), and \( a_n = b \delta_n \) where \( b \) is any complex number. Then:

1. \( \sum a_n \in \text{MR}(\sum a_n) \) if and only if \( b = Q/(Q-1) \).
2. \( \sum a_n \) converges to \( S \) with the same rapidity as \( \sum a_n \) if, and only if, \( b \neq Q/(Q-1) \).

Proof: Part (1). From Theorem 4.25, \( \sum a_n \in \text{MR}(\sum a_n) \) if and only if \( b \delta_n/n \to 1/(Q-1) \), i.e., \( b/Q_n \to 1/(Q-1) \). But this is equivalent to \( b/Q = 1/(Q-1) \), i.e., \( b = Q/(Q-1) \).
Part (2). Suppose that \( b \neq Q/(Q-1) \). From Lemma 4.21, \( \Sigma a_n \) converges. From Theorem 4.23, \( n/T_n \rightarrow Q-1 \).

Thus, since \( r_n \rightarrow 1 \), \( (S-S_{a(n-1)})/(S-S_{n-1}) \)
\[ = (S-S_{n-1}-a_n \delta_n)/(S-S_{n-1}) = 1-r_n \delta_n/T_n = 1-b \delta_n/T_n \]
\[ = 1-(b \delta_n)/(T_n Q_n) \rightarrow 1-b(Q-1)/Q \neq 0. \]
Consequently \( \Sigma a_n \) converges to \( S \) with the same rapidity as \( \Sigma a_n \).

The converse follow from (1). Q.E.D.

**Corollary 4.28.** If \( Q_n \rightarrow Q \) where \( \text{Re} \ Q > 1 \), then \( \Sigma a_{b_n} \) converges to \( S \) with the same rapidity as \( \Sigma a_n \).

**Proof:** Setting \( b=1 \), we have \( \delta_n = b \delta_n \) and \( b \neq Q/(Q-1) \).

Now apply (2) of Theorem 4.27. Q.E.D.

**Corollary 4.29.** Suppose that \( \Sigma a_n \) is a real series such that \(-1 < r_n \leq r_{n+1} \) and \( Q_n \leq Q_{n+1} \). Then a n.a.s.c. that \( \Sigma a_{b_n} \in \text{MR}(\Sigma a_n) \) is that \( Q_n \rightarrow +\infty \).

**Proof:** The sufficiency is a restatement of Theorem 4.17.

For the necessity, since \( \Sigma a_n \) converges and \( Q_n \leq Q_{n+1} \), we see that \( Q_n \rightarrow Q \) where \( 1 < Q \leq +\infty \).

From Corollary 4.28, we cannot have \( Q < +\infty \). Thus, \( Q = +\infty \). Q.E.D.
Lubkin (17, p. 232) has proved the following theorem.

**Theorem 8.** If $\Sigma a_n$ is a convergent real series, $Q$ exists $\neq 1$, and $n(Q_n - Q_{n-1}) \to 0$ as $n \to \infty$, then the series $U = \Sigma u_n$ converges more rapidly to $S$ than $\Sigma a_n$, where $u_n = (Qa_n - a_n)/(Q-1)$ for $n \geq 0$.

In Theorem 8, the convergence of $\Sigma a_n$ and the existence of $Q \neq 1$ implies that $Q > 1$. With this in mind, we presently show that the condition $n(Q_n - Q_{n-1}) \to 0$ can be omitted from the hypothesis of Theorem 8 and, at the same time, generalize into the complex plane. Pflanz (18, p. 25) proved this fact for real series.

Before extending Theorem 8, we note that Shanks (23, p. 39) suggests the transform $e_1^{(s)}(A_n)$

$$= (s B_n - A_n)/(s-1),$$

where $s = \lim_{n \to \infty} (\Delta A_n)/(\Delta B_n)$ and $B_n = e_1(A_n)$, be applied for acceleration in the critical case $r_n \to 1$. In our notation, this transform becomes

$$e_1^{(s)}(S_n) = S_\alpha n = (s S_\delta n - S_n)/(s-1) = [s(S_n + a_{n+1, \delta n+1} - S_n)]/(s-1)$$

$$= [(s-1) S_n + s a_n + a_{n+1, \delta n+1}]/(s-1) = S_n + a_{n+1, \delta n+1}/(s-1)$$

$$= S_n + a_{n+1, \sigma n+1},$$

where $\alpha_n = s \delta_n/(s-1)$ and $s = \lim a_n/a_\delta n$. Shanks (23, p. 40) appears to be unaware of Lubkin's transform given in Theorem 8, or, at least, that
the two transforms are identical, if \( n(Q_n-Q_{n-1}) \to 0 \) and \( Q \) exists with \( \text{Re } Q > 1 \). In fact, we will see in Theorem 4.32 that if \( Q \) exists with \( \text{Re } Q > 1 \), then \( e^{(s)}(S_n) \) converges more rapidly to \( S \) than \( S_n \) if and only if \( n(Q_n-Q_{n-1}) \to 0 \); consequently Lubkin's transform, given in Theorem 8, has a broader applicability if \( \text{Re } Q > 1 \), since the condition \( n(Q_n-Q_{n-1}) \to 0 \) is irrelevant.

We now extend Lubkin's Theorem 8.

**Theorem 4.30.** If \( \Sigma a_n \) is a series such that \( Q_n \to Q \) where \( \text{Re } Q > 1 \), and \( u_n = (Qa_{\delta n} - a_n)/(Q-1) \) for \( n \geq 0 \), then \( \Sigma u_n \in \text{MR}(\Sigma a_n) \).

**Proof:** Set \( a_n = Q\delta_n/(Q-1) \) for \( n \geq 1 \). Then

\[
U_n = \sum_{k=0}^{n} u_k = \sum_{k=0}^{n} \frac{(Qa_{\delta k} - a_k)}{(Q-1)} = \frac{(Q \sum_{k=0}^{n} a_{\delta k} - S_n)}{(Q-1)} = \frac{(Q S_{\delta n} - S_n)}{(Q-1)} = \frac{[Q(S_n + a_{n+1,\delta n+1}) - S_n]}{(Q-1)} = \frac{[(Q-1)S_n + Q a_{n+1,\delta n+1}]/(Q-1)}{(Q-1)} = \frac{S_n + a_{n+1}[Q\delta_{n+1}]/(Q-1)}{(Q-1)} = \frac{S_n + a_{n+1} a_{\delta n+1}}{Q-1}.
\]

From (1) of Theorem 4.27, \( \Sigma a_n \in \text{MR}(\Sigma a_n) \), so that \( (S-U_n)/(S-S_n) = (S-S_{a_n})/(S-S_n) \to 0 \). Q.E.D.

**Lemma 4.31.** Suppose that \( Q_n \to Q \) for some complex number \( Q \neq 0 \). Then \( a_n/a_{\delta n} \to Q \) if and only if \( n(Q_n-Q_{n-1}) \to 0 \).
Proof: Since $Q_n \neq 0$,

(1) \[ \frac{a_{\delta n}}{a_n} = \frac{(n+1)(Q_n - Q_{n+1})}{Q_n Q_{n+1}} + \frac{1}{Q_n} \]

and

(2) \[ (n+1)(Q_n - Q_{n+1}) = Q_n Q_{n+1} \frac{a_{\delta n}}{a_n} - Q_{n+1}. \]

Thus, if $n(Q_n - Q_{n-1}) \to 0$, then from (1), \[ \frac{a_{\delta n}}{a_n} \to \frac{1}{Q}. \]

Hence, \[ \frac{a_n}{a_{\delta n}} \to Q. \] Conversely, if \[ \frac{a_n}{a_{\delta n}} \to Q, \] then \[ \frac{a_{\delta n}}{a_n} \to \frac{1}{Q}. \] Thus from (2), \[ n(Q_n - Q_{n-1}) \to 0. \] Q.E.D.

Theorem 4.32. Suppose that $Q_n \to Q$ where $\Re Q > 1$,

\[ s = \lim \frac{a_n}{a_{\delta n}} \neq 1, \text{ and } \alpha_n = \frac{s}{s/(s-1)}. \]

Then

\[ \Sigma a_n \in MR(\Sigma a_n) \text{ if and only if } n(Q_n - Q_{n-1}) \to 0. \]

Proof: From Theorem 4.27, \[ \Sigma a_n \in MR(\Sigma a_n) \text{ if and only if } \frac{s}{s/(s-1)} = \frac{Q}{Q-1}, \text{ i.e., } Q = s = \lim \frac{a_n}{a_{\delta n}}. \] But, from

Lemma 4.31, \[ Q = \lim \frac{a_n}{a_{\delta n}} \text{ if and only if } n(Q_n - Q_{n-1}) \to 0. \]

Q.E.D.

It is very easy to construct a series $\Sigma a_n$ satisfying the hypothesis of Theorem 4.30, while \[ n(Q_n - Q_{n-1}) \to 0. \] In particular, we mention the following example.

Example 4.33. Let $Q$ be any number such that $\Re Q > 1$.

Set $\gamma_{2n} = 0, \gamma_{2n-1} = \frac{1}{\sqrt{n}}, \text{ and } Q_n = Q + \gamma_n$. Then
\[ n(Q_n - Q_{n-1}) = n[(Q + \gamma_n) - (Q + \gamma_{n-1})] = n(\gamma_n - \gamma_{n-1}), \]
\[ 2n(Q_{2n} - Q_{2n-1}) = 2n(\gamma_{2n} - \gamma_{2n-1}) = -2\sqrt{n} \to -\infty, \quad \text{and} \]
\[ (2n-1)(Q_{2n-1} - Q_{2n-2}) = (2n-1)(\gamma_{2n-1} - \gamma_{2n-2}) \]
\[ = (2n-1)/\sqrt{n} \to +\infty. \quad \text{Clearly, Q} \to Q \quad \text{so that the hypothesis of Theorem 4.30 is satisfied while} \quad n(Q_n - Q_{n-1}) \to 0. \]

Thus, Lubkin's transformation \( \sum a_n \), given in Theorem 4.30, converges rapidly to \( S \) than \( \sum a_n \). However, as we have just observed, \( |n(Q_n - Q_{n-1})| \to +\infty \); thus, according to Theorem 4.32, Daniel Shank's transform \( e_1(s)(S_n) \)
\[ = S_n + s \delta_{n+1}/(s-1) \] must fail to converge more rapidly to \( S \) than \( S_n \). Here, \( s = \lim a_n/a_{n+1} = 0 \) since
\[ \lim |a_n/a_{n+1}| = \lim |(n+1)(Q_n - Q_{n+1})/Q_n Q_{n+1} + 1/Q_n| = +\infty. \]
Hence, we have in fact \( e_1(s)(S_n) = e_1(0)(S_n) = S_n \), and

thus \( e_1(s)(S_n) \) clearly converges with the same rapidity as \( S_n \). We could have also applied Theorem 4.27 to arrive at this conclusion. If we carry our analysis a little deeper in this example, a very surprising phenomenon arises. In particular, \( u_n/a_n = (Q a_{\delta n}/a_n - 1)/(Q-1), \)
\[ a_{\delta n}/a_n = 1/Q_n - (n+1)(Q_{n+1} - Q_n)/Q_n Q_{n+1}, \quad Q_n \to Q, \] and, as shown above,
\[ (n+1)|Q_{n+1} - Q_n| \to +\infty. \quad \text{Consequently,} \quad |u_n/a_n| \to +\infty \] even though \( \sum u_n \in \text{MR}(\Sigma a_n) \).
Lubkin (17, p. 232-233) has proven the following theorem.

**Theorem 9.** If $\Sigma a_n$ is a convergent real series, $Q \neq 1$, $n(Q_n - Q_{n-1}) \to 0$, and $n[(n+1)(Q_{n+1} - Q_n) - n(Q_n - Q_{n-1})] \to 0$, then the transform $\Sigma w_n$ converges more rapidly to $S$ than $\Sigma a_n$, where $W_0 = 0$ and

$$W_n = w_0 + \cdots + w_n = S_n + a_{n+1}(1-r_n)/(1-2r_{n+1}+r_n r_{n+1})$$

for $n > 0$.

As previously noted, we must have $Q > 1$. With this in mind, we will show in Theorem 4.35 that the condition $n[(n+1)(Q_{n+1} - Q_n) - n(Q_n - Q_{n-1})] \to 0$ can be omitted from the hypothesis of Theorem 9 and, at the same time, generalize into the complex plane.

**Lemma 4.34.** Suppose that $Q_n \to Q$ for some complex number $Q \neq 0$ or $1$, and $a_n = (1-r_{n-1})/(1-2r_n+r_{n-1}r_n)$. Then $a_n/n \to 1/(Q-1)$ if and only if $n(Q_n - Q_{n-1}) \to 0$.

**Proof:** From Lemma 4.31, $n(Q_n - Q_{n-1}) \to 0$ if and only if $a_{\delta n}/a_n \to 1/Q$. As shown in the proof of Theorem 3.4,

$$1-2r_{n+1}+r_n r_{n+1} = (1-r_{n})(1-r_{n+1})(1-a_{\delta n}/a_n).$$

Thus,

$$a_{n+1}/(n+1) = [1/(n+1)][(1-r_n)/(1-2r_{n+1}+r_n r_{n+1})]$$

$$= 1/[Q_{n+1}(1-a_{\delta n}/a_n)],$$

so that $a_{\delta n}/a_n \to 1/Q$ if and only
if \( \alpha_{n+1}/(n+1) \to 1/(Q-1) \). Q.E.D.

**Theorem 4.35.** Suppose that \( Q_n \to Q \) where \( \text{Re } Q > 1 \), and
\[
\alpha_n = (1-r_{n-1})/(1-2r_n + r_{n-1}^2 r_n).
\]
Then \( \sum \alpha_n \in \text{MR}(\Sigma a_n) \) if and only if \( n(Q_n - Q_{n-1}) \to 0 \).

**Proof:** From Theorem 4.25, \( \sum \alpha_n \in \text{MR}(\Sigma a_n) \) if and only if
\[
\alpha_n/n \to 1/(Q-1);
\]
and according to Lemma 4.34, this is equivalent to \( n(Q_n - Q_{n-1}) \to 0 \). Q.E.D.
CHAPTER V

NONALTERNATING SERIES

A real series $\Sigma a_n$ will be called nonalternating iff $r_n > 0$ for every $n$, and $N$-nonalternating iff $r_n > 0$ for $n \geq N$, where $N$ is some integer.

Shortly, it will be shown that E. E. Kummer's criterion for the convergence of a nonalternating series is not only sufficient, but also necessary. We now prove a slight generalization of this fact.

Theorem 5.1. Let $L$ be any real number and $c$ be any positive number. Then a n.a.s.c. that an $N$-nonalternating series $\Sigma a_n$ converge is that there exist a sequence $\{\beta_n\}$ such that,

(1) $a_n \beta_n \to L$,

and

(2) $\beta_n \geq c + r_n \beta_n + 1$, $n \geq N$.

Moreover, if (1) and (2) hold, then for $n \geq N$,

(a) $0 < r_n < T_n \leq r_n \beta_n / c - L/c a_{n-1}$.

And in general, for $n \geq N$ and $k \geq 1$,

(b) $T_{n, k-2} < T_n \leq T_{n, k-2} + (r_n \cdots r_{n+k-1}) \beta_{n+k-1} / c - L/c a_{n-1}$.
Proof: For the necessity, define \( \beta_n = c + c T_{n+1} + L/a_n \) for \( n \geq N \). Consequently, \( a_n \beta_n = c a_n + c T_{n+1} + L = c a_n + c (S-S_n) + L \to L \) as \( n \to \infty \). For \( n \geq N \),
\[ c + r_{n+1} \beta_{n+1} = c + r_{n+1} (c + c T_{n+2} + L/a_{n+1}) = c + c r_{n+1} (1 + T_{n+2}) + L/a_n \]
\[ = c + T_{n+1} + L/a_n = \beta_n, \] so that (2) hold with equality.

For the sufficiency, assume that (1) and (2) hold. Let \( n \) be any integer \( \geq N \), and define \( P_k = T_{n,k-2} + (r_n \cdots r_{n+k-1}) \beta_{n+k-1}/c \) for \( k \geq 1 \). From (2),
\[ P_{k+1} - P_k = (r_n \cdots r_{n+k-1}) (1 + r_{n+k} \beta_{n+k}/c - \beta_{n+k-1}/c) \leq 0 \] for \( k \geq 1 \). Also, \( P_k = a_{n+k-1} \beta_{n+k-1}/c a_{n-1} \to L/c a_{n-1} \) as \( k \to \infty \). Thus \( \{P_k\} \) is a monotone bounded sequence, so that \( P_k \to P \) as \( k \to \infty \), for some number \( P \). Consequently, \( T_{n,k-2} = P_k - a_{n+k-1} \beta_{n+k-1}/c a_{n-1} \to P - L/c a_{n-1} \) as \( k \to \infty \). Hence \( T_n = P - (L/c a_{n-1}) \leq P_k - (L/c a_{n-1}) \) for \( k \geq 1 \). Obviously, \( T_{n,k-2} < T_n \) for \( k \geq 1 \). Thus (b) holds, and (a) follows from (b). Q.E.D.

Condition (1) of Theorem 5.1 can be somewhat weakened, as is now proven.

Corollary 5.2. Let \( c \) be any positive number. Then a n.a.s.c. that an N-nonalternating series \( \Sigma a_n \) converge is that there exist a sequence \( \{\beta_n\} \) such that,
(1) some subsequence of \( \{a_n \beta_n\} \) is bounded,
and

(2) \( \beta_n \geq c + r_{n+1} \beta_{n+1} \), \( n \geq N \).

Moreover, if (1) and (2) hold, then \( \{a_n \beta_n\} \) converges.

**Proof:** The necessity follows from Theorem 5.1.

For the sufficiency, we may assume that \( a_n > 0 \) for \( n \geq N-1 \). From (2), \( a_n \beta_n \geq c a_n + a_{n+1} \beta_{n+1} > a_{n+1} \beta_{n+1} \)
for \( n \geq N \). Thus \( \{a_n \beta_n\} \) converges because of (1). Now apply Theorem 5.1. Q.E.D.

**Corollary 5.3.** Let \( c \) be any positive number. Then a n.a.s.c. that a series \( \sum a_n \) of positive terms converge
is that there exist a sequence \( \{\beta_n\} \) such that,

(1) some subsequence of \( \{a_n \beta_n\} \) is bounded below,
and

(2) \( \beta_n \geq c + r_{n+1} \beta_{n+1} \).

Moreover, if (1) and (2) hold, then \( \{a_n \beta_n\} \) converges.

**Proof:** The necessity follows from Theorem 5.1.

For the sufficiency, from (2) we have \( a_n \beta_n \geq c a_n + a_{n+1} \beta_{n+1} \geq a_{n+1} \beta_{n+1} \). Thus \( \{a_n \beta_n\} \) converges because of (1). From Theorem 5.1, \( \sum a_n \) converges. Q.E.D.
Corollary 5.4. Let $L$ be any real number. Then a n.a.s.c. that an $N$-nonalternating $\Sigma a_n$ converge is that there exist a sequence $\{\beta_n\}$ such that,

1. $a_n\beta_n \to L$,

and

2. $\beta_n \geq 1 + r_{n+1}\beta_{n+1}$, $n \geq N$.

Moreover, if (1) and (2) hold, then for $n \geq N$,

(a) $0 < r_n < T_n \leq r_n\beta_n - (L/a_{n-1})$.

And in general, for $n \geq N$ and $k \geq 1$,

(b) $T_{n,k-2} < T_n \leq T_{n,k-2} + (r_n \cdots r_{n+k-1})\beta_{n+k-1} - (L/a_{n-1})$.

Proof: Choose $c = 1$ in Theorem 5.1. Q.E.D.

Let $\Sigma a_n$ be any divergent nonalternating series such that $a_n \to 0$. Let $\beta_1$ be any real number, and define $\{\beta_n\}$ recursively by $\beta_n = 1 + r_{n+1}\beta_{n+1}$. Then $a_n\beta_n - a_{n+1}\beta_{n+1} = a_n \to 0$, and $\beta_n \geq 1 + r_{n+1}\beta_{n+1}$ for $n \geq 1$. Thus, we cannot replace (1) of Corollary 5.4 by the condition that $a_n\beta_n - a_{n+1}\beta_{n+1} \to 0$.

Theorem 5.5. (Kummer's criterion) Let $c$ be any positive number. Then a n.a.s.c. that an $N$-nonalternating series $\Sigma a_n$ converge is that there exist a sequence $\{\beta_n\}$ such that,
(1) $\beta_n \geq 0$, $n \geq N$, 

and 

(2) $\beta_n \geq c + r_{n+1} \beta_{n+1}$, $n \geq N$.

Moreover, if (1) and (2) hold, then for $n \geq N$,

(a) $0 < r_n < T_n \leq r_n \beta_n / c - (\lim_{k \to \infty} a_k \beta_k) / c a_{n-1} \leq r_n \beta_n / c$,

and

(b) $\{a_n \beta_n\}$ converges.

**Proof:** We may assume throughout that $a_{n-1} > 0$ for $n \geq N$.

For the necessity, choose $L > 0$ in Theorem 5.1.

From (a) of Theorem 5.1, $\beta_n \geq 0$ for $n \geq N$.

For the sufficiency, according to Theorem 5.1 we need only show that $a_n \beta_n \to L$ for some number $L > 0$. From (1) and (2) above, $a_n \beta_n \geq c a_n + a_{n+1} \beta_{n+1} > a_{n+1} \beta_{n+1} \geq 0$ for $n \geq N$, which implies the existence of the required number $L$. Q.E.D.

The fact that Kummer's criterion, Theorem 5.5, is also necessary was first published by Shanks (24, p. 338-341). In (24, p. 338-341), Shanks employs Theorem 5.5 in an equivalent form to serve as a general framework for short proofs of the sufficient conditions of many of the known tests for convergence or divergence of series with positive terms. On the other hand, we are interested in Theorem 5.5
also as furnishing bounds for $T_n$ and $S-S_{n-1}$, and consequently exhibiting the convergence of \{T_n\} under certain conditions.

It should be noted that Theorem 5.1, as a criterion for convergence of $\Sigma a_n$, is more general than Theorem 5.5 in the sense that for every convergent non-alternating series $\Sigma a_n$ there is a sequence $\{\beta_n\}$ satisfying (1) and (2) of Theorem 5.1 with $N=1$, while condition (1) of Theorem 5.5 fails to hold for the same sequence $\{1/n\}$. In particular, let $\Sigma a_n$ be a convergent non-alternating series and $\{\beta_n\}$ be any sequence satisfying (1) and (2) of Theorem 5.1 with $N=1$. Let $L'$ be any number such that $(L-L')/a_n < 0$ for $n \geq 0$, and define $\beta'_n = \beta_n - L'/a_n$. Then $a_n\beta'_n = a_n\beta_n - L' \rightarrow L-L'$, so that $\beta'_n \rightarrow -\infty$ and $\beta_n < 0$. Moreover, for $n \geq 1$, $\beta'_n = \beta_n - L'/a_n \geq c + r_{n+1}\beta'_{n+1}$. Thus, (1) $a_n\beta'_n \rightarrow L-L'$ and (2) $\beta'_n \geq c + r_{n+1}\beta'_{n+1}$, while the condition $\beta'_n \geq 0$ fails for large $n$.

Theorem 5.6. A n.a.s.c. that an $N$-nonalternating series $\Sigma a_n$ converge is that there exist a sequence $\{\beta_n\}$ such that
(1) \( \beta_n \geq 0, \ n \geq N, \)

and

(2) \( \beta_n \geq 1 + r_{n+1}\beta_{n+1}, \ n \geq N. \)

Moreover, if (1) and (2) hold, then for \( n \geq N, \)

(a) \( 0 < r_n < T_n \leq r_n\beta_n - (\lim_{k \to \infty} a_k\beta_k)/a_{n-1} \leq r_n\beta_n. \)

**Proof:** Choose \( c = 1 \) in Theorem 5.5. Q.E.D.

**Example 5.7.** Let \( \Sigma a_n = 1 + 1/2^2 + 1/3^2 + \cdots. \) Then,

\[
a_n = 1/(n+1)^2 \quad \text{for} \quad n \geq 0, \quad \text{and} \quad r_n = [n/(n+1)]^2 \quad \text{for} \quad n \geq 1. \]

Defining \( \beta_n = (n+2)^2 \quad \text{for} \quad n \geq 1, \quad \beta_n \geq 1 + r_{n+1}\beta_{n+1} \quad \text{for} \quad n \geq 1, \quad \text{and, for} \quad k \geq 1, \quad a_k\beta_k = [(k+2)/(k+1)]^2 - 1. \) From Theorem 5.6, \( \Sigma a_n \) converges.

Some of the known tests for convergence are now proven by exhibiting a sequence \( \{\beta_n\} \) satisfying the conditions of the preceding theorem.

**Theorem 5.8.** (Comparison test) If \( 0 < a_n' \leq a_n \) and \( \Sigma a_n \) converges, then \( \Sigma a_n' \) converges.

**Proof:** From Theorem 5.6, there is a sequence \( \{\beta_n\} \) such that \( \beta_n \geq 0 \) and \( \beta_n \geq 1 + r_{n+1}\beta_{n+1}. \) Accordingly,

\[
a_n\beta_n/a_n' \geq a_n/a_n' + (a_{n+1}'/a_n')(a_{n+1}\beta_{n+1}/a_{n+1}') \geq 1 + r_{n+1}'(a_{n+1}\beta_{n+1}/a_{n+1}') \geq 0. \]  

Now apply Theorem 5.6. Q.E.D.
Theorem 5.9. (Ratio comparison test) If \( 0 < r_n' \leq r_n \) and \( \Sigma a_n \) converges, then \( \Sigma a_n' \) converges.

Proof: From Theorem 5.6, there is a sequence \( \{\beta_n\} \) such that \( \beta_n \geq 0 \) and \( \beta_n \geq 1 + r_{n+1}\beta_{n+1} \). Accordingly, \( \beta_n \geq 1 + r_{n+1}\beta_{n+1} \geq 1 + r_{n+1}'\beta_{n+1} \), since \( 0 < r_n' \leq r_n \) and \( \beta_n \geq 0 \). Now apply Theorem 5.6. Q.E.D.

Theorem 5.10. (Root test) If \( a_n > 0 \) and \( \lim \sup \sqrt[n]{a_n} < 1 \), then \( \Sigma a_n \) converges.

Proof: Let \( t \) be any number satisfying \( \lim \sup \sqrt[n]{a_n} < t < 1 \).

Then \( a_n \leq t_n \). Defining \( \beta_n = \frac{t^n}{a_n(1-t)} \), \( \beta_n - r_{n+1}\beta_{n+1} = \frac{t^n}{a_n(1-t)} - r_{n+1} t^{n+1}/a_{n+1}(1-t) = \frac{t^n}{a_n(1-t)} - t^{n+1}/a_{n+1}(1-t) = \frac{[t^n/a_n(1-t)](1-t)}{a_n} \geq 1 \).

Thus \( \beta_n \geq 0 \) and \( \beta_n \geq 1 + r_{n+1}\beta_{n+1} \). Now apply Theorem 5.6. Q.E.D.

Theorem 5.11. (Ratio test) If \( 0 < r_n \) and \( \lim \sup r_n < 1 \), then \( \Sigma a_n \) converges.

Proof: Let \( t \) be any number for which \( \lim \sup r_n < t < 1 \).

Defining \( \beta_n = 1/(1-t) \), we have \( \beta_n = 1 + t\beta_n \geq 1 + r_{n+1}\beta_{n+1} \) since \( 0 < r_n < t \). Now apply
Theorem 5.6. Q.E.D.

Theorem 5.12. (Raabe's test) If $0 < r_n \leq 1 - a/n$ where $1 < a$, then $\Sigma a_n$ converges.

**Proof:** Set $\beta_n = n/(a-1)$. Then $\beta_n > 0$ and

$$1 + r_{n+1}\beta_{n+1} \leq 1 + [1-a/(n+1)]\beta_{n+1} = 1 + (n+1)/(a-1) - a/(a-1) = n/(a-1) = \beta_n,$$ so that $\beta_n \geq 1 + r_{n+1}\beta_{n+1}$.

Now apply Theorem 5.6. Q.E.D.

Theorem 5.13. Let $L$ be any real number and $c$ be any positive number. Then a necessary condition that an $N$-nonalternating series $\Sigma a_n$ converge is that there exist a sequence $\{a_n\}$ such that,

1. $a_n a_n \to L$,

and

2. $a_n \leq c + r_{n+1}a_{n+1}$, $n \geq N$.

Moreover, if (1) and (2) hold, then for $n \geq N$,

(a) $r_n a_n / c - L / ca_n - 1 \leq T_n'$

and in general, for $n \geq N$ and $k \geq 1$,

(b) $T_{n, k-1} + (r_n \cdots r_{n+k-1}) a_{n+k-1} / c - L / ca_{n-1} \leq T_n$.

