AN ABSTRACT OF THE THESIS OF

Richard Ray Tucker for the Doctor of Philosophy
(Name) (Degree)

in Mathematics .
(Major)

Date thesis is presented May 11, 1963

Title ERROR ANALYSIS, CONVERGENCE, DIVERGENCE, AND THE ACCELERATION OF CONVERGENCE.

Abstract approved Redacted for privacy
(Major Professor)

The problem of acceleration or speed-up of a convergent complex series Σa_n, i.e., finding a series Σb_n which converges more rapidly than a given series Σa_n, and which has the same sum, has occupied the interest of various mathematicians, dating back at least to E.E. Kummer in 1837. In many cases, only real series have been considered; in particular, series of positive terms or alternating series.

To the author's knowledge, there is no basic treatment of this subject in the literature to date, and it is hoped that this paper will serve, at least as a beginning, to fill this gap. Such an exposition should present some of the methods in some type of unified setting and, at

the same time, bring new information to light. The author believes that both of these objectives have been "partially" fulfilled, while presenting a more or less self-contained introduction to some of the aspects of speed-up.

ERROR ANALYSIS, CONVERGENCE, DIVERGENCE, AND THE ACCELERATION OF CONVERGENCE

by

RICHARD RAY TUCKER

A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

DOCTOR OF PHILOSOPHY

June 1963

APPROVED

Redacted for privacy

Professor of Mathematics

in Charge of Major

Redacted for privacy

Chairman of Department of Mathematics

Redacted for privacy

Dean of Graduate School

Date thesis is presented May 11, 1963

Typed by Jolán Erőss

ACKNOWLEDGMENT

Many persons have contributed to the completion of this thesis. The author is above all indebted to Professor A. T. Lonseth, without whose guidance, encouragement, and patience the completion of this thesis would not have been possible.

To his wife, for her patience, encouragement, and faith, the author is deeply grateful.

TABLE OF CONTENTS

ERROR ANALYSIS, CONVERGENCE, DIVERGENCE, AND THE ACCELERATION OF CONVERGENCE

CHAPTER I

INTRODUCTION

Given a complex series $\sum_0^\infty a_n$, we shall write Σa_n for $\sum_0^\infty a_n$, $S_n = \sum_0^n a_k$, and, if Σa_n converges, $S = \Sigma a_n$. Similarly, if $\Sigma a'_n$ converges, then $S' = \Sigma a'_n$ Given two convergent series Σa_n and $\Sigma a'_n$, the latter is said to converge more rapidly than the former iff $(S'-S'_n)/(S-S_n) \to 0$ as $n \to \infty$. If Σa_n converges, "$MR(\Sigma a_n)$" will denote the class of all series Σb_n which converge more rapidly to S than Σa_n, i.e., $\Sigma b_n \in MR(\Sigma a_n)$ iff Σb_n converges more rapidly to S than Σa_n. The concept of "acceleration" or "speed-up" can now be defined as the problem of finding a series Σb_n such that $\Sigma b_n \in MR(\Sigma a_n)$. We will say that $\Sigma a'_n$ converges with the same rapidity as Σa_n iff there are numbers A and B such $0 < A <. |S'-S'_n|/|S-S_n| <. B$. The notation "<." means that $<$ holds for all sufficiently large n. If "*" denotes any relation, "*."

will be used in the same manner, while "$*$:" means that $*$ holds for infinitely many positive integers n. Similarly, $f(x) \leq . \, g(x)$ iff $f(x) \leq g(x)$ for all sufficiently large values of the real variable x.

Various methods, found in the literature, for obtaining a series $\Sigma a'_n \in MR(\Sigma a_n)$ may be summarized as follows. A sequence $\{b_n\}$ is proposed, and then the partial sums S'_n are specified by the equation $S'_n = S_n + b_{n+1}$ for $n \geq 0$. It is immediate that $a'_0 = a_0 + b_1$, and $a'_n = a_n + b_{n+1} - b_n$ for $n \geq 1$.

It seems somewhat advantageous to set $b_n = a_n \alpha_n$ for $n \geq 1$, and specify the "transform sequence" $\{\alpha_n\}$. In doing so, we set $S_{\alpha n} = S_n + a_{n+1}\alpha_{n+1}$ for $n \geq 0$, $a_{\alpha 0} = S_{\alpha 0} = a_0 + a_1\alpha_1$, and $a_{\alpha n} = S_{\alpha n} - S_{\alpha(n-1)}$ $= a_n + a_{n+1}\alpha_{n+1} - a_n\alpha_n$ for $n \geq 1$. It follows that if Σa_n converges, and $a_n =: 0$ or $\alpha_n =: 0$, then $S_{\alpha n} =: S_n$, and thus $\Sigma a_{\alpha n} \notin MR(\Sigma a_n)$. Consequently, we shall usually consider only series Σa_n for which $a_n \neq . \, 0$. If $\Sigma a_{\alpha n}$ converges, its sum will be denoted by S_α.

Suppose that Σa_n converges and $a_n \neq 0$ for $n \geq 0$.

The optimal choice of $\{\alpha_n\}$ for acceleration should yield $S_{\alpha n} = S$ for $n \geq 0$. Thus $S_n + a_{n+1}\alpha_{n+1} = S$ and we must have $\alpha_{n+1} = (S-S_n)/a_{n+1}$ for $n \geq 0$. We easily verify that $S_{\alpha n} = S_n + a_{n+1}\alpha_{n+1} = S_n + a_{n+1}(S-S_n)/a_{n+1} = S$ for $n \geq 0$, with $\alpha_n = (S-S_{n-1})/a_n$ for $n \geq 1$. Hence this transform sequence is the "exact" solution to our problem of speed-up. In general we must satisfy ourselves with an approximation to this solution. We now turn to some of these "approximations".

For each n such that $a_{n-1} \neq 0$ we write $r_n = a_n/a_{n-1}$. The notation $Q_n = n(1-r_n)$, $Q = \lim Q_n$, and $r = \lim r_n$ of Lubkin (17, p. 228-229) will be used (Lubkin uses "R" in place of our "r").

Aitken's δ^2-process will be treated in detail in this paper and can be obtained by defining its transform sequence $\{\delta_n\}$ as follows:

1.1 $\quad \delta_n = 1/(1-r_n)$ if $r_n \neq 1$, $\delta_n = 0$ otherwise.

The notation in 1.1 will be adhered to throughout this paper. Various other processes considered in this paper can be described by defining their corresponding transform sequence. We enumerate some of them as follows:

1.2 $\quad \alpha_n = 1/(1-r)$.

1.3 $\quad \alpha_n = (1-r_{n-1})/(1-2r_n+r_{n-1}r_n) \quad \text{for} \quad n \geq 2, \ \alpha_1 = -1/r_1$.

1.4 $\quad \alpha_n = n/(Q-1)$.

1.5 $\quad \alpha_n = Q/(Q-1)(1-r_n) = nQ/(Q-1)Q_n = Q\delta_n/(Q-1)$.

1.6 $\quad \alpha_n = s/(s-1)(1-r_n), \ s = \lim a_n/a_{\delta n}$.

Among publications in which 1.1 is found are the following: Aitken (1,p.301), Forsythe (11, p. 310), Hartree (12, p. 233), Householder (13, p. 117), Isakson (14, p. 443), Lubkin (17, p. 228), Pflanz (18, p. 27), Samuelson (20, p. 131), Schmidt (21, p. 376), Shanks (23, p. 233), Todd (28, p. 5, 86, 115, 187, 197, 260). We find 1.2 in Lubkin (17, p. 232), Shanks (22, p. 39) and (23, p. 25-26); 1.3 in Lubkin (17, p. 229); 1.4 in Szász (26, p. 274); 1.5 in Lubkin (17, p. 232), Pflanz (18, p. 25); 1.6 in Shanks (23, p. 39).

Lubkin calls $\Sigma a_{\delta n}$ the T transformation, $\Sigma a_{\alpha n}$ of 1.2 the Ratio transformation, and $\Sigma a_{\alpha n}$ of 1.3 the W transformation. The transformation defined by 1.5 is found in Lubkin's Theorem 8 (17, p. 232). Daniel Shanks calls $\Sigma a_{\alpha n}$ of 1.6 the $e_1^{(s)}$ transformation.

The author suggests the use of the following transform sequences for acceleration.

1.7 $\alpha_n = (n+a)/(Q-1)$, a some complex number.

1.8 $\alpha_n = (n+a)/(Q_n-1)$, a some complex number.

The sequence 1.7 reduces to 1.4, if $a = 0$. A method for determining the most appropriate value for a in 1.7 will be indicated by an example at the end of Chapter V. The sequence 1.8, with $a = 0$, is suggested for application to power series Σa_n where $a_n = b_n z^n$ for $n \geq 0$.

Given any sequence $\{x_n\}$ we define, for every n, $\Delta x_n = x_{n+1} - x_n$ and $\Delta^2 x_n = \Delta(\Delta x_n) = \Delta x_{n+1} - \Delta x_n = x_{n+2} - 2x_{n+1} + x_n$. No use will be made of the higher order differences $\Delta^k x_n$, $k \geq 3$.

Aitken's δ^2-process can be formulated in various ways. In particular, assuming that division by zero is excluded, we have:

1.9 $S_{\delta n} = S_n + a_{n+1}\delta_{n+1} = S_n + a_{n+1}/(1-r_{n+1})$, $n \geq 0$.

1.10 $S_{\delta n} = (S_{n-1}S_{n+1} - S_n^2)/(S_{n-1} - 2S_n + S_{n+1})$, $n \geq 1$.

1.11 $S_{\delta n} = \begin{vmatrix} S_{n-1} & S_n \\ \Delta S_{n-1} & \Delta S_n \end{vmatrix} \div \begin{vmatrix} 1 & 1 \\ \Delta S_{n-1} & \Delta S_n \end{vmatrix}$, $n \geq 1$.

1.12 $S_{\delta n} = S_{n-1} - (\Delta S_{n-1})^2/\Delta^2 S_{n-1}$, $n \geq 1$.

1.13 $S_{\delta n} = S_n - (\Delta S_{n-1}\Delta S_n)/\Delta^2 S_{n-1}$, $n \geq 1$.

1.14 $\quad S_{\delta n} = S_{n+1} - (\Delta S_n)^2/\Delta^2 S_{n-1}, \ n \geq 1.$

Moreover, if we define $F(x,y,z) = (xz-y^2)/(x-2y+z)$, $x-2y+z \neq 0$, we have $F(x+a, y+a, z+a) = a + F(x,y,z)$, for every a, and 1.10 becomes,

1.15 $\quad S_{\delta n} = F(S_{n-1}, S_n, S_{n+1}), \ n \geq 1.$

The function F also satisfies $F(c,x,cy,cz) = cF(x,y,z)$. We see that these two properties of F may be of some use in actual numerical calculations. For example, suppose that $S_1 = 15.001418373$, $S_2 = 15.000304169$, and $S_3 = 15.000065221$. Then, $S_{\delta 2} = F(S_1,S_2,S_3) = 15.000065221 + 10^{-9} F(1353152, 238948, 0) = 15.000065221 + (10^{-9})[-(238948)^2]/[1353152-2(238948)-0] =$ etc.

The δ^2-process has the following geometrical interpretation. Suppose that $S_n \to S$, so that $(S_n,S_{n+1}) \to (S,S)$. The points (S,S) and (S_n,S_{n+1}), $n \geq 0$, are graphed. The straight line through two successive points (S_{n-1},S_n) and (S_n,S_{n+1}) is intersected with the line $y = x$. Denoting this point of intersection by $(S_{\delta n},S_{\delta n})$ yields Aitken's δ^2-process. This interpretation is found in Todd (28, p. 260), but no mention is made of the δ^2-process there. Also, Todd (28, p. 5) credits the δ^2-process to Kummer (16, p. 206-214).

Returning to the exact solution for speed-up $\alpha_n = (S-S_{n-1})/a_n$, $n \geq 1$, we have $\alpha_n = (a_n+(S-S_n))/a_n$ $= 1 + (S-S_n)/a_n = 1 + T_{n+1}$, if we set $T_{n+1} = (S-S_n)/a_n$ for $n \geq 1$. Hence $1 + T_{n+1}$, $n \geq 1$, is the exact solution.

Suppose that Σa_n converges and n is any integer ≥ 1 such that $a_{n-1} \neq 0$. We then formally define

1.16 $\quad T_n = (S-S_{n-1})/a_{n-1}$.

Various relations are satisfied by the quantities T_n, some of which we now state and prove:

1.17 $\quad T_n = r_n(1+T_{n+1})$, if $a_{n-1}a_n \neq 0$.

1.18 $\quad (1-r_n)(1+T_{n+1}) = 1 + T_{n+1} - T_n$, if $a_{n-1}a_n \neq 0$.

1.19 $\quad [(1-r_n)/a_n](S-S_{n-1}) = 1+T_{n+1}-T_n$, if $a_{n-1}a_n \neq 0$.

1.20 $\quad T_{n+1} = r_n/(1-r_n) + (T_{n+1}-T_n)/(1-r_n)$, if $r_n \neq 0$ or 1.

1.21 $\quad T_n = r_n+r_nr_{n+1}+\cdots+ (r_nr_{n+1}\cdots r_{n+k})+\cdots$, if $a_m \neq 0$ for $m \geq n-1$.

For 1.17, $T_n = (S-S_{n-1})/a_{n-1} = (a_n+S-S_n)/a_{n-1} = a_n/a_{n-1}$ $+ (a_n/a_{n-1})[(S-S_n)/a_n] = r_n+r_nT_{n+1} = r_n(1+T_{n+1})$. Thus, $(1-r_n)(1+T_{n+1}) = 1+T_{n+1}-r_n(1+T_{n+1}) = 1+T_{n+1}-T_n$, i.e., 1.18

holds. Consequently, $[(1-r_n)/a_n](S-S_{n-1})$ $= (1-r_n)[(S-S_{n-1})/a_n] = (1-r_n)(T_n/r_n) = (1-r_n)(1+T_{n+1})$ $= 1+T_{n+} -T_n$, and thus 1.19 holds. From 1.18, $1+T_{n+1}$ $= 1/(1-r_n) + (T_{n+1}-T_n)/(1-r_n)$, so that T_{n+1} $= 1/(1-r_n) - 1 + (T_{n+1}-T_n)/(1-r_n) = r_n/(1-r_n)$ $+ (T_{n+1}-T_n)/(1-r_n)$, i e., 1.20 holds. Finally, $T_n = (S-S_{n-1})/a_{n-1} = (a_n+a_{n+1}+\cdots+ a_{n+k} +\cdots)/a_{n-1}$ $= a_n/a_{n-1} + a_{n+1}/a_{n-1} +\cdots+ a_{n+k}/a_{n-1}+\cdots= a_n/a_{n-1}$ $+ (a_na_{n+1})/(a_{n-1}a_n) +\cdots+ (a_na_{n+1}\cdots a_{n+k})/(a_{n-1}a_n\cdots a_{n+k-1})$ $+\cdots= r_n+r_nr_{n+1}+\cdots+(r_nr_{n+1}\cdots r_{n+k})+\cdots$, i.e., 1.21 holds.

Given a series Σa_n, not necessarily convergent, we define

1.22 $\quad T_{n,k} = (S_{n+k}-S_{n-1})/a_{n-1}$, for $k \geq -1$ and $a_{n-1} \neq 0$.

We note that $T_{n,-1} = 0$. Also, if k is any integer ≥ 0, and n is any integer such that $a_m \neq 0$ for $n - 1 \leq m \leq n + k$, then

1.23 $\quad T_{n,k} = r_n + r_nr_{n+1} +\cdots+ (r_nr_{n+1}\cdots r_{n+k})$.

We also define $\alpha_n \sim \beta_n$ iff $\alpha_n/\beta_n \to 1$ as $n \to \infty$. The abreviation "n.a.s.c." is used both for "necessary and sufficient condition" and "necessary and sufficient conditions."

Instead of a convergent series Σa_n, one may desire to accelerate the convergence of a sequence of complex numbers S_n. We then set $S_{\alpha n} = S_n + a_{n+1}\alpha_{n+1}$, where $a_n = \Delta S_{n-1} = S_n - S_{n-1}$, $r_n = a_n / a_{n-1}$, and $\{\alpha_n\}$ is a prescribed transform sequence. If $s = \lim S_n$, we require that $(S-S_{\alpha n})/(S-S_n) \to 0$ in order that $\{S_{\alpha n}\}$ converge more rapidly to S than $\{S_n\}$. Thus we may view acceleration from either the series or sequential viewpoint. They are clearly one and the same thing.

CHAPTER II

ACCELERATION, RAPIDITY OF CONVERGENCE, AITKEN'S δ^2-PROCESS, AND DIVERGENCE

All series in this chapter are assumed complex unless explicitly stated to the contrary.

<u>Theorem 2.1</u>. The conditions (1) $r_n \to 0$, (2) $T_n \to 0$, and (3) $T_n/r_n \to 1$ are equivalent.

<u>Proof</u>: If $T_n \to 0$, then $a_n \neq. 0$ so that $r_n =. T_n/(1+T_{n+1}) \to 0$. Conversely, assume that $r_n \to 0$. Let $0 < \varepsilon < 1$. Then $|r_n| \leq. \varepsilon$, so that $|T_n| =. |r_n+r_nr_{n+1}+\cdots| \leq. |r_n|+|r_n||r_{n+1}|+\cdots \leq. \varepsilon/(1-\varepsilon)$ and thus $T_n \to 0$.

If $T_n \to 0$, then $T_n/r_n =. 1+T_{n+1} \to 1$. Conversely, if $T_n/r_n \to 1$, then $T_{n+1} =. T_n/r_n - 1 \to 0$. Q.E.D.

<u>Theorem 2.2</u>. If $T_n \to t$ for some complex number t, then:

(1) $r = t/(1+t)$, $|r| \leq 1$, and $r \neq 1$.

(2) $t = r/(1-r)$ and $-\frac{1}{2} \leq \mathrm{Re}\,t$.

If, in addition, $\{\alpha_n\}$ is a sequence of complex numbers such that $\alpha_n \to \alpha_0$ for some complex number α_0, then:

(3) $S_\alpha = S$.

(4) $\Sigma a_{\alpha n} \; \varepsilon \; MR(\Sigma a_n)$ if and only if $\alpha_0 = 1/(1-r)$.

(5) $\Sigma a_{\alpha n}$ converges with the same rapidity as Σa_n if and only if $\alpha_0 \neq 1/(1-r)$.

<u>Proof</u>: Since $\{T_n\}$ converges and $T_n \mathrel{=\cdot} r_n(1+T_{n+1})$, $T_n \mathrel{\neq\cdot} 0$ and $T_n \mathrel{\neq\cdot} -1$. Consequently $t \neq -1$, since otherwise $|r_n| \mathrel{=\cdot} |T_n/(1+T_{n+1})| \to +\infty$, which is impossible since $a_n \to 0$. Thus, $r_n \mathrel{=\cdot} T_n/(1+T_{n+1}) \to t/(1+t)$, i.e., $r = t/(1+t) \neq 1$. Clearly, $|r| \leq 1$ so that (1) holds. From (1), $t = r/(1-r)$ and $|t|/|(-1)-t| = |t/(1+t)| = |r| \leq 1$. Thus, $|t| \leq |(-1)-t|$, which is equivalent to $-\frac{1}{2} \leq \operatorname{Re} t$, so that (2) holds. (3) holds since $S_{\alpha n} = S_n + a_{n+1}\alpha_{n+1} \to S + 0\alpha_0 = S$. Since $T_n \mathrel{\neq\cdot} 0$, we have $(S-S_{n-1}) \mathrel{\neq\cdot} 0$. If $t = 0$, then $r_n/T_n \to 1 = 1-r$, according to (1), (2) and Theorem 2.1. If $t \neq 0$, then $r_n/T_n \to r/t = (1-r)$ from (1) and (2). In either case,

$(S-S_{\alpha n})/(S-S_n) \mathrel{=\cdot} [S-(S_n+a_{n+1}\alpha_{n+1})]/(S-S_n)$

$\mathrel{=\cdot} 1-a_{n+1}\alpha_{n+1}/(S-S_n) \mathrel{=\cdot} 1-\alpha_{n+1}r_{n+1}/T_{n+1} \to 1-\alpha_0(1-r)$.

Hence, (4) and (5) hold, since $1-\alpha_0(1-r) = 0$ is equivalent to $\alpha_0 = 1/(1-r)$. Q.E.D.

<u>Corollary 2.3</u>. If $\{T_n\}$ converges, then $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

<u>Proof</u>: Suppose $T_n \to t$. From (1) of Theorem 2.2, $r_n \to r$ where $r \neq 1$. Thus $\delta_n =. 1/(1-r_n) \to 1/(1-r)$, so that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ according to (4) of Theorem 2.2. Q.E.D.

We inquire if the convergence of $\{T_n\}$ is also necessary for $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. In the following chapter, we shall see that the answer is in the negative. There it will be proven that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if and only if $T_{n+1}-T_n \to 0$.

<u>Theorem 2.4</u>. If Σa_n and $\Sigma a_{\delta n}$ are convergent real series, then $S = S_\delta$.

<u>Proof</u>: Assume that $S \neq S_\delta$. Since $a_n\delta_n =. S_{\delta(n-1)}-S_{(n-1)}$ $\to S_\delta - S \neq 0$, $\delta_n \neq. 0$ and $a_n/(1-r_n) =. a_n\delta_n \to S_\delta - S \neq 0$. Thus $a_n \to 0$ implies that $1-r_n \to 0$, i.e., $r_n \to r = 1$ so that $0 <. r_n$ and $0 <. T_n$. From $1+T_{n+1}-T_n$ $=. [(1-r_n)/a_n](S-S_{n-1}) \to 0$, we have $1+T_{n+1}-T_n <. \frac{1}{2}$ and $0 <. T_{n+1} <. T_n$, which implies that $\{T_n\}$ converges. From (1) of Theorem 2.2, $r \neq 1$, which contradicts $r = 1$.

Thus our assumption is false, and $S = S_\delta$. Q.E.D.

Lubkin (17, p. 230) gave the first published proof of Theorem 2.4 for real series. The proof of this theorem for the complex case is given in Theorem 2.6, and to the author's knowledge is the first such proof.

<u>Theorem 2.5</u>. If $(1-r_n)/a_n \to L \neq 0$, then Σa_n diverges.

<u>Proof</u>: Assume that Σa_n converges. We may suppose that $L = 1-i$; since otherwise $\Sigma a'_n$ converges where $a'_n = a_n L/(1-i)$ and $(1-r'_n)/a'_n \doteq (1-r_n)/[a_n L/(1-i)] \to 1-i$. Accordingly, $(1-r_n)/a_n \doteq [(\mathrm{Re}\ a_n)/|a_n|^2 - (\mathrm{Re}\ a_{n-1})/|a_{n-1}|^2] + i\,[(\mathrm{Im}\ a_{n-1})/|a_{n-1}|^2 - (\mathrm{Im}\ a_n)/|a_n|^2] \to 1-i$. Consequently, $(\mathrm{Re}\ a_{n-1})/|a_{n-1}|^2 \mathrel{<\!\!\cdot} (\mathrm{Re}\ a_n)/|a_n|^2$ so that $(\mathrm{Re}\ a_n)/|a_n|^2 \to L_1$ for some $L_1 \leq +\infty$. If $L_1 < +\infty$, then $\mathrm{Re}\ [(1-r_n)/a_n] \to L_1 - L_1 = 0$, which is impossible since $\mathrm{Re}\ [(1-r_n)/a_n] \to 1$. Thus $L_1 = +\infty$ and $0 \mathrel{<\!\!\cdot} \mathrm{Re}\ a_n$. Similarly, $(\mathrm{Im}\ a_{n-1})/|a_{n-1}|^2 \mathrel{<\!\!\cdot} (\mathrm{Im}\ a_n)/|a_n|^2$ and $0 \mathrel{<\!\!\cdot} \mathrm{Im}\ a_n$. Hence setting $a_n = |a_n| e^{i\theta_n}$ we may chose θ_n such that $0 \mathrel{<\!\!\cdot} \theta_n \mathrel{<\!\!\cdot} \pi/2$. From

$$T_n \doteq a_n/a_{n-1} + a_{n+1}/a_{n-1} + \cdots + a_{n+k}/a_{n-1} + \cdots$$

$$\doteq |a_n/a_{n-1}| e^{i(\theta_n - \theta_{n-1})} + |a_{n+1}/a_{n-1}| e^{i(\theta_n - \theta_{n-1})} + \cdots$$

$$=. \; [|a_n|\cos(\theta_n-\theta_{n-1})+\cdots+|a_{n+k}|\cos(\theta_{n+k}-\theta_{n-1})$$
$$+\cdots]/|a_{n-1}| + (\mathrm{Im}\ T_n)i$$

and $0 <. \; \theta_n <. \; \pi/2$, we have $0 <.$ Re T_n. Since $1+T_{n+1}-T_n =. \; [(1-r_n)/a_n](S-S_{n-1}) \to 0$, we have $1+\mathrm{Re}\ T_{n+1}-\mathrm{Re}\ T_n =. \; \mathrm{Re}\ (1+T_{n+1}-T_n) \to 0$. Thus Re T_{n+1} - Re $T_n < -\frac{1}{2}$ for $n \geq N$, where N is some positive integer. Consequently,

$$\mathrm{Re}\ T_{N+n} =. \; \mathrm{Re}\ T_N + \sum_{i=1}^{n} \mathrm{Re}[T_{N+i}-T_{N+i-1}] <. \; \mathrm{Re}\ T_N - \frac{n}{2} \to -\infty$$

as $n \to \infty$. Hence, Re $T_n <. \; 0$ which contradicts $0 <.$ Re T_n. Consequently our initial assumption cannot hold, i.e., Σa_n must diverge. Q.E.D.

Theorem 2.6. If Σa_n and $\Sigma a_{\delta n}$ both converge, then $S = S_\delta$.

Proof: Assume that $S \neq S_\delta$. Then $a_n\delta_n =. \; S_{\delta(n-1)}-S_{n-1} \to S_\delta - S \neq 0$ so that $\delta_n \neq. \; 0$ and $a_n/(1-r_n) =. \; a_n\delta_n \to S_\delta - S \neq 0$. Thus $(1-r_n)/a_n \to 1/(S_\delta-S) \neq 0$, which implies, in view of Theorem 2.5, that Σa_n diverges, a contradiction. Therefore our assumption cannot hold, i.e., $S = S_\delta$. Q.E.D.

It should be kept in mind throughout the remainder of this paper that, according to the preceeding theorem, the statements "$\Sigma a_{\delta n} \in MR(\Sigma a_n)$" and "$\Sigma a_{\delta n}$ converges more rapidly than Σa_n" are equivalent.

Lemma 2.7. Suppose that Σa_n is a convergent series, $a_n \neq. 0$, and $c_n = c + S_n - S$ for $n \geq 0$ where c is some complex number. Then,

$$1 + c\left(\frac{1-r_n}{a_n}\right) + \frac{c_{n-1}}{a_{n-1}} - \frac{c_n}{a_n} =. \frac{1-r_n}{a_n}(S-S_{n-1}).$$

Proof: We have

$$1 + c\left(\frac{1-r_n}{a_n}\right) + \frac{c_{n-1}}{a_{n-1}} - \frac{c_n}{a_n} =. 1+c\left(\frac{1}{a_n} - \frac{1}{a_{n-1}}\right) + \frac{c+S_{n-1}-S}{a_{n-1}}$$

$$- \frac{c+S_n-S}{a_n} =. 1 + \frac{S-S_n}{a_n} - \frac{S-S_{n-1}}{a_{n-1}} =. \frac{S-S_{n-1}}{a_n} - \frac{S-S_{n-1}}{a_{n-1}}$$

$$=. \left(\frac{1}{a_n} - \frac{1}{a_{n-1}}\right)(S-S_{n-1}) =. \left(\frac{1-r_n}{a_n}\right)(S-S_{n-1}). \quad \text{Q.E.D.}$$

Theorem 2.8. If $\{(1-r_n)/a_n\}$ is bounded, then the complex series Σa_n diverges.

Proof: Assume that Σa_n converges. Since $\{(1-r_n)/a_n\}$ is bounded, there is an $\varepsilon > 0$ such that $|\varepsilon(1-r_n)/a_n| <. 1/4$. Let c be any complex number satisfying $|c| = \varepsilon$ so that

(1) $\quad -\mathrm{Re}\ c(1-r_n)/a_n <. \tfrac{1}{4}.$

Setting $c_n = c + S_n - S$, for $n \geq 0$, we have $c_n \to c$.

From Lemma 2.7,

$$\mathrm{Re}\ [1 + c\ (\frac{1-r_n}{a_n}) + \frac{c_{n-1}}{a_{n-1}} - \frac{c_n}{a_n}] =. \ \mathrm{Re}\ \frac{1-r_n}{a_n}(S-S_{n-1}) \to 0$$

and thus,

$$(2) \qquad 1 + \mathrm{Re}\ c\ (\frac{1-r_n}{a_n}) + \mathrm{Re}\ \frac{c_{n-1}}{a_{n-1}} - \mathrm{Re}\ \frac{c_n}{a_n} <. \tfrac{1}{4}.$$

Using (1) and (2),

$$\tfrac{1}{2} + \mathrm{Re}\ \frac{c_{n-1}}{a_{n-1}} <. \ \mathrm{Re}\ \frac{c_n}{a_n} - \mathrm{Re}\ c\ (\frac{1-r_n}{a_n}) - \tfrac{1}{4} <. \ \mathrm{Re}\ \frac{c_n}{a_n},$$

from which it is easily seen that $\mathrm{Re}\ c_n/a_n \to +\infty$ and $\mathrm{Re}\ c_n/a_n >. 0$. Since $\mathrm{Re}\ c_n/a_n >. 0$ and $c_n \to c$, we conclude that

(3) $\quad a_n \not\in. \{z: \arg c + 3\pi/4 \leq \arg z \leq \arg c + 5\pi/4\}.$

Chosing $\arg c$ successively in (3) as $0, \pi/2, \pi$, and $3\pi/2$, we conclude that a_n is not in the complex plane for large n, which is absurd. Hence, our initial assumption cannot hold, i.e., Σa_n must diverge. Q.E.D.

For the series Σa_n where $a_n = 1/\ell n\ n$ for $n \geq 2$, we have $(1-r_n)/a_n =. 1/a_n - 1/a_{n-1}$ $=. \ell n\ n - \ell n(n-1) \to 0$ so that, from Theorem 2.8, Σa_n diverges. Similarly, with $a_n = 1/(n+1)$ for $n \geq 0$, we have $1/a_n - 1/a_{n-1} = (n+1)-n = 1$ for $n \geq 1$, and thus Σa_n diverges. For the divergent series Σa_n where $a_n \doteq 1/(n \ln n)$, we have $1/a_n - 1/a_{n-1} \doteq n \ln n - (n-1) \ln (n-1) \doteq (n-1)[\ln n - \ln(n-1)] + \ln n \to \infty$, so that Theorem 2.8 is not applicable, and thus appears to be a very limited criterion for divergence.

Theorem 2.9. If Σa_n is a convergent series, then some subsequence of $\{S_{\delta n}\}$ converges to S.

Proof: Suppose Σa_n is convergent and assume that no subsequence of $\{S_{\delta n}\}$ converges to S. Since $S_{\delta n} - S_n = a_{n+1}\delta_{n+1}$, our assumption holds if and only if no subsequence of $\{a_n\delta_n\}$ converges to zero, and this is equivalent to $|a_n\delta_n| \gtrdot B$ for some $B > 0$. Thus $|(1-r_n)/a_n| \doteq 1/|a_n\delta_n| \lessdot 1/B$. From Theorem 2.8, Σa_n diverges, a contradiction. Therefore our assumption cannot be true, i.e., some subsequence of $\{S_{\delta n}\}$ converges to S. Q.E.D.

Theorem 2.9 clearly yields a second proof of Theorem 2.6.

Example 2.10. It is not necessarily true that if Σa_n converges, $\Sigma a_{\delta n}$ will also converge. In particular, Lubkin (1, p. 240) considers the series $\Sigma a_n = 1 + 1/2 - 1/3 - 1/4 + 1/5 + 1/6 - 1/7 - 1/8 + 1/9 + \cdots$ which converges while $\Sigma a_{\delta n}$ diverges. However, according to Theorem 2.9 some subsequence of $\{S_{\delta n}\}$ must converge to S. Hence, of course, this is evident since $r_n <: 0$ and $S_{\delta n} =. S_n + a_{n+1}/(1-r_{n+1})$. This particular series shows that the δ^2-process is not regular.

<u>Example 2.11</u>. Lubkin (17, p. 240) also shows that the series $\Sigma a_n = 1+1/(1+1) + 1/2^2 + 2^2/(2^4+1) + 1/3^2 + 3^2/(3^4+1) + \cdots$ converges while $\Sigma a_{\delta n}$ diverges. Again, according to Theorem 2.9, some subsequence of $\{S_{\delta n}\}$ must converge to S. This is not so obvious by inspection as was the case in Example 2.10.

<u>Theorem 2.12</u>. If Σa_n is a series such that $\Sigma a_{\delta n}$ is properly divergent, i.e., $|S_{\delta n}| \to \infty$, as $n \to \infty$, then Σa_n diverges.

<u>Proof</u>: Assume that Σa_n is convergent. From Theorem 2.9 some subsequence of $\{S_{\delta n}\}$ converges to S, so that $|S_{\delta n}| \not\to \infty$ as $n \to \infty$, i.e., $\Sigma a_{\delta n}$ is not properly divergent. Q.E.D.

Theorem 2.13. A n.a.s.c. that $\{T_n\}$ converge is that $r_n \to r \neq 1$ and $T_{n+1} - T_n \to 0$.

Proof: The necessity follows from (1) of Theorem 2.2 and the fact that $\{T_n\}$ converges implies that $T_{n+1} - T_n \to 0$.

For the sufficiency, $r \neq 1$ implies that $r_n(1-r_n) \neq 0$. From 1.20, $T_{n+1} = r_n/(1-r_n) + (T_{n+1}-T_n)/(1-r_n) \to r/(1-r)$. Q.E.D.

Theorem 2.14. If $r_n \to r$ where $|r| < 1$, then $T_n \to r/(1-r)$.

Proof: Since $|r| < 1$, $r \neq 1$ and Σa_n converges, so that T_n exists for large n. Let $\varepsilon > 0$ and ρ be any number such that $|r| < \rho < 1$. There exists an integer N such that for $n \geq N$ and $m \geq N$ we have $|r_n| < \rho$ and $|r_n - r_n| < \varepsilon(1-\rho)$. Thus, for each $n \geq N$ we have $|T_{n+1}-T_n| = |[r_{n+1}-r_n] + [r_{n+1}r_{n+2}-r_nr_{n+1}] +\cdots+ [(r_{n+1}\cdots r_{n+k+1}) - (r_n\cdots r_{n+k})] +\cdots|$

$\leq |r_{n+1}-r_n| + |r_{n+1}||r_{n+2}-r_n| +\cdots+ |r_{n+1}\cdots r_{n+k}||r_{n+k+1}-r_n| +\cdots$

$< \varepsilon(1-\rho) + \rho\,\varepsilon(1-\rho) +\cdots+ \rho^k\varepsilon(1-\rho) +\cdots = \varepsilon.$

Hence, $|T_{n+1}-T_n| \to 0$, i.e., $T_{n+1}-T_n \to 0$. From

Theorem 2.13, $\{T_n\}$ converges. Consequently, $T_n \to r/(1-r)$ according to (2) of Theorem 2.2. Q.E.D.

Theorem 2.15. Suppose that $r_n \to r$ where $|r| < 1$, and let $\{\alpha_n\}$ be a complex sequence converging to some complex number α_0. Then $T_n \to t$ for some complex number t, and conditions (1) through (5) of Theorem 2.2 hold.

Proof: From Theorem 2.14, $\{T_n\}$ converges. Now apply Theorem 2.2. Q.E.D.

According to Theorem 2.15, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if $r = 0$. Nevertheless, the reader should be forewarned in case $r = 0$. In particular, let $\Sigma a_n = \sum_0^\infty (-1)^n/n! = 1/e$. We have $r_n = -1/n$ for $n \geq 1$, and $\delta_n = 1/(1-r_n)$ $= 1/[1+(1/n)] = n/(n+1) = 1-1/(n+1) = 1+r_{n+1}$ for $n \geq 2$. Consequently, $S_{\delta n} = S_n + a_{n+1}\delta_{n+1} = S_n + a_{n+1}(1+r_{n+2}) = S_{n+2}$ for $n \geq 1$. Hence $\{\delta_n\}$ appears to be a poor selection for accelerating the convergence of Σa_n.

Lemma 2.16. If $|r| < 1$, then $T_n/r_n \to 1/(1-r)$.

Proof: If $r = 0$, then $T_n/r_n \to 1 = 1/(1-r)$ according to Theorem 2.1. If $r \neq 0$, then $T_n/r_n \to [r/(1-r)]/r$ $= 1/(1-r)$ according to Theorem 2.14. Q.E.D.

Theorem 2.17. Suppose that Σa_n and $\Sigma a'_n$ are series such that $|r| < 1$ and $|r'| < 1$. Then:

(1) $\Sigma a'_n$ converges more rapidly than Σa_n if and only if $a'_n/a_n \to 0$.

(2) $\Sigma a'_n$ converges with the same rapidity as Σa_n if and only if there are numbers a and b such that $0 < a \mathrel{<.} |a'_n/a_n| \mathrel{<.} b$.

Proof: From Lemma 2.16, $T_n/r_n \to 1/(1-r)$ and $T'_n/r'_n \to 1/(1-r')$.

If $a'_n/a_n \to 0$,

$$\frac{S'-S'_{n-1}}{S-S_{n-1}} \mathrel{=.} \frac{a'_n}{a_n}\frac{T'_n/r'_n}{T_n/r_n} \to 0 \cdot \frac{1/(1-r')}{1/(1-r)} = 0.$$

Conversely, if $\Sigma a'_n$ converges more rapidly than Σa_n,

$$\frac{a'_n}{a_n} \mathrel{=.} \frac{T_n/r_n}{T'_n/r'_n}\frac{S'-S'_{n-1}}{S-S_{n-1}} \to \frac{1/(1-r)}{1/(1-r')} \cdot 0 = 0.$$

This proves (1).

Assume that a and b are numbers such that $0 < a \mathrel{<.} |a'_n/a_n| \mathrel{<.} b$. Since $|T'_n/r'_n)/(T_n/r_n)| \to |(1-r)/(1-r')| \neq 0$, there are numbers c and d such that $0 < c \mathrel{<.} |(T'_n/r'_n)/(T_n/r_n)| \mathrel{<.} d$. Thus,

$$0 < ac \mathrel{<.} \left|\frac{S'-S_{n-1}}{S-S_{n-1}}\right| \mathrel{=.} \left|\frac{a'_n}{a_n}\right| \left|\frac{T'_n/r'_n}{T_n/r_n}\right| \mathrel{<.} bd.$$

Assume that A and B are numbers such that $0 < A \mathrel{<\!\!.} |(S'-S'_{n-1})/(S-S_{n-1})| \mathrel{<\!\!.} B$. As above, there are numbers c and d such that $0 < c \mathrel{<\!\!.} |(T_n/r_n)/(T'_n/r_n)| \mathrel{<\!\!.} d$. Thus,

$$0 < Ac \mathrel{<\!\!.} \left|\frac{a'_n}{a_n}\right| \mathrel{=\!\!.} \left|\frac{T_n/r_n}{T'_n/r'_n}\right| \left|\frac{S'-S'_{n-1}}{S-S_{n-1}}\right| \mathrel{<\!\!.} Bd. \quad \text{Q.E.D.}$$

<u>Lemma 2.18</u>. If $|r_n| \mathrel{\leq\!\!.} \rho < 1/2$ for some number ρ, then $0 < (1-2\rho)/(1-\rho) \mathrel{\leq\!\!.} |T_n/r_n| \mathrel{\leq\!\!.} 1/(1-\rho)$.

