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Eugene Zhang

3D symmetric tensor fields have a wide range of applications, such as in solid and fluid mechanics, medical imaging, meteorology, molecular dynamics, geophysics and computer graphics. There has been much research carried out in this field, yet our knowledge of the tensor field is still at its initial stage to completely understand the behavior of 3D linear tensor fields. To understand the behavior and to design such applications, topology plays an important role. The degenerate points are the most studied topological feature of symmetric tensor fields. Though several attempts have been made to understand such features, still none of them seems to be complete. In this work, we provide an interactive interface to study such features. We also study the maximum number of transition points in a linear tensor field and classification of wedge and trisector along the degenerate curves. Finally, we provide an insight over the upper and lower bound on the number of transition points in a linear tensor field.

Interactive Design and Transition Point Analysis of 3D Linear Symmetric Tensor Fields

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Ritesh Sharma

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Ritesh Sharma, Author

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Chapter 1: Introduction

3D symmetric tensor fields have a wide range of applications, such as in solid and fluid mechanics, medical imaging, meteorology, molecular dynamics, geophysics and computer graphics. There has been much research carried out in tensor fields, yet our knowledge of the tensor field is still at its initial stage to completely understand the behavior of 3D linear tensor fields. To understand the behavior and to design aforementioned applications, topology plays an important role. The degenerate points are the most studied topological feature of symmetric tensor fields. Though several attempts have been made to effectively visualize tensor field topology, still none of them can completely visualize all the topological features. This work is an attempt to design and study topological features of a tensor fields, and in particular, understand their behavior more effectively.

The main theme of this thesis is to design 3D linear symmetric tensor field and analyze transition points using our interactive interface. The work presented here will help in understanding the behavior of linear tensor fields and studying tensor field topology in detail. The term tensor, in general, is a linear relation between scalar, vector, and other tensors. A tensor field is a tensor valued function in space. It is often difficult to understand the behavior of tensor fields due to its nine components. As a result, the visualization of tensor field is cluttered and difficult to understand. To simplify such visualization and to make them intuitive, we study tensor field topology. Tensor field topology helps in understanding the behavior of tensor fields by focusing only on the required information. Usually, researchers are mostly interested in symmetric tensor fields and sometimes the data is inherently symmetric. A symmetric tensor can be decomposed into an isotropic and a deviatoric tensors. We can neglect the isotropic part as it only provides information about the uniform scaling, whereas deviatoric part provides the orientation and direction of eigenvector fields. So from here onwards whenever we are citing tensor fields, the reader should assume that we are talking about symmetric traceless tensor fields. In this work, we study linear tensor field topology and propose method for robust extraction of tensor field topology. The first term of Taylor expansion of tensor field shows that the behavior of tensor field near the point of interest is dictated by the linearization at the point [9]. This also means that generic tensor field can be easily understood by linear tensor fields. The main contributions of this work are as follows:

- Interactive interface to design any 3D linear tensor fields.
- Observe and analyze 3D linear tensor fields.
- Provide theoretical as well as observed lower and upper bound on the number of transition points.

1.1 Previous Work

There has been much research carried out on the analysis and visualization of 2D and 3D linear tensor fields. The detailed survey can be found at [5]. Delmarcelle and Hesselink [1, 2] introduce the topology of 2D symmetric tensor fields. They point out that there are two fundamental types of degenerate points in a 2D symmetric tensor field, i.e., wedges and trisectors, which have a tensor index of $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Hesselink et al. later extended this work to 3D symmetric tensor field [3] and study triple degenerate points, i.e., all eigenvalues are the same. Zheng et al. [12] point out that triple degeneracy are not structurally stable features. They further show that double degeneracies, i.e., tensor with only two equal eigenvalues, form lines in the domain. In this work and subsequent research [13], they provide a number of degenerate curve extraction methods based on the analysis of the discriminant function of the tensor field. Furthermore, Zheng et al. [14] point out that near degenerate curves the tensor field exhibits 2D degenerate patterns and define separating surfaces which are extensions of separatrices from 2D symmetric tensor field topology. Tricoche et al. [7] convert the problem of extracting degenerate curves in a 3D tensor field to that of finding the ridge and valley lines of an invariant of the tensor field, thus leading to a more robust extraction algorithm. Perhaps the most relevant research is the work of Zhang et al.[9], which showed that there are at least one and at most four degenerate curves in a 3D linear tensor field under structurally stable conditions.

Chapter 2: Mathematical Background

In this chapter, we review the mathematical background on tensor fields. For the sake of understanding, we will start with general definition of tensor and tensor fields and then narrow down our study from general tensor fields to linear symmetric tensor fields, so that the readers have a better understanding of different mathematical concepts explained in this thesis. The mathematical definitions and concepts presented below are taken from the thesis of Kratz [6].

2.1 Tensor

Tensor is defined as an array of numbers whose entries changes in a certain fashion with respect to the change of basis. Basically, a tensor is a generalization of the concepts of scalar, vector, and matrix values from linear and multi-linear algebra. The zeroth order tensor is known as a scalar, whereas the first order tensor is known as a vector. A two-dimensional matrix is known as a second order tensor. All tensors greater than second order tensor are higher order tensors. Geometrically, a tensor is a quantity having magnitude and two opposite directions, which is analogous to a vector having a magnitude and a direction. The most important feature of tensor which makes it powerful to study is its invariant property which does not change with the change of basis.

2.1.1 Tensor definition

Definition 2.1.1 (Tensor as a multilinear map)**.** *Let V and V^* be n-dimensional vector space and its dual vector space over $\mathbb{R}$ respectively. A multilinear map which takes k copies of vector space V and l copies of its dual space V^* into the space $\mathbb{R}$ of real numbers*

$$T : \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{l \text{ times}} \to \mathbb{R} \tag{2.1}$$

is known as tensor of order $k + l$.

Definition 2.1.2 (Second-order tensor). *Let V be an n-dimensional vector space over $\mathbb{R}$. A bilinear map which takes two copies of vector space V onto the space $\mathbb{R}$ of real numbers*

$$T : V \otimes V \to \mathbb{R} \tag{2.2}$$

is known as second order tensor. In other words, it can be said that the tensor T is a bilinear map which takes a vector $v \in V$ and maps onto another vector $w \in V$. In such a case, T is a map from the vector space V onto itself. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of vector space V, then matrix notation for the second-order tensor T is given as,

$$T(v, w) = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} M \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \tag{2.3}$$

where $v = \sum_{i=1}^{n} v_i e_i$, $w = \sum_{i=1}^{n} w_i e_i$, M is the n x n matrix representing T. From now onwards, the term tensor in this thesis will be used for second order tensor.

2.1.2 Tensor properties

This section provides some of the properties of second order tensors.

Definition 2.1.3 (Second-order symmetric tensor). *A second-order tensor is said to be symmetric if $T(u, v) = T(v, u)$ for all $u, v \in V$. In matrix notation, $T_{ij} = T_{ji}$ for all $i, j \in \{1, \ldots, n\}$.*

Definition 2.1.4 (Second-order antisymmetric tensor). *A second-order tensor is said to be antisymmetric if $T(u, v) = -T(v, u)$ for all $u, v \in V$. In matrix notation, $T_{ij} = -T_{ji}$ for all $i, j \in \{1, \ldots, n\}$.*

Definition 2.1.5 (Second-order traceless tensor). *The trace $tr(T)$ of a tensor is defined as the sum of its diagonal element. A second-order tensor T is said to be traceless if the trace $tr(T)$ is zero.*

Definition 2.1.6 (Second-order positive (semi-) definitive tensor). *A second-order tensor T is said to be positive (semi-) definite if $T(v, v) \geq 0$. for any non zero vector $v \in V$.*

This means that all the eigenvalues and the determinant of tensor T are greater than zero or equal to zero.

Definition 2.1.7 (Second-order negative (semi-)definitive tensor)**.** *A second-order tensor T is said to be negative (semi-) definite if $T(v, v) \leq 0$. for any non zero vector $v \in V$. This means that all the eigenvalues and the determinant of tensor T are smaller than zero or equal to zero.*

Definition 2.1.8 (Second-order indefinitive tensor)**.** *A second-order tensor T is said to be indefinite if it is neither positive nor negative definite.*

The decomposition of tensors in different parts reveals important information which is otherwise difficult to understand from the original tensor. A tensor T can be decomposed into symmetric S and antisymmetric tensor A as follows:

$$T = \underbrace{\frac{1}{2}(T + T^T)}_{\text{S}} + \underbrace{\frac{1}{2}(T - T^T)}_{\text{A}} \tag{2.4}$$

where T^T is transpose of the tensor T.
The symmetric part S of tensor T can further be decomposed into isotropic Iso and deviatoric part D,

$$S = \underbrace{\frac{1}{3}tr(S)}_{I_{so}} + \underbrace{(S - I_{so})}_{D} \tag{2.5}$$

where $tr(S)$ is the trace of the symmetric tensor S.
Zhang et al. pointed that these decomposed part reveals different information in the context of flow visualization[8]. The isotropic part I_{so} represents isotropic scaling (dilation), deviatoric part D represents anisotropic stretching and antisymmetric part A represents rotation (vorticity).

2.1.3 Tensor diagonalization

In this thesis, we will always discuss about second-order 3x3 symmetric tensors. The diagonalization of symmetric tensor is well explained in the thesis by Kratz[6]. Symmetric tensor T can be represented as diagonal matrices where the basis for such representation is given by the eigenvectors of the diagonal matrix.

$$UTU^T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2.6}$$

The diagonal elements λ_1, λ_2 and λ_3 are the eigenvalues and U is the orthogonal matrix composed of the eigenvectors. In case of symmetric tensor, all the eigenvalues are real and all the eigenvectors are orthogonal to each other. The diagonalization of a symmetric tensor is numerically computed by Singular Value Decomposition method. In this work, we will assume eigenvalues are ordered as $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

2.1.4 Tensor projection

The projection $P(n)$ of a second order 3D tensor T onto the tangent plane of a given surface defined by normal n is given by

$$\widehat{T} = P(n) \cdot T \cdot P^T(n) \tag{2.7}$$

where $\widehat{T}$ is a projected tensor. Zheng et al. also pointed that the projection of 3D tensor onto the tangent plane results into 2D tensor[14]. The projected tensor has one eigenvector in the direction of the normal n and other two orthogonal eigenvectors along the tangent plane. It should be noted here that the eigenvectors of tensor T are, in general, not the eigenvector of projected tensor $\widehat{T}$.

2.1.5 Degenerate tensors

A tensor T is called degenerate when it consists of repeating eigenvalues. In 2D case, a degenerate tensor is simply multiple of identity matrix, whereas in 3D case, there are two types of degenerate tensors: double degenerate tensor and triple degenerate tensor. Let λ_1, λ_2 and λ_3 be the eigenvalues of a 3D tensor such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. λ_1, λ_2 and λ_3 are known as major, medium and minor eigenvalues respectively.