**Proof:** For the necessity, we may use the proof of the necessity of Theorem 5.1, replacing "$\beta$" by "$\alpha$" throughout.
Next, assume that (1) and (2) hold. Let \( n \) be any integer \( \geq N \), and define \( P_k = T_n, k-2 + (r_n \cdots r_{n+k-1}) a_{n+k-1}/c \) for \( k \leq 1 \). From (2), \( P_{k+1} - P_k = (r_n \cdots r_{n+k-1}) (1 + r_{n+1} a_{n+k-1}/c - a_{n+k-1}/c) \geq 0 \) for \( k \leq 1 \).

Also, \( P_k = T_n, k-2 + a_{n+k-1} a_{n+k-1}/a_{n-1} - c = T_n + L/c a_{n-1} \).

Thus, \( P_k - L/c a_{n-1} \leq T_n \) for \( k \leq 1 \), i.e., (b) holds.

With \( k = 1 \), (b) reduces to (a). Q.E.D.

Theorem 5.14. Let \( L \) be any real number. Then a necessary condition that an \( N \)-nonalternating series \( \sum a_n \) converge is that there exist a sequence \( \{a_n\} \) such that,

1. \( a_n a_n \to L \),

and

2. \( a_n \leq 1 + r_{n+1} a_{n+1} \), \( n \geq N \).

Moreover, if (1) and (2) hold, then for \( n \geq N \),

(a) \( r_n a_n - (L/a_{n-1}) \leq T_n \),

and in general, for \( n \geq N \) and \( k \geq 1 \),

(b) \( T_n, k-2 + (r_n \cdots r_{n+k-1}) a_{n+k-1} - (L/a_{n-1}) \leq T_n \).

Proof: Choose \( c = 1 \) in Theorem 5.13. Q.E.D.

Theorem 5.15. Let \( c \) be any positive number. Then a n.a.s.c. that an \( N \)-nonalternating series \( \sum a_n \) diverge is that there exist a sequence \( \{a_n\} \) such that,
(1) $|a_n a_n| \to \infty$,

and

(2) $a_n \leq c + r_{n+1} a_{n+1} \leq c + a_n$, $n \geq N$.

**Proof:** We may assume that $a_{n-1} > 0$ for $n \geq N$.

For the necessity, let $a_N$ be any real number, and define $\{a_n\}$ recursively by the equation $a_n = c + r_{n+1} a_{n+1}$. Accordingly, $a_n = c + r_{n+1} a_{n+1} < c + a_n$ for $n \geq N$, i.e., (2) holds. For $k \geq 1$, $a_{N+k} a_{N+k} = a_N a_N$

$\cdots + a_{N+k-1} \to -\infty$ as $k \to \infty$, i.e., (1) holds.

For the sufficiency, from (2) we have $a_{n+1} a_{n+1} \leq a_n a_n$ for $n \geq N$. Thus, (1) implies that $a_n a_n \to -\infty$.

From (2), $(a_n a_n - a_n a_n a_n a_n) / c \leq a_n + a_{n+1} + \cdots + a_{N+n-1} \to +\infty$

as $k \to \infty$, since $-a_n a_n \to +\infty$ as $n \to \infty$. Thus $\sum a_n$ diverges. Q.E.D.

**Corollary 5.16.** Let $c$ be any positive number. Then a n.a.s.c. that a series $\sum a_n$ of positive terms diverge is that there exist a sequence $\{a_n\}$ such that,

(1) some subsequence of $\{a_n a_n\}$ is unbounded, and

(2) $a_n \leq c + r_{n+1} a_{n+1} \leq c + a_n$, $n \geq 1$. 

Moreover, if (1) and (2) hold, then $a_n a_n \to -\infty$.

**Proof:** The necessity follows from Theorem 5.15.

For the sufficiency, from (2) we have

$$a_{n+1}a_{n+1} \leq a_na_n \text{ for } n \geq 1.$$ Thus from (1), $a_na_n \to -\infty$.

Hence $|a_na_n| \to +\infty$ and, according to Theorem 5.15, $\Sigma a_n$ diverges. Q.E.D.

Clearly, (1) of Corollary 5.16 may be replaced by the condition $a_na_n \to -\infty$.

**Theorem 5.17.** If $\Sigma a_n$ is an N-nonalternating series such that $0 < p < r_n < q < 1$ for $n \geq N$, where $p$ and $q$ are constants, then

(1) $p/(1-p) \leq r_n/(1-p) \leq T_n \leq r_n/(1-q) \leq q/(1-q)$,

for $n \geq N$.

**Proof:** Set $\alpha_n = 1/(1-p)$ and $\beta_n = 1/(1-q)$ for $n \geq N$.

For $n \geq N$, $\alpha_n = 1 + p\alpha_{n+1} \leq 1 + r_{n+1} \alpha_{n+1}$ and $\beta_n = 1 + q\beta_{n+1}$, $\geq 1 + r_{n+1} \beta_{n+1}$.

From Theorem 5.6, $\Sigma a_n$ converges, so that

$$\lim a_n a_n = \lim a_n \beta_n = 0.$$ From (a) of Theorems 5.6 and 5.14, we obtain (1). Q.E.D.

**Theorem 5.18.** If $\Sigma a_n$ is an N-nonalternating series and $0 \leq r < 1$, then $T_n \to r/(1-r)$. 
Proof: We implicitly restrict \( n \) to large values throughout. There is a monotone increasing series \( \{p_n\} \) such that \( 0 \leq p_n \leq r_n \) and \( p_n \to r \). Define a monotone increasing sequence \( \{\alpha_n\} \) by the equation \( \alpha_n = 1/(1-p_{n+1}) \).

Accordingly, \( \alpha_n = 1 + p_{n+1}\alpha_n \leq 1 + r_{n+1}\alpha_{n+1} \), i.e.,
\[
\alpha_n \leq 1 + r_{n+1}\alpha_{n+1}.
\]
Similarly, there is a monotone decreasing sequence \( \{q_n\} \) such that \( r_n \leq q_n < 1 \) and \( q_n \to r \).

Define a monotone decreasing sequence \( \{\beta_n\} \) by the equation \( \beta_n = 1/(1-q_{n+1}) \). We then have
\[
\beta_n = 1 + q_{n+1}\beta_n \geq 1 + r_{n+1}\beta_{n+1}, \quad \text{i.e.,} \quad \beta_n \geq 1 + r_{n+1}\beta_{n+1} \geq 0. \]

From Theorems 5.6 and 5.14, \( r_n\alpha_n \leq T_n \leq r_n\beta_n \). Also \( \lim r_n\alpha_n = \lim r_n\beta_n = r/(1-r) \), so that \( T_n \to r/(1-r) \). Q.E.D.

We now turn to the critical case \( r_n \to 1 \). Suppose that \( \Sigma a_n \) is a positive term series and \( Q_n \to Q > 1 \).

According to Theorem 4.25, \( \Sigma a_n \in MR(\Sigma a_n) \) if and only if \( a_n \sim n/(Q-1) \). As we have seen, Szász suggests \( a_n = n/(Q-1) \)
for \( n \geq 1 \). Now for a fixed number \( k \), \((n+k)/(Q-1) \sim n/(Q-1) \), so that, with \( \beta_n = (n+k)/(Q-1) \) for \( n \geq 1 \),
\[
\Sigma a_n \in MR(\Sigma a_n). \]
Thus, why should we restrict ourselves to \( k = 0 ? \) We shall see that we should not make this restriction.
Suppose that \( \{a_n\} \) and \( \{\beta_n\} \) have been determined such that

1. \( a_n \to 0 \) and \( 0 < a_n \leq 1 + r_{n+1}a_{n+1} \), \( n \geq N \),

and

2. \( a_n \to 0 \) and \( 0 < 1 + r_{n+1}\beta_{n+1} \leq \beta_n \), \( n \geq N \).

From Theorems 5.4 and 5.14,

3. \( a_n \leq T_n/r_n = 1 + T_{n+1} \leq \beta_n \) for \( n \geq N \).

From (3), it is clear that we wish to maximize the \( a_n \) and minimize the \( \beta_n \), in order to obtain sharp bounds for \( 1 + T_{n+1} \). Also, we desire \( a_n \sim \beta_n \sim n/(Q-1) \). Multiplying (3) by \( a_n \), we obtain

4. \( a_n a_n \leq S - S_{n-1} \leq a_n \beta_n \) for \( n \geq N \).

Thus,

5. \( S_{\alpha(n-1)} = S_{n-1} + a_n a_n \leq S \leq S_{\beta(n-1)} + a_n \beta_n \), \( n \geq N \).

From (1) and (2), for \( n \geq N \), \( a_n \alpha_n/a_n = 1 + r_{n+1}a_{n+1} - a_n \geq 0 \), and \( a_n \beta_n/a_n = 1 + r_{n+1}\beta_{n+1} - \beta_n \leq 0 \). Hence for \( n \geq N \), \( a_n \geq 0, \beta_n \leq 0, S_{\alpha(n-1)} \leq S_{\alpha n}, \) and \( S_{\beta n} \leq S_{\beta(n-1)} \). In order to obtain fairly sharp bounds by (4), we will give only one example to show the general procedure.

Example 5.19.

\[
\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{(4n+1)(4n+3)} = \frac{1}{1 \cdot 3} + \frac{1}{5 \cdot 7} + \cdots = \pi/8.
\]
This series is considered by Szász (26,p.275). He takes
\( k=0 \) in \( \alpha'_n = (n+k)/(Q-1) \), and sets \( t_n = S_n^{\alpha_{n+1}} \alpha'_{n+1} \)
for \( n \geq 0 \). Thus, \( t_n = S_{\alpha'_n} \) for \( n \geq 0 \). The numbers \( t_n \),
\( 2 \leq n \leq 7 \), in (26,p.275) are in error. They should read:

\[
\begin{align*}
t_2 &= .38739, \\
t_3 &= .38952, \\
t_4 &= .39056 \\
t_5 &= .39116, \\
t_6 &= .39153, \\
t_7 &= .39183.
\end{align*}
\]

Now .39269908 < \( \pi/8 \) < .39269909. Setting \( \pi/8 = .39270 \),
\( \pi/8 - t_7 = .00087 \).

We have \( a_n = 1/(4n+1)(4n+3) \) for \( n \geq 0 \), and for
\( n \geq 1 \),
\( r_n = a_n/a_{n-1} = (4n-3)(4n-1)/(4n+1)(4n+3) = 1 - Q_n/n \). Thus
\( Q_n = 32n^2/(4n+1)(4n+3) \rightarrow Q = 2 \) and \( \alpha'_n = (n+k)/(Q-1) = n+k \).

We have, for \( n \geq 1 \),
\( (6) \ a_n/a_{n-1} = 1+r_{n-1}\alpha'_{n-1} - \alpha'_n = [32n(1-k) - 32k + 38]/(16n^2 + 48n + 35) \).

From (6), it is obvious \( k=1 \) yields the best sequence \( \{\alpha'_n\} \) for the acceleration of \( \Sigma a_n \). Thus, setting
\( \alpha_n = n+1 \) for \( n \geq 1 \),
\( (7) \ a_0 + a_1 = 1/3 + 2/(5 \cdot 7) = 1/3 + 6/(1 \cdot 3 \cdot 5 \cdot 7) \)

and from (6), for \( n \geq 1 \),
\( (8) \ a_\alpha_n = [6/(4n+5)(4n+7)] a_n = 6/(4n+1)(4n+3)(4n+5)(4n+7). \)

Thus,
\[(9) \sum_{n=0}^{\infty} a_n = \left[1/3+6/(1\cdot3\cdot5\cdot7)\right] + \sum_{n=0}^{\infty} 6/(4n+1)(4n+3)(4n+5) \times (4n+7)\]

or

\[(10) \sum_{n=0}^{\infty} a_n = 1/3 + \sum_{n=0}^{\infty} 6/(4n+1)(4n+3)(4n+5)(4n+7) = 1/3 + \sum_{n=0}^{\infty} b_n.\]

In (10) we have absorbed part of \(a_0\) into the summation, i.e., \(a_0 = 1/3 + b_0\) and \(a_n = b_n\) for \(n \geq 1\). No use will be made of (10), although it is suggestive for application of the above procedure to \(\Sigma b_n\).

At this point we have the following alternatives:

\[(11) S_{\Sigma a_n} = S_n + a_{n+1} a_{n+1} \sum_{i=0}^{n} = 1/(4i+1)(4i+3) + (n+2)/(4n+5)(4n+7)\]

or

\[(12) S_{\Sigma a_i} = a_0 + \sum_{i=1}^{n} [1/3+2/35] + \sum_{i=1}^{n} 6/(4i+1)(4i+3)(4i+5) \times (4i+7).\]

Clearly, (1) is preferable for actual numerical calculation. Leaving \(\Sigma a_n\) in the form (11), we have a so-called "modified series" of Bradshaw (9,p.486-492). In applying (11) as an approximation to \(S\), we have no information, assuming no previous calculations for \(\pi/8\) as known, as to the error involved, i.e., \(S - S_{\Sigma a_n}\). We now turn to the resolution of this problem.

Comparing (1) with (6), we require

\[(1.3) l \pm r_{n+1} \alpha'_{n+1} - \alpha'_n \geq 0 \quad \text{for} \quad n \geq N.\]

From (6), (13) is seen to be equivalent to
From (14), we must have \( k \leq 1 \), since \( 1 + \frac{3}{(16n+16)} \to 1 \) as \( n \to \infty \). Thus, we are led to set \( k = 1 \) and \( \alpha_n' = n+k = n+1 = \alpha_n \), \( \alpha_n \) as defined for (9) and (11). We now see from (4) that

(15) \( a_n \alpha_n \leq S-S_{n-1} \) for \( n \geq 1 \), \( \alpha_n = n+1 \) for \( n \geq 1 \).

Comparing (2) with (6), with \( \beta_n = \alpha_n' \), we require

(16) \( 1 + r_{n+1} \beta_{n+1} - \beta_n \leq 0 \) for \( n \geq N \).

From (6), (16) is seen to be equivalent to

(17) \( k \geq 1 + \frac{3}{(16n+16)} \), \( n \geq N \).

Recalling that \( \beta_n = n+k \) is to be minimized and noting that \( \{1+3/(16n+16)\} \) is monotone decreasing, we set \( k = 1 + 3/(16N+16) \) as the optimal choice satisfying (17). From (4), we then have,

(18) \( S-S_{n-1} \leq a_n \beta_n \) and \( \beta_n = n+1 + \frac{3}{(16N+16)} \), \( n \geq N \).

Setting \( n = N \) in (18) and noting that (18) holds for \( N \geq 1 \), we have

(19) \( S-S_{n-1} \leq a_n \beta_n \) and \( \beta_n = n+1 + \frac{3}{(16n+16)} \), \( n \geq 1 \).

From (15) and (19), we obtain the desired bounds for \( S-S_{\alpha_n} \), i.e.,

(20) \( 0 \leq S-S_{\alpha_{n-1}} \leq a_n (\beta_n - \alpha_n) = \frac{3}{(4n+1)(4n+3)(16n+16)} \), \( n \geq 1 \).
With \( n = 1 \) in (20), \( 0 \leq S - S_{\alpha_0} \leq 3/(5 \cdot 7 \cdot 32) < 0.0027 \).

With \( n = 8 \) in (20), \( 0 \leq S - S_{\alpha_7} \leq 3/(33 \cdot 35 \cdot 144) = 1/55440 < 0.000019 \). Using \( a^- \iff a < a^- \) and \( a^+ \iff a < a^+ \), we have \( S^- = 0.3848938, S^+ = 0.3848946, (a_{\alpha_8})^- = 0.0077922, \)
and \( (a_{\alpha_8})^+ = 0.0078102 \). Thus, \( S^- + (a_{\alpha_8})^- = 0.3926860 \)
\( < S < 0.3927050 = S^+ + (a_{\alpha_8})^+ \). Letting \( S' \) be the average
of these two bounds for \( S = \pi/8 \), we find \( S' = 0.3926955 \)
and we must have \( |S - S'| = |\pi/8 - 0.3926955| \)
\( \leq (0.3927050 - 0.3926860)/2 = 0.0000095 \).
CHAPTER VI

CONVERGENCE AND DIVERGENCE OF REAL SERIES

Throughout this chapter, all series are assumed to be real. We now state and prove some of the theorems, corresponding to those of Chapter V.

Theorem 6.1. Let \( L \) be any real number and \( c \) be any positive number. Then a n.a.s.c. that a series \( \Sigma a_n \) converge is that there exist a convergent series \( \Sigma b_n \) and a sequence \( \{\beta_n\} \) such that,

1. \( (a_n + b_n)\beta_n \to L \),
2. \( 0 < \frac{(a_{n+1} + b_{n+1})}{(a_n + b_n)} \),

and

3. \( \beta_n \geq c + \frac{(a_{n+1} + b_{n+1})}{(a_n + b_n)}\beta_{n+1} \).

Proof: For the necessity, let \( \Sigma c_n \) be any convergent non-alternating series, and define \( b_n = c_n - a_n \) for \( n \geq 0 \). The series \( \Sigma (a_n + b_n) = \Sigma c_n \) is a convergent nonalternating series, so that (2) holds. According to Theorem 5.1, there is a sequence \( \{\beta_n\} \) which satisfies conditions (1) and (3) above. Clearly, \( \Sigma b_n \) converges.

For the sufficiency, we see that \( \Sigma (a_n + b_n) \) converges according to Theorem 5.1. Consequently, \( \Sigma a_n \)
converges since $\Sigma b_n$ converges. Q.E.D.

**Theorem 6.2.** Let $L$ be any real number and $c$ be any positive number. Then a n.a.s.c. that a series $\Sigma a_n$ diverge is that there exist a divergent series $\Sigma b_n$ and a sequence $\{\beta_n\}$ such that,

1. $(a_n + b_n)\beta_n \to L$,
2. $0 < (a_{n+1} + b_{n+1})/(a_n + b_n)$,

and

3. $\beta_n \geq c + [(a_{n+1} + b_{n+1})/(a_n + b_n)]\beta_{n+1}$.

**Proof:** For the necessity, let $\Sigma c_n$ be any convergent non-alternating series and define $b_n = c_n - a_n$ for $n \geq 0$. The series $\Sigma (a_n + b_n) = \Sigma c_n$ is a convergent nonalternating series so that (2) holds. From Theorem 5.1, there is a sequence $\{\beta_n\}$ such that (1) and (3) hold. Also, $\Sigma b_n$ must diverge.

For the sufficiency, $\Sigma a_n$ must diverge, since otherwise $\Sigma b_n$ would converge according to Theorem 6.1.

**Theorem 6.3.** Let $c$ be any positive number. Then a n.a.s.c. that a series $\Sigma a_n$ converge is that there exist a convergent series $\Sigma b_n$ and a sequence $\{\beta_n\}$ such that,
(1) \( \beta_n \geq 0 \),

(2) \( 0 < \frac{(a_{n+1} + b_{n+1})}{(a_n + b_n)} \),

and

(3) \( \beta_n \geq c + \left[\frac{(a_{n+1} + b_{n+1})}{(a_n + b_n)}\right] \beta_{n+1} \).

**Proof:** For the necessity, let \( \Sigma c_n \) be any convergent nonalternating series, and define \( b_n = c_n - a_n \) for \( n \geq 0 \). The series \( \Sigma (a_n + b_n) = \Sigma c_n \) is a convergent nonalternating series so that (2) holds. According to Theorem 5.5, there is a sequence \( \{\beta_n\} \) satisfying conditions (1) and (3) above. Also, \( \Sigma b_n \) converges.

For the sufficiency, Theorem 5.5 implies that \( \Sigma (a_n + b_n) \) converges. Thus, \( \Sigma a_n \) converges since \( \Sigma b_n \) converges. Q.E.D.

**Theorem 6.4.** Let \( c \) be any positive number. Then a n.a.s.c. that a series \( \Sigma a_n \) diverges is that there exist a divergent series \( \Sigma b_n \) and a sequence \( \{\beta_n\} \) such that,

(1) \( \beta_n \geq 0 \),

(2) \( 0 < \frac{(a_{n+1} + b_{n+1})}{(a_n + b_n)} \),

and

(3) \( \beta_n \geq c + \left[\frac{(a_{n+1} + b_{n+1})}{(a_n + b_n)}\right] \beta_{n+1} \).

**Proof:** For the necessity, let \( \Sigma c_n \) be convergent
nonalternating series and define \( b_n = c_n - a_n \) for \( n \geq 0 \).

The series \( \sum (a_n + b_n) = \sum c_n \) is a convergent nonalternating series so that (2) holds. From Theorem 5.5, there is a sequence \( \{ \beta_n \} \) satisfying conditions (1) and (3). Moreover, \( \Sigma b_n \) must diverge.

For the sufficiency, \( \Sigma a_n \) must diverge since otherwise \( \Sigma b_n \) would converge according to Theorem 6.3. Q.E.D.

**Theorem 6.5.** Let \( c \) be any positive number. Then a n.a.s.c. that a series \( \Sigma a_n \) converge is that there exist a convergent series \( \Sigma b_n \) and a sequence \( \{ \beta_n \} \) such that,

\[
(1) \quad \beta_n \geq 0,
\]

and

\[
(2) \quad \beta_n \geq c + \frac{|a_{n+1} + b_{n+1}|}{(a_n + b_n)}|\beta_{n+1}|
\]

**Proof:** The necessity follows from Theorem 6.3.

For the sufficiency, Theorem 5.5 implies that \( \Sigma |a_n + b_n| \) converges. Consequently, \( \Sigma (a_n + b_n) \) converges, so that \( \Sigma a_n \) converges since \( \Sigma b_n \) converges. Q.E.D.

**Theorem 6.6.** Let \( c \) be any positive number. Then a n.a.s.c. that a series \( \Sigma a_n \) diverge is that there exist a divergent series \( \Sigma b_n \) and a sequence \( \{ \beta_n \} \) such that,
\( \beta_n \geq 0, \)

and

\( \beta_n \geq c + \frac{a_{n+1} + b_{n+1}}{(a_n + b_n)} \beta_{n+1}. \)

**Proof:** The necessity follows from Theorem 6.4.

For the sufficiency, \( \Sigma a_n \) must diverge, since otherwise \( \Sigma b_n \) would converge according to Theorem 6.5.

Q.E.D.

**Theorem 6.7.** Let \( c \) be any positive number. Then a n.a.s.c. that a series \( \Sigma a_n \) converge is that there exist a convergent series \( \Sigma b_n \) and a sequence \( \{ \beta_n \} \) such that,

1. \( \beta_n \geq 0, \)
2. \( 0 < a_n + b_n, \)

and

3. \( \beta_n \geq c + \frac{(a_{n+1} + b_{n+1})}{(a_n + b_n)} \beta_{n+1}. \)

**Proof:** For the necessity, let \( \Sigma c_n \) be any convergent series of positive terms, and define \( b_n = c_n - a_n \) for \( n \geq 0. \)

Clearly, \( \Sigma b_n \) converges and (2) above holds. The existence of a sequence \( \{ \beta_n \} \) satisfying (1) and (3) follows from Theorem 5.5.

The sufficiency follows from Theorem 6.3. Q.E.D.
CHAPTER VII

CONVERGENCE AND DIVERGENCE OF COMPLEX SERIES

Throughout this chapter, all series are assumed to be complex.

A complex series \( \sum a_n \) will be called restricted iff \( r_n \neq 0 \) for every \( n \), and \( N \)-restricted iff \( r_n \neq 0 \) for \( n \geq N \), where \( N \) is some integer. We now generalize some of the theorems in Chapters V and VI.

**Theorem 7.1.** Let \( L \) be any real number and \( c \) be any positive number. Then a n.a.s.c. that an \( N \)-restricted series \( \sum a_n \) converge absolutely is that there exist a sequence \( \{\beta_n\} \) such that

\[
(1) \quad |a_n| \beta_n \to L,
\]

and

\[
(2) \quad \beta_n \geq c + |r_{n+1}| \beta_{n+1}, \quad n \geq N.
\]

**Proof:** Apply Theorem 5.1 to \( \sum |a_n| \). Q.E.D.

**Theorem 7.2.** (Kummer's criterion) Let \( c \) be any positive number. Then a n.a.s.c. that an \( N \)-restricted series \( \sum a_n \) converge absolutely is that there exist a sequence \( \{\beta_n\} \) such that
\( \beta_n \geq 0, \ n \geq N, \)

and

\( \beta_n \geq c + |r_{n+1}| \beta_{n+1}, \ n \geq N. \)

**Proof:** Apply Theorem 5.5 to \( \Sigma |a_n| \). Q.E.D.

**Theorem 7.3.** Let \( c \) be any positive number. Then a n.a.s.c. that a series \( \Sigma a_n \) converge is that there exist a convergent series \( \Sigma b_n \) and a sequence \( \{ \beta_n \} \) such that,

1. \( \beta_n \geq 0, \)

and

2. \( \beta_n \geq c + (a_{n+1} + b_{n+1})/(a_n + b_n) |\beta_{n+1}|. \)

**Proof:** For the necessity, let \( \Sigma c_n \) be any restricted series which converges absolutely and define \( b_n = c_n - a_n \) for every \( n \). Since \( a_n + b_n = c_n \) for all \( n \), \( \Sigma (a_n + b_n) \) is a restricted series which converges absolutely. From Theorem 7.2, there is a sequence \( \{ \beta_n \} \) satisfying conditions (1) and (2) above. Clearly, \( \Sigma b_n \) converges.

For the sufficiency, \( \Sigma |a_n + b_n| \) converges according to Theorem 7.2 so that \( \Sigma (a_n + b_n) \) converges. Thus, \( \Sigma a_n \) converges since \( \Sigma b_n \) converges. Q.E.D.

**Corollary 7.4.** Suppose that \( c > 0 \) and \( \{ \beta_n \} \) is a
sequence such that,

(1) \( \beta_n \geq 0 \),

and

(2) \( \beta_n \geq c + \frac{|(a_{n+1} + b_{n+1})/(a_n + b_n)|}{|\beta_{n+1}|} \).

Then \( \sum a_n \) converges if and only if \( \sum b_n \) converges.

**Proof:** Apply Theorem 7.3. Q.E.D.

**Theorem 7.5.** Let \( c \) be any positive number. Then a n.a.s.c. that a series \( \sum a_n \) diverge is that there exist a divergent series \( \sum b_n \) and a sequence \( \{\beta_n\} \) such that,

(1) \( \beta_n \geq 0 \),

and

(2) \( \beta_n \geq c + \frac{|(a_{n+1} + b_{n+1})/(a_n + b_n)|}{|\beta_{n+1}|} \).

**Proof:** For the necessity, let \( \sum c_n \) be any restricted series which converges absolutely and define \( b_n = c_n - a_n \) for \( n \geq 0 \). The series \( \sum (a_n + b_n) = \sum c_n \) is a restricted series which converges absolutely. From Theorem 7.2, there is a sequence \( \{\beta_n\} \) satisfying conditions (1) and (2) above. Clearly, \( \sum b_n \) diverges.

For the sufficiency, \( \sum |a_n + b_n| \) converges according to Theorem 7.2. From Theorem 7.3, \( \sum a_n \) must diverge since otherwise \( \sum b_n \) would converge. Q.E.D.
CHAPTER VIII

ALTERNATING SERIES

A real series \( \sum a_n \) is called alternating iff
\[ r_n < 0 \] for every \( n \), and \( N \)-alternating iff \( r_n < 0 \) for \( n > N \), where \( N \) is some integer.

Various theorems stating necessary and sufficient conditions for the convergence of an \( N \)-alternating series will be proven, along with corresponding error bounds for the quantities \( T_n \). In many such theorems, it will be proven that all inequalities, excluding those between indices, may be reversed. Calling any such theorem and the derived theorem duals, a duality structure become apparent, but fails in at least one case. In particular, Theorem 8.32 has no dual according to Counterexample 8.10. Because of this duality, if the sequence \( \{r_n\} \) is fairly smooth, the difficulty in satisfying the required inequalities involving \( \{a_n\} \) or \( \{\beta_n\} \) is reduced considerably. Of course, the more judicious the choice of \( \{a_n\} \) or \( \{\beta_n\} \), the better the resulting bounds for the quantities \( T_n \).

Several theorems proven in this chapter will contain explicitly, or implicitly, in their conclusion that \( \{T_n\} \) converges. As we have previously seen, this implies
Lemma 8.1. If \{p_{2n-1}\} is monotone decreasing, \{p_{2n}\} is monotone increasing, and some subsequence of \{p_{2n-1}-p_{2n}\} is bounded below, then \{p_{2n-1}\} and \{p_{2n}\} both converge.