<u>Proof</u>: We have $|T_n| \mathrel{\leq\!\!.} |r_n|+|r_n r_{n+1}|+\cdots+|r_n \cdots r_{n+k}| +\cdots \mathrel{\leq\!\!.} |r_n|/(1-\rho) \mathrel{\leq\!\!.} \rho/(1-\rho) < 1$. Thus, $|T_n/r_n| \mathrel{\leq\!\!.} 1/(1-\rho)$ and $|T_n/r_n| \mathrel{=\!\!.} |1+T_{n+1}| \mathrel{\geq\!\!.} ||1| - |T_{n+1}|| \mathrel{=\!\!.} 1-|T_{n+1}| \mathrel{\geq\!\!.} 1-\rho/(1-\rho) = (1-2\rho)/(1-\rho) > 0$. Q.E.D.

<u>Theorem 2.19</u>. Suppose that Σa_n, $\Sigma a'_n$ are series such that $a'_n/a_n \to 0$, and $|r_n| \mathrel{\leq\!\!.} \rho_1 < 1/2$, $|r'_n| \mathrel{\leq\!\!.} \rho_2 < 1$ for some numbers ρ_1, ρ_2. Then $\Sigma a'_n$ converges more rapidly than Σa_n.

<u>Proof</u>: From Lemma 2.18, $0 < (1-2\rho_1)/(1-\rho_1) \mathrel{\leq\!\!.} |T_n/r_n|$. Also, $|T'_n/r'_n| \mathrel{=\!\!.} |1+r'_{n+1}+r'_{n+1}\, r'_{n+2}+\cdots| \mathrel{\leq\!\!.} 1/(1-\rho_2)$. Thus,

$$\frac{|S'-S'_{n-1}|}{|S-S_{n-1}|} \doteq \frac{|a'_n|}{|a_n|}\,\frac{|T'_n/r'_n|}{|T_n/r_n|} \lessdot \frac{|a'_n|}{|a_n|}\,\frac{1/(1-\rho_2)}{(1-2\rho_1)/(1-\rho_1)} \to 0.$$

Q.E.D.

According to the following counterexample, Theorem 2.19 fails to hold if we replace "$\rho_1 < \frac{1}{2}$" by "$\rho_1 \leq 1$" and "$\rho_2 < 1$" by "$\rho_2 \leq 1$".

Counterexample 2.20. For $n \geq 0$, define $a_n = (-1)^n/(n+1)$ and $a'_n = 1/(n+1)(n+2)$. Then $a'_n/a_n \to 0$, $r'_n \to r' = 1$, and $r_n \to r = -1$. Since $S'-S'_n \doteq 1/(n+2)$ and $|S-S_n| \lessdot |a_{n+1}| \doteq 1/(n+2)$, we have $|S'-S'_n|/|S-S_n| \gtrdot 1$, and thus $\Sigma a'_n$ does not converge more rapidly than Σa_n.

CHAPTER III

BASIC THEOREMS FOR ACCELERATION, AITKEN'S δ^2-PROCESS, AND LUBKIN'S W TRANSFORMATION

All series in this chapter are assumed to be complex. The first two theorems of this chapter, the second theorem in particular, are basic for a study of acceleration.

<u>Theorem 3.1</u>. Suppose that Σa_n is a complex series $\{b_n\}$ is a complex sequence, and $\Sigma a'_n$ is a series with partial sums $S'_n \doteq S_n + b_{n+1}$. Then $\Sigma a'_n \in MR(\Sigma a_n)$ if and only if $b_{n+1} \sim S-S_n \to 0$.

<u>Proof</u>: If either condition holds, then $S-S_n \doteq S-S'_n+b_{n+1}$ $\dot{\neq} 0$, so that $b_{n+1}/(S-S_n) + (S-S'_n)/(S-S_n) \doteq 1$. Thus $(S-S'_n)/(S-S_n) \to 0$ and $S-S_n \to 0$ if, and only if, $b_{n+1}/(S-S_n) \to 1$ and $S-S_n \to 0$; but this is equivalent to $b_{n+1} \sim S-S_n \to 0$. Q.E.D.

From Theorem 3.1, we see that the class of all sequences $\{c_n\}$ such that $\Sigma a'_n \in MR(\Sigma a_n)$, where $S'_n = S_n + c_{n+1}$, is completely determined by one such sequence $\{b_n\}$; the required condition being that $c_n \sim b_n$.

Similarly, we now show that if $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$, then $\Sigma a_{\beta n} \varepsilon MR(\Sigma a_n)$, if and only if $\beta_n \sim \alpha_n$.

<u>Theorem 3.2</u>. Suppose that $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$. Then $\Sigma a_{\beta n} \varepsilon MR(\Sigma a_n)$ if and only if $\beta_n \sim \alpha_n$.

<u>Proof</u>: From Theorem 3.1, $a_{n+1}\alpha_{n+1} \sim S-S_n \to 0$. Hence, from Theorem 3.1, $\Sigma a_{\beta n} \varepsilon MR(\Sigma a_n)$ if and only if $a_{n+1}\beta_{n+1} \sim S-S_n$, and this is equivalent to $a_{n+1}\beta_{n+1} \sim a_{n+1}\beta_{n+1}$, that is, $\beta_{n+1} \sim \alpha_{n+1}$. Q.E.D.

<u>Lemma 3.3</u>. If $(1-r_n)(1-r_{n+1}) \neq 0$, then $a_{\delta n}/a_n = 1/(1-r_{n+1}) - 1/(1-r_n) = r_{n+1}/(1-r_{n+1}) - r_n/(1-r_n) = (r_{n+1}-r_n)/(1-r_n)(1-r_{n+1})$.

<u>Proof</u>: Since $r_n \neq 1$ and $r_{n+1} \neq 1$, we have $\delta_n = 1/(1-r_n)$ and $\delta_{n+1} = 1/(1-r_{n+1})$. Thus, $a_{\delta n}/a_n = (a_n+a_{n+1}\delta_{n+1}-a_n\delta_n)/a_n = 1+r_{n+1}\delta_{n+1}-\delta_n = r_{n+1}/(1-r_{n+1}) + 1-1/(1-r_n) = r_{n+1}/(1-r_{n+1}) - r_n/(1-r_n) = [r_{n+1}(1-r_n) - r_n(1-r_{n+1})]/(1-r_n)(1-r_{n+1}) = (r_{n+1}-r_n)/(1-r_n)(1-r_{n+1}) = 1/(1-r_{n+1}) - 1/(1-r_n)$. Q.E.D.

<u>Theorem 3.4</u>. Suppose that $a_{\delta n}/a_n \to 0$. Then

$\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if and only if $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ where $\alpha_n \doteq (1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$.

<u>Proof</u>: Suppose that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. From Lemma 3.3, $1-2r_{n+1}+r_n r_{n+1} \doteq (1-r_n)(1-r_{n+1}) - (r_{n+1}-r_n)$ $\doteq (1-r_n)(1-r_{n+1})[1-(r_{n+1}-r_n)/(1-r_n)(1-r_{n+1})]$ $\doteq (1-r_n)(1-r_{n+1})(1-a_{\delta n}/a_n) \not\doteq 0$. Hence, α_n/δ_n $\doteq (1-r_n)(1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1}) \doteq 1/(1-a_{\delta n}/a_n) \to 1$. From Theorem 3.2, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$.

Suppose that $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$. Then $r_n \not\doteq 1$, so that $\alpha_n/\delta_n \doteq 1/(1-a_{\delta n}/a_n) \to 1$ and, from Theorem 3.2, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. Q.E.D.

<u>Theorem 3.5</u>. Suppose that $a_{\delta n}/a_n \to 0$. Then $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if and only if $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, where $\alpha_n \doteq (1-r_{n-1})/(1-2r_n+r_{n-1}r_n)$.

<u>Proof</u>: Suppose that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. As in the proof of Theorem 3.4, $1-2r_n+r_{n-1}r_n \doteq (1-r_{n-1})(1-r_n)[1-a_{\delta(n-1)}/a_{n-1}]$ $\not\doteq 0$. Hence, $\alpha_n/\delta_n \doteq (1-r_{n-1})(1-r_n)/(1-2r_n+r_{n-1}r_n)$ $\doteq 1/(1-a_{\delta(n-1)}/a_{n-1}) \to 1$. From Theorem 3.2, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$.

Suppose that $\Sigma a_{\alpha n} \varepsilon\ MR(\Sigma a_n)$. Then $r_n \neq. 1$, and thus $\alpha_n/\delta_n =. 1/(1-a_{\delta(n-1)}/a_{n-1}) \to 1$. From Theorem 3.2, $\Sigma a_{\delta n} \varepsilon\ MR(\Sigma a_n)$.

<u>Theorem 3.6</u>. $\Sigma a_{\alpha n} \varepsilon\ MR(\Sigma a_n)$, $\alpha_n \sim T_n/r_n$, and $\alpha_n \sim 1+T_{n+1}$ are equivalent.

<u>Proof</u>: From Theorem 3.1, $\Sigma a_{\alpha n} \varepsilon\ MR(\Sigma a_n)$ if and only if $a_{n+1}\alpha_{n+1} \sim S-S_n \to 0$; and this is equivalent to $\alpha_{n+1} \sim (S-S_n)/a_{n+1} =. T_{n+1}/r_{n+1}$. Moreover, $\alpha_n \sim T_n/r_n$ is equivalent to $\alpha_n \sim 1+T_{n+1}$, since $T_n/r_n = 1+T_{n+1}$. Q.E.D.

<u>Lemma 3.7</u>. If Σa_n is a convergent series and n is a positive integer such that $T_{n+1}-T_n \neq -1$, then $(S-S_{\delta(n-1)})/(S-S_{n-1}) = (T_{n+1}-T_n)/(1+T_{n+1}-T_n)$.

<u>Proof</u>: From $(1-r_n)(1+T_{n+1}) = 1+T_{n+1}-T_n \neq 0$, $T_{n+1} \neq -1$ and $r_n \neq 1$. Thus $S-S_{n-1} = a_n(1+T_{n+1}) \neq 0$. We then have

$$(S-S_{\delta(n-1)})/(S-S_{n-1}) = (S-S_{n-1}-a_n\delta_n)/(S-S_{n-1})$$

$$= 1-a_n\delta_n/(S-S_{n-1})$$

$$= 1 - \frac{a_n}{S-S_{n-1}}\,\frac{1}{1-r_n} = 1 - \frac{1}{T_n}\,\frac{r_n}{1-r_n} = 1 - \frac{1}{T_n}\,\frac{T_n/(1+T_{n+1})}{1-T_n/(1+T_{n+1})}$$

$= 1 - 1/(1+T_{n+1}-T_n) = (T_{n+1}-T_n)/(1+T_{n+1}-T_n)$. Q.E.D.

Theorem 3.8. $\Sigma a_{\delta n} \varepsilon MR(\Sigma a_n)$ if and only if $T_{n+1}-T_n \to 0$.

1st Proof: From Theorem 3.6, $\Sigma a_{\delta n} \varepsilon MR(\Sigma a_n)$ if and only if $\delta_n \sim 1+T_{n+1}$, and this is equivalent to $(1+T_{n+1})(1-r_n) \to 1$, since $\delta_n =. 1/(1-r_n)$. Finally, $(1+T_{n+1})(1-r_n) \to 1$ if and only if $T_{n+1}-T_n \to 0$, since $T_{n+1}-T_n =. (1+T_{n+1})(1-r_n) - 1$. Q.E.D.

2nd Proof: If $T_{n+1}-T_n \to 0$, then $T_{n+1}-T_n \neq. -1$. Thus, from Lemma 3.7, $(S-S_{\delta(n-1)})/(S-S_{n-1})$ $=. (T_{n+1}-T_n)/(1+T_{n+1}-T_n) \to 0$. Conversely, suppose that $(S-S_{\delta(n-1)})/(S-S_{n-1}) \to 0$. Then $a_n \neq. 0$ and $r_n \neq. 1$, since $\delta_n \neq. 0$. We must have $1+T_{n+1}-T_n \neq. 0$, since otherwise $(1-r_n)(T_n/r_n) =. 1+T_{n+1}-T_n =: 0$, $T_n =: 0$, and $S-S_{n-1} =: 0$; a contradiction. From Lemma 3.7, $(T_{n+1}-T_n)/(1+T_{n+1}-T_n) =. (S-S_{\delta(n-1)})/(S-S_{n-1}) \to 0$, and thus $T_{n+1}-T_n \to 0$. Q.E.D.

The preceeding theorem immediately yields the corollary, also proven in the previous chapter, that the convergence of $\{T_n\}$ implies $\Sigma a_{\delta n} \varepsilon$ MR(Σa_n).

Lemma 3.9. If Σa_n is a convergent series and n is a positive integer such that $a_{n-1} a_n a_{n+1} \neq 0$, then
$r_{n+1} - r_n = (T_{n+2} - T_{n+1})(1-r_n)(1-r_{n+1}) - (T_{n+2} - T_{n+1})(1-r_n)$
$+ \; (T_{n+1} - T_n)(1-r_{n+1})$.

Proof: We have $(1-r_n)(1+T_{n+1}) = 1-r_n+T_{n+1}-r_nT_{n+1}$
$= 1+T_{n+1}-r_n(1+T_{n+1}) = 1+T_{n+1}-T_n$, so that $T_{n+1}-T_n$
$= (1-r_n)(1+T_{n+1})-1$. Similarly, $T_{n+2}-T_{n+1}$
$= (1-r_{n+1})(1+T_{n+2})-1$. Thus, $(T_{n+2}-T_{n+1})(1-r_n)(1-r_{n+1})$
$- \; (T_{n+2}-T_{n+1})(1-r_n) + (T_{n+1}-T_n)(1-r_{n+1})$
$= (T_{n+2}-T_{n+1})(1-r_n)(1-r_{n+1}) - (1-r_n)[(1-r_{n+1})(1+T_{n+2})-1]$
$+ \; (1-r_{n+1})[(1-r_n)(1+T_{n+1})-1] = (T_{n+2}-T_{n+1})(1-r_n)(1-r_{n+1})$
$+ \; (1-r_n) - (1-r_n)(1-r_{n+1})(1+T_{n+2}) - (1-r_{n+1})$
$+ \; (1-r_n)(1-r_{n+1})(1+T_{n+1}) = (1-r_n)(1-r_{n+1})[(T_{n+2}-T_{n+1})$
$- \; (1+T_{n+2}) + (1+T_{n+1})] + r_{n+1}-r_n = r_{n+1}-r_n$. Q.E.D.

Lemma 3.10. If Σa_n is a convergent series and n is a positive integer such that $(1-r_n)(1-r_{n+1})a_{n+1} \neq 0$, then
$a_{\delta n}/a_n = (T_{n+2}-T_{n+1}) - (T_{n+2}-T_{n+1})/(1-r_{n+1})$
$+ \; (T_{n+1}-T_n)/(1-r_n)$.

<u>Proof</u>: We have $a_{n-1}a_n a_{n+1} \neq 0$, and $a_{\delta n}/a_n$ $= (r_{n+1}-r_n)/(1-r_n)(1-r_{n+1})$ according to Lemma 3.3. Now apply Lemma 3.9. Q.E.D.

<u>Lemma 3.11</u>. If $\Sigma a_{\delta n}$ ε $MR(\Sigma a_n)$ and $0 < B \underline{<}. |1-r_n|$ for some number B, then $a_{\delta n}/a_n \to 0$.

<u>Proof</u>: From Theorem 3.8, $T_{n+1}-T_n \to 0$. Using Lemma 3.10 and $0 < B \underline{<}. |1-r_n|$, it is obvious that $a_{\delta n}/a_n \to 0$. Q.E.D.

<u>Theorem 3.12</u>. Suppose that $\Sigma a_{\delta n}$ ε $MR(\Sigma a_n)$ and $0 < B \underline{<}. |1-r_n|$. Then $\Sigma a_{\alpha n}$ ε $MR(\Sigma a_n)$, where α_n $=. (1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$ or α_n $=. (1-r_{n-1})/(1-2r_n+r_{n-1}r_n)$.

<u>Proof</u>: From Lemma 3.11, $a_{\delta n}/a_n \to 0$. We now apply Theorem 3.4, if $\alpha_n =. (1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$; or Theorem 3.5, if $\alpha_n =. (1-r_{n-1})/(1-2r_n+r_{n-1}r_n)$. Q.E.D.

<u>Theorem 3.13</u>. If $\Sigma a_{\delta n}$ ε $MR(\Sigma a_n)$ and $|r_n| \underline{<}. B$ for some number B, then $r_{n+1}-r_n \to 0$.

<u>Proof</u>: From Theorem 3.8, Lemma 3.9, and $|r_n| \underline{<}. B$, it

is obvious that $r_{n+1}-r_n \to 0$. Q.E.D.

Theorem 3.14. Suppose that $|r_n| \leq. \rho < 1$ for some number ρ. Then a n.a.s.c. that $\Sigma a_{\delta n} \varepsilon$ MR(Σa_n) is that $r_{n+1}-r_n \to 0$.

Proof: Since $|r_n| \leq. \rho < 1$, Σa_n converges.

The necessity follows from Theorem 3.13.

For the sufficiency, let $\varepsilon' > 0$. Since $r_{n+1}-r_n \to 0$, $|r_{n+1}-r_n| \leq. \varepsilon'/(1-\rho)^2$. With $\varepsilon = \varepsilon'/(1-\rho)^2$,

$$|T_{n+1}-T_n| =. |(r_{n+1}-r_n)+r_{n+1}(r_{n+2}-r_n)+r_{n+1}r_{n+2}(r_{n+3}-r_n) +\cdots+(r_{n+1}\cdots r_{n+k-1})(r_{n+k}-r_n)+\cdots|$$

$$\leq. |r_{n+1}-r_n|+|r_{n+1}||r_{n+2}-r_n|+\cdots+|r_{n+1}\cdots r_{n+k-1}||r_{n+k}-r_n|+\cdots$$

$$\leq. \varepsilon + 2\varepsilon|r_{n+1}|+\cdots+ k\varepsilon|r_{n+1}\cdots r_{n+k-1}|+\cdots$$

$$\leq. \varepsilon\,[1+2\rho+3\rho^2+\cdots+k\rho^{k-1}+\cdots] = \varepsilon/(1-\rho^2) = \varepsilon'.$$

Hence $T_{n+1}-T_n \to 0$, and thus, from Theorem 3.8, $\Sigma a_{\delta n}\ \varepsilon$ MR(Σa_n). Q.E.D.

Corollary 3.15. Suppose that $|r_n| \leq. \rho < 1$ for some number ρ, and $\Sigma a_{\delta n}\ \varepsilon$ MR(Σa_n). Suppose, in addition, that q is an integer and $a_n' = a_n z^{n+q}$ for every n. Then $\Sigma a_{\delta n}'\ \varepsilon$ MR$(\Sigma a_n')$, for each complex number z

satisfying $0 < |z| < 1/\rho$.

Proof: From Theorem 3.14, $r_{n+1} - r_n \to 0$. Let z be any complex number such that $0 < |z| < 1/\rho$. Then $|r'_n| \doteq |r_n z| \mathrel{\dot{\leq}} \rho|z| < 1$ and $r'_{n+1} - r'_n = r_{n+1}z - r_n z$ $= z(r_{n+1} - r_n) \to 0$. Thus $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$, according to Theorem 3.14. Q.E.D.

Corollary 3.16. Suppose that $|r_n| \mathrel{\dot{\leq}} \rho < 1$ for some number ρ, and $r_{n+1} - r_n \to 0$. Suppose, in addition, that q is an integer and $a'_n = a_n z^{n+q}$ for every n. Then $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$, for each complex number z satisfying $0 < |z| < 1/\rho$.

Proof: From Theorem 3.14, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. We now apply Corollary 3.15. Q.E.D.

Lemma 3.17. If $0 < A \mathrel{\dot{\leq}} |1 - r_n| \mathrel{\dot{\leq}} B$, then $a_{\delta n}/a_n$ $\doteq (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1})$, and $a_{\delta n}/a_n \to 0$ if and only if $r_{n+1} - r_n \to 0$.

Proof: Since $0 < A \mathrel{\dot{\leq}} |1 - r_n| \mathrel{\dot{\leq}} B$, $0 < A^2 \mathrel{\dot{\leq}} |(1 - r_n)(1 - r_{n+1})| \leq B^2$. Hence from Lemma 3.3, $a_{\delta n}/a_n \doteq (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1})$. Thus, from

$0 < A^2 \leq. |(1-r_n)(1-r_{n+1})| \leq. B^2$, $a_{\delta n}/a_n \to 0$ if and only if $r_{n+1}-r_n \to 0$. Q.E.D.

<u>Lemma 3.18</u>. If $|r_n| \leq. \rho < 1$, then $a_{\delta n}/a_n =. (r_{n+1}-r_n)/(1-r_n)(1-r_{n+1})$, and $a_{\delta n}/a_n \to 0$ if and only if $r_{n+1}-r_n \to 0$.

<u>Proof</u>: From $|r_n| \leq. \rho < 1$, $0 < 1-\rho \leq. |1-r_n| \leq. 2$. We now apply Lemma 3.17. Q.E.D.

<u>Theorem 3.19</u>. Suppose that $|r_n| \leq. \rho < 1$. Then $\Sigma a_{\delta n}$ ε $MR(\Sigma a_n)$ if and only if $a_{\delta n}/a_n \to 0$.

<u>Proof</u>: Lemma 3.18, $a_{\delta n}/a_n \to 0$ if and only if $r_{n+1}-r_n \to 0$. From Theorem 3.14, $\Sigma a_{\delta n}$ ε $MR(\Sigma a_n)$ if and only if $r_{n+1}-r_n \to 0$. Consequently, $\Sigma a_{\delta n}$ ε $MR(\Sigma a_n)$ if and only if $a_{\delta n}/a_n \to 0$. Q.E.D.

<u>Theorem 3.20</u>. If $|r_n| \leq. \rho < 1$ and $a_{\delta n}/a_n \to 0$, then $\Sigma a_{\alpha n}$ ε $MR(\Sigma a_n)$, where $\alpha_n =. (1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$ or $\alpha_n =. (1-r_{n-1})/(1-2r_n+r_{n-1}r_n)$.

<u>Proof</u>: From Theorem 3.19, $\Sigma a_{\delta n}$ ε $MR(\Sigma a_n)$. From Theorem 3.4, $\Sigma a_{\alpha n}$ ε $MR(\Sigma a_n)$ if $\alpha_n =. (1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$.

If $\alpha_n =. (1-r_{n-1})/(1-2r_n+r_{n-1}\, r_n)$, we may apply Theorem 3.5 to obtain $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$. Q.E.D.

Theorem 3.21. If $|r_n| \leq. \rho < 1$ and $r_{n+1}-r_n \to 0$, then $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$, where $\alpha_n =. (1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$ or $\alpha_n =. (1-r_{n-1})/(1-2r_n+r_{n-1}r_n)$.

Proof: From Lemma 3.18, $a_{\delta n}/a_n \to 0$. We now apply Theorem 3.20. Q.E.D.

CHAPTER IV

RAPIDITY OF CONVERGENCE AND VARIOUS METHODS FOR ACCELERATING CONVERGENCE. A VACUOUS THEOREM

In this chapter, both real and complex series will be considered. Various methods for accelerating convergence will be treated. That part of Lubkin's Theorem 6 (17, p. 231) concerning acceleration will be shown to have no application if $r_n \to 1$. That part of his Theorem 7 (17, p. 232) concerning acceleration will be proven to be vacuous.

If α, β are real numbers and $0 \leq \beta < \pi/2$, the notation $\langle\alpha,\beta\rangle$ will be used to denote the set of complex numbers z such that $|\arg z - \alpha| \leq \beta$ for some $\arg z$. Thus $\langle\alpha,\beta\rangle$ is the infinite sector in the complex plane, subtending the angle 2β and bisected by the ray $\theta = \alpha$. If $\beta = 0$, $\langle\alpha,\beta\rangle$ degenerates to the ray $\theta = \alpha$.

The following theorem appears to be the only one of general character, concerning rapidity of convergence, which is found in Knopp (15, p. 279-280).

<u>Theorem 4.1</u>. Suppose that Σa_n and Σb_n are convergent series of positive terms. Then Σa_n converges more rapidly than Σb_n if $a_n/b_n \to 0$.

According to Counterexample 2.20, Theorem 4.1 fails to hold for arbitrary convergent complex series Σa_n, Σb_n.

The converse of Theorem 4.1 is false. That is, if Σa_n and Σb_n are series of positive terms, and Σa_n converges more rapidly than Σb_n, then it is not necessarily true that $a_n/b_n \to 0$. This is made obvious by the following theorem.

Theorem 4.2. Suppose that Σa_n and Σb_n are series of positive terms, and that Σa_n converges more rapidly than Σb_n. Then $a_0+a_0+a_1+a_1+\cdots+a_n+a_n+\cdots$ converges more rapidly than $a_0+b_0+a_1+b_1+\cdots+a_n+b_n+\cdots$.

Proof: We have

$$\frac{a_n+a_n+a_{n+1}+a_{n+1}+\cdots}{a_n+b_n+a_{n+1}+b_{n+1}+\cdots} \doteq \frac{2(a_n+a_{n+1}+\cdots)/(b_n+b_{n+1}+\cdots)}{(a_n+a_{n+1}+\cdots)/(b_n+b_{n+1}+\cdots)+1} \to 0$$

as $n \to \infty$, and

$$\frac{a_n+a_{n+1}+a_{n+1}+a_{n+2}+a_{n+2}+\cdots}{b_n+a_{n+1}+b_{n+1}+a_{n+2}+b_{n+2}+\cdots} \lessdot \frac{2(a_n+a_{n+1}+\cdots)}{(a_{n+1}+a_{n+2}+\cdots)+(b_n+b_{n+1}+\cdots)}$$

$$\doteq \frac{2(a_n+a_{n+1}+\cdots)/(b_n+b_{n+1}+\cdots)}{(a_{n+1}+a_{n+2}+\cdots)/(b_n+b_{n+1}+\cdots)+1} \to 0$$

as $n \to \infty$. Q.E.D.

As previously noted, Theorem 4.2 shows that the converse of Theorem 4.1 is false; however, we do have the following theorem.

<u>Theorem 4.3</u>. Suppose that Σa_n and Σb_n are convergent series of positive terms. Then $a_n/b_n \to 0$ if, and only if $\Sigma a_{n'}$ converges more rapidly than $\Sigma b_{n'}$, for each subsequence $\{n'\}$ of $\{n\}$.

<u>Proof</u>: If $a_n/b_n \to 0$ and $\{n'\}$ is any subsequence of $\{n\}$, then $a_{n'}/b_{n'} \to 0$ and, according to Theorem 4.1, $\Sigma a_{n'}$ converges more rapidly than $\Sigma b_{n'}$.

Assume that $a_n/b_n \not\to 0$. Then there is an $\varepsilon > 0$ and a subsequence $\{n'\}$ of $\{n\}$ such that $a_{n'}/b_{n'} \geq. \varepsilon$. Consequently, $\sum_{k=n}^{\infty} a_{k'} \geq. \varepsilon \sum_{k=n}^{\infty} b_{k'}$, and thus $\Sigma a_{n'}$ does not converge more rapidly than $\Sigma b_{n'}$. Q.E.D.

<u>Lemma 4.4</u>. If Σa_n is a convergent complex series such that $a_n \ \varepsilon.\ \langle\alpha,\beta\rangle$ for some set $\langle\alpha,\beta\rangle$, then $\sum_{k=n}^{\infty} |a_k| \leq. |\sum_{k=n}^{\infty} a_k|/\cos\beta$.

<u>Proof</u>: We may assume that $\alpha = 0$, since with $b_n = a_n e^{-i\alpha}$ for $n \geq 0$, we have $b_n \ \varepsilon.\ \langle 0,\beta\rangle$, $|\sum_{k=n}^{\infty} a_k| =. |\sum_{k=n}^{\infty} b_k|$, and $\sum_{k=n}^{\infty} |a_k| =. \sum_{k=n}^{\infty} |b_k|$. Since $a_n \ \varepsilon.\ \langle 0,\beta\rangle$, we may

set $a_n = |a_n| e^{i\theta_n}$ where $|\theta_n| \leq. \beta < \pi/2$. Thus, $\cos\theta_n \geq. \cos\beta$ and $|\sum_{k=n}^{\infty} a_k| =. |\sum_{k=n}^{\infty} |a_k| \cos\theta_k + i \sum_{k=n}^{\infty} |a_k| \sin\theta_k| \geq. |\sum_{k=n}^{\infty} |a_k| \cos\theta_k| =. \sum_{k=n}^{\infty} |a_k| \cos\theta_k \geq. \sum_{k=n}^{\infty} |a_k| \cos\beta =. (\cos\beta) \sum_{k=n}^{\infty} |a_k|$. Q.E.D.

Theorem 4.5. Suppose that Σa_n, Σb_n are complex series such that Σa_n converges and $a_n \varepsilon. \langle\alpha,\beta\rangle$ for some set $\langle\alpha,\beta\rangle$. Then $b_n/a_n \to 0$ if and only if $\Sigma b_{n'}$ converges more rapidly than $\Sigma a_{n'}$ for every subsequence $\{n'\}$ of $\{n\}$.

Proof: If $a_n =: 0$, then $a_{n'} =. 0$ for some subsequence $\{n'\}$ of $\{n\}$, and both conditions in the conclusion of our theorem fail to hold. Thus we may assume that $a_n \neq. 0$.

Suppose that $b_n/a_n \to 0$, $\varepsilon > 0$, and $\{n'\}$ is any subsequence of $\{n\}$. Then $|b_{n'}| \leq. \varepsilon |a_{n'}| \cos\beta$, and $\Sigma |b_{n'}|$, $\Sigma |a_{n'}|$ both converge, since $\Sigma |a_n|$ converges according to Lemma 4.4. Hence, $|\sum_{k=n}^{\infty} b_{k'}| \leq. \sum_{k=n}^{\infty} |b_{k'}| \leq. (\varepsilon\cos\beta) \sum_{k=n}^{\infty} |a_{k'}| \leq. \varepsilon |\sum_{k=n}^{\infty} a_{k'}|$, the last inequality following from Lemma 4.4. Thus $\Sigma b_{n'}$ converges more rapidly than $\Sigma a_{n'}$.

Suppose that $b_n/a_n \not\to 0$. Then there is an $\varepsilon > 0$ and a subsequence $\{n'\}$ of $\{n\}$ such that $|b_{n'}| \geq. \varepsilon |a_{n'}|$. Since $b_{n'}$ ε: $<\alpha', \pi/4>$ for some real α', there is a subsequence $\{n^*\}$ of $\{n'\}$ such that b_{n*} $\varepsilon.$ $<\alpha', \pi/4>$ and $|b_{n*}| \geq. \varepsilon |a_{n*}|$. If Σb_{n*} does not converge, there is nothing to prove. Hence, assume that Σb_{n*} converges. From $|b_{n*}| \geq. \varepsilon |a_{n*}|$ and Lemma 4.4,

$$\left|\sum_{k=n}^{\infty} b_{k*}\right| \geq. (\cos \pi/4) \sum_{k=n}^{\infty} |b_{k*}| \geq. (\varepsilon \cos \pi/4) \sum_{k=n}^{\infty} |a_{k*}|$$

$\geq. (\varepsilon \cos \pi/4)\left|\sum_{k=n}^{\infty} a_{k*}\right|$, and thus Σb_{n*} does not converge more rapidly than Σa_{n*}. Q.E.D.

<u>Corollary 4.6</u>. Suppose that Σa_n is a convergent series such that a_n $\varepsilon.$ $<\alpha,\beta>$ for some set $<\alpha,\beta>$. Then a n.a.s.c. that $\Sigma a_{\delta n'}$ converge more rapidly than $\Sigma a_{n'}$ for each subsequence $\{n'\}$ of $\{n\}$, is that $a_{\delta n}/a_n \to 0$.

<u>Proof</u>: Set $a_{\delta n} = b_n$ and apply Theorem 4.5. Q.E.D.

Theorem 4.7. Suppose that Σa_n is a convergent real series such that $r_n \leq. r_{n+1}$ and $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. Suppose, in addition, that q is an integer and $a'_n = a_n z^{n+q}$ for every n. Then $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$ for each complex number z satisfying $0 < |z| \leq 1$.

Proof: Let $0 < |z| \leq 1$. Since Σa_n converges and $r_n \leq. r_{n+1}$, $r_n \to r$ where $-1 < r \leq 1$. If $r < 1$, then $|r_n| \leq. \rho < 1$ for some number ρ, and $0 < |z| < 1/\rho$. Since $r_{n+1} - r_n \to 0$, Corollary 3.16 implies $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$. Suppose that $r = 1$. We note that $0 <. r_n$, so that $0 <. a_n$ or $a_n <. 0$. In either case, $\Sigma|a_n|$ converges. Also, $|r'_n| =. |r_n z| \leq. |r_n|$, and thus $\Sigma a'_n$ converges absolutely. In view of Theorem 3.8, $T_{n+1}-T_n \to 0$. Since $r'_n =. r_n z$, $T'_n = r'_n + r'_n r'_{n+1} + \cdots$
$+ (r'_n \cdots r'_{n+k}) + \cdots = r_n z + r_n r_{n+1} z^2 + \cdots + (r_n \cdots r_{n+k}) z^{k+1} + \cdots$.
Thus, $|T'_{n+1} - T'_n| =. |(r_{n+1}-r_n)z + r_{n+1}(r_{n+2}-r_n)z^2 + \cdots$
$+ (r_{n+1} \cdots r_{n+k-1})(r_{n+k}-r_n)z^k + \cdots|$
$\leq. |r_{n+1}-r_n| + |r_{n+1}(r_{n+2}-r_n)| + \cdots + |(r_{n+1} \cdots r_{n+k-1})(r_{n+k}-r_n)|$
$+ \cdots$
$=. (r_{n+1}-r_n) + r_{n+1}(r_{n+2}-r_n) + \cdots + (r_{n+1} \cdots r_{n+k-1})(r_{n+k}-r_n) + \cdots$

$=. \quad T_{n+1} - T_n \to 0$

as $n \to \infty$. Hence $T'_{n+1} - T'_n \to 0$, and thus $\Sigma a'_{\delta n} \; \varepsilon \; MR(\Sigma a'_n)$ according to Theorem 3.8. Q.E.D.

Theorem 4.8. If Σa_n is a real series, $0 <. \; r_n$, and $\Sigma a_{\delta n} \; \varepsilon \; MR(\Sigma a_n)$, then $r_n <. \; 1$ and $0 <. \; Q_n$.

Proof: Since $0 <. \; r_n$, $T_n >. \; 0$. From Theorem 3.6, $\delta_n =. \; 1/(1-r_n) \sim T_n/r_n >. \; 0$, so that $1 - r_n >. \; 0$. Thus, $r_n <. \; 1$ and $0 <. \; n(1-r_n) = Q_n$. Q.E.D.

Lemma 4.9. Suppose that Σa_n is a real convergent series such that $a_{\delta n}/a_n \to 0$ and $0 \leq. \; r_n$. Then $r_n <. \; 1$, $r_{n+1} - r_n \to 0$, and $\Sigma a_{\delta n} \; \varepsilon \; MR(\Sigma a_n)$.

Proof: Since $0 \leq. \; r_n$, $a_n \; \varepsilon. \; \langle 0,0 \rangle$ or $a_n \; \varepsilon. \; \langle \pi,0 \rangle$. From Corollary 4.6 and Theorem 2.6, $\Sigma a_{\delta n} \; \varepsilon \; MR(\Sigma a_n)$ since $a_{\delta n}/a_n \to 0$. Thus, according to Theorem 4.8, $r_n <. \; 1$, so that $|r_n| \leq. \; 1$. Hence $r_{n+1} - r_n \to 0$ in view of Theorem 3.13. Q.E.D.

Theorem 4.10. Suppose that Σa_n is a convergent real series such that $r_n \leq. \; r_{n+1}$ and $a_{\delta n}/a_n \to 0$. Suppose, in addition, that q is an integer and $a'_n = a_n z^{n+q}$ for every n. Then $\Sigma a'_{\delta n} \varepsilon MR(\Sigma a'_n)$ for every complex number z such that $0 < |z| \leq 1$.

Proof: Since Σa_n converges, $r_n \to r$ where $-1 < r \leq 1$. If $r < 1$, we may complete the proof in the same manner as in the proof of Theorem 4.7. If $r = 1$, then $0 \leq. r_n$, and $\Sigma a_{\delta n} \varepsilon MR(\Sigma a_n)$ according to Lemma 4.9. We may now apply Theorem 4.7 to complete the proof. Q.E.D.

Theorem 4.11. Suppose that Σa_n is a convergent series such that $a_n \varepsilon. \langle\alpha,\beta\rangle$ for some set $\langle\alpha,\beta\rangle$. Then a n.a.s.c. that $\Sigma a_{\delta n'}$ converge more rapidly than $\Sigma a_{n'}$, for each subsequence $\{n'\}$ of $\{n\}$, is that $(r_{n+1}-r_n)/(1-r_n)(1-r_{n+1}) \to 0$.

Proof: For the sufficiency, $\delta_n =. 1/(1-r_n)$ since $(r_{n+1}-r_n)/(1-r_n)(1-r_{n+1})$ exists for large n. Thus $a_{\delta n}/a_n =. (r_{n+1}-r_n)/(1-r_n)(1-r_{n+1}) \to 0$. From Corollary 4.6, $\Sigma a_{\delta n'}$ converges more rapidly than $\Sigma a_{n'}$ for each subsequence $\{n'\}$ of $\{n\}$.

For the necessity, $\delta_n \neq. 0$; since if $\delta_n =: 0$, then $S_{\delta n} =: S_n$, and thus, $\Sigma a_{\delta n}$ does not converge more rapidly than Σa_n, a contradiction. Hence,

$\delta_n \doteq 1/(1-r_n)$ and, from Corollary 4.6, $(r_{n+1}-r_n)/(1-r_n)(1-r_{n+1}) \doteq a_{\delta n}/a_n \to 0$. Q.E.D.

Theorem 4.12. If Σa_n is a real series such that $r = 1$ and $|n(n+1)(r_{n+1}-r_n)| \,\underline{\lessdot}\, 1$, then Σa_n diverges.

Proof: By hypothesis, $1-r_n \to 0$ and $|r_{n+1}-r_n| \,\underline{\lessdot}\, 1/n(n+1)$. Thus,

$$1-r_n \doteq \sum_{k=n}^{\infty} [(1-r_k)-(1-r_{k+1})] \,\underline{\lessdot}\, \sum_{k=n}^{\infty} |r_{k+1}-r_k|$$

$$\underline{\lessdot}\, \sum_{k=n}^{\infty} 1/k(k+1) \doteq 1/n,$$

from which $1-1/n \,\underline{\lessdot}\, r_n$. Since $\Sigma a_n'$, $a_n' \doteq 1/n$, diverges and $r_n' \doteq (n-1)/n \doteq 1-1/n \,\underline{\lessdot}\, r_n$, Σa_n must diverge. Q.E.D.

Corollary 4.13. If Σa_n is a real series such that $r = 1$ and $n^2(r_{n+1}-r_n) \to 0$, then Σa_n diverges.

Proof: Since $n^2(r_{n+1}-r_n) \to 0$, $n(n+1)(r_{n+1}-r_n) \to 0$ so that $|n(n+1)(r_{n+1}-r_n)| \,\underline{\lessdot}\, 1$. We now apply Theorem 4.12. Q.E.D.

Lubkin (17, p. 231-232) has proven the following two theorems.