Definition 2.1.9 (Double degenerate tensor)**.** *A tensor T is said to be double degenerate if two of its eigenvalues are same. In other words, T has two repeating eigenvalues.*

Definition 2.1.10 (Planar degenerate tensor). *A tensor T is said to be planar degenerate if its medium and minor eigenvalues are same.*

Definition 2.1.11 (Linear degenerate tensor). *A tensor T is said to be linear degenerate if its major and medium eigenvalues are same.*

Definition 2.1.12 (Triple degenerate tensor). *A tensor T is said to be triple degenerate if all three of its eigenvalues are same.*

2.2 Tensor Field Topology and Features

A tensor field is a tensor-valued function over some domain. The topology of a tensor field is defined as the set of degenerate points, i.e., points in the domain where the tensor field becomes degenerate. The main objective of this section is to make readers aware of different terminologies and definitions of tensor field topology. The topology of tensor fields differs based on its dimension. We will give a brief overview of 2D and 3D symmetric tensor field topology in this section.

2.2.1 2D Tensor field topology

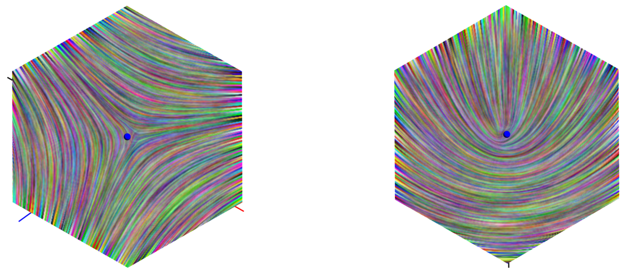
A second-order symmetric tensor field $T(x, y)$ over some domain in $\mathbb{R}^2$ is given as,

$$T(x,y) = \begin{bmatrix} T_{11}(x,y) & T_{12}(x,y) \\ T_{12}(x,y) & T_{22}(x,y) \end{bmatrix} \tag{2.8}$$

Tensor field $T(x, y)$ have eigenvalues $\lambda_i(x, y)$ and eigenvectors $e_i(x, y)$ at every point in $\mathbb{R}^2$ where $i = 1, 2$. Each of these eigenvectors are orthogonal to each other. To get the continuous representation of tensor fields, Delmarcelle et al. considered these eigenvectors as a bidirectional vector fields and integrated a series of curves along one of the eigenvectors [2]. They named these curves as Hyperstreamlines.

Similar to critical points in vector fields, the singularities in tensor field are known as degenerate points. Delmarcelle et al. gave the definition of degenerate points as follows:

Definition 2.2.1 (Degenerate point). *A point $P_0(x, y)$ is a degenerate point of the tensor*



(a) Trisector (b) Wedge

Figure 2.1: Trisector (left) and Wedge (right).

field $T(x, y)$ iff the two eigenvalues of $T(x, y)$ are equal to each other at $P_0(x, y)$ i.e., $\lambda_1(x, y) = \lambda_2(x, y)$[2].

In a 2D tensor field, there are two fundamental types of degenerate point i.e., wedges and trisectors as shown in figure 2.1. These degenerate points are basically local patterns at a point, determined by taking the gradients of tensor field at that point. The partial derivative of $T(x, y)$ is given as,

$$\begin{array}{cc} a = \frac{1}{2}\frac{\partial(T_{11}-T_{22})}{\partial x} & b = \frac{1}{2}\frac{\partial(T_{11}-T_{22})}{\partial y} \\ c = \frac{\partial T_{12}}{\partial x} & d = \frac{\partial T_{12}}{\partial y} \end{array} \tag{2.9}$$

The degenerate point can be classified based on the invariant δ given as,

$$\delta = ad - bc \tag{2.10}$$

A degenerate point p_0 is a wedge when $\delta > 0$ and a trisector when $\delta < 0$. When $\delta = 0$, p_0 is a higher-order degenerate point, which is structurally unstable. Delmarcelle et al. also defined degenerate point based on tensor index[2].

Definition 2.2.2 (Tensor Index)**.** *The index T_{index} at the degenerate point $P_0(x, y)$ of a tensor field is the number of counter-clockwise revolutions made by the eigenvectors when travelling along a closed path encompassing $P_0(x, y)$. The path is chosen close enough to $P_0(x, y)$ so that it does not encompass any other degenerate points.*

If the tensor index T_{index} is $-\frac{1}{2}$, the degenerate point is called trisector point and if the tensor index T_{index} is $\frac{1}{2}$, the degenerate point is called wedge point. The pattern of hyperstreamlines in the tensor field is characterized by the tensor index at the degenerate point. In the neighborhood of the degenerate points, when the hyperstreamlines sweeps near the degenerate point in both direction, the region is known as hyperbolic sector and on the contrary, when the hyperstreamlines touches the degenerate point, the region is known as parabolic sector.

2.2.2 3D Tensor field topology

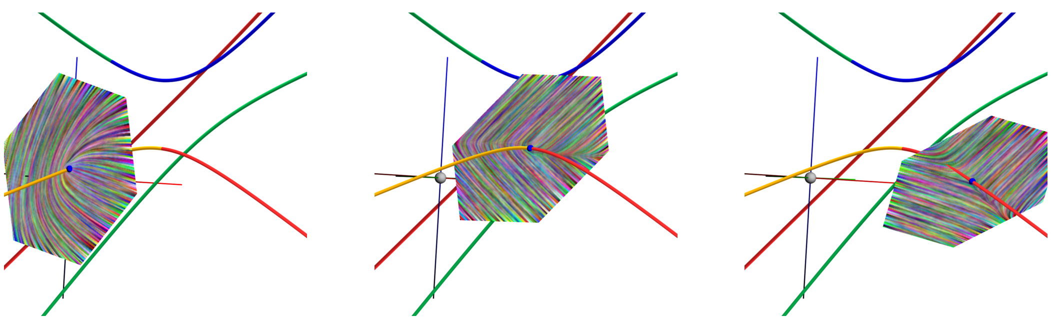
In order to study 3D symmetric tensor field, we only study deviatoric part of the tensor. Though isotropic part of symmetric tensor field is important for providing the uniform scaling information, still it is not considered in study of symmetric tensor field because it does not provide directional information which is important to study topological behavior of 3D symmetric tensor field. A 3D symmetric traceless tensor field $T(x, y, z)$ over some domain in $\mathbb{R}^3$ is given as,

$$T(x, y, z) = \begin{bmatrix} T_{11}(x, y, z) & T_{12}(x, y, z) & T_{13}(x, y, z) \\ T_{12}(x, y, z) & T_{22}(x, y, z) & T_{23}(x, y, z) \\ T_{13}(x, y, z) & T_{23}(x, y, z) & -T_{12}(x, y, z) - T_{22}(x, y, z) \end{bmatrix} \tag{2.11}$$

In 3D symmetric tensor field $T(x, y, z)$, a point $P_0(x, y, z)$ is known as degenerate point if the tensor at the point $P_0(x, y, z)$ is degenerate. The topology of 3D tensor field consists of three types of degenerate points: linear degenerate point, planar degenerate point and triple degenerate point. A degenerate point is said to be linear, planar or triple degenerate point if the tensor at that point is either linear degenerate, planar degenerate or triple degenerate. Zheng et al. [9] point out that while triple degeneracies can exist, they are structurally unstable, i.e., they can disappear under arbitrarily small perturbations. In contrast, linear and planar degenerate points are structurally stable, i.e., they persist under small enough perturbations in the tensor field. Moreover, under structural stable conditions such points form curves, along which the tensor field is either always linear degenerate or always planar degenerate. While it is possible that linear and planar degenerate points are isolated points or form surfaces and volumes, these three scenarios do not persist under arbitrarily small perturbation in the field, i.e., structurally unstable.

A degenerate point can also be classified on the basis of wedge-trisector classification[10]. Given a degenerate point P_0, let n be the non-repeating eigenvector at P_0. The plane P that passes through P_0, whose normal is n, is known as the non-repeating plane at P_0. The projection of 3D tensor field onto P results in a 2D symmetric tensor field on the plane P which, under structurally stable conditions, has exactly one degenerate point, P_0. In the 2D tensor field, P_0 can be either a wedge, a trisector, or a higher-order degenerate point which is structurally unstable. By following the convention of Zheng et al. [9], we will refer to P_0 as a wedge or trisector degenerate point in the 3D tensor field. When P_0 is a higher-order degenerate point in the projected tensor field, it becomes a transition point in the 3D tensor field. Note that while a higher-order degenerate point is structurally unstable, a transition point is structurally stable. Moreover, a transition point is not the same as triple degenerate points. At the transition point, the non-repeating plane is tangent to the degenerate curve. Figure 2.2 shows the wedge, trisector and transition point along a degenerate curve.

Palacios et al. also added feature surfaces based on eigenvalue manifold (parametrization of 3D degenerate tensor based on eigenvalues) to the topological analysis of 3D tensor field[4]. They showed that apart from degenerate points or degenerate curves, feature surfaces, such as, neutral surface and mode surface also help us in understand-



(a) Wedge (b) Transition point (c) Trisector

Figure 2.2: The degenerate pattern at the degenerate points (blue in color) along the degenerate curve from left to right is wedge, transition point and a trisector. The design patterns shown on the nonrepeating plane are well-known visualization technique known as line integral convolution (LIC) developed by Cabral et al.

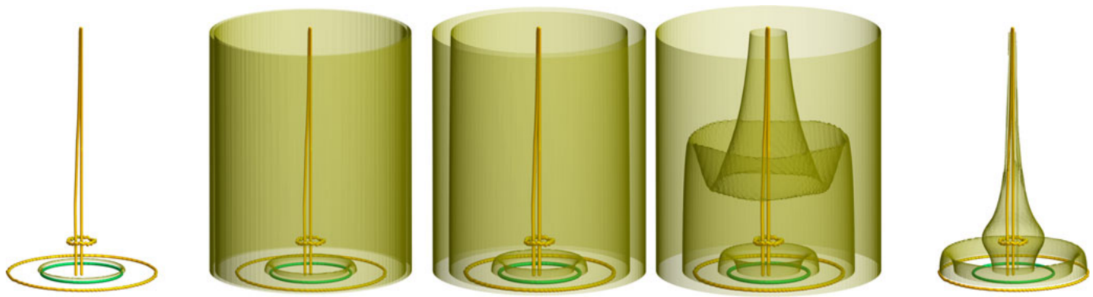


Figure 2.3: The mode surfaces of the Sullivan vortex. From left to right, the mode values are 0.8, 0.4, 0 (the neutral surface), -0.4, and -0.94 respectively. Notice that the linear degenerate curves (green) and the planar degenerate curves (yellow) are separated by the neutral surface. Moreover, the topology of the mode surface changes. As the mode increases, the mode surfaces converge toward linear degenerate curves. In contrast, when the mode decreases, the mode surfaces converge toward planar degenerate curves.

ing the behavior of 3D tensor field. A neutral surface is the special level set surface when medium eigenvalue becomes zero i.e., the surface which divide the domain into two halves: planar tensor and linear tensor. A mode surface is the level set surface of mode of a 3D tensor field. The Figure 2.3 taken from [4] shows neutral surface and mode surface.