Proof: Suppose that L is a lower bound of some subsequence \{p_{2n'-1}-p_{2n'}\} of \{p_{2n-1}-p_{2n}\}. It is easily seen that \{p_{2n-1}-p_{2n}\} is monotone decreasing. Consequently, 

\[ L \leq p_{2n'-1}-p_{2n'} \leq p_{2n-1}-p_{2n} \quad \text{for } n \geq 1. \]

We then have 

\[ L+p_2 \leq L+p_{2n} \leq p_{2n-1} \leq p_1. \]

and 

\[ p_2 \leq p_{2n} \leq p_{2n-1}-L \leq p_1-L, \]

for \( n \geq 1 \). Accordingly, \{p_{2n-1}\} and \{p_{2n}\} are bounded monotone sequences, and thus converge. Q.E.D.

Theorem 8.2. Let \( L_1 \) and \( L_2 \) be any real numbers. Then a n.a.s.c. that an N-alternating series \( \Sigma a_n \) converge is that

\[(0) \quad a_n \to 0,\]

and there exist a sequence \( \{a_n\} \) such that,

\[(1) \quad a_{2n-1}a_{2n-1} \to L_1 \quad \text{and} \quad a_{2n}a_{2n} \to L_2\]

and

\[(2) \quad a_n \leq l+r_{n+1}+r_{n+1}r_{n+1}a_{n+2}, \quad n \geq N.\]
Moreover, if (0), (1), and (2) hold, then, for \( n \geq N \),

\[
\begin{align*}
\text{(a)} \quad & r_n + r_n r_{n+1} a_{n+1} - (L_2/a_{n-1}) \leq T_n \leq r_n a_n - (L_1/a_{n-1}) \\
\text{or} \quad & r_n + r_n r_{n+1} a_{n+1} - (L_1/a_{n-1}) \leq T_n \leq r_n a_n - (L_2/a_{n-1}),
\end{align*}
\]

accordingly as \( n \) is odd or even, respectively. And in general, for \( n \geq N \) and \( k \geq 1 \),

\[
\begin{align*}
\text{(b)} \quad & T_{n,2k-2} + (r_n \cdots r_{n+2k-1}) a_{n+2k-1} - (L_2/a_{n-1}) \leq T_n \\
& \leq T_{n,2k-2} + (r_n \cdots r_{n+2k-2}) a_{n+2k-2} - (L_1/a_{n-1}) \\
& \leq T_{n,2k-2} + (r_n \cdots r_{n+2k-2}) a_{n+2k-2} - (L_2/a_{n-1}),
\end{align*}
\]

accordingly as \( n \) is odd or even, respectively.

**Proof:** Assume that \( \sum a_n \) converges. Accordingly (0) holds.

Define \( L_{2n-1} = L_1 \) and \( L_{2n} = L_2 \) for every \( n \), and

\[
a_n = 1 + T_{n+1} + L_n/a_n \quad \text{for} \quad n \geq N.
\]

We then have \( a_n a_n \)

\[
= a_n + a_n T_{n+1} + L_n = a_n + (S-S_n) + L_n = S - S_{n-1} + L_n.
\]

Thus \( a_{2n-1} a_{2n-1} \)

\[
= S - S_{2n-2} + L_{2n-1} \rightarrow L_1 \quad \text{and} \quad a_{2n} a_{2n} = S - S_{2n-1} + L_{2n} \rightarrow L_2,
\]

so that (1) holds. For \( n \geq N \),

\[
\begin{align*}
& a_n - r_{n+1} - r_{n+1} r_{n+2} a_{n+2} \\
& = 1 + T_{n+1} + L_n/a_n - r_{n+1} - r_{n+1} r_{n+2} (1 + T_{n+3} + L_{n+2}/a_{n+2}) \\
& = T_{n+1} + L_n/a_n - r_{n+1} - r_{n+1} r_{n+2} - r_{n+1} r_{n+2} T_{n+3} - L_{n+2}/a_n \\
& = T_{n+1} + L_n/a_n - T_{n+1} - L_n/a_n = 0,
\end{align*}
\]

so that
\[ a_n = 1 + r_{n+1} + r_{n+1} r_{n+2} a_{n+2} \text{ for } n \geq N. \] Thus (2) holds with equality. This proves the necessity.

For the sufficiency, assume that (0), (1), and (2) hold, and let \( n \) be any integer \( \geq N \). We now define
\[ P_k = T_{n, k-2} + (r_n \cdots r_{n+k-1}) a_{n+k-1} \text{ for } k \geq 1. \] Accordingly
\begin{equation}
(3) \quad P_k - P_{k+2} = (r_n \cdots r_{n+k-1}) [a_{n+k-1} - (1 + r_{n+k} + r_{n+k} r_{n+k+1} a_{n+k+1})],
\end{equation}
\( k \geq 1. \)

From (2) and (3) it can be seen that \( P_{2k} - P_{2k+2} \leq 0 \) and \( P_{2k-1} - P_{2k+1} \geq 0 \) for \( k \geq 1 \), so that \( \{P_{2k}\} \) is monotone increasing and \( \{P_{2k-1}\} \) is monotone decreasing. Moreover,
\[ P_k - P_{k+1} = (r_n \cdots r_{n+k-1}) [a_{n+k-1} - (1 + r_{n+k} + r_{n+k} a_{n+k})] = [a_{n+k-1} a_{n+k-1}
- a_{n+k-1} - a_{n+k} a_{n+k+1}] / a_{n-1}, \] so that, by (0) and (1), the sequence \( \{P_k - P_{k+1}\} \) is bounded. Consequently \( \{P_{2k-1} - P_{2k}\} \) is bounded. By Lemma 8.1, \( P_{2k-1} \to P' \) and \( P_{2k} \to P'' \), for some numbers \( P' \) and \( P'' \). We then have
\[ T_{n, 2k-2} = r_n + \cdots + (r_n \cdots r_{n+2k-2}) = P_{2k} - (r_n \cdots r_{n+2k-1} a_{n+2k-1} = P_{2k-1} a_{n+2k-1} / a_{n-1} \to P'' - (L_2 / a_{n-1}) \text{ or}
\]
\[ P'' - (L_1 / a_{n-1}) \text{, accordingly as } n \text{ is odd or even. Similarly,}
\]
\[ T_{n, 2k-1} = r_n + r_{n+1} \cdots + (r_{n} \cdots r_{n+2k-1}) = P_{2k+1}
-(r_n \cdots r_{n+2k}) a_{n+2k} = P_{2k+1} a_{n+2k} a_{n+2k} / a_{n-1} \to P' - (L_1 / a_{n-1}) \text{.} \]
or \( P'(L_2/a_{n-1}) \), accordingly as \( n \) is odd or even. Also, 
\[
T_{n,2k-1} - T_{n,2k-2} = \cdot (r_n \cdots r_{n+2k-1}) = a_{n+2k-1}/a_{n-1} \to 0 \text{ as } k \to \infty, 
\]
so that \( T_{n,k} \to T_n \) as \( k \to \infty \). Using the monotoneity of \( \{P_{2k-1}\} \) and \( \{P_{2k}\} \), we have, for \( k \geq 1 \), 
\[
P_{2k} - (L_2/a_{n-1}) \leq T_n \leq P_{2k-1} - (L_1/a_{n-1}), \text{ if } n \text{ is odd, or}
\]
\[
P_{2k} - (L_1/a_{n-1}) \leq T_n \leq P_{2k-1} - (L_2/a_{n-1}), \text{ if } n \text{ is even.}
\]
With \( k = 1 \), we obtain (a), and with \( k \geq 1 \), we obtain (b). Q.E.D.

The dual of Theorem 8.2 is Theorem 8.25.

Choosing \( L_1 = L_2 = 0 \) in Theorem 8.2, we obtain the following theorem.

**Theorem 8.3.** A n.a.s.c. that an \( N \)-alternating series \( \sum a_n \) converge is that
\[
(0) \quad a_n \to 0,
\]
and there exist a sequence \( \{a_n\} \) such that,
\[
(1) \quad a_n a_n \to 0
\]
and
\[
(2) \quad a_n \leq 1 + r_{n+1}^+ r_{n+1}^+ r_{n+2} a_{n+2}, \quad n \geq N.
\]
Moreover, if (0), (1), and (2) hold, then
\[
(a) \quad r_{n}^+ r_{n+1}^+ a_{n+1} \leq T_n \leq r_{n}^+ a_n, \quad n \geq N.
\]
And in general, for \( n \geq N \) and \( k \geq 1 \),
\[
(b) \quad T_{n,2k-2}^+ (r_n \cdots r_{n+2k-1}) a_{n+2k-1} \leq T_n \leq T_{n,2k-3}^+ (r_n \cdots r_{n+2k-2}) a_{n+2k-2}.
\]
The dual of Theorem 8.3 is Theorem 8.27.

The following example shows that condition (2) of Theorem 8.3 cannot be replaced by the condition

\[(2') \quad a_n \leq c + a_{n+1} + r_n + a_{n+2}, \quad 1 < c.\]

**Example 8.4.** Let \(1 < c\). Define \(a = (1+c)/2\), so that \(1 < a < c\). Define \(a_{2n} = 1/(n+1)\) and \(a_{2n+1} = -a/(n+1)\)

\[= -aa_{2n}\text{ for } n \geq 0.\] Clearly \(a_n \to 0\). Also, \(S_{2n-1}\)

\[= (a_0 + a_1) + (a_2 + a_3) + \cdots + (a_{2n-2} + a_{2n-1}) = (1-a)a_0 + (1-a)a_2 + \cdots + (1-a)a_{2n-2} = (1-a)[1 + 1/2 + 1/3 + \cdots + 1/n] \to -\infty, \text{ i.e.,}\]

\[\Sigma a_n \text{ diverges.}\] We have \(r_{2n} = -n/a(n+1) \to -1/a, \quad r_{2n+1} = -a, \quad r_{2n}r_{2n+1} = n/(n+1), \quad r_{2n+1}r_{2n+2} = (n+1)/(n+2),\]

\[c+r_{2n} \to c-1/a > 0, \quad \text{and} \quad c+r_{2n+1} \to c-a > 0.\] Thus,

\[(c+r_{n+1})/(1-r_{n+1}r_{n+2}) \to +\infty \quad \text{and} \quad a \leq (c+r_{n+1})/(1-r_{n+1}r_{n+2})\]

for any real number \(a\). Consequently, \(a(1-r_{n+1}r_{n+2}) \leq (c+r_{n+1}) \text{ and } a \leq c + r_{n+1} + r_{n+2}a.\) With \(a_n = a,\) condition \((2')\) holds. We conclude that conditions (0) and (1) of Theorem 8.3, and \((2')\) are necessary, but not sufficient, for the convergence of \(\Sigma a_n.\)

**Theorem 8.5.** Let \(c\) be any number \(< 1\). Then a n.a.s.c. that an alternating series \(\Sigma a_n\) converge absolutely is that
\( a_n \to 0, \)

and there exist a sequence \( \{a_n\} \) such that,

(1) \( a_n a_n \to 0 \)

and

(2) \( a_n \leq c + r_{n+1} r_{n+2} a_{n+2}, \quad n \geq 1. \)

**Proof:** For the necessity, define \( a_n, n \geq 1, \) by the equation

\[ a_n a_n = c \left(a_{n+1} + a_{n+2} + \cdots + a_{n+3} + \cdots\right). \]

Then \( a_n a_n \to 0. \) Also \( a_n a_n = c a_n + a_{n+1} + a_{n+2} a_{n+2}, \) and thus

\[ a_n = c + r_{n+1} r_{n+2} a_{n+2} \quad \text{for} \quad n \geq 1. \]

For the sufficiency, we first note that \( \Sigma a_n \) converges according to Theorem 8.3. Define \( a_n' = 1 + r_{n+1} + r_{n+2} a_{n+2}, \) for \( n \geq 1, \) and \( a_n a_n' \to 0, \) so that

\[ |a_n| \beta_n = |a_n| (\frac{a_n' - a_n}{1-c}) \to 0. \]

Also, \( (1-c)(1-\beta_n \beta_n') = (1-c)(1-(\frac{a_n' - a_n}{1-c})) \to 1-c. \)

\[ + \frac{a_n + a_{n+2}}{a_n} = \frac{(1-c) a_n' - a_n}{1-c} \]

\[ + (a_n' - a_n) r_{n+1} r_{n+2} / (1-c) = 1-c - a_n' + a_n + (a_n' - a_n) r_{n+1} r_{n+2} \]

\[ = -a_n' + (a_n' - a_n) r_{n+1} r_{n+2} a_{n+2} - c r_{n+1} r_{n+2} a_{n+2} = a_n - c \]

\[ - r_{n+1} r_{n+2} a_{n+2} \leq 0 \quad \text{for} \quad n \geq 1. \]

Thus, \( \beta_n \geq 1 \)

\[ + \left(\frac{a_n + a_{n+2}}{a_n}\right) \beta_n^2 = 1 + (\frac{|a_{n+2}|}{|a_n|}) \beta_n^2 \quad \text{for} \quad n \geq 1. \]

From Theorem 5.1, \( \Sigma |a_{2n}| \) and \( \Sigma |a_{2n+1}| \) converge, and thus \( \Sigma a_n \) is absolutely convergent. Q.E.D.
The dual of Theorem 8.5 is Theorem 8.29.

**Theorem 8.6.** Let \( c, L_1, L_2 \) be any real numbers where \( c < 1 \). Then a n.a.s.c. that an alternating series \( \sum a_n \) converge absolutely is that

1. \( a_n \to 0 \),

and there exist a sequence \( \{a_n\} \) such that,

2. \( a_{2n-1} a_{2n-1} \to L_1 \) and \( a_{2n} a_{2n} \to L_2 \)

and

3. \( a_n \leq c + r_{n+1} r_{n+2} a_{n+2}, n \geq 1 \).

**Proof:** For the necessity, there is a sequence \( \{a_n\} \) satisfying (1) and (2) of Theorem 8.5. Define \( \{a'_n\} \) by the equations

\[
a_{2n-1} a_{2n-1} = a_{2n-1} a_{2n-1} + L_1 \quad \text{and} \quad a_{2n} a_{2n} = a_{2n} a_{2n} + L_2.
\]

It may be seen that \( \{a'_n\} \) satisfies (1) and (2) above.

For the sufficiency, define \( \{a'_n\} \) by the equations

\[
a_{2n-1} a_{2n-1} = a_{2n-1} a_{2n-1} - L_1 \quad \text{and} \quad a_{2n} a_{2n} = a_{2n} a_{2n} - L_2.
\]

It may be seen that \( \{a'_n\} \) satisfies (1) and (2) of Theorem 8.5, and thus \( \sum a_n \) converges absolutely. Q.E.D.

The dual of Theorem 8.6 is Theorem 8.30.

**Theorem 8.7.** Suppose that \( \sum a_n \) is an N-alternating series such that \( a_n \to 0, r_{n+1} r_{n+2} < 1 \) for \( n \geq N \), and \( \alpha \) is a real number such that \( \alpha \leq (1+r_{n+1})/(1-r_{n+1} r_{n+2}) \) for
n > N. Then \( r_n r_{n+1} a \leq T_n \leq r_n a \) for \( n \geq N \).

**Proof:** For \( n \geq N \), \( a(1-r_{n+1} r_{n+2}) \leq 1+r_{n+1} \) and \( a \leq 1 + r_{n+1} r_{n+2} a \). Setting \( a_n = a \) for \( n \geq N \), we may use (a) of Theorem 8.3 to complete the proof. Q.E.D.

Taking \( N = 1 \) in Theorem 8.3, we have the following theorem.

**Theorem 8.8.** A n.a.s.c. that an alternating series \( \sum a_n \) converge is that

1. \( a_n \to 0 \),
2. and there exist a sequence \( \{a_n\} \) such that,
3. \( a_n \leq 1 + r_{n+1} r_{n+2} a_{n+2} \), \( n \geq 1 \).

Moreover, if (0), (1), and (2) hold, then

(a) \( r_n r_{n+1} a_{n+1} \leq T_n \leq r_n a_n \), \( n \geq 1 \).

And in general, for \( n \geq 1 \) and \( k \geq 1 \),

(b) \( T_n, a_{k-2} + (r_n \cdots r_{n+2k-2}) a_{n+2k-1} \leq T_n \leq T_n, a_{k-2} + (r_n \cdots r_{n+2k-2}) a_{n+2k-2} \).

The dual of Theorem 8.8 is Theorem 8.3.

**Remark 8.9.** We will show that if any of the three conditions (0), (1), or (2) of Theorem 8.2 are omitted, the
remaining two are not sufficient for the convergence of $\Sigma a_n$. We may do this by making the same considerations of Theorem 8.8, since condition (0), (1), or (2) of Theorem 8.8 implies the corresponding condition of Theorem 8.2. We will show even more. In particular, condition (a) of Theorem 8.8 implies that $a_n \leq 1+r_{n+1}a_{n+1}$ for $n \geq 1$.

We thus consider the four conditions:

(0) $a_n \to 0$,

(1) $a_n a_n \to 0$,

(2) $a_n \leq 1+r_{n+1}a_n+1$, $n \geq 1$,

(3) $a_n \leq r_{n+1}a_{n+1}$, $n \geq 1$.

We will show if (0), (1), or (2) is omitted, the remaining three conditions are not sufficient for the convergence of $\Sigma a_n$. We will also show that if (1) is replaced by the two weaker conditions that $a_n a_n a_{n+1} a_{n+1} \to 0$ and that $\{a_n a_n\}$ be bounded, the resulting four conditions are not sufficient for the convergence of $\Sigma a_n$.

**Counterexample 8.10.** Let $\Sigma a_n$ be the divergent series $1-1+1-1-\cdots$. We have $a_n = (-1)^n$ for $n \geq 0$, and $r_n = -1$ for $n \geq 1$. Defining $a_n = 0$ for $n \geq 1$, the following three conditions obviously hold:
(1) \( a_n a_n \to 0, \)

(2) \( a_n \leq 1 + r_{n+1} + r_{n+1} r_{n+2} a_{n+2}, \ n \geq 1, \)

(3) \( a_n \leq 1 + r_{n+1} a_{n+1}, \ n \geq 1. \)

We have shown that conditions (1), (2), and (3) are not sufficient for the convergence of \( \Sigma a_n. \)

Counterexample 8.11. Let \( \Sigma a_n = 1 - 1/2 + 1/2 - 1/(2.2) + \cdots + 1/(n+1) - 1/2(n+1) + \cdots. \)

This series is divergent, since for \( n \geq 1, \)

\[
S_{2n-1} = (1 - 1/2) + (1/2 - 1/(2.2)) + \cdots + (1/n - 1/2n)
= (1/2)(1 + 1/2 + 1/3 + \cdots + 1/n).
\]

Let \( \alpha_1 \) be any real number, and define the sequence \( \{\alpha_n\} \) recursively by the equation \( \alpha_n = 1 + r_{n+1} \alpha_{n+1}. \) The following conditions are seen to hold:

(0) \( a_n \to 0, \)

(2) \( a_n \leq 1 + r_{n+1} + r_{n+1} r_{n+2} a_{n+2}, \ n \geq 1, \)

(3) \( a_n \leq 1 + r_{n+1} a_{n+1}, \ n \geq 1. \)

We conclude that conditions (0), (2), and (3) are not sufficient for the convergence of \( \Sigma a_n. \) Moreover, \( a_n a_n \)

\(- a_{n+1} a_{n+1} = a_n \to 0, \) so that the four conditions \( a_n a_n \)

\(- a_{n+1} a_{n+1} \to 0, \) (0), (2), and (3) are not sufficient for the convergence of \( \Sigma a_n. \)
Counterexample 8.12. Let $\Sigma a_n$ be the divergent series given in Counterexample 8.11. Defining $a_n = 0$ for $n \geq 1$, it is obvious that the following conditions hold:

(0) $a_n \to 0$,

(1) $a_n a_n \to 0$,

(3) $a_n \leq 1 + r_{n+1} a_{n+1}$, $n \geq 1$.

Thus conditions (0), (1), and (3) are not sufficient for the convergence of $\Sigma a_n$. Also, Theorem 8.8 implies that the condition

(2) $a_n \leq 1 + r_{n+1} + r_{n+1} r_{n+2} a_{n+2}$, $n \geq 1$,

is false. Indeed, (2) must fail to hold for infinitely many values of $n$ according to Theorem 8.3.

Counterexample 8.13. Let $\Sigma a_n$ be any divergent alternating series whose partial sums are bounded, and such that $a_n \to 0$. Let $a_1$ be any real number, and define the sequence $\{a_n\}$ recursively by the equation $a_n = 1 + r_{n+1} a_{n+1}$. We easily see that $a_{n+1} a_{n+1} = a_1 a_1 - (a_1 + a_2 + \cdots + a_n)$ for $n \geq 1$. Consequently, the sequence $\{a_n a_n\}$ is bounded, since the partial sums $S_n$ are bounded. Conditions (0), (2), and (3) of Remark 8.9 are easily seen to hold. Consequently, these three conditions along with
the condition that \( \{a_n \sigma_n\} \) be bounded are not sufficient for the convergence of \( \Sigma a_n \). Moreover, it is of no avail to also require that \( a_n \sigma_n - a_{n+1} \sigma_{n+1} \to 0 \), since
\[
a_n = 1 + r_{n+1} \sigma_{n+1}
\]
yields \( a_n \sigma_n - a_{n+1} \sigma_{n+1} = a_n \to 0 \) in the present counterexample.

**Theorem 8.14.** Let \( L \) be any real number and \( \Sigma a_n \) be any \( N \)-alternating series such that \( a_{2n} > 0 \). Then a
n.a.s.c. that \( \Sigma a_n \) converge is that

1. \( a_n \to 0 \),
2. some subsequence of \( \{a_{2n-1}, a_{2n-1}\} \) is bounded below and \( a_{2n-1} a_{2n} \to L \),
3. \( a_n \leq 1 + r_{n+1} r_{n+1} r_{n+2} \sigma_{n+2} \), \( n \geq N \).

Moreover, if conditions (0), (1), and (2) hold, then \( \{a_{2n-1}, a_{2n-1}\} \) converges.

**Proof:** The necessity is immediate from Theorem 8.2.

For the sufficiency, let \( m \) be any odd integer \( \geq N+1 \). Define
\[
P_k = T_{m,k-2} + (r_m \cdots r_{m+k-1}) a_{m+k-1} \quad \text{for} \quad k \geq 1.
\]
Then,
(3) \[ P_{k+2} - P_k = (r_m \cdots r_{m+k-1})[\alpha_{m+k-1} - (1 + r_{m+k} + r_{m+k-1}\alpha_{m+k-1})], \quad k \geq 1. \]

From (2) and (3), we see that 
\[ P_{2k} - P_{2k+2} \leq 0 \]
and 
\[ P_{2k-1} - P_{2k+1} \geq 0 \]
for \( k \geq 1 \), so that \( \{P_{2k}\} \) is monotone increasing and \( \{P_{2k-1}\} \) is monotone decreasing.

Also,

(4) \[ P_{2k-1} - P_{2k} = (a_{m+2k-2} - a_{m+2k-2}) \]
\[ - a_{m+2k-1} \alpha_{m+2k-1} / a_{m-1} \]
for \( k \geq 1 \), so that by (0), (1), and the fact that \( a_{m-1} > 0 \), some subsequence of \( \{P_{2k-1} - P_{2k}\} \) is bounded below. By Lemma 8.1, \( P_{2k-1} \rightarrow P' \) and \( P_{2k} \rightarrow P'' \) for some numbers \( P' \) and \( P'' \). Also, according to (1), 
\[ a_{m+2k-1} \alpha_{m+2k-1} \rightarrow L \quad \text{as} \quad k \rightarrow \infty. \]

From (4), 
\[ a_{m+2k-2} - a_{m+2k-2} \]
\[ \alpha_{m+2k-1} + a_{m+2k-1} a_{m-1} (P_{2k-1} - P_{2k}) \rightarrow L + a_{m-1} (P' - P'') \]
as \( k \rightarrow \infty \). Consequently, \( m \) being odd, we see that \( \{a_{2n-1} \alpha_{2n-1}\} \) converges. Theorem 8.2 now implies that 
\( \Sigma a_n \) converges. Q.E.D.

The dual of Theorem 8.14 is Theorem 8.40.

Theorem 8.15. Let \( L \) be any real number and \( \Sigma a_n \) be any \( N \)-alternating series such that \( a_{2n} < 0 \). Then a n.a.s.c. that \( \Sigma a_n \) converge is that
(0) \( a_n \to 0 \),
and there exist a sequence \( \{a_n\} \) such that,
(1) some subsequence of \( \{a_{2n-1}a_{2n-1}\} \) is bounded
above and \( a_{2n}a_{2n} \to L \)
and
(2) \( a_n \leq 1 + r_{n+1} + r_{n+1}r_{n+2}a_{n+2} \), \( n \geq N \).

Moreover, if conditions (0), (1) and (2) hold, then
\( \{a_{2n-1}a_{2n-1}\} \) converges.

**Proof:** The necessity follows from Theorem 8.2.

For the sufficiency, define \( a'_n = -a_n \) for \( n \geq 0 \).

Accordingly, \( r'_n = a'_n/a'_{n-1} = a_n/a_{n-1} = r_n \) for \( n \geq N \).

It is obvious that Theorem 8.14 is applicable, yielding
the convergence of \( \Sigma a'_n \) and \( \{a'_{2n-1}a'_{2n-1}\} \). Thus, \( \Sigma a_n \)
and \( \{a_{2n-1}a_{2n-1}\} \) both converge. Q.E.D.

The dual of Theorem 8.15 is Theorem 8.39.

It has been shown that (1) of Theorem 8.2 cannot be
omitted, or replaced by the weaker condition that \( \{a_n'a_n\} \)
be bounded and \( a_n'a_n - a_{n+1}'a_{n+1} \to 0 \). The following theorem
shows that (1) can be replaced by the weaker condition that
some subsequence of \( \{a_{2n-1}a_{2n-1}\} \) be bounded and
\( \{a_{2n}a_{2n}\} \) converge.
Theorem 8.16. Let $L$ be any real number. Then a n.a.s.c. that an N-alternating series $\Sigma a_n$ converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{a_n\}$ such that,

(1) some subsequence of $\{a_{2n-1}a_{2n-1}\}$ is bounded and $a_{2n}a_{2n} \to L$

and

(2) $a_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}a_{n+2}$, $n \geq N$.

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n-1}a_{2n-1}\}$ converges.

Proof: The necessity follows from Theorem 8.2.

For the sufficiency, we need only note that $a_{2n} > 0$ or $a_{2n} < 0$, and then apply Theorem 8.14 or Theorem 8.15, respectively. Q.E.D.

The dual of Theorem 8.16 is Theorem 8.41.

The following counterexample shows that (1) of Theorem 8.14 or Theorem 8.16 cannot be replaced by the condition

(1') $\{a_{2n-1}a_{2n-1}\}$ is bounded above and $a_{2n}a_{2n} \to L$.

Counterexample 8.17. Let $\Sigma a_n$ be the divergent series given in Counterexample 8.11. We have $a_{2n} = 1/(n+1)$ and $a_{2n+1} = -1/2(n+1)$ for $n \geq 0$. Define $a_{2n} = 0$
for \( n \geq 1 \). Define \( \{a_{2n-1}\} \) recursively by the equation
\[
a_{2n-1} = 1 + r_{2n} + r_{2n-1} a_{2n+1}, \quad n \geq 1,
\]
where \( a_1 \) is any real number. It can be seen that (0) \( a_n \to 0 \), (1) \( a_{2n} a_{2n} \to 0 \),
and (2) \( a_n \leq 1 + r_{n+1} + r_{n+2} a_{n+2} \) for \( n \geq 1 \). Also,
\[
a_{2n+1} a_{2n+1} = a_{1} a_{1} -(a_{1} + a_{2} + \cdots + a_{2n}) \to -\infty,
\]
so that
\[
\{a_{2n-1}, a_{2n-1}\} \text{ is bounded above.}
\]

The following counterexample shows that (1) of Theorem 8.15 or Theorem 8.16 cannot be replaced by the condition
\[
(1') \quad \{a_{2n-1} a_{2n-1}\} \text{ is bounded below and } a_{2n} a_{2n} \to L.
\]

Counterexample 8.18. Let \( \Sigma a_n \) be the divergent series
whose terms are the negatives of those of the series given in Counterexample 8.17, i.e., \( a_{2n} = -1/(n+1) \) and \( a_{2n+1} = 1/(n+1) \) for \( n \geq 0 \). Define \( a_{2n} = 0 \) for \( n \geq 1 \).
Define \( \{a_{2n-1}\} \) recursively by the equation
\[
a_{2n-1} = 1 + r_{2n} + r_{2n-1} a_{2n+1}, \quad n \geq 1,
\]
where \( a_1 \) is any real number.
Then (0) \( a_n \to 0 \), (1) \( a_{2n} a_{2n} \to 0 \), and (2) \( a_n \leq 1 + r_{n+1} + r_{n+2} a_{n+2} \) for \( n \geq 1 \). Also,
\[
a_{2n+1} a_{2n+1} = a_{1} a_{1} -(a_{1} + a_{2} + \cdots + a_{2n}) \to +\infty,
\]
so that \( \{a_{2n-1}, a_{2n-1}\} \) is bounded below.