Theorem 6. If Σa_n is a convergent real series, $r_n \geq 0$, $Q_n \geq K > 0$, and $n^2(r_{n+1}-r_n) \to 0$, as $n \to \infty$, then $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

Theorem 7. If Σa_n is a convergent real series, Q exists (as a finite limit), and $n^2(r_{n+1}-r_n) \to 0$, then $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

If Σa_n is a real series such that $\{n^2(r_{n+1}-r_n)\}$ is bounded, then $\Sigma|r_{n+1}-r_n|$ converges since $|r_{n+1}-r_n| \leq B/n^2$ for some number B. Thus $\Sigma(r_{n+1}-r_n)$ converges, from which $r_n \to r$ for some number r. In view of Corollary 4.13, it is now evident that $0 \leq r < 1$, if the hypothesis of Theorem 6 is satisfied. Consequently if $r = 1$, the hypothesis of Theorem 6 cannot be satisfied. On the other hand, $r = 1$ if Q exists. Hence, according to Corollary 4.13, the hypothesis of Theorem 7 can never be fulfilled.

Theorem 4.14.

(1) If Re $Q_n \to Q'$ and Re $n^2(r_{n+1}-r_n) \to P'$, then $P' = Q'$.

(2) If Im $Q_n \to Q''$ and Im $n^2(r_{n+1}-r_n) \to P''$, then $P'' = Q''$.

(3) If $Q_n \to Q$ and $n^2(r_{n+1}-r_n) \to P$, then $P = Q$.

Proof: We first note that $Q_{n+1}-Q_n \doteq (n+1)(1-r_{n+1})-n(1-r_n)$ $\doteq n(1-r_{n+1})+(1-r_{n+1})-n(1-r_n) = (1-r_{n+1})-n(r_{n+1}-r_n)$ and

$$n(Q_{n+1}-Q_n)\doteq n(1-r_{n+1})-n^2(r_{n+1}-r_n)\doteq(n+1)(1-r_{n+1})-(1-r_{n+1})$$
$$-n^2(r_{n+1}-r_n).$$

Assume that $P' \neq Q'$. Set $Q'_n \doteq \operatorname{Re} Q_n$. Since $\operatorname{Re} n(1-r_n) \to Q'$, $\operatorname{Re}(1-r_n) \to 0$. Thus, $n(Q'_{n+1}-Q'_n)$ $\doteq Q'_{n+1}-\operatorname{Re}(1-r_{n+1})-\operatorname{Re} n^2(r_{n+1}-r_n) \to Q'-0-P'=Q'-P'\neq 0$. Let $L = (Q'-P')/2$. If $L > 0$, then $n \Delta Q'_n \mathrel{\underline{\dot{>}}} L$. Hence there is a positive integer m such that $Q'_{m+n} \doteq Q'_m + \Delta Q'_m + \Delta Q'_{m+1} + \cdots + \Delta Q'_{m+n-1} \to +\infty$, so that $Q'_n \to +\infty$, a contradiction. If $L < 0$, then $n \Delta Q'_n \mathrel{\underline{\dot{<}}} L$. Hence there is a positive integer m such that $Q'_{m+n} \doteq Q'_m + \Delta Q'_m + \cdots + \Delta Q'_{m+n-1} \to -\infty$, so that $Q'_n \to -\infty$, a contradiction. Thus we must have $P' = Q'$. This proves (1). The proof of (2) follows in a similar manner, and (3) is an immediate consequence of (1) and (2). Q.E.D.

Theorem 4.14 again shows that the hypothesis of Lubkin's Theorem 7, previously mentioned, can never be fulfilled, since we would have $Q = 0$ and Σa_n would diverge.

Theorem 4.15. If $0 < K \mathrel{\underline{\dot{<}}} \operatorname{Re} Q_n$ and $\operatorname{Re}[n^2(r_{n+1}-r_n)] \to 0$, then $\operatorname{Re} Q_n \mathrel{\dot{<}} \operatorname{Re} Q_{n+1}$ and $\operatorname{Re} Q_n \to +\infty$.

Proof: Since $\operatorname{Re} n^2(r_{n+1}-r_n) \to 0$, $\operatorname{Re} n(n+1)(r_{n+1}-r_n) \to 0$. Also, $(n+1)(Q_n-Q_{n+1}) \doteq -Q_{n+1} + (n+1)\,Q_n - nQ_{n+1}$ $\doteq -Q_{n+1} + n(n+1)(r_{n+1}-r_n)$. Thus, with $Q'_n \doteq \operatorname{Re} Q_n$, $(n+1)(Q'_n-Q'_{n+1}) \doteq -Q'_{n+1} + \operatorname{Re} n(n+1)(r_{n+1}-r_n) \mathrel{\dot{\le}} -K$ $+ \operatorname{Re} n(n+1)(r_{n+1}-r_n) \mathrel{\dot{<}} 0$ from which $Q'_n \mathrel{\dot{<}} Q'_{n+1}$. Hence, $Q'_n \to Q'$ where $K < Q' \le +\infty$. If $Q' < +\infty$, $Q' = 0$ according to (1) of Theorem 4.14; this is a contradiction. Thus, $Q' = +\infty$. Q.E.D.

Theorem 4.16. Suppose that Σa_n is a convergent series such that (1) $a_n \mathrel{\dot{\varepsilon}} \langle\alpha,\beta\rangle$ for some set $\langle\alpha,\beta\rangle$ and (2) $Q_n \to \infty$. Suppose further that $\{P_n\}$ is a sequence such that (3) $P_n/Q_{n+1} \to 0$ and (4) $n|Q_{n+1}-Q_n| \mathrel{\dot{\le}} |P_nQ_n|$. Then $a_{\delta n}/a_n \to 0$ and $\Sigma a_{\delta n} \;\varepsilon\; MR(\Sigma a_n)$.

Proof: From (2), $\delta_n \doteq 1/(1-r_n)$ and $a_{\delta n}/a_n$ $\doteq n(Q_n-Q_{n+1})/Q_nQ_{n+1} + 1/Q_{n+1}$. From (2), $1/Q_n \to 0$. From (3) and (4), $|n(Q_n-Q_{n+1})/Q_nQ_{n+1}| \mathrel{\dot{\le}} |P_nQ_n/Q_nQ_{n+1}|$ $\doteq |P_n/Q_{n+1}| \to 0$. Thus $a_{\delta n}/a_n \to 0$. Hence $\Sigma a_{\delta n} \;\varepsilon\; MR(\Sigma a_n)$ according to Corollary 4.6 and Theorem 2.6. Q.E.D.

Theorem 4.17. Suppose that Σa_n is a real series such that $-1 <. r_n \leq. r_{n+1}$, $Q_n \leq. Q_{n+1}$, and $Q_n \to +\infty$. Then $a_{\delta n}/a_n \to 0$ and $\Sigma a_{\delta n} \varepsilon\ MR(\Sigma a_n)$.

Proof: Since $Q_n =. n(1-r_n) \to +\infty$, $r_n <. 1$. Hence $-1 <. r_n \leq. r_{n+1} <. 1$ and thus $r_n \to r$ where $-1 < r \leq 1$. If $r < 1$, it is obvious that $|r_n| \leq. \rho < 1$ for some number ρ. Also $r_{n+1}-r_n \to 0$. Thus from Theorem 3.14, $\Sigma a_{\delta n} \varepsilon\ MR(\Sigma a_n)$. Suppose that $r = 1$. Then $0 <. r_n \leq. r_{n+1} <. 1$, and $a_n \varepsilon. \langle 0,0\rangle$ or $a_n \varepsilon. \langle \pi,0\rangle$. Also, $a_{\delta n}/a_n =. 1/(1-r_{n+1}) - 1/(1-r_n) \geq. 0$ and $0 \leq. a_{\delta n}/a_n =. n(Q_n-Q_{n+1})/Q_nQ_{n+1} + 1/Q_{n+1}$. Hence, with $P_n =. 1$, we have $0 \leq. n(Q_{n+1}-Q_n)/Q_nQ_{n+1} \leq. 1/Q_{n+1}$, $n|Q_{n+1}-Q_n| \leq. |P_nQ_n|$, and $P_n/Q_{n+1} \to 0$. Since $Q_n \to +\infty$, Σa_n converges. Thus, from Theorem 4.16, $a_{\delta n}/a_n \to 0$ and $\Sigma a_{\delta n} \varepsilon\ MR(\Sigma a_n)$. Q.E.D.

As previously noted, Lubkin's Theorem 6 is not applicable if $r_n \to r = 1$, and his Theorem 7, in which $r_n \to 1$, is vacuous. This is not the case with Theorem 4.17. In particular, if $Q_n =. an^p$ where $a > 0$ and $0 < p < 1$, it can be verified that $r_n \to 1$ and Theorem

4.17 is applicable. The same is true with $Q_n =. an/(\ell n\ n)^p$ where $a > 0$ and $p > 0$. Moreover, the proof of Theorem 4.17 shows that the theorem itself is a special case of Theorem 4.16. Consequently, Theorem 4.16 is also applicable with $r_n \to 1$.

Theorem 4.18. If Σa_n is a complex series such that $\Sigma a_{\alpha n}$, $\alpha_n =. n/(Q_n-1)$, and $\Sigma a_{\delta n}$ both converge more rapidly to S than Σa_n, then $Q_n \to \infty$.

Proof: From Theorem 3.2, $\alpha_n \sim \delta_n$, i.e., $n/(Q_n-1) \sim n/Q_n$. Hence, $(Q_n-1)/Q_n =. 1-1/Q_n \to 1$, and thus $Q_n \to \infty$. Q.E.D.

Theorem 4.19. Suppose that Σa_n is a complex series such that $Q_n \to \infty$. Then $\Sigma a_{\alpha n}$, $\alpha_n =. n/(Q_n-1)$, converges more rapidly to S than Σa_n if and only if $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

Proof: Since $Q_n \to \infty$, $\delta_n/\alpha_n =. [n/Q_n][(Q_n-1)/n]$ $=. 1-1/Q_n \to 1$, i.e., $\delta_n \sim \alpha_n$. Thus, from Theorem 3.2, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ if and only if $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. Q.E.D.

Theorem 4.20. Suppose that Σa_n is a real series such that $-1 <. r_n \leq. r_{n+1}$, $Q_n \leq. Q_{n+1}$, and $Q_n \to +\infty$. Suppose, in addition, that q is an integer and $a'_n = a_n z^{n+q}$ for every n. Then for each complex number z satisfying $0 < | z | \leq 1$, $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$ and $\Sigma a'_{\alpha n} \in MR(\Sigma a'_n)$, where $\alpha_n \doteq (1-r'_{n-1})/(1-2r'_n+r'_{n-1}r'_n)$ or $\alpha_n \doteq n/(Q'_n-1)$.

<u>Proof</u>: From Theorem 4.17, $a_{\delta n}/a_n \to 0$ and $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. Let z be any complex number such that $0 < | z | \leq 1$. From Theorem 4.7, $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$.

Suppose $\alpha_n \doteq (1-r'_{n-1})/(1-2r'_n-r'_{n-1}r'_n)$. If $z = 1$, $a'_{\delta n}/a'_n \doteq a_{\delta n}/a_n \to 0$. If $z \neq 1$, $a'_{\delta n}/a'_n$ $\doteq (r'_{n+1}-r'_n)/(1-r'_n)(1-r'_{n+1}) \doteq (zr_{n+1}-zr_n)/(1-zr_n)(1-zr_{n+1})$ $\to 0/(1-zr)(1-zr) = 0$, since $r_n \to r$ where $-1 < r \leq 1$. In either case, Theorem 3.5 implies $\Sigma a'_{\alpha n} \in MR(\Sigma a'_n)$.

Suppose that $\alpha_n \doteq n/(Q'_n-1)$. Then $Q'_n \doteq n(1-r'_n)$ $\doteq n(1-zr_n) \to \infty$. From Theorem 4.19 and $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$, $\Sigma a'_{\alpha n} \in MR(\Sigma a'_n)$. Q.E.D.

<u>Lemma 4.21</u>. If Σa_n is a complex series such that $Q_n \to Q$ where $\text{Re } Q > 1$, then $n(1-|r_n|) \to \text{Re } Q$, Σa_n converges absolutely, $na_n \to 0$, and $\Sigma a_{\alpha n} = S$ where $\alpha_n \doteq n/(Q-1)$.

<u>Proof</u>: Let a, b be any numbers satisfying $1 < a < \text{Re } Q < b$. Geometrically, it can be seen that $|n-b| \leq. |n-Q_n| \leq. |n-a|$ so that $1-b/n \leq. |1-Q_n/n| \doteq |r_n| \leq. 1-a/n$, and thus

$a \leq. n(1-|r_n|) \leq. b$ and $|\mathrm{Re}\, Q - n(1-|r_n|)| \leq. |b-a|$. With $|b-a| > 0$ taken arbitrarily small, we thus conclude that $n(1-|r_n|) \to \mathrm{Re}\, Q$. Since $|r_n| \leq. 1-a/n$, Σa_n converges absolutely. Since $|r_n| \leq. 1$ and $\Sigma |a_n|$ converges, $n|a_n| \to 0$, i.e., $na_n \to 0$ (15, p. 124). Consequently, $S_{\alpha n} =. S_n - a_{n+1}\alpha_{n+1} =. S_n - a_{n+1}(n+1)/(Q-1) \to S$, i.e., $\Sigma a_{\alpha n} = S$. Q.E D.

<u>Theorem 4.22</u>. If Σa_n is a complex series such that $a_n \,\varepsilon.\, \langle \alpha', \beta \rangle$ for some set $\langle \alpha', \beta \rangle$ and $Q_n \to Q$ where $\mathrm{Re}\, Q > 1$, then $T_n/n \to 1/(Q-1)$ and $\Sigma a_{\alpha n} \,\varepsilon\, MR(\Sigma a_n)$ where $\alpha_n =. n/(Q-1)$.

<u>Proof</u>: From Lemma 4.21, $\Sigma a_{\alpha n} = S$. Also, $a_{\alpha n}/a_n$ $=. 1+r_{n+1}\alpha_{n+1}-\alpha_n =. 1+[1-Q_{n+1}/(n+1)][(n+1)/(Q-1)]-n/(Q-1)$ $=. 1+(n+1)/(Q-1)-Q_{n+1}/(Q-1)-n/(Q-1) = 1+1/(Q-1)-Q_{n+1}/(Q-1)$ $=. (Q-Q_{n+1})/(Q-1) \to 0$. Thus, from Theorem 4.5, $\Sigma a_{\alpha n}$ converges more rapidly than Σa_n. From Theorem 3.6, $n/(Q-1)$ $=. \alpha_n \sim T_n/r_n$, so that $T_n/n \sim r_n/(Q-1) \to 1/(Q-1)$ and $T_n/n \to 1/(Q-1)$. Q.E.D.

Szász (26, p. 274) has proven Theorem 4.22 in the following form for real series: If $u_n > 0$, $a > 1$, and

$u_n/u_{n-1} = 1-a/n + \gamma_{n-1}/n$ where $\gamma_n \to 0$, then the transform $t_n = s_n + (n+1)u_{n+1}/(a-1)$ converges more rapidly than $s_n = u_0+u_1+u_2+\ldots+u_n$, and $|s-t_n| < \bar{\gamma}_{n+1}(s-s_n)/(a-1)$ where $\bar{\gamma}_n = \max_{k\geq n} |\gamma_k|$. A slight error is evident here, since strict equality cannot hold if $\gamma_n \doteq 0$. We now generalize Theorem 4.22 by removing the condition $a_n \,\dot{\varepsilon}\, \langle\alpha',\beta\rangle$.

<u>Theorem 4.23</u>. If $Q_n \to Q$ where Re $Q > 1$, then $T_n/n \to 1/(Q-1)$, and $\Sigma a_{\alpha n} \,\varepsilon\, MR(\Sigma a_n)$ where $\alpha_n \doteq n/(Q-1)$.

<u>Proof</u>: We have $r_n \doteq 1-Q_n/n \doteq 1-Q/n-(Q_n-Q)/n$. Setting $\gamma_{n-1} \doteq Q_n-Q$, $r_n \doteq 1-Q/n-\gamma_{n-1}/n$ where $\gamma_n \to 0$. Hence, $na_n \doteq na_{n-1}-Qa_{n-1}-\gamma_{n-1}a_{n-1} \doteq (n-1)a_{n-1}+(1-Q)a_{n-1}-\gamma_{n-1}\,a_{n-1}$ and, replacing n by $n+1$, $(n+1)a_{n+1} \doteq na_n+(1-Q)a_n-\gamma_n a_n$. Consequently $na_n-(n+1)a_{n+1} \doteq (Q-1)a_n + \gamma_n a_n$. From Lemma 4.21, $na_n \to 0$ and Σa_n converges. Thus $na_n \doteq \sum_{k=n}^{\infty} [ka_k-(k+1)a_{k+1}] \doteq (Q-1) \sum_{k=n}^{\infty} a_k + \sum_{k=n}^{\infty} \gamma_k a_k$. From Lemma 4.21, $\Sigma|a_n|$ converges, so that $|na_n-(Q-1)\sum_{k=n}^{\infty} a_k| \doteq |\sum_{k=n}^{\infty} \gamma_k a_k| \mathrel{\dot{\leq}} \sum_{k=n}^{\infty} |\gamma_k a_k| \mathrel{\dot{\leq}} \bar{\gamma}_n \sum_{k=n}^{\infty} |a_k|$ where $\bar{\gamma}_n \doteq \max_{k\geq n} |\gamma_k| \to 0$. Dividing by $|na_{n-1}|$, $|r_n-(Q-1)T_n/n| \mathrel{\dot{\leq}} \bar{\gamma}_n \sum_{k=n}^{\infty} |a_k|/|na_{n-1}|$. Setting $a_n' \doteq |a_n|$,

$r'_n \doteq a'_n/a'_{n-1} \doteq |r_n|$, $Q'_n \doteq n(1-r'_n) \doteq n(1-|r_n|)$, and $T'_n \doteq \sum_{k=n}^{\infty} |a_k|/|a_{n-1}|$, we have $Q'_n \to Q' = \operatorname{Re}Q$ from Lemma 4.21, and $\sum_{k=n}^{\infty} |a_k|/|na_{n-1}| \doteq T'_n/n \to 1/(Q'-1)$ from Theorem 4.22. Thus, $|r_n-(Q-1)T_n/n| \mathrel{\dot{\leq}} \bar{\gamma}_n T'_n/n \to 0$, so that $(Q-1)T_n/n \to 1$ since $r_n \to 1$. Hence $T_n/n \to 1/(Q-1)$, and $n/(Q-1) \sim T_n \sim T_n/r_n$, i.e., $\alpha_n \sim T_n/r_n$. From Theorem 3.6, $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$. Q.E.D.

<u>Corollary 4.24</u>. If $Q_n \to Q$ where $\operatorname{Re} Q > 1$, then $T_{n+1}-T_n \to 1/(Q-1)$.

<u>Proof</u>: Using Theorem 4.23, $T_{n+1}-T_n \doteq T_{n+1}-r_n(1+T_{n+1})$ $\doteq (1-r_n)T_{n+1}-r_n \doteq Q_nT_{n+1}/n-r_n \to Q/(Q-1) - 1 = 1/(Q-1)$. Q.E.D.

Suppose that $Q_n \to Q$ where $\operatorname{Re} Q > 1$. Recalling that $\alpha_n = 1+T_{n+1}$, $n \geq 2$, yields the best transform for accelerating convergence, we are led quite naturally to the transform sequence 1.5 in the Introduction by Corollary 4.24 and the following estimate: $1+T_{n+1} \doteq 1/(1-r_n)$ $+ (T_{n+1}-T_n)/(1-r_n) \mathrel{\dot{\approx}} 1/(1-r_n) + [1/(Q-1)]/(1-r_n)$ $= Q/(Q-1)(1-r_n)$.

Theorem 4.25. Suppose that $Q_n \to Q$ where $\text{Re } Q > 1$. Then $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$ if and only if $\alpha_n/n \to 1/(Q-1)$.

Proof: From Theorem 4.23, $\Sigma a_{\beta n} \varepsilon MR(\Sigma a_n)$ where $\beta_n \doteq n/(Q-1)$. Thus, from Theorem 3.2, $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$ if and only if $\alpha_n \sim \beta_n$, i.e., $\alpha_n \sim n/(Q-1)$. But this is equivalent to $\alpha_n/n \to 1/(Q-1)$. Q.E.D.

Corollary 4.26. Suppose that $Q_n \to Q$ where $\text{Re } Q > 1$, and that $\alpha_n \doteq n/(Q_n-1)$. Then $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$.

Proof: We have $\alpha_n/n \doteq 1/(Q_n-1) \to 1/(Q-1)$. Thus, from Theorem 4.25, $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$. Q.E.D.

Theorem 4.27. Suppose that $Q_n \to Q$ where $\text{Re } Q > 1$, and $\alpha_n \doteq b\delta_n$ where b is any complex number. Then:

(1) $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$ if and only if $b = Q/(Q-1)$.

(2) $\Sigma a_{\alpha n}$ converges to S with the same rapidity as Σa_n if, and only if, $b \neq Q/(Q-1)$.

Proof: Part (1). From Theorem 4.25, $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$ if and only if $b\delta_n/n \to 1/(Q-1)$, i.e., $b/Q_n \to 1/(Q-1)$. But this is equivalent to $b/Q = 1/(Q-1)$, i.e., $b = Q/(Q-1)$.

Part (2). Suppose that $b \neq Q/(Q-1)$. From Lemma 4.21, Σa_n converges. From Theorem 4.23, $n/T_n \to Q-1$. Thus, since $r_n \to 1$, $(S-S_{\alpha(n-1)})/(S-S_{n-1})$ $\doteq (S-S_{n-1}-a_n\alpha_n)/(S-S_{n-1}) \doteq 1-r_n\alpha_n/T_n \doteq 1-br_n\delta_n/T_n$ $\doteq 1-(bnr_n)/(T_nQ_n) \to 1-b(Q-1)/Q \neq 0$. Consequently $\Sigma a_{\alpha n}$ converges to S with the same rapidity as Σa_n.

The converse follow from (1). Q.E.D.

<u>Corollary 4.28</u>. If $Q_n \to Q$ where $\mathrm{Re}\, Q > 1$, then $\Sigma a_{\delta n}$ converges to S with the same rapidity as Σa_n.

<u>Proof</u>: Setting $b=1$, we have $\delta_n \doteq b\delta_n$ and $b \neq Q/(Q-1)$. Now apply (2) of Theorem 4.27. Q.E.D.

<u>Corollary 4.29</u>. Suppose that Σa_n is a real series such that $-1 \mathrel{<\!\cdot} r_n \mathrel{\leq\!\cdot} r_{n+1}$ and $Q_n \mathrel{\leq\!\cdot} Q_{n+1}$. Then a n.a.s.c. that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ is that $Q_n \to +\infty$.

<u>Proof</u>: The sufficiency is a restatement of Theorem 4.17.

For the necessity, since Σa_n converges and $Q_n \mathrel{\leq\!\cdot} Q_{n+1}$, we see that $Q_n \to Q$ where $1 < Q \leq +\infty$. From Corollary 4.28, we cannot have $Q < +\infty$. Thus, $Q = +\infty$. Q.E.D.

Lubkin (17, p. 232) has proves the following theorem.

Theorem 8. If Σa_n is a convergent real series, Q exists $\neq 1$, and $n(Q_n - Q_{n-1}) \to 0$ as $n \to \infty$, then the series $U = \Sigma u_n$ converges more rapidly to S than Σa_n, where $u_n = (Qa_{\delta n} - a_n)/(Q-1)$ for $n \geq 0$.

In Theorem 8, the convergence of Σa_n and the existence of $Q \neq 1$ implies that $Q > 1$. With this in mind, we presently show that the condition $n(Q_n - Q_{n-1}) \to 0$ can be omitted from the hypothesis of Theorem 8 and, at the same time, generalize into the complex plane. Pflanz (18, p. 25) proved this fact for real series.

Before extending Theorem 8, we note that Shanks (23, p. 39) suggests the transform $e_1^{(s)}(A_n) = (s\, B_n - A_n)/(s-1)$, where $s = \lim_{n \to \infty} (\Delta A_n)/(\Delta B_n)$ and $B_n = e_1(A_n)$, be applied for acceleration in the critical case $r_n \to 1$. In our notation, this transform becomes

$$e_1^{(s)}(S_n) = S_{\alpha n} = (s\, S_{\delta n} - S_n)/(s-1) = [s(S_n + a_{n+1}\delta_{n+1}) - S_n]/(s-1)$$
$$= [(s-1)\, S_n + s a_{n+1}\delta_{n+1}]/(s-1) = S_n + a_{n+1}\ s\, \delta_{n+1}/(s-1)$$
$$= S_n + a_{n+1}\alpha_{n+1},$$

where $\alpha_n = s\, \delta_n/(s-1)$ and $s = \lim a_n/a_{\delta n}$. Shanks (23, p. 40) appears to be unaware of Lubkin's transform given in Theorem 8, or, at least, that the two transforms are identical, if $n(Q_n - Q_{n-1}) \to 0$ and Q exists with $\operatorname{Re} Q > 1$. In fact, we will see in Theorem 4.32 that if Q exists with $\operatorname{Re} Q > 1$, then $e_1^{(s)}(S_n)$ converges more rapidly to S than S_n if and only if $n(Q_n - Q_{n-1}) \to 0$; consequently Lubkin's transform, given in Theorem 8, has a broader applicability if $\operatorname{Re} Q > 1$, since the condition $n(Q_n - Q_{n-1}) \to 0$ is irrelevant.

We now extend Lubkin's Theorem 8.

Theorem 4.30. If Σa_n is a series such that $Q_n \to Q$ where $\operatorname{Re} Q > 1$, and $u_n = (Qa_{\delta n} - a_n)/(Q-1)$ for $n \geq 0$, then $\Sigma u_n \ \varepsilon\ \mathrm{MR}(\Sigma a_n)$.

Proof: Set $\alpha_n = Q\delta_n/(Q-1)$ for $n \geq 1$. Then

$$U_n \doteq \sum_{k=0}^{n} u_k \doteq \sum_{k=0}^{n} (Qa_{\delta k} - a_k)/(Q-1) \doteq (Q \sum_{k=0}^{n} a_{\delta k} - S_n)/(Q-1)$$

$$\doteq (Q\, S_{\delta n} - S_n)/(Q-1) \doteq [Q(S_n + a_{n+1}\delta_{n+1}) - S_n]/(Q-1)$$

$$\doteq [(Q-1)S_n + Q\, a_{n+1}\delta_{n+1}]/(Q-1) \doteq S_n + a_{n+1}[Q\delta_{n+1}/(Q-1)]$$

$\doteq S_n + a_{n+1}\alpha_{n+1}$. From (1) of Theorem 4.27, $\Sigma a_{\alpha n} \ \varepsilon\ \mathrm{MR}(\Sigma a_n)$, so that $(S-U_n)/(S-S_n) \doteq (S-S_{\alpha n})/(S-S_n) \to 0$. Q.E.D.

Lemma 4.31. Suppose that $Q_n \to Q$ for some complex number $Q \neq 0$. Then $a_n/a_{\delta n} \to Q$ if and only if $n(Q_n - Q_{n-1}) \to 0$.

<u>Proof</u>: Since $Q_n \neq. 0$,

(1) $a_{\delta n}/a_n =. (n+1)(Q_n - Q_{n+1})/Q_nQ_{n+1} + 1/Q_n$

and

(2) $(n+1)(Q_n - Q_{n+1}) =. Q_nQ_{n+1}\ a_{\delta n}/a_n - Q_{n+1}$.

Thus, if $n(Q_n - Q_{n-1}) \to 0$, then from (1), $a_{\delta n}/a_n \to 1/Q$. Hence, $a_n/a_{\delta n} \to Q$. Conversely, if $a_n/a_{\delta n} \to Q$, then $a_{\delta n}/a_n \to 1/Q$. Thus from (2), $n(Q_n - Q_{n-1}) \to 0$. Q.E.D.

<u>Theorem 4.32</u>. Suppose that $Q_n \to Q$ where $\operatorname{Re} Q > 1$, $s = \lim a_n/a_{\delta n} \neq 1$, and $\alpha_n =. s\,\delta_n/(s-1)$. Then $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ if and only if $n(Q_n - Q_{n-1}) \to 0$.

<u>Proof</u>: From Theorem 4.27, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ if and only if $s/(s-1) = Q/(Q-1)$, i.e., $Q = s = \lim a_n/a_{\delta n}$. But, from Lemma 4.31, $Q = \lim a_n/a_{\delta n}$ if and only if $n(Q_n - Q_{n-1}) \to 0$. Q.E.D.

It is very easy to construct a series Σa_n satisfying the hypothesis of Theorem 4.30, while $n(Q_n - Q_{n-1}) \not\to 0$. In particular, we mention the following example.

<u>Example 4.33</u>. Let Q be any number such that $\operatorname{Re} Q > 1$. Set $\gamma_{2n} =. 0$, $\gamma_{2n-1} =. 1/\sqrt{n}$, and $Q_n =. Q + \gamma_n$. Then

$n(Q_n - Q_{n-1}) \doteq n[(Q+\gamma_n) - (Q+\gamma_{n-1})] \doteq n(\gamma_n - \gamma_{n-1})$,

$2n(Q_{2n} - Q_{2n-1}) \doteq 2n(\gamma_{2n} - \gamma_{2n-1}) \doteq -2\sqrt{n} \to -\infty$, and

$(2n-1)(Q_{2n-1} - Q_{2n-2}) = (2n-1)(\gamma_{2n-1} - \gamma_{2n-2})$

$= (2n-1)/\sqrt{n} \to +\infty$. Clearly, $Q \to Q$ so that the hypothesis of Theorem 4.30 is satisfied while $n(Q_n - Q_{n-1}) \not\to 0$. Thus, Lubkin's transformation Σa_n, given in Theorem 4.30, converges rapidly to S than Σa_n. However, as we have just observed, $|n(Q_n - Q_{n-1}| \to +\infty$; thus, according to Theorem 4.32, Daniel Shank's transform $e_1^{(s)}(S_n)$ $\doteq S_n + s\,\delta_{n+1}/(s-1)$ must fail to converge more rapidly to S than S_n. Here, $s = \lim a_n/a_{\delta n} = 0$ since $\lim|a_{\delta n}/a_n| = \lim|(n+1)(Q_n - Q_{n+1})/Q_n Q_{n+1} + 1/Q_n| = +\infty$. Hence, we have in fact $e_1^{(s)}(S_n) \doteq e_1^{(0)}(S_n) \doteq S_n$, and thus $e_1^{(s)}(S_n)$ clearly converges with the same rapidity as S_n. We could have also applied Theorem 4.27 to arrive at this conclusion. If we carry our analysis a little deeper in this example, a very surprising phenomenon arises. In particular, $u_n/a_n \doteq (Q\,a_{\delta n}/a_n - 1)/(Q-1)$, $a_{\delta n}/a_n \doteq 1/Q_n - (n+1)(Q_{n+1} - Q_n)/Q_n Q_{n+1}$, $Q_n \to Q$, and, as shown above, $(n+1)|Q_{n+1} - Q_n| \to +\infty$. Consequently, $|u_n/a_n| \to +\infty$ even though $\Sigma u_n \in MR(\Sigma a_n)$.

Lubkin (17, p. 232-233) has proven the following theorem.

<u>Theorem 9</u>. If Σa_n is a convergent real series, Q exists $\neq 1$, $n(Q_n - Q_{n-1}) \to 0$, and $n[(n+1)(Q_{n+1} - Q_n) - n(Q_n - Q_{n-1})] \to 0$, then the transform Σw_n converges more rapidly to S than Σa_n, where $W_0 = 0$ and $W_n = w_0 + \cdots + w_n = S_n + a_{n+1}(1-r_n)/(1-2r_{n+1}+r_n r_{n+1})$ for $n > 0$.

As previously noted, we must have $Q > 1$. With this in mind, we will show in Theorem 4.35 that the condition $n[(n+1)(Q_{n+1}-Q_n)-n(Q_n-Q_{n-1})] \to 0$ can be omitted from the hypothesis of Theorem 9 and, at the same time, generalize into the complex plane.

<u>Lemma 4.34</u>. Suppose that $Q_n \to Q$ for some complex number $Q \neq 0$ or 1, and $\alpha_n =. (1-r_{n-1})/(1-2r_n+r_{n-1}r_n)$. Then $\alpha_n/n \to 1/(Q-1)$ if and only if $n(Q_n-Q_{n-1}) \to 0$.

<u>Proof</u>: From Lemma 4.31, $n(Q_n-Q_{n-1}) \to 0$ if and only if $a_{\delta n}/a_n \to 1/Q$. As shown in the proof of Theorem 3.4, $1-2r_{n+1}+r_n r_{n+1} =. (1-r_n)(1-r_{n+1})(1-a_{\delta n}/a_n)$. Thus, $\alpha_{n+1}/(n+1) = [1/(n+1)][(1-r_n)/(1-2r_{n+1}+r_n r_{n+1})]$ $=. 1/[Q_{n+1}(1-a_{\delta n}/a_n)]$, so that $a_{\delta n}/a_n \to 1/Q$ if and only

if $\alpha_{n+1}/(n+1) \to 1/(Q-1)$. Q.E.D.

<u>Theorem 4.35</u>. Suppose that $Q_n \to Q$ where $\operatorname{Re} Q > 1$, and $\alpha_n =. (1-r_{n-1})/(1-2r_n+r_{n-1}r_n)$. Then $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ if and only if $n(Q_n-Q_{n-1}) \to 0$.

<u>Proof</u>: From Theorem 4.25, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ if and only if $\alpha_n/n \to 1/(Q-1)$; and according to Lemma 4.34, this is equivalent to $n(Q_n-Q_{n-1}) \to 0$. Q.E.D.

CHAPTER V

NONALTERNATING SERIES

A real series Σa_n will be called nonalternating iff $r_n > 0$ for every n, and N-nonalternating iff $r_n > 0$ for $n \geq N$, where N is some integer.

Shortly, it will be shown that E. E. Kummer's criterion for the convergence of a nonalternating series is not only sufficient, but also necessary. We now prove a slight generalization of this fact.

Theorem 5.1. Let L be any real number and c be any positive number. Then a n.a.s.c. that an N-nonalternating series Σa_n converge is that there exist a sequence $\{\beta_n\}$ such that,

(1) $\quad a_n\beta_n \to L,$

and

(2) $\quad \beta_n \geq c + r_{n+1}\beta_{n+1},\ n \geq N.$

Moreover, if (1) and (2) hold, then for $n \geq N$,

(a) $\quad 0 < r_n < T_n \leq r_n\beta_n/c - L/ca_{n-1}.$

And in general, for $n \geq N$ and $k \geq 1$,

(b) $$T_{n,k-2} < T_n \leq T_{n,k-2} + (r_n \cdots r_{n+k-1})\beta_{n+k-1}/c - L/c\, a_{n-1}.$$

Proof: For the necessity, define $\beta_n = c + c\,T_{n+1} + L/a_n$ for $n \geq N$. Consequently, $a_n\beta_n \doteq ca_n + ca_nT_{n+1} + L \doteq ca_n + c(S-S_n) + L \to L$ as $n \to \infty$. For $n \geq N$, $c + r_{n+1}\beta_{n+1} = c + r_{n+1}(c+cT_{n+2}+L/a_{n+1}) = c + cr_{n+1}(1+T_{n+2}) + L/a_n = c + T_{n+1} + L/a_n = \beta_n$, so that (2) hold with equality.

For the sufficiency, assume that (1) and (2) hold. Let n be any integer $\geq N$, and define $P_k = T_{n,k-2} + (r_n \cdots r_{n+k-1})\beta_{n+k-1}/c$ for $k \geq 1$. From (2), $P_{k+1} - P_k = (r_n \cdots r_{n+k-1})(1+r_{n+k}\beta_{n+k}/c - \beta_{n+k-1}/c) \leq 0$ for $k \geq 1$. Also, $P_k \geq. a_{n+k-1}\beta_{n+k-1}/ca_{n-1} \to L/ca_{n-1}$ as $k \to \infty$. Thus $\{P_k\}$ is a monotone bounded sequence, so that $P_k \to P$ as $k \to \infty$, for some number P. Consequently, $T_{n,k-2} \doteq P_k - a_{n+k-1}\beta_{n+k-1}/ca_{n-1} \to P - L/ca_{n-1}$ as $k \to \infty$. Hence $T_n = P - (L/ca_{n-1}) \leq P_k - (L/ca_{n-1})$ for $k \geq 1$. Obviously, $T_{n,k-2} < T_n$ for $k \geq 1$. Thus (b) holds, and (a) follows from (b). Q.E.D.

Condition (1) of Theorem 5.1 can be somewhat weakened, as is now proven.

Corollary 5.2. Let c be any positive number. Then a n.a.s.c. that an N-nonalternating series Σa_n converge is that there exist a sequence $\{\beta_n\}$ such that,

(1) some subsequence of $\{a_n\beta_n\}$ is bounded,

and

(2) $\beta_n \geq c + r_{n+1}\beta_{n+1}$, $n \geq N$.

Moreover, if (1) and (2) hold, then $\{a_n\beta_n\}$ converges.

Proof: The necessity follows from Theorem 5.1.

For the sufficiency, we may assume that $a_n > 0$ for $n \geq N-1$. From (2), $a_n\beta_n \geq c\ a_n + a_{n+1}\beta_{n+1} > a_{n+1}\beta_{n+1}$ for $n \geq N$. Thus $\{a_n\beta_n\}$ converges because of (1). Now apply Theorem 5.1. Q.E.D.

Corollary 5.3. Let c be any positive number. Then a n.a.s.c. that a series Σa_n of positive terms converge is that there exist a sequence $\{\beta_n\}$ such that,

(1) some subsequence of $\{a_n\beta_n\}$ is bounded below,

and

(2) $\beta_n \geq\cdot\ c + r_{n+1}\beta_{n+1}$.

Moreover, if (1) and (2) hold, then $\{a_n\beta_n\}$ converges.

Proof: The necessity follows from Theorem 5.1.

For the sufficiency, from (2) we have $a_n\beta_n \geq\cdot\ ca_n + a_{n+1}\beta_{n+1} \geq\cdot\ a_{n+1}\beta_{n+1}$. Thus $\{a_n\beta_n\}$ converges because of (1). From Theorem 5.1, Σa_n converges. Q.E.D.

Corollary 5.4. Let L be any real number. Then a n.a.s.c. that an N-nonalternating Σa_n converge is that there exist a sequence $\{\beta_n\}$ such that,

(1) $a_n\beta_n \to L$,

and

(2) $\beta_n \geq 1 + r_{n+1}\beta_{n+1}$, $n \geq N$.

Moreover, if (1) and (2) hold, then for $n \geq N$,

(a) $0 < r_n < T_n \leq r_n\beta_n - (L/a_{n-1})$.

And in general, for $n \geq N$ and $k \geq 1$,

(b) $T_{n,k-2} < T_n \leq T_{n,k-2} + (r_n \cdots r_{n+k-1})\beta_{n+k-1} - (L/a_{n-1})$.

Proof: Choose $c = 1$ in Theorem 5.1. Q.E.D.

Let Σa_n be any divergent nonalternating series such that $a_n \to 0$. Let β_1 be any real number, and define $\{\beta_n\}$ recursively by $\beta_n = 1 + r_{n+1}\beta_{n+1}$. Then $a_n\beta_n - a_{n+1}\beta_{n+1} = a_n \to 0$, and $\beta_n \geq 1 + r_{n+1}\beta_{n+1}$ for $n \geq 1$. Thus, we cannot replace (1) of Corollary 5.4 by the condition that $a_n\beta_n - a_{n+1}\beta_{n+1} \to 0$.

Theorem 5.5. (Kummer's criterion) Let c be any positive number. Then a n.a.s.c. that an N-nonalternating series Σa_n converge is that there exist a sequence $\{\beta_n\}$ such that,

(1) $\beta_n \geq 0$, $n \geq N$,

and

(2) $\beta_n \geq c + r_{n+1}\beta_{n+1}$, $n \geq N$.

Moreover, if (1) and (2) hold, then for $n \geq N$,

(a) $0 < r_n < T_n \leq r_n\beta_n/c - (\lim_{k\to\infty} a_k\beta_k)/ca_{n-1} \leq r_n\beta_n/c$,

and

(b) $\{a_n\beta_n\}$ converges.