2.3 3D Linear Symmetric Tensor Fields

In this section, we present an overview of 3D linear symmetric tensor fields and their important properties. These properties helps us in studying the topology of 3D linear tensor field. The topology of 3D linear tensor field consists of degenerate points which eventually forms degenerate curves. The brief review provided in this section is the work of [9, 11]. The behavior of tensor field near the point of interest is mostly dictated by the linearization of the tensor field. As a result, it is intuitive to study the linear terms of Taylor expansion of tensor fields. Given a tensor $LT(x, y, z)$, the taylor expansion of

$LT(x, y, z)$ is given as,

$$
\begin{aligned}
LT(x,y,z) = T(x_0,y_0,z_0) + (x-x_0)\frac{\partial T(x_0,y_0,z_0)}{\partial x} + (y-y_0)\frac{\partial T(x_0,y_0,z_0)}{\partial y} \\
+(z-z_0)\frac{\partial T(x_0,y_0,z_0)}{\partial z} + \dots
\end{aligned}
\tag{2.12}
$$

Since the behavior only depends on linear terms, we neglect the higher order terms in the expansion. $\frac{\partial T(x_0,y_0,z_0)}{\partial x}$, $\frac{\partial T(x_0,y_0,z_0)}{\partial y}$ and $\frac{\partial T(x_0,y_0,z_0)}{\partial z}$ are the gradient of tensor field along x, y, and z axes, and is given as T_x, T_y and T_z respectively. (x_0, y_0, z_0) is the origin.

$$
LT(x,y,z) = T(x_0,y_0,z_0) + (x-x_0)T_x + (y-y_0)T_y + (z-z_0)T_z \tag{2.13}
$$

$$
\Rightarrow LT(x,y,z) = \underbrace{T(x_0,y_0,z_0) - x_0T_x - y_0T_y - z_0T_z}_{T_0} + xT_x + yT_y + zT_z \tag{2.14}
$$

where T_0 is the 3D symmetric tensor at the origin (x_0, y_0, z_0).
Therefore, a 3D linear symmetric tensor field is given as,

$$
LT(x,y,z) = T_0 + xT_x + yT_y + zT_z \tag{2.15}
$$

2.3.1 3D Linear symmetric tensor field overview

A 3D symmetric traceless tensor field have following form $LT(x,y,z) = T_0 + xT_x + yT_y + zT_z$ where, $T_0 = \begin{bmatrix} a_0 & b_0 & c_0 \\ b_0 & d_0 & e_0 \\ c_0 & e_0 & -a_0-d_0 \end{bmatrix}$, $T_x = \begin{bmatrix} a_x & b_x & c_x \\ b_x & d_x & e_x \\ c_x & e_x & -a_x-d_x \end{bmatrix}$, $T_y = \begin{bmatrix} a_y & b_y & c_y \\ b_y & d_y & e_y \\ c_y & e_y & -a_y-d_y \end{bmatrix}$ and, $T_z = \begin{bmatrix} a_z & b_z & c_z \\ b_z & d_z & e_z \\ c_z & e_z & -a_z-d_z \end{bmatrix}$ are symmetric traceless matrices. A 3D linear symmetric traceless tensor field is preserved under change of basis and also when projected over a plane. This property helps us to understand the topological behaviors of linear tensor fields.

Lemma 2.3.1. *Given a linear symmetric tensor field $LT(x,y,z) = T_0 + xT_x + yT_y + zT_z$, its linearity is preserved under change of coordinate systems[9]*

Proof. The proof can be found in [9]. In this work, we borrow part of it for our use. Let $C = (o, e_1, e_2, e_3)$ and $C^{'} = (o^{'}, e_1^{'}, e_2^{'}, e_3^{'})$ be two coordinate systems where o and $o^{'}$ are the respective origins and the e_i's and $e_i^{'}$'s are the basis vectors. Let (x, y, z) and $(x^{'}, y^{'}, z^{'})$ be the coordinates of a point p under C and $C^{'}$, respectively. The two sets of coordinate are related by

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = M \begin{bmatrix} x^{'} \\ y^{'} \\ z^{'} \end{bmatrix} \tag{2.16}$$

where $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ is the origin of C, and M is the unique linear transformation such that $M(e_i^{'}) = \sum_{j=1}^{3} M_{ij} e_j^{'} = e_i$ for $1 \leq i \leq 3$.

Given a tensor field $LT(x, y, z) = T_0 + xT_x + yT_y + zT_z$ under C, its formula under $C^{'}$:

$$LT(x^{'}, y^{'}, z^{'}) = T_0^{'} + x^{'}T_x^{'} + y^{'}T_y^{'} + z^{'}T_z^{'} \tag{2.17}$$

where,

$$T_0 = M^{-1}(T_0 + xT_x + yT_y + zT_z)M \tag{2.18}$$

$$T_x^{'} = M^{-1}(M_{11}T_x + M_{12}T_y + M_{13}zT_z)M \tag{2.19}$$

$$T_y^{'} = M^{-1}(M_{21}T_x + M_{22}T_y + M_{23}zT_z)M \tag{2.20}$$

$$T_z^{'} = M^{-1}(M_{31}T_x + M_{32}T_y + M_{33}zT_z)M \tag{2.21}$$

□

Now, we consider the projection of a linear tensor field onto a plane. The following proof is taken from [9].

Lemma 2.3.2. *Given a linear symmetric tensor field $LT(x, y, z) = T_0 + xT_x + yT_y + zT_z$, its projection onto a plane is a 2D symmetric, linear tensor field inside the plane[9].*

Proof. Let N be a normal to the plane and $p_0 = (x_0, y_0, z_0)$ be a point on the plane. We construct a new coordinate system (p_0, X, Y, N) such that p_0 is the new origin and X and Y form a basis for the plane. Based on Lemma 2.3.1, the linear tensor field under the new basis has the form (Equations 2.18 - 2.21).

Given a point p in the plane. Under the new coordinate systems, p has the form

$$LT^{''}(x^{'}, y^{'}) = T_0^{''} + x^{'}T_x^{''} + y^{'}T_y^{''} \tag{2.22}$$

where $LT^{''}$ is the projected tensor of LT on the plane, and $T_0^{''}$, $T_x^{''}$, and $T_y^{''}$ are respectively the 2x2 subblock of $T_0^{'}$, $T_x^{'}$ and $T_y^{'}$ corresponding to the plane. It is clear that $T^{''}$ remains a symmetric linear tensor field.

□

2.3.2 3D Linear symmetric tensor field topology

The topology of 3D linear tensor field consists of degenerate points. The degenerate points are the point in space where the tensor is degenerate. In this thesis, we will study about different types of degenerate points and curves, their numbers and the location in the linear tensor fields. In 3D linear symmetric tensor field, a degenerate points is classified into double degenerate points and triple degenerate points. Double degenerate points are classified into linear and planar types based on the eigenvalues, whereas Triple degenerate points are the points where all the eigenvalues are same. Double degenerate points are said to occur in structurally stable conditions, whereas triple degenerate points are said to be structurally unstable.

It is to be noted here that the set of all traceless, symmetric tensors with configuration $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & -a-d \end{bmatrix}$ forms a five-dimensional linear space $\mathbb{T}$ spanned by the basis $T_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $T_b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $T_c = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $T_d = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and , $T_e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Any tensor in the linear space $\mathbb{T}$ can be expressed as $t_aT_a+t_bT_b+t_cT_c+t_dT_d+t_eT_e$ for some $t_a, t_b, t_c, t_d, t_e \in \mathbb{R}$. It can also be written in the vector form $(t_a, t_b, t_c, t_d, t_e)$.

T_0, T_x, T_y and T_z are linearly independent vector in $\mathbb{T}$ under structurally stable conditions. $LT(x, y, z)$ leads to the injective mapping $LT : \mathbb{R}^3 \to \mathbb{T}$. The image U of LT is given as,

$$U = \{xT_x + yT_y + zT_z | x, y, z \in \mathbb{R}\}$$

is a linear three dimensional space of $\mathbb{T}$. Further it can be said that LT is isomorphism between $\mathbb{R}^3$ and U. As a consequence of this isomorphism, LT is also isomorphism between the set of degenerate points of $LT(x, y, z)$ and $U \bigcap D$ where $D \subset \mathbb{T}$ is the set of all degenerate tensors. Since U is a three dimensional (codimension two) subspace of $\mathbb{T}$, there exists two linear homogeneous functions F and G such that $F(a, d, b, c, e) = F(a_0, d_0, b_0, c_0, e_0)$ and $G(a, d, b, c, e) = G(a_0, d_0, b_0, c_0, e_0)$. In fact, F and G are vector in the space $\mathbb{R}$ and are perpendicular to the space spanned by the vector corresponding to T_x, T_y and T_z. Also, F and G are linearly independent and together with T_x, T_y and T_z they form basis in $\mathbb{T}$. D is a non-linear subspace of $\mathbb{T}$ consisting of tensors of the following format $k \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} - \frac{k}{3} I - T_0$ for some $k \in \mathbb{R}$ and some unit vector $\begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}$ which is equivalent to

$$k \begin{bmatrix} \alpha^2 - \frac{1}{3} & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2 - \frac{1}{3} & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2 - \frac{1}{3} \end{bmatrix} - T_0, \tag{2.23}$$

where $\alpha^2 + \beta^2 + \gamma^2 = 1$. When $k < 0$, the tensor T is linear i.e., one repeating positive eigenvalue and one non-repeating negative eigenvalue. When $k > 0$, the tensor T is planar i.e., one repeating negative eigenvalue and one non-repeating positive eigenvalue. A degenerate tensor in U therefore must satisfy:

$$F\Big(k\big(\alpha^2 - \frac{1}{3}\big), k\big(\beta^2 - \frac{1}{3}\big), k\big(\alpha\beta\big), k\big(\alpha\gamma\big), k\big(\beta\gamma\big)\Big) = F(a_0, d_0, b_0, c_0, e_0) \tag{2.24}$$

$$G\Big(k\big(\alpha^2 - \frac{1}{3}\big), k\big(\beta^2 - \frac{1}{3}\big), k\big(\alpha\beta\big), k\big(\alpha\gamma\big), k\big(\beta\gamma\big)\Big) = G(a_0, d_0, b_0, c_0, e_0) \tag{2.25}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{2.26}$$