Theorem 8.19. Let \( L \) be any real number and \( \Sigma a_n \) be
any N-alternating series such that \( a_{2n} > 0 \). Then a
n.a.s.c. that \( \Sigma a_n \) converge is that

(0) \( a_n \to 0 \),

and there exist a sequence \( \{a_n\} \) such that,

(1) some subsequence of \( \{a_{2n}^2n\} \) is bounded above
    and \( a_{2n-1}a_{2n-1} \to L \)

and

(2) \( a_n \leq 1 + r_{n+1} + r_{n+1} + r_{n+2}a_{n+2} \), \( n \geq N \).

Moreover, if conditions (0), (1), and (2) hold, then
\( \{a_{2n}a_{2n}\} \) converges.

**Proof:** The necessity follows from Theorem 8.2.

According to Theorem 8.2, for the sufficiency we
need only show that \( \{a_{2n}^2n\} \) converges. Define

\( a_n' = a_{n+1} \) for \( n \geq 0 \), and \( a_n' = a_{n+1} \) for \( n \geq N \). Then

\( a_n' \to 0 \) and \( a_{2n}^2n = a_{2n+1}a_{2n+1} \to L \). Since some subse-
quence of \( \{a_{2n}^2n\} \) is bounded above and \( a_{2n-1}a_{2n-1} \)

= \( a_{2n}a_{2n} \), it follows that some subsequence of \( \{a_{2n-1}a_{2n-1}\} \)
is bounded above. We have \( a_{2n}' = a_{2n+1} < 0 \). Also,

\( r_n' = a_n'/a_{n-1}' = a_{n+1}/a_n = r_{n+1} \) for \( n \geq N \). From (2), for

\( n \geq N \), \( a_n' = a_{n+1} \leq 1 + r_{n+2}r_{n+2}r_{n+2}a_{n+3} = 1 + r_{n+1}' + 
+ r_{n+1}'r_{n+2}r_{n+2}' \). Applying Theorem 8.15, \( \{a_{2n-1}a_{2n-1} \} \)
converges. Thus, \( \{a_{2n}a_{2n}\} \) converges. Q.E.D.

The dual of Theorem 8.19 is Theorem 8.43.

**Theorem 8.20.** Let \( L \) be any real number and \( \Sigma a_n \) any N-alternating series such that \( a_{2n} < 0 \). Then a n.a.s.c. that \( \Sigma a_n \) converge is that

1. \( a_n \to 0 \),
2. and there exist a sequence \( \{a_n\} \) such that,
3. some subsequence of \( \{a_{2n}a_{2n}\} \) is bounded below and \( a_{2n-1}a_{2n-1} \to L \)

Moreover, if conditions (0), (1), and (2) hold, then \( \{a_{2n}a_{2n}\} \) converges.

**Proof:** The necessity follows from Theorem 8.2.

For the sufficiency, define \( a'_n = -a_n \) for \( n \geq 0 \). Accordingly, \( r'_n = a'_n/a'_{n-1} = a_n/a_{n-1} = r_n \) for \( n \geq N \).

It is easily seen that Theorem 8.19 is applicable, yielding the convergence of \( \Sigma a'_n \) and \( \{a'_{2n}a'_{2n}\} \). Thus, \( \Sigma a_n \) and \( \{a_{2n}a_{2n}\} \) both converge. Q.E.D.

The dual of Theorem 8.20 is Theorem 8.42.
Theorem 8.21. Let $L$ be any real number. Then a n.a.s.c. that an $N$-alternating series $\sum a_n$ converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{a_n\}$ such that,

(1) some subsequence of $\{a_{2n}a_{2n}\}$ is bounded and

$$a_{2n-1}a_{2n-1} \to L$$

and

(2) $a_n \leq 1 + r_{n+1} + r_{n+2} + r_{n+3} + \cdots, \quad n \geq N$.

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n}a_{2n}\}$ converges.

Proof: The necessity follows from Theorem 8.2.

For the sufficiency, we need only note that $a_{2n} > 0$ or $a_{2n} < 0$, and then apply Theorem 8.19 or Theorem 8.20, respectively. Q.E.D.

The dual of Theorem 8.21 is Theorem 8.44.

The following counterexample shows that (1) of Theorem 8.19 or Theorem 8.21 cannot be replaced by the condition

(1') $\{a_{2n}a_{2n}\}$ is bounded below and $a_{2n-1}a_{2n-1} \to L$.

Counterexample 8.22. Define $a_{2n} = 1/2(n+1)$ and

$$a_{2n+1} = -1/(n+1)$$

for $n \geq 0$. Since $a_{2n}a_{2n+1} = 1/2(n+1)$
for \( n \geq 0 \), \( S_n \to -\infty \). Define \( a_{2n-1} = 0 \) for \( n \geq 1 \). Define \( \{a_{2n}\} \) recursively by the equation \( a_{2n} = 1 + r_{2n+1} + r_{2n+1} r_{2n+2} a_{2n+2} \), \( n \geq 1 \), where \( a_2 \) is any real number. We then have (0) \( a_n \to 0 \), (1) \( a_{2n-1} a_{2n-1} \to 0 \), and (2) 
\[
\alpha_n \leq 1 + r_{n+1} + r_{n+1} r_{n+2} a_{n+2} \quad \text{for} \quad n \geq 1.
\]
Also, \( a_{2n} a_{2n} = a_2 a_2 - (a_2 + a_3 + \cdots + a_{2n-1}) \to +\infty \), so that \( \{a_{2n} a_{2n}\} \) is bounded below.

The following counterexample shows that (1) of Theorem 8.20 or Theorem 8.21 cannot be replaced by the condition

\( (1') \quad \{a_{2n} a_{2n}\} \) is bounded above and \( a_{2n-1} a_{2n-1} \to L \).

**Counterexample 8.23.** Let \( \Sigma a_n \) be the divergent series whose terms are the negatives of those of the series given in Counterexample 8.22, i.e., \( a_{2n} = -1/(n+1) \) and \( a_{2n+1} = 1/(n+1) \) for \( n \geq 0 \). Define \( a_{2n-1} = 0 \) for \( n \geq 1 \). Define \( \{a_{2n}\} \) recursively by the equation \( a_{2n} = 1 + r_{2n+1} + r_{2n+1} r_{2n+2} a_{2n+2} \), \( n \geq 1 \), where \( a_2 \) is any real number. Accordingly, (0) \( a_n \to 0 \), (1) \( a_{2n-1} a_{2n-1} \to 0 \), and (2) \( \alpha_n \leq 1 + r_{n+1} + r_{n+1} r_{n+2} a_{n+2} \) for \( n \geq 1 \). Also, \( a_{2n} a_{2n} = a_2 a_2 - (a_2 + a_3 + \cdots + a_{2n-1}) \to -\infty \), and thus \( \{a_{2n} a_{2n}\} \) is bounded above.
Lemma 8.24. Let $\Sigma a_n$ be an $N$-alternating series and 
\{\beta_n\} be a sequence such that

(0) $a_n \to 0,$

(1) $a_{2n-1} \beta_{2n-1} \to L_1$ and $a_{2n} \beta_{2n} \to L_2,$ for some $L_1$ and $L_2,$

and

(2) $\beta_n \geq 1 + r_{n+1} r_{n+1} r_{n+2} \beta_{n+2}, \ n \geq N.$

Defining $\alpha_n = 1 + r_{n+1} \beta_{n+1},$ for $n \geq N,$ we have

(3) $a_{2n-1} \alpha_{2n-1} \to L_2$ and $a_{2n} \alpha_{2n} \to L_1$

and

(4) $\alpha_n \leq 1 + r_{n+1} r_{n+1} r_{n+2} \alpha_{n+2}, \ n \geq N.$

Moreover, for $n \geq N$ and $k \geq 1,$

(5) $T_{n,2k-2} + (r_n \cdots r_{n+2k-1}) \beta_{n+2k-1}$

$= T_{n,2k-2} + (r_n \cdots r_{n+2k-2}) \alpha_{n+2k-2}$

and

(6) $T_{n,2k-3} + (r_n \cdots r_{n+2k-2}) \beta_{n+2k-2}$

$\leq T_{n,2k-2} + (r_n \cdots r_{n+2k-1}) \alpha_{n+2k-1}.$

Proof: Since $\alpha_n = 1 + r_{n+1} \beta_{n+1},$ $a_{2n-1} \alpha_{2n-1} = a_{2n-1} + a_{2n} \beta_{2n}$

$\to L_2$ and $a_{2n} \alpha_{2n} = a_{2n} + a_{2n+1} \beta_{2n+1} \to L_1.$ Using (2),
$\alpha_n - (1 + r_{n+1}r_{n+2}a_{n+2}) = 1 + r_{n+1}\beta_{n+1} + (1 + r_{n+1}r_{n+2}a_{n+2})$

so that (4) holds. Next, $T_{n, 2k-3} + (r_n \cdots r_{n+2k-2})\alpha_{n+2k-2}$

$= T_{n, 2k-3} + (r_n \cdots r_{n+2k-2})(1 + r_{n+2k-1}r_{n+2k-2}) = T_{n, 2k-3}$

+ $(r_n \cdots r_{n+2k-2})\beta_{n+2k-1}$. Thus (5) holds. Again using (2),

$T_{n, 2k-3} + (r_n \cdots r_{n+2k-2})\beta_{n+2k-2} \leq T_{n, 2k-3}$

+ $(r_n \cdots r_{n+2k-2})(1 + r_{n+2k-1}r_{n+2k-2}) = T_{n, 2k-3}$

+ $(r_n \cdots r_{n+2k-2})\beta_{n+2k-1}$. Consequently (6) holds. Q.E.D.

**Theorem 8.25.** Let $L_1$ and $L_2$ be any real numbers. Then an n.a.s.c. that an N-alternating series $\sum a_n$ converge is that

(0) $a_n \rightarrow 0$,

and there exist a sequence $\{\beta_n\}$ such that,

(1) $a_{2n-1}\beta_{2n-1} \rightarrow L_1$ and $a_{2n}\beta_{2n} \rightarrow L_2$

and

(2) $\beta_n \geq 1 + r_{n+1}r_{n+2}b_{n+2}$, $n \geq N$.

Moreover, if (0), (1), and (2) hold, then, for $n \geq N$,

$$r_n + r_{n+1}\beta_{n+1} - (L_2/a_{n-1}) \geq T_n \geq r_n\beta_n - (L_1/a_{n-1})$$

or

$$r_n + r_{n+1}\beta_{n+1} - (L_1/a_{n-1}) \geq T_n \geq r_n\beta_n - (L_2/a_{n-1}),$$
accordingly as \( n \) is odd or even, respectively. And in general, for \( n \geq N \) and \( k \geq 1 \),

\[
\begin{align*}
T_{n,2k-2} + (r_n \cdots r_{n+2k-3}) & \beta_{n+2k-2} - (L_2/a_{n-1}) \\
\geq T_{n,2k-3} + (r_n \cdots r_{n+2k-2}) & \beta_{n+2k-2} - (L_1/a_{n-1}) \\
or \\
T_{n,2k-2} + (r_n \cdots r_{n+2k-3}) & \beta_{n+2k-2} - (L_2/a_{n-1}) \\
\geq T_{n,2k-3} + (r_n \cdots r_{n+2k-2}) & \beta_{n+2k-2} - (L_1/a_{n-1}),
\end{align*}
\]

accordingly as \( n \) is odd or even, respectively.

Proof: For the necessity, we may use the proof of the necessity of Theorem 8.2, replacing "\( \alpha \)" by "\( \beta \)" throughout. For the sufficiency, assume that (0), (1), and (2) hold, and define \( \alpha_n = 1 + r_{n+1} \beta_{n+1} \) for \( n \geq N \). According to Lemma 8.24, conditions (0), (1), and (2) of Theorem 8.2 hold, with \( L_1 \) and \( L_2 \) interchanged. Using (b) of Theorem 8.2, and (5) and (6) of Lemma 8.24, we obtain (b) of the present theorem, from which (a) follows with \( k = 1 \). Q.E.D.

The dual of Theorem 8.25 is Theorem 8.2.

Choosing \( L_1 = L_2 = L \) in Theorem 8.25, we obtain the following theorem.

Theorem 8.26. Let \( L \) be any real number. Then a n.a.s.c. that an \( N \)-alternating series \( \sum a_n \) converge is that
(0) \( a_n \to 0 \),

and there exist a sequence \( \{\beta_n\} \) such that,

(1) \( a_n\beta_n \to L \)

and

(2) \( \beta_n \geq 1 + r_{n+1} + r_{n+1}r_{n+2}\beta_{n+2}, \ n \geq N \).

Moreover, if (0), (1), and (2) hold, then, for \( n \geq N \),

(a) \( r_n + r_n r_{n+1} \beta_{n+1} - (L/a_{n-1}) \geq T_n \geq r_n \beta_n - (L/a_{n-1}) \).

And in general, for \( n \geq N \) and \( k \geq 1 \),

(b) \( T_{n,2k-2} + (r_n \cdots r_{n+2k-1}) \beta_{n+2k-1} - (L/a_{n-1}) \geq T_n \)

\( \geq T_{n,2k-2} + (r_n \cdots r_{n+2k-2}) \beta_{n+2k-2} - (L/a_{n-1}) \).

Theorem 8.26 can be seen to have a dual by setting \( L_1 = L_2 = L \) in Theorem 8.2.

The following example shows that condition (2) of Theorem 8.27 cannot be replaced by the condition

(2') \( \beta_n \geq c + r_{n+1} + r_{n+1}r_{n+2}\beta_{n+2}, \ c < 1 \).

**Example 8.28.** Let \( 0 < c < 1 \), so that \( 1 < l/c \). Let

\( \Sigma a_n \) be the divergent series defined in Example 8.4.

According to that example, \( a_n \to 0 \), and there is a sequence

\( \{\alpha_n\} \) such that \( a_n\alpha_n \to 0 \) and \( \alpha_n \leq 1/c + r_{n+1} + r_{n+1}r_{n+2}\alpha_{n+2} \).

Defining \( \beta_n = c(1 + r_{n+1}\alpha_{n+1}), a_n\beta_n = c(a_n + a_{n+1}\alpha_{n+1}) \to 0 \).

From the preceding inequality it is easily seen that
(2') holds. We conclude that (0) and (1) of Theorem 8.27 and (2') are necessary, but not sufficient, for the convergence of $\Sigma a_n$.

Choosing $L_1 = L_2 = 0$ in Theorem 8.25, we obtain the following theorem.

Theorem 8.27. A n.a.s.c. that an N-alternating series $\Sigma a_n$ converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\beta_n\}$ such that,

(1) $a_n \beta_n \to 0$

and

(2) $\beta_n \geq 1 + r_{n+1} r_{n+2}^2, \quad n \geq N$.

Moreover, if (0), (1), and (2) hold, then, for $n \geq N$,

(a) $r_n + r_n \beta_{n+1} \geq T_n \geq r_n \beta_n$.

And in general, for $n \geq N$ and $k \geq 1$,

(b) $T_{n,2k-2} + (r_n \cdots r_{n+2k-1}) \beta_{n+2k-1} \geq T_n \geq T_{n,2k-3}$

$+ (r_n \cdots r_{n+2k-2}) \beta_{n+2k-2}$.

The dual of Theorem 8.27 is Theorem 8.3.

Theorem 8.29. Let $c$ be any number $> 1$. Then a n.a.s.c. that an alternating series $\Sigma a_n$ converge absolutely is that

(0) $a_n \to 0$, 
and there exist a sequence \( \{ \beta_n \} \) such that,

1. \( a_n \beta_n \to 0 \)

and

2. \( \beta_n \geq c + r_{n+1} + r_{n+1}r_{n+2} + r_{n+2} \), \( n \geq 1 \).

**Proof:** For the necessity, we may use the proof of the necessity of Theorem 8.5, replacing "\( a \)" by "\( \beta \)" throughout.

For the sufficiency, define \( \alpha_n, n \geq 1 \), by the equation \( c\alpha_n = 1 + r_{n+1} + \beta_{n+1} \). Then \( a_n \to 0 \) and \( a_n \beta_n \Rightarrow \frac{(a_n + a_{n+1})\beta_{n+1}}{c} \to 0 \). From (2),

\[
c[\alpha_n - (1/c + r_{n+1} + r_{n+1} + r_{n+2} \beta_{n+2})] = r_{n+1} [\beta_{n+1} - (c + r_{n+2} + r_{n+2} + r_{n+3} \beta_{n+3})] \leq 0,
\]

so that \( \alpha_n \leq 1/c + r_{n+1} + r_{n+2} + r_{n+3} \beta_{n+3} \) for \( n \geq 1 \), where \( 1/c < 1 \). According to Theorem 8.5, \( \Sigma |a_n| \) converges. Q.E.D.

The dual of Theorem 8.29 is Theorem 8.5.

**Theorem 8.30.** Let \( c, L_1, L_2 \) be any real numbers where \( 1 < c \). Then a n.a.s.c. that an alternating series \( \Sigma a_n \) converge absolutely is that

1. \( a_n \to 0, \)

and there exist a sequence \( \{ \beta_n \} \) such that,

\[
(1) \quad a_{2n-1}\beta_{2n-1} \to L_1 \quad \text{and} \quad a_{2n}\beta_{2n} \to L_2
\]
and

\( \beta_n \geq c + r_{n+1} r_{n+1} r_{n+2} \beta_{n+2}, \ n \geq 1. \)

**Proof:** For the necessity, there is a sequence \( \{\beta_n\} \) satisfying (1), (2) of Theorem 8.29. Define \( \{\beta'_n\} \) by the equations

\[ a_{2n-1}\beta'_{2n-1} = a_{2n-1}\beta_{2n-1} + L_1 \quad \text{and} \quad a_{2n}\beta'_{2n} = a_{2n}\beta_{2n} + L_2. \]

It is easily seen that \( \{\beta'_n\} \) satisfies (1) and (2) above.

For the sufficiency, define \( \{\beta'_n\} \) by the equations

\[ a_{2n-1}\beta'_{2n-1} = a_{2n-1}\beta_{2n-1} - L_1 \quad \text{and} \quad a_{2n}\beta'_{2n} = a_{2n}\beta_{2n} - L_2. \]

We easily verify that \( \{\beta'_n\} \) satisfies (1) and (2) of Theorem 8.29, and thus \( \Sigma a_n \) converges absolutely. Q.E.D.

The dual of Theorem 8.30 is Theorem 8.6.

With \( N = 1 \) in Theorem 8.27, we obtain the following theorem.

**Theorem 8.31.** A n.a.s.c. that an alternating series \( \Sigma a_n \) converge is that

(0) \( a_n \to 0, \)

and there exist a sequence \( \{\beta_n\} \) such that,

(1) \( a_n\beta_n \to 0 \)

and

(2) \( \beta_n \geq 1 + r_{n+1} r_{n+1} r_{n+2} \beta_{n+2}, \ n \geq 1. \)
Moreover, if (0), (1), and (2) hold, then, for \( n \geq 1 \),

(a) \( r_n + r_n r_{n+1} \beta_{n+1} \geq T_n \geq r_n \beta_n \).

And in general, for \( n \geq 1 \) and \( k \geq 1 \),

(b) \( T_{n,2k-2} + (r_n \cdots r_{n+2k-3}) \beta_{n+2k-1} \geq T_n \geq T_{n,2k-2} + (r_n \cdots r_{n+2k-3}) \beta_{n+2k-3} \).

The dual of Theorem 8.31 is Theorem 8.8.

**Theorem 8.32.** Let \( L \) be any real number. Then a n.a.s.c. that an \( N \)-alternating series \( \sum a_n \) converge is that there exist a sequence \( \{\beta_n\} \) such that

1. \( a_n \beta_n \rightarrow L \),
2. \( \beta_n \geq 1 + r_n \beta_{n+1} - (L/a_{n-1}) \), \( n \geq N \),
3. \( \beta_n \geq 1 + r_n \beta_{n+1} \), \( n \geq N \).

Moreover, if (1), (2), and (3) hold, then, for \( n \geq N \),

(a) \( r_n + r_n r_{n+1} \beta_{n+1} - (L/a_{n-1}) \geq T_n \geq r_n \beta_n - (L/a_{n-1}) \).

And in general, for \( n \geq N \) and \( k \geq 1 \),

(b) \( T_{n,2k-2} + (r_n \cdots r_{n+2k-3}) \beta_{n+2k-1} - (L/a_{n-1}) \geq T_n \geq T_{n,2k-2} + (r_n \cdots r_{n+2k-3}) \beta_{n+2k-3} - (L/a_{n-1}) \).

**Proof:** For the necessity, Theorem 8.26 implies the existence of a sequence \( \{\beta_n\} \) such that conditions (0), (1), and (2) are satisfied. Also, by (a) of Theorem 8.26, we have \( r_n + r_n r_{n+1} \beta_{n+1} - (L/a_{n-1}) \geq r_n \beta_n - (L/a_{n-1}) \) for
For the sufficiency, assume that (1), (2), and (3) hold. Using (1), (3), and the fact that $|a_n|/a_n$, $n \geq N$, is bounded, we have $0 < |a_n| \leq |a_n|/(\beta_n \beta_{n+1} r_{n+1})$

$$= (|a_n|/a_n)(a_n \beta_n - a_{n+1} \beta_{n+1}) \to 0,$$

so that $|a_n| \to 0$, i.e., $a_n \to 0$. Now apply Theorem 8.26. Q.E.D.

According to Counterexample 8.10, Theorem 8.32 has no dual.

Remark 8.33. We now consider the four conditions:

(0) $a_n \to 0$,

(1) $a_n \beta_n \to 0$,

(2) $\beta_n \geq 1 + r_{n+1} + r_{n+1} \beta_{n+1} + \beta_{n+2}$, $n \geq 1$,

(3) $\beta_n \geq 1 + r_{n+1} \beta_{n+1}$, $n \geq 1$.

We have seen that if (0) or (3) is omitted, the remaining three conditions are necessary and sufficient for the convergence of an alternating series $\Sigma a_n$. It will be shown that if condition (1) or (2) is omitted, the remaining three are not sufficient for the convergence of $\Sigma a_n$. We will see that conditions (1) and (2) are not sufficient for the convergence of $\Sigma a_n$. It will also be seen that if (1) is replaced by the weaker conditions that
anβn - an+1βn+1 → 0 and that \{a_nβ_n\} be bounded, the resulting four conditions are not sufficient for the convergence of \(\Sigma a_n\).

**Counterexample 8.34.** We use Counterexample 8.11 with \(a_n, n \geq 1\), as defined there. Defining \(β_n = α_n\) for \(n \geq 1\), the following conditions are obvious:

1. \(a_n \to 0\),
2. \(β_n \geq 1 + r_{n+1}^r r_{n+1}^r r_{n+2}^r r_{n+2}^r, n \geq 1\),
3. \(β_n \geq 1 + r_{n+1}^r β_{n+1}, n \geq 1\).

Also, \(a_nβ_n - a_{n+1}β_{n+1} \to 0\) so that the four conditions \(a_n^r β_n^r - a_{n+1}^r β_{n+1}^r \to 0\), (0), (2), and (3) are not sufficient for the convergence of \(\Sigma a_n\).

**Counterexample 8.35.** Let \(\Sigma a_n\) be the divergent series given in Counterexample 8.11. Defining \(β_n = 1\) for \(n \geq 1\), it is obvious that the following conditions hold:

1. \(a_n \to 0\),
2. \(a_n β_n \to 0\),
3. \(β_n \geq 1 + r_{n+1}^r β_{n+1}, n \geq 1\).

Thus conditions (0), (1), and (3) are not sufficient for the convergence of \(\Sigma a_n\).
Counterexample 8.36. Let $\sum a_n$ be the divergent series in Counterexample 8.10 and $\{\beta_n\}$ be any monotone decreasing sequence such that $\beta_n \to 0$. We then have

(1) $a_n\beta_n \to 0$

and

(2) $\beta_n \geq 1 + r_{n+1} + r_{n+1}^2 \beta_{n+2}$, $n \geq 1$.

Thus conditions (1) and (2) are not sufficient for the convergence of $\sum a_n$.

Counterexample 8.37. Let $\sum a_n$ be the divergent series in Counterexample 8.10, $L$ be any number $\geq 1/2$, and $\{\beta_n\}$ be any monotone decreasing sequence converging to $L$. We then have

(1) $a_{2n-1}\beta_{2n-1} \to -L$ and $a_{2n}\beta_{2n} \to L$,

(2) $\beta_n \geq 1 + r_{n+1} + r_{n+1}^2 \beta_{n+2}$, $n \geq 1$,

and

(3) $\beta_n \geq 1 + r_{n+1}^2 \beta_{n+1}$, $n \geq 1$.

Consequently, (1) of Theorem 8.32 cannot be replaced by the weaker condition that $a_{2n-1}\beta_{2n-1} \to L_1$ and $a_{2n}\beta_{2n} \to L_2$, for some numbers $L_1$ and $L_2$. The corresponding replacement in Theorem 8.26 was valid according to Theorem 8.25.
Counterexample 8.38. We use Counterexample 8.13 with \( a_n \), \( n \geq 1 \), as defined there. Defining \( \beta_n = a_n \), for \( n \geq 1 \), the following conditions hold:

1. \( a_n \to 0 \),
2. \( \beta_n \geq 1 + r_{n+1} + r_{n+1} r_{n+2} \beta_{n+2} \), \( n \geq 1 \),
3. \( \beta_n \geq 1 + r_{n+1} \beta_{n+1} \), \( n \geq 1 \).

According to Counterexample 8.13, the sequence \( \{a_n \beta_n\} \) is bounded and \( a_n \beta_n - a_{n+1} \beta_{n+1} \to 0 \). Thus, replacing (1) of Remark 8.33 by these two conditions, the resulting conditions are not sufficient for the convergence of \( \Sigma a_n \).

Theorem 8.39. Let \( L \) be any real number and \( \Sigma a_n \) be any \( N \)-alternating series such that \( a_{2n} > 0 \). Then a n.a.s.c. that \( \Sigma a_n \) converge is that

1. \( a_n \to 0 \),
2. there exist a sequence \( \{\beta_n\} \) such that, some subsequence of \( \{a_{2n-1} \beta_{2n-1}\} \) is bounded above and \( a_{2n} \beta_{2n} \to L \)

and

(2) \( \beta_n \geq 1 + r_{n+1} + r_{n+1} r_{n+2} \beta_{n+2} \), \( n \geq N \).

Moreover, if conditions (0), (1), and (2) hold, then \( \{a_{2n-1} \beta_{2n-1}\} \) converges.
Proof: The necessity follows from Theorem 8.25.

For the sufficiency, define \( a_n = 1 + r_{n+1} \beta_{n+1} \) for \( n \geq N \). Then \( a_{2n-1} a_{2n-1} = a_{2n-1} + a_{2n} \beta_{2n} \rightarrow L \). Since

\[
a_{2n} a_{2n} = a_{2n} + a_{2n+1} \beta_{2n+1},
\]

some subsequence of \( \{a_{2n} a_{2n}\} \) is bounded above. Also, \( \sigma_n = 1 + r_{n+1} \beta_{n+1} \leq 1 + r_{n+1} \)

\[
+ r_{n+1} r_{n+2} (1 + r_{n+3} \beta_{n+3}) = 1 + r_{n+1} + r_{n+1} r_{n+2} \sigma_{n+2} \quad \text{for} \ n \geq N.
\]

From Theorem 8.19, both \( \Sigma a_n \) and \( \{a_{2n} a_{2n}\} \) converge. Consequently, \( a_{2n+1} \beta_{2n+1} = a_{2n} a_{2n} - a_{2n} \rightarrow \lim a_{2n} a_{2n} \), i.e., \( \{a_{2n-1} \beta_{2n-1}\} \) converges. Q.E.D.

The dual of Theorem 8.39 is Theorem 8.15.

**Theorem 8.40.** Let \( L \) be any real number and \( \Sigma a_n \) be any \( N \)-alternating series such that \( a_{2n} < 0 \). Then a n.a.s.c. that \( \Sigma a_n \) converge is that

1. \( a_n \rightarrow 0 \),

and there exist a sequence \( \{\beta_n\} \) such that,

2. some subsequence of \( \{a_{2n-1} \beta_{2n-1}\} \) is bounded below and \( a_{2n} \beta_{2n} \rightarrow L \)

and

3. \( \beta_n \geq 1 + r_{n+1} + r_{n+1} r_{n+2} \beta_{n+2}, \ n \geq N. \)

Moreover, if conditions (0), (1), and (2) hold, then \( \{a_{2n-1} \beta_{2n-1}\} \) converges.
Proof: The necessity follows from Theorem 8.25.