Proof: We may assume throughout that $a_{n-1} > 0$ for $n \geq N$.

For the necessity, choose $L \geq 0$ in Theorem 5.1. From (a) of Theorem 5.1, $\beta_n \geq 0$ for $n \geq N$.

For the sufficiency, according to Theorem 5.1 we need only show that $a_n\beta_n \to L$ for some number $L \geq 0$. From (1) and (2) above, $a_n\beta_n \geq ca_n + a_{n+1}\beta_{n+1} > a_{n+1}\beta_{n+1} \geq 0$ for $n \geq N$, which implies the existence of the required number L. Q.E.D.

The fact that Kummer's criterion, Theorem 5.5, is also necessary was first published by Shanks (24, p. 338-341). In (24, p. 338-341), Shanks employs Theorem 5.5 in an equivalent form to serve as a general framework for short proofs of the sufficient conditions of many of the known tests for convergence or divergence of series with positive terms. On the other hand, we are interested in Theorem 5.5

also as furnishing bounds for T_n and $S-S_{n-1}$, and consequently exhibiting the convergence of $\{T_n\}$ under certain conditions.

It should be noted that Theorem 5.1, as a criterion for convergence of Σa_n, is more general than Theorem 5.5 in the sense that for every convergent non-alternating series Σa_n there is a sequence $\{\beta_n\}$ satisfying (1) and (2) of Theorem 5.1 with $N = 1$, while condition (1) of Theorem 5.5 fails to hold for the same sequence $\{\beta_n\}$. In particular, let Σa_n be a convergent non-alternating series and $\{\beta_n\}$ be any sequence satisfying (1) and (2) of Theorem 5.1 with $N = 1$. Let L' be any number such that $(L-L')/a_n < 0$ for $n \geq 0$, and define $\beta_n' = \beta_n - L'/a_n$. Then $a_n\beta_n' = a_n\beta_n - L' \to L-L'$, so that $\beta_n' \to -\infty$ and $\beta_n < 0$. Moreover, for $n \geq 1$, $\beta_n' = \beta_n - L'/a_n \geq c + r_{n+1}\beta_{n+1} - L'/a_n = c + r_{n+1}(\beta_{n+1} - L'/a_{n+1}) = c + r_{n+1}\beta_{n+1}$. Thus, (1) $a_n\beta_n' \to L-L'$ and (2) $\beta_n' \geq c + r_{n+1}\beta_{n+1}'$, while the condition $\beta_n' \geq 0$ fails for large n.

Theorem 5.6. A n.a.s.c. that an N-nonalternating series Σa_n converge is that there exist a sequence $\{\beta_n\}$ such that,

(1) $\beta_n \geq 0$, $n \geq N$,

and

(2) $\beta_n \geq 1 + r_{n+1}\beta_{n+1}$, $n \geq N$.

Moreover, if (1) and (2) hold, then for $n \geq N$,

(a) $0 < r_n < T_n \leq r_n\beta_n - (\lim_{k\to\infty} a_k\beta_k)/a_{n-1} \leq r_n\beta_n$.

Proof: Choose $c = 1$ in Theorem 5.5. Q.E.D.

Example 5.7. Let $\Sigma a_n = 1 + 1/2^2 + 1/3^2 + \cdots$. Then, $a_n = 1/(n+1)^2$ for $n \geq 0$, and $r_n = [n/(n+1)]^2$ for $n \geq 1$. Defining $\beta_n = (n+2)^2$ for $n \geq 1$, $\beta_n \geq 1 + r_{n+1}\beta_{n+1}$ for $n \geq 1$, and, for $k \geq 1$, $a_k\beta_k = [(k+2)/(k+1)]^2 \to 1$. From Theorem 5.6, Σa_n converges.

Some of the known tests for convergence are now proven by exhibiting a sequence $\{\beta_n\}$ satisfying the conditions of the preceeding theorem.

Theorem 5.8. (Comparison test) If $0 < a'_n \leq. a_n$ and Σa_n converges, then $\Sigma a'_n$ converges.

Proof: From Theorem 5.6, there is a sequence $\{\beta_n\}$ such that $\beta_n \geq 0$ and $\beta_n \geq. 1 + r_{n+1}\beta_{n+1}$. Accordingly, $a_n\beta_n/a'_n \geq. a_n/a'_n + (a'_{n+1}/a'_n)(a_{n+1}\beta_{n+1}/a'_{n+1}) \geq. 1 + r'_{n+1}(a_{n+1}\beta_{n+1}/a'_{n+1}) \geq. 0$. Now apply Theorem 5.6. Q.E.D.

Theorem 5.9. (Ratio comparison test) If $0 <. r'_n \leq. r_n$ and Σa_n converges, then $\Sigma a'_n$ converges.

Proof: From Theorem 5.6, there is a sequence $\{\beta_n\}$ such that $\beta_n \geq. 0$ and $\beta_n \geq. 1 + r_{n+1}\beta_{n+1}$. Accordingly, $\beta_n \geq. 1 + r_{n+1}\beta_{n+1} \geq. 1 + r'_{n+1}\beta_{n+1}$, since $0 <. r'_n \leq. r_n$ and $\beta_n \geq. 0$. Now apply Theorem 5.6. Q.E.D.

Theorem 5.10. (Root test) If $a_n >. 0$ and $\limsup \sqrt[n]{a_n} < 1$, then Σa_n converges.

Proof: Let t be any number satisfying $\limsup \sqrt[n]{a_n} < t < 1$. Then $a_n \leq. t^n$. Defining $\beta_n =. t^n/a_n(1-t)$, $\beta_n - r_{n+1}\beta_{n+1}$
$=. t^n/a_n(1-t) - r_{n+1}\, t^{n+1}/a_{n+1}(1-t)$
$=. t^n/a_n(1-t) - t^{n+1}/a_n(1-t) =. [t^n/a_n(1-t)](1-t) =. t^n/a_n \geq. 1$.
Thus $\beta_n \geq. 0$ and $\beta_n \geq. 1 + r_{n+1}\beta_{n+1}$. Now apply Theorem 5.6. Q.E.D.

Theorem 5.11. (Ratio test) If $0 <. r_n$ and $\limsup r_n < 1$, then Σa_n converges.

Proof: Let t be any number for which $\limsup r_n < t < 1$. Defining $\beta_n =. 1/(1-t)$, we have $\beta_n =. 1 + t\beta_n$ $\geq. 1 + r_{n+1}\beta_{n+1}$ since $0 <. r_n <. t$. Now apply

Theorem 5.6. Q.E.D.

Theorem 5.12. (Raabe's test) If $0 <. r_n \leq. 1-a/n$ where $1 < a$, then Σa_n converges.

Proof: Set $\beta_n =. n/(a-1)$. Then $\beta_n >. 0$ and $1 + r_{n+1}\beta_{n+1} \leq. 1 + [1-a/(n+1)]\beta_{n+1} =. 1 + (n+1)/(a-1) - a/(a-1) =. n/(a-1) =. \beta_n$, so that $\beta_n \geq. 1 + r_{n+1}\beta_{n+1}$. Now apply Theorem 5.6. Q.E.D.

Theorem 5.13. Let L be any real number and c be any positive number. Then a necessary condition that an N-nonalternating series Σa_n converge is that there exist a sequence $\{\alpha_n\}$ such that,

(1) $a_n\alpha_n \to L$,

and

(2) $\alpha_n \leq c + r_{n+1}\alpha_{n+1}$, $n \geq N$.

Moreover, if (1) and (2) hold, then for $n \geq N$,

(a) $r_n\alpha_n/c - L/ca_{n-1} \leq T_n$,

and in general, for $n \geq N$ and $k \geq 1$,

(b) $T_{n,k-1} + (r_n \cdots r_{n+k-1})\alpha_{n+k-1}/c - L/ca_{n-1} \leq T_n$.

Proof: For the necessity, we may use the proof of the necessity of Theorem 5.1, replacing "β" by "α" throughout.

Next, assume that (1) and (2) hold. Let n be any integer $\geq N$, and define $P_k = T_{n,k-2}$ $+ (r_n \cdots r_{n+k-1})\, \alpha_{n+k-1}/c$ for $k \geq 1$. From (2), $P_{k+1} - P_k$ $= (r_n \cdots r_{n+k-1})(1 + r_{n+1}\alpha_{n+k}/c - \alpha_{n+k-1}/c) \geq 0$ for $k \geq 1$. Also, $P_k =. T_{n,k-2} + a_{n+k-1}\alpha_{n+k-1}/a_{n-1}c \to T_n + L/ca_{n-1}$. Thus, $P_k - L/ca_{n-1} \leq T_n$ for $k \geq 1$, i.e., (b) holds. With $k = 1$, (b) reduces to (a). Q.E.D.

<u>Theorem 5.14</u>. Let L be any real number. Then a necessary condition that an N-nonalternating series Σa_n converge is that there exist a sequence $\{\alpha_n\}$ such that,

(1) $a_n\alpha_n \to L$,

and

(2) $\alpha_n \leq 1 + r_{n+1}\alpha_{n+1}$, $n \geq N$.

Moreover, if (1) and (2) hold, then for $n \geq N$,

(a) $r_n\alpha_n - (L/a_{n-1}) \leq T_n$,

and in general, for $n \geq N$ and $k \geq 1$,

(b) $T_{n,k-2} + (r_n \cdots r_{n+k-1})\alpha_{n+k-1} - (L/a_{n-1}) \leq T_n$.

<u>Proof</u>: Choose $c = 1$ in Theorem 5.13. Q.E.D.

<u>Theorem 5.15</u>. Let c be any positive number. Then a n.a.s.c. that an N-nonalternating series Σa_n diverge is that there exist a sequence $\{\alpha_n\}$ such that,

(1) $|a_n\alpha_n| \to \infty$,

and

(2) $\alpha_n \leq c + r_{n+1}\alpha_{n+1} \leq c + \alpha_n$, $n \geq N$.

Proof: We may assume that $a_{n-1} > 0$ for $n \geq N$.

For the necessity, let α_N be any real number, and define $\{\alpha_n\}$ recursively by the equation $\alpha_n = c+r_{n+1}\alpha_{n+1}$. Accordingly, $\alpha_n = c + r_{n+1}\alpha_{n+1} < c + \alpha_n$ for $n \geq N$, i.e., (2) holds. For $k \geq 1$, $a_{N+k}\alpha_{N+k} = a_N\alpha_N - c(a_N+a_{N+1}+\cdots+ a_{N+k-1}) \to -\infty$ as $k \to \infty$, i.e., (1) holds.

For the sufficiency, from (2) we have $a_{n+1}\alpha_{n+1} \leq a_n\alpha_n$ for $n \geq N$. Thus, (1) implies that $a_n\alpha_n \to -\infty$. From (2), $(a_N\alpha_N - a_{N+n}\alpha_{N+n})/c \leq a_N + a_{N+1} + \cdots + a_{N+n-1} \to +\infty$ as $k \to \infty$, since $-a_n\alpha_n \to +\infty$ as $n \to \infty$. Thus Σa_n diverges. Q.E.D.

Corollary 5.16. Let c be any positive number. Then a n.a.s.c. that a series Σa_n of positive terms diverge is that there exist a sequence $\{\alpha_n\}$ such that,

(1) some subsequence of $\{a_n\alpha_n\}$ is unbounded,

and

(2) $\alpha_n \leq c + r_{n+1}\alpha_{n+1} \leq c + \alpha_n$, $n \geq 1$.

Moreover, if (1) and (2) hold, then $a_n\alpha_n \to -\infty$.

Proof: The necessity follows from Theorem 5.15.

For the sufficiency, from (2) we have $a_{n+1}\alpha_{n+1} \leq a_n\alpha_n$ for $n \geq 1$. Thus from (1), $a_n\alpha_n \to -\infty$. Hence $|a_n\alpha_n| \to +\infty$ and, according to Theorem 5.15, Σa_n diverges. Q.E.D.

Clearly, (1) of Corollary 5.16 may be replaced by the condition $a_n\alpha_n \to -\infty$.

Theorem 5.17. If Σa_n is an N-nonalternating series such that $0 \leq p \leq r_n \leq q < 1$ for $n \geq N$, where p and q are constants, then

(1) $p/(1-p) \leq r_n/(1-p) \leq T_n \leq r_n/(1-q) \leq q/(1-q)$,

for $n \geq N$.

Proof: Set $\alpha_n = 1/(1-p)$ and $\beta_n = 1/(1-q)$ for $n \geq N$. For $n \geq N$, $\alpha_n = 1 + p\alpha_{n+1} \leq 1 + r_{n+1}\alpha_{n+1}$ and $\beta_n = 1 + q\beta_{n+1} \geq 1 + r_{n+1}\beta_{n+1}$. From Theorem 5.6, Σa_n converges, so that $\lim a_n\alpha_n = \lim a_n\beta_n = 0$. From (a) of Theorems 5.6 and 5.14, we obtain (1). Q.E.D.

Theorem 5.18. If Σa_n is an N-nonalternating series and $0 \leq r < 1$, then $T_n \to r/(1-r)$.

Proof: We implicitly restrict n to large values throughout. There is a monotone increasing series $\{p_n\}$ such that $0 \leq p_n \leq r_n$ and $p_n \to r$. Define a monotone increasing sequence $\{\alpha_n\}$ by the equation $\alpha_n = 1/(1-p_{n+1})$. Accordingly, $\alpha_n = 1 + p_{n+1}\alpha_n \leq 1 + r_{n+1}\alpha_{n+1}$, i.e., $\alpha_n \leq 1 + r_{n+1}\alpha_{n+1}$. Similarly, there is a monotone decreasing sequence $\{q_n\}$ such that $r_n \leq q_n < 1$ and $q_n \to r$. Define a monotone decreasing sequence $\{\beta_n\}$ by the equation $\beta_n = 1/(1-q_{n+1})$. We then have $\beta_n = 1+q_{n+1}\beta_n \geq 1+r_{n+1}\beta_{n+1}$, i.e., $\beta_n \geq 1+r_{n+1}\beta_{n+1} \geq 0$. From Theorems 5.6 and 5.14, $r_n\alpha_n \leq T_n \leq r_n\beta_n$. Also $\lim r_n\alpha_n = \lim r_n\beta_n = r/(1-r)$, so that $T_n \to r/(1-r)$. Q.E.D.

We now turn to the critical case $r_n \to 1$. Suppose that Σa_n is a positive term series and $Q_n \to Q > 1$. According to Theorem 4.25, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ if and only if $\alpha_n \sim n/(Q-1)$. As we have seen, Szász suggests $\alpha_n = n/(Q-1)$ for $n \geq 1$. Now for a fixed number k, $(n+k)/(Q-1) \sim n/(Q-1)$, so that, with $\beta_n = (n+k)/(Q-1)$ for $n \geq 1$, $\Sigma a_n \in MR(\Sigma a_n)$. Thus, why should we restrict ourselves to $k=0$? We shall see that we should not make this restriction.

Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ have been determined such that

(1) $a_n\alpha_n \to 0$ and $0 < \alpha_n \leq 1+r_{n+1}\alpha_{n+1}$, $n \geq N$,

and

(2) $a_n\beta_n \to 0$ and $0 < 1+r_{n+1}\beta_{n+1} \leq \beta_n$, $n \geq N$.

From Theorems 5.4 and 5.14,

(3) $\alpha_n \leq T_n/r_n = 1+T_{n+1} \leq \beta_n$ for $n \geq N$.

From (3), it is clear that we wish to maximize the α_n and minimize the β_n, in order to obtain sharp bounds for $1+T_{n+1}$. Also, we desire $\alpha_n \sim \beta_n \sim n/(Q-1)$. Multiplying (3) by a_n, we obtain

(4) $a_n\alpha_n \leq S-S_{n-1} \leq a_n\beta_n$ for $n \geq N$.

Thus,

(5) $S_{\alpha(n-1)} = S_{n-1}+a_n\alpha_n \leq S \leq S_{\beta(n-1)}+a_n\beta_n$, $n \geq N$.

From (1) and (2), for $n \geq N$, $a_{\alpha n}/a_n = 1+r_{n+1}\alpha_{n+1}-\alpha_n \geq 0$ and $a_{\beta n}/a_n = 1+r_{n+1}\beta_{n+1}-\beta_n \leq 0$. Hence for $n \geq N$, $a_{\alpha n} \geq 0$, $a_{\beta n} \leq 0$, $S_{\alpha(n-1)} \leq S_{\alpha n}$, and $S_{\beta n} \leq S_{\beta(n-1)}$. In order to obtain fairly sharp bounds by (4), we will give only one example to show the general procedure.

Example 5.19.

$$\sum_0^\infty a_n = \sum_0^\infty 1/(4n+1)(4n+3) = 1/(1\cdot 3)+1/(5\cdot 7)+\cdots = \pi/8.$$

This series is considered by Szász (26,p.275). He takes $k=0$ in $\alpha'_n = (n+k)/(Q-1)$, and sets $t_n = S_n + a_{n+1}\alpha'_{n+1}$ for $n \geq 0$. Thus, $t_n = S_{\alpha' n}$ for $n \geq 0$. The numbers t_n, $2 \leq n \leq 7$, in (26,p.275) are in error. They should read:

$$t_2 = .38739, \quad t_3 = .38952, \quad t_4 = .39056$$
$$t_5 = .39116, \quad t_6 = .39153, \quad t_7 = .39183.$$

Now $.39269908 < \pi/8 < .39269909$. Setting $\pi/8 = .39270$, $\pi/8 - t_7 = .00087$.

We have $a_n = 1/(4n+1)(4n+3)$ for $n \geq 0$, and for $n \geq 1$, $r_n = a_n/a_{n-1} = (4n-3)(4n-1)/(4n+1)(4n+3)$ $= 1 - 32n/(4n+1)(4n+3) = 1 - Q_n/n$. Thus $Q_n = 32n^2/(4n+1)(4n+3) \to Q=2$ and $\alpha'_n = (n+k)/(Q-1) = n+k$. We have, for $n \geq 1$,

(6) $a_{\alpha' n}/a_n = 1 + r_{n+1}\alpha'_{n+1} - \alpha'_n = [32n(1-k)-32k+38]/(16n^2+48n+35)$.

From (6), it is obvious $k=1$ yields the best sequence $\{\alpha'_n\}$ for the acceleration of Σa_n. Thus, setting $\alpha_n = n+1$ for $n \geq 1$,

(7) $a_{\alpha 0} = a_0 + a_1\alpha_1 = 1/3 + 2/(5\cdot 7) = 1/3 + 6/(1\cdot 3\cdot 5\cdot 7)$

and from (6), for $n \geq 1$,

(8) $a_{\alpha n} = [6/(4n+5)(4n+7)]a_n = 6/(4n+1)(4n+3)(4n+5)(4n+7)$.

Thus,

$$(9)\quad \sum_0^\infty a_{\alpha n} = [1/3+6/(1\cdot3\cdot5\cdot7)] + \sum_0^\infty 6/(4n+1)(4n+3)(4n+5) \times (4n+7)$$

or

$$(10)\quad \sum_0^\infty a_{\alpha n}=1/3+\sum_0^\infty 6/(4n+1)(4n+3)(4n+5)(4n+7)=1/3+\sum_0^\infty b_n .$$

In (10) we have absorbed part of $a_{\alpha o}$ into the summation, i.e., $a_{\alpha o} = 1/3+b_o$ and $a_{\alpha n} = b_n$ for $n \geq 1$. No use will be made of (10), although it is suggestive for application of the above procedure to Σb_n.

At this point we have the following alternatives:

$$(11)\quad S_{\alpha n} = S_n+a_{n+1}\alpha_{n+1}=\sum_0^n 1/(4i+1)(4i+3) + (n+2)/(4n+5)(4n+7)$$

or

$$(12)\quad S_{\alpha n}=\sum_0^n a_{\alpha i}=a_{\alpha o}+\sum_1^n a_{\alpha i}=[1/3+2/35]+\sum_1^n 6/(4i+1)(4i+3)(4i+5) \times (4i+7).$$

Clearly, (1) is preferable for actual numerical calculation. Leaving $\Sigma a_{\alpha n}$ in the form (11), we have a so-called "modified series" of Bradshaw (9,p.486-492). In applying (11) as an approximation to S, we have no information, assuming no previous calculations for $\pi/8$ as known, as to the error involved, i.e., $S-S_{\alpha n}$. We now turn to the resolution of this problem.

Comparing (1) with (6), we require

$$(1.3)\quad 1+r_{n+1}\alpha'_{n+1}-\alpha'_n \geq 0 \quad \text{for} \quad n \geq N.$$

From (6), (13) is seen to be equivalent to

(14) $k \leq 1+3/(16n+16), \quad n \geq N.$

From (14), we must have $k \leq 1$, since $1+3/(16n+16) \to 1$ as $n \to \infty$. Thus, we are led to set $k = 1$ and $\alpha'_n = n+k = n+1 = \alpha_n$, α_n as defined for (9) and (11). We now see from (4) that

(15) $a_n\alpha_n \leq S-S_{n-1}$ for $n \geq 1$, $\alpha_n = n+1$ for $n \geq 1$.

Comparing (2) with (6), with $\beta_n = \alpha'_n$, we require

(16) $1 + r_{n+1}\beta_{n+1} - \beta_n \leq 0$ for $n \geq N$.

From (6), (16) is seen to be equivalent to

(17) $k \geq 1+3/(16n+16), \quad n \geq N.$

Recalling that $\beta_n = n+k$ is to be minimized and noting that $\{1+3/(16n+16)\}$ is monotone decreasing, we set $k = 1+3/(16N+16)$ as the optimal choice satisfying (17). From (4), we then have,

(18) $S-S_{n-1} \leq a_n\beta_n$ and $\beta_n = n+1+3/(16N+16)$, $n \geq N$.

Setting $n = N$ in (18) and noting that (18) holds for $N \geq 1$, we have

(19) $S-S_{n-1} \leq a_n\beta_n$ and $\beta_n = n+1+3/(16n+16)$, $n \geq 1$.

From (15) and (19), we obtain the desired bounds for $S-S_{\alpha n}$, i.e.,

(20) $0 \leq S-S_{\alpha(n-1)} \leq a_n(\beta_n - \alpha_n) = 3/(4n+1)(4n+3)(16n+16),$

$n \geq 1.$

With $n = 1$ in (20), $0 \leq S-S_{\alpha_0} \leq 3/(5\cdot 7\cdot 32) < .0027$.

With $n=8$ in (20), $0 \leq S-S_{\alpha_7} \leq 3/(33\cdot 35\cdot 144) = 1/55440$ $< .000019$. Using a^- iff $a^- < a$ and a^+ iff $a < a^+$, we have $S_7^- = .3848938$, $S_7^+ = .3848946$, $(a_8\alpha_8)^- = .0077922$, and $(a_8\alpha_8)^+ = .0078102$. Thus, $S_7^- + (a_8\alpha_8)^- = .3926860$ $< S < .3927050 = S_7^+ + (a_8\alpha_8)^+$. Letting S' be the average of these two bounds for $S = \pi/8$, we find $S' = .3926955$ and we must have $|S-S'| = |\pi/8 - .3926955|$ $\leq (.3927050 - .3926860)/2 = .0000095$.

CHAPTER VI

CONVERGENCE AND DIVERGENCE OF REAL SERIES

Throughout this chapter, all series are assumed to be real. We now state and prove some of the theorems, corresponding to those of Chapter V.

Theorem 6.1. Let L be any real number and c be any positive number. Then a n.a.s.c. that a series Σa_n converge is that there exist a convergent series Σb_n and a sequence $\{\beta_n\}$ such that,

(1) $(a_n+b_n)\beta_n \to L$,

(2) $0 <. (a_{n+1}+b_{n+1})/(a_n+b_n)$,

and

(3) $\beta_n \geq. c + [(a_{n+1}+b_{n+1})/(a_n+b_n)]\beta_{n+1}$.

Proof: For the necessity, let Σc_n be any convergent nonalternating series, and define $b_n = c_n - a_n$ for $n \geq 0$. The series $\Sigma(a_n+b_n) = \Sigma c_n$ is a convergent nonalternating series, so that (2) holds. According to Theorem 5.1, there is a sequence $\{\beta_n\}$ which satisfies conditions (1) and (3) above. Clearly, Σb_n converges.

For the sufficiency, we see that $\Sigma(a_n+b_n)$ converges according to Theorem 5.1. Consequently, Σa_n

converges since Σb_n converges. Q.E.D.

Theorem 6.2. Let L be any real number and c be any positive number. Then a n.a.s.c. that a series Σa_n diverge is that there exist a divergent series Σb_n and a sequence $\{\beta_n\}$ such that,

(1) $(a_n+b_n)\beta_n \to L$,

(2) $0 <. (a_{n+1}+b_{n+1})/(a_n+b_n)$,

and

(3) $\beta_n \geq. c + [(a_{n+1}+b_{n+1})/(a_n+b_n)]\beta_{n+1}$.

Proof: For the necessity, let Σc_n be any convergent nonalternating series and define $b_n = c_n - a_n$ for $n \geq 0$. The series $\Sigma(a_n+b_n) = \Sigma c_n$ is a convergent nonalternating series so that (2) holds. From Theorem 5.1, there is a sequence $\{\beta_n\}$ such that (1) and (3) hold. Also, Σb_n must diverge.

For the sufficiency, Σa_n must diverge, since otherwise Σb_n would converge according to Theorem 6.1.

Theorem 6.3. Let c be any positive number. Then a n.a.s.c. that a series Σa_n converge is that there exist a convergent series Σb_n and a sequence $\{\beta_n\}$ such that,

(1) $\beta_n \geq. 0$,

(2) $0 <. (a_{n+1}+b_{n+1})/(a_n+b_n)$,

and

(3) $\beta_n \geq. c + [(a_{n+1}+b_{n+1})/(a_n+b_n)]\beta_{n+1}$.

Proof: For the necessity, let Σc_n be any convergent nonalternating series, and define $b_n = c_n - a_n$ for $n \geq 0$. The series $\Sigma(a_n+b_n) = \Sigma c_n$ is a convergent nonalternating series so that (2) holds. According to Theorem 5.5, there is a sequence $\{\beta_n\}$ satisfying conditions (1) and (3) above. Also, Σb_n converges.

For the sufficiency, Theorem 5.5 implies that $\Sigma(a_n+b_n)$ converges. Thus, Σa_n converges since Σb_n converges. Q.E.D.

Theorem 6.4. Let c be any positive number. Then a n.a.s.c. that a series Σa_n diverge is that there exist a divergent series Σb_n and a sequence $\{\beta_n\}$ such that,

(1) $\beta_n \geq. 0$,

(2) $0 <. (a_{n+1}+b_{n+1})/(a_n+b_n)$,

and

(3) $\beta_n \geq. c + [(a_{n+1}+b_{n+1})/(a_n+b_n)]\beta_{n+1}$.

Proof: For the necessity, let Σc_n be convergent nonalternating series and define $b_n = c_n - a_n$ for $n \geq 0$. The series $\Sigma(a_n+b_n) = \Sigma c_n$ is a convergent nonalternating series so that (2) holds. From Theorem 5.5, there is a sequence $\{\beta_n\}$ satisfying conditions (1) and (3). Moreover, Σb_n must diverge.

For the sufficiency, Σa_n must diverge since otherwise Σb_n would converge according to Theorem 6.3. Q.E.D.

Theorem 6.5. Let c be any positive number. Then a n.a.s.c. that a series Σa_n converge is that there exist a convergent series Σb_n and a sequence $\{\beta_n\}$ such that,

(1) $\beta_n \geq 0$,

and

(2) $\beta_n \geq c + |(a_{n+1}+b_{n+1})/(a_n+b_n)|\beta_{n+1}$.

Proof: The necessity follows from Theorem 6.3.

For the sufficiency, Theorem 5.5 implies that $\Sigma|a_n+b_n|$ converges. Consequently, $\Sigma(a_n+b_n)$ converges, so that Σa_n converges since Σb_n converges. Q.E.D.

Theorem 6.6. Let c be any positive number. Then a n.a.s.c. that a series Σa_n diverge is that there exist a divergent series Σb_n and a sequence $\{\beta_n\}$ such that,

(1) $\beta_n \geq . 0$,

and

(2) $\beta_n \geq . c + |(a_{n+1}+b_{n+1})/(a_n+b_n)|\beta_{n+1}$.

<u>Proof</u>: The necessity follows from Theorem 6.4.

For the sufficiency, Σa_n must diverge, since otherwise Σb_n would converge according to Theorem 6.5. Q.E.D.

<u>Theorem 6.7</u>. Let c be any positive number. Then a n.a.s.c. that a series Σa_n converge is that there exist a convergent series Σb_n and a sequence $\{\beta_n\}$ such that,

(1) $\beta_n \geq . 0$,

(2) $0 <. a_n + b_n$,

and

(3) $\beta_n \geq . c + [(a_{n+1}+b_{n+1})/(a_n+b_n)]\beta_{n+1}$.

<u>Proof</u>: For the necessity, let Σc_n be any convergent series of positive terms, and define $b_n = c_n - a_n$ for $n \geq 0$. Clearly, Σb_n converges and (2) above holds. The existence of a sequence $\{\beta_n\}$ satisfying (1) and (3) follows from Theorem 5.5.

The sufficiency follows from Theorem 6.3. Q.E.D.

CHAPTER VII

CONVERGENCE AND DIVERGENCE OF COMPLEX SERIES

Throughout this chapter, all series are assumed to be complex.

A complex series Σa_n will be called restricted iff $r_n \neq 0$ for every n, and N-restricted iff $r_n \neq 0$ for $n \geq N$, where N is some integer. We now generalize some of the theorems in Chapters V and VI.

Theorem 7.1. Let L be any real number and c be any positive number. Then a n.a.s.c. that an N-restricted series Σa_n converge absolutely is that there exist a sequence $\{\beta_n\}$ such that

(1) $\quad |a_n|\beta_n \to L,$

and

(2) $\quad \beta_n \geq c + |r_{n+1}|\beta_{n+1}, \quad n \geq N.$

Proof: Apply Theorem 5.1 to $\Sigma|a_n|$. Q.E.D.

Theorem 7.2. (Kummer's criterion) Let c be any positive number. Then a n.a.s.c. that an N-restricted series Σa_n converge absolutely is that there exist a sequence $\{\beta_n\}$ such that

(1) $\beta_n \geq 0, \quad n \geq N,$

and

(2) $\beta_n \geq c + |r_{n+1}|\beta_{n+1}, \quad n \geq N.$

<u>Proof</u>: Apply Theorem 5.5 to $\Sigma|a_n|$. Q.E.D.

<u>Theorem 7.3</u>. Let c be any positive number. Then a n.a.s.c. that a series Σa_n converge is that there exist a convergent series Σb_n and a sequence $\{\beta_n\}$ such that,

(1) $\beta_n \geq 0,$

and

(2) $\beta_n \geq c + |(a_{n+1}+b_{n+1})/(a_n+b_n)|\beta_{n+1}.$

<u>Proof</u>: For the necessity, let Σc_n be any restricted series which converges absolutely and define $b_n = c_n - a_n$ for every n. Since $a_n+b_n = c_n$ for all n, $\Sigma(a_n+b_n)$ is a restricted series which converges absolutely. From Theorem 7.2, there is a sequence $\{\beta_n\}$ satisfying conditions (1) and (2) above. Clearly, Σb_n converges.

For the sufficiency, $\Sigma|a_n+b_n|$ converges according to Theorem 7.2 so that $\Sigma(a_n+b_n)$ converges. Thus, Σa_n converges since Σb_n converges. Q.E.D.

<u>Corollary 7.4</u>. Suppose that $c > 0$ and $\{\beta_n\}$ is a

sequence such that,

(1) $\beta_n \geq\cdot 0$,

and

(2) $\beta_n \geq\cdot c + |(a_{n+1}+b_{n+1})/(a_n+b_n)|\beta_{n+1}$.

Then Σa_n converges if and only if Σb_n converges.

Proof: Apply Theorem 7.3. Q.E.D.

Theorem 7.5. Let c be any positive number. Then a n.a.s.c. that a series Σa_n diverge is that there exist a divergent series Σb_n and a sequence $\{\beta_n\}$ such that,

(1) $\beta_n \geq\cdot 0$,

and

(2) $\beta_n \geq\cdot c + |(a_{n+1}+b_{n+1})/(a_n+b_n)|\beta_{n+1}$.

Proof: For the necessity, let Σc_n be any restricted series which converges absolutely and define $b_n = c_n - a_n$ for $n \geq 0$. The series $\Sigma(a_n+b_n) = \Sigma c_n$ is a restricted series which converges absolutely. From Theorem 7.2, there is a sequence $\{\beta_n\}$ satisfying conditions (1) and (2) above. Clearly, Σb_n diverges.

For the sufficiency, $\Sigma|a_n+b_n|$ converges according to Theorem 7.2. From Theorem 7.3, Σa_n must diverge since otherwise Σb_n would converge. Q.E.D.

CHAPTER VIII

ALTERNATING SERIES

A real series Σa_n is called alternating iff $r_n < 0$ for every n, and N-alternating iff $r_n < 0$ for $n \geq N$, where N is some integer.

Various theorems stating necessary and sufficient conditions for the convergence of an N-alternating series will be proven, along with corresponding error bounds for the quantities T_n. In many such theorems, it will be proven that all inequalities, excluding those between indices, may be reversed. Calling any such theorem and the derived theorem duals, a duality structure become apparent, but fails in at least one case. In particular, Theorem 8.32 has no dual according to Counterexample 8.10. Because of this duality, if the sequence $\{r_n\}$ is fairly smooth, the difficulty in satisfying the required inequalities involving $\{\alpha_n\}$ or $\{\beta_n\}$ is reduced considerably. Of course, the more judicious the choice of $\{\alpha_n\}$ or $\{\beta_n\}$, the better the resulting bounds for the quantities T_n.

Several theorems proven in this chapter will contain explicitly, or implicitly, in their conclusion that $\{T_n\}$ converges. As we have previously seen, this implies

$\Sigma a_{\delta n} \in MR(\Sigma a_n)$, but this will usually be omitted from the conclusion.

<u>Lemma 8.1</u>. If $\{P_{2n-1}\}$ is monotone decreasing, $\{P_{2n}\}$ is monotone increasing, and some subsequence of $\{P_{2n-1}-P_{2n}\}$ is bounded below, then $\{P_{2n-1}\}$ and $\{P_{2n}\}$ both converge.

<u>Proof</u>: Suppose that L is a lower bound of some subsequence $\{P_{2n'-1}-P_{2n'}\}$ of $\{P_{2n-1}-P_{2n}\}$. It is easily seen that $\{P_{2n-1}-P_{2n}\}$ is monotone decreasing. Consequently, $L \leq P_{2n'-1}-P_{2n'} \leq P_{2n-1}-P_{2n}$ for $n \geq 1$. We then have $L+P_2 \leq L+P_{2n} \leq P_{2n-1} \leq P_1$ and $P_2 \leq P_{2n} \leq P_{2n-1}-L \leq P_1-L$, for $n \geq 1$. Accordingly, $\{P_{2n-1}\}$ and $\{P_{2n}\}$ are bounded monotone sequences, and thus converge. Q.E.D.

<u>Theorem 8.2</u>. Let L_1 and L_2 be any real numbers. Then a n.a.s.c. that an N-alternating series Σa_n converge is that

(0) $\quad a_n \to 0,$

and there exist a sequence $\{\alpha_n\}$ such that,

(1) $\quad a_{2n-1}\alpha_{2n-1} \to L_1$ and $a_{2n}\alpha_{2n} \to L_2$

and

(2) $\quad \alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}, \quad n \geq N.$

Moreover, if (0), (1), and (2) hold, then, for $n \geq N$,

$$\text{(a)} \quad \begin{cases} r_n + r_n r_{n+1}\alpha_{n+1} - (L_2/a_{n-1}) \leq T_n \leq r_n\alpha_n - (L_1/a_{n-1}) \\ \text{or} \\ r_n + r_n r_{n+1}\alpha_{n+1} - (L_1/a_{n-1}) \leq T_n \leq r_n\alpha_n - (L_2/a_{n-1}), \end{cases}$$

accordingly as n is odd or even, respectively. And in general, for $n \geq N$ and $k \geq 1$,

$$\text{(b)} \quad \begin{cases} T_{n,2k-2} + (r_n \cdots r_{n+2k-1})\alpha_{n+2k-1} - (L_2/a_{n-1}) \leq T_n \\ \qquad \leq T_{n,2k-3} + (r_n \cdots r_{n+2k-2})\alpha_{n+2k-2} - (L_1/a_{n-1}) \\ \text{or} \\ T_{n,2k-2} + (r_n \cdots r_{n+2k-1})\alpha_{n+2k-1} - (L_1/a_{n-1}) \leq T_n \\ \qquad \leq T_{n,2k-3} + (r_n \cdots r_{n+2k-2})\alpha_{n+2k-2} - (L_2/a_{n-1}), \end{cases}$$

accordingly as n is odd or even, respectively.

<u>Proof</u>: Assume that Σa_n converges. Accordingly (0) holds. Define $L_{2n-1} = L_1$ and $L_{2n} = L_2$ for every n, and $\alpha_n = 1 + T_{n+1} + L_n/a_n$ for $n \geq N$. We then have $a_n\alpha_n$ $= a_n + a_n T_{n+1} + L_n = a_n + (S - S_n) + L_n = S - S_{n-1} + L_n$. Thus $a_{2n-1}\alpha_{2n-1}$ $=. S - S_{2n-2} + L_{2n-1} \to L_1$ and $a_{2n}\alpha_{2n} =. S - S_{2n-1} + L_{2n} \to L_2$, so that (1) holds. For $n \geq N$, $\alpha_n - 1 - r_{n+1} - r_{n+1}r_{n+2}\alpha_{n+2}$

$= 1 + T_{n+1} + L_n/a_n - 1 - r_{n+1} - r_{n+1}r_{n+2}(1 + T_{n+3} + L_{n+2}/a_{n+2})$

$= T_{n+1} + L_n/a_n - r_{n+1} - r_{n+1}r_{n+2} - r_{n+1}r_{n+2}T_{n+3} - L_{n+2}/a_n$

$= T_{n+1} + L_n/a_n - T_{n+1} - L_n/a_n = 0$, so that

$\alpha_n = 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$ for $n \geq N$. Thus (2) holds with equality. This proves the necessity.

For the sufficiency, assume that (0), (1), and (2) hold, and let n be any integer $\geq N$. We now define $P_k = T_{n,k-2}+(r_n\cdots r_{n+k-1})\alpha_{n+k-1}$ for $k \geq 1$. Accordingly

(3) $P_k - P_{k+2}$

$$= (r_n\cdots r_{n+k-1})[\alpha_{n+k-1}-(1+r_{n+k}+r_{n+k}r_{n+k+1}\alpha_{n+k+1})],$$

$k \geq 1$.

From (2) and (3) it can be seen that $P_{2k}-P_{2k+2} \leq 0$ and $P_{2k-1}-P_{2k+1} \geq 0$ for $k \geq 1$, so that $\{P_{2k}\}$ is monotone increasing and $\{P_{2k-1}\}$ is monotone decreasing. Moreover, $P_k-P_{k+1} = (r_n\cdots r_{n+k-1})[\alpha_{n+k-1}-(1+r_{n+k}\alpha_{n+k})] = [a_{n+k-1}\alpha_{n+k-1} - a_{n+k-1}-a_{n+k}\alpha_{n+k}]/a_{n-1}$, so that, by (0) and (1), the sequence $\{P_k-P_{k+1}\}$ is bounded. Consequently $\{P_{2k-1}-P_{2k}\}$ is bounded. By Lemma 8.1, $P_{2k-1} \to P'$ and $P_{2k} \to P''$, for some numbers P' and P''. We then have $T_{n,2k-2}$ $= r_n+\cdots+(r_n\cdots r_{n+2k-2}) \doteq P_{2k}-(r_n\cdots r_{n+2k-1})\alpha_{n+2k-1}$ $\doteq P_{2k}-a_{n+2k-1}\alpha_{n+2k-1}/a_{n-1} \to P'' - (L_2/a_{n-1})$ or $P'' - (L_1/a_{n-1})$, accordingly as n is odd or even. Similarly, $T_{n,2k-1} \doteq r_n+r_nr_{n+1}+\cdots+(r_n\cdots r_{n+2k-1}) \doteq P_{2k+1}$ $-(r_n\cdots r_{n+2k})\alpha_{n+2k} \doteq P_{2k+1}-a_{n+2k}\alpha_{n+2k}/a_{n-1} \to P'-(L_1/a_{n-1})$

or $P'-(L_2/a_{n-1})$, accordingly as n is odd or even. Also, $T_{n,2k-1}-T_{n,2k-2} \doteq (r_n \cdots r_{n+2k-1}) \doteq a_{n+2k-1}/a_{n-1} \to 0$ as $k \to \infty$, so that $T_{n,k} \to T_n$ as $k \to \infty$. Using the monotoneity of $\{P_{2k-1}\}$ and $\{P_{2k}\}$, we have, for $k \geq 1$,

$P_{2k}-(L_2/a_{n-1}) \leq T_n \leq P_{2k-1}-(L_1/a_{n-1})$, if n is odd, or

$P_{2k}-(L_1/a_{n-1}) \leq T_n \leq P_{2k-1}-(L_2/a_{n-1})$, if n is even.