Let $f_0 = F(a_0, d_0, b_0, c_0, e_0)$ and $g_0 = G(a_0, d_0, b_0, c_0, e_0)$. Then above equations are written as,

$$F\Big(k\big(\frac{2\alpha^2-\beta^2-\gamma^2}{3}\big), k\big(\frac{2\beta^2-\alpha^2-\gamma^2}{3}\big), k\big(\alpha\beta\big), k\big(\alpha\gamma\big), k\big(\beta\gamma\big)\Big) = f_0 \tag{2.27}$$

$$G\Big(k\big(\alpha^2-\frac{1}{3}\big), k\big(\beta^2-\frac{1}{3}\big), k\big(\alpha\beta\big), k\big(\alpha\gamma\big), k\big(\beta\gamma\big)\Big) = g_0 \tag{2.28}$$

$$\alpha^2+\beta^2+\gamma^2 = 1 \tag{2.29}$$

since both F and G are homogeneous polynomial of degree one, they satisfy following conditions: $F(ka, kd, kc, kd, ke) = kF(a, d, b, c, e)$ and $G(ka, kd, kc, kd, ke) = kG(a, d, b, c, e)$ for any $k \in \mathbb{R}$. Therefore, above equations are equivalent to,

$$kF\Big(\frac{2\alpha^2-\beta^2-\gamma^2}{3}, \frac{2\beta^2-\alpha^2-\gamma^2}{3}, \alpha\beta, \alpha\gamma, \beta\gamma\Big) = f_0 \tag{2.30}$$

$$kG\Big(\frac{2\alpha^2-\beta^2-\gamma^2}{3}, \frac{2\beta^2-\alpha^2-\gamma^2}{3}, \alpha\beta, \alpha\gamma, \beta\gamma\Big) = g_0 \tag{2.31}$$

$$\alpha^2+\beta^2+\gamma^2 = 1 \tag{2.32}$$

or,

$$F\Big(\frac{2\alpha^2-\beta^2-\gamma^2}{3}, \frac{2\beta^2-\alpha^2-\gamma^2}{3}, \alpha\beta, \alpha\gamma, \beta\gamma\Big) = \frac{f_0}{k} \tag{2.33}$$

$$G\Big(\frac{2\alpha^2-\beta^2-\gamma^2}{3}, \frac{2\beta^2-\alpha^2-\gamma^2}{3}, \alpha\beta, \alpha\gamma, \beta\gamma\Big) = \frac{g_0}{k} \tag{2.34}$$

$$\alpha^2+\beta^2+\gamma^2 = 1 \tag{2.35}$$

Let us define f and g to make the above equation simple enough for our further study,

$$f(\alpha, \beta, \gamma) = F\Big(\frac{2\alpha^2-\beta^2-\gamma^2}{3}, \frac{2\beta^2-\alpha^2-\gamma^2}{3}, \alpha\beta, \alpha\gamma, \beta\gamma\Big) \tag{2.36}$$

$$g(\alpha, \beta, \gamma) = G\Big(\frac{2\alpha^2-\beta^2-\gamma^2}{3}, \frac{2\beta^2-\alpha^2-\gamma^2}{3}, \alpha\beta, \alpha\gamma, \beta\gamma\Big) \tag{2.37}$$

The above definitions lead us to following equations,

$$f(\alpha, \beta, \gamma) = \frac{f_0}{k} \tag{2.38}$$

$$g(\alpha, \beta, \gamma) = \frac{g_0}{k} \tag{2.39}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{2.40}$$

Above equations lead to the following equation which characterize degenerate points in the form of its non-repeating eigenvector (α,β and γ).

$$h(\alpha, \beta, \gamma) = 0 \tag{2.41}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{2.42}$$

where $h(\alpha, \beta, \gamma) = g_0 f(\alpha, \beta, \gamma) - f_0 g(\alpha, \beta, \gamma)$. The solution to above system of equations are degenerate points which eventually form degenerate curves [11].

Another result given by Zhang et al. [9] is as follows:

Theorem 2.3.3. *Given a linear symmetric tensor field $LT(x, y, z) = xT_x + yT_y + zT_y$, a point (x_0, y_0, z_0) is degenerate if and only if (kx_0, ky_0, kz_0) is also degenerate for any $k \neq 0$. Moreover, $k(x_0, y_0, z_0)$ is triple degenerate if and only if (x_0, y_0, z_0) is triple degenerate. If (x_0, y_0, z_0) is a degenerate point, then $k(x_0, y_0, z_0)$ is linear if $k > 0$ and planar if $k < 0$.[9]*

Proof. The proof can be found in [9]. □

Zhang et al. further gave corollary 2.3.3.1 and proved that all degenerate curves ends at infinity[9].

Corollary 2.3.3.1. *Given a numerically stable linear tensor field $LT(x, y, z) = xT_x + yT_y + zT_y$, there are no degenerate loops.*

Proof. We will present the same proof as presented in the work of [9]. If a point $p_0 = (x_0, y_0, z_0)$ is degenerate, then all the points on the line passing through p_0 and the origin.

So if there is a loop of degenerate points, the each point on the loop will lead to a line of degenerate points. As a result, the loop will lead to a cylindrical surface of degenerate point which is structurally unstable. □

The above corollary proves that all degenerate curves in such a tensor field must end at infinity.

From our earlier discussion, we know that a degenerate point can be also classified as either a wedge, a trisector, or, a transition point. This classification is based on the 2D degenerate pattern around the degenerate point inside the repeated plane. Zhang et al. showed that the choice of the normal of a repeated plane does not change wedge/trisector classification.

Theorem 2.3.4. *The wedge/trisector classification of a degenerate point p in a 3D linear, symmetric tensor field T is independent of the choice of the normal in the repeated plane and the coordinate systems for the repeated plane.*

Proof. The proof can be found in [9]. □

The above theorem establishes the fact that wedge/trisector classification is well defined. As a result, a degenerate point can be classified as a Linear-Wedge (LW), a Linear-Trisector (LT), a Planar-Wedge (PW) and a Planar-Trisector (PT). Zhang et al. in their work in [9] gave another important result (Theorem 2.3.5) about linear tensor fields.

Theorem 2.3.5. *Given a linear symmetric tensor field $LT(x, y, z) = T_0 + xT_x + yT_y + zT_y$, a point $p_0 = (x_0, y_0, z_0)$ is a wedge if and only if $p_0^{'} = (kx_0, ky_0, kz_0)$ is also a wedge for any $k \neq 0$. Similarly, a point $p_0 = (x_0, y_0, z_0)$ is a trisector if and only if $p_0^{'} = (kx_0, ky_0, kz_0)$ is also a trisector for any $k \neq 0$. Moreover the orientations of the degenerate patterns remain constant regardless of k. [9]*

Proof. The proof can be found in [9]. □

Above theorem says that the local orientation of degenerate patterns does not change along the degenerate curve. This means that if a point $p = (x, y, z)$ is a degenerate point, then any point on the ray emanating from origin and containing p is also degenerate.

This gives us an important fact that the wedge/trisector and linear/planar classification does not change along the ray. Also, the rays in opposite direction from the origin has the same wedge/trisector but opposite linear/planar classification.

Based on the discussion above, it can be said that the linear/planar classification does not change along the degenerate curve when $T_0 \neq 0$ and wedge/trisector classification does not change along the degenerate curve when $T_0 = 0$. Another fact given by Zhang et al. needs to be mentioned here:

Theorem 2.3.6. *Given a linear symmetric tensor field* $LT(x, y, z) = T_0 + xT_x + yT_y + zT_y$ *where* $T_0 = 0$*, there are same number of LWs as PWs and same number of LTs as PTs.*

Proof. When $T_0 = 0$, all the curves pass through the origin which is also a triple degenerate point. At triple degenerate point, the degenerate curve changes from linear to planar and vice versa. As a result, we always have same number of LWs as PWs and same number of LTs as PTs. □

Chapter 3: 3D Linear Symmetric Tensor Field Design

This chapter provides an overview of tensor field design both algebraically and geometrically, and presents the strength of new interface compared to the interface provided in the work of Zhang et al.[9]. The interface presented in this research has been built over the interface provided by Dr. Jonathan Palacios. The interface has been built using wxWidget framework.

3.1 Different Tensor Field Design Approaches

In this section, we will discuss algebraic and geometric approach for designing linear tensor fields. These approaches also help in creating simple, but interactive interfaces to study linear tensor fields with better controls.

3.1.1 Algebraic approach

A linear tensor can be algebraically represented as a degree-one polynomial where the entries of the polynomial are the coefficient of the tensor. In order to change the tensor, we need to change each of its entries. Since this thesis is about 3D linear symmetric traceless tensor field, we will adhere our discussion to 3D linear symmetric traceless tensor field for our further discussion. The tensor in this case is traceless and symmetric. As a result, we have five degrees of freedom to change individual tensor T_0, T_x, T_y and T_z by changing their entries. For each entries, both our interface and the interface developed by Zhang et al. have text boxes. To design linear tensor field, the user just needs to change the entries corresponding to individual tensor.

3.1.2 Geometric approach

A tensor can be geometrically represented by its eigenvalues and eigenvectors. Since the tensor in our case is traceless and symmetric, we can change two of its eigenvalues and third will be negative of the sum of the first two eigenvalues. Also the eigenvector will form an orthogonal basis. So if we know the first eigenvectors, we can find other two eigenvectors of tensor. To find second and third eigenvector, we can follow steps below:

- Pick a vector V from the standard basis which is not the first eigenvector E_1
- Compute the second eigenvector E_2 , $E_2 = V \times E_1$
- Compute the third eigenvector E_3 using E_1 and E_2, $E_3 = E_1 \times E_2$

So, in order to change the tensor, we just need to know the eigenvector and the angle formed by the eigenvector. In the interface designed by Zhang et al., they have provided six text boxes for changing the tensor geometrically [9]: Two text boxes for eigenvalues, three text boxes for eigenvector and one text box for the angle formed by the eigenvector. The interface developed by Zhang et al. is shown in Figure 3.1.

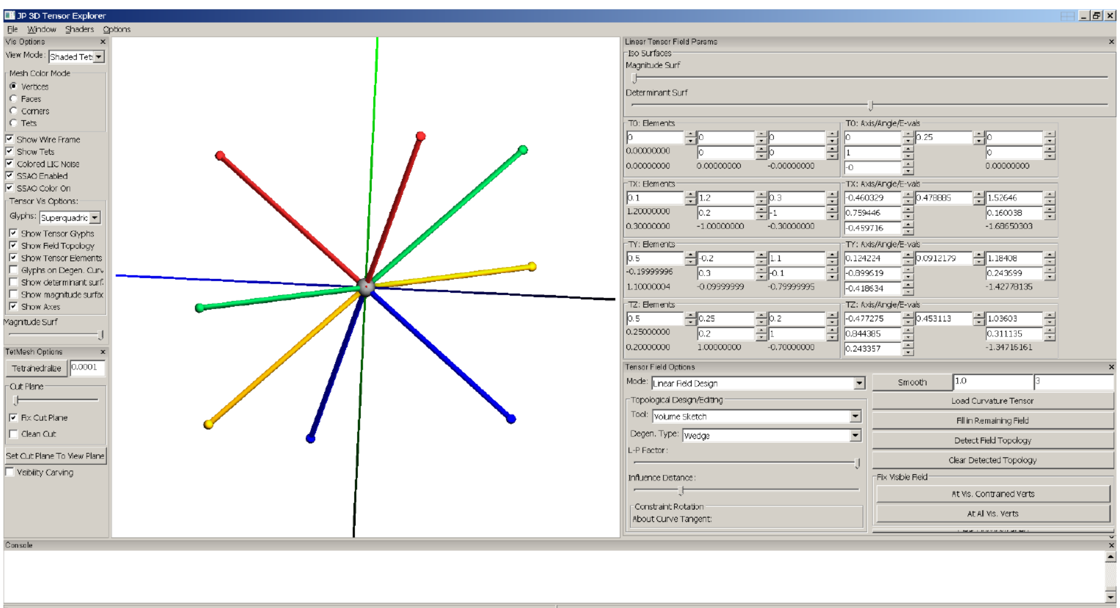


Figure 3.1: Linear tensor field design interface developed by Zhang et al.[9].