For the sufficiency, define \( a_n' = -a_n \) for \( n \geq 0 \). Accordingly, \( r_n' = a_n'/a_{n-1}' = a_n/a_{n-1} = r_n \) for \( n \geq N \). It is easily seen that Theorem 8.39 is applicable, yielding the convergence of \( \Sigma a_n \) and \( \{ a_{2n-1}' \beta_{2n-1}' \} \). Thus, \( \Sigma a_n \) and \( \{ a_{2n-1}' \beta_{2n-1}' \} \) both converge. Q.E.D.

The dual of Theorem 8.40 is Theorem 8.14.

Theorem 8.41. Let \( L \) be any real number. Then an n.a.s.c. that an N-alternating series \( \Sigma a_n \) converge is that

\[ (0) \quad a_n \to 0, \]

and there exist a sequence \( \{ \beta_n \} \) such that,

\[ (1) \quad \text{some subsequence of } \{ a_{2n-1}' \beta_{2n-1}' \} \text{ is bounded and } a_{2n}' \beta_{2n} \to L \]

and

\[ (2) \quad \beta_n \geq 1 + r_{n+1} + r_{n+1} r_{n+2} \beta_{n+2}, \quad n \geq N. \]

Moreover, if conditions (0), (1), and (2) hold, then \( \{ a_{2n-1}' \beta_{2n-1}' \} \) converges.

Proof: The necessity follows from Theorem 8.25.

For the sufficiency, we need only note that \( a_{2n} > 0 \) or \( a_{2n} < 0 \), and then apply Theorem 8.39 or Theorem 8.40, respectively. Q.E.D.
The dual of Theorem 8.41 is Theorem 8.16.

**Theorem 8.42.** Let $L$ be any real number and $\Sigma a_n$ any $N$-alternating series such that $a_{2n} > 0$. Then a n.a.s.c. that $\Sigma a_n$ converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\beta_n\}$ such that,

(1) some subsequence of $\{a_{2n}\beta_{2n}\}$ is bounded below and $a_{2n-1}\beta_{2n-1} \to L$

and

(2) $\beta_n \geq 1+r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq N$.

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n}\beta_{2n}\}$ converges.

**Proof:** The necessity follows from Theorem 8.25.

For the sufficiency, define $\alpha_n = 1+r_{n+1}\beta_{n+1}$ for $n \geq N$. Then $a_{2n}\alpha_{2n} = a_{2n} + a_{2n+1}\beta_{2n+1} \to L$. Since $a_{2n-1}\alpha_{2n-1} = a_{2n-1} + a_{2n}\beta_{2n}$, some subsequence of $\{a_{2n-1}\alpha_{2n-1}\}$ is bounded below. Also, $\alpha_n = 1+r_{n+1}\beta_{n+1}$ $\leq 1+r_{n+1}r_{n+2}(1+r_{n+3}\beta_{n+3}) = 1+r_{n+1}r_{n+2}\beta_{n+2}$ for $n \geq N$. From Theorem 8.14, both $\Sigma a_n$ and $\{a_{2n-1}\alpha_{2n-1}\}$ converge. Consequently, $a_{2n}\beta_{2n} = a_{2n-1}\alpha_{2n-1} - a_{2n-1}$ $\to \lim a_{2n-1}\alpha_{2n-1}$, i.e., $\{a_{2n}\beta_{2n}\}$ converges. Q.E.D.
The dual of Theorem 8.42 is Theorem 8.20.

Theorem 8.43. Let $L$ be any real number and $\Sigma a_n$ be any $N$-alternating series such that $a_{2n} < 0$. Then a n.a.s.c. that $\Sigma a_n$ converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\beta_n\}$ such that,

(1) some subsequence of $\{a_{2n}\}$ is bounded above and $a_{2n-1}\beta_{2n-1} \to L$

and

(2) $\beta_n > 1 + r_{n+1}r_{n+2}^2$, $n \geq N$.

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n}\}$ converges.

Proof: The necessity follows from Theorem 8.25.

For the sufficiency, define $a'_n = -a_n$ for $n \geq 0$. Then $r'_n = a'_n/a'_{n-1} = a_n/a_{n-1} = r_n$ for $n \geq N$.

From Theorem 8.42, both $\Sigma a'_n$ and $\{a'_n\beta_{2n}\}$ converge.

Thus, $\Sigma a_n$ and $\{a_{2n}\}$ converge. Q.E.D.

The dual of Theorem 8.43 is Theorem 8.19.

Theorem 8.44. Let $L$ be any real number. Then a n.a.s.c. that an $N$-alternating series $\Sigma a_n$ converge is that
(0) \( a_n \to 0, \)

and there exist a sequence \( \{\beta_n\} \) such that,

(1) some subsequence of \( \{a_{2n}\_{2n}\} \) is bounded and

\[ a_{2n-1}\beta_{2n-1} \to L \]

and

(2) \( \beta_n \geq 1 + r_{n+1} + r_{n+1} r_n \beta_{n+2}, \quad n \geq N. \)

Moreover, if conditions (0), (1), and (2) hold, then \( \{a_{2n}\_{2n}\} \) converges.

**Proof:** The necessity follows from Theorem 8.25.

For the sufficiency, we need only note that

\( a_{2n} > 0 \) or \( a_{2n} < 0, \)

and then apply Theorem 8.42 or Theorem 8.43, respectively. Q.E.D.

The dual of Theorem 8.44 is Theorem 8.21.

**Theorem 8.45.** (Leibnitz's Theorem for alternating series.)

Let \( \Sigma a_n \) be an alternating series such that \(-1 \leq r_n, \)

for \( n \geq 2, \) and \( a_n \to 0. \) Then \( \Sigma a_n \) converges, and moreover \( |S - S_{n-1}| \leq |a_n| \) for \( n \geq 1. \)

**1st Proof:** Choosing \( a_n = 0 \) for \( n \geq 1, \) we may use (a)

of Theorem 8.8 to obtain

\[ r_n + r_n r_{n+1} \cdot 0 \leq (S - S_{n-1}) / a_{n-1} \leq r_n \cdot 0, \quad n \geq 1, \]

and this immediately yields the desired inequality. Q.E.D.
2nd Proof: Choosing $\beta_n = 1$ for $n \geq 1$, we may use (a) of Theorem 8.31 to obtain

$$0 \geq r_n + r_n r_{n+1} \cdot 1 \geq (S-S_{n-1})/a_{n-1} \geq r_n \cdot 1, \quad n \geq 1,$$
from which the desired inequality follows. Q.E.D.

Lemma 8.46. Suppose that $p, x, y,$ and $q$ are numbers such that $-1 < p \leq q \leq 0$, $p \leq x \leq q$, and $p \leq y \leq q$. Setting $\alpha = (1+p)/(1-pq)$ and $\beta = (1+q)/(1-pq)$, we have

1. $p\beta \leq x\beta \leq x + xy\alpha \leq x + y\beta \leq x\alpha \leq q\alpha,$
2. $\alpha \leq 1 + x + xy\alpha$ and $\beta \geq 1 + x + y\beta,$
and
3. $p\beta \leq x/(1-x) \leq q\alpha.$

Proof: It is easily seen that $0 < \alpha = 1 + p\beta \leq \beta = 1 + q\alpha$.
Accordingly, $p\beta \leq x\beta = x(1 + q\alpha) \leq x(1 + y\alpha) \leq x(1 + y\beta)$
$\leq x(1 + p\beta) = x\alpha \leq q\alpha, \quad \alpha = 1 + p\beta \leq 1 + x + xy\alpha,$ and $\beta = 1 + q\alpha$
$\geq 1 + x + y\beta$. This proves (1) and (2). For (3), we have

$$[x/(1-x)] - p\beta = [(x-p) + p(x-q)]/[(1-x)(1-pq)] \geq 0$$ and
$$q\alpha - [x/(1-x)] = [(q-x) + q(p-x)]/[(1-x)(1-pq)] \geq 0.$$ Q.E.D.

Theorem 8.47. Suppose that $\Sigma a_n$ is an $N$-alternating series such that $-1 < p \leq r_n \leq q \leq 0$ for $n \geq N$, where $p$ and $q$ are constants. Setting $\alpha = (1+p)/(1-pq)$ and $\beta = (1+q)/(1-pq),$

1. $p\beta \leq r_n \beta \leq r_n + r_n r_{n+1} \alpha \leq T_n \leq r_n + r_n r_{n+1} \beta \leq r_n \alpha$
$\leq q\alpha, \quad n \geq N.$
Proof: Define \( \alpha_n = \alpha \) and \( \beta_n = \beta \) for \( n \geq N \). Since \( |r_n| \leq |p| < 1 \) for \( n \geq N \), \( a_n \to 0 \), \( a_n \alpha_n \to 0 \), and \( a_n \beta_n \to 0 \). By Lemma 8.46, \( \alpha_n \leq 1 + r_{n+1} + r_{n+1} r_{n+2} \alpha_{n+2} \) and \( \beta_n \geq 1 + r_{n+1} + r_{n+1} r_{n+2} \beta_{n+2} \) for \( n \geq N \). Let \( n \) be any integer \( \geq N \). Using (1) of Lemma 8.46, \( p \beta \leq r_n \beta \leq r_n \) and \( + r_n r_{n+1} \alpha \leq r_n + r_n r_{n+1} \beta \leq r_n \alpha \leq q \alpha \). Also Theorem 8.8 and Theorem 8.27 yield the respective inequalities \( r_{n+1} r_n \alpha \leq T_n \) and \( T_n \leq r_{n+1} r_n \beta \). (1) of the present theorem is now evident. Q.E.D.

Suppose that \( p, q \) are constants such that \(-1 < p \leq q < 0\). We now exhibit a series \( \Sigma a_n \) satisfying the hypotheses of Theorem 8.47, and for which \( p \beta \) and \( q \alpha \) are the corresponding largest and smallest constants such that \( p \beta \leq T_n \leq q \alpha \) for \( n \geq N = 1 \). In particular, let \( \Sigma a_n = 1 + p + pq + p^2 q + p^2 q^2 + p^3 q^2 + \cdots \). Then \( r_{2n-1} = p \) and \( r_{2n} = q \) for \( n \geq 1 \), so that \( T_{2n-1} = r_{2n-1} r_{2n-1} r_{2n} + \cdots = p \beta \) and \( T_{2n} = r_{2n} r_{2n} r_{2n+1} + \cdots = q \alpha \), for \( n \geq 1 \).

Lemma 8.48. If \(-1 < x, \alpha < 1, \) and \( \alpha \leq x(1+y)/(1+x) \), then \( 1/(1-\alpha) \leq 1+x+xy/(1-\alpha) \).

Proof: We have \( 0 < 1-\alpha \) and \( 0 < 1+x \). Thus, \( \alpha(1+x) \leq x(1+y) \), \( 1 \leq (1-\alpha)+x(1-\alpha)+xy \), and \( 1/(1-\alpha) \leq 1+x+xy/(1-\alpha) \). Q.E.D.
Lemma 8.49. If \(-1 < x\) and \(1 > \beta \geq x(1+y)/(1+x)\), then 
\[
1/(1-\beta) \geq 1+x+xy/(1-\beta).
\]

Proof: We have \(0 < 1-\beta\) and \(0 < 1+x\). The following
inequalities are now obvious: \(\beta(1+x) \geq x(1+y)\),
\[
1 \geq (1-\beta)x(l-\beta)+xy,
\]
\[
1/(1-\beta) \geq 1+x+xy/(1-\beta).
\] Q.E.D.

We give three proofs of the following theorem.

Theorem 8.50. If \(r_n \to r\), \(-1 < r < 0\), then \(T_n \to r/(1-r)\).

1st Proof: Let \(\varepsilon > 0\). Since \((y-x)/(1-xy) \to 0\) as
\((x,y) \to (r,r)\), there are numbers \(p,q\) such that
\(-1 < p < r < q < 0\) and \((q-p)/(1-pq) < \varepsilon\). Using (3) of
Lemma 8.46, \(p\beta \leq r/(1-r) \leq qa\) where \(\alpha = (1+p)/(1-pq)\)
and \(\beta = (1+q)/(1-pq)\). Also, there is a positive integer
\(N\) such that \(p \leq r_n \leq q\) for \(n \geq N\). By Theorem 8.47,
\(p\beta \leq T_n \leq qa\) for \(n \geq N\). Hence, \(|T_n - r/(1-r)| \leq qa - p\beta
= (q-p)/(1-pq) < \varepsilon\) for \(n \geq N\). Q.E.D.

2nd Proof: Since \(r_n(1+r_{n+1})/(1+r_n) \to r\), there is a po-
sitive integer \(N\) and a monotone increasing sequence
\(\{\alpha_n\}\) such that \(\alpha_n \to r\) and, for \(n \geq N\), \(-1 < r_n < 0\)
and \(\alpha_n \leq r_{n+1}(1+r_{n+1})/(1+r_{n+1})\). We now use Lemma 8.48
and the inequality \(1/(1-\alpha_n) \leq 1/(1-\alpha_{n+1})\) for \(n \geq N\) to
obtain
\[ \frac{1}{1-\alpha_n} \leq 1 + r_{n+1} + r_{n+1} r_{n+2} \frac{1}{1-\alpha_n} \leq 1 + r_{n+1} \]

\[ + r_{n+1} r_{n+2} \frac{1}{1-\alpha_n+2} \]

for \( n \geq N \). Since \( |r| < 1 \), \( a_n \rightarrow 0 \) and \( a_n/(1-\alpha_n) \rightarrow 0 \).

According to Theorem 8.3, \( r_n + r_n r_{n+1}/(1-\alpha_{n+1}) \leq T_n \)
\[ \leq r_n/(1-\alpha_n) \] for \( n \geq N \). The conclusion now follows since \( r_n + r_n r_{n+1}/(1-\alpha_{n+1}) \rightarrow r + r^2/(1-r) = r/(1-r) \) and \( r_n/(1-\alpha_n) \rightarrow r/(1-r) \). Q.E.D.

3rd Proof: Since \( r_n (1+r_{n+1})/(1+r_n) \rightarrow r \), there is a positive integer \( N \) and a monotone decreasing sequence \( \{\beta_n\} \) such that \( \beta_n \rightarrow r \) and, for \( n \geq N \), \( -1 < r_n < 0 \) and \( 1 > \beta_n \geq r_{n+1}(1+r_{n+2})/(1+r_{n+1}) \). We now use Lemma 8.49 and the inequality \( 1/(1-\beta_n) \geq 1/(1-\beta_{n+2}) \) for \( n \geq N \) to obtain

\[ 1/(1-\beta_n) \geq 1 + r_{n+1} + r_{n+1} r_{n+2}/(1-\beta_n) \]
\[ \geq 1 + r_{n+1} + r_{n+1} r_{n+2}/(1-\beta_n+2) \]

for \( n \geq N \). Since \( |r| < 1 \), \( a_n \rightarrow 0 \) and \( a_n/(1-\beta_n) \rightarrow 0 \).

According to Theorem 8.27, \( r_n + r_n r_{n+1}/(1-\beta_{n+1}) \geq T_n \)
\[ \geq r_n/(1-\beta_n) \] for \( n \geq N \). The conclusion now follows since \( r_n + r_n r_{n+1}/(1-\beta_{n+1}) \rightarrow r + r^2/(1-r) = r/(1-r) \) and \( r_n/(1-\beta_n) \)
\[ \rightarrow r/(1-r) \). Q.E.D.
Theorem 8.51. If $\Sigma a_n$ is an $N$-alternating series, $-1 < r < 0$, and $1/(1-r) \leq 1+r_{n+1}+r_{n+2}/(1-r)$ for $n \geq N$, then $r_n+r_{n+1}/(1-r) \leq T_n \leq r_n/(1-r)$ for $n \geq N$.

Proof: Since $|r| < 1$, $a_n \to 0$ and $a_n/(1-r) \to 0$. Now apply Theorem 8.3 with $a_n = 1/(1-r)$ for $n \geq N$. Q.E.D.

Theorem 8.52. If $\Sigma a_n$ is an $N$-alternating series, $-1 < r < 0$, and $r_{n+2} < r_{n+1}$ for $n \geq N$, then $r_n+r_{n+1}/(1-r) \leq T_n < r_n/(1-r)$ for $n \geq N$.

Proof: Let $n \geq N$. Then $-1 < r \leq r_{n+2} < r_{n+1}$, so that $r \leq r_{n+1} \leq r_{n+1}(1+r_{n+2})/(1+r_{n+1})$. By Lemma 8.48, $1/(1-r) \leq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-r)$. Now apply Theorem 8.51. Q.E.D.

Theorem 8.53. If $-1 < r < r_{n+1} \leq r_n < 0$ for $n \geq N$, then, for $n \geq N$, $r_n+r_{n+1}(1+r)/(1-rr_n) \leq T_n \leq r_n+r_{n+1}(1+r)/(1-rr_n)$.

Proof: Let $m$ be any integer $\geq N$, $p = r$, $q = r_m$, $\alpha = (1+p)/(1-pq)$, and $\beta = (1+q)/(1-pq)$. Then $-1 < p \leq r_n \leq q < 0$ for $n \geq m$. From (1) of Theorem 8.47, $r_n+r_{n+1}\alpha \leq T_n \leq r_n+r_{n+1}\beta$ for $n \geq m$. Setting $n = m$, the desired inequality obtains. Q.E.D.
Assuming the hypotheses of Theorem 8.53, the lower bound given there for $T_n$ and that given by Theorem 8.52 will now be compared. No comparison of upper bounds appears evident.

The following inequalities are equivalent:
\[ r_n + r_n r_{n+1} / (1-r) \geq r_n + r_n r_{n+1} (1+r)/(1-rr_n), \quad 1/(1-r) \geq (1+r)/(1-rr_n), \quad 1-rr_n \geq 1-r^2, \quad r_n \geq r. \]
Consequently, the lower bound for $T_n$ given by Theorem 8.52 appears better. It is also simpler in form.

Theorem 8.54. Let $\Sigma a_n$ be an $N$-alternating series. Then a n.a.s.c. that $T_n \to -1/2$ is that $a_n \to 0$, $r = -1$, and there exist a sequence $\{a_n\}$ such that

1. $\alpha_n \to 1/2$,

and

2. $\alpha_n \leq 1 + r_n + r_{n+1} r_{n+2} a_{n+2}, \quad n \geq N$.

Proof: For the necessity, assume that $T_n \to -1/2$. Accordingly, $\Sigma a_n$ converges and $a_n \to 0$. Thus, $r_n = T_n/(1+T_{n+1}) \to (-1/2)/(1-1/2) = -1$, i.e., $r = -1$. Defining $\alpha_n = 1 + T_{n+1}$ for $n \geq N$, $\alpha_n \to 1-1/2 = 1/2$ and $\alpha_n = 1 + r_{n+1} + r_{n+1} r_{n+2} a_{n+2}$ for $n \geq N$.

For the sufficiency, Theorem 8.3 yields
\[ r_n + r_n r_{n+1}^{a_{n+1}} \leq T_n \leq r_n a_n \quad \text{for} \quad n \geq N. \] Also,
\[
\lim (r_n + r_n r_{n+1}^{a_{n+1}}) = \lim r_n a_n = -1/2, \quad \text{which implies that} \quad T_n + -1/2. \quad \text{Q.E.D.}
\]

**Theorem 8.55.** Let \( \Sigma a_n \) be an N-alternating series. Then a n.a.s.c. that \( T_n + -1/2 \) is that \( a_n + 0, \quad r = -1, \) and there exist a sequence \( \{\beta_n\} \) such that

1. \( \beta_n + 1/2 \)

and

2. \( \beta_n + 1 + r_{n+1}^{a_{n+1}} r_{n+2} \beta_{n+2}, \quad n \geq N. \)

**Proof:** For the necessity we may use the proof of the necessity of Theorem 8.54, replacing "a" by "\( \beta \)" throughout.

For the sufficiency, we use Theorem 8.27 to obtain
\[
r_n \beta_n \leq T_n \leq r_n + r_n r_{n+1}^{a_{n+1}} \beta_{n+1} \quad \text{for} \quad n \geq N. \] Also, \( r_n \beta_n + -1/2 \) and \( r_n + r_n r_{n+1}^{a_{n+1}} \beta_{n+1} + -1/2, \) so that \( T_n + -1/2. \quad \text{Q.E.D.} \)

**Lemma 8.56.** If \( x_n \to x, \quad -\infty < x < 0, \) and \( \lim \sup y_n = y, \) \(-\infty \leq y \leq +\infty, \) then \( \lim \inf x_n y_n = (\lim x_n)(\lim \sup y_n). \)

**Proof:** Suppose that \( y = +\infty. \) Then \( y_n \to +\infty \) for some subsequence \( \{n'\} \) of \( \{n\}, \) \( x_n y_n \to x (+\infty) = -\infty, \) and \( \lim \inf x y = -\infty. \) Also \( (\lim x_n)(\lim \sup y_n) = x(+\infty) = -\infty, \)
and thus \( \lim \inf x_n y_n = (\lim x_n)(\lim \sup y_n) \).

Suppose that \( y = -\infty \). Then \( \lim y_n = -\infty \), \( \lim \inf x_n y_n = +\infty \), and \( (\lim x_n)(\lim \sup y_n) = x(-\infty) = +\infty \). Hence \( \lim \inf x_n y_n = (\lim x_n)(\lim \sup y_n) \).

Suppose that \( -\infty < y < +\infty \) and let \( \lim \inf x_n y_n = L \). Then \( -\infty < L < +\infty \) and \( y_n \to y \) for some subsequence \( \{n'\} \) of \( \{n\} \). Hence \( x_n, y_{n'} \to xy \), and thus \( L \leq xy \).

Since \( \lim \inf x_n y_n = L \), there is a subsequence \( \{n^*\} \) of \( \{n\} \) such that \( x_{n^*} y_{n^*} \to L \), and thus \( y_{n^*} = \frac{x_{n^*} y_{n^*}}{x_{n^*}} \to \frac{L}{x} \leq y \). Consequently, \( L \geq xy \). Hence, \( L = xy \). Q.E.D.

**Theorem 8.57.** If \( -1 < r_n \) and \( \lim \sup (1+r_{n+1})/(1+r_n) < 1 \), then \( r_n \to r = -1 \), \( |a_n| \to a \) for some \( a > 0 \), \( \Sigma a_n \) diverges, and there is a positive integer \( m \) such that \( \prod_{n=m}^{\infty} |r_n| \) converges.

**Proof:** By hypothesis, \( 0 < l+r_n \) and \( (1+r_{n+1})/(1+r_n) < 1 \).

Thus \( -1 < r_{n+1} < r_n \) and \( r_n \to r \) where \( -1 \leq r \). We must have \( r = -1 \); since otherwise, \( \lim \sup (1+r_{n+1})/(1+r_n) = \lim (1+r_{n+1})/(1+r_n) = 1 \), a contradiction. Since \( r = -1 \), we have \( -1 < r_n < 0 \), \( |r_n| = |a_n/a_{n-1}| < 1 \), and
\[ |a_n| < |a_{n-1}|. \] Consequently, \( |a_n| \to a \) for some \( a \geq 0 \).

Assume that \( a = 0 \). Setting \( L = \lim \sup (1+r_{n+1})/(1+r_n) \), \( 0 \leq L < 1 \). From Lemma 8.56, \( \lim \inf r_n(1+r_{n+1})/(1+r_n) = (\lim r_n) [\lim \sup (1+r_{n+1})/(1+r_n)] = -L, -1 < -L \leq 0 \).

Hence, there is a positive integer \( N \) and a monotone increasing sequence \( \{a_n\} \) such that \( \sigma_n \to -L \) and, for \( n \geq N \), \( -1 < r_n < 0 \) and \( a_n \leq r_{n+1}(1+r_{n+2})/(1+r_{n+1}) \). From Lemma 8.48 and the inequality \( 1/(1-\sigma_n) \leq 1/(1-\sigma_{n+2}) \) for \( n \geq N \), \( 1/(1-\sigma_n) \leq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-\sigma_n) \leq 1+r_{n+1}+r_{n+2}/(1-\sigma_{n+2}) \) for \( n \geq N \). Also, \( a_n/(1-\sigma_n) \to 0 \).

From (a) of Theorem 8.3, \( r_n+r_{n+1}/(1-\sigma_{n+1}) \leq r_n/(1-\sigma_n) \) for \( n \geq N \). Letting \( n \to \infty \), we obtain \( -1+1/(1+L) \leq -1/(1+L), -(1+L)+1 \leq -1, \) and \( 1 \leq L \); a contradiction. Thus, \( a > 0 \) and \( \Sigma a_n \) must diverge. Since \( r_n < 0 \), there is a positive integer \( m \) such that \( r_n \neq 0 \) for \( n \geq m \), and thus \( \prod |r_m| |r_{m+1}| \ldots |r_{m+n}| = |a_{m+n}|/|a_{m-1}| \to a/|a_{m-1}| > 0 \) as \( n \to \infty \). Hence \( \prod |r_n| \) converges to \( a/|a_{m-1}| \). Q.E.D.

The preceding proof of Theorem 8.57 involved only the theory of \( N \)-alternating series. By use of known theorems for series of positive terms, and alternate proof is now given.
Proof: By hypothesis, $0 < l+r_n$ and $(1+r_{n+1})/(1+r_n) < 1$. Thus $-1 < r_{n+1} < r_n$ and $r_n \to r$ where $-1 \leq r$.

We must have $r = -1$; since otherwise, 
\[ \limsup (1+r_{n+1})/(1+r_n) = \lim (1+r_{n+1})/(1+r_n) = 1, \]
a contradiction. Since $r = -1$, $-1 < r_n < 0$ and there is a positive integer $m$ such that $-1 < r_n < 0$ for $n \geq m$.

Consequently, $\sum_{m}^{\infty} (1-|r_n|) = \sum_{m}^{\infty} (1+r_n)$ is a series of positive terms, which converges since $\limsup (1+r_{n+1})/(1+r_n) < 1$. Thus $1+r_n \to 0$ and $r_n \to r = -1$. Also with $1-|r_n| > 0$, for $n \geq m$, it is known (5, p. 382) that 
\[ \sum_{m}^{\infty} (1-|r_n|) \text{ converges if and only if } \prod_{m}^{\infty} [1-(1-|r_n|)] \]
\[ = \prod_{m}^{\infty} |r_n| \text{ converges; thus } \prod_{m}^{\infty} |r_k| = a \text{ for some } a > 0. \]

Hence, for $n > m$, $|a_n| = |a_m| |r_{m+1} r_{m+2} \cdots r_n|$
\[ \to |a_m| (\prod_{m}^{\infty} |r_k|) = |a_m|(a) > 0. \]
Consequently, $\sum_{m} a_n$ diverges. Q.E.D.

Corollary 8.58. If $a_n \to 0$ and $-1 < r_n$, then 
\[ \limsup (1+r_{n+1})/(1+r_n) \geq 1. \]

Proof: Assume that $\limsup (1+r_{n+1})/(1+r_n) < 1$. Then from Theorem 8.57, $|a_n| \to a > 0$ which contradicts
an → 0. Thus, \( \limsup \frac{1+r_{n+1}}{1+r_n} \geq 1 \). Q.E.D.

**Theorem 8.59.** If \( a_n \to 0 \), \( r = -1 < r_n \), and 
\( \limsup \frac{1+r_{n+1}}{1+r_n} = 1 \), then \( \lim T_n = r/(1-r) = -1/2 \).

**Proof:** From Lemma 8.56, 
\( \liminf \frac{1+r_{n+1}}{1+r_n} = r_n \limsup \frac{1+r_{n+1}}{1+r_n} = r \cdot 1 = r \). Consequently, there is a positive integer \( N \) and a monotone increasing sequence \( \{a_n\} \) such that \( a_n \to r \) and, for \( n \geq N \), 
\(-1 < r_n < 0 \) and \( a_n \leq r_{n+1}(1+r_{n+2})/(1+r_{n+1}) \). Using Lemma 8.48 and the inequality \( 1/(1-a_n) \leq 1/(1-a_{n+2}) \) for \( n \geq N \), 
\( 1/(1-a_n) \leq 1+r_{n+1}+r_{n+2}/(1-a_n) \leq 1+r_{n+1}+r_{n+2}/(1-a_{n+2}) \) for \( n \geq N \). Also, \( 1/(1-a_n) \to 1/2 \). Now apply Theorem 8.54. Q.E.D.

**Corollary 8.60.** If \( a_n \to 0 \), \( r = -1 < r_n \), and 
\( \limsup \frac{1+r_{n+1}}{1+r_n} \leq 1 \), then \( \limsup \frac{1+r_{n+1}}{1+r_n} = 1 \) and \( \lim T_n = r/(1-r) = -1/2 \).

**Proof:** From Corollary 8.58, \( \limsup \frac{1+r_{n+1}}{1+r_n} \geq 1 \), and thus \( \limsup \frac{1+r_{n+1}}{1+r_n} = 1 \). Now apply Theorem 8.59. Q.E.D.