With $k = 1$, we obtain (a), and with $k \geq 1$, we obtain (b). Q.E.D.

The dual of Theorem 8.2 is Theorem 8.25.

Choosing $L_1 = L_2 = 0$ in Theorem 8.2, we obtain the following theorem.

Theorem 8.3. A n.a.s.c. that an N-alternating series Σa_n converge is that

(0) $\quad a_n \to 0$,

and there exist a sequence $\{\alpha_n\}$ such that,

(1) $\quad a_n\alpha_n \to 0$

and

(2) $\quad \alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}, \quad n \geq N.$

Moreover, if (0), (1), and (2) hold, then

(a) $\quad r_n+r_nr_{n+1}\alpha_{n+1} \leq T_n \leq r_n\alpha_n, \quad n \geq N.$

And in general, for $n \geq N$ and $k \geq 1$,

(b) $\quad T_{n,2k-2}+(r_n \cdots r_{n+2k-1})\alpha_{n+2k-1} \leq T_n \leq T_{n,2k-3} + (r_n \cdots r_{n+2k-2})\alpha_{n+2k-2}$.

The dual of Theorem 8.3 is Theorem 8.27.

The following example shows that condition (2) of Theorem 8.3 cannot be replaced by the condition

(2') $\alpha_n \mathrel{\dot{\leq}} c+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$, $1 < c$.

Example 8.4. Let $1 < c$. Define $a = (1+c)/2$, so that $1 < a < c$. Define $a_{2n} = 1/(n+1)$ and $a_{2n+1} = -a/(n+1)$ $= -aa_{2n}$ for $n \geq 0$. Clearly $a_n \to 0$. Also, S_{2n-1} $\dot{=} (a_0+a_1)+(a_2+a_3)+\cdots+(a_{2n-2}+a_{2n-1}) \dot{=} (1-a)a_0+(1-a)a_2$ $+\cdots+(1-a)a_{2n-2} \dot{=} (1-a)[1+1/2+1/3+\cdots+1/n] \to -\infty$, i.e., Σa_n diverges. We have $r_{2n} \dot{=} -n/a(n+1) \to -1/a$, r_{2n+1} $= -a$, $r_{2n}r_{2n+1} \dot{=} n/(n+1)$, $r_{2n+1}r_{2n+2} \dot{=} (n+1)/(n+2)$, $c+r_{2n} \to c-1/a > 0$, and $c+r_{2n+1} \to c-a > 0$. Thus, $(c+r_{n+1})/(1-r_{n+1}r_{n+2}) \to +\infty$ and $\alpha \mathrel{\dot{\leq}} (c+r_{n+1})/(1-r_{n+1}r_{n+2})$ for any real number α. Consequently, $\alpha(1-r_{n+1}r_{n+2})$ $\mathrel{\dot{\leq}} (c+r_{n+1})$ and $\alpha \mathrel{\dot{\leq}} c+r_{n+1}+r_{n+1}r_{n+2}\alpha$. With $\alpha_n \dot{=} \alpha$, condition (2') holds. We conclude that conditions (0) and (1) of Theorem 8.3, and (2') are necessary, but not sufficient, for the convergence of Σa_n.

Theorem 8.5. Let c be any number < 1. Then a n.a.s.c. that an alternating series Σa_n converge absolutely is that

(0) $a_n \to 0,$

and there exist a sequence $\{\alpha_n\}$ such that,

(1) $a_n\alpha_n \to 0$

and

(2) $\alpha_n \leq c+r_{n+1}r_{n+1}r_{n+2}\alpha_{n+2}, \quad n \geq 1.$

Proof: For the necessity, define α_n, $n \geq 1$, by the equation $a_n\alpha_n = c(a_n+a_{n+2}+\cdots)+(a_{n+1}+a_{n+3}+\cdots)$. Then $a_n\alpha_n \to 0$. Also $a_n\alpha_n = ca_n+a_{n+1}+a_{n+2}\alpha_{n+2}$, and thus $\alpha_n = c+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$ for $n \geq 1$.

For the sufficiency, we first note that Σa_n converges according to Theorem 8.3. Define $\alpha'_n = 1+T_{n+1}$ and $\beta_n = (\alpha'_n-\alpha_n)/(1-c)$ for $n \geq 1$. Then $\alpha'_n = 1+r_{n+1} + r_{n+1}r_{n+2}\alpha'_{n+2}$, for $n \geq 1$, and $a_n\alpha'_n \to 0$, so that $|a_n|\beta_n =. |a_n|(\alpha'_n-\alpha_n)/(1-c) \to 0$. Also, $(1-c)[1-\beta_n + \beta_{n+2}a_{n+2}/a_n] = (1-c)[1-(\alpha'_n-\alpha_n)/(1-c) +(\alpha'_{n+2}-\alpha_{n+2})r_{n+1}r_{n+2}/(1-c)] = 1-c-\alpha'_n+\alpha_n+(\alpha'_{n+2}-\alpha_{n+2})r_{n+1}r_{n+2} = -\alpha'_n+1+r_{n+1}+r_{n+1}r_{n+2}\alpha'_{n+2}+\alpha_n-c-r_{n+1}-r_{n+1}r_{n+2}\alpha_{n+2} = \alpha_n-c -r_{n+1}-r_{n+1}r_{n+2}\alpha_{n+2} \leq 0$ for $n \geq 1$. Thus, $\beta_n \geq 1 +(a_{n+2}/a_n)\beta_{n+2} = 1+(|a_{n+2}|/|a_n|)\beta_{n+2}$ for $n \geq 1$. From Theorem 5.1, $\Sigma|a_{2n}|$ and $\Sigma|a_{2n+1}|$ converge, and thus Σa_n is absolutely convergent. Q.E.D.

The dual of Theorem 8.5 is Theorem 8.29.

<u>Theorem 8.6</u>. Let c, L_1, L_2 be any real numbers where $c < 1$. Then a n.a.s.c. that an alternating series Σa_n converge absolutely is that

(0) $a_n \to 0$,

and there exist a sequence $\{\alpha_n\}$ such that,

(1) $a_{2n-1}\alpha_{2n-1} \to L_1$ and $a_{2n}\alpha_{2n} \to L_2$

and

(2) $\alpha_n \leq c + r_{n+1} + r_{n+1}r_{n+2}\alpha_{n+2}$, $n \geq 1$.

<u>Proof</u>: For the necessity, there is a sequence $\{\alpha_n\}$ satisfying (1) and (2) of Theorem 8.5. Define $\{\alpha'_n\}$ by the equations $a_{2n-1}\alpha'_{2n-1} = a_{2n-1}\alpha_{2n-1} + L_1$ and $a_{2n}\alpha'_{2n} = a_{2n}\alpha_{2n} + L_2$. It may be seen that $\{\alpha'_n\}$ satisfies (1) and (2) above.

For the sufficiency, define $\{\alpha'_n\}$ by the equations $a_{2n-1}\alpha'_{2n-1} = a_{2n-1}\alpha_{2n-1} - L_1$ and $a_{2n}\alpha'_{2n} = a_{2n}\alpha_{2n} - L_2$. It may be seen that $\{\alpha'_n\}$ satisfies (1) and (2) of Theorem 8.5, and thus Σa_n converges absolutely. Q.E.D.

The dual of Theorem 8.6 is Theorem 8.30.

<u>Theorem 8.7</u>. Suppose that Σa_n is an N-alternating series such that $a_n \to 0$, $r_{n+1}r_{n+2} < 1$ for $n \geq N$, and α is a real number such that $\alpha \leq (1+r_{n+1})/(1-r_{n+1}r_{n+2})$ for

$n \geq N$. Then $r_n + r_n r_{n+1}\alpha \leq T_n \leq r_n\alpha$ for $n \geq N$.

Proof: For $n \geq N$, $\alpha(1 - r_{n+1}r_{n+2}) \leq 1 + r_{n+1}$ and $\alpha \leq 1 + r_{n+1} + r_{n+1}r_{n+2}\alpha$. Setting $\alpha_n = \alpha$ for $n \geq N$, we may use (a) of Theorem 8.3 to complete the proof. Q.E.D.

Taking $N = 1$ in Theorem 8.3, we have the following theorem.

Theorem 8.8. A n.a.s.c. that an alternating series Σa_n converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\alpha_n\}$ such that,

(1) $a_n\alpha_n \to 0$

and

(2) $\alpha_n \leq 1 + r_{n+1} + r_{n+1}r_{n+2}\alpha_{n+2}$, $n \geq 1$.

Moreover, if (0), (1), and (2) hold, then

(a) $r_n + r_n r_{n+1}\alpha_{n+1} \leq T_n \leq r_n\alpha_n$, $n \geq 1$.

And in general, for $n \geq 1$ and $k \geq 1$,

(b) $$T_{n,2k-2} + (r_n \cdots r_{n+2k-1})\alpha_{n+2k-1} \leq T_n \leq T_{n,2k-3} + (r_n \cdots r_{n+2k-2})\alpha_{n+2k-2}.$$

The dual of Theorem 8.8 is Theorem 8.31.

Remark 8.9. We will show that if any of the three conditions (0), (1), or (2) of Theorem 8.2 are omitted, the remaining two are not sufficient for the convergence of Σa_n. We may do this by making the same considerations of Theorem 8.8, since condition (0), (1), or (2) of Theorem 8.8 implies the corresponding condition of Theorem 8.2. We will show even more. In particular, condition (a) of Theorem 8.8 implies that $\alpha_n \leq 1+r_{n+1}\alpha_{n+1}$ for $n \geq 1$. We thus consider the four conditions:

(0) $a_n \to 0$,

(1) $a_n\alpha_n \to 0$,

(2) $\alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$, $n \geq 1$,

(3) $\alpha_n \leq r_{n+1}\alpha_{n+1}$, $n \geq 1$.

We will show if (0), (1), or (2) is omitted, the remaining three conditions are not sufficient for the convergence of Σa_n. We will also show that if (1) is replaced by the two weaker conditions that $a_n\alpha_n-a_{n+1}\alpha_{n+1} \to 0$ and that $\{a_n\alpha_n\}$ be bounded, the resulting four conditions are not sufficient for the convergence of Σa_n.

Counterexample 8.10. Let Σa_n be the divergent series $1-1+1-1+-\cdots$. We have $a_n = (-1)^n$ for $n \geq 0$, and $r_n = -1$ for $n \geq 1$. Defining $\alpha_n = 0$ for $n \geq 1$, the following three conditions obviously hold:

(1) $a_n\alpha_n \to 0$,

(2) $\alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$, $n \geq 1$,

(3) $\alpha_n \leq 1+r_{n+1}\alpha_{n+1}$, $n \geq 1$.

We have shown that conditions (1), (2), and (3) are not sufficient for the convergence of Σa_n.

<u>Counterexample 8.11</u>. Let $\Sigma a_n = 1-1/2+1/2-1/(2.2)$

$+\cdots+ 1/(n+1)-1/2(n+1)+\cdots$.

This series is divergent, since for $n \geq 1$,

$$S_{2n-1} = (1-1/2)+(1/2-1/(2.2))+\cdots+(1/n-1/2n)$$

$$= (1/2)(1+1/2+1/3+\cdots+1/n).$$

Let α_1 be any real number, and define the sequence $\{\alpha_n\}$ recursively by the equation $\alpha_n = 1+r_{n+1}\alpha_{n+1}$. The following conditions are seen to hold:

(0) $a_n \to 0$,

(2) $\alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$, $n \geq 1$,

(3) $\alpha_n \leq 1+r_{n+1}\alpha_{n+1}$, $n \geq 1$.

We conclude that conditions (0), (2), and (3) are not sufficient for the convergence of Σa_n. Moreover, $a_n\alpha_n - a_{n+1}\alpha_{n+1} =. a_n \to 0$, so that the four conditions $a_n\alpha_n - a_{n+1}\alpha_{n+1} \to 0$, (0), (2), and (3) are not sufficient for the convergence of Σa_n.

Counterexample 8.12. Let Σa_n be the divergent series given in Counterexample 8.11. Defining $\alpha_n = 0$ for $n \geq 1$, it is obvious that the following conditions hold:

(0) $a_n \to 0$,

(1) $a_n \alpha_n \to 0$,

(3) $\alpha_n \leq 1 + r_{n+1}\alpha_{n+1}$, $n \geq 1$.

Thus conditions (0), (1), and (3) are not sufficient for the convergence of Σa_n. Also, Theorem 8.8 implies that the condition

(2) $\alpha_n \leq 1 + r_{n+1} + r_{n+1}r_{n+2}\alpha_{n+2}$, $n \geq 1$,

is false. Indeed, (2) must fail to hold for infinitely many values of n according to Theorem 8.3.

Counterexample 8.13. Let Σa_n be any divergent alternating series whose partial sums are bounded, and such that $a_n \to 0$. Let α_1 be any real number, and define the sequence $\{\alpha_n\}$ recursively by the equation $\alpha_n = 1 + r_{n+1}\alpha_{n+1}$. We easily see that $a_{n+1}\alpha_{n+1} = a_1\alpha_1 - (a_1 + a_2 + \cdots + a_n)$ for $n \geq 1$. Consequently, the sequence $\{a_n\alpha_n\}$ is bounded, since the partial sums S_n are bounded. Conditions (0), (2), and (3) of Remark 8.9 are easily seen to hold. Consequently, these three conditions along with the condition that $\{a_n\alpha_n\}$ be bounded are not sufficient for the convergence of Σa_n. Moreover, it is of no avail to also require that $a_n\alpha_n - a_{n+1}\alpha_{n+1} \to 0$, since $\alpha_n \doteq 1 + r_{n+1}\alpha_{n+1}$ yields $a_n\alpha_n - a_{n+1}\alpha_{n+1} \doteq a_n \to 0$ in the present counterexample.

Theorem 8.14. Let L be any real number and Σa_n be any N-alternating series such that $a_{2n} \mathrel{\dot{>}} 0$. Then a n.a.s.c. that Σa_n converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\alpha_n\}$ such that,

(1) some subsequence of $\{a_{2n-1}\alpha_{2n-1}\}$ is bounded below and $a_{2n}\alpha_{2n} \to L$

and

(2) $a_n \leq 1 + r_{n+1} + r_{n+1}r_{n+2}\alpha_{n+2}$, $n \geq N$.

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n-1}\alpha_{2n-1}\}$ converges.

Proof: The necessity is immediate from Theorem 8.2.

For the sufficiency, let m be any odd integer $\geq N+1$. Define $P_k = T_{m,k-2} + (r_m \cdots r_{m+k-1})\alpha_{m+k-1}$ for $k \geq 1$. Then,

(3) $\quad P_k - P_{k+2} = (r_m \cdots r_{m+k-1})[\alpha_{m+k-1} - (1+r_{m+k}$

$+ r_{m+k} r_{m+k-1} \alpha_{m+k-1})], \quad k \geq 1.$

From (2) and (3), we see that $P_{2k} - P_{2k+2} \leq 0$ and $P_{2k-1} - P_{2k+1} \geq 0$ for $K \geq 1$, so that $\{P_{2k}\}$ is monotone increasing and $\{P_{2k-1}\}$ is monotone decreasing.

Also,

(4) $\quad P_{2k-1} - P_{2k} = (a_{m+2k-2}\alpha_{m+2k-2} - a_{m+2k-2}$

$- a_{m+2k-1}\alpha_{m+2k-1})/a_{m-1}$

for $k \geq 1$, so that by (0), (1), and the fact that $a_{m-1} > 0$, some subsequence of $\{P_{2k-1} - P_{2k}\}$ is bounded below. By Lemma 8.1, $P_{2k-1} \to P'$ and $P_{2k} \to P''$ for some numbers P' and P''. Also, according to (1), $a_{m+2k-1}\alpha_{m+2k-1} \to L$ as $k \to \infty$. From (4), $a_{m+2k-2}\alpha_{m+2k-2}$ $=. \; a_{m+2k-2} + a_{m+2k-1}\alpha_{m+2k-1} + a_{m-1}(P_{2k-1} - P_{2k}) \to L + a_{m-1}(P' - P'')$ as $k \to \infty$. Consequently, m being odd, we see that $\{a_{2n-1}\alpha_{2n-1}\}$ converges. Theorem 8.2 now implies that Σa_n converges. Q.E.D.

The dual of Theorem 8.14 is Theorem 8.40.

<u>Theorem 8.15</u>. Let L be any real number and Σa_n be any N-alternating series such that $a_{2n} <. \; 0$. Then a n.a.s.c. that Σa_n converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\alpha_n\}$ such that,

(1) some subsequence of $\{a_{2n-1}\alpha_{2n-1}\}$ is bounded above and $a_{2n}\alpha_{2n} \to L$

and

(2) $\alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$, $n \geq N$.

Moreover, if conditions (0), (1) and (2) hold, then $\{a_{2n-1}\alpha_{2n-1}\}$ converges.

Proof: The necessity follows from Theorem 8.2.

For the sufficiency, define $a'_n = -a_n$ for $n \geq 0$. Accordingly, $r'_n = a'_n/a'_{n-1} = a_n/a_{n-1} = r_n$ for $n \geq N$. It is obvious that Theorem 8.14 is applicable, yielding the convergence of $\Sigma a'_n$ and $\{a'_{2n-1}\alpha_{2n-1}\}$. Thus, Σa_n and $\{a_{2n-1}\alpha_{2n-1}\}$ both converge. Q.E.D.

The dual of Theorem 8.15 is Theorem 8.39.

It has been shown that (1) of Theorem 8.2 cannot be omitted, or replaced by the weaker condition that $\{a_n\alpha_n\}$ be bounded and $a_n\alpha_n-a_{n+1}\alpha_{n+1} \to 0$. The following theorem shows that (1) can be replaced by the weaker condition that some subsequence of $\{a_{2n-1}\alpha_{2n-1}\}$ be bounded and $\{a_{2n}\alpha_{2n}\}$ converge.

Theorem 8.16. Let L be any real number. Then a n.a.s.c. that an N-alternating series Σa_n converge is that

(0) $\quad a_n \to 0,$

and there exist a sequence $\{\alpha_n\}$ such that,

(1) some subsequence of $\{a_{2n-1}\alpha_{2n-1}\}$ is bounded and

$a_{2n}\alpha_{2n} \to L$

and

(2) $\quad \alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}, \quad n \geq N.$

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n-1}\alpha_{2n-1}\}$ converges.

Proof: The necessity follows from Theorem 8.2.

For the sufficiency, we need only note that $a_{2n} >. 0$ or $a_{2n} <. 0$, and then apply Theorem 8.14 or Theorem 8.15, respectively. Q.E.D.

The dual of Theorem 8.16 is Theorem 8.41.

The following counterexample shows that (1) of Theorem 8.14 or Theorem 8.16 cannot be replaced by the condition

(1') $\quad \{a_{2n-1}\alpha_{2n-1}\}$ is bounded above and $a_{2n}\alpha_{2n} \to L.$

Counterexample 8.17. Let Σa_n be the divergent series given in Counterexample 8.11. We have $a_{2n} = 1/(n+1)$ and $a_{2n+1} = -1/2(n+1)$ for $n \geq 0$. Define $\alpha_{2n} = 0$

for $n \geq 1$. Define $\{\alpha_{2n-1}\}$ recursively by the equation $\alpha_{2n-1} = 1+r_{2n}+r_{2n}r_{2n+1}\alpha_{2n+1}$, $n \geq 1$, where α_1 is any real number. It can be seen that (0) $a_n \to 0$, (1) $a_{2n}\alpha_{2n} \to 0$, and (2) $\alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$ for $n \geq 1$. Also, $a_{2n+1}\alpha_{2n+1} \doteq a_1\alpha_1-(a_1+a_2+\cdots+a_{2n}) \to -\infty$, so that $\{a_{2n-1}\alpha_{2n-1}\}$ is bounded above.

The following counterexample shows that (1) of Theorem 8.15 or Theorem 8.16 cannot be replaced by the condition

(1') $\{a_{2n-1}\alpha_{2n-1}\}$ is bounded below and $a_{2n}\alpha_{2n} \to L$.

Counterexample 8.18. Let Σa_n be the divergent series whose terms are the negatives of those of the series given in Counterexample 8.17, i.e., $a_{2n} = -1/(n+1)$ and $a_{2n+1} = 1/2(n+1)$ for $n \geq 0$. Define $\alpha_{2n} = 0$ for $n \geq 1$. Define $\{\alpha_{2n-1}\}$ recursively by the equation $\alpha_{2n-1} = 1 + r_{2n}+r_{2n}r_{2n+1}\alpha_{2n+1}$, $n \geq 1$, where α_1 is any real number. Then (0) $a_n \to 0$, (1) $a_{2n}\alpha_{2n} \to 0$, and (2) $\alpha_n \leq 1+r_{n+1} + r_{n+1}r_{n+2}\alpha_{n+2}$ for $n \geq 1$. Also, $a_{2n+1}\alpha_{2n+1} \doteq a_1\alpha_1 - (a_1+a_2+\cdots+a_{2n}) \to +\infty$, so that $\{a_{2n-1}\alpha_{2n-1}\}$ is bounded below.

Theorem 8.19. Let L be qny real number and Σa_n be any N-alternating series such that $a_{2n} >. 0$. Then a n.a.s.c. that Σa_n converge is that

(0) $\quad a_n \to 0,$

and there exist a sequence $\{\alpha_n\}$ such that,

(1) $\quad$ some subsequence of $\{a_{2n}\alpha_{2n}\}$ is bounded above

$\quad$ and $a_{2n-1}\alpha_{2n-1} \to L$

and

(2) $\quad \alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}, \quad n \geq N.$

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n}\alpha_{2n}\}$ converges.

<u>Proof</u>: The necessity follows from Theorem 8.2.

According to Theorem 8.2, for the sufficiency we need only show that $\{a_{2n}\alpha_{2n}\}$ converges. Define $a'_n = a_{n+1}$ for $n \geq 0$, and $\alpha'_n = \alpha_{n+1}$ for $n \geq N$. Then $a'_n \to 0$ and $a'_{2n}\alpha'_{2n} = a_{2n+1}\alpha_{2n+1} \to L$. Since some subsequence of $\{a_{2n}\alpha_{2n}\}$ is bounded above and $a'_{2n-1}\alpha'_{2n-1} =. a_{2n}\alpha_{2n}$, it follows that some subsequence of $\{a'_{2n-1}\alpha'_{2n-1}\}$ is bounded above. We have $a'_{2n} =. a_{2n+1} <. 0$. Also, $r'_n = a'_n/a'_{n-1} = a_{n+1}/a_n = r_{n+1}$ for $n \geq N$. From (2), for $n \geq N$, $\alpha'_n = \alpha_{n+1} \leq 1+r_{n+2}+ r_{n+2}r_{n+3}\alpha_{n+3} = 1+r'_{n+1} + r'_{n+1}r'_{n+2}\alpha'_{n+2}$. Applying Theorem 8.15, $\{a'_{2n-1}\alpha'_{2n-1}\}$

converges. Thus, $\{a_{2n}\alpha_{2n}\}$ converges. Q.E.D.

The dual of Theorem 8.19 is Theorem 8.43.

Theorem 8.20. Let L be any real number and Σa_n any N-alternating series such that $a_{2n} <\cdot 0$. Then a n.a.s.c. that Σa_n converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\alpha_n\}$ such that,

(1) some subsequence of $\{a_{2n}\alpha_{2n}\}$ is bounded below and

$$a_{2n-1}\alpha_{2n-1} \to L$$

and

(2) $\alpha_n \leq 1 + r_{n+1} + r_{n+1}r_{n+2}\alpha_{n+2}, \quad n \geq N.$

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n}\alpha_{2n}\}$ converges.

Proof: The necessity follows from Theorem 8.2.

For the sufficiency, define $a_n' = -a_n$ for $n \geq 0$. Accordingly, $r_n' = a_n'/a_{n-1}' = a_n/a_{n-1} = r_n$ for $n \geq N$. It is easily seen that Theorem 8.19 is applicable, yielding the convergence of $\Sigma a_n'$ and $\{a_{2n}'\alpha_{2n}\}$. Thus, Σa_n and $\{a_{2n}\alpha_{2n}\}$ both converge. Q.E.D.

The dual of Theorem 8.20 is Theorem 8.42.

Theorem 8.21. Let L be any real number. Then a n.a.s.c. that an N-alternating series Σa_n converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\alpha_n\}$ such that,

(1) some subsequence of $\{a_{2n}\alpha_{2n}\}$ is bounded and

$a_{2n-1}\alpha_{2n-1} \to L$

and

(2) $\alpha_n \leq 1 + r_{n+1} + r_{n+1}r_{n+2}\alpha_{n+2}$, $n \geq N$.

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n}\alpha_{2n}\}$ converges.

Proof: The necessity follows from Theorem 8.2.

For the sufficiency, we need only note that $a_{2n} >. 0$ or $a_{2n} <. 0$, and then apply Theorem 8.19 or Theorem 8.20, respectively. Q.E.D.

The dual of Theorem 8.21 is Theorem 8.44.

The following counterexample shows that (1) of Theorem 8.19 or Theorem 8.21 cannot be replaced by the condition

(1') $\{a_{2n}\alpha_{2n}\}$ is bounded below and $a_{2n-1}\alpha_{2n-1} \to L$.

Counterexample 8.22. Define $a_{2n} = 1/2(n+1)$ and $a_{2n+1} = -1/(n+1)$ for $n \geq 0$. Since $a_{2n}+a_{2n+1} = 1/2(n+1)$

for $n \geq 0$, $S_n \to -\infty$. Define $\alpha_{2n-1} = 0$ for $n \geq 1$. Define $\{\alpha_{2n}\}$ recursively by the equation $\alpha_{2n} = 1+r_{2n+1} +r_{2n+1}r_{2n+2}\alpha_{2n+2}$, $n \geq 1$, where α_2 is any real number. We then have (0) $a_n \to 0$, (1) $a_{2n-1}\alpha_{2n-1} \to 0$, and (2) $\alpha_n \leq 1 +r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$ for $n \geq 1$. Also, $a_{2n}\alpha_{2n} \doteq a_2\alpha_2-(a_2+a_3+\cdots+a_{2n-1}) \to +\infty$, so that $\{a_{2n}\alpha_{2n}\}$ is bounded below.

The following counterexample shows that (1) of Theorem 8.20 or Theorem 8.21 cannot be replaced by the condition

(1') $\{a_{2n}\alpha_{2n}\}$ is bounded above and $a_{2n-1}\alpha_{2n-1} \to L$.

<u>Counterexample 8.23</u>. Let Σa_n be the divergent series whose terms are the negatives of those of the series given in Counterexample 8.22, i.e., $a_{2n} = -1/2(n+1)$ and $a_{2n+1} = 1/(n+1)$ for $n \geq 0$. Define $\alpha_{2n-1} = 0$ for $n \geq 1$. Define $\{\alpha_{2n}\}$ recursively by the equation $\alpha_{2n} = 1+r_{2n+1}+r_{2n+1}r_{2n+2}\alpha_{2n+2}$, $n \geq 1$, where α_2 is any real number. Accordingly, (0) $a_n \to 0$, (1) $a_{2n-1}\alpha_{2n-1} \to 0$, and (2) $\alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$ for $n \geq 1$. Also, $a_{2n}\alpha_{2n} \doteq a_2\alpha_2-(a_2+a_3+\cdots+a_{2n-1}) \to -\infty$, and thus $\{a_{2n}\alpha_{2n}\}$ is bounded above.

Lemma 8.24. Let Σa_n be an N-alternating series and $\{\beta_n\}$ be a sequence such that

(0) $a_n \to 0$,

(1) $a_{2n-1}\beta_{2n-1} \to L_1$ and $a_{2n}\beta_{2n} \to L_2$, for some L_1 and L_2,

and

(2) $\beta_n \geq 1 + r_{n+1} + r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq N$.

Defining $\alpha_n = 1+r_{n+1}\beta_{n+1}$, for $n \geq N$, we have

(3) $a_{2n-1}\alpha_{2n-1} \to L_2$ and $a_{2n}\alpha_{2n} \to L_1$

and

(4) $\alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$, $n \geq N$.

Moreover, for $n \geq N$ and $k \geq 1$,

(5)
$$T_{n,2k-2}+(r_n\cdots r_{n+2k-1})\beta_{n+2k-1}$$
$$= T_{n,2k-3}+(r_n\cdots r_{n+2k-2})\alpha_{n+2k-2}$$

and

(6)
$$T_{n,2k-3}+(r_n\cdots r_{n+2k-2})\beta_{n+2k-2}$$
$$\leq T_{n,2k-2}+(r_n\cdots r_{n+2k-1})\alpha_{n+2k-1} \cdot$$

Proof: Since $\alpha_n =. 1+r_{n+1}\beta_{n+1}$, $a_{2n-1}\alpha_{2n-1} =. a_{2n-1}+a_{2n}\beta_{2n} \to L_2$ and $a_{2n}\alpha_{2n} =. a_{2n}+a_{2n+1}\beta_{2n+1} \to L_1$. Using (2),

$\alpha_n-(1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}) = 1+r_{n+1}\beta_{n+1}-(1+r_{n+1}+r_{n+1}r_{n+2}$ $+r_{n+1}r_{n+2}r_{n+3}\beta_{n+3}) = r_{n+1}[\beta_{n+1}-(1+r_{n+2}+r_{n+2}r_{n+3}\beta_{n+3})]\leq 0$, so that (4) holds. Next, $T_{n,2k-3} + (r_n\cdots r_{n+2k-2})\alpha_{n+2k-2}$ $= T_{n,2k-3}+(r_n\cdots r_{n+2k-2})(1+r_{n+2k-1}\beta_{n+2k-1}) = T_{n,2k-2}$ $+ (r_n\cdots r_{n+2k-1})\beta_{n+2k-1}$. Thus (5) holds. Again using (2), $T_{n,2k-3}+(r_n\cdots r_{n+2k-2})\beta_{n+2k-2} \leq T_{n,2k-3}$ $+ (r_n\cdots r_{n+2k-2})(1+r_{n+2k-1}+r_{n+2k-1}r_{n+2k}\beta_{n+2k})] = T_{n,2k-2}$ $+ (r_n\cdots r_{n+2k-1})(1+r_{n+2k}\beta_{n+2k}) = T_{n,2k-2}$ $+ (r_n\cdots r_{n+2k-1})\alpha_{n+2k-1}$. Consequently (6) holds. Q.E.D.

<u>Theorem 8.25</u>. Let L_1 and L_2 be any real numbers. Then a n.a.s.c. that an N-alternating series Σa_n converge is that

(0) $\quad a_n \to 0,$

and there exist a sequence $\{\beta_n\}$ such that,

(1) $\quad a_{2n-1}\beta_{2n-1} \to L_1$ and $a_{2n}\beta_{2n} \to L_2$

and

(2) $\quad \beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq N$.

Moreover, if (0), (1), and (2) hold, then, for $n \geq N$,

$$(a)\quad \begin{cases} r_n+r_nr_{n+1}\beta_{n+1}-(L_2/a_{n-1}) \geq T_n \geq r_n\beta_n-(L_1/a_{n-1}) \\ \text{or} \\ r_n+r_nr_{n+1}\beta_{n+1}-(L_1/a_{n-1}) \geq T_n \geq r_n\beta_n-(L_2/a_{n-1}), \end{cases}$$

accordingly as n is odd or even, respectively. And in general, for $n \geq N$ and $k \geq 1$,

$$
\text{(b)} \begin{cases} T_{n,2k-2}+(r_n\cdots r_{n+2k-1})\beta_{n+2k-1}-(L_2/a_{n-1}) \\ \qquad \geq T_{n,2k-3}+(r_n\cdots r_{n+2k-2})\beta_{n+2k-2}-(L_1/a_{n-1}) \\ \text{or} \\ T_{n,2k-2}+(r_n\cdots r_{n+2k-1})\beta_{n+2k-1}-(L_1/a_{n-1}) \geq T_n \\ \qquad \geq T_{n,2k-3}+(r_n\cdots r_{n+2k-2})\beta_{n+2k-2}-(L_2/a_{n-1}), \end{cases}
$$

accordingly as n is odd or even, respectively.

Proof: For the necessity, we may use the proof of the necessity of Theorem 8.2, replacing "α" by "β" throughout.

For the sufficiency, assume that (0), (1), and (2) hold, and define $\alpha_n = 1+r_{n+1}\beta_{n+1}$ for $n \geq N$. According to Lemma 8.24, conditions (0), (1), and (2) of Theorem 8.2 hold, with L_1 and L_2 interchanged. Using (b) of Theorem 8.2, and (5) and (6) of Lemma 8.24, we obtain (b) of the present theorem, from which (a) follows with $k=1$. Q.E.D.

The dual of Theorem 8.25 is Theorem 8.2. Choosing $L_1 = L_2 = L$ in Theorem 8.25, we obtain the following theorem.

Theorem 8.26. Let L be any real number. Then a n.a.s.c. that an N-alternating series Σa_n converge is that

(0) $\quad a_n \to 0$,

and there exist a sequence $\{\beta_n\}$ such that,

(1) $\quad a_n\beta_n \to L$

and

(2) $\quad \beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}, \quad n \geq N.$

Moreover, if (0), (1), and (2) hold, then, for $n \geq N$,

(a) $\quad r_n+r_nr_{n+1}\beta_{n+1}-(L/a_{n-1}) \geq T_n \geq r_n\beta_n-(L/a_{n-1}).$

And in general, for $n \geq N$ and $k \geq 1$,

(b) $\quad T_{n,2k-2}+(r_n\cdots r_{n+2k-1})\beta_{n+2k-1}-(L/a_{n-1}) \geq T_n$

$$\geq T_{n,2k-3}+(r_n\cdots r_{n+2k-2})\beta_{n+2k-2}-(L/a_{n-1}).$$

Theorem 8.26 can be seen to have a dual by setting $L_1 = L_2 = L$ in Theorem 8.2.

The following example shows that condition (2) of Theorem 8.27 cannot be replaced by the condition

(2') $\quad \beta_n \geq c+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}, \quad c < 1.$

Example 8.28. Let $0 < c < 1$, so that $1 < 1/c$. Let Σa_n be the divergent series defined in Example 8.4. According to that example, $a_n \to 0$, and there is a sequence $\{\alpha_n\}$ such that $a_n\alpha_n \to 0$ and $\alpha_n \leq. 1/c+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$. Defining $\beta_n =. c(1+r_{n+1}\alpha_{n+1})$, $a_n\beta_n =. c(a_n+a_{n+1}\alpha_{n+1}) \to 0$. From the preceeding inequality it is easily seen that

(2') holds. We conclude that (0) and (1) of Theorem 8.27 and (2') are necessary, but not sufficient, for the convergence of Σa_n.

Choosing $L_1 = L_2 = 0$ in Theorem 8.25, we obtain the following theorem.

Theorem 8.27. A n.a.s.c. that an N-alternating series Σa_n converge is that

(0) $\quad a_n \to 0,$

and there exist a sequence $\{\beta_n\}$ such that,

(1) $\quad a_n \beta_n \to 0$

and

(2) $\quad \beta_n \geq 1 + r_{n+1} + r_{n+1} r_{n+2} \beta_{n+2}, \quad n \geq N.$

Moreover, if (0), (1), and (2) hold, then, for $n \geq N$,

(a) $\quad r_n + r_n r_{n+1} \beta_{n+1} \geq T_n \geq r_n \beta_n.$

And in general, for $n \geq N$ and $k \geq 1$,

(b) $$T_{n,2k-2} + (r_n \cdots r_{n+2k-1})\beta_{n+2k-1} \geq T_n \geq T_{n,2k-3} + (r_n \cdots r_{n+2k-2})\beta_{n+2k-2}.$$

The dual of Theorem 8.27 is Theorem 8.3.

<u>Theorem 8.29</u>. Let c be any number > 1. Then a n.a.s.c. that an alternating series Σa_n converge absolutely is that

(0) $\quad a_n \to 0,$

and there exist a sequence $\{\beta_n\}$ such that,

(1) $a_n\beta_n \to 0$

and

(2) $\beta_n \geq c + r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq 1$.

Proof: For the necessity, we may use the proof of the necessity of Theorem 8.5, replacing "α" by "β" throughout.

For the sufficiency, define α_n, $n \geq 1$, by the equation $c\alpha_n = 1+r_{n+1}\beta_{n+1}$. Then $a_n \to 0$ and $a_n\alpha_n = (a_n+a_{n+1}\beta_{n+1})/c \to 0$. From (2),

$c[\alpha_n - (1/c+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2})] = r_{n+1}[\beta_{n+1} - (c+r_{n+2}+r_{n+2}r_{n+3}\beta_{n+3})] \leq 0$, so that $\alpha_n \leq 1/c + r_{n+1} + r_{n+1}r_{n+2}\alpha_{n+2}$ for $n \geq 1$, where $1/c < 1$. According to Theorem 8.5, $\Sigma|a_n|$ converges. Q E.D.

The dual of Theorem 8.29 is Theorem 8.5.

Theorem 8.30. Let c, L_1, L_2 be any real numbers where $1 < c$. Then a n.a.s.c. that an alternating series Σa_n converge absolutely is that

(0) $a_n \to 0$,

and there exist a sequence $\{\beta_n\}$ such that,

(1) $a_{2n-1}\beta_{2n-1} \to L_1$ and $a_{2n}\beta_{2n} \to L_2$

and

(2) $\beta_n \geq c + r_{n+1} + r_{n+1} r_{n+2} \beta_{n+2}, \quad n \geq 1.$

Proof: For the necessity, there is a sequence $\{\beta_n\}$ satisfying (1), (2) of Theorem 8.29. Define $\{\beta_n'\}$ by the equations $a_{2n-1}\beta_{2n-1}' = a_{2n-1}\beta_{2n-1} + L_1$ and $a_{2n}\beta_{2n}' = a_{2n}\beta_{2n} + L_2$. It is easily seen that $\{\beta_n'\}$ satisfies (1) and (2) above.

For the sufficiency, define $\{\beta_n'\}$ by the equations $a_{2n-1}\beta_{2n-1}' = a_{2n-1}\beta_{2n-1} - L_1$ and $a_{2n}\beta_{2n}' = a_{2n}\beta_{2n} - L_2$. We easily verify that $\{\beta_n'\}$ satisfies (1) and (2) of Theorem 8.29, and thus Σa_n converges absolutely. Q.E.D.

The dual of Theorem 8.30 is Theorem 8.6. With $N = 1$ in Theorem 8.27, we obtain the following theorem.