We believe that our interface provides more intuitive geometric design for linear tensor fields than the interface provided by Zhang et al. [9]. With our interface, user have more control over how the eigenvalues and eigenvectors changes relative to one another. The eigenvalues of tensor can also be changed using mode of the tensor. Mode is one of the invariant quantity of tensor and the value of mode provides the relative strength of eigenvalues. The magnitude of eigenvalues can also be changed using tensor magnitude. The change in magnitude scales both the tensor and eigenvalues without changing its basis. Therefore, we have two controls to change the eigenvalues i.e. mode and magnitude. There are same number of controls in our interface compared to the interface provided by Zhang et al. [9], we believe that our method provides better control in changing the eigenvalues. The user can control the relative strength of the eigenvalues without worrying about how these changes will make the tensor degenerate or non degenerate. The orientation of tensor is given by eigenvectors. There have been much study over tensor field visualization based on its orientation. The tensor glyph was always supposed to be one of the choice. The orientation and shape of glyph represents the eigenvector and eigenvalues of tensor respectively. In our study, we have used the glyph visualization developed by Palacios et al.[4] shown in figure 3.2. The eigenvectors of tensor can be

Figure 3.2: Tensor glyph corresponding to mode values -1 (left most), zero (middle) and 1 (right most).

changed by rotating the glyphs. This approach is more intuitive to user as user always knows how the orthonormal basis of tensor is changing given the orientation of glyph after rotation. It is rather difficult to precisely change the entries of eigenvector and angle to desired orthonormal basis using text boxes. Zhang et al. interface shows text boxes(figure 3.1) to change the basis[9]. Based on our analysis above, we feel that our interface has better tensor field design approach with fewer control and better precision. The next section provides the detailed computation of tensor based on magnitude, mode and glyph rotation.

3.2 Tensor Field Design using Mode and Magnitude of Tensor

The mode μ of a tensor T is given as,

$$\mu = 3\sqrt{6}\frac{|T|}{||T||^3} \tag{3.1}$$

where $|T|$ is determinant of the deviatoric tensor and $||T||$ is the frobenius norm, also known as magnitude of the tensor.
The magnitude $||T||$ of a tensor T is given as,

$$||T|| = \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}^2} \tag{3.2}$$

A new tensor $T^{'}$ can be created by changing either the mode or magnitude of a tensor individually. T can be diagonalized as follows:

$$T = RDR^T \tag{3.3}$$

Where R is the 3 x 3 matrix containing eigenvector as columns and D is the the diagonal 3 x 3 matrix containing eigenvalues λ_1, λ_2 and λ_3.
When tensor mode μ is changed to $\mu^{'}$, the eigenvalues change relative to one another while keeping the eigenvector same. The eigenvalues for the new tensor mode are calculated as follows:

$$\lambda_k = \frac{2}{\sqrt{6}}||T||cos\left(\frac{1}{3}acos(\mu^{'}) - \frac{2\pi k}{3}\right) \tag{3.4}$$

Where $1 \leq k \leq 3$ for 3 x 3 symmetric traceless matrix. As a result, diagonal matrix D changes to $D^{'}$. $D^{'}$ contains new eigenvalues along the diagonal. The new tensor $T^{'}$ is then calculated as shown below:

$$T^{'} = RD^{'}R^T \tag{3.5}$$

Similarly, when magnitude changes from $||T||$ to $||T^{'}||$, the eigenvalues are calculated using the same formula as 3.15,

$$\lambda_i = \frac{2}{\sqrt{6}}||T^{'}||cos\left(\frac{1}{3}acos(\mu) - \frac{2\pi k}{3}\right) \tag{3.6}$$

The 3D linear tensor field can be changed by changing the mode or magnitude of either of the tensors T_0, T_x, T_y and T_z. The interface (shown in figure 3.3) provides slider as well as spinbox for the said operations.

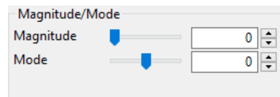


Figure 3.3: Snapshot showing the controls for changing tensor field using mode and magnitude.

3.3 Tensor Field Design using Eigenvectors of Tensor

A new tensor field T' can be created from the original tensor field T by rotating the glyphs (shown in figure 3.2) corresponding to the particular tensor. The shape and orientation of glyph are represented by the eigenvalue and eigenvectors of the glyph. For more details, readers can refer to [4].

A tensor T can be diagonalized as PDP^T. Where P is the matrix containing eigenvectors as column vector and D is the diagonal matrix containing eigenvalues. If the matrix containing eigenvectors changes from P to Q, then T can be written as QDQ^T. In our work, we change the eigenvector by rotating the glyph corresponding to the tensor.

All the above methods provide user with a flexibility to create as many tensor field as needed, to study behavior of tensor field, using the controls shown in the figure 3.4.

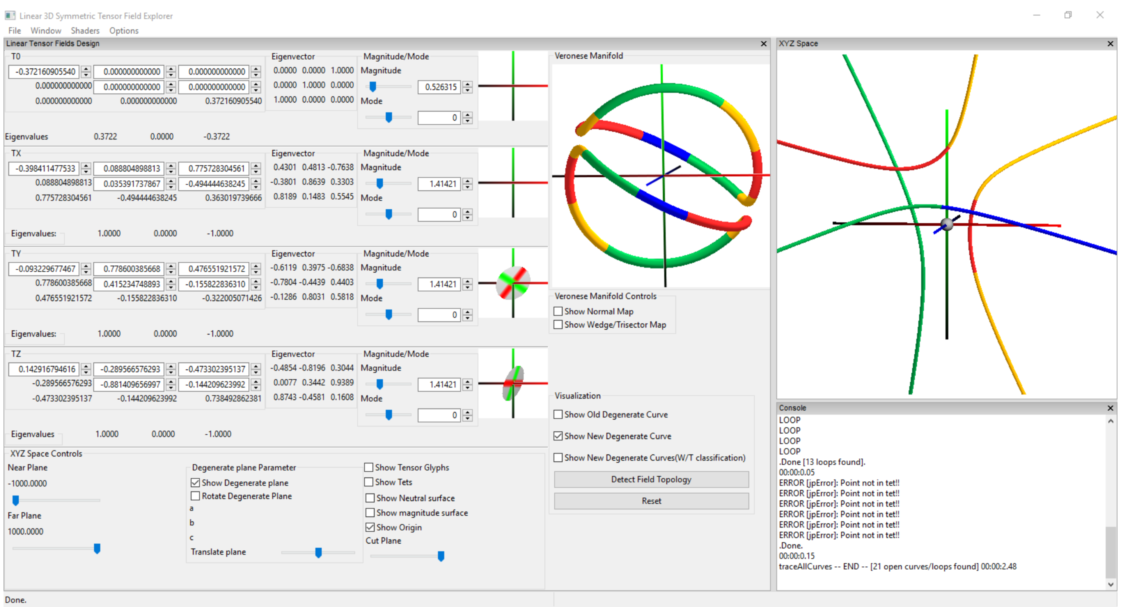


Figure 3.4: 3D Linear Symmetric Traceless Tensor Field Design Interface.

Chapter 4: Transition Point Overview, Analysis and Observation

This chapter provides detailed study of transition points, its analysis, and the upper and lower bound on the number of transition points under different 3D linear tensor field. Some of the relevant results and observations related to the number of transition points are also provided to facilitate the analysis of transition points under different linear tensor field.

Based on the type of classification of degenerate points, a degenerate curve can be divided into different segments:

- Linear-Wedge (LW) i.e., $k > 0$ and $\delta > 0$ (Green in color)
- Planar-Wedge (PW) i.e., $k < 0$ and $\delta > 0$ (Yellow in color)
- Linear-Trisector (LT) i.e., $k > 0$ and $\delta < 0$ (Blue in color)
- Planar-Trisector (PT) i.e., $k < 0$ and $\delta < 0$ (Red in color)

Where k is mode of tensor and δ is the discriminant of projected tensor on the plane whose normal is non-repeating eigenvector of the degenerate tensor (discussed in section 2.2.2).

A degenerate curve can be classified as three different types of curves based on their ends being on wedge or in trisector.

Definition 4.0.1 (Wedge-Wedge (WW) Curve)**.** *A degenerate curve is said to be WW curve if the ends of the curve are in wedge region.*

Definition 4.0.2 (Trisector-Trisector (TT) Curve)**.** *A degenerate curve is said to be TT curve if the ends of the curve are in trisector region.*

Definition 4.0.3 (Wedge-Trisector (WT) Curve)**.** *A degenerate curve is said to be WT curve if it has one end in wedge region and the other end in trisector region.*

In addition to above definitions, we have following result:

Theorem 4.0.1. *Under structurally stable conditions, a WW curve or a TT curve have an even number of transition points.*

Proof. The transition points are defined by $\delta = 0$. The δ is continuous function along the curve and have same sign at the ends in case of WW and TT curves. In order to have the same sign on both ends, the δ have to cross an even number of times. This proves that a WW curve or a TT curve have an even number of transition points. □

Theorem 4.0.2. *Under structurally stable conditions, a WT curve have an odd number of transition points.*

Proof. As the transition points are defined by $\delta = 0$ and δ is continuous function along the curve. In case WT curves, both the ends of WT curve have different signs. In order to have different signs on different ends, the δ have to cross odd number of times. Thus, a WT curve have an odd number of transition points. □

Theorem 4.0.3. *Under structurally stable conditions, there are an even number of WT curves in a linear symmetric tensor field.*

Proof. In case of two or four curves when $T_0 = 0$, each of the curves passes through the origin which is also a triple degenerate point. At triple degenerate point the degenerate curve changes from linear to planar and vice versa but does not change from wedge to trisector. As a result, we always have same number of LWs as PWs and the same number of LTs as PTs on a degenerate curve. Since for each curve there are an even number of wedges and trisectors, the total number of wedges and trisectors are also even (i.e, an even number of curves multiplied by an even number of wedge/trisector is always even). This mean that there are even number of wedges and trisectors in a linear tensor field. When $T_0 \neq 0$, they form WW, TT and WT curves. WW curve contributes to even number of wedges and TT curves contributes to even number of trisectors. If even number of wedges is taken by WW and even number of trisectors by TT curves, then there must be an even number wedge/trisector left. Since, WT curve have odd number of wedges and odd number of trisectors, then there should be even number of curves to satisfy the left over wedges and trisectors. This proves that there are an even number of WT curves and they always occur in pairs. □

We would like to present another important result:

Theorem 4.0.4. *Under structurally stable conditions, a linear 3D tensor fields has an even number of transition points.*

Proof. Under structurally stable conditions, there are either two or four curves. WW and TT curves always have an even number of transition points and WT curves have an odd number of transition points. Also, WT curve always occurs in pairs (i.e., in even number). If we take any two or four curves, such that WT if exists, are in pairs, we will have an even number of transition points. □

4.1 Transition Point Overview

This section provides methodology to find the transition point and also comment on maximum number of transition points possible in a linear tensor field. The same equation which were used for characterizing degenerate points along with one more constraint ie., $\delta = 0$ can be used to locate the transition points. The study in this section starts with the previous work of [11, 10] and then move onto finding theoretical bound on the number of transition points in a 3D linear symmetric tensor field[10]. Before we go any further, let us discuss some results obtained from the work of [11, 10] which will eventually lead us to the equation satisfying our constraint $\delta = 0$.