**Lemma 8.61.** If \( a_n \to 0 \) and \( \liminf \frac{1+r_{n+1}}{1+r_n} = L \), 
\( 0 < L \leq +\infty \), then \( -1 < r_n \).
Proof: Since $0 < L$, $0 < \frac{(1+r_{n+1})}{(1+r_n)}$. Hence $1+r_n < 0$ or $0 < 1+r_n$. If $1+r_n < 0$, then $r_n < -1$, $1 < |r_n|$, and $|a_{n-1}| < |a_n|$. This is impossible since $a_n \to 0$. Thus $0 < 1+r_n$ and $-1 < r_n$. Q.E.D.

Lemma 8.62. If $x_n \to x$, $-\infty < x < 0$, and $\lim \inf y_n = y$, $-\infty \leq y \leq +\infty$, then $\lim \sup x_n y_n = (\lim x_n)(\lim \inf y_n)$.

Proof: Suppose that $y = +\infty$. Then $\lim y_n = +\infty$, $\lim \sup x_n y_n = -\infty$, and $(\lim x_n)(\lim \inf y_n) = x(+\infty) = +\infty$.

Suppose that $y = -\infty$. Then $y_n' \to -\infty$ for some subsequence $\{n\}'$ of $\{n\}$, $x_n'y_n' \to x(-\infty) = +\infty$, and $\lim \sup x_n y_n = +\infty$. Also, $(\lim x_n)(\lim \inf y_n) = x(-\infty) = +\infty$.

Suppose that $-\infty < y < +\infty$ and let $\lim \sup x_n y_n = L$. Then $-\infty < L < +\infty$ and $y_n' \to y$ for some subsequence $\{n\}'$ of $\{n\}$. Hence $x_n'y_n' \to xy$, and thus $xy \leq L$. Since $\lim \sup x_n y_n = L$, there is a subsequence $\{n\}^*$ of $\{n\}$ such that $x_{n^*}y_{n^*} \to L$, and thus $y_{n^*} = \frac{x_{n^*}y_{n^*}}{x_{n^*}} \to L/x \geq y$. Thus $L \leq xy$. Hence $L = xy$. Q.E.D.

Theorem 8.63. If $a_n \to 0$, $r = -1$, and
\[ \lim \inf \frac{1+ r_{n+1}}{1+r_n} = 1, \text{ then } -1 < r_n \text{ and } T_n \rightarrow r/(1-r) = -1/2. \]

**Proof:** Using Lemma 8.61 and the fact that \( r_n \rightarrow r = -1 \), \(-1 < r_n < 0 \). From Lemma 8.62, 
\[ \lim \sup r_n \frac{1+ r_{n+1}}{1+r_n} = (\lim r_n)[\lim \inf \frac{1+ r_{n+1}}{1+r_n}] = r \cdot 1 = r. \] Consequently, there is a positive integer \( N \) and a monotone decreasing sequence \( \{\beta_n\} \) such that \( \beta_n \rightarrow r \) and, for \( n \geq N \), \(-1 < \beta_n < 0 \) and \( 1 > \beta_n \geq \frac{r_{n+1}(1+r_{n+2})}{(1+r_{n+1})} \). Using Lemma 8.49 and the inequality \( 1/(1-\beta_n) \geq 1/(1-\beta_{n+2}) \) for \( n \geq N \), \( \frac{1}{1-\beta_n} \geq \frac{1+r_{n+1}+r_{n+1}r_{n+2}}{(1-\beta_n)} > 1+r_{n+1} + r_{n+2} \) for \( n \geq N \). Also, \( 1/(1-\beta_n) \rightarrow 1/2. \)

Now apply Theorem 8.55. Q.E.D.

**Theorem 8.64.** If \( a_n \rightarrow 0, r = -1, \) and 
\[ \lim \frac{1+ r_{n+1}}{1+r_n} = 1, \text{ then } -1 < r_n \text{ and } \lim T_n = r/(1-r) = -1/2. \]

**Proof:** Since \( \lim \inf \frac{1+ r_{n+1}}{1+r_n} = \lim \frac{1+ r_{n+1}}{1+r_n} = 1 \), the conclusion follows from Theorem 8.63. Q.E.D.

Pflanz (18, p. 27) has proven that if \( \sum a_n \) is an alternating series such that \( r_n = -1+a/n+\gamma_n/n \), where \( a > 0 \)
and $\gamma_n \to 0$, then $\Sigma a_{\delta n} \epsilon MR(\Sigma a_n)$. We now give a short proof of this fact.

**Theorem 8.65.** If $r_n = -1 + a/n + \gamma_n/n$ where $a > 0$ and $\gamma_n \to 0$, then $T_n \to -1/2$ and $\Sigma a_{\delta n} \epsilon MR(\Sigma a_n)$.

**Proof:** By hypothesis, $r = \lim r_n = -1$ and $-1 < r_n < 0$. Thus, $|r_n| = |a_n/a_{n-1}| < 1$, $|a_n| < |a_{n-1}|$, and $|a_n| \to c$ for some $c > 0$. Also, $|r_n| = 1 - (a + \gamma_n)/n$, $(a + \gamma_n)/n > 0$, and $\Sigma (a + \gamma_n)/n$ diverges. Consequently, from Apostol (5, p.238), $\Pi |r_n|$ diverges to zero so that $c = 0$, i.e., $a_n \to 0$. Moreover,

$$(1 + r_{n+1})/(1 + r_n) = [(a + \gamma_{n+1})/(n+1)]/[(a + \gamma_n)/n] = [n/(n+1)][(a + \gamma_{n+1})/(a + \gamma_n)] \to 1.$$ From Theorem 8.64, $T_n \to -1/2$, and thus $T_{n+1} - T_n \to 0$. We now apply Theorem 3.8. Q.E.D.

**Lemma 8.66.** If $-1 < r_n < a$ for some number $a$, then $0 \leq \lim \inf (1 + r_{n+1})/(1 + r_n) \leq 1$.

**Proof:** From $-1 < r_n$, $0 < (1 + r_{n+1})/(1 + r_n)$. Thus setting $L = \lim \inf (1 + r_{n+1})/(1 + r_n)$, $0 \leq L \leq +\infty$. Suppose $1 < L$. Then $1 < (1 + r_{n+1})/(1 + r_n)$, $-1 < r_n < r_{n+1} < a$, and $r$ exists with $-1 < r \leq a$. Hence
\[ L = \lim \inf \frac{1+r_{n+1}}{1+r_n} = \lim \frac{1+r_{n+1}}{1+r_n} = 1, \]
a contradiction. Thus \( 0 \leq L \leq 1. \) Q.E.D.

**Theorem 8.67.** If \( a_n \to 0, \) \( r = -1 < r_n, \) and
\[
\lim \frac{1+r_{n+1}}{1+r_n} = L \quad \text{where} \quad -\infty \leq L \leq +\infty,
\]
then \( L = 1 \) and \( T_n \to \frac{r}{1-r} = -1/2. \)

**Proof:** Since \( r = -1 < r_n, \) \( -1 < r_n < 0. \) From Corollary 8.58 and Lemma 8.66, \( L \geq 1 \) and \( L \leq 1, \) respectively. Hence \( L = 1, \) and thus, from Theorem 8.64,
\[ T_n \to \frac{r}{1-r} = -1/2. \] Q.E.D.

**Theorem 8.68.** If \( a_n \to 0, \) \( r = -1, \) and \( \lim \)
\[
\frac{1+r_{n+1}}{1+r_n} = L \quad \text{where} \quad -\infty \leq L \leq +\infty,
\]
then exactly one of the following statements is true:

1. \( -1 < r_n \) and \( L = 1. \)
2. \( 1+r_n \) is alternately positive and negative, for large \( n, \) and \( L = -1. \)

**Proof:** Since \( r_n \to -1 \) we may assume that \( -2 < r_n < 0 \) for \( n \geq 1. \) Exactly one of the following statements is true:

(i) \( -1 < r_n. \)

(ii) \( r_n < -1. \)
If (i) holds, then \( L = 1 \) according to Theorem 8.67.

Suppose that (ii) is true. For each integer \( n \geq 1 \), define \( r'_n = r_n \) if \(-1 \leq r_n\), or \( r'_n = -2 - r_n \) if \( r_n < -1 \).

Accordingly, for \( n \geq 1 \) we have \(-2 < r_n \leq r'_n < 0\) and \( 0 \leq l+r'_n \). Define \( a'_0 = 1 \) and \( a'_n = r'_1 r'_2 \cdots r'_n \) for \( n \geq 1 \). Since \( 0 < |r'_n| \leq |r_n| \) for \( n \geq 1 \),

\[
|a'_n| = |r'_1| |r'_2| \cdots |r'_n| \leq |r_1| |r_2| \cdots |r_n| = |a_n/a_0| \to 0,
\]

i.e., \( a'_n \to 0 \). Also, \( l+ r'_n = l+ r_n \) or \( l+ r'_n = -1 - r_n \),

i.e., \( l+ r'_n = |l+ r_n| \) for \( n \geq 1 \), so that

\[
\lim (l+ r'_{n+1})/(l+ r'_n) = \lim |(l+ r_{n+1})/(l+ r_n)| = |L|.
\]

Moreover, \( l+ r'_n = |l+ r_n| \to 0 \), i.e., \( r'_n \to -1 \). We now have \( a'_n \to 0 \), \( r' = \lim r'_n = -1 \), \(-1 < r'_n \), and

\[
\lim (l+ r'_{n+1})/(l+ r'_n) = |L|. \quad \text{From Theorem 8.67, } |L| = 1,
\]

i.e., \( L = -1 \) or \( L = 1 \). Assume that \( L = 1 \). Then \( l+ r_n \) is of constant sign for large \( n \). Hence, according to (ii), \( l+ r_n < 0 \), i.e., \( r_n < -1 \). This contradicts \( a_n \to 0 \); thus \( L = -1 \) and \( l+ r_n \) is alternately positive and negative for large \( n \). Q.E.D.

Corollary 8.69. If \( a_n \to 0 \), \( r = -1 \), and \( \lim (l+ r_{n+1})/(l+ r_n) = L \) where \(-\infty \leq L \leq \infty \) and \( L \neq -1 \), then
-1 < \( r_n \), \( L = 1 \), and \( T_n \to r/(1-r) = -1/2 \).

**Proof:** From Theorem 8.68, \( -1 < r_n \) and \( L = 1 \). We may now apply Theorem 8.64 or Theorem 8.67 to complete the proof. Q.E.D.

**Lemma 8.70.** If \( (1+r_n)(1+r_{n+1}) < 0 \), some subsequence of \( \{r_{2n-1}\} \) converges to \(-1\), and some subsequence of \( \{r_{2n}\} \) converges to \(-1\), then \(-1 \leq \limsup \frac{1+r_{n+1}}{1+r_n} \leq 0\).

**Proof:** By hypothesis, \( (1+r_{n+1})/(1+r_n) < 0 \). Thus, setting \( L = \limsup (1+r_{n+1})/(1+r_n) \), we have \(-\infty \leq L \leq 0\).

Suppose that \( L < -1 \). Then \( (1+r_{n+1})/(1+r_n) < -1 \) and \( (1+r_{n+2})/(1+r_n) = ([1+r_{n+2}]/[1+r_{n+1}])[(1+r_{n+1})/(1+r_n)] \geq 1 \). Either \( 1+r_{2n} < 0 \), or \( 1+r_{2n-1} < 0 \). In the former case, \( 1+r_{2n+2} < 1+r_{2n} \), so that \( r_{2n+2} < r_{2n} < -1 \). This is impossible since some subsequence of \( \{r_{2n}\} \) converges to \(-1\). In the latter case, \( 1+r_{2n+1} < 1+r_{2n-1} \), so that \( r_{2n+1} < r_{2n-1} < -1 \). This is impossible since some subsequence of \( \{r_{2n-1}\} \) converges to \(-1\). Thus, \(-1 \leq L \leq 0\). Q.E.D.
Lemma 8.71. If \( a_{2n} \to 0 \), \( r_{2n-1} \to -1 < r_{2n-1} \), and 
\[ \limsup \frac{1+r_{2n}}{1+r_{2n-1}} = L \] 
where \(-\infty \leq L \leq -1\), then \( r_{2n} < -1 \), some subsequence of \( \{r_{2n}\} \) converges to \(-1\), and \( L = -1 \).

Proof: By hypothesis, 
\[ \frac{1+r_{2n}}{1+r_{2n-1}} < 0 < 1 + r_{2n-1} \], and thus, \( r_{2n} < -1 \). Clearly, 
\[ (1+r_n)(1+r_{n+1}) < 0. \] Assume that no subsequence of \( \{r_{2n}\} \) converges to \(-1\). Then there is a number \( \alpha \) such that \( r_{2n} < \alpha < -1 \). Since \( r_{2n-1} \alpha \to -\alpha > 1 \), \( |a_{2n}/a_{2n-2}| \) 
\[ = r_{2n-1} r_{2n} > r_{2n-1} \alpha > 1. \] Thus, \( |a_{2n}| > |a_{2n-2}| \), which contradicts \( a_{2n} \to 0 \). If follows that some subsequence of \( \{r_{2n}\} \) converges to \(-1\). From Lemma 8.70, 
\(-1 \leq L \leq 0\), and thus \( L = -1 \). Q.E.D.

Theorem 8.72. If \( \sum a_n \) converges, \( r_{2n-1} \to -1, -1 < r_{2n-1} \), and 
\[ \limsup \frac{1+r_{2n}}{1+r_{2n-1}} = L \] 
where \(-\infty \leq L \leq 1\), then \( r_{2n} < -1 \), some subsequence of \( \{r_{2n}\} \) converges to \(-1\), \( L = -1, T_{2n-1} \to +\infty \).

Proof: From Lemma 8.71, \( r_{2n} < -1 \), some subsequence of 
\( \{r_{2n}\} \) converges to \(-1\), and \( L = -1 \). Let \( \alpha \) be any 
number \(< 1\). From Lemma 8.56, \( \liminf r_{2n-1} (1+r_{2n})/(1+r_{2n-1}) \)
Thus, \( a_{2n} \leq \frac{1}{1-a} \) for \( n \geq 1 \), so \( a_{2n-2} \leq 1 + r_{2n-1}^2a_{2n} \). Clearly, \( a_{2n-1}a_{2n} \to 0 \). From Theorem 8.3, there is a sequence \( \{a_{2n-1}\} \) such that \( a_{2n-1}a_{2n-1} \to 0 \) and \( a_{2n-1} \leq 1 + r_{2n-1}^2a_{2n}a_{2n+1} \).

We now have \( a_n a_n \to 0 \) and \( a_n \leq 1 + r_{n+1}^2a_{n+1}a_{n+2} \).

From Theorem 8.3, \(-1 - r_{2n-1}a_{2n} \leq r_{2n-1}^2a_{2n}a_{2n} \leq T_{2n-1} \). Accordingly, \( \liminf (-1 - r_{2n-1}a_{2n}) = -1 + \frac{1}{1-a} = a/(1-a) \leq \liminf T_{2n-1} \). Since \( a/(1-a) \to +\infty \) as \( a \to 1- \), \( \liminf T_{2n-1} = +\infty \); thus, \( T_{2n-1} \to +\infty \).

Since \( r_{2n} < -1 \), \( T_{2n} = r_{2n}(1 + T_{2n+1}) \leq -(1 + T_{2n+1}) \to -\infty \), which yields \( T_{2n} \to -\infty \). Q.E.D.

The series \( \Sigma a_n \) defined in Example 8.82 satisfies the hypothesis of Theorem 8.72.

According to the following counterexample, we cannot replace "\(-\infty \leq L \leq -1\)" in Theorem 8.72 by "\(-\infty \leq L \leq -\frac{1}{2}\)".

**Counterexample 8.73.** Set \( a_{2n} = 1/(n+1) \) and \( a_{2n+1} = -1/(n+3) \) for \( n \geq 0 \). Then \( S = 3/2 \), \( r = -1 \), \( r_{2n} < -1 < r_{2n-1} \), \( \lim (1+r_{2n})/(1+r_{2n-1}) = -1/2 \).
\[
\lim \frac{(1+r_{2n+1})}{(1+r_{2n})} = -2, \quad T_{2n} = \frac{-(2n+3)}{(n+1)} \to -2, \quad \text{and} \\
T_{2n+1} = \frac{(n+1)}{(n+2)} \to 1.
\]

According to the following counterexample, we cannot replace "\(r_{2n-1} \to -1\)" and "\(-1 < r_{2n-1}\)" in Theorem 8.72 by "\(r_{2n} \to -1\)" and "\(-1 < r_{2n}\)", respectively, and obtain as a conclusion that \(L = -1, T_{2n-1} \to +\infty\), or \(T_{2n} \to +\infty\).

Counterexample 8.74. Set \(a'_n = a_{n+1}\) for \(n \geq 0\), where \(a_n\) is defined as in Counterexample 8.73. Accordingly, \(S' = 1/2, r' = -1, r'_{2n-1} = r_{2n} < -1 < r'_{2n} = r_{2n+1}\), \(\lim \frac{(1+r'_{2n})}{(1+r'_{2n-1})} = \lim \frac{(1+r_{2n+1})}{(1+r_{2n})} = -2\), \(\lim \frac{(1+r'_{2n+1})}{(1+r'_{2n})} = \lim \frac{(1+r_{2n+2})}{(1+r_{2n+1})} = -1/2\), \(T'_{2n} = T_{2n+1} \to 1\), and \(T'_{2n-1} = T_{2n} \to -2\).

Theorem 8.75. If \(\Sigma a_n\) converges, \(r_{2n} \to -1, -1 < r_{2n}\), and \(\limsup \frac{(1+r_{2n+1})}{(1+r_{2n})} = L\) where \(-\infty \leq L \leq -1\), then \(r_{2n-1} < -1\), some subsequence of \(\{r_{2n-1}\}\) converges to \(-1\), \(L = -1, T_{2n-1} \to -\infty\), and \(T_{2n} \to +\infty\).

Proof: Define \(a'_n = a_{n+1}\) for \(n \geq 0\). Then \(-1 < r'_{2n-1} = r_{2n}, r'_{2n-1} \to -1\), and
\[ \lim \sup \left( \frac{1+r'_{2n}}{1+r'_{2n-1}} \right) = \lim \sup \left( \frac{1+r_{2n+1}}{1+r_{2n}} \right) = L \leq -1. \]

We may apply Theorem 8.72 to \( \Sigma_{a_n'} \), obtaining

\[ r_{2n+1} = r_{2n} < -1, \]

some subsequence of \( \{r_{2n}'\} = \{r_{2n+1}\} \) converges to \(-1\), \( T_{2n+1} = T_{2n} \to -\infty \), and \( T_{2n} = T_{2n-1} \to +\infty \). Q.E.D.

Theorem 8.76. If \( \Sigma_{a_n} \) converges, \( r = -1 \), and

\[ \lim \sup \left( \frac{1+r_{n+1}}{1+r_n} \right) = L \] where \(-\infty \leq L \leq -1\), then \( L = -1 \), and exactly one of the following statements is true:

(1) \( r_{2n} < -1 < r_{2n-1}, \ T_{2n-1} \to +\infty, \) and \( T_{2n} \to -\infty. \)

(2) \( r_{2n-1} < -1 < r_{2n}, \ T_{2n-1} \to -\infty, \) and \( T_{2n} \to +\infty. \)

Proof: Exactly one of the following statements is true:

(i) \( r_{2n} < -1 < r_{2n-1}. \)

(ii) \( r_{2n-1} < -1 < r_{2n}. \)

Suppose that (i) is true. Then

\[ \lim \sup \left( \frac{1+r_{2n}}{1+r_{2n-1}} \right) \leq \lim \sup \left( \frac{1+r_{n+1}}{1+r_n} \right) \leq L \leq -1. \]

From Theorem 8.72, \( L = -1, \ T_{2n-1} \to +\infty, \) and \( T_{2n} \to -\infty. \)

Suppose that (ii) is true. Then

\[ \lim \sup \left( \frac{1+r_{2n+1}}{1+r_{2n}} \right) \leq \lim \sup \left( \frac{1+r_{n+1}}{1+r_n} \right) \leq L \leq -1. \]

From Theorem 8.75, \( L = -1, \ T_{2n-1} \to -\infty, \) and
Lemma 8.77. If \( x < -1 \), \( 1 < \beta \), and \( \beta \geq x(1+y)/(1+x) \), then \( 1/(1-\beta) \geq 1+x+xy/(1-\beta) \).

Proof: By hypothesis, \( 1+x < 0 \) and \( 1-\beta < 0 \). Thus, \( \beta(1+x) \leq x(1+y) \), \( 1 \leq (1-\beta)+x(1-\beta)+xy \), and \( 1/(1-\beta) \geq 1+x+xy/(1-\beta) \). Q.E.D.

Theorem 8.78. If \( \Sigma_{n} \) converges, \( r_{2n-1} \rightarrow -1 \), \( r_{2n-1} < -1 \), \( r_{2n} \rightarrow -1 \), and \( \lim \inf (1+r_{2n})/(1+r_{2n-1}) = L \geq -1 \), then \( r = -1 \), \( T_{2n-1} \rightarrow -\infty \), and \( T_{2n} \rightarrow +\infty \).

Proof: Let \( \alpha \) be any number \( < -1 \). By hypothesis, \( \alpha \leq (1+r_{2n})/(1+r_{2n-1}) \), \( \alpha(1+r_{2n-1}) \geq 1+r_{2n} \), and \( -1 \leq r_{2n} \leq -1+\alpha(1+r_{2n-1}) \). Also, \( \lim [-1+\alpha(1+r_{2n-1})] = -1 \), so that \( \lim r_{2n} = -1 \). Thus, \( r = -1 \).

Let \( \beta \) be any number \( > 1 \). From Lemma 8.62, 
\[
\lim \sup r_{2n-1}(1+r_{2n})/(1+r_{2n-1}) = (\lim r_{2n-1})
\]
\[
[\lim \inf (1+r_{2n})/(1+r_{2n-1})] = (-1)(L) = -L \text{ where 0 \leq -L \leq 1. Consequently, } \beta \geq r_{2n-1}(1+r_{2n})/(1+r_{2n-1}).
\]

From Lemma 8.77, \( 1/(1-\beta) \geq 1+r_{2n-1}+r_{2n-1}r_{2n}/(1-\beta) \).

Defining \( \beta_{2n} = 1/(1-\beta) \) for \( n \geq 1 \), \( \beta_{2n} \geq 1+r_{2n+1}+r_{2n+1}r_{2n+2}r_{2n+2} \). From Theorem 8.27, there is a sequence
\( \{ \beta_{2n-1} \} \) such that \( a_{2n-1} \beta_{2n-1} \to 0 \) and \( \beta_{2n-1} \geq 1 + r_{2n} + r_{2n+1} \beta_{2n+1} \). We now have \( a_n > 0 \) and \( \beta_n \geq 1 + r_{n+1} + r_{n+2} \beta_{n+2} \). From Theorem 8.27, \( r_{2n-1} + r_{2n-1} r_{2n} \beta_{2n} \geq T_{2n-1} \). Accordingly, \( \limsup (r_{2n-1} + r_{2n-1} r_{2n} \beta_{2n}) = -1 \) +1/(1-\( \beta \)) = \( \beta/(1-\beta) \) \( \geq \limsup T_{2n-1} \). Also, \( \beta/(1-\beta) \to -\infty \) as \( \beta \to 1^- \), so that \( \limsup T_{2n-1} = -\infty \). Thus, 

\[ T_{2n-1} \to -\infty. \]

Consequently, \( T_{2n} = r_{2n}(1+T_{2n+1}) \) \( \to (-1)(1-\infty) = +\infty \). Q.E.D.

**Theorem 8.79.** If \( \Sigma a_n \) converges, \( r_{2n} \to -1, r_{2n} < -1 < r_{2n-1}, \liminf \frac{1+r_{2n+1}}{1+r_{2n}} = L \geq -1, \) then \( r = -1, T_{2n-1} \to +\infty, \) and \( T_{2n} \to -\infty. \)

**Proof:** Define \( a'_n = a_{n+1} \) for \( n \geq 0 \). Then \( r'_n = r_{n+1} \).

Thus, \( r'_{2n-1} < -1 < r'_2, r'_{2n-1} \to -1, \) and 

\[ \liminf \frac{1+r_{2n}}{1+r'_{2n-1}} = \liminf \frac{1+r_{2n+1}}{1+r_{2n}} = L. \]

Applying Theorem 8.78 to \( \Sigma a'_n, r_{2n+1} = r'_2, r'_{2n} \to -1, T_{2n+1} = T'_2 \to +\infty, \) and \( T_{2n} = T'_{2n-1} \to -\infty. \) Q.E.D.

**Theorem 8.80.** If \( \Sigma a_n \) converges, \( r = -1, (1+r_n)(1+r_{n+1}) < 0, \) and \( \liminf \frac{1+r_{n+1}}{1+r_n} \geq -1, \) then exactly one of the following statements is true:
(1) \[ r_{2n-1} < -1 < r_{2n}, \quad T_{2n-1} \to -\infty, \quad \text{and} \quad T_{2n} \to +\infty. \]

(2) \[ r_{2n} < -1 < r_{2n-1}, \quad T_{2n-1} \to +\infty, \quad \text{and} \quad T_{2n} \to -\infty. \]

**Proof:** Exactly one of the following statements is true:

(i) \[ r_{2n-1} < -1 < r_{2n}. \]

(ii) \[ r_{2n} < -1 < r_{2n-1}. \]

Suppose that (i) is true. By hypothesis,

\[ -1 \leq \lim \inf \frac{1+r_{n+1}}{1+r_n} \leq \lim \inf \frac{1+r_{2n}}{1+r_{2n-1}}. \]

From Theorem 8.78, \( T_{2n-1} \to -\infty \) and \( T_{2n} \to +\infty. \)

Suppose that (ii) is true. Then

\[ -1 \leq \lim \inf \frac{1+r_{n+1}}{1+r_n} \leq \lim \inf \frac{1+r_{2n+1}}{1+r_{2n}}. \]

From Theorem 8.79, \( T_{2n-1} \to +\infty \) and \( T_{2n} \to -\infty. \) Q.E.D.

**Theorem 8.81.** If \( \Sigma a_n \) converges, \( r = -1, \) and

\[ \lim \frac{1+r_{n+1}}{1+r_n} = L \quad \text{where} \quad -\infty \leq L \leq +\infty \quad \text{and} \quad L \neq 1, \]

then \( L = -1, \) and exactly one of the following statements is true:

(1) \[ r_{2n-1} < -1 < r_{2n}, \quad T_{2n-1} \to -\infty, \quad \text{and} \quad T_{2n} \to +\infty. \]

(2) \[ r_{2n} < -1 < r_{2n-1}, \quad T_{2n-1} \to +\infty, \quad \text{and} \quad T_{2n} \to -\infty. \]

**Proof:** From Theorem 8.68, \( L = -1 \) and \( (1+r_n)(1+r_{n+1}) < 0. \)

Now apply Theorem 8.76 or Theorem 8.80. Q.E.D.

If \( \Sigma a_n \) is a series satisfying the hypothesis of Theorem 8.68 with \( L = 1, \) according to Theorem 8.64, \( \Sigma a_n \)
converges and \( T_n \to -1/2 \). With \( L = -1 \), \( \sum a_n \) may or may not converge, as is shown in the following two examples. Consequently, we cannot replace the requirement in Theorem 8.81 that \( \sum a_n \) converge by the condition that \( a_n \to 0 \).

**Example 8.82.** Set \( a_{2n} = 1/(n+2) \) and \( a_{2n+1} = 1/(n+2)^{3/2} - 1/(n+2) \) for \( n \geq 0 \). Then \( a_n \to 0 \) and, for \( n \geq 0 \), \( a_{2n} + a_{2n+1} = 1/(n+2)^{3/2} \). Thus,

\[
S = \sum_{n=0}^{\infty} 1/(n+2)^{3/2} = z(3/2) - 1, \text{ where } z(s) = \sum_{n=1}^{\infty} 1/n^s, \quad s > 1, \text{ is the Riemann zeta function. It can be verified that } r = -1, (1+r_{n+1})/(1+r_n) \to -1, \quad \text{and } r_{2n} < -1 < r_{2n-1} \text{ for } n \geq 1. \quad \text{Thus, } \sum a_n \text{ is a convergent series satisfying the hypothesis of Theorem 8.68 with } L = -1.

From Theorem 8.81, \( T_{2n} \to -\infty \) and \( T_{2n-1} \to +\infty \).

**Example 8.83.** Set \( a_{2n} = 1/(n+1)^{1/2} \) and \( a_{2n+1} = [1-(n+2)^{1/2}] / [(n+1)(n+2)]^{1/2} \) for \( n \geq 0 \). We have \( a_n \to 0 \) and, for \( n \geq 0 \), \( a_{2n} + a_{2n+1} = 1/[(n+1)(n+2)]^{1/2} > 1/(n+2) \). Thus \( \sum a_n \) diverges. Also, \( r = -1 \) and \( (1+r_{n+1})/(1+r_n) \to -1 \). Consequently, the hypothesis of Theorem 8.68 is satisfied by the given divergent series
where $L = -1$. Moreover, we see that the requirement in Theorem 8.81 that $\Sigma a_n$ converge cannot be replaced by the condition that $a_n \to 0$.