Theorem 8.31. A n.a.s.c. that an alternating series Σa_n converge is that

(0) $a_n \to 0,$

and there exist a sequence $\{\beta_n\}$ such that,

(1) $a_n\beta_n \to 0$

and

(2) $\beta_n \geq 1 + r_{n+1} + r_{n+1} r_{n+2} \beta_{n+2}, \quad n \geq 1.$

Moreover, if (0), (1), and (2) hold, then, for $n \geq 1$,

(a) $\quad r_n + r_n r_{n+1}\beta_{n+1} \geq T_n \geq r_n\beta_n.$

And in general, for $n \geq 1$ and $k \geq 1$,

(b) $$T_{n,2k-2} + (r_n \cdots r_{n+2k-1})\beta_{n+2k-1} \geq T_n \geq T_{n,2k-3} + (r_n \cdots r_{n+2k-2})\beta_{n+2k-2} .$$

The dual of Theorem 8.31 is Theorem 8.8.

Theorem 8.32. Let L be any real number. Then a n.a.s.c. that an N-alternating series Σa_n converge is that there exist a sequence $\{\beta_n\}$ such that

(1) $\quad a_n\beta_n \to L,$

(2) $\quad \beta_n \geq 1 + r_{n+1} + r_{n+1}r_{n+2}\beta_{n+2}, \quad n \geq N,$

and

(3) $\quad \beta_n \geq 1 + r_{n+1}\beta_{n+1}, \quad n \geq N.$

Moreover, if (1), (2), and (3) hold, then, for $n \geq N$,

(a) $\quad r_n + r_n r_{n+1}\beta_{n+1} - (L/a_{n-1}) \geq T_n \geq r_n\beta_n - (L/a_{n-1}).$

And in general, for $n \geq N$ and $k \geq 1$,

(b) $$T_{n,2k-2} + (r_n \cdots r_{n+2k-1})\beta_{n+2k-1} - (L/a_{n-1}) \geq T_n \geq T_{n,2k-3} + (r_n \cdots r_{n+2k-2})\beta_{n+2k-2} - (L/a_{n-1}).$$

Proof: For the necessity, Theorem 8.26 implies the existence of a sequence $\{\beta_n\}$ such that conditions (0), (1), and (2) are satisfied. Also, by (a) of Theorem 8.26, we have $r_n + r_n r_{n+1}\beta_{n+1} - (L/a_{n-1}) \geq r_n\beta_n - (L/a_{n-1})$ for

$n \geq N$, from which (3) follows.

For the sufficiency, assume that (1), (2), and (3) hold. Using (1), (3), and the fact that $|a_n|/a_n$, $n \geq N$, is bounded, we have $0 \mathrel{<\!\cdot} |a_n| \mathrel{\leq\!\cdot} |a_n|(\beta_n - \beta_{n+1} r_{n+1})$ $\mathrel{=\!\cdot} (|a_n|/a_n)(a_n\beta_n - a_{n+1}\beta_{n+1}) \to 0$, so that $|a_n| \to 0$, i.e., $a_n \to 0$. Now apply Theorem 8.26. Q.E.D.

According to Counterexample 8.10, Theorem 8.32 has no dual.

<u>Remark 8.33</u>. We now consider the four conditions:

(0) $\quad a_n \to 0$,

(1) $\quad a_n\beta_n \to 0$,

(2) $\quad \beta_n \geq 1 + r_{n+1} + r_{n+1}r_{n+2}\beta_{n+2}$, $\quad n \geq 1$,

(3) $\quad \beta_n \geq 1 + r_{n+1}\beta_{n+1}$, $\quad n \geq 1$.

We have seen that if (0) or (3) is omitted, the remaining three conditions are necessary and sufficient for the convergence of an alternating series Σa_n. It will be shown that if condition (1) or (2) is omitted, the remaining three are not sufficient for the convergence of Σa_n. We will see that conditions (1) and (2) are not sufficient for the convergence of Σa_n. It will also be seen that if (1) is replaced by the weaker conditions that

$a_n\beta_n - a_{n+1}\beta_{n+1} \to 0$ and that $\{a_n\beta_n\}$ be bounded, the resulting four conditions are not sufficient for the convergence of Σa_n.

<u>Counterexample 8.34</u>. We use Counterexample 8.11 with α_n, $n \geq 1$, as defined there. Defining $\beta_n = \alpha_n$ for $n \geq 1$, the following conditions are obvious:

(0) $a_n \to 0$,

(2) $\beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq 1$,

(3) $\beta_n \geq 1+r_{n+1}\beta_{n+1}$, $n \geq 1$.

Also, $a_n\beta_n - a_{n+1}\beta_{n+1} = a_n \to 0$ so that the four conditions $a_n\beta_n - a_{n+1}\beta_{n+1} \to 0$, (0), (2), and (3) are not sufficient for the convergence of Σa_n.

<u>Counterexample 8.35</u>. Let Σa_n be the divergent series given in Counterexample 8.11. Defining $\beta_n = 1$ for $n \geq 1$, it is obvious that the following conditions hold:

(0) $a_n \to 0$,

(1) $a_n\beta_n \to 0$,

(3) $\beta_n \geq 1+r_{n+1}\beta_{n+1}$, $n \geq 1$.

Thus conditions (0), (1), and (3) are not sufficient for the convergence of Σa_n.

Counterexample 8.36. Let Σa_n be the divergent series in Counterexample 8.10 and $\{\beta_n\}$ be any monotone decreasing sequence such that $\beta_n \to 0$. We then have

(1) $\quad a_n\beta_n \to 0$

and

(2) $\quad \beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}, \quad n \geq 1.$

Thus conditions (1) and (2) are not sufficient for the convergence of Σa_n.

Counterexample 8.37. Let Σa_n be the divergent series in Counterexample 8.10, L be any number $\geq 1/2$, and $\{\beta_n\}$ be any monotone decreasing sequence converging to L. We then have

(1) $\quad a_{2n-1}\beta_{2n-1} \to -L$ and $a_{2n}\beta_{2n} \to L,$

(2) $\quad \beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}, \quad n \geq 1,$

and

(3) $\quad \beta_n \geq 1+r_{n+1}\beta_{n+1}, \quad n \geq 1.$

Consequently, (1) of Theorem 8.32 cannot be replaced by the weaker condition that $a_{2n-1}\beta_{2n-1} \to L_1$ and $a_{2n}\beta_{2n} \to L_2$, for some numbers L_1 and L_2. The corresponding replacement in Theorem 8.26 was valid according to Theorem 8.25.

Counterexample 8.38. We use Counterexample 8.13 with α_n, $n \geq 1$, as defined there. Defining $\beta_n = \alpha_n$, for $n \geq 1$, the following conditions hold:

(0) $a_n \to 0$,

(2) $\beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq 1$,

(3) $\beta_n \geq 1+r_{n+1}\beta_{n+1}$, $n \geq 1$.

According to Counterexample 8.13, the sequence $\{a_n\beta_n\}$ is bounded and $a_n\beta_n - a_{n+1}\beta_{n+1} \to 0$. Thus, replacing (1) of Remark 8.33 by these two conditions, the resulting conditions are not sufficient for the convergence of Σa_n.

Theorem 8.39. Let L be any real number and Σa_n be any N-alternating series such that $a_{2n} >. 0$. Then a n.a.s.c. that Σa_n converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\beta_n\}$ such that,

(1) some subsequence of $\{a_{2n-1}\beta_{2n-1}\}$ is bounded above and $a_{2n}\beta_{2n} \to L$

and

(2) $\beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq N$.

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n-1}\beta_{2n-1}\}$ converges.

Proof: The necessity follows from Theorem 8.25.

For the sufficiency, define $\alpha_n = 1+r_{n+1}\beta_{n+1}$ for $n \geq N$. Then $a_{2n-1}\alpha_{2n-1} \doteq a_{2n-1}+a_{2n}\beta_{2n} \to L$. Since $a_{2n}\alpha_{2n} = a_{2n}+a_{2n+1}\beta_{2n+1}$, some subsequence of $\{a_{2n}\alpha_{2n}\}$ is bounded above. Also, $\alpha_n = 1+r_{n+1}\beta_{n+1} \leq 1+r_{n+1}$ $+r_{n+1}r_{n+2}(1+r_{n+3}\beta_{n+3}) = 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$ for $n \geq N$. From Theorem 8.19, both Σa_n and $\{a_{2n}\alpha_{2n}\}$ converge. Consequently, $a_{2n+1}\beta_{2n+1} \doteq a_{2n}\alpha_{2n}-a_{2n} \to \lim a_{2n}\alpha_{2n}$, i.e., $\{a_{2n-1}\beta_{2n-1}\}$ converges. Q.E.D.

The dual of Theorem 8.39 is Theorem 8.15.

Theorem 8.40. Let L be any real number and Σa_n be any N-alternating series such that $a_{2n} \lessdot 0$. Then a n.a.s.c. that Σa_n converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\beta_n\}$ such that,

(1) some subsequence of $\{a_{2n-1}\beta_{2n-1}\}$ is bounded below and $a_{2n}\beta_{2n} \to L$

and

(2) $\beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq N$.

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n-1}\beta_{2n-1}\}$ converges.

Proof: The necessity follows from Theorem 8.25.

For the sufficiency, define $a'_n = -a_n$ for $n \geq 0$. Accordingly, $r'_n = a'_n/a'_{n-1} = a_n/a_{n-1} = r_n$ for $n \geq N$. It is easily seen that Theorem 8.39 is applicable, yielding the convergence of $\Sigma a'_n$ and $\{a'_{2n-1}\beta_{2n-1}\}$. Thus, Σa_n and $\{a_{2n-1}\beta_{2n-1}\}$ both converge. Q.E.D.

The dual of Theorem 8.40 is Theorem 8.14.

Theorem 8.41. Let L be any real number. Then a n.a.s.c. that an N-alternating series Σa_n converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\beta_n\}$ such that,

(1) some subsequence of $\{a_{2n-1}\beta_{2n-1}\}$ is bounded and

$$a_{2n}\beta_{2n} \to L$$

and

(2) $\beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq N$.

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n-1}\beta_{2n-1}\}$ converges.

Proof: The necessity follows from Theorem 8.25.

For the sufficiency, we need only note that $a_{2n} >. 0$ or $a_{2n} <. 0$, and then apply Theorem 8.39 or Theorem 8.40, respectively. Q.E.D.

The dual of Theorem 8.41 is Theorem 8.16.

Theorem 8.42. Let L be any real number and Σa_n any N-alternating series such that $a_{2n} >. 0$. Then a n.a.s.c. that Σa_n converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\beta_n\}$ such that,

(1) some subsequence of $\{a_{2n}\beta_{2n}\}$ is bounded below and $a_{2n-1}\beta_{2n-1} \to L$

and

(2) $\beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq N$.

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n}\beta_{2n}\}$ converges.

Proof: The necessity follows from Theorem 8.25.

For the sufficiency, define $\alpha_n = 1+r_{n+1}\beta_{n+1}$ for $n \geq N$. Then $a_{2n}\alpha_{2n} =. a_{2n}+a_{2n+1}\beta_{2n+1} \to L$. Since $a_{2n-1}\alpha_{2n-1} =. a_{2n-1}+a_{2n}\beta_{2n}$, some subsequence of $\{a_{2n-1}\alpha_{2n-1}\}$ is bounded below. Also, $\alpha_n = 1+r_{n+1}\beta_{n+1}$ $\leq 1+r_{n+1}+r_{n+1}r_{n+2}(1+r_{n+3}\beta_{n+3}) = 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$ for $n \geq N$. From Theorem 8.14, both Σa_n and $\{a_{2n-1}\alpha_{2n-1}\}$ converge. Consequently, $a_{2n}\beta_{2n} =. a_{2n-1}\alpha_{2n-1}-a_{2n-1}$ $\to \lim a_{2n-1}\alpha_{2n-1}$, i.e., $\{a_{2n}\beta_{2n}\}$ converges. Q.E.D.

The dual of Theorem 8.42 is Theorem 8.20.

Theorem 8.43. Let L be any real number and Σa_n be any N-alternating series such that $a_{2n} <. 0$. Then a n.a.s.c. that Σa_n converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\beta_n\}$ such that,

(1) some subsequence of $\{a_{2n}\beta_{2n}\}$ is bounded above and $a_{2n-1}\beta_{2n-1} \to L$

and

(2) $\beta_n \geq 1 + r_{n+1} + r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq N$.

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n}\beta_{2n}\}$ converges.

Proof: The necessity follows from Theorem 8.25.

For the sufficiency, define $a'_n = -a_n$ for $n \geq 0$. Then $r'_n = a'_n/a'_{n-1} = a_n/a_{n-1} = r_n$ for $n \geq N$. From Theorem 8.42, both $\Sigma a'_n$ and $\{a'_{2n}\beta_{2n}\}$ converge. Thus, Σa_n and $\{a_{2n}\beta_{2n}\}$ converge. Q.E.D.

The dual of Theorem 8.43 is Theorem 8.19.

Theorem 8.44. Let L be any real number. Then a n.a.s.c. that an N-alternating series Σa_n converge is that

(0) $a_n \to 0$,

and there exist a sequence $\{\beta_n\}$ such that,

(1) some subsequence of $\{a_{2n}\beta_{2n}\}$ is bounded and

$$a_{2n-1}\beta_{2n-1} \to L$$

and

(2) $\beta_n \geq 1 + r_{n+1} + r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq N$.

Moreover, if conditions (0), (1), and (2) hold, then $\{a_{2n}\beta_{2n}\}$ converges.

Proof: The necessity follows from Theorem 8.25.

For the sufficiency, we need only note that $a_{2n} >. 0$ or $a_{2n} <. 0$, and then apply Theorem 8.42 or Theorem 8.43, respectively. Q.E.D.

The dual of Theorem 8.44 is Theorem 8.21.

Theorem 8.45. (Leibnitz's Theorem for alternating series.) Let Σa_n be an alternating series such that $-1 \leq r_n$, for $n \geq 2$, and $a_n \to 0$. Then Σa_n converges, and moreover $|S-S_{n-1}| \leq |a_n|$ for $n \geq 1$.

1st Proof: Choosing $\alpha_n = 0$ for $n \geq 1$, we may use (a) of Theorem 8.8 to obtain

$$r_n + r_n r_{n+1} \cdot 0 \leq (S-S_{n-1})/a_{n-1} \leq r_n \cdot 0, \quad n \geq 1,$$

and this immediately yields the desired inequality. Q.E.D.

<u>2nd Proof</u>: Choosing $\beta_n = 1$ for $n \geq 1$, we may use (a) of Theorem 8.31 to obtain

$$0 \geq r_n + r_n r_{n+1} \cdot 1 \geq (S - S_{n-1})/a_{n-1} \geq r_n \cdot 1, \quad n \geq 1,$$

from which the desired inequality follows. Q.E.D.

<u>Lemma 8.46</u>. Suppose that $p, x, y,$ and q are numbers such that $-1 < p \leq q \leq 0$, $p \leq x \leq q$, and $p \leq y \leq q$. Setting $\alpha = (1+p)/(1-pq)$ and $\beta = (1+q)/(1-pq)$, we have

(1) $\quad p\beta \leq x\beta \leq x + xy\alpha \leq x + xy\beta \leq x\alpha \leq q\alpha,$

(2) $\quad \alpha \leq 1 + x + xy\alpha$ and $\beta \geq 1 + x + xy\beta,$

and

(3) $\quad p\beta \leq x/(1-x) \leq q\alpha.$

<u>Proof</u>: It is easily seen that $0 < \alpha = 1 + p\beta \leq \beta = 1 + q\alpha$. Accordingly $p\beta \leq x\beta = x(1+q\alpha) \leq x(1+y\alpha) \leq x(1+y\beta) \leq x(1+p\beta) = x\alpha \leq q\alpha$, $\alpha = 1 + p\beta \leq 1 + x + xy\alpha$, and $\beta = 1 + q\alpha \geq 1 + x + y\beta$. This proves (1) and (2). For (3), we have $[x/(1-x)] - p\beta = [(x-p) + p(x-q)]/[(1-x)(1-pq)] \geq 0$ and $q\alpha - [x/(1-x)] = [(q-x) + q(p-x)]/[(1-x)/(1-pq)] \geq 0$. Q.E.D.

<u>Theorem 8.47</u>. Suppose that Σa_n is an N-alternating series such that $-1 < p \leq r_n \leq q \leq 0$ for $n \geq N$, where p and q are constants. Setting $\alpha = (1+p)/(1-pq)$ and $\beta = (1+q)/(1-pq)$,

(1) $\quad p\beta \leq r_n\beta \leq r_n + r_n r_{n+1}\alpha \leq T_n \leq r_n + r_n r_{n+1}\beta \leq r_n\alpha$

$$\leq q\alpha, \quad n \geq N.$$

<u>Proof</u>: Define $\alpha_n = \alpha$ and $\beta_n = \beta$ for $n \geq N$. Since $|r_n| \leq |p| < 1$ for $n \geq N$, $a_n \to 0$, $a_n\alpha_n \to 0$, and $a_n\beta_n \to 0$. By Lemma 8.46, $\alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$ and $\beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$ for $n \geq N$. Let n be any integer $\geq N$. Using (1) of Lemma 8.46, $p\beta \leq r_n\beta \leq r_n + r_nr_{n+1}\alpha \leq r_n+r_nr_{n+1}\beta \leq r_n\alpha \leq q\alpha$. Also Theorem 8.8 and Theorem 8.27 yield the respective inequalities $r_n+r_nr_{n+1}\alpha \leq T_n$ and $T_n \leq r_n+r_nr_{n+1}\beta$. (1) of the present theorem is now evident. Q.E.D.

Suppose that p,q are constants such that $-1 < p \leq q < 0$. We now exhibit a series Σa_n satisfying the hypotheses of Theorem 8.47, and for which $p\beta$ and $q\alpha$ are the corresponding largest and smallest constants such that $p\beta \leq T_n \leq q\alpha$ for $n \geq N = 1$. In particular, let $\Sigma a_n = 1+p+pq+p^2q+p^2q^2+p^3q^2+\cdots$. Then $r_{2n-1} = p$ and $r_{2n} = q$ for $n \geq 1$, so that $T_{2n-1} = r_{2n-1}+r_{2n-1}r_{2n}+\cdots = p\beta$ and $T_{2n} = r_{2n}+r_{2n}r_{2n+1}+\cdots = q\alpha$, for $n \geq 1$.

<u>Lemma 8.48</u>. If $-1 < x, \alpha < 1$, and $\alpha \leq x(1+y)/(1+x)$, then $1/(1-\alpha) \leq 1+x+xy/(1-\alpha)$.

<u>Proof</u>: We have $0 < 1-\alpha$ and $0 < 1+x$. Thus, $\alpha(1+x) \leq x(1+y)$, $1 \leq (1-\alpha)+x(1-\alpha)+xy$, and $1/(1-\alpha) \leq 1+x+xy/(1-\alpha)$. Q.E.D.

Lemma 8.49. If $-1<x$ and $1>\beta\geq x(1+y)/(1+x)$, then $1/(1-\beta) \geq 1+x+xy/(1-\beta)$.

Proof: We have $0 < 1-\beta$ and $0 < 1+x$. The following inequalities are now obvious: $\beta(1+x) \geq x(1+y)$, $1 \geq (1-\beta)+x(1-\beta)+xy$, $1/(1-\beta) \geq 1+x+xy/(1-\beta)$. Q.E.D.

We give three proofs of the following theorem.

Theorem 8.50. If $r_n \to r$, $-1 < r < 0$, then $T_n \to r/(1-r)$.

1st Proof: Let $\varepsilon > 0$. Since $(y-x)/(1-xy) \to 0$ as $(x,y) \to (r,r)$, there are numbers p,q such that $-1 < p < r < q < 0$ and $(q-p)/(1-pq) < \varepsilon$. Using (3) of Lemma 8.46, $p\beta \leq r/(1-r) \leq q\alpha$ where $\alpha = (1+p)/(1-pq)$ and $\beta = (1+q)/(1-pq)$. Also, there is a positive integer N such that $p \leq r_n \leq q$ for $n \geq N$. By Theorem 8.47, $p\beta \leq T_n \leq q\alpha$ for $n \geq N$. Hence, $|T_n-r/(1-r)| \leq q\alpha-p\beta = (q-p)/(1-pq) < \varepsilon$ for $n \geq N$. Q.E.D.

2nd Proof: Since $r_n(1+r_{n+1})/(1+r_n) \to r$, there is a positive integer N and a monotone increasing sequence $\{\alpha_n\}$ such that $\alpha_n \to r$ and, for $n \geq N$, $-1 < r_n < 0$ and $\alpha_n \leq r_{n+1}(1+r_{n+2})/(1+r_{n+1})$. We now use Lemma 8.48 and the inequality $1/(1-\alpha_n) \leq 1/(1-\alpha_{n+2})$ for $n \geq N$ to obtain

$$\frac{1}{1-\alpha_n} \leq 1+r_{n+1}+r_{n+1}r_{n+2}\,\frac{1}{1-\alpha_n} \leq 1+r_{n+1} + r_{n+1}r_{n+2}\,\frac{1}{1-\alpha_{n+2}}$$

for $n \geq N$. Since $|r| < 1$, $a_n \to 0$ and $a_n/(1-\alpha_n) \to 0$. According to Theorem 8.3, $r_n+r_nr_{n+1}/(1-\alpha_{n+1}) \leq T_n \leq r_n/(1-\alpha_n)$ for $n \geq N$. The conclusion now follows since $r_n+r_nr_{n+1}/(1-\alpha_{n+1}) \to r+r^2/(1-r) = r/(1-r)$ and $r_n/(1-\alpha_n) \to r/(1-r)$. Q.E.D.

<u>3rd Proof</u>: Since $r_n(1+r_{n+1})/(1+r_n) \to r$, there is a positive integer N and a monotone decreasing sequence $\{\beta_n\}$ such that $\beta_n \to r$ and, for $n \geq N$, $-1 < r_n < 0$ and $1 > \beta_n \geq r_{n+1}(1+r_{n+2})/(1+r_{n+1})$. We now use Lemma 8.49 and the inequality $1/(1-\beta_n) \geq 1/(1-\beta_{n+2})$ for $n \geq N$ to obtain

$$\begin{aligned} 1/(1-\beta_n) &\geq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-\beta_n) \\ &\geq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-\beta_{n+2}) \end{aligned}$$

for $n \geq N$. Since $|r| < 1$, $a_n \to 0$ and $a_n/(1-\beta_n) \to 0$. According to Theorem 8.27, $r_n+r_nr_{n+1}/(1-\beta_{n+1}) \geq T_n \geq r_n/(1-\beta_n)$ for $n \geq N$. The conclusion now follows since $r_n+r_nr_{n+1}/(1-\beta_{n+1}) \to r+r^2/(1-r) = r/(1-r)$ and $r_n/(1-\beta_n) \to r/(1-r)$. Q.E.D.

Theorem 8.51. If Σa_n is an N-alternating series, $-1 < r < 0$, and $1/(1-r) \leq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-r)$ for $n \geq N$, then $r_n+r_nr_{n+1}/(1-r) \leq T_n \leq r_n/(1-r)$ for $n \geq N$.

Proof: Since $|r| < 1$, $a_n \to 0$ and $a_n/(1-r) \to 0$. Now apply Theorem 8.3 with $\alpha_n = 1/(1-r)$ for $n \geq N$. Q.E.D.

Theorem 8.52. If Σa_n is an N-alternating series, $-1 < r < 0$, and $r_{n+2} \leq r_{n+1}$ for $n \geq N$, then $r_n+r_nr_{n+1}/(1-r) \leq T_n \leq r_n/(1-r)$ for $n \geq N$.

Proof: Let $n \geq N$. Then $-1 < r \leq r_{n+2} \leq r_{n+1}$, so that $r \leq r_{n+1} \leq r_{n+1}(1+r_{n+2})/(1+r_{n+1})$. By Lemma 8.48, $1/(1-r)$ $\leq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-r)$. Now apply Theorem 8.51. Q.E.D.

Theorem 8.53. If $-1 < r \leq r_{n+1} \leq r_n < 0$ for $n \geq N$, then, for $n \geq N$, $r_n+r_nr_{n+1}(1+r)/(1-rr_n) \leq T_n \leq r_n$ $+r_nr_{n+1}(1+r_n)/(1-rr_n)$.

Proof: Let m be any integer $\geq N$, $p = r$, $q = r_m$, $\alpha = (1+p)/(1-pq)$, and $\beta = (1+q)/(1-pq)$. Then $-1 < p \leq r_n \leq q < 0$ for $n \geq m$. From (1) of Theorem 8.47, $r_n+r_nr_{n+1}\alpha \leq T_n \leq r_n+r_nr_{n+1}\beta$ for $n \geq m$. Setting $n = m$, the desired inequality obtains. Q.E.D.

Assuming the hypotheses of Theorem 8.53, the lower bound given there for T_n and that given by Theorem 8.52 will now be compared. No comparison of upper bounds appears evident.

The following inequalities are equivalent: $r_n+r_nr_{n+1}/(1-r) \geq r_n+r_nr_{n+1}(1+r)/(1-rr_n)$, $1/(1-r) \geq (1+r)/(1-rr_n)$, $1-rr_n \geq 1-r^2$, $r_n \geq r$. Consequently, the lower bound for T_n given by Theorem 8.52 appears better. It is also simpler in form.

Theorem 8.54. Let Σa_n be an N-alternating series. Then a n.a.s.c. that $T_n \to -1/2$ is that $a_n \to 0$, $r = -1$, and there exist a sequence $\{\alpha_n\}$ such that

(1) $\alpha_n \to 1/2,$

and

(2) $\alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}, \quad n \geq N.$

Proof: For the necessity, assume that $T_n \to -1/2$. Accordingly, Σa_n converges and $a_n \to 0$. Thus, $r_n = T_n/(1+T_{n+1}) \to (-1/2)/(1-1/2) = -1$, i.e., $r = -1$. Defining $\alpha_n = 1+T_{n+1}$ for $n \geq N$, $\alpha_n \to 1-1/2 = 1/2$ and $\alpha_n = 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$ for $n \geq N$.

For the sufficiency, Theorem 8.3 yields

$r_n + r_n r_{n+1}\alpha_{n+1} \leq T_n \leq r_n\alpha_n$ for $n \geq N$. Also, $\lim\,(r_n + r_n r_{n+1}\alpha_{n+1}) = \lim r_n\alpha_n = -1/2$, which implies that $T_n \to -1/2$. Q.E.D.

<u>Theorem 8.55</u>. Let Σa_n be an N-alternating series. Then a n.a.s.c. that $T_n \to -1/2$ is that $a_n \to 0$, $r = -1$, and there exist a sequence $\{\beta_n\}$ such that

(1) $\quad \beta_n \to 1/2$

and

(2) $\quad \beta_n \geq 1 + r_{n+1} + r_{n+1}r_{n+2}\beta_{n+2}, \quad n \geq N.$

<u>Proof</u>: For the necessity we may use the proof of the necessity of Theorem 8.54, replacing "α" by "β" throughout.

For the sufficiency, we use Theorem 8.27 to obtain $r_n\beta_n \leq T_n \leq r_n + r_n r_{n+1}\beta_{n+1}$ for $n \geq N$. Also, $r_n\beta_n \to -1/2$ and $r_n + r_n r_{n+1}\beta_{n+1} \to -1/2$, so that $T_n \to -1/2$. Q.E.D.

<u>Lemma 8.56</u>. If $x_n \to x$, $-\infty < x < 0$, and $\limsup y_n = y$, $-\infty \leq y \leq +\infty$, then $\liminf x_n y_n = (\lim x_n)(\limsup y_n)$.

<u>Proof</u>: Suppose that $y = +\infty$. Then $y_{n'} \to +\infty$ for some subsequence $\{n'\}$ of $\{n\}$, $x_{n'}y_{n'} \to x\,(+\infty) = -\infty$, and $\liminf x\,y = -\infty$. Also $(\lim x_n)(\limsup y_n) = x(+\infty) = -\infty$,

and thus $\liminf x_n y_n = (\lim x_n)(\limsup y_n)$.

Suppose that $y = -\infty$. Then $\lim y_n = -\infty$, $\liminf x_n y_n = +\infty$, and $(\lim x_n)(\limsup y_n) = x(-\infty) = +\infty$. Hence $\liminf x_n y_n = (\lim x_n)(\limsup y_n)$.

Suppose that $-\infty < y < +\infty$ and let $\liminf x_n y_n = L$. Then $-\infty < L < +\infty$ and $y_{n'} \to y$ for some subsequence $\{n'\}$ of $\{n\}$. Hence $x_{n'} y_{n'} \to xy$, and thus $L \leq xy$. Since $\liminf x_n y_n = L$, there is a subsequence $\{n^*\}$ of $\{n\}$ such that $x_{n^*} y_{n^*} \to L$, and thus $y_{n^*} =. x_{n^*} y_{n^*}/x_{n^*} \to L/x \leq y$. Consequently, $L \geq xy$. Hence, $L = xy$. Q.E.D.

<u>Theorem 8.57</u>. If $-1 <. r_n$ and $\limsup (1+r_{n+1})/(1+r_n) < 1$, then $r_n \to r = -1$, $|a_n| \to a$ for some $a > 0$, Σa_n diverges, and there is a positive integer m such that $\prod_m^\infty |r_n|$ converges.

<u>Proof</u>: By hypothesis, $0 <. 1+r_n$ and $(1+r_{n+1})/(1+r_n) <. 1$. Thus $-1 <. r_{n+1} <. r_n$ and $r_n \to r$ where $-1 \leq r$. We must have $r = -1$; since otherwise, $\limsup (1+r_{n+1})/(1+r_n) = \lim (1+r_{n+1})/(1+r_n) = 1$, a contradiction. Since $r = -1$, we have $-1 <. r_n <. 0$, $|r_n| =. |a_n/a_{n-1}| <. 1$, and

$|a_n| <. |a_{n-1}|$. Consequently, $|a_n| \to a$ for some $a \geq 0$. Assume that $a = 0$. Setting $L = \limsup\, (1+r_{n+1})/(1+r_n)$, $0 \leq L < 1$. From Lemma 8.56, $\liminf\, r_n(1+r_{n+1})/(1+r_n)$ $= (\lim r_n)\,[\limsup\, (1+r_{n+1})/(1+r_n)] = -L$, $-1 < -L \leq 0$. Hence, there is a positive integer N and a monotone increasing sequence $\{\alpha_n\}$ such that $\alpha_n \to -L$ and, for $n \geq N$, $-1 < r_n < 0$ and $\alpha_n \leq r_{n+1}(1+r_{n+2})/(1+r_{n+1})$. From Lemma 8.48 and the inequality $1/(1-\alpha_n) \leq 1/(1-\alpha_{n+2})$ for $n \geq N$, $1/(1-\alpha_n) \leq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-\alpha_n) \leq 1+r_{n+1}$ $+r_{n+1}r_{n+2}/(1-\alpha_{n+2})$ for $n \geq N$. Also, $a_n/(1-\alpha_n) \to 0$. From (a) of Theorem 8.3, $r_n+r_nr_{n+1}/(1-\alpha_{n+1}) \leq r_n/(1-\alpha_n)$ for $n \geq N$. Letting $n \to \infty$, we obtain $-1+1/(1+L)$ $\leq -1/(1+L)$, $-(1+L)+1 \leq -1$, and $1 \leq L$; a contradiction. Thus, $a > 0$ and Σa_n must diverge. Since $r_n <. 0$, there is a positive integer m such that $r_n \neq 0$ for $n \geq m$, and thus $|r_m||r_{m+1}|\cdots|r_{m+n}| =. |a_{m+n}|/|a_{m-1}|$ $\to a/|a_{m-1}| > 0$ as $n \to \infty$. Hence $\prod_m^\infty |r_n|$ converges to $a/|a_{m-1}|$. Q.E.D.

The preceeding proof of Theorem 8.57 involved only the theory of N-alternating series. By use of known theorems for series of positive terms, and alternate proof is now given.

Proof: By hypothesis, $0 \lessdot 1+r_n$ and $(1+r_{n+1})/(1+r_n) \lessdot 1$. Thus $-1 \lessdot r_{n+1} \lessdot r_n$ and $r_n \to r$ where $-1 \leq r$. We must have $r = -1$; since otherwise, $\limsup (1+r_{n+1})/(1+r_n) = \lim (1+r_{n+1})/(1+r_n) = 1$, a contradiction. Since $r = -1$, $-1 \lessdot r_n \lessdot 0$ and there is a positive integer m such that $-1 < r_n < 0$ for $n \geq m$. Consequently, $\sum_m^\infty (1-|r_n|) = \sum_m^\infty (1+r_n)$ is a series of positive terms, which converges since $\limsup (1+r_{n+1})/(1+r_n) < 1$. Thus $1+r_n \to 0$ and $r_n \to r = -1$. Also with $1-|r_n| > 0$, for $n \geq m$, it is known (5, p. 382) that $\sum_m^\infty (1-|r_n|)$ converges if and only if $\prod_m^\infty [1-(1-|r_n|)] = \prod_m^\infty |r_n|$ converges; thus $\prod_m^\infty |r_k| = a$ for some $a > 0$. Hence, for $n > m$, $|a_n| = |a_m||r_{m+1}r_{m+2}\cdots r_n| \to |a_m| \left(\prod_m^\infty |r_k|\right) = |a_m|(a) > 0$. Consequently, Σa_n diverges. Q.E.D.

Corollary 8.58. If $a_n \to 0$ and $-1 \lessdot r_n$, then $\limsup (1+r_{n+1})/(1+r_n) \geq 1$.

Proof: Assume that $\limsup (1+r_{n+1})/(1+r_n) < 1$. Then from Theorem 8.57, $|a_n| \to a > 0$ which contradicts

$a_n \to 0$. Thus, $\limsup (1+r_{n+1})/(1+r_n) \geq 1$. Q.E.D.

Theorem 8.59. If $a_n \to 0$, $r = -1 <. r_n$, and $\limsup (1+r_{n+1})/(1+r_n) = 1$, then $T_n \to r/(1-r) = -1/2$.

Proof: From Lemma 8.56, $\liminf r_n(1+r_{n+1})/(1+r_n)$ $= \lim r_n \cdot \limsup (1+r_{n+1})/(1+r_n) = r \cdot 1 = r$. Consequently, there is a positive integer N and a monotone increasing sequence $\{\alpha_n\}$ such that $\alpha_n \to r$ and, for $n \geq N$, $-1 < r_n < 0$ and $\alpha_n \leq r_{n+1}(1+r_{n+2})/(1+r_{n+1})$. Using Lemma 8.48 and the inequality $1/(1-\alpha_n) \leq 1/(1-\alpha_{n+2})$ for $n \geq N$, $1/(1-\alpha_n) \leq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-\alpha_n) \leq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-\alpha_{n+2})$ for $n \geq N$. Also, $1/(1-\alpha_n) \to 1/2$. Now apply Theorem 8.54. Q.E.D.

Corollary 8.60. If $a_n \to 0$, $r = -1 <. r_n$, and $\limsup (1+r_{n+1})/(1+r_n) \leq 1$, then $\limsup (1+r_{n+1})/(1+r_n)$ $= 1$ and $T_n \to r/(1-r) = -1/2$.

Proof: From Corollary 8.58, $\limsup (1+r_{n+1})/(1+r_n) \geq 1$, and thus $\limsup (1+r_{n+1})/(1+r_n) = 1$. Now apply Theorem 8.59. Q.E.D.

Lemma 8.61. If $a_n \to 0$ and $\liminf (1+r_{n+1})/(1+r_n) = L$, $0 < L \leq +\infty$, then $-1 <. r_n$.

Proof: Since $0 < L$, $0 <. (1+r_{n+1})/(1+r_n)$. Hence $1+r_n <. 0$ or $0 <. 1+r_n$. If $1+r_n <. 0$, then $r_n <. -1$, $1 <. |r_n|$, and $|a_{n-1}| <. |a_n|$. This is impossible since $a_n \to 0$. Thus $0 <. 1+r_n$ and $-1 <. r_n$. Q.E.D.

Lemma 8.62. If $x_n \to x$, $-\infty < x < 0$, and $\liminf y_n = y$, $-\infty \leq y \leq +\infty$, then $\limsup x_n y_n = (\lim x_n)(\liminf y_n)$.

Proof: Suppose that $y = +\infty$. Then $\lim y_n = +\infty$, $\limsup x_n y_n = -\infty$, and $(\lim x_n)(\liminf y_n) = x(+\infty) = -\infty$.

Suppose that $y = -\infty$. Then $y_{n'} \to -\infty$ for some subsequence $\{n'\}$ of $\{n\}$, $x_{n'} y_{n'} \to x(-\infty) = +\infty$, and $\limsup x_n y_n = +\infty$. Also, $(\lim x_n)(\liminf y_n)$ $= x(-\infty) = +\infty$.

Suppose that $-\infty < y < +\infty$ and let $\limsup x_n y_n$ $= L$. Then $-\infty < L < +\infty$ and $y_{n'} \to y$ for some subsequence $\{n'\}$ of $\{n\}$. Hence $x_{n'} y_{n'} \to xy$, and thus $xy \leq L$. Since $\limsup x_n y_n = L$, there is a subsequence $\{n^*\}$ of $\{n\}$ such that $x_{n^*} y_{n^*} \to L$, and thus y_{n^*} $=. x_{n^*} y_{n^*} / x_{n^*} \to L/x \geq y$. Thus $L \leq xy$. Hence $L = xy$. Q.E.D.

Theorem 8.63. If $a_n \to 0$, $r = -1$, and

$\liminf (1+r_{n+1})/(1+r_n) = 1$, then $-1 <. r_n$ and $T_n \to r/(1-r) = -1/2$.

Proof: Using Lemma 8.61 and the fact that $r_n \to r = -1$, $-1 <. r_n <. 0$. From Lemma 8.62, $\limsup r_n(1+r_{n+1})/(1+r_n) = (\lim r_n)[\liminf (1+r_{n+1})/(1+r_n)] = r\cdot 1 = r$. Consequently, there is a positive integer N and a monotone decreasing sequence $\{\beta_n\}$ such that $\beta_n \to r$ and, for $n \geq N$, $-1 < r_n < 0$ and $1 > \beta_n \geq r_{n+1}(1+r_{n+2})/(1+r_{n+1})$. Using Lemma 8.49 and the inequality $1/(1-\beta_n) \geq 1/(1-\beta_{n+2})$ for $n \geq N$, $1/(1-\beta_n) \geq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-\beta_n) \geq 1+r_{n+1}$ $+r_{n+1}r_{n+2}/(1-\beta_{n+2})$ for $n \geq N$. Also, $1/(1-\beta_n) \to 1/2$. Now apply Theorem 8.55. Q.E.D.

Theorem 8.64. If $a_n \to 0$, $r = -1$, and $\lim (1+r_{n+1})/(1+r_n) = 1$, then $-1 <. r_n$ and $\lim T_n = r/(1-r) = -1/2$.

Proof: Since $\liminf (1+r_{n+1})/(1+r_n) = \lim (1+r_{n+1})/(1+r_n) = 1$, the conclusion follows from Theorem 8.63. Q.E.D.

Pflanz (18, p. 27) has proven that if Σa_n is an alternating series such that $r_n = -1+a/n+\gamma_n/n$, where $a > 0$

and $\gamma_n \to 0$, then $\Sigma a_{\delta n} \varepsilon MR(\Sigma a_n)$. We now give a short proof of this fact.

Theorem 8.65. If $r_n =. -1+a/n+\gamma_n/n$ where $a > 0$ and $\gamma_n \to 0$, then $T_n \to -1/2$ and $\Sigma a_{\delta n} \varepsilon MR(\Sigma a_n)$.

Proof: By hypothesis, $r = \lim r_n = -1$ and $-1 <. r_n <. 0$. Thus, $|r_n| =. |a_n/a_{n-1}| <. 1$, $|a_n| <. |a_{n-1}|$, and $|a_n| \to c$ for some $c \geq 0$. Also, $|r_n| =. 1-(a+\gamma_n)/n$, $(a+\gamma_n)/n >. 0$, and $\Sigma (a+\gamma_n)/n$ diverges. Consequently, from Apostol (5, p.238), $\Pi|r_n|$ diverges to zero so that $c = 0$, i.e., $a_n \to 0$. Moreover,
$(1+r_{n+1})/(1+r_n) =. [(a+\gamma_{n+1})/(n+1)]/[(a_n+\gamma_n)/n]$
$=. [n/(n+1)][(a+\gamma_{n+1})/(a+\gamma_n)] \to 1$. From Theorem 8.64, $T_n \to -1/2$, and thus $T_{n+1}-T_n \to 0$. We now apply Theorem 3.8. Q.E.D.