Based on lemma 2.3.2, under structurally stable condition, the projection of a tensor onto a plane can have at most one degenerate point. This degenerate point, if exist, can be classified as a wedge, or a trisector, or a higher order degenerate point. It should be noted that in a 2D linear tensor field, the δ function characterizing a degenerate point is always constant.

Another lemma given by Zhang et al., in their work [10] describes that the normal of the projection plane classifies the type of degenerate point i.e., wedge or trisector.

Lemma 4.1.1. *The wedge/trisector classification of a degenerate point p in the projection of a 3D linear, symmetric tensor field T onto a plane P is independent of the choice of normal of P and the coordinate system of P.*

The proof of this lemma is essentially the same as that to Theorem 2 in [9]. The only difference is that this time the proof is based on computing the δ and showing that the value of δ is constant. Under structurally stable conditions, the projected tensor field can have at most one degenerate point and if there is one, then it must be wedge, trisector or transition point if $\delta > 0$, $\delta < 0$ or $\delta = 0$ respectively.

4.1.1 Lower bound on the number of transition points

The theorem 4.0.2 provides us an important results on the lower bound on the total number of transition points.

Theorem 4.1.2. *Under structurally stable conditions, the number of WT curves in a linear tensor field provides the lower bound on total number of transition points in a linear tensor field.*

In our observation, we are able to achieve this lower bound. The results are presented in the observation section.

4.1.2 Upper bound on the number of transition points

This section presents the theoretical upper bound on the number of transition points in a 3D linear tensor field. Zhang et al. proved that there are maximum of 20 transition points in a 3D linear tensor field[10]. They followed the theorem 2 in [9] and deduced the corollary 4.1.3.1 [10]. We will present the proof of corollary here to show the formulation of δ function. The theorem and corresponding corollary is given below:

Theorem 4.1.3. *Given a 3D linear tensor field $LT = T_0 + T_x + T_y + T_z$ and a plane P, the discriminant function δ of the projection of LT onto P, is a function of T_x, T_y and T_z.*

Proof. The proof of the theorem can be found in [9]. □

Corollary 4.1.3.1. *Given a 3D linear tensor field $LT = T_0 + T_x + T_y + T_z$ and two parallel planes P_1 and P_2, the discriminant function δ of the projection of LT onto $P1$ and $P2$ are identical [10].*

According to the above corollary, the function δ is a function of possible plane normals in $\mathbb{R}^3$, which can be modelled on to the space of $\mathbb{RP}^2$, the two dimensional real projective space. Let (α, β, γ) be unit vector, it is sufficient to compute δ in the plane P given as,

$$\alpha x + \beta y + \gamma y = 0 \tag{4.1}$$

Without loss of generality, we can compute a basis for P using the column vector of the matrix $M = \begin{bmatrix} \frac{-\beta}{\sqrt{\alpha^2+\beta^2}} & \frac{-\alpha\gamma}{\sqrt{\alpha^2+\beta^2}} & \alpha \\ \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} & \frac{-\beta\gamma}{\sqrt{\alpha^2+\beta^2}} & \beta \\ 0 & \frac{\alpha^2+\beta^2}{\sqrt{\alpha^2+\beta^2}} & \gamma \end{bmatrix}$. Under this basis, a 3D linear tensor field $LT(x, y, z)$ becomes $LT(x^{'}, y^{'}, z^{'}) = T_0^{'} + x^{'}T_x^{'} + y^{'}T_y^{'} + z^{'}T_z^{'}$, where

$$T_x^{'} = \frac{1}{\sqrt{\alpha^2 + \beta^2}} M^T(-\beta T_x + \alpha T_y)M \tag{4.2}$$

$$T_y^{'} = \frac{1}{\sqrt{\alpha^2 + \beta^2}} M^T(-\alpha\gamma T_x - \beta\gamma T_y + (\alpha^2 + \beta^2)T_z)M \tag{4.3}$$

Recall the definition of δ,

$$\delta = (a_x^{'} - d_x^{'})b_y^{'} - (a_y^{'} - d_y^{'})b_x^{'} \tag{4.4}$$

The calculation shows that,

$$\begin{array}{lll} a_x^{'} = \frac{p_a(\alpha,\beta,\gamma)}{(\alpha^2+\beta^2)^{\frac{3}{2}}} & b_x^{'} = \frac{p_b(\alpha,\beta,\gamma)}{(\alpha^2+\beta^2)^{\frac{3}{2}}} & d_x^{'} = \frac{p_d(\alpha,\beta,\gamma)}{(\alpha^2+\beta^2)^{\frac{3}{2}}} \\ a_y^{'} = \frac{q_a(\alpha,\beta,\gamma)}{(\alpha^2+\beta^2)^{\frac{3}{2}}} & b_y^{'} = \frac{q_b(\alpha,\beta,\gamma)}{(\alpha^2+\beta^2)^{\frac{3}{2}}} & d_y^{'} = \frac{q_d(\alpha,\beta,\gamma)}{(\alpha^2+\beta^2)^{\frac{3}{2}}} \end{array} \tag{4.5}$$

where $p_a(\alpha, \beta, \gamma)$, $p_b(\alpha, \beta, \gamma)$ and $p_d(\alpha, \beta, \gamma)$ are polynomial of degrees 3, 4 and 5 respectively. $q_a(\alpha, \beta, \gamma)$, $q_b(\alpha, \beta, \gamma)$ and $q_d(\alpha, \beta, \gamma)$ are polynomial of degrees 4, 5 and 6 respectively. As a result of these polynomial, $\delta(\alpha, \beta, \gamma)$ is a polynomial of degree 10.
A transition point thus satisfy following system of polynomial equations:

$$h(\alpha, \beta, \gamma) = 0 \tag{4.6}$$

$$\delta(\alpha, \beta, \gamma) = 0 \tag{4.7}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{4.8}$$

The solution of above system of equation gives the location of a degenerate point which is also a transition point in the form of non-repeating eigenvector. Recall that $h(\alpha, \beta, \gamma) = 0$ is a degree two polynomial, $\delta(\alpha, \beta, \gamma)$ is a degree-ten polynomial, and $\alpha^2 + \beta^2 + \gamma^2 = 1$ is a degree two polynomial. According to B$\acute{e}$zout's theorem, there are at most 2 x 10 x 2 = 40 real solutions. It can be seen here that all three equations in the system contains only monomial of even degree. Consequently, if (α, β, γ) is a solution, then $(-\alpha, -\beta, -\gamma)$ is also a solution to the system of equations. Under structurally stable condition, there are up to maximum of 20 transition points in any 3D linear tensor field.

4.2 Transition Point Analysis and Observation

This section provides observation on the number of transition points both on a single degenerate curve and in a 3D linear tensor field. Zhang et al. proved that there are always two or four degenerate curves in a linear tensor field under structurally stable conditions[11]. When T_x, T_y and T_z are linearly independent, T_x, T_y and T_z spans a subspace of $\mathbb{R}^3$ (codimension two). So, there are two quadratic polynomials forming a system of polynomials. The real solution to the system of these polynomials either have zero, two, or four real solutions. The system of polynomial equations cannot have zero solution as the tensor field is traceless. Therefore, two or four degenerate curves are the only possible cases under structurally stable conditions.

In case of two curves, when T_0 is linearly independent, there are 3 different cases possible: (1) WW, WW, (2) TT, TT, and (3) WW, TT.

(1) WW, WW: We know from our previous discussion that there are same number of LWs as PWs. Therefore, we have 2 LWs and 2 PWs as the only combinations. We also know that along the degenerate curve linear/planar classification does not change. So, LW can only be connected to LW and PW can only be connected to PW. As a result, we have two degenerate curves connected as LW$\leftrightarrow$ LW and PW $\leftrightarrow$ PW. This gives us two WW curves.

(2) TT, TT: We know from our previous discussion that there are same number of LTs as PTs. Therefore, we have 2 LTs and 2 PTs as the only combinations. We also know that along the degenerate curve linear/planar classification does not change. So, LT can only be connected to LT and PT can only be connected to PT. As a result, we have two degenerate curves connected as LT$\leftrightarrow$ LT and PT $\leftrightarrow$ PT. This gives us two TT curves.

(3) WW, TT: We know from our previous discussion that there are same number of LWs as the PWs and same number of LTs as PTs. Therefore, we have 1 LW, 1 PW, 1 LT and 1 PT as the only combinations. We also know that along the degenerate curve linear/planar classification does not change. So, LT can only be connected to LW and PT can only be connected to PW. As a result, we have two degenerate curves connected as LW $\leftrightarrow$ LT and PW $\leftrightarrow$ PT which gives us two WT curves.

Similarly, in case of four degenerate curves, we have 5 different possible cases: (1) WW, WW, WW, WW, (2) WW, WW, WW, TT, (2) WW, WW, TT, TT, (4) WW, TT, TT, TT, and (5) TT, TT, TT, TT.

(1) WW, WW, WW, WW: Same analysis can be applied as case 1 for two degenerate curves. In this case, we have 4 LWs and 4 PWs as the only combinations possible, and LW can only be connected to LW and PW can only be connected to PW. As a result, we have four degenerate curves connected as LW$\leftrightarrow$ LW, LW$\leftrightarrow$ LW, PW$\leftrightarrow$ PW and PW $\leftrightarrow$ PW. This gives us four WW curves.