**Theorem 8.84.** If $\Sigma a_n$ is an $N$-alternating series, $a_n \to 0$, and $1/2 \leq 1 + r_n + r_n r_{n+1}/2$ for $n \geq N$, then, for $n \geq N$, $-1 < r_n$, $-1/2 \leq r_n + r_n r_{n+1}/2 \leq T_n \leq r_n/2$, and $|a_n|/2 \leq |S - S_{n-1}| \leq |a_{n-1}|/2$. If, in addition, $r = -1$, then $T_n \to r/(1-r) = -1/2$.

**Proof:** Since $1/2 \leq 1 + r_n + r_n r_{n+1}/2$ for $n \geq N$, we have $-1/2 \leq r_n + r_n r_{n+1}/2$. For $n \geq N$, we use Theorem 8.3 with $\alpha_n = 1/2$ to obtain $-1/2 \leq r_n + r_n r_{n+1}/2 \leq T_n \leq r_n/2$ and $-1 \leq r_n$. For $n \geq N$, $-1/2 \leq T_n \leq r_n/2 < 0$, from which $|r_n|/2 \leq |T_n| \leq 1/2$ and $|a_n|/2 \leq |S - S_{n-1}| \leq |a_{n-1}|/2$.

Suppose that $r_m = -1$ for some integer $m \geq N$. Assume that $n$ is any integer $\geq m$ such that $r_n = -1$. Then $1/2 \leq 1 + r_n + r_n r_{n+1}/2 = -r_{n+1}/2$ and $r_{n+1} \leq -1$. Consequently, $r_{n+1} = -1$ since $-1 \leq r_{n+1}$. By induction, $r_n = -1$ for $n \geq m$ which contradicts $a_n \to 0$. Thus, $-1 < r_n$ for $n \geq N$. If, in addition, $r = -1$, then from $-1/2 \leq T_n \leq r_n/2 \to -1/2$, we have $\lim T_n = -1/2$. Q.E.D.
Corollary 8.85. If $\Sigma a_n$ is an $N$-alternating series, $a_n \to 0$, and $r_{n+1} \leq r_n$ for $n \geq N$, then, for $n \geq N$,

$$-1 < r_n, \quad -1/2 < r_n + r_n r_{n+1}/2 \leq T_n \leq r_n/2,$$

and

$$|a_n|/2 \leq |S - S_{n-1}| < |a_{n-1}|/2. \quad \text{If, in addition, } r = -1, \quad \text{then } T_n \to r/(1-r) = 1/2.$$

Proof: The inequality $1/2 < 1+x+x^2/2$ holds for all real $x$. Consequently, since $r_{n+1} \leq r_n < 0$ for $n \geq N$, it follows that $1/2 < 1+r_n + r_n^2/2 \leq 1 + r_n + r_n r_{n+1}/2$ for $n \geq N$. Now apply Theorem 8.84. Q.E.D.

Corollary 8.86. If $\Sigma a_n$ is an $N$-alternating series, $a_n \to 0$, and $\Delta^2 |a_{n-1}| \geq 0$ for $n \geq N$, then, for $n \geq N$,

$$-1 < r_n, \quad -1/2 < r_n + r_n r_{n+1}/2 \leq T_n \leq r_n/2,$$

and

$$|a_n|/2 < |S - S_{n-1}| < |a_{n-1}|/2. \quad \text{If, in addition, } r = -1, \quad \text{then } T_n \to r/(1-r) = -1/2.$$

Proof: Let $n \geq N$. Then $1 + r_n + r_n r_{n+1}/2 - 1/2$

$$= (1 + 2r_n + r_n r_{n+1}/2 - 1/2) = (1 - 2|a_n|/|a_{n-1}| + |a_{n+1}|/|a_{n-1}|)/2$$

$$= (|a_{n-1}| - 2|a_n|/|a_{n+1}|/|a_{n-1}|)/2 |a_{n-1}| = (\Delta^2 |a_{n-1}|)/2 |a_{n-1}| \geq 0,$$

and thus $1/2 < 1 + r_n + r_n r_{n+1}/2$. We now apply Theorem 8.84. Q.E.D.
Calabrese (10, p. 215-217) appears to be the first to publish a result similar to our Corollary 8.86. In particular, he states that if \( \sum_{1}^{\infty} a_{n} \) is a convergent alternating series, \( |a_{n}| - |a_{n+1}| > |a_{n+1}| - |a_{n+2}| \), i.e., \( \Delta^{2}|a_{n}| > 0 \) for all \( n \), and \( |a_{k}| \leq 2\varepsilon \) for some integer \( k \), then \( |S_{k} - S| \leq \varepsilon \). His proof is incorrect since he uses the fact that in "every" convergent alternating series the sum \( S \) must lie between any two successive sums \( S_{n-1} \) and \( S_{n} \).

It would be very convenient if the conditions \( a_{n} \to 0 \) and \( r = -1 < \cdot r_{n} \) implied that \( T_{n} \to r(1-r) = -1/2 \), but the following counterexample shows that this is not the case.

**Counterexample 8.87.** Let \( S' = a_{0} + a_{1} + a_{2} + \cdots \) be any alternating series such that \( a_{n} \to 0 \) and \( r' = -1 < r_{n+1} < r_{n} < -1/2 \) for \( n \geq 1 \). For \( n \geq 1 \), set \( r_{2n-1} = r_{2n-1}' \) and \( r_{2n} = -1 + 2(1 + r_{2n-1}) \). Define \( a_{0} = a'_{0} \) and \( a_{n} = a_{0} r_{1} r_{2} \cdots r_{n} \) for \( n \geq 1 \). It can be verified that \( \sum a_{n} \) is a convergent alternating series such that \( r = -1 < r_{n} \) for \( n \geq 1 \). Defining \( \beta_{2n} = 2r_{2n+1} \) for \( n \geq 1 \), we have \( \beta_{2n} = -1 + r_{2n+2} > -1 + r_{2n+4} = \beta_{2n+2} \).
for \( n \geq 1 \). Also, \( \beta_{2n} = \frac{r_{2n+1}(1+r_{2n+2})}{(1+r_{2n+1})} \) for \( n \geq 1 \), so that \( \frac{1}{(1-\beta_{2n})} = \frac{1}{1+r_{2n+1}} + \frac{r_{2n+1}}{1+r_{2n+2}} \) for \( n \geq 1 \). Consequently, it can be seen that \( \frac{1}{(1-\beta_{2n})} \geq 1 + T_{2n+1} \), i.e., \( T_{2n+1} \leq \frac{\beta_{2n}}{1-\beta_{2n}} \) for \( n \geq 1 \). For \( n \geq 1 \), \(-2 < \beta_{2n} = \frac{r_{2n+1}(1+r_{2n+2})}{(1+r_{2n+1})} \), from which \( 1/3 \leq 1+r_{2n+1} + r_{2n+1} r_{2n+2}/3 \). Consequently, \( 1/3 \leq 1 + T_{2n+1} \) for \( n \geq 1 \), and thus \( -2/3 \leq T_{2n+1} \leq \frac{\beta_{2n}}{1-\beta_{2n}} \) for \( n \geq 1 \). Since \( \beta_{2n}/(1-\beta_{2n}) \to -2/3 \), \( T_{2n-1} \to -2/3 \) and \( T_{2n} = r_{2n}(1+T_{2n+1}) \to -1/3 \). An example of such a series \( \sum_{n} \) is \( 1/3 - 1/5 + 1/7 = 1/9 + \cdots = 1 - \pi/4 \).

**Theorem 8.88.** Let \( \sum_{n} \) be a convergent series and \( n \) be any positive integer such that \( r_n < 0 \). Then we either have

(1) \( T_{n+1} < r_n/(1-r_n), \quad T_{n+1} < T_n, \quad \text{and} \quad r_n/(1-r_n) < T_n, \)

(2) \( T_{n+1} = r_n/(1-r_n), \quad T_{n+1} = T_n, \quad \text{and} \quad r_n/(1-r_n) = T_n, \)

or

(3) \( T_{n+1} > r_n/(1-r_n), \quad T_{n+1} > T_n, \quad \text{and} \quad r_n/(1-r_n) > T_n. \)

**Proof:** Since \( T_n = r_n(1+T_{n+1}) \) and \( T_{n+1} = T_n/r_n - 1 \), the following inequalities are equivalent:
\[ T_{n+1} < \frac{r_n}{1-r_n}, \quad T_{n+1} = T_n \frac{T_{n+1}}{T_n} < r_n, \quad T_{n+1} < r_n(1+T_{n+1}) , \]
\[ T_{n+1} < T_n, \quad T_n \frac{1}{r_{n-1}} < T_n, \quad T_n > r_n T_n, \quad T_n > T_{n-1} > T_n > r_n, \]
\[ T_n > r_n/(1-r_n), \quad r_n/(1-r_n) < T_n. \]
Consequently, the inequalities in (1) are equivalent. Similarly, the equalities in (2) are equivalent and the inequalities in (3) are equivalent. Q.E.D.

**Theorem 8.89.** Let \( \Sigma a_n \) be an N-alternating series.

Then the following three conditions are equivalent:

1. \( T_{n+1} \leq T_n, \quad n \geq N, \)

2. \( T_{n+1} \leq \frac{r_n}{1-r_n}, \quad n \geq N, \)

3. \( \frac{r_n}{1-r_n} \leq T_n, \quad n \geq N. \)

Moreover, if (1), (2), or (3) holds, then

4. \( r_{n+1} \leq r_n, \quad n \geq N, \)

and

5. \( T_n \leq \frac{r_n}{1-r_{n+1}}, \quad n \geq N. \)

**Proof:** According to Theorem 8.88, if equality holds in (1), (2), or (3), it also holds in the other two, and likewise for inequality. Thus, (1), (2), and (3) are equivalent.

Assume that (1), (2), or (3) holds, and let \( n \) be any integer \( \geq N. \) From (3) and (2), \( \frac{r_{n+1}}{1-r_{n+1}} \leq T_{n+1} \leq r_n/(1-r_n). \) Then \( r_{n+1}(1-r_n) \leq r_n (1-r_{n+1}) \) and
\[ r_{n+1} \leq r_n, \text{ i.e., (4) holds.} \]

Finally, since
\[ r_{n+1}/(1-r_{n+1}) \leq T_{n+1} = T_n/r_n - 1, \]
we have \( T_n/r_n \geq 1 \)
\[ + r_{n+1}/(1-r_{n+1}) = 1/(1-r_{n+1}) \]
and \( T_n \leq r_n/(1-r_{n+1}) \).

**Q.E.D.**

**Theorem 8.90.** Let \( \Sigma a_n \) be an \( N \)-alternating series. Then

a n.a.s.c. that \( T_{n+1} \leq T_n \) for \( n \geq N \) is that

(0) \( a_n \to 0 \),

and there exist a sequence \( \{ \beta_n \} \) such that

(1) \( a_n \beta_n \to 0 \),

(2) \( \beta_n > 1 + r_{n+1} + r_{n+1} r_{n+2} \beta_{n+2}, n \geq N \),

and

(3) \( r_{n+1} + r_{n+1} r_{n+2} \beta_{n+2} \leq r_n \beta_n, n \geq N \).

Moreover, if (0), (1), (2), and (3) hold, then for \( n \geq N \),

(4) \( T_{n+1} \leq r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1}) \)

and

(5) \( 1/(1-r_{n+1}) \leq \beta_n \leq r_{n+1}/r_n (1-r_{n+1}) \).

**Proof:** For the necessity, define \( \beta_n = 1 + T_{n+1}, n \geq N \).

Then \( a_n \beta_n = a_n + a_n T_{n+1} = a_n + (S-S_n) \to 0 \). Also, \( \beta_n = 1 + T_{n+1} = 1 + r_{n+1} + r_{n+1} r_{n+2} (1+T_{n+3}) = 1 + r_{n+1} + r_{n+1} r_{n+2} \beta_{n+2} \) for \( n \geq N \), so that (2) holds with equality. Moreover,

\[ r_{n+1} + r_{n+1} r_{n+2} \beta_{n+2} = T_{n+1} \leq T_n = r_n (1+T_{n+1}) = r_n \beta_n \]
n ≥ N, i.e., (3) holds.

For the sufficiency, according to (a) of Theorem 8.27 and (3) of the present theorem, we have

\[ T_{n+1} \leq r_{n+1} + r_{n+2} \beta_{n+2} \leq r_{n} \beta_{n} \leq T_{n} \]

for \( n ≥ N \), so that \( T_{n+1} ≤ T_{n} \)

for \( n ≥ N \). Theorem 8.89 implies (4) of the present theorem. We now have

\[ \frac{r_{n+1}}{1-r_{n+1}} \leq T_{n+1} \leq \frac{r_{n} \beta_{n}}{1-r_{n+1}} \leq \frac{r_{n}}{1-r_{n+1}} \]

for \( n ≥ N \), from which (5) of the present theorem is immediate. Q.E.D.

**Theorem 8.91.** Let \( \Sigma a_{n} \) be an \( N \)-alternating series. Then a n.a.s.c. that \( T_{n+1} ≤ T_{n} \) for \( n ≥ N \) is that

(0) \( a_{n} → 0 \),

and there exist a sequence \( \{\beta_{n}\} \) such that

(1) \( a_{n} \beta_{n} → 0 \),

(2) \( \beta_{n} ≥ 1+r_{n+1}+r_{n+1} r_{n+2} \beta_{n+2} \), \( n ≥ N \),

and

(3) \( \beta_{n} ≤ 1/(1-r_{n}) \), \( n ≥ N \).

Moreover, if (0), (1), (2), and (3) hold, then, for \( n ≥ N \),

(4) \( T_{n+1} ≤ \frac{r_{n}}{1-r_{n}} \leq \frac{r_{n} \beta_{n}}{1-r_{n+1}} ≤ T_{n} ≤ \frac{r_{n}+r_{n} r_{n+1} \beta_{n+1}}{1-r_{n+1}} \)

and

(5) \( 1/(1-r_{n+1}) ≤ \beta_{n} \).
Proof: Define $\beta_n = 1 + T_{n+1}$ for $n \geq N$. As in the proof of the necessity of Theorem 8.90, conditions (0), (1), and (2) hold. Using Theorem 8.89, $\beta_n = 1 + T_{n+1} \leq 1 + r_n/(1 - r_n) = 1/(1 - r_n)$, $n \geq N$, so that (3) holds.

For the sufficiency, assume that (0), (1), (2), and (3) hold. Using (3), we have for $n \geq N$, $(1 - r_n)\beta_n \leq 1$, $\beta_n - r_n \beta_n \leq 1$, and $\beta_n - 1 \leq r_n \beta_n$. Consequently, from (2), $r_{n+1} + r_{n+1}r_{n+2} \beta_{n+2} \leq \beta_n - 1 \leq r_n \beta_n$ for $n \geq N$. From Theorem 8.90, we obtain, for $n \geq N$, $T_{n+1} \leq T_n$, $T_{n+1} \leq r_n/(1 - r_n)$, and $1/(1 - r_{n+1}) \leq \beta_n$. From (3), for $n \geq N$, we have $r_n/(1 - r_n) \leq r_n \beta_n$ and $r_n + r_n r_{n+1} \beta_{n+1} \leq r_n + r_n r_{n+1}/(1 - r_{n+1}) = r_n/(1 - r_{n+1})$. Applying (a) of Theorem 8.27, $r_n \beta_n \leq T_n \leq r_n + r_n r_{n+1} \beta_{n+1}$ for $n \geq N$. Q.E.D.

Theorem 8.92. If $\Sigma a_n$ is an N-alternating series, then a n.a.s.c. that $T_{n+1} \leq T_n$ for $n \geq N$ is that

(0) $a_n \to 0$,

and there exist a sequence $\{p_n\}$ such that, for $n \geq N$,

(1) $1/(1 - p_n) \geq 1 + r_{n+1} + r_{n+1} r_{n+2}/(1 - p_{n+2})$

and

(2) $p_n \leq r_n$.

Moreover, if (0), (1), and (2) hold, then for $n \geq N$,
(3) \[ T_{n+1} \leq \frac{r_n}{1-r_n} \leq \frac{r_n}{1-p_n} \leq T_n \leq r_n \]
\[ + r_n \frac{r_{n+1}}{1-p_{n+1}} \leq r_n / (1-r_{n+1}) \]
and
(4) \[ r_{n+1} \leq p_n. \]

**Proof:** For the necessity, there is a sequence \( \{\beta_n\} \) satisfying (1), (2), (3), and (5) of Theorem 8.91. Defining \( p_n = 1-1/\beta_n \) for \( n \geq N \), we easily verify that \( p_n \leq r_n \) for \( n \geq N \). Also, for \( n \geq N \), \( \beta_n = 1/(1-p_n) \), so that (2) of Theorem 8.91 reduces to (1) above.

For the sufficiency, define \( \beta_n = 1/(1-p_n) \) for \( n \geq N \). Condition (1) above thus yields (2) of Theorem 8.91. From (2) and \( r_n < 0 \) for \( n \geq N \), we have
\[ 0 < \frac{1}{1-p_n} = \beta_n \leq \frac{1}{1-r_n} < 1 \text{ for } n \geq N, \]
and thus
\[ a_n \beta_n \to 0, \text{ i.e., (1) and (3) of Theorem 8.91 hold.} \]

Finally, (3) and (4) above follow respectively from (4) and (5) of Theorem 8.91. Q.E.D.

**Theorem 8.93.** Let \( \Sigma a_n \) be an N-alternating series. Then a n.a.s.c. that \( T_{n+1} \leq T_n \) for \( n \geq N \) is that
\[ (0) \quad a_n \to 0, \]
and there exist a sequence \( \{\alpha_n\} \) such that
(1) \( a_n \alpha_n \to 0 \),

(2) \( a_n \leq 1 + r_{n+1} + r_{n+1} r_{n+2} \alpha_{n+2} \), \( n \geq N+1 \),

and

(3) \( r_n / r_{n+1} (1-r_n) \leq \alpha_{n+1} \), \( n \geq N \).

Moreover, if (0), (1), (2), and (3) hold, then for \( n \geq N \),

(4) \( T_{n+1} \leq r_{n+1} \alpha_{n+1} \leq r_n / (1-r_n) \leq r_n + r_n r_{n+1} \alpha_{n+1} \leq T_n \)

\( \leq r_n / (1-r_{n+1}) \)

and

(5) \( \alpha_{n+1} \leq 1 / (1-r_{n+1}) \).

**Proof:** For the necessity, define \( \alpha_n = 1 + T_{n+1} \), \( n \geq N \).

Then \( a_n \alpha_n = a_n + a_n T_{n+1} = a_n + (S-S_n) \to 0 \). Also, \( \alpha_n = 1 + T_{n+1} = 1 + r_{n+1} + r_{n+1} r_{n+2} (1+T_{n+2}) = 1 + r_{n+1} + r_{n+1} r_{n+2} \alpha_{n+2} \)

for \( n \geq N \) so that (2) holds with equality. Moreover,

\( r_{n+1} \alpha_{n+1} = r_{n+1} (1+T_{n+2}) = T_{n+1} \leq T_n = r_n + r_n r_{n+1} \alpha_{n+1} \), for \( n \geq N \), from which (3) is immediate.

For the sufficiency, define \( \alpha_N = 1 + r_{N+1} \),

\( + r_{N+1} r_{N+2} \alpha_{N+2} \). From (3), \( r_{n+1} \alpha_{n+1} \leq r_n / (1-r_n) \leq r_n \)

\( + r_n r_{n+1} \alpha_{n+1} \) for \( n \geq N \). From (a) of Theorem 8.3,

\( T_{n+1} \leq r_{n+1} \alpha_{n+1} \leq r_n + r_n r_{n+1} \alpha_{n+1} \leq T_n \) for \( n \geq N \). From (5) of Theorem 8.89, \( T_n \leq r_n / (1-r_{n+1}) \) for \( n \geq N \).
Consequently, (4) holds. (5) is a consequence of (4).
Q.E.D.

Lemma 8.94. If $r_n$, $r_{n+1}$, $r_{n+2}$ are any real numbers such that $(1-r_n)(1-r_{n+2}) \neq 0$, then

$$1 + r_{n+1} + r_{n+1} r_{n+2} / (1-r_{n+2}) - 1/(1-r_n) = r_{n+1} / (1-r_{n+2}) - r_n / (1-r_n)$$

$$= (\Delta r_n + r_n \Delta r_{n+1}) / [(1-r_n)(1-r_{n+2})].$$

Proof: We have

$$1 + r_{n+1} + r_{n+1} r_{n+2} / (1-r_{n+2}) - 1/(1-r_n)$$

$$= [1 - 1/(1-r_n)] + r_{n+1} [1 + r_{n+2} / (1-r_{n+2})] = -r_n / (1-r_n)$$

$$+ r_{n+1} / (1-r_{n+2}) = [r_{n+1}(1-r_n) - r_n(1-r_{n+2})] / (1-r_n)(1-r_{n+2})$$

$$= [r_{n+1} - r_n + r_n(r_{n+2} - r_{n+1})] / (1-r_n)(1-r_{n+2})$$

$$= (\Delta r_n + r_n \Delta r_{n+1}) / [(1-r_n)(1-r_{n+2})].$$

Q.E.D.

Lemma 8.95. If $r_n$, $r_{n+1}$, $r_{n+2}$ are any real numbers, then the following inequalities are equivalent:

(1) $1/(1-r_n) \geq 1 + r_{n+1} + r_{n+1} r_{n+2} / (1-r_{n+2})$

(2) $r_n / (1-r_n) \geq r_{n+1} / (1-r_{n+2})$

(3) $0 \geq [\Delta r_n + r_n \Delta r_{n+1}] / [(1-r_n)(1-r_{n+2})].$

Proof: The equivalence follows immediately from Lemma 8.94. Q.E.D.

Theorem 8.96. If $\Sigma a_n$ is an N-alternating series, $a_n \to 0$,
and \( r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n) \) for \( n \geq N \), then, for \( n \geq N \), (1) \( \Delta r_n \leq 0 \) and (2) \( T_{n+1} \leq r_{n+1}/(1-r_{n+2}) \) 
\leq r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1}). \)

1st Proof: Defining \( \beta_n = 1/(1-r_n) \) for \( n \geq N \), we see that \( 0 < \beta_n < 1 \) for \( n \geq N \) and thus \( a_n \beta_n \to 0 \). From (1) and (2) of Lemma 8.95, \( \beta_n \geq 1+r_{n+1}^+r_{n+2}/(1-r_{n+1}^2) \) for \( n \geq N \). From (4) of Theorem 8.91, (2) of the present theorem holds. (1) follows from (2). We could also obtain (1) from (4) of Theorem 8.89. Q.E.D.

2nd Proof: Define \( p_n = r_n \) for \( n \geq N \). From (1) and (2) of Lemma 8.95, \( 1/(1-p_n) \geq 1+r_{n+1}^+r_{n+2}/(1-p_{n+1}^2) \) for \( n \geq N \). Now apply Theorem 8.92 and Theorem 8.89. Q.E.D.

Theorem 8.97. If \( \Sigma a_n \) is an \( N \)-alternating series, \( a_n \to 0 \), and \( \Delta r_n + r_n \Delta r_{n+1} \leq 0 \) for \( n \geq N \), then, for \( n \geq N \), \( \Delta r_n \leq 0 \) and \( T_{n+1} \leq r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n) \) 
\leq T_n \leq r_n/(1-r_{n+1}). \)

Proof: If \( n \geq N \), then \( \Delta r_n + r_n \Delta r_{n+1} \leq 0 \), \( (1-r_n)(1-r_{n+2}) > 0 \), and \( (\Delta r_n + r_n \Delta r_{n+1})/(1-r_n)(1-r_{n+2}) \leq 0 \). Thus from Lemma 8.95, \( r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n) \). We now apply Theorem 8.96. Q.E.D.
Theorem 8.98. If $\sum a_n$ is an N-alternating series and
\[
\frac{r_n}{1-r_n} \leq T_n \leq \frac{r_n}{1-r_{n+1}} \quad \text{for } n \geq N,
\]
then
\[
(1) \quad 0 < (-1)^n a_n/(1-r_{n+1}) \leq (-1)^n(S-S_{n-1}) < (-1)^n a_n/(1-r_n), \quad n \geq N,
\]
or
\[
(2) \quad (-1)^n a_n/(1-r_n) \leq (-1)^n(S-S_{n-1}) \leq (-1)^n a_n/(1-r_{n+1}) < 0, \quad n \geq N,
\]
according as $a_{2n} > 0$ or $a_{2n} < 0$, respectively.

Proof: Multiplying the inequality $r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1})$ throughout by $|a_n|$, we have
\[
\frac{|a_{n-1}|}{a_{n-1}} \frac{a_n}{1-r_n} \leq \frac{|a_{n-1}|}{a_{n-1}} (S-S_{n-1}) \leq \frac{|a_{n-1}|}{a_{n-1}} \frac{a_n}{1-r_{n+1}} < 0,
\]
and this reduces to (1) if $a_{2n} > 0$, or (2) if $a_{2n} < 0$. Q.E.D.

Theorem 8.99. If $\sum a_n$ is an N-alternating series such that $a_n \to 0$ and $\Delta r_n \leq \Delta r_{n+1}$ for $n \geq N$, then, for $n \geq N$, $\Delta r_n \leq 0$, $\Delta r_n r_{n+1} \leq 0$, and $T_{n+1}$
\[
\leq \frac{r_{n+1}}{1-r_{n+2}} \leq \frac{r_n}{1-r_n} \leq T_n \leq \frac{r_n}{1-r_{n+1}}.
\]

Proof: We first show that $\Delta r_n \leq 0$ for $n \geq N$. In particular, assume that $0 < \Delta r_m$ for some $m \geq N$. Then
\[ \Delta r_m \leq \Delta r_n \text{ for } n \geq m, \text{ and thus } x_{m+k} = x_m + \Delta r_m + \Delta r_{m+1} + \cdots \]
\[ + \Delta r_{m+k-1} \geq x_m + k\Delta r_m \to \infty \text{ as } k \to \infty; \text{ hence } x_n \to \infty. \] This contradicts \( a_n \to 0 \), so that \( \Delta r_n \leq 0 \), i.e., \( a_{n+1} \leq a_n < 0 \) for \( n \geq N \).
Consequently, \( -1 < r_n \) for \( n \geq N \), since \( a_n \to 0 \). Therefore, \( \Delta r_n + r_n \Delta r_{n+1} \leq \Delta r_{n+1} + r_n \Delta r_{n+1} = (1+r_n)\Delta r_{n+1} \leq 0 \) for \( n \geq N \). We may now apply Theorem 8.97. Q.E.D.

**Theorem 8.100.** Suppose that \( \Sigma a_n \) is a series such that \( a_n \to 0 \), and that \( f \) is a function and \( N \) is a positive integer such that:

1. \( f(x) < 0 \) for \( N \leq x \),
2. \( f' \) is increasing on \([N, \infty)\), or \( f''(x) \geq 0 \) for \( N \leq x \),
3. \( r_n = f(n) \) for \( n \geq N \).

Then, for \( n \geq N \), \( \Delta r_n \leq \Delta r_{n+1} \) and
\[ T_{n+1} \leq r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n) \leq r_n/(1-r_{n+1}). \]

**Proof:** Let \( n \) be any integer \( \geq N \). By the Mean Value Theorem for derivatives there exist \( u,v \) such that
\( n < u < n+1 < v < n+2 \) and \( \Delta r_n = f(n+1)-f(n) = f'(u)[(n+1)-n] = f'(u) \leq f'(v) = f'(v)[(n+2)-(n+1)] = f(n+2)-f(n+1) = \Delta r_{n+1} \). We now apply Theorem 8.99 to complete the proof. Q.E.D.
We now illustrate Theorem 8.100 with some examples.

**Example 8.101.** \( \ln 2 = 1 - 1/2 + 1/3 - 1/4 + \cdots \). Here \( a_n = (-1)^n/(n+1) \) for \( n \geq 0 \), \( r_n = a_n/a_{n-1} = -n/(n+1) \) for \( n \geq 1 \), and we set \( f(x) = -x/(x+1) \) for \( x \geq N = 1 \).

Accordingly, for \( 1 \leq x \), we have \( f(x) < 0 \), \( f'(x) = -1/(x+1)^2 \), and \( f''(x) = 2/(x+1)^3 > 0 \). Thus \( \Delta r_n \leq \Delta r_{n+1} \) for \( n \geq 1 \), and Theorem 8.100 is applicable with \( N = 1 \). (1) of Theorem 8.98 reduces to

\[
\frac{(n+2)}{(n+1)(2n+3)} \leq \frac{1}{(n+1)} - \frac{1}{(n+2)} + 1/(n+3) - \frac{1}{(n+4)} + \cdots
\]

\( = (-1)^n(S - S_{n-1}) \leq 1/(2n+1) \) for \( n \geq 1 \).