Lemma 8.66. If $-1 <. r_n <. a$ for some number a, then $0 \leq \liminf (1+r_{n+1})/(1+r_n) \leq 1$.

Proof: From $-1 <. r_n$, $0 <. (1+r_{n+1})/(1+r_n)$. Thus setting $L = \liminf (1+r_{n+1})/(1+r_n)$, $0 \leq L \leq +\infty$. Suppose $1 < L$. Then $1 <. (1+r_{n+1})/(1+r_n)$, $-1 <. r_n <. r_{n+1} <. a$, and r exists with $-1 < r \leq a$. Hence

$L = \liminf (1+r_{n+1})/(1+r_n) = \lim (1+r_{n+1})/(1+r_n) = 1$, a contradiction. Thus $0 \le L \le 1$. Q.E.D.

<u>Theorem 8.67</u>. If $a_n \to 0$, $r = -1 <. r_n$, and $\lim (1+r_{n+1})/(1+r_n) = L$ where $-\infty \le L \le +\infty$, then $L = 1$ and $T_n \to r/(1-r) = -1/2$.

<u>Proof</u>: Since $r = -1 <. r_n$, $-1 <. r_n <. 0$. From Corollary 8.58 and Lemma 8.66, $L \ge 1$ and $L \le 1$, respectively. Hence $L = 1$, and thus, from Theorem 8.64, $T_n \to r/(1-r) = -1/2$. Q.E.D.

<u>Theorem 8.68</u>. If $a_n \to 0$, $r = -1$, and $\lim (1+r_{n+1})/(1+r_n) = L$ where $-\infty \le L \le +\infty$, then exactly one of the following statements is true:

(1) $-1 <. r_n$ and $L = 1$.

(2) $1+r_n$ is alternately positive and negative, for large n, and $L = -1$.

<u>Proof</u>: Since $r_n \to -1$ we may assume that $-2 < r_n < 0$ for $n \ge 1$. Exactly one of the following statements is true:

(i) $-1 <. r_n$.

(ii) $r_n <: -1$.

If (i) holds, then $L = 1$ according to Theorem 8.67.

Suppose that (ii) is true. For each integer $n \geq 1$, define $r'_n = r_n$ if $-1 \leq r_n$, or $r'_n = -2-r_n$ if $r_n < -1$. Accordingly, for $n \geq 1$ we have $-2 < r_n \leq r'_n < 0$ and $0 \leq 1+r'_n$. Define $a'_0 = 1$ and $a'_n = r'_1 r'_2 \cdots r'_n$ for $n \geq 1$. Since $0 < |r'_n| \leq |r_n|$ for $n \geq 1$,
$|a'_n| = |r'_1||r'_2|\cdots|r'_n| \leq |r_1||r_2|\cdots|r_n| = |a_n/a_0| \to 0$,
i.e., $a'_n \to 0$. Also, $1+r'_n = 1+r_n$ or $1+r'_n = -1-r_n$, i.e., $1+r'_n = |1+r_n|$ for $n \geq 1$, so that
$\lim\,(1+r'_{n+1})/(1+r'_n) = \lim\,|(1+r_{n+1})/(1+r_n)| = |L|$.
Moreover, $1+r'_n =.\ |1+r_n| \to 0$, i.e., $r'_n \to -1$. We now have $a'_n \to 0$, $r' = \lim r'_n = -1$, $-1 <.\ r'_n$, and $\lim\,(1+r'_{n+1})/(1+r'_n) = |L|$. From Theorem 8.67, $|L| = 1$, i.e., $L = -1$ or $L = 1$. Assume that $L = 1$. Then $1+r_n$ is of constant sign for large n. Hence, according to (ii), $1+r_n <.\ 0$, i.e., $r_n <.\ -1$. This contradicts $a_n \to 0$; thus $L = -1$ and $1+r_n$ is alternately positive and negative for large n. Q.E.D.

<u>Corollary 8.69</u>. If $a_n \to 0$, $r = -1$, and $\lim\,(1+r_{n+1})/(1+r_n) = L$ where $-\infty \leq L \leq \infty$ and $L \neq -1$, then

$-1 <. r_n$, $L = 1$, and $T_n \to r/(1-r) = -1/2$.

<u>Proof</u>: From Theorem 8.68, $-1 <. r_n$ and $L = 1$. We may now apply Theorem 8.64 or Theorem 8.67 to complete the proof. Q.E.D.

<u>Lemma 8.70</u>. If $(1+r_n)(1+r_{n+1}) <. 0$, some subsequence of $\{r_{2n-1}\}$ converges to -1, and some subsequence of $\{r_{2n}\}$ converges to -1, then $-1 \leq \lim\sup (1+r_{n+1})/(1+r_n) \leq 0$.

<u>Proof</u>: By hypothesis, $(1+r_{n+1})/(1+r_n) <. 0$. Thus, setting $L = \lim\sup (1+r_{n+1})/(1+r_n)$, we have $-\infty \leq L \leq 0$. Suppose that $L < -1$. Then $(1+r_{n+1})/(1+r_n) <. -1$ and $(1+r_{n+2})/(1+r_n) =. [(1+r_{n+2})/(1+r_{n+1})][(1+r_{n+1})/(1+r_n)] >. 1$. Either $1+r_{2n} <. 0$, or $1+r_{2n-1} <. 0$. In the former case, $1+r_{2n+2} <. 1+r_{2n}$, so that $r_{2n+2} <. r_{2n} <. -1$. This is impossible since some subsequence of $\{r_{2n}\}$ converges to -1. In the latter case, $1+r_{2n+1} <. 1+r_{2n-1}$, so that $r_{2n+1} <. r_{2n-1} <. -1$. This is impossible since some subsequence of $\{r_{2n-1}\}$ converges to -1. Thus, $-1 \leq L \leq 0$. Q.E.D.

Lemma 8.71. If $a_{2n} \to 0$, $r_{2n-1} \to -1 <. r_{2n-1}$, and $\limsup (1+r_{2n})/(1+r_{2n-1}) = L$ where $-\infty \leq L \leq -1$, then $r_{2n} <. -1$, some subsequence of $\{r_{2n}\}$ converges to -1, and $L = -1$.

Proof: By hypothesis, $(1+r_{2n})/(1+r_{2n-1}) <. 0 <. 1 + r_{2n-1}$, and thus, $r_{2n} <. -1$. Clearly, $(1+r_n)(1+r_{n+1}) <. 0$. Assume that no subsequence of $\{r_{2n}\}$ converges to -1. Then there is a number α such that $r_{2n} <. \alpha < -1$. Since $r_{2n-1}\alpha \to -\alpha > 1$, $|a_{2n}/a_{2n-2}| =. r_{2n-1}r_{2n} >. r_{2n-1}\alpha >. 1$. Thus, $|a_{2n}| >. |a_{2n-2}|$, which contradicts $a_{2n} \to 0$. If follows that some subsequence of $\{r_{2n}\}$ converges to -1. From Lemma 8.70, $-1 \leq L \leq 0$, and thus $L = -1$. Q.E.D.

Theorem 8.72. If Σa_n converges, $r_{2n-1} \to -1$, $-1 <. r_{2n-1}$, and $\limsup (1+r_{2n})/(1+r_{2n-1}) = L$ where $-\infty \leq L \leq 1$, then $r_{2n} <. -1$, some subsequence of $\{r_{2n}\}$ converges to -1, $L = -1$, $T_{2n-1} \to +\infty$.

Proof: From Lemma 8.71, $r_{2n} <. -1$, some subsequence of $\{r_{2n}\}$ converges to -1, and $L = -1$. Let α be any number < 1. From Lemma 8.56, $\liminf r_{2n-1}(1+r_{2n})/(1+r_{2n-1})$

$= \lim r_{2n-1} \cdot \limsup (1+r_{2n})/(1+r_{2n-1}) = 1$. Thus, $\alpha \mathrel{\underline{<}\cdot} r_{2n-1}(1+r_{2n})/(1+r_{2n-1})$. From Lemma 8.48, $1/(1-\alpha) \mathrel{\underline{<}\cdot} 1+r_{2n-1}+r_{2n-1}r_{2n}/(1-\alpha)$. Defining $\alpha_{2n} = 1/(1-\alpha)$ for $n \geq 1$, $\alpha_{2n-2} \mathrel{\underline{<}\cdot} 1+r_{2n-1}+r_{2n-1}r_{2n}\alpha_{2n}$. Clearly, $a_{2n}\alpha_{2n} \to 0$. From Theorem 8.3, there is a sequence $\{\alpha_{2n-1}\}$ such that $a_{2n-1}\alpha_{2n-1} \to 0$ and $\alpha_{2n-1} \mathrel{\underline{<}\cdot} 1+r_{2n}+r_{2n}r_{2n+1}\alpha_{2n+1}$. We now have $a_n\alpha_n \to 0$ and $\alpha_n \mathrel{\underline{<}\cdot} 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$.

From Theorem 8.3, $-1-r_{2n-1}\alpha_{2n} \mathrel{\underline{<}\cdot} r_{2n-1}+r_{2n-1}r_{2n}\alpha_{2n} \mathrel{\underline{<}\cdot} T_{2n-1}$. Accordingly, $\liminf (-1-r_{2n-1}\alpha_{2n}) = -1 + 1/(1-\alpha) = \alpha/(1-\alpha) \leq \liminf T_{2n-1}$. Since $\alpha/(1-\alpha) \to +\infty$ as $\alpha \to 1-$, $\liminf T_{2n-1} = +\infty$; thus, $T_{2n-1} \to +\infty$. Since $r_{2n} \mathrel{<\cdot} -1$, $T_{2n} \mathrel{=\cdot} r_{2n}(1+T_{2n+1}) \mathrel{\underline{<}\cdot} -(1+T_{2n+1}) \to -\infty$, which yields $T_{2n} \to -\infty$. Q.E.D.

The series Σa_n defined in Example 8.82 satisfies the hypothesis of Theorem 8.72.

According to the following counterexample, we cannot replace "$-\infty \leq L \leq -1$" in Theorem 8.72 by "$-\infty \leq L \leq -\frac{1}{2}$".

<u>Counterexample 8.73</u>. Set $a_{2n} = 1/(n+1)$ and $a_{2n+1} = -1/(n+3)$ for $n \geq 0$. Then $S = 3/2$, $r = -1$, $r_{2n} \mathrel{<\cdot} -1 \mathrel{<\cdot} r_{2n-1}$, $\lim (1+r_{2n})/(1+r_{2n-1}) = -1/2$,

$\lim\,(1+r_{2n+1})/(1+r_{2n}) = -2$, $T_{2n} = -(2n+3)/(n+1) \to -2$, and $T_{2n+1} = (n+1)/(n+2) \to 1$.

According to the following counterexample, we cannot replace " $r_{2n-1} \to -1$" and " $-1 <. r_{2n-1}$" in Theorem 8.72 by " $r_{2n} \to -1$" and " $-1 <. r_{2n}$", respectively, and obtain as a conclusion that $L = -1$, $T_{2n-1} \to \pm\infty$, or $T_{2n} \to \mp\infty$.

Counterexample 8.74. Set $a'_n = a_{n+1}$ for $n \geq 0$, where a_n is defined as in Counterexample 8.73. Accordingly, $S' = 1/2$, $r' = -1$, $r'_{2n-1} =. r_{2n} <. -1 <. r'_{2n} =. r_{2n+1}$, $\lim\,(1+r'_{2n})/(1+r'_{2n-1}) = \lim\,(1+r_{2n+1})/(1+r_{2n}) = -2$, $\lim\,(1+r'_{2n+1})/(1+r'_{2n}) = \lim\,(1+r_{2n+2})/(1+r_{2n+1}) = -1/2$, $T'_{2n} =. T_{2n+1} \to 1$, and $T'_{2n-1} =. T_{2n} \to -2$.

Theorem 8.75. If Σa_n converges, $r_{2n} \to -1$, $-1 <. r_{2n}$, and $\lim\sup\,(1+r_{2n+1})/(1+r_{2n}) = L$ where $-\infty \leq L \leq -1$, then $r_{2n-1} <. -1$, some subsequence of $\{r_{2n-1}\}$ converges to -1, $L = -1$, $T_{2n-1} \to -\infty$, and $T_{2n} \to +\infty$.

Proof: Define $a'_n = a_{n+1}$ for $n \geq 0$. Then $-1 <. r'_{2n-1} =. r_{2n}$, $r'_{2n-1} \to -1$, and

$\limsup (1+r'_{2n})/(1+r'_{2n-1}) = \limsup (1+r_{2n+1})/(1+r_{2n})$ $= L \leq -1$. We may apply Theorem 8.72 to $\Sigma a'_n$, obtaining $r_{2n+1} \doteq r'_{2n} <\cdot -1$, some subsequence of $\{r'_{2n}\} = \{r_{2n+1}\}$ converges to -1, $T_{2n+1} \doteq T'_{2n} \to -\infty$, and $T_{2n} \doteq T'_{2n-1}$ $\to +\infty$. Q.E.D.

<u>Theorem 8.76</u>. If Σa_n converges, $r = -1$, and $\limsup (1+r_{n+1})/(1+r_n) = L$ where $-\infty \leq L \leq -1$, then $L = -1$, and exactly one of the following statements is true:

(1) $r_{2n} <\cdot -1 <\cdot r_{2n-1}$, $T_{2n-1} \to +\infty$, and $T_{2n} \to -\infty$.

(2) $r_{2n-1} <\cdot -1 <\cdot r_{2n}$, $T_{2n-1} \to -\infty$, and $T_{2n} \to +\infty$.

<u>Proof</u>: Exactly one of the following statements is true:

(i) $r_{2n} <\cdot -1 <\cdot r_{2n-1}$.

(ii) $r_{2n-1} <\cdot -1 <\cdot r_{2n}$.

Suppose that (i) is true. Then $\limsup (1+r_{2n})/(1+r_{2n-1}) \leq \limsup (1+r_{n+1})/(1+r_n)$ $\leq L \leq -1$. From Theorem 8.72, $L = -1$, $T_{2n-1} \to +\infty$, and $T_{2n} \to -\infty$.

Suppose that (ii) is true. Then $\limsup (1+r_{2n+1})/(1+r_{2n}) \leq \limsup (1+r_{n+1})/(1+r_n) \leq L$ ≤ -1. From Theorem 8.75, $L = -1$, $T_{2n-1} \to -\infty$, and

$T_{2n} \to +\infty$. Q.E.D.

<u>Lemma 8.77</u>. If $x < -1$, $1 < \beta$, and $\beta \geq x(1+y)/(1+x)$, then $1/(1-\beta) \geq 1+x+xy/(1-\beta)$.

<u>Proof</u>: By hypothesis, $1+x < 0$ and $1-\beta < 0$. Thus, $\beta(1+x) \leq x(1+y)$, $1 \leq (1-\beta)+x(1-\beta)+xy$, and $1/(1-\beta)$ $\geq 1+x+xy/(1-\beta)$. Q.E.D.

<u>Theorem 8.78</u>. If Σa_n converges, $r_{2n-1} \to -1$, $r_{2n-1} <\cdot -1$ $<\cdot\ r_{2n}$, and $\liminf\, (1+r_{2n})/(1+r_{2n-1}) = L \geq -1$, then $r = -1$, $T_{2n-1} \to -\infty$, and $T_{2n} \to +\infty$.

<u>Proof</u>: Let α be any number < -1. By hypothesis, $\alpha \leq\cdot\ (1+r_{2n})/(1+r_{2n-1})$, $\alpha(1+r_{2n-1}) \geq\cdot\ 1+r_{2n}$, and $-1 \leq\cdot\ r_{2n} \leq\cdot\ -1+\alpha(1+r_{2n-1})$. Also, $\lim\, [-1+\alpha(1+r_{2n-1})]$ $= -1$, so that $\lim r_{2n} = -1$. Thus, $r = -1$.

Let β be any number > 1. From Lemma 8.62, $\limsup r_{2n-1}(1+r_{2n})/(1+r_{2n-1}) = (\lim r_{2n-1})$ $[\liminf\, (1+r_{2n})/(1+r_{2n-1})] = (-1)(L) = -L$ where $0 \leq -L \leq 1$. Consequently, $\beta \geq\cdot\ r_{2n-1}(1+r_{2n})/(1+r_{2n-1})$. From Lemma 8.77, $1/(1-\beta) \geq\cdot\ 1+r_{2n-1}+r_{2n-1}r_{2n}/(1-\beta)$. Defining $\beta_{2n} = 1/(1-\beta)$ for $n \geq 1$, $\beta_{2n} \geq\cdot\ 1+r_{2n+1}$ $+r_{2n+1}r_{2n+2}\beta_{2n+2}$. From Theorem 8.27, there is a sequence

$\{\beta_{2n-1}\}$ such that $a_{2n-1}\beta_{2n-1} \to 0$ and $\beta_{2n-1} \geq. 1+r_{2n} +r_{2n}r_{2n+1}\beta_{2n+1}$. We now have $a_n\beta_n \to 0$ and $\beta_n \geq. 1 +r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$. From Theorem 8.27, $r_{2n-1}+r_{2n-1}r_{2n}\beta_{2n} \geq. T_{2n-1}$. Accordingly, $\limsup\,(r_{2n-1}+r_{2n-1}r_{2n}\beta_{2n}) = -1 +1/(1-\beta) = \beta/(1-\beta) \geq \limsup\, T_{2n-1}$. Also, $\beta/(1-\beta) \to -\infty$ as $\beta \to 1-$, so that $\limsup\, T_{2n-1} = -\infty$. Thus, $T_{2n-1} \to -\infty$. Consequently, $T_{2n} =. r_{2n}(1+T_{2n+1}) \to (-1)(1-\infty) = +\infty$. Q.E.D.

<u>Theorem 8.79.</u> If Σa_n converges, $r_{2n} \to -1$, $r_{2n} <. -1 <. r_{2n-1}$, $\liminf\,(1+r_{2n+1})/(1+r_{2n}) = L \geq -1$, then $r = -1$, $T_{2n-1} \to +\infty$, and $T_{2n} \to -\infty$.

<u>Proof</u>: Define $a'_n = a_{n+1}$ for $n \geq 0$. Then $r'_n =. r_{n+1}$. Thus, $r'_{2n-1} <. -1 <. r'_{2n}$, $r'_{2n-1} \to -1$, and $\liminf\,(1+r'_{2n})/(1+r'_{2n-1}) = \liminf\,(1+r_{2n+1})/(1+r_{2n}) = L$. Applying Theorem 8.78 to $\Sigma a'_n$, $r_{2n+1} =. r'_{2n} \to -1$, $T_{2n+1} =. T'_{2n} \to +\infty$, and $T_{2n} =. T'_{2n-1} \to -\infty$. Q.E.D.

<u>Theorem 8.80</u>. If Σa_n converges, $r = -1$, $(1+r_n)(1+r_{n+1}) <. 0$, and $\liminf\,(1+r_{n+1})/(1+r_n) \geq -1$, then exactly one of the following statements is true:

(1) $r_{2n-1} <. -1 <. r_{2n}$, $T_{2n-1} \to -\infty$, and $T_{2n} \to +\infty$.

(2) $r_{2n} <. -1 <. r_{2n-1}$, $T_{2n-1} \to +\infty$, and $T_{2n} \to -\infty$.

<u>Proof</u>: Exactly one of the following statements is true:

(i) $r_{2n-1} <. -1 <. r_{2n}$.

(ii) $r_{2n} <. -1 <. r_{2n-1}$.

Suppose that (i) is true. By hypothesis,
$-1 \leq \liminf (1+r_{n+1})/(1+r_n) \leq \liminf (1+r_{2n})/(1+r_{2n-1})$.
From Theorem 8.78, $T_{2n-1} \to -\infty$ and $T_{2n} \to +\infty$.

Suppose that (ii) is true. Then
$-1 \leq \liminf (1+r_{n+1})/(1+r_n) \leq \liminf (1+r_{2n+1})/(1+r_{2n})$.
From Theorem 8.79, $T_{2n-1} \to +\infty$ and $T_{2n} \to -\infty$. Q.E.D.

<u>Theorem 8.81.</u> If Σa_n converges, $r = -1$, and $\lim (1+r_{n+1})/(1+r_n) = L$ where $-\infty \leq L \leq +\infty$ and $L \neq 1$, then $L = -1$, and exactly one of the following statements is true:

(1) $r_{2n-1} <. -1 <. r_{2n}$, $T_{2n-1} \to -\infty$, and $T_{2n} \to +\infty$.

(2) $r_{2n} <. -1 <. r_{2n-1}$, $T_{2n-1} \to +\infty$, and $T_{2n} \to -\infty$.

<u>Proof</u>: From Theorem 8.68, $L = -1$ and $(1+r_n)(1+r_{n+1}) <. 0$. Now apply Theorem 8.76 or Theorem 8.80. Q.E.D.

If Σa_n is a series satisfying the hypothesis of Theorem 8.68 with $L = 1$, according to Theorem 8.64, Σa_n

converges and $T_n \to -1/2$. With $L = -1$, Σa_n may or may not converge, as is shown in the following two examples. Consequently, we cannot replace the requirement in Theorem 8.81 that Σa_n converge by the condition that $a_n \to 0$.

<u>Example 8.82</u>. Set $a_{2n} = 1/(n+2)$ and $a_{2n+1} = 1/(n+2)^{3/2} - 1/(n+2)$ for $n \geq 0$. Then $a_n \to 0$ and, for $n \geq 0$, $a_{2n} + a_{2n+1} = 1/(n+2)^{3/2}$. Thus, $S = \sum_0^\infty 1/(n+2)^{3/2} = z(3/2) - 1$, where $z(s) = \sum_1^\infty 1/n^s$, $s > 1$, is the Riemann zeta function. It can be verified that $r = -1$, $(1+r_{n+1})/(1+r_n) \to -1$, and $r_{2n} < -1 < r_{2n-1}$ for $n \geq 1$. Thus, Σa_n is a convergent series satisfying the hypothesis of Theorem 8.68 with $L = -1$. From Theorem 8.81, $T_{2n} \to -\infty$ and $T_{2n-1} \to +\infty$.

<u>Example 8.83</u>. Set $a_{2n} = 1/(n+1)^{1/2}$ and $a_{2n+1} = [1-(n+2)^{1/2}]/[(n+1)(n+2)]^{1/2}$ for $n \geq 0$. We have $a_n \to 0$ and, for $n \geq 0$, $a_{2n} + a_{2n+1} = 1/[(n+1)(n+2)]^{1/2} > 1/(n+2)$. Thus Σa_n diverges. Also, $r = -1$ and $(1+r_{n+1})/(1+r_n) \to -1$. Consequently, the hypothesis of Theorem 8.68 is satisfied by the given divergent series

where $L = -1$. Moreover, we see that the requirement in Theorem 8.81 that Σa_n converge cannot be replaced by the condition that $a_n \to 0$.

Theorem 8.84. If Σa_n is an N-alternating series, $a_n \to 0$, and $1/2 \leq 1+r_n+r_n r_{n+1}/2$ for $n \geq N$, then, for $n \geq N$, $-1 < r_n$, $-1/2 \leq r_n+r_n r_{n+1}/2 \leq T_n \leq r_n/2$, and $|a_n|/2 \leq |S-S_{n-1}| \leq |a_{n-1}|/2$. If, in addition, $r = -1$, then $T_n \to r/(1-r) = -1/2$.

Proof: Since $1/2 \leq 1+r_n+r_n r_{n+1}/2$ for $n \geq N$, we have $-1/2 \leq r_n+r_n r_{n+1}/2$. For $n \geq N$, we use Theorem 8.3 with $\alpha_n = 1/2$ to obtain $-1/2 \leq r_n+r_n r_{n+1}/2 \leq T_n \leq r_n/2$ and $-1 \leq r_n$. For $n \geq N$, $-1/2 \leq T_n \leq r_n/2 < 0$, from which $|r_n|/2 \leq |T_n| \leq 1/2$ and $|a_n|/2 \leq |S-S_{n-1}| \leq |a_{n-1}|/2$. Suppose that $r_m = -1$ for some integer $m \geq N$. Assume that n is any integer $\geq m$ such that $r_n = -1$. Then $1/2 \leq 1+r_n+r_n r_{n+1}/2 = -r_{n+1}/2$ and $r_{n+1} \leq -1$. Consequently, $r_{n+1} = -1$ since $-1 \leq r_{n+1}$. By induction, $r_n = -1$ for $n \geq m$ which contradicts $a_n \to 0$. Thus, $-1 < r_n$ for $n \geq N$. If, in addition, $r = -1$, then from $-1/2 \leq T_n \leq r_n/2 \to -1/2$, we have $\lim T_n = -1/2$. Q.E.D.

Corollary 8.85. If Σa_n is an N-alternating series, $a_n \to 0$, and $r_{n+1} \leq r_n$ for $n \geq N$, then, for $n \geq N$, $-1 < r_n$, $-1/2 \leq r_n + r_n r_{n+1}/2 \leq T_n \leq r_n/2$, and $|a_n|/2 \leq |S-S_{n-1}| \leq |a_{n-1}|/2$. If, in addition, $r = -1$, then $T_n \to r/(1-r) = 1/2$.

Proof: The inequality $1/2 \leq 1+x+x^2/2$ holds for all real x. Consequently, since $r_{n+1} \leq r_n < 0$ for $n \geq N$, it follows that $1/2 \leq 1+r_n+r_n^2/2 \leq 1+r_n+r_n r_{n+1}/2$ for $n \geq N$. Now apply Theorem 8.84. Q.E.D.

Corollary 8.86. If Σa_n is an N-alternating series, $a_n \to 0$, and $\Delta^2 |a_{n-1}| \geq 0$ for $n \geq N$, then, for $n \geq N$, $-1 < r_n$, $-1/2 \leq r_n + r_n r_{n+1}/2 \leq T_n \leq r_n/2$, and $|a_n|/2 \leq |S-S_{n-1}| \leq a_{n-1}|/2$. If, in addition, $r = -1$, then $T_n \to r/(1-r) = -1/2$.

Proof: Let $n \geq N$. Then $1+r_n+r_n r_{n+1}/2-1/2$
$= (1+2r_n+r_n r_{n+1})/2 = (1-2|a_n|/|a_{n-1}|+|a_{n+1}|/|a_{n-1}|)/2$
$= (|a_{n-1}|-2|a_n|+|a_{n+1}|)/2|a_{n-1}| = (\Delta^2|a_{n-1}|)/2|a_{n-1}| \geq 0$, and thus $1/2 \leq 1+r_n+r_n r_{n+1}/2$. We now apply Theorem 8.84. Q.E.D.

Calabrese (10, p.215-217) appears to be the first to publish a result similar to our Corollary 8.86. In particular, he states that if $\sum_1^\infty a_n$ is a convergent alternating series, $|a_n|-|a_{n+1}| > |a_{n+1}|-|a_{n+2}|$, i.e., $\Delta^2|a_n| > 0$ for al n, and $|a_k| \leq 2\varepsilon$ for some integer k, then $|S_k-S| \leq \varepsilon$. His proof is incorrect since he uses the fact that in "every" convergent alternating series the sum S must lie between any two successive sums S_{n-1} and S_n.

It would be very convenient if the conditions $a_n \to 0$ and $r = -1 <. r_n$ implied that $T_n \to r/(1-r) = -1/2$, but the following counterexample shows that this is not the case.

<u>Counterexample 8.87</u>. Let $S' = a_0'+a_1'+a_2'+\cdots$ be any alternating series such that $a_n' \to 0$ and $r' = -1 < r_{n+1}' < r_n' < -1/2$ for $n \geq 1$. For $n \geq 1$, set $r_{2n-1} = r_{2n-1}'$ and $r_{2n} = -1+2(1+r_{2n-1})$. Define $a_0 = a_0'$ and $a_n = a_0 r_1 r_2 \cdots r_n$ for $n \geq 1$. It can be verified that Σa_n is a convergent alternating series such that $r = -1 < r_n$ for $n \geq 1$. Defining $\beta_{2n} = 2r_{2n+1}$ for $n \geq 1$, we have $\beta_{2n} = -1+r_{2n+2} > -1 + r_{2n+4} = \beta_{2n+2}$

for $n \geq 1$. Also, $\beta_{2n} = r_{2n+1}(1+r_{2n+2})/(1+r_{2n+1})$ for $n \geq 1$, so that $1/(1-\beta_{2n}) = 1+r_{2n+1}+r_{2n+1}r_{2n+2}/(1-\beta_{2n})$ $\geq 1+r_{2n+1}+r_{2n+1}r_{2n+2}/(1-\beta_{2n+2})$ for $n \geq 1$. Consequently, it can be seen that $1/(1-\beta_{2n}) \geq 1+T_{2n+1}$, i.e., T_{2n+1} $\leq \beta_{2n}/(1-\beta_{2n})$ for $n \geq 1$. For $n \geq 1$, $-2 < \beta_{2n}$ $= r_{2n+1}(1+r_{2n+2})/(1+r_{2n+1})$, from which $1/3 \leq 1+r_{2n+1}$ $+r_{2n+1}r_{2n+2}/3$. Consequently, $1/3 \leq 1+T_{2n+1}$ for $n \geq 1$, and thus $-2/3 \leq T_{2n+1} \leq \beta_{2n}/(1-\beta_{2n})$ for $n \geq 1$. Since $\beta_{2n}/(1-\beta_{2n}) \to -2/3$, $T_{2n-1} \to -2/3$ and $T_{2n} = r_{2n}(1+T_{2n+1})$ $\to -1/3$. An example of such a series $\Sigma a_n'$ is $1/3-1/5$ $+ 1/7=1/9+\cdots = 1-\pi/4$.

Theorem 8.88. Let Σa_n be a convergent series and n be any positive integer such that $r_n < 0$. Then we either have

(1) $\quad T_{n+1} < r_n/(1-r_n)$, $T_{n+1} < T_n$, and $r_n/(1-r_n) < T_n$,

(2) $\quad T_{n+1} = r_n/(1-r_n)$, $T_{n+1} = T_n$, and $r_n/(1-r_n) = T_n$,

or

(3) $\quad T_{n+1} > r_n/(1-r_n)$, $T_{n+1} > T_n$, and $r_n/(1-r_n) > T_n$.

Proof: Since $T_n = r_n(1+T_{n+1})$ and $T_{n+1} = T_n/r_n - 1$, the following inequalities are equivalent:

$T_{n+1} < r_n/(1-r_n)$, $T_{n+1}-r_nT_{n+1} < r_n$, $T_{n+1} < r_n(1+T_{n+1})$, $T_{n+1} < T_n$, $T_n/r_n - 1 < T_n$, $T_n - r_n > r_nT_n$, $T_n - r_nT_n > r_n$, $T_n > r_n/(1-r_n)$, $r_n/(1-r_n) < T_n$. Consequently, the inequalities in (1) are equivalent. Similarly, the equalities in (2) are equivalent and the inequalities in (3) are equivalent. Q.E.D.

<u>Theorem 8.89</u>. Let Σa_n be an N-alternating series. Then the following three conditions are equivalent:

(1) $T_{n+1} \leq T_n$, $n \geq N$,

(2) $T_{n+1} \leq r_n/(1-r_n)$, $n \geq N$,

(3) $r_n/(1-r_n) \leq T_n$, $n \geq N$.

Moreover, if (1), (2), or (3) holds, then

(4) $r_{n+1} \leq r_n$, $n \geq N$,

and

(5) $T_n \leq r_n/(1-r_{n+1})$, $n \geq N$.

<u>Proof</u>: According to Theorem 8.88, if equality holds in (1), (2), or (3), it also holds in the other two, and likewise for inequality. Thus, (1), (2), and (3) are equivalent.

Assume that (1), (2), or (3) holds, and let n be any integer $\geq N$. From (3) and (2), $r_{n+1}/(1-r_{n+1}) \leq T_{n+1} \leq r_n/(1-r_n)$. Then $r_{n+1}(1-r_n) \leq r_n\,(1-r_{n+1})$ and

$r_{n+1} \leq r_n$, i.e., (4) holds. Finally, since $r_{n+1}/(1-r_{n+1}) \leq T_{n+1} = T_n/r_n - 1$, we have $T_n/r_n \geq 1 + r_{n+1}/(1-r_{n+1}) = 1/(1-r_{n+1})$ and $T_n \leq r_n/(1-r_{n+1})$. Q.E.D.

<u>Theorem 8.90</u>. Let Σa_n be an N-alternating series. Then a n.a.s.c. that $T_{n+1} \leq T_n$ for $n \geq N$ is that

(0) $\quad a_n \to 0,$

and there exist a sequence $\{\beta_n\}$ such that

(1) $\quad a_n\beta_n \to 0,$

(2) $\quad \beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}, \; n \geq N,$

and

(3) $\quad r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2} \leq r_n\beta_n, \quad n \geq N.$

Moreover, if (0), (1), (2), and (3) hold, then for $n \geq N$,

(4) $\quad T_{n+1} \leq r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1})$

and

(5) $\quad 1/(1-r_{n+1}) \leq \beta_n \leq r_{n+1}/r_n(1-r_{n+1}).$

<u>Proof</u>: For the necessity, define $\beta_n = 1+T_{n+1}$, $n \geq N$. Then $a_n\beta_n = a_n + a_nT_{n+1} = a_n + (S-S_n) \to 0$. Also, $\beta_n = 1+T_{n+1} = 1+r_{n+1}+r_{n+1}r_{n+2}(1+T_{n+3}) = 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$ for $n \geq N$, so that (2) holds with equality. Moreover, $r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2} = T_{n+1} \leq T_n = r_n(1+T_{n+1}) = r_n\beta_n$ for

$n \geq N$, i.e., (3) holds.

For the sufficiency, according to (a) of Theorem 8.27 and (3) of the present theorem, we have $T_{n+1} \leq r_{n+1} + r_{n+1}r_{n+2}\beta_{n+2} \leq r_n\beta_n \leq T_n$ for $n \geq N$, so that $T_{n+1} \leq T_n$ for $n \geq N$. Theorem 8.89 implies (4) of the present theorem. We now have $r_{n+1}/(1-r_{n+1}) \leq T_{n+1} \leq r_n\beta_n \leq T_n \leq r_n/(1-r_{n+1})$ for $n \geq N$, from which (5) of the present theorem is immediate. Q.E.D.

Theorem 8.91. Let Σa_n be an N-alternating series. Then a n.a.s.c. that $T_{n+1} \leq T_n$ for $n \geq N$ is that

(0) $\quad a_n \to 0$,

and there exist a sequence $\{\beta_n\}$ such that

(1) $\quad a_n\beta_n \to 0$,

(2) $\quad \beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$, $n \geq N$,

and

(3) $\quad \beta_n \leq 1/(1-r_n)$, $n \geq N$.

Moreover, if (0), (1), (2), and (3) hold, then, for $n \geq N$,

(4) $\quad T_{n+1} \leq r_n/(1-r_n) \leq r_n\beta_n \leq T_n \leq r_n+r_nr_{n+1}\beta_{n+1}$
$\qquad \leq r_n/(1-r_{n+1})$

and

(5) $\quad 1/(1-r_{n+1}) \leq \beta_n$.

Proof: Define $\beta_n = 1+T_{n+1}$ for $n \geq N$. As in the proof of the necessity of Theorem 8.90, conditions (0), (1), and (2) hold. Using Theorem 8.89, $\beta_n = 1+T_{n+1} \leq 1+r_n/(1-r_n)$ $= 1/(1-r_n)$, $n \geq N$, so that (3) holds.

For the sufficiency, assume that (0), (1), (2), and (3) hold. Using (3), we have for $n \geq N$, $(1-r_n)\beta_n \leq 1$, $\beta_n - r_n\beta_n \leq 1$, and $\beta_n - 1 \leq r_n\beta_n$. Consequently, from (2), $r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2} \leq \beta_n - 1 \leq r_n\beta_n$ for $n \geq N$. From Theorem 8.90, we obtain, for $n \geq N$, $T_{n+1} \leq T_n$, $T_{n+1} \leq r_n/(1-r_n)$, and $1/(1-r_{n+1}) \leq \beta_n$. From (3), for $n \geq N$, we have $r_n/(1-r_n) \leq r_n\beta_n$ and $r_n+r_nr_{n+1}\beta_{n+1} \leq r_n+r_nr_{n+1}/(1-r_{n+1})$ $= r_n/(1-r_{n+1})$. Applying (a) of Theorem 8.27, $r_n\beta_n \leq T_n$ $\leq r_n+r_nr_{n+1}\beta_{n+1}$ for $n \geq N$. Q.E.D.

Theorem 8.92. If Σa_n is an N-alternating series, then a n.a.s.c. that $T_{n+1} \leq T_n$ for $n \geq N$ is that

(0) $\quad a_n \to 0$,

and there exist a sequence $\{p_n\}$ such that, for $n \geq N$,

(1) $\quad 1/(1-p_n) \geq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-p_{n+2})$

and

(2) $\quad p_n \leq r_n$.

Moreover, if (0), (1), and (2) hold, then for $n \geq N$,

(3) $\quad T_{n+1} \le r_n/(1-r_n) \le r_n/(1-p_n) \le T_n \le r_n$
$+r_n r_{n+1}/(1-p_{n+1}) \le r_n/(1-r_{n+1})$

and

(4) $\quad r_{n+1} \le p_n.$

Proof: For the necessity, there is a sequence $\{\beta_n\}$ satisfying (1), (2), (3), and (5) of Theorem 8.91. Defining $p_n = 1-1/\beta_n$ for $n \ge N$, we easily verify that $p_n \le r_n$ for $n \ge N$. Also, for $n \ge N$, $\beta_n = 1/(1-p_n)$, so that (2) of Theorem 8.91 reduces to (1) above.

For the sufficiency, define $\beta_n = 1/(1-p_n)$ for $n \ge N$. Condition (1) above thus yields (2) of Theorem 8.91. From (2) and $r_n < 0$ for $n \ge N$, we have $0 < 1/(1-p_n) = \beta_n \le 1/(1-r_n) < 1$ for $n \ge N$, and thus $a_n\beta_n \to 0$, i.e., (1) and (3) of Theorem 8.91 hold. Finally, (3) and (4) above follow respectively from (4) and (5) of Theorem 8.91. Q.E.D.

Theorem 8.93. Let Σa_n be an N-alternating series. Then a n.a.s.c. that $T_{n+1} \le T_n$ for $n \ge N$ is that

(0) $\quad a_n \to 0,$

and there exist a sequence $\{\alpha_n\}$ such that

(1) $a_n\alpha_n \to 0$,

(2) $\alpha_n \leq 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$, $n \geq N+1$,

and

(3) $r_n/\ r_{n+1}(1-r_n) \leq \alpha_{n+1}$, $n \geq N$.

Moreover, if (0), (1), (2), and (3) hold, then for $n \geq N$,

(4) $$T_{n+1} \leq r_{n+1}\alpha_{n+1} \leq r_n/(1-r_n) \leq r_n+r_nr_{n+1}\alpha_{n+1} \leq T_n \leq r_n/(1-r_{n+1})$$

and

(5) $\alpha_{n+1} \leq 1/(1-r_{n+1})$.

<u>Proof</u>: For the necessity, define $\alpha_n = 1+T_{n+1}$, $n \geq N$. Then $a_n\alpha_n = a_n+a_nT_{n+1} = a_n+(S-S_n) \to 0$. Also, $\alpha_n = 1 + T_{n+1} = 1+r_{n+1}+r_{n+1}r_{n+2}(1+T_{n+3}) = 1+r_{n+1}+r_{n+1}r_{n+2}\alpha_{n+2}$ for $n \geq N$ so that (2) holds with equality. Moreover, $r_{n+1}\alpha_{n+1} = r_{n+1}(1+T_{n+2}) = T_{n+1} \leq T_n = r_n+r_nr_{n+1}\alpha_{n+1}$, for $n \geq N$, from which (3) is immediate.