(2) WW, WW, WW, TT: Since there are same number of LWs as PWs and same number of LTs as PTs. The only possible combination is 3 LWs, 3 PWs, 1 LT and 1 PT and they are connected as LW$\leftrightarrow$ LT, LW$\leftrightarrow$ LW, PW$\leftrightarrow$ PW and PW $\leftrightarrow$ PT. Therefore, we have 2 WW and 2 WT curves.

(3) WW, WW, TT, TT: In the work of Zhang et al. [11], they showed that if two singularities are connected by a linear degenerate curve, then the same singularities cannot be connected by a planar degenerate curve. Keeping this fact in mind, and, also the fact that there are same number of LWs as PWs and same number of LTs as PTs,

and linear/planar classification does not change along the degenerate curve, we have 2 LWs, 2 PWs, 2 LTs, and 2 PTs as the only combinations and they can be connected in following two ways to give different type of curves:
(a) If the wedges and trisectors are connected as: PT $\leftrightarrow$ PW, LW $\leftrightarrow$ LT, PT $\leftrightarrow$ PW, and LW $\leftrightarrow$ LT. Then, we have 4 WT curves.
(b) If the wedges and trisectors are connected as: PT $\leftrightarrow$ PW, LW $\leftrightarrow$ LW, PW $\leftrightarrow$ PT, and LT $\leftrightarrow$ LT. Then, we have 1 WW, 1 TT and 2 WT curves.
(c) If the wedges and trisectors are connected as: LT $\leftrightarrow$ LW, PW $\leftrightarrow$ PW, LW $\leftrightarrow$ LT, and PT $\leftrightarrow$ PT. Then, we have 1 WW, 1 TT and 2 WT curves similar to case 3(b). The only difference here is that the tensor field is negated.

(4) WW, TT, TT, TT: The only possible combination in this case is 1 LW, 1 PW, 3 LTs, and 3 PTs, and they are connected as LW$\leftrightarrow$ LT, LT$\leftrightarrow$ LT, PT$\leftrightarrow$ PT and PT $\leftrightarrow$ PW. Therefore, we have 2 TT and 2 WT curves.

(5) TT, TT, TT, TT: This case is symmetric to case 1 for four degenerate curves. In this case, we have four TT curves.

From our analysis for two degenerate curves, we have 3 possible scenarios: (1) Two WW curves, (2) Two TT curves, and (3) Two WT curves. Also, from the analysis of four degenerate curves, we have 6 possible scenarios: (1) Four WW curves, (2) Four TT curves, (3) Two WW curves and two WT curves, (4) Two TT curves and Two WT curves, (5) Four WT curves, and, (6) One WW curve, one TT curve and two WT curves. Combining the analysis for two curves and four curves, we have 9 possible scenarios in a linear tensor field, shown in Table 4.1.

The theoretical bound on the number of transition points is found to be 20 according to the corollary 4.1.3.1. In our observation, we found that there are at most 8 transition points under structurally stable conditions in a 3D linear tensor field. To validate our claim, we analyze all 9 structurally stable scenarios with different configurations. A table for each scenario is also provided to show all possible cases under structurally stable conditions. To understand the table provided in each scenario, the reader should assume following points:

Scenarios	Number of WW curves	Number of TT curves	Number of WT curves
1	2	0	0
2	0	2	0
3	0	0	2
4	4	0	0
5	0	4	0
6	2	0	2
7	0	2	2
8	0	0	4
9	1	1	2

Table 4.1: Different combinations of degenerate curves under structurally stable conditions.

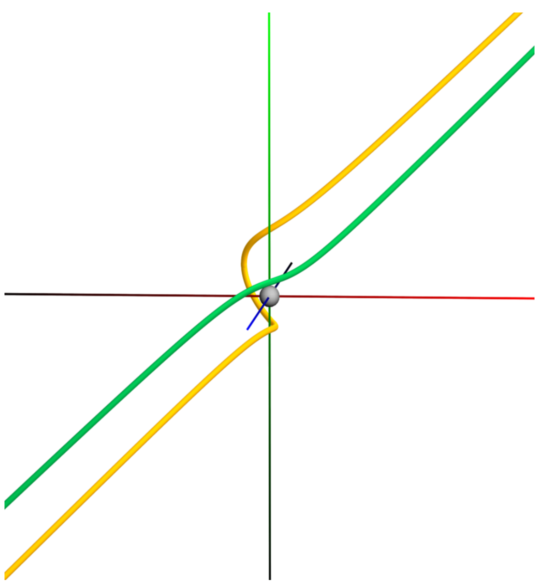
- a, b means a transition points on one degenerate curve and b transition points on the other degenerate curve.
- The number of element separated by comma represents number of degenerate curves in the particular scenario.
- The comment 'Asymmetry' followed by scenario number means that this configuration was not observed in the current scenario but was observed in the symmetrical scenario or vice versa.

4.2.1 Scenario 1: Two WW curves

When there are two WW curves, there are zero or even number of transition points on each of the degenerate curve. The reason behind these even number of transition points is due to the curve having their both ends in wedge. Each of these curves cannot switch from wedge to trisector unless there is a transition point along the curve. Therefore, the local pattern i.e, wedge and trisector either have to switch twice on each curve or remain same throughout the curve to have its end in wedge. In our analysis, we search for the following configuration (Table 4.2) under scenario 1.

Number of transition points	**Zero**	**Two**	**Four**	**Six**	**Eight**	**Ten**
Possible Combinations	0,0 Figure 4.1	2,0 Figure 4.2	4,0 Figure 4.3	6, 0 Figure 4.5	8,0 Not Found	10,0 Not Found
			2,2 Figure 4.4	4,2 Figure 4.6	6,2 Figure 4.7	8,2 Not Found
					4,4 Not Found Asymmetry with scenario 2	6,4 Not Found

Table 4.2: Combination of different configurations in scenario 1.



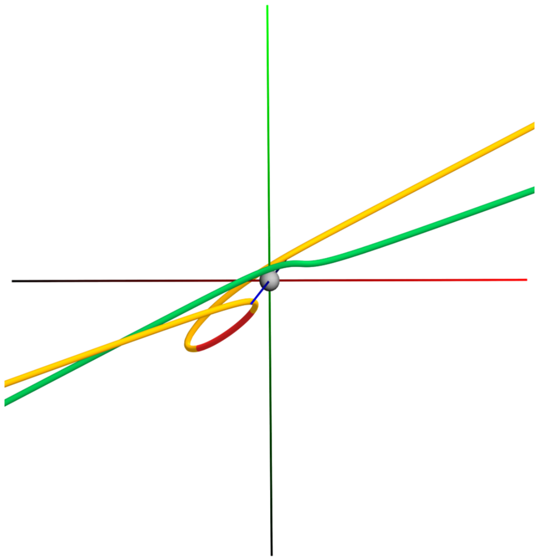
$$T_0 = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.100000 \\ 0.0000 & 0.1000 & 0.0000 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 \\ 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.9739 & -0.0114 & 0.1770 \\ -0.0114 & 0.6823 & -0.4680 \\ 0.1770 & -0.46800 & 0.2916 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.3636 & 0.4061 & -0.6208 \\ 0.4061 & 0.7348 & -0.2113 \\ -0.6208 & -0.2113 & -0.3712 \end{bmatrix}$$

Figure 4.1: Zero transition point in scenario 1.



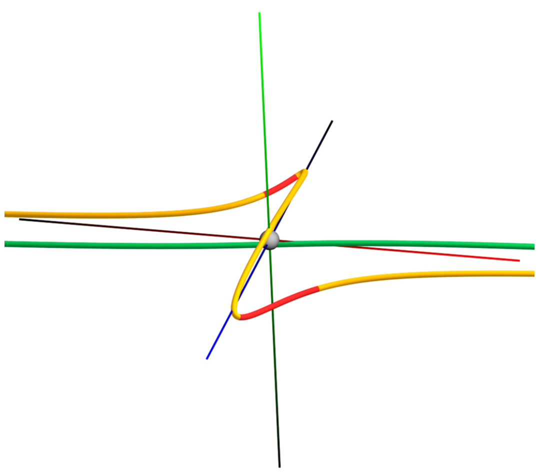
$$T_0 = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.1000 \\ 0.0000 & 0.1000 & 0.0000 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 \\ 1.0000 & 0.0000 & 0.1000 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.9739 & -1.3114 & 0.1770 \\ -1.3114 & 1.5823 & -0.5680 \\ 0.1770 & -0.5680 & -0.6084 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.3636 & 0.4061 & -0.6208 \\ 0.4061 & 0.7348 & -0.2113 \\ -0.6208 & -0.2113 & -0.3712 \end{bmatrix}$$

Figure 4.2: Two transition points in scenario 1.



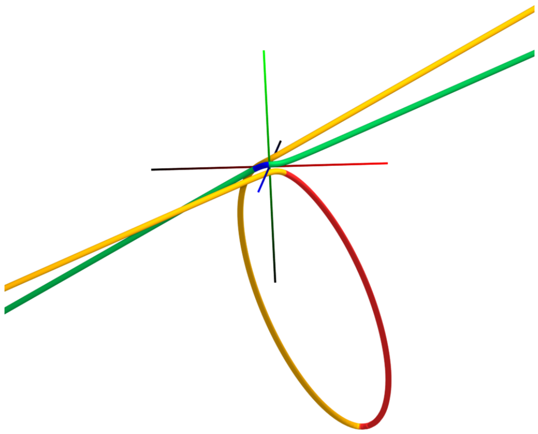
$$T_0 = \begin{bmatrix} 0.4000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.1000 \\ 0.0000 & 0.1000 & -0.4000 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.0000 & 1.000000 & 0.0000 \\ 1.0000 & -1.0000 & 0.1000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

$$T_y = \begin{bmatrix} 0.0000 & 0.0000 & 1.0000 \\ 0.0000 & 10.0000 & 0.0000 \\ 1.0000 & 0.0000 & -10.0000 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 \\ 1.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

Figure 4.3: Four Transition points: Four transition points on one degenerate curve and zero transition point on the other degenerate curve in scenario 1.



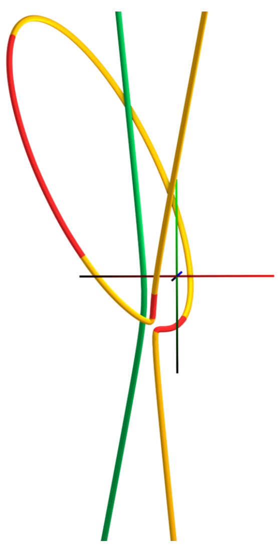
$$T_0 = \begin{bmatrix} 0.4000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.1000 \\ 0.0000 & 0.1000 & -0.4000 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 \\ 1.0000 & -1.0000 & 0.1000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.9739 & -1.3114 & 1.770 \\ -1.3114 & 1.6823 & -0.5680 \\ 1.770 & -0.5680 & -0.7084 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.3636 & 0.4061 & -0.6208 \\ 0.4061 & 0.7348 & -0.2113 \\ -0.6208 & -0.2113 & -0.3712 \end{bmatrix}$$

Figure 4.4: Four Transition points: Two transition points each on each of the two degenerate curves in scenario 1.