**Example 8.102.** \( \pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots \). Here \( a_n = (-1)^n/(2n+1) \) for \( n \geq 0 \), \( r_n = a_n/a_{n-1} = -(2n-1)(2n+1) \) for \( n \geq 1 \), and we set \( f(x) = -(2x-1)(2x+1) \) for \( x \geq N = 1 \).

For \( 1 \leq x \), \( f(x) < 0 \), \( f'(x) = -4/(2x+1)^2 \), and \( f''(x) = 16/(2x+1)^3 > 0 \). From Theorem 8.100 and (1) of Theorem 8.98 we obtain, with \( N = 1 \), \( (2n+3)/(2n+1)(4n+4) \leq (-1)^n(S - S_{n-1}) \leq 1/(2n+1) - 1/(2n+3) + 1/(2n+5) - 1/(2n+7) + \cdots \leq 1/4n \) for \( n \geq 1 \).

**Example 8.103.** \( \ln 3/2 = 1/2 - 1/(2 \cdot 2^2) + 1/(3 \cdot 2^3) - 1/(4 \cdot 2^4) + \cdots \). Here \( a_n = (-1)^n/(n+1)2^{n+1} \) for \( n \geq 0 \), \( r_n = a_n/a_{n-1} = -n/2(n+1) \) for \( n \geq 1 \), and we set \( f(x) = -x/2(x+1) \) for \( x \geq N = 1 \). For \( 1 \leq x \), \( f(x) < 0 \), \( f'(x) = -1/2(x+1)^2 \), and \( f''(x) = 1/(x+1)^3 > 0 \).
From Theorem 8.100 and (1) of Theorem 8.98, we have, with
\( N = 1 \),
\[
\frac{(n+2)}{2^n(n+1)(3n+5)} \leq (-1)^n(S - S_{n-1})
\]
\[
= \frac{1}{2^{n+1}} \left[ \frac{1}{(n+1)} - 1/2(n+2) + 1/2^2(n+3) - 1/2^3(n+4) + \cdots \right]
\]
\[
\leq \frac{1}{2^n(3n+2)} \quad \text{for} \ n \geq 1.
\]

**Example 8.104.** \( (1 - \sqrt{2})z(1/2) = 1 - 1/\sqrt{2} + 1/\sqrt{3} - 1/\sqrt{4} + \cdots \).

Here \( z \) is the Riemann zeta function, \( a_n = (-1)^n/\sqrt{n+1} \) for \( n \geq 0 \), \( r_n = a_n/a_{n-1} = -\sqrt{n/(n+1)} \) for \( n \geq 1 \), and we set \( f(x) = -\sqrt{x/(x+1)} \) for \( x \geq N = 1 \). For \( 1 \leq x \), we have \( f(x) < 0 \), \( f'(x) = -1/[2x^{1/2}(x+1)^{3/2}] \), and \( f''(x) = (4x+1)/[4x^{3/2}(x+1)^{5/2}] > 0 \). We may now use Theorem 8.100 and (1) of Theorem 8.98, obtaining, with \( N = 1 \),
\[
\left[ \frac{(n+2)}{(n+1)^2} \right]^{1/2} \left( \sqrt{n+2} - \sqrt{n+1} \right) \leq (-1)^n(S - S_{n-1})
\]
\[
= 1/\sqrt{n+1} - 1/\sqrt{n+2} + 1/\sqrt{n+3} - 1/\sqrt{n+4} + \cdots \leq \sqrt{n+1} - \sqrt{n}
\]
for \( n \geq 1 \).

**Example 8.105.** \( \pi^2/12 = 1 - 1/2^2 + 1/3^2 - 1/4^2 + \cdots \).

Here \( a_n = (-1)^n/(n+1)^2 \) for \( n \geq 0 \), \( r_n = -n^2/(n+1)^2 \) for \( n \geq 1 \), and we set \( f(x) = -x^2/(x+1)^2 \) for \( x \geq N = 1 \). For \( x \geq 1 \), \( f(x) < 0 \) and \( f''(x) = 2(2x-1)/(x+1)^4 > 0 \).

Applying Theorem 8.100 and (1) of Theorem 8.98, with \( N = 1 \), we have
\[
\frac{(n+2)^2}{(n+1)^2 + (n+2)^2} \leq \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2}
\]
\[
+ \frac{1}{(n+3)^2} - \frac{1}{(n+4)^2} + \cdots \leq \frac{1}{n^2 + (n+1)^2}.
\]
for \( n \geq 1 \). We note that \( f(x) = -1 + 2/(x+1) - 1/(x+1)^2 \), suggesting Theorem 8.107 which follows shortly.

**Example 8.106.** \( 1/\sqrt{2} = 1 - 1/2 + (1 \cdot 3)/(2 \cdot 4) - (1 \cdot 3 \cdot 5)/(2 \cdot 4 \cdot 6) + (1 \cdot 3 \cdot 5 \cdot 7)/(2 \cdot 4 \cdot 6 \cdot 8) - \cdots \). Here

\[
 a_n = (-1)^n [1 \cdot 3 \cdots (2n-1)]/[2 \cdot 4 \cdots (2n)] \text{ for } n \geq 1, \\
 a_0 = 1, r_n = -(2n-1)/(2n) \text{ for } n \geq 1, \text{ and we set } f(x) = -(2x-1)/(2x) \text{ for } x \geq N = 1. \text{ For } x \geq 1, f(x) < 0 \text{ and } f''(x) = 1/x^3 > 0. \text{ From Theorem 8.100 and (1) of Theorem 8.98 with } N = 1,
\]

\[
 \frac{2n+2}{4n+3} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \leq (-1)^n (S - S_{n-1}) \\
 = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} - \frac{1 \cdot 3 \cdots (2n+1)}{2 \cdot 4 \cdots (2n+2)} \\
 + \cdots \leq \frac{2n}{4n-1} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}
\]

for \( n \geq 1 \).

**Theorem 8.107.** Suppose that \( \Sigma a_n \) is a series such that

\( a_n \to 0, r_n = b + b_1/n + b_2/n^2 + \cdots \), where \( b < 0 \), and the first non-zero \( b_k \), if such exists, is positive. Then

\( \Delta r_n \leq \Delta r_{n+1} \text{ and } T_{n+1} \leq r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1}) \).

**Proof:** If \( b_k = 0 \) for all \( k > 0 \), then \( r_n = b \), \(-1 < b < 0 \) since \( a_n \to 0 \), and each inequality in the conclusion of our theorem holds with equality.
Suppose on the other hand that $b_p$ is the first non-zero $b_k$, so that $b_p > 0$ and $r_n = \frac{b + b_p}{n^p}$

$$+\frac{b_{p+1}}{n^{p+1}} + \frac{b_{p+2}}{n^{p+2}} + \cdots.$$ Setting $f(x) = \frac{b + b_p}{x^p}$

$$+\frac{b_{p+1}}{x^{p+1}} + \frac{b_{p+2}}{x^{p+2}} + \cdots,$$ we see that $f$ is an analytic function of $\frac{1}{x}$ for large $x$, $f(x) < 0$, and $f(n) = r_n$. Differentiating twice, we have $f''(x) = \frac{p(p+1)b + (p+1)(p+2)b_p}{x^{p+2}} > 0$, since $b_p > 0$. We may now apply Theorem 8.100. Q.E.D.

**Theorem 8.108.** Suppose that (1) $\sum a_n$ is an $N$-alternating series such that $a_n \to 0$ and $\Delta r_n \leq \Delta r_{n+1}$ for $n \geq N$,

(2) $\sum a'_n$ is a series such that $a'_n \to 0$, and (3) $f$ is a function such that $r'_n = -f(|r_n|)$, for $n \geq N$, and $f'(x) \geq 0$ and $f''(x) \leq 0$, for $|r_n| \leq x$. Then, for $n \geq N$, $\Delta r'_n \leq \Delta r'_{n+1}$ and $T'_{n+1} \leq r'_{n+1}/(1-r'_{n+2})$

$$\leq r'_n/(1-r'_n) \leq T'_n \leq r'_n/(1-r'_{n+1}).$$

**Proof:** Let $n$ be any integer $\geq N$. As shown in the proof of Theorem 8.99, $r_{n+2} \leq r_{n+1} \leq r_n < 0$, i.e.,

$$0 < |r_n| \leq |r_{n+1}| \leq |r_{n+2}|.$$ By the Mean Value Theorem for derivatives there is a $u$ such that
\[ |r_n| \leq u \leq |r_{n+1}| \quad \text{and} \quad \Delta r_n' = r_{n+1}' - r_n' = f(|r_n|) - f(|r_{n+1}|) = f'(u)(|r_n| - |r_{n+1}|) = f'(u)(r_{n+1} - r_n) = f'(u)\Delta r_n. \]

Similarly, there is a \( v \) such that \( |r_{n+1}| \leq v \leq |r_{n+2}| \) and \( \Delta r_{n+1}' = f'(v)\Delta r_{n+1}. \) Thus from \( f'(u) \geq f'(v) \geq 0 \) and \( \Delta r_n \leq 0, \Delta r_n' = f'(u)\Delta r_n \leq f'(v)\Delta r_n \leq f'(v)\Delta r_{n+1}, \)

\[ = \Delta r_{n+1}' \quad \text{and} \quad \Delta r_n \leq \Delta r_{n+1}'. \] We may now apply Theorem 8.99 to complete the proof. Q.E.D.

**Corollary 8.109.** If \( \Sigma a_n \) is an \( N \)-alternating series such that \( a_n \to 0, \Delta r_n \leq \Delta r_{n+1} \) for \( n \geq N \), and \( \Sigma a_n' \) is an \( N \)-alternating series such that \( |a_n'| = |a_n|^p \) for \( n \geq N-1 \), where \( 0 < p < 1 \); then, for \( n \geq N, \) \( \Delta r_n' \leq \Delta r_{n+1}' \) and

\[ T_{n+1}' \leq r_{n+1}'/(1-r_{n+2}') \leq r_n'(1-r_n) \leq T_n \leq r_n'(1-r_{n+1}). \]

**Proof:** It is obvious that \( a_n' \to 0 \). Set \( f(x) = x^p \) for \( |r_n| \leq x \). Then for \( n \geq N, r_n' = -|a_n'|/|a_{n-1}'| = -|a_n|^p/|a_{n-1}|^p = -|a_n/a_{n-1}|^p = -|r_n|^p = -f(|r_n|). \) Also for \( |r_n| \leq x, f'(x) = px^{p-1} > 0 \) and \( f''(x) = p(p-1)x^{p-2} < 0. \) We now apply Theorem 8.108. Q.E.D.

**Example 8.110.** \((1-2^{1-P})z(p) = 1-1/2^p+1/3^p-1/4^p+\cdots,\)

\( 0 < p < 1. \) Here \( z \) is the Riemann zeta function and
\[ a_n^i = (-1)^n/(n+1)^P \] for \( n \geq 0 \). With \( a_n = (-1)^n/(n+1) \) for \( n \geq 0 \), Example 8.101 and Theorem 8.100 show that \( \Delta r_n \leq \Delta r_{n+1} \) for \( n \geq 1 \). Noting that \( |a_n^i| = |a_n|^P \) for \( n \geq 0 \), we may apply Corollary 8.109 to obtain
\[ T_{n+1}^i \leq r_n^i/(1-r_n^{i+2}) \leq r_n^i/(1-r_n^i) \leq T_n^i \leq r_n^i/(1-r_n^{i+1}) \] for \( n \geq 1 \). The case \( p = 1/2 \) was previously considered in Example 8.104, but the above procedure, requiring the second derivative of \(-x/(x+1)\), is preferable to differentiating \(-x^P/(1+x)^P\) twice, as was done in Example 8.104.

**Lemma 8.111.** Suppose that \( f \) is a function and \( N \) is a positive integer such that (1) \( f(x) > 0 \), (2) \( f'(x) \geq 0 \), (3) \( f''(x) \leq 0 \), and (4) \( f'''(x) \geq 0 \), for \( N-1 < x \). Then the function \( g(x) = -f(x-1)/f(x) \) satisfies the conditions \( g(x) < 0 \) and \( g''(x) > 0 \), for \( N \leq x \).

**Proof:** Let \( N \leq x \). Clearly \( g(x) < 0 \) and, differentiating twice, \( g''(x) = \frac{[f(x)[f(x-1)f''(x)-f(x)f''(x-1)] + 2f'(x)[f(x)f'(x-1)-f(x-1)f'(x)]]}{f^3(x)} \).

From (2), \( f(x-1) \leq f(x) \) and thus \( f(x-1)f''(x) \leq f(x)f''(x) \) according to (3). From (4), \( f''(x)-f''(x-1) \geq 0 \), so that
\[ f(x-1)f''(x)-f(x)f''(x-1) \geq f(x)f''(x)-f(x)f''(x-1) = f(x)[f''(x)-f''(x-1)] \geq 0, \] since \( f(x) > 0 \). From (2), \( f(x)f'(x-1) \geq f(x-1)f'(x-1) \). From (3), \( f'(x-1)-f'(x) \geq 0, \)
and thus \( f(x)f'(x-1)f'(x) \geq f(x-1)f'(x-1) \) 
\(-f(x-1)f'(x) = f(x-1)[f'(x-1)-f'(x)] \geq 0. \) The inequality \( g''(x) \geq 0 \) is now evident. Q.E.D.

**Theorem 8.112.** Suppose that \( \Sigma a_n \) is a series such that \( a_n \to 0. \) Suppose that \( f \) is a function and \( N \) is a positive integer such that: \( f(x) > 0, f'(x) \geq 0, f''(x) \leq 0, \) and \( f'''(x) \geq 0, \) for \( N-1 \leq x; \) and \( r_n = -f(n-1)/f(n) \) for \( N \leq n. \) Then, for \( n \geq N, \Delta r_n \leq \Delta r_{n+1} \) and

\[
T_{n+1} \leq r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n) \leq r_n/(1-r_{n+1}).
\]

**Proof:** Define \( g(x) = -f(x-1)/f(x) \) for \( N \leq x. \) Then \( r_n = g(n) \) for \( n \geq N. \) Also \( g(x) < 0 \) and \( g''(x) > 0 \) for \( N \leq x \) according to Lemma 8.111. We may now use Theorem 8.100 to complete the proof. Q.E.D.

**Theorem 8.113.** Suppose that \( \Sigma a_n \) is an \( N \)-alternating series such that \( a_n \to 0. \) Suppose that \( f \) is a function and \( N \) is a positive integer such that: \( f(x) > 0, f'(x) \geq 0, f''(x) \leq 0, \) and \( f'''(x) \geq 0, \) for \( N-1 \leq x; \) and \( |a_n| = 1/f(n) \) for \( N-1 \leq n. \) Then, for \( N \leq n, \)

\[
\Delta r_n \leq \Delta r_{n+1} \text{ and } T_{n+1} \leq r_{n+1}/(1-r_{n+1}) \leq r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1}).
\]
Proof: For $N \leq n$, $r_n = a_n/a_{n-1} = -|a_n|/|a_{n-1}|$

$= -f(n-1)/f(n)$. Now apply Theorem 8.112. Q.E.D.

We now apply Theorem 8.113 to some of the series considered previously.

Example 8.114. \( \ln 2 = 1 - 1/2 + 1/3 - 1/4 + \cdots \). We have

\[ a_n = (-1)^n(n+1), \text{ for } n \geq 0, \text{ and we set } f(x) = x+1, \]

for $x \geq 0$. Clearly, $|a_n| = 1/f(n)$ for $0 \leq n$. For $0 \leq x, f(x) > 0, f'(x) = 1 \geq 0, f''(x) = 0 \leq 0, \text{ and } f'''(x) = 0 \geq 0$. Theorem 8.113 is now applicable with $N = 1$. This series was previously treated in Example 8.101.

Example 8.115. \( \pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots \) (see Example 8.102). We have $a_n = (-1)^n/(2n+1)$, for $n \geq 0$, and we set $f(x) = 2x+1$, for $x \geq 0$, so that $|a_n| = 1/f(n)$ for $n \geq 0$. If $x \geq 0$, then $f(x) > 0, f'(x) = 2 \geq 0, f''(x) = 0 \leq 0, \text{ and } f'''(x) = 0$. We may now apply Theorem 8.113 with $N = 1$.

Example 8.116. \( \ln 3/2 = \sum a_n; a_n = (-1)^n/(n+1)2^{n+1} \) for $n \geq 0$. Setting $f(x) = (x+1)2^{x+1}$, for $x \geq 0$, we find $f''(x) = 2^{x+1}[2+(x+1)\ln 2] \ln 2 > 0$, for $x \geq 0$, so that Theorem 8.113 is not applicable. In Example 8.103, Theorem 8.100 was shown to be applicable.
Example 8.117. \((1-2^{-p}) z(p) = \sum a_n; \quad a_n = (-1)^n/(n+1)^p\),

for \(n \geq 0\), where \(0 < p < 1\). Setting \(f(x) = (x+1)^p\), for \(x \geq 0\), \(|a_n| = 1/f(n)\) for \(n \geq 0\). For \(x \geq 0\),

\(f(x) > 0, \quad f'(x) = p(x+1)^{p-1} > 0, \quad f''(x) = p(p-1)(x+1)^{p-2} < 0, \quad \text{and} \quad f'''(x) = p(p-1)(p-2)(x+1)^{p-3} > 0\). Theorem 8.113 is thus applicable with \(N = 1\). This series was also considered in Example 8.110.

The function \(f\) in Theorem 8.113 satisfies the condition

\((A) \quad f(x) \to \infty \text{ as } x \to \infty, \quad f'(x) \geq 0, \quad f''(x) \leq 0, \quad f'''(x) \geq 0.\)

We now prove that if \(f\) and \(g\) are functions satisfying condition \((A)\), then so does the composite function \(h\) where \(h(x) = f(g(x))\). This will allow us to build up, or easily recognize, a wide variety of series \(\sum a_n\) for which Theorem 8.113 is applicable.

**Theorem 8.118.** If \(f\) and \(g\) are functions which satisfy condition \((A)\), then the composite function \(h = f \circ g\) also satisfies condition \((A)\).

**Proof:** Clearly \(h(x) = f(g(x)) \to \infty \text{ as } x \to \infty\). Also \(h'(x) = f'(g(x)) \cdot g'(x) \geq 0\) since \(g(x) \to \infty \text{ as } x \to \infty\), \(f'(x) \geq 0, \text{ and } g'(x) \geq 0.\) Moreover, \(h''(x) = f''(g(x))[g'(x)]^2 + f'(g(x)) \cdot g''(x) \leq 0\) is quite evident.
Finally, \( h''(x) = f''(g(x)) [g'(x)]^3 + f''(g(x)) \cdot 2g'(x)g''(x) + f''(g(x))g'(x)g''(x) + f'(g(x)) \cdot g''(x) \geq 0 \). Q.E.D.

**Corollary 8.119.** Suppose that \( f \) and \( g \) are functions satisfying condition (A), and that \( \Sigma a_n \) is a series for which \( a_n = (-1)^n / f(g(n)) \). Then \( \Delta r_n \leq \Delta r_{n+1} \) and \( r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1}) \).

**Proof:** Defining \( h(x) = f(g(x)) \), \( h \) satisfies condition (A), according to Theorem 8.118. Thus \( f(x) > 0 \) and \( |a_n| = 1/h(n) \to 0 \). We may now apply Theorem 8.113. Q.E.D.

**Theorem 8.120.** Suppose that \( \Sigma a_n \) is an \( N \)-alternating series, \( a_n \to 0 \), and \( \Delta r_n + r_n \Delta r_{n+1} \leq 0 \) for \( n \geq N \). Let \( \Sigma a'_n \) be the power series defined by \( a'_n = a_n x^{n+p} \), where \( p \) is some fixed real number. Then, for \( 0 < x < 1 \) and \( n \geq N \), \( \Delta r'_n + r'_n \Delta r'_{n+1} \leq 0 \) and \( T_{n+1}' \leq r'_{n+1}/(1-r'_{n+2}) \leq r'_{n+1}/(1-r'_{n+1}) \).

**Proof:** Let \( x \) be any number satisfying \( 0 < x < 1 \) and \( n \) be any integer \( \geq N \). Clearly, \( a'_n = a_k x^{k+p} \to 0 \) as \( k \to \infty \). From Theorem 8.97, \( \Delta r_{n+1} \leq 0 \) so that \( x^2 r_n \Delta r_{n+1} \leq x r_n \Delta r_{n+1} \). Thus \( r'_n = a_n x^{n+p} / a_{n-1} x^{n-1+p} = x r_n \).
\( \Delta r'_n = r'_{n+1} - r'_n = x_{n+1} - x_n = x \Delta r_n' \) and \( \Delta r'_n + r'_n \Delta r'_{n+1} \)

\[
x \Delta r_n + x^2 r_n \Delta r'_{n+1} \leq x \Delta r_n + x r_n \Delta r'_{n+1} = x(\Delta r_n + r_n \Delta r'_{n+1}) \leq 0.
\]

Now apply Theorem 8.97 to \( \Sigma a'_n \). Q.E.D.

Theorem 8.121. Suppose that \( \Sigma a_n \) is an \( N \)-alternating series, \( a_n \to 0 \), and \( \Delta r_n \leq \Delta r_{n+1} \) for \( n \geq N \). Let \( \Sigma a'_n \) be the series defined by \( a'_n = a_n x^{n+p} \), where \( p \) is some fixed real number. Then, for \( 0 < x < 1 \) and \( n \geq N \),

\[
\Delta r'_n \leq \Delta r'_{n+1} \quad \text{and} \quad T'_{n+1} \leq r'_{n+1}/(1-r'_{n+1}) \leq r'/(1-r') \leq T'_n \leq r'/(1-r'_{n+1}).
\]

Proof: Let \( x \) be any number satisfying \( 0 < x < 1 \) and \( n \) be any integer \( \geq N \). Clearly, \( a'_k \to 0 \) as \( k \to \infty \).

Also, \( \Delta r'_n = x \Delta r_n \leq x \Delta r'_{n+1} = \Delta r'_{n+1} \). We now apply Theorem 8.99 to \( \Sigma a'_n \). Q.E.D.

Example 8.122. \( \ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots \), \( 0 < x \leq 1 \). We have \( a_n = (-1)^n/(n+1) \) and \( a'_n = a_n x^{n+1} \)

for \( n \geq 0 \). As shown in Example 8.101 or 8.114,

\( \Delta r_n \leq \Delta r_{n+1} \) for \( n \geq 1 \), so that Theorem 8.121 is applicable to \( \Sigma a'_n \), where \( N = p = 1 \).
CHAPTER IX

SUMMARY

In Chapter I, definitions and notations are introduced. In particular, the quantities $T_n$ are defined by the equation $T_n = (S - S_{n-1})/a_{n-1}$, if $\Sigma a_n$ converges to $S$ and $n$ is any integer such that $a_{n-1} \neq 0$. Various algebraic properties of $T_n$ are proven. A geometrical interpretation of Aitken's $\delta^2$-process is given, and several formulas are set forth, each of which yields this method of acceleration. Also, the notion of "transform sequence" is introduced to set up a unifying framework for investigating various methods of acceleration.

In Chapter II, the convergence of $\{T_n\}$ is treated and corresponding n.a.s.c. for $\Sigma a_n \in MR(\Sigma a_n)$ are proven. Divergence theorems are proven, which are used to prove that if $\Sigma a_n$ and $\Sigma a_\delta n$ are convergent complex series, then $S = S_\delta$. This fact was first published by Lubkin (17, p. 230) for real series. We are then led in a natural manner to some theorems on rapidity of convergence.

In Chapter III, n.a.s.c for $\Sigma a_n \in MR(\Sigma a_n)$ are
established. It is shown that any sequence \( \{a_n\} \) such that \( \sum a_n \in MR(\sum a_n) \) determines all such sequences \( \{\beta_n\} \) by the simple condition \( \beta_n \sim a_n \). This is then used, along with algebraic properties of \( T_n \), to prove that 

\[ \sum a_{\delta n} \in MR(\sum a_n) \] if and only if \( T_{n+1} - T_n \to 0 \). With the added condition \( |r_n| \leq \rho < 1 \), it is proven that 

\[ \sum a_{\delta n} \in MR(\sum a_n) \] if and only if \( r_{n+1} - r_n \to 0 \). It is also proven that if \( |r_n| \leq \rho < 1 \) and \( r_{n+1} - r_n \to 0 \), then Lubkin's \( W \) transformation and a slight variant of the \( W \) transformation may be used for accelerating the convergence of \( \sum a_n \). The relationship between the \( \delta^2 \)-process and the \( W \) transformation, as concerns acceleration, is shown under the restriction \( a_{\delta n}/a_n \to 0 \); in particular, 

\[ a_{\delta n}/a_n \to 0 \] implies that \( \sum a_{\delta n} \in MR(\sum a_n) \) if, and only if, 

\[ \sum a_{\delta n} \in MR(\sum a_n), \] where \( a_n = (1-r_{n-1})/(1-2r_n+r_{n-1}r_n) \).

The application of the \( \delta^2 \)-process to power series is also considered.

In Chapter IV, rapidity of convergence is again considered. Methods for accelerating convergence published by various authors, previously cited, are extended to complex series. In extending Lubkin's Theorems 8 and 9 (17, p. 232-233), it is shown that part of each hypotheses may be omitted. Pflanz (18, p. 25) established this fact for
the former theorem where $\Sigma a_n$ is real.

If $\Sigma a_n$ is a convergent series such that $|r_n| \to 1$, the application of Aitken's $\delta^2$-process becomes critical. In particular, that part of Lubkin's Theorem 6 (17, p. 231) concerning acceleration is shown to have no application if $r_n \to 1$. Similarly, that part of his Theorem 7 (17, p. 232) concerning acceleration is proven to be vacuous. The letter "C" in Theorem 7 is in error and should be replaced by "Q". At this point, one wonders if the $\delta^2$-process is ever practicable if $|r_n| \to 1$. The answer is in the affirmative, as is shown by Theorem 4.17, Theorem 4.20, and the discussion following the former theorem. Theorems on the acceleration of power series are also established.

Kummer's criterion, known to be sufficient for the convergence of a series $\Sigma a_n$ of positive terms, is proven to also be necessary in Chapter V. The necessity was first published by Shanks (24, p. 340). The criterion is that there exists a sequence $\{\beta_n\}$ and a positive number $c$ such that $\beta_n > 0$, for $n > 0$, and $\beta_n \geq c + r_{n+1}\beta_{n+1}$ for $n \geq 1$. It is proven in this paper that "$\beta_n > 0$" can be replaced by any one of the conditions "$\beta_n \geq 0$", "$\{a_n\beta_n\}$ converges", or "some subsequence of $\{a_n\beta_n\}$ is bounded below". Proofs of the sufficiency of
the comparison test, ratio comparison test, root test, ratio test, and Raabe's test, are given by exhibiting a sequence \( \{\beta_n\} \) such that \( \beta_n > 0 \) and \( \beta_n \geq l+r_{n+1}\beta_{n+1} \).

At the end of Chapter V, a method for applying the previously developed error analysis is indicated by one example.

Chapter VI gives the analogues of some of the theorems of Chapter V for real series, and Chapter VII does likewise for complex series.

In Chapter VIII, theorems, similar to Kummer's criterion for the convergence of series of positive terms, stating n.a.s.c. for an alternating series to converge are proven. Some of these theorems lead to fairly sharp bounds for the quantities \( T_n \). In many such theorems, it is proven that all inequalities, excluding those between indices, may be reversed. Calling any such theorem and the derived theorem duals, we encounter a duality structure, which unhappily fails in at least one case.

The theory of alternating series in this paper resulted from an initial study of Aitken's \( \delta^2 \)-process in the critical case \( r_n \to -1 \). Lubkin's Theorem 5 (17, p. 231) states that if \( \Sigma a_n \) is a real convergent series, \( r = -1 \), and \( \frac{l+r_{n+1}}{(1+r_n)} \to 1 \), then \( \Sigma a_{\delta n} \in MR(\Sigma a_n) \). Generalizations of this theorem are proven; one involves
lim inf \( \frac{1+r_{n+1}}{1+r_n} \) \( \to 1 \), while another involves
\lim \sup \frac{1+r_{n+1}}{1+r_n} = 1.
Another theorem along this line involves the inequality
\[ \frac{1}{2} \leq \frac{1+r_{n+1}+r_{n+1}r_{n+2}}{2}, \]
actually the first theorem discovered by the author. A detailed analysis of bounds for \( T_n \) is considered throughout, which immediately yield bounds for \( S-S_{n-1} \).

Calabrese (10, p. 216) appears to be the only one to publish any result along the lines developed in our chapter on alternating series. His theorem is true, but the proof which he gives contains an error. The final part of Chapter VIII is devoted to finding simple tests for applying the developed error bounds for \( T_n \).
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