For the sufficiency, define $\alpha_N = 1+r_{N+1} +r_{N+1}r_{N+2}\alpha_{N+2}$. From (3), $r_{n+1}\alpha_{n+1} \leq r_n/(1-r_n) \leq r_n +r_nr_{n+1}\alpha_{n+1}$ for $n \geq N$. From (a) of Theorem 8.3, $T_{n+1} \leq r_{n+1}\alpha_{n+1} \leq r_n+r_nr_{n+1}\alpha_{n+1} \leq T_n$ for $n \geq N$. From (5) of Theorem 8.89, $T_n \leq r_n/(1-r_{n+1})$ for $n \geq N$.

Consequently, (4) holds. (5) is a consequence of (4). Q.E.D.

Lemma 8.94. If r_n, r_{n+1}, r_{n+2} are any real numbers such that $(1-r_n)(1-r_{n+2}) \neq 0$, then
$1+r_{n+1}+r_{n+1}r_{n+2}/(1-r_{n+2})-1/(1-r_n) = r_{n+1}/(1-r_{n+2})-r_n/(1-r_n)$
$= (\Delta r_n+r_n\Delta r_{n+1})/[(1-r_n)(1-r_{n+2})]$.

Proof: We have $1+r_{n+1}+r_{n+1}r_{n+2}/(1-r_{n+2})-1/(1-r_n)$
$= [1-1/(1-r_n)]+r_{n+1}[1+r_{n+2}/(1-r_{n+2})] = -r_n/(1-r_n)$
$+r_{n+1}/(1-r_{n+2}) = [r_{n+1}(1-r_n)-r_n(1-r_{n+2})]/(1-r_n)(1-r_{n+2})$
$= [(r_{n+1}-r_n)+r_n(r_{n+2}-r_{n+1})]/(1-r_n)(1-r_{n+2})$
$= [\Delta r_n+r_n\Delta r_{n+1}]/(1-r_n)(1-r_{n+2})$. Q.E.D.

Lemma 8.95. If r_n, r_{n+1}, r_{n+2} are any real numbers, then the following inequalities are equivalent:

(1) $1/(1-r_n) \geq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-r_{n+2})$

(2) $r_n/(1-r_n) \geq r_{n+1}/(1-r_{n+2})$

(3) $0 \geq [\Delta r_n+r_n\Delta r_{n+1}]/[(1-r_n)(1-r_{n+2})]$.

Proof: The quivalence follows immediately from Lemma 8.94. Q.E.D.

Theorem 8.96. If Σa_n is an N-alternating series, $a_n \to 0$, and $r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n)$ for $n \geq N$, then, for $n \geq N$, (1) $\Delta r_n \leq 0$ and (2) $T_{n+1} \leq r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1})$.

<u>1st Proof</u>: Defining $\beta_n = 1/(1-r_n)$ for $n \geq N$, we see that $0 < \beta_n < 1$ for $n \geq N$ and thus $a_n\beta_n \to 0$. From (1) and (2) of Lemma 8.95, $\beta_n \geq 1+r_{n+1}+r_{n+1}r_{n+2}\beta_{n+2}$ for $n \geq N$. From (4) of Theorem 8.91, (2) of the present theorem holds. (1) follows from (2). We could also obtain (1) from (4) of Theorem 8.89. Q.E.D.

<u>2nd Proof</u>: Define $p_n = r_n$ for $n \geq N$. From (1) and (2) of Lemma 8.95, $1/(1-p_n) \geq 1+r_{n+1}+r_{n+1}r_{n+2}/(1-p_{n+2})$ for $n \geq N$. Now apply Theorem 8.92 and Theorem 8.89. Q.E.D.

<u>Theorem 8.97</u>. If Σa_n is an N-alternating series, $a_n \to 0$, and $\Delta r_n+r_n\Delta r_{n+1} \leq 0$ for $n \geq N$, then, for $n \geq N$, $\Delta r_n \leq 0$ and $T_{n+1} \leq r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1})$.

<u>Proof</u>: If $n \geq N$, then $\Delta r_n+r_n\Delta r_{n+1} \leq 0$, $(1-r_n)(1-r_{n+2}) > 0$, and $(\Delta r_n+r_n\Delta r_{n+1})/(1-r_n)(1-r_{n+2}) \leq 0$. Thus from Lemma 8.95, $r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n)$. We now apply Theorem 8.96. Q.E.D.

Theorem 8.98. If Σa_n is an N-alternating series and $r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1})$ for $n \geq N$, then

(1) $$0 < (-1)^n a_n/(1-r_{n+1}) \leq (-1)^n(S-S_{n-1}) \leq (-1)^n a_n/(1-r_n),\ n \geq N,$$

or

(2) $$(-1)^n a_n/(1-r_n) \leq (-1)^n(S-S_{n-1}) \leq (-1)^n a_n/(1-r_{n+1}) < 0,\ n \geq N,$$

according as $a_{2n} >. 0$ or $a_{2n} <. 0$, respectively.

Proof: Multiplying the inequality $r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1})$ throughout by $|a_{n-1}|$,

$$\frac{|a_{n-1}|}{a_{n-1}}\,\frac{a_n}{1-r_n} \leq \frac{|a_{n-1}|}{a_{n-1}}\,(S-S_{n-1}) \leq \frac{|a_{n-1}|}{a_{n-1}}\,\frac{a_n}{1-r_{n+1}} < 0,$$

and this reduces to (1) if $a_{2n} >. 0$, or (2) if $a_{2n} <. 0$. Q.E.D.

Theorem 8.99. If Σa_n is an N-alternating series such that $a_n \to 0$ and $\Delta r_n \leq \Delta r_{n+1}$ for $n \geq N$, then, for $n \geq N$, $\Delta r_n \leq 0$, $\Delta r_n + r_n \Delta r_{n+1} \leq 0$, and $T_{n+1} \leq r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1})$.

Proof: We first show that $\Delta r_n \leq 0$ for $n \geq N$. In particular, assume that $0 < \Delta r_m$ for some $m \geq N$. Then

$\Delta r_m \le \Delta r_n$ for $n \ge m$, and thus $r_{m+k} = r_m + \Delta r_m + \Delta r_{m+1} + \cdots + \Delta r_{m+k-1} \ge r_m + k\Delta r_m \to \infty$ as $k \to \infty$; hence $r_n \to \infty$. This contradicts $a_n \to 0$, so that $\Delta r_n \le 0$, i.e., $r_{n+1} \le r_n < 0$ for $n \ge N$. Consequently, $-1 < r_n$ for $n \ge N$, since $a_n \to 0$. Therefore, $\Delta r_n + r_n \Delta r_{n+1} \le \Delta r_{n+1} + r_n \Delta r_{n+1} = (1+r_n)\Delta r_{n+1} \le 0$ for $n \ge N$. We may now apply Theorem 8.97. Q.E.D.

<u>Theorem 8.100</u>. Suppose that Σa_n is a series such that $a_n \to 0$, and that f is a function and N is a positive integer such that:

(1) $f(x) < 0$ for $N \le x$,

(2) f' is increasing on $[N, \infty)$, or $f''(x) \ge 0$ for $N \le x$,

(3) $r_n = f(n)$ for $n \ge N$.

Then, for $n \ge N$, $\Delta r_n \le \Delta r_{n+1}$ and $T_{n+1} \le r_{n+1}/(1-r_{n+2}) \le r_n/(1-r_n) \le T_n \le r_n/(1-r_{n+1})$.

<u>Proof</u>: Let n be any integer $\ge N$. By the Mean Value Theorem for derivatives there exist u, v such that $n < u < n+1 < v < n+2$ and $\Delta r_n = f(n+1) - f(n) = f'(u)[(n+1)-n] = f'(u) \le f'(v) = f'(v)[(n+2)-(n+1)] = f(n+2) - f(n+1) = \Delta r_{n+1}$. We now apply Theorem 8.99 to complete the proof. Q.E.D.

We now illustrate Theorem 8.100 with some examples.

<u>Example 8.101</u>. $\ln 2 = 1-1/2+1/3-1/4+\cdots$. Here $a_n = (-1)^n/(n+1)$ for $n \geq 0$, $r_n = a_n/a_{n-1} = -n/(n+1)$ for $n \geq 1$, and we set $f(x) = -x/(x+1)$ for $x \geq N = 1$. Accordingly, for $1 \leq x$, we have $f(x) < 0$, $f'(x) = -1/(x+1)^2$, and $f''(x) = 2/(x+1)^3 > 0$. Thus $\Delta r_n \leq \Delta r_{n+1}$, for $n \geq 1$, and Theorem 8.100 is applicable with $N = 1$. (1) of Theorem 8.98 reduces to $(n+2)/(n+1)(2n+3) \leq 1/(n+1)-1/(n+2)+1/(n+3)-1/(n+4)+\cdots = (-1)^n(S-S_{n-1}) \leq 1/(2n+1)$ for $n \geq 1$.

<u>Example 8.102</u>. $\pi/4 = 1-1/3+1/5-1/7+\cdots$. Here $a_n = (-1)^n/(2n+1)$ for $n \geq 0$, $r_n = a_n/a_{n-1} = -(2n-1)(2n+1)$ for $n \geq 1$, and we set $f(x) = -(2x-1)(2x+1)$ for $x \geq N=1$. For $1 \leq x$, $f(x) < 0$, $f'(x) = -4/(2x+1)^2$, and $f''(x) = 16/(2x+1)^3 > 0$. From Theorem 8.100 and (1) of Theorem 8.98 we obtain, with $N = 1$, $(2n+3)/(2n+1)(4n+4) \leq (-1)^n(S-S_{n-1}) = 1/(2n+1)-1/(2n+3)+1/(2n+5)-1/(2n+7)+\cdots \leq 1/4n$ for $n \geq 1$.

<u>Example 8.103</u>. $\ln 3/2 = 1/2-1/(2\cdot 2^2)+1/(3\cdot 2^3)-1/(4\cdot 2^4)+\cdots$. Here $a_n = (-1)^n/(n+1)2^{n+1}$ for $n \geq 0$, $r_n = a_n/a_{n-1} = -n/2(n+1)$ for $n \geq 1$, and we set $f(x) = -x/2(x+1)$ for $x \geq N = 1$. For $1 \leq x$, $f(x) < 0$, $f'(x) = -1/2(x+1)^2$, and $f''(x) = 1/(x+1)^3 > 0$.

From Theorem 8.100 and (1) of Theorem 8.98, we have, with $N = 1$, $(n+2)/2^n(n+1)(3n+5) \le (-1)^n(S-S_{n-1})$

$= (1/2^{n+1})[1/(n+1)-1/2(n+2)+1/2^2(n+3)-1/2^3(n+4)+\cdots]$

$\le 1/2^n(3n+2)$ for $n \ge 1$.

Example 8.104. $(1-\sqrt{2})z(1/2) = 1-1/\sqrt{2} + 1/\sqrt{3} - 1/\sqrt{4}+\cdots$. Here z is the Riemann zeta function, $a_n = (-1)^n/\sqrt{n+1}$ for $n \ge 0$, $r_n = a_n/a_{n-1} = -\sqrt{n/(n+1)}$ for $n \ge 1$, and we set $f(x) = -\sqrt{x/(x+1)}$ for $x \ge N = 1$. For $1 \le x$, we have $f(x) < 0$, $f'(x) = -1/[2x^{1/2}(x+1)^{3/2}]$, and $f''(x) = (4x+1)/[4x^{3/2}(x+1)^{5/2}] > 0$. We may now use Theorem 8.100 and (1) of Theorem 8.98, obtaining, with $N = 1$, $[(n+2)/(n+1)]^{1/2}(\sqrt{n+2} - \sqrt{n+1}) \le (-1)^n(S-S_{n-1})$

$= 1/\sqrt{n+1} - 1/\sqrt{n+2} + 1/\sqrt{n+3} - 1/\sqrt{n+4} + \cdots \le \sqrt{n+1} - \sqrt{n}$

for $n \ge 1$.

Example 8.105. $\pi^2/12 = 1-1/2^2+1/3^2-1/4^2+\cdots$. Here $a_n = (-1)^n/(n+1)^2$ for $n \ge 0$, $r_n = -n^2/(n+1)^2$ for $n \ge 1$, and we set $f(x) = -x^2/(x+1)^2$ for $x \ge N = 1$. For $x \ge 1$, $f(x) < 0$ and $f''(x) = 2(2x-1)/(x+1)^4 > 0$. Applying Theorem 8.100 and (1) of Theorem 8.98, with $N = 1$, we have

$$\left(\frac{n+2}{n+1}\right)^2 \frac{1}{(n+1)^2+(n+2)^2} \le (n+1)^{-2} - (n+2)^{-2}$$

$$+ (n+3)^{-2} - (n+4)^{-2} + \cdots \le \frac{1}{n^2+(n+1)^2}$$

for $n \geq 1$. We note that $f(x) \doteq -1+2/(x+1)-1/(x+1)^2$, suggesting Theorem 8.107 which follows shortly.

Example 8.106. $1/\sqrt{2} = 1-1/2+(1\cdot3)/(2\cdot4)-(1\cdot3\cdot5)/(2\cdot4\cdot6)+(1\cdot3\cdot5\cdot7)/(2\cdot4\cdot6\cdot8)-\cdots$. Here $a_n = (-1)^n[1\cdot3\cdots(2n-1)]/[2\cdot4\cdots(2n)]$ for $n \geq 1$, $a_0 = 1$, $r_n = -(2n-1)/(2n)$ for $n \geq 1$, and we set $f(x) = -(2x-1)/(2x)$ for $x \geq N = 1$. For $x \geq 1$, $f(x) < 0$ and $f''(x) = 1/x^3 > 0$. From Theorem 8.100 and (1) of Theorem 8.98 with $N = 1$,

$$\frac{2n+2}{4n+3}\ \frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)} \leq (-1)^n(S-S_{n-1})$$
$$= \frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)} - \frac{1\cdot3\cdots(2n+1)}{2\cdot4\cdots(2n+2)}$$
$$+ - \cdots \leq \frac{2n}{4n-1}\ \frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}$$

for $n \geq 1$.

Theorem 8.107. Suppose that Σa_n is a series such that $a_n \to 0$, $r_n \doteq b+b_1/n+b_2/n^2+\cdots$, where $b < 0$, and the first non-zero b_k, if such exists, is positive. Then $\Delta r_n \lesssim \Delta r_{n+1}$ and $T_{n+1} \lesssim r_{n+1}/(1-r_{n+2}) \lesssim r_n/(1-r_n) \lesssim T_n \lesssim r_n/(1-r_{n+1})$.

Proof: If $b_k = 0$ for all $k > 0$, then $r_n \doteq b$, $-1 < b < 0$ since $a_n \to 0$, and each inequality in the conclusion of our theorem holds with equality.

Suppose on the other hand that b_p is the first non-zero b_k, so that $b_p > 0$ and $r_n \doteq b+b_p/n^p +b_{p+1}/n^{p+1}+b_{p+2}/n^{p+2}+\cdots$. Setting $f(x) = b+b_p/x^p +b_{p+1}/x^{p+1}+b_{p+2}/x^{p+2}+\cdots$, we see that f is an analytic function of $1/x$ for large x, $f(x) \lessdot 0$, and $f(n) \doteq r_n$. Differentiating twice, we have $f''(x) \doteq [p(p+1)b_p +(p+1)(p+2)b_{p+1}/x+\cdots]/x^{p+2} \gtrdot 0$, since $b_p > 0$. We may now apply Theorem 8.100. Q.E.D.

<u>Theorem 8.108</u>. Suppose that (1) Σa_n is an N-alternating series such that $a_n \to 0$ and $\Delta r_n \leq \Delta r_{n+1}$ for $n \geq N$, (2) $\Sigma a'_n$ is a series such that $a'_n \to 0$, and (3) f is a function such that $r'_n = -f(|r_n|)$, for $n \geq N$, and $f'(x) \geq 0$ and $f''(x) \leq 0$, for $|r_N| \leq x$. Then, for $n \geq N$, $\Delta r'_n \leq \Delta r'_{n+1}$ and $T'_{n+1} \leq r'_{n+1}/(1-r'_{n+2}) \leq r'_n/(1-r'_n) \leq T'_n \leq r'_n/(1-r'_{n+1})$.

<u>Proof</u>: Let n be any integer $\geq N$. As shown in the proof of Theorem 8.99, $r_{n+2} \leq r_{n+1} \leq r_n < 0$, i.e., $0 < |r_n| \leq |r_{n+1}| \leq |r_{n+2}|$. By the Mean Value Theorem for derivatives there is a u such that

$|r_n| \leq u \leq |r_{n+1}|$ and $\Delta r'_n = r'_{n+1} - r'_n = f(|r_n|) - f(|r_{n+1}|)$ $= f'(u)(|r_n| - |r_{n+1}|) = f'(u)(r_{n+1} - r_n) = f'(u)\Delta r_n$. Similarly, there is a v such that $|r_{n+1}| \leq v \leq |r_{n+2}|$ and $\Delta r'_{n+1} = f'(v)\Delta r_{n+1}$. Thus from $f'(u) \geq f'(v) \geq 0$ and $\Delta r_n \leq 0$, $\Delta r'_n = f'(u)\Delta r_n \leq f'(v)\Delta r_n \leq f'(v)\Delta r_{n+1}$ $= \Delta r'_{n+1}$ and $\Delta r'_n \leq \Delta r'_{n+1}$. We may now apply Theorem 8.99 to complete the proof. Q.E.D.

Corollary 8.109. If Σa_n is an N-alternating series such that $a_n \to 0$, $\Delta r_n \leq \Delta r_{n+1}$ for $n \geq N$, and $\Sigma a'_n$ is an N-alternating series such that $|a'_n| = |a_n|^p$ for $n \geq N-1$, where $0 < p < 1$; then, for $n \geq N$, $\Delta r'_n \leq \Delta r'_{n+1}$ and $T'_{n+1} \leq r'_{n+1}/(1-r'_{n+2}) \leq r'_n/(1-r'_n) \leq T'_n \leq r'_n/(1-r'_{n+1})$.

Proof: It is obvious that $a'_n \to 0$. Set $f(x) = x^p$ for $|r_N| \leq x$. Then for $n \geq N$, $r'_n = -|a'_n|/|a'_{n-1}|$ $= -|a_n|^p/|a_{n-1}|^p = -|a_n/a_{n-1}|^p = -|r_n|^p = -f(|r_n|)$. Also for $|r_N| \leq x$, $f'(x) = px^{p-1} > 0$ and $f''(x)$ $= p(p-1)x^{p-2} < 0$. We now apply Theorem 8.108. Q.E.D.

Example 8.110. $(1-2^{1-p})z(p) = 1-1/2^p+1/3^p-1/4^p+\cdots$, $0 < p < 1$. Here z is the Rieman zeta function and

$a_n' = (-1)^n/(n+1)^p$ for $n \geq 0$. With $a_n = (-1)^n/(n+1)$ for $n \geq 0$, Example 8.101 and Theorem 8.100 show that $\Delta r_n \leq \Delta r_{n+1}$ for $n \geq 1$. Noting that $|a_n'| = |a_n|^p$ for $n \geq 0$, we may apply Corollary 8.109 to obtain $T_{n+1}' \leq r_{n+1}'/(1-r_{n+2}') \leq r_n'/(1-r_n') \leq T_n' \leq r_n'/(1-r_{n+1}')$ for $n \geq 1$. The case $p = 1/2$ was previously considered in Example 8.104, but the above procedure, requiring the second derivative of $-x/(x+1)$, is preferable to differentiating $-x^p/(1+x)^p$ twice, as was done in Example 8.104.

Lemma 8.111. Suppose that f is a function and N is a positive integer such that (1) $f(x) > 0$, (2) $f'(x) \geq 0$, (3) $f''(x) \leq 0$, and (4) $f'''(x) \geq 0$, for $N-1 \leq x$. Then the function $g(x) = -f(x-1)/f(x)$ satisfies the conditions $g(x) < 0$ and $g''(x) \geq 0$, for $N \leq x$.

Proof: Let $N \leq x$. Clearly $g(x) < 0$ and, differentiating twice, $g''(x) = \{f(x)[f(x-1)f''(x)-f(x)f''(x-1)] +2f'(x)[f(x)f'(x-1)-f(x-1)f'(x)]\}/f^3(x)$. From (2), $f(x-1) \leq f(x)$ and thus $f(x-1)f''(x) \geq f(x)f''(x)$ according to (3). From (4), $f''(x)-f''(x-1) \geq 0$, so that $f(x-1)f''(x)-f(x)f''(x-1) \geq f(x)f''(x)-f(x)f''(x-1)$ $= f(x)[f''(x)-f''(x-1)] \geq 0$, since $f(x) > 0$. From (2), $f(x)f'(x-1) \geq f(x-1)f'(x-1)$. From (3), $f'(x-1)-f'(x) \geq 0$, and thus $f(x)f'(x-1)-f(x-1)f'(x) \geq f(x-1)f'(x-1)$ $-f(x-1)f'(x) = f(x-1)[f'(x-1)-f'(x)] \geq 0$. The inequality $g''(x) \geq 0$ is now evident. Q.E.D.

Theorem 8.112. Suppose that Σa_n is a series such that $a_n \to 0$. Suppose that f is a function and N is a positive integer such that: $f(x) > 0$, $f'(x) \geq 0$, $f''(x) \leq 0$, and $f'''(x) \geq 0$, for $N-1 \leq x$; and $r_n = -f(n-1)/f(n)$ for $N \leq n$. Then, for $n \geq N$, $\Delta r_n \leq \Delta r_{n+1}$ and $T_{n+1} \leq r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1})$.

Proof: Define $g(x) = -f(x-1)/f(x)$ for $N \leq x$. Then $r_n = g(n)$ for $n \geq N$. Also $g(x) < 0$ and $g''(x) \geq 0$ for $N \leq x$ according to Lemma 8.111. We may now use Theorem 8.100 to complete the proof. Q.E.D.

Theorem 8.113. Suppose that Σa_n is an N-alternating series such that $a_n \to 0$. Suppose that f is a function and N is a positive integer such that: $f(x) > 0$, $f'(x) \geq 0$, $f''(x) \leq 0$, and $f'''(x) \geq 0$, for $N-1 \leq x$; and $|a_n| = 1/f(n)$ for $N-1 \leq n$. Then, for $N \leq n$, $\Delta r_n \leq \Delta r_{n+1}$ and $T_{n+1} \leq r_{n+1}/(1-r_{n+1}) \leq r_n/(1-r_n) \leq T_n$ $\leq r_n/(1-r_{n+1})$.

Proof: For $N \leq n$, $r_n = a_n/a_{n-1} = -|a_n|/|a_{n-1}|$ $= -f(n-1)/f(n)$. Now apply Theorem 8.112. Q.E.D.

We now apply Theorem 8.113 to some of the series considered previously.

Example 8.114. $\ln 2 = 1-1/2+1/3-1/4+\cdots$. We have $a_n = (-1)^n(n+1)$, for $n \geq 0$, and we set $f(x) = x+1$, for $x \geq 0$. Clearly, $|a_n| = 1/f(n)$ for $0 \leq n$. For $0 \leq x$, $f(x) > 0$, $f'(x) = 1 \geq 0$, $f''(x) = 0 \leq 0$, and $f'''(x) = 0 \geq 0$. Theorem 8.113 is now applicable with $N = 1$. This series was previously treated in Example 8.101.

Example 8.115. $\pi/4 = 1-1/3+1/5-1/7+\cdots$ (see Example 8.102). We have $a_n = (-1)^n/(2n+1)$, for $n \geq 0$, and we set $f(x) = 2x+1$, for $x \geq 0$, so that $|a_n| = 1/f(n)$, for $n \geq 0$. If $x \geq 0$, then $f(x) > 0$, $f'(x) = 2 \geq 0$, $f''(x) = 0 \leq 0$, and $f'''(x) = 0$. We may now apply Theorem 8.113 with $N = 1$.

Example 8.116. $\ln 3/2 = \Sigma a_n$; $a_n = (-1)^n/(n+1)2^{n+1}$ for $n \geq 0$. Setting $f(x) = (x+1)2^{x+1}$, for $x \geq 0$, we find $f''(x) = 2^{x+1}[2+(x+1)\ln 2]\ln 2 > 0$, for $x \geq 0$, so that Theorem 8.113 is not applicable. In Example 8.103, Theorem 8.100 was shown to be applicable.

Example 8.117. $(1-2^{1-p})\ z\ (p) = \Sigma a_n$; $a_n = (-1)^n/(n+1)^p$, for $n \geq 0$, where $0 < p < 1$. Setting $f(x) = (x+1)^p$, for $x \geq 0$, $|a_n| = 1/f(n)$ for $n \geq 0$. For $x \geq 0$, $f(x) > 0$, $f'(x) = p(x+1)^{p-1} > 0$, $f''(x) = p(p-1)(x+1)^{p-2} < 0$, and $f'''(x) = p(p-1)(p-2)(x+1)^{p-3} > 0$. Theorem 8.113 is thus applicable with $N = 1$. This series was also considered in Example 8.110.

The function f in Theorem 8.113 satisfies the condition

(A) $\quad f(x) \to \infty$ as $x \to \infty$, $f'(x) \geq. 0$, $f''(x) \leq. 0$, $f'''(x) \geq 0$.

We now prove that if f and g are functions satisfying condition (A), then so does the composite function h where $h(x) = f(g(x))$. This will allow us to build up, or easily recognize, a wide variety of series Σa_n for which Theorem 8.113 is applicable.

Theorem 8.118. If f and g are functions which satisfy condition (A), then the composite function $h = f \circ g$ also satisfies condition (A).

Proof: Clearly $h(x) =. f(g(x)) \to \infty$ as $x \to \infty$. Also $h'(x) =. f'(g(x)) \cdot g'(x) \geq. 0$ since $g(x) \to \infty$ as $x \to \infty$, $f'(x) \geq. 0$, and $g'(x) \geq. 0$. Moreover, $h''(x) =. f''(g(x))[g'(x)]^2 + f'(g(x)) \cdot g''(x) \leq. 0$ is quite evident.

Finally, $h'''(x) \doteq f'''(g(x))\,[g'(x)]^3 + f''(g(x))\cdot 2g'(x)g''(x) + f''(g(x))g'(x)g''(x) + f'(g(x))\cdot g'''(x) \geq 0$. Q.E.D.

Corollary 8.119. Suppose that f and g are functions satisfying condition (A), and that Σa_n is a series for which $a_n \doteq (-1)^n/f(g(n))$. Then $\Delta r_n \leq \Delta r_{n+1}$ and $r_{n+1}/(1-r_{n+2}) \leq r_n/(1-r_n) \leq T_n \leq r_n/(1-r_{n+1})$.

Proof: Defining $h(x) \doteq f(g(x))$, h satisfies condition (A), according to Theorem 8.118. Thus $f(x) > 0$ and $|a_n| \doteq 1/h(n) \to 0$. We may now apply Theorem 8.113. Q.E.D.

Theorem 8.120. Suppose that Σa_n is an N-alternating series, $a_n \to 0$, and $\Delta r_n + r_n \Delta r_{n+1} \leq 0$ for $n \geq N$. Let $\Sigma a'_n$ be the power series defined by $a'_n = a_n x^{n+p}$, where p is some fixed real number. Then, for $0 < x \leq 1$ and $n \geq N$, $\Delta r'_n + r'_n \Delta r'_{n+1} \leq 0$ and $T'_{n+1} \leq r'_{n+1}/(1-r'_{n+2}) \leq r'_n(1-r'_n) \leq T'_n \leq r'_n/(1-r'_{n+1})$.

Proof: Let x be any number satisfying $0 < x \leq 1$ and n be any integer $\geq N$. Clearly, $a'_k = a_k x^{k+p} \to 0$ as $k \to \infty$. From Theorem 8.97, $\Delta r_{n+1} \leq 0$ so that $x^2 r_n \Delta r_{n+1} \leq x r_n \Delta r_{n+1}$. Thus $r'_n = a_n x^{n+p}/a_{n-1}x^{n-1+p} = x r_n$,

$\Delta r'_n = r'_{n+1} - r'_n = xr_{n+1} - xr_n = x\Delta r_n$, and $\Delta r'_n + r'_n \Delta r'_{n+1}$

$= x\Delta r_n + x^2 r_n \Delta r_{n+1} \leq x\Delta r_n + xr_n \Delta r_{n+1} = x(\Delta r_n + r_n \Delta r_{n+1}) \leq 0.$

Now apply Theorem 8.97 to $\Sigma a'_n$. Q.E.D.

Theorem 8.121. Suppose that Σa_n is an N-alternating series, $a_n \to 0$, and $\Delta r_n \leq \Delta r_{n+1}$ for $n \geq N$. Let $\Sigma a'_n$ be the series defined by $a'_n = a_n x^{n+p}$, where p is some fixed real number. Then, for $0 < x \leq 1$ and $n \geq N$, $\Delta r'_n \leq \Delta r'_{n+1}$ and $T'_{n+1} \leq r'_{n+1}/(1-r'_{n+1}) \leq r'_n/(1-r'_n) \leq T'_n \leq r'_n/(1-r'_{n+1})$.

Proof: Let x be any number satisfying $0 < x \leq 1$ and n be any integer $\geq N$. Clearly, $a'_k \to 0$ as $k \to \infty$. Also, $\Delta r'_n = x\Delta r_n \leq x\Delta r_{n+1} = \Delta r'_{n+1}$. We now apply Theorem 8.99 to $\Sigma a'_n$. Q.E.D.

Example 8.122. $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$, $0 < x \leq 1$. We have $a_n = (-1)^n/(n+1)$ and $a'_n = a_n x^{n+1}$ for $n \geq 0$. As shown in Example 8.101 or 8.114, $\Delta r_n \leq \Delta r_{n+1}$ for $n \geq 1$, so that Theorem 8.121 is applicable to $\Sigma a'_n$, where $N = p = 1$.

CHAPTER IX

SUMMARY

In Chapter I, definitions and notations are introduced. In particular, the quantities T_n are defined by the equation $T_n = (S-S_{n-1})/a_{n-1}$, if Σa_n converges to S and n is any integer such that $a_{n-1} \neq 0$. Various algebraic properties of T_n are proven. A geometrical interpretation of Aitken's δ^2-process is given, and several formulas are set forth, each of which yields this method of acceleration. Also, the notion of "transform sequence" is introduced to set up a unifying framework for investigating various methods of acceleration.

In Chapter II, the convergence of $\{T_n\}$ is treated and corresponding n.a.s.c. for $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ are proven. Divergence theorems are proven, which are used to prove that if Σa_n and $\Sigma a_{\delta n}$ are convergent complex series, then $S = S_\delta$. This fact was first published by Lubkin (17, p. 230) for real series. We are then led in a natural manner to some theorems on rapidity of convergence.

In Chapter III, n.a.s.c for $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ are

established. It is shown that any sequence $\{\alpha_n\}$ such that $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$ determines all such sequences $\{\beta_n\}$ by the simple condition $\beta_n \sim \alpha_n$. This is then used, along with algebraic properties of T_n, to prove that $\Sigma a_{\delta n} \varepsilon MR(\Sigma a_n)$ if and only if $T_{n+1}-T_n \to 0$. With the added condition $|r_n| \leq. \rho < 1$, it is proven that $\Sigma a_{\delta n} \varepsilon MR(\Sigma a_n)$ if and only if $r_{n+1}-r_n \to 0$. It is also proven that if $|r_n| \leq. \rho < 1$ and $r_{n+1}-r_n \to 0$, then Lubkin's W transformation and a slight variant of the W transformation may be used for accelerating the convergence of Σa_n. The relationship between the δ^2-process and the W transformation, as concerns acceleration, is shown under the restriction $a_{\delta n}/a_n \to 0$; in particular, $a_{\delta n}/a_n \to 0$ implies that $\Sigma a_{\delta n} \varepsilon MR(\Sigma a_n)$ if, and only if, $\Sigma a_{\alpha n} \varepsilon MR(\Sigma a_n)$, where $\alpha_n =. (1-r_{n-1})/(1-2r_n+r_{n-1}r_n)$. The application of the δ^2-process to power series is also considered.

In Chapter IV, rapidity of convergence is again considered. Methods for accelerating convergence published by various authors, previously cited, are extended to complex series. In extending Lubkin's Theorems 8 and 9 (17, p. 232-233), it is shown that part of each hypotheses may be omitted. Pflanz (18, p. 25) established this fact for the former theorem where Σa_n is real.

If Σa_n is a convergent series such that $|r_n| \to 1$, the application of Aitken's δ^2-process becomes critical. In particular, that part of Lubkin's Theorem 6 (17, p. 231) concerning acceleration is shown to have no application if $r_n \to 1$. Similarly, that part of his Theorem 7 (17, p. 232) concerning acceleration is proven to be vacuous. The letter "C" in Theorem 7 is in error and should be replaced by "Q". At this point, one wonders if the δ^2-process is ever practicable if $|r_n| \to 1$. The answer is in the affirmative, as is shown by Theorem 4.17, Theorem 4.20, and the discussion following the former theorem. Theorems on the acceleration of power series are also established.

Kummer's criterion, known to be sufficient for the convergence of a series Σa_n of positive terms, is proven to also be necessary in Chapter V. The necessity was first published by Shanks (24, p. 340). The criterion is that there exists a sequence $\{\beta_n\}$ and a positive number c such that $\beta_n > 0$, for $n > 0$, and $\beta_n \geq c + r_{n+1}\beta_{n+1}$ for $n \geq 1$. It is proven in this paper that "$\beta_n > 0$" can be replaced by any one of the conditions "$\beta_n \geq 0$", "$\{a_n\beta_n\}$ converges", or "some subsequence of $\{a_n\beta_n\}$ is bounded below". Proofs of the sufficiency of the comparison test, ratio comparison test, root test, ratio test, and Raabe's test, are given by exhibiting a sequence $\{\beta_n\}$ such that $\beta_n \geq. 0$ and $\beta_n \geq. 1+r_{n+1}\beta_{n+1}$. At the end of Chapter V, a method for applying the previously developed error analysis is indicated by one example.

Chapter VI gives the analogues of some of the theorems of Chapter V for real series, and Chapter VII does likewise for complex series.

In Chapter VIII, theorems, similar to Kummer's criterion for the convergence of series of positive terms, stating n.a.s.c. for an alternating series to converge are proven. Some of these theorems lead to fairly sharp bounds for the quantities T_n. In many such theorems, it is proven that all inequalities, excluding those between indices, may be reversed. Calling any such theorem and the derived theorem duals, we encounter a duality structure, which unhappily fails in at least one case.

The theory of alternating series in this paper resulted from an initial study of Aitken's δ^2-process in the critical case $r_n \to -1$. Lubkin's Theorem 5 (17, p. 231) states that if Σa_n is a real convergent series, $r = -1$, and $(1+r_{n+1})/(1+r_n) \to 1$, then $\Sigma a_{\delta n} \in \mathrm{MR}(\Sigma a_n)$. Generalizations of this theorem are proven; one involves lim inf $(1+r_{n+1})/(1+r_n) \to 1$, while another involves lim sup $(1+r_{n+1})/(1+r_n) = 1$. Another theorem along this line involves the inequality $1/2 \leq 1+r_{n+1}+r_{n+1}r_{n+2}/2$, actually the first theorem discovered by the author. A detailed analysis of bounds for T_n is considered throughout, which immediately yield bounds for $S-S_{n-1}$. Calabrese (10, p. 216) appears to be the only one to publish any result along the lines developed in our chapter on alternating series. His theorem is true, but the proof which he gives contains an error. The final part of Chapter VIII is devoted to finding simple tests for applying the developed error bounds for T_n.

BIBLIOGRAPHY

1. Aitken, A. C. On Bernoulli's numerical solution of algebraic equations. Proceedings of the Royal Society of Edinburgh 46:289-305. 1926.

2. Aitken, A. C. Further studies in algebraic equations and matrices. Proceedings of the Royal Society of Edinburgh 51:80-90. 1931.

3. Aitken, A. C. Studies in practical mathematics. Proceedings of the Royal Society of Edinburgh 57: 269-304. 1937.

4. Aitken, A. C. Studies in practical mathematics. V. On the iterative solution of a system of linear equations. Proceedings of the Royal Society of Edinburgh 63:52-60. 1950.

5. Apostol, Tom M. Mathematical analysis. Reading, Massachusetts, Addison-Wesley, 1957. 553 p.

6. Bickley, W. G. and J. C. P. Miller. The numerical summation of slowly convergent series of positive terms. Philosophical Magazine 22: 754-767. 1936.

7. Blanch, G. and L. K. Jackson. Computation of harmonic measure by L. Ahlfor's method. National Bureau of Standards Applied Mathematics Series 42: 53-61. 1955. (Experiments in the Computation of Conformal Maps).

8. Bradshaw, J. W. Modified continued fractions for certain series. American Mathematical Monthly 45: 352-362. 1938.

9. Bradshaw, J. W. Modified series. American Mathematical Monthly 46:486-492. 1939.

10. Calabrese, Philip. A note on alternating series. American Mathematical Monthly 69:215-217. 1962.

11. Forsythe, George E. Solving linear algebraic equations can be interesting. Bulletin of the American Mathematical Society 59:299-329. 1953.

12. Hartree, D. R. Notes on iterative processes. Cambridge Philosophical Society 45:230-236. 1949.

13. Householder, Alton S. Principals of numerical analysis. New York, McGraw-Hill, 1953. 274 p.

14. Isakson, Gabriel. A method for accelerating the convergence of an iterative process. Journal of the Aeronautical Sciences 16:443. 1949.

15. Knopp, Konrad. Theory and application of infinite series. 2d. ed. New York, Hafner, n.d. 563 p.

16. Kummer, E. E. Eine neue Methode, die numerischen Summen langsam convergirenden Reihen zu berechnen. Journal für die Reine und Angewandte Mathematik 16: 206-214. 1837.

17. Lubkin, Samuel. A method of summing infinite series. Journal of Research of the National Bureau of Standards 48:228-254. 1952.

18. Pflanz, Erwin. Über die Beschleunigung der Konvergenz langsam konvergenter unendlicher Reihen. Archiv der Mathematik 3:24-30. 1952.

19. Rosser, J. Barkley. Transformations to speed the convergence of series. Journal of Research of the National Bureau of Standards 46:56-64. 1951.

20. Samuelson, Paul A. A convergent iterative process. Journal of Mathematics and Physics 24:131-134. 1945.

21. Schmidt, R. J. On the numerical solution of linear simultaneous equations by an iterative method. Philosophical magazine 32:369-383. 1941.

22. Shanks, Daniel. An analogy between transients and mathematical sequences and some nonlinear sequence-to-sequence transforms suggested by it. White Oak, Md., 1949. 42 p. (Naval Ordnance Laboratory Memorandum 9994. Part I. Project NOL-4-Re 9d-21-2. 26 July 1949).

23. Shanks, Daniel. Non-linear transformations of divergent and slowly convergent sequences. Journal of Mathematics and Physics 34:1-42. 1955.

24. Shanks, E. Baylis. Commvergence of series with positive terms. American Mathematical Monthly 64: 338-341. 1957.

25. Shohat, J. A. On a certain transformation of infinite series. American Mathematical Monthly 40:226-229. 1933.

26. Szász, Otto. Summation of slowly convergent series. Journal of Mathematical Physics 28:272-279. 1950.

27. Thielman, Henry P. Theory of functions of real variables. New York, Prentice-Hall, 1953. 209 p.

28. Todd, John. (ed.) Survey of numerical analysis. New York, McGraw-Hill, 1962. 589 p.

29. Todd, John and S. E. Warshawski. On the solution of the Lichtenstein-Gershgorin integral equation in conformal mapping: II. Computational experiments. National Bureau of Standards Applied Mathematics Series 42:31-44. 1955. (Experiments in the Computation of Conformal Maps).

30. Wynn, P. On a device for computing $e_m(S_n)$ transformation. Mathematical Tables and Other Aids to computation 10:91-96. 1956.