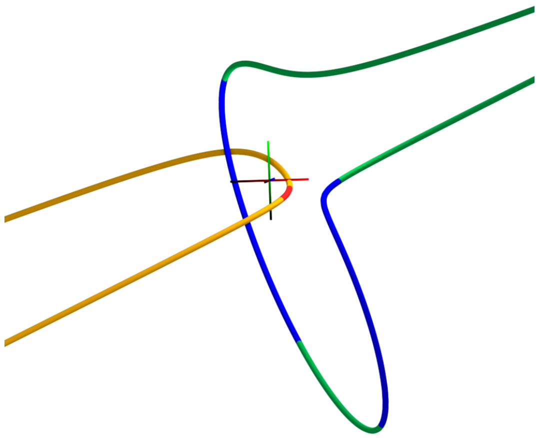
$$T_0 = \begin{bmatrix} -1.3255 & 0.2754 & 0.6118 \\ 0.2754 & 0.4000 & -0.1401 \\ 0.6118 & -0.1401 & 0.9255 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -1.3613 & 0.9225 & 2.1393 \\ 0.9225 & -0.3564 & 0.4082 \\ 2.1393 & 0.4082 & 1.7177 \end{bmatrix}$$

$$T_y = \begin{bmatrix} 0.3777 & -0.2103 & 0.7597 \\ -0.2103 & -0.3438 & -1.5334 \\ 0.7597 & -1.5334 & -0.0339 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -2.4063 & -0.8692 & 0.2452 \\ -0.8692 & 1.5207 & 1.5363 \\ 0.2452 & 1.5363 & 0.8856 \end{bmatrix}$$

Figure 4.5: Six transition points: Six transition points on one degenerate curve and zero transition point on the other degenerate curve in scenario 1.



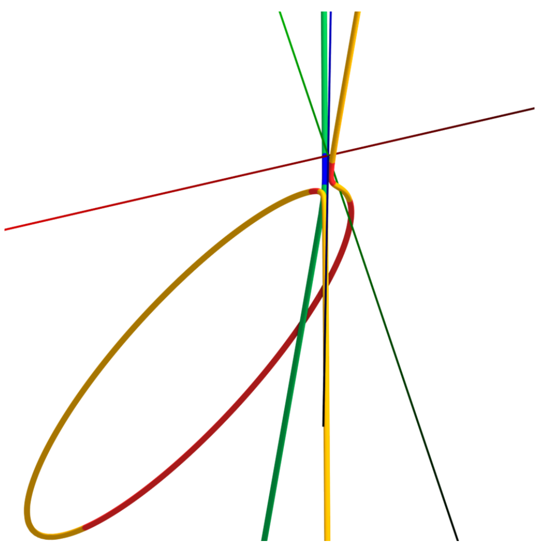
$$T_0 = \begin{bmatrix} 0.7425 & 0.3580 & -0.9072 \\ 0.3580 & -1.0637 & -1.4240 \\ -0.9072 & -1.4240 & 0.3212 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.1446 & 0.6549 & 2.1568 \\ 0.6549 & -1.5726 & 1.6439 \\ 2.1568 & 1.6439 & 1.4280 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -1.889 & -0.6525 & 1.0078 \\ -0.6525 & 1.9779 & -1.6968 \\ 1.0078 & -1.6968 & -0.0889 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.7365 & -1.2447 & 0.532 \\ -1.2447 & 1.5137 & -1.1979 \\ 0.532 & -1.1979 & -0.7772 \end{bmatrix}$$

Figure 4.6: Six transition points: Four transition points on one degenerate curve and two transition points on the other degenerate curve in scenario 1.



$$T_0 = \begin{bmatrix} 0.0595 & -0.0305 & 0.0404 \\ -0.0305 & 0.0716 & 0.0598 \\ 0.6118 & 0.0598 & -0.1311 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -2.2608 & -1.2583 & 0.6668 \\ -1.2583 & 0.6141 & 1.7100 \\ 0.6668 & 1.7100 & 1.6467 \end{bmatrix}$$

$$T_y = \begin{bmatrix} 0.0607 & -0.5707 & 0.1859 \\ -0.5707 & 0.8933 & 1.7705 \\ 0.1859 & 1.7705 & -0.9540 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.9947 & -0.9103 & -0.4214 \\ -0.9103 & -0.2301 & 0.2761 \\ -0.4214 & 0.2761 & -0.7646 \end{bmatrix}$$

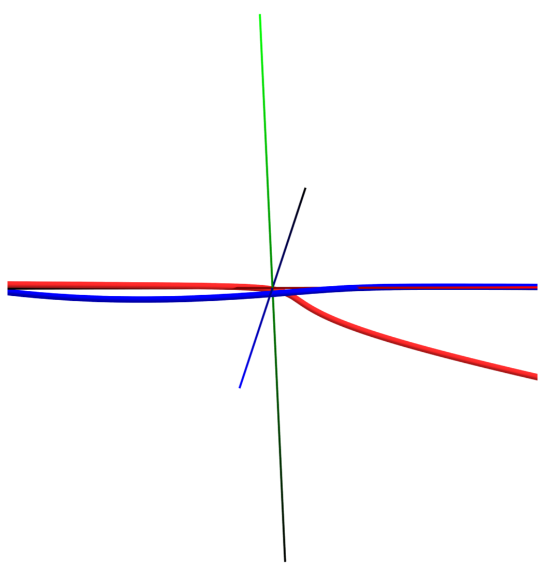
Figure 4.7: Eight transition points when there are two degenerate curves in scenario 1.

4.2.2 Scenario 2: Two TT curves

This scenario is symmetrical to scenario 1. So, we look for the similar configurations (Table 4.3) similar to scenario 1:

Number of transition points	**Zero**	**Two**	**Four**	**Six**	**Eight**	**Ten**
Possible Combinations	0,0 Figure 4.8	2,0 Figure 4.9	4,0 Figure 4.10	6,0 Figure 4.12	8,0 Not Found	10,0 Not Found
			2,2 Figure 4.11	4,2 Figure 4.13	6,2 Figure 4.14	8,2 Not Found
					4,4 Figure 4.15 Asymmetry with scenario 1	6,4 Not Found

Table 4.3: Combination of different configurations in scenario 2.



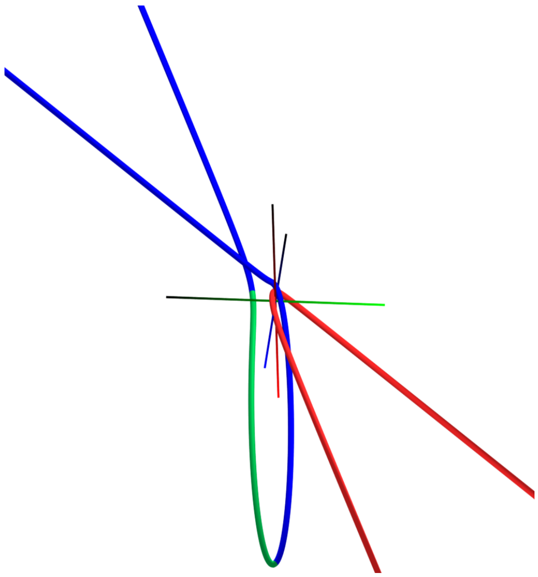
$$T_0 = \begin{bmatrix} -0.0723 & 0.0074 & 0.0527 \\ 0.0074 & 0.0336 & -0.0569 \\ 0.0527 & 0.0336 & 0.0387 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.1238 & -0.2017 & -0.0300 \\ -0.2017 & 0.0134 & -0.1645 \\ -0.0300 & 0.0134 & -0.1372 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.9850 & 2.1346 & 0.0075 \\ 2.1346 & 2.5524 & -1.7452 \\ 0.0075 & 2.5524 & -1.5674 \end{bmatrix}$$

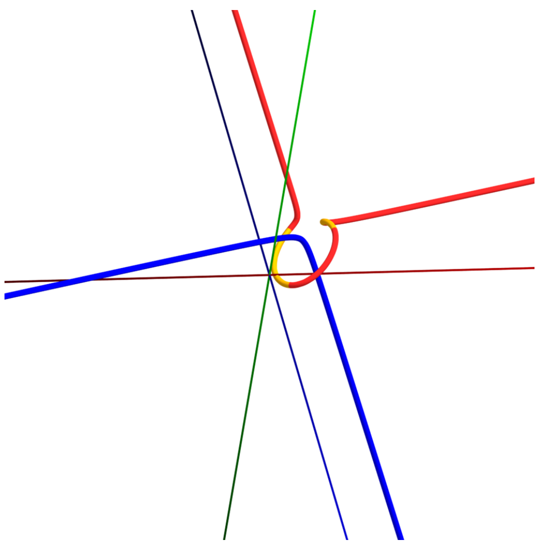
$$T_z = \begin{bmatrix} 0.3305 & 0.3829 & -0.4246 \\ 0.3829 & -2.0064 & -1.9651 \\ -0.4246 & -2.0064 & 1.6759 \end{bmatrix}$$

Figure 4.8: Zero transition point in scenario 2.



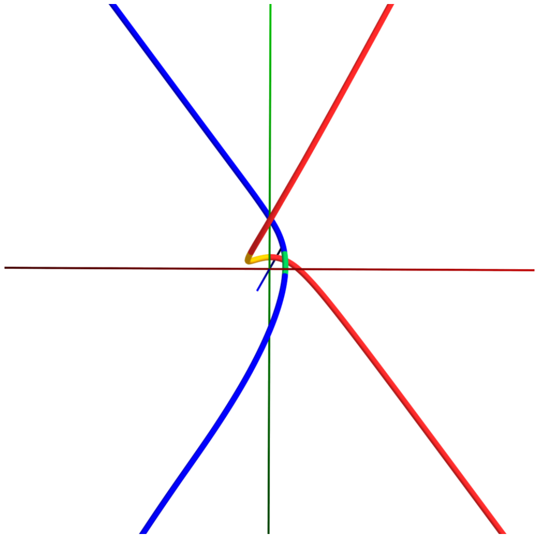
$$T_0 = \begin{bmatrix} 0.1000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.1000 \end{bmatrix}$$
$$T_x = \begin{bmatrix} 0.5000 & 1.0000 & 0.1000 \\ 1.0000 & 0.1000 & -1.3000 \\ 0.1000 & 0.1000 & -0.6000 \end{bmatrix}$$
$$T_y = \begin{bmatrix} 1.2446 & -1.2305 & 0.4739 \\ -1.2305 & -1.6873 & 0.4599 \\ 0.4739 & -1.6873 & 0.4427 \end{bmatrix}$$
$$T_z = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \\ 0.0000 & 0.0000 & -0.0000 \end{bmatrix}$$

Figure 4.9: Two transition points on one degenerate curve and zero transition point on the other degenerate curve in scenario 2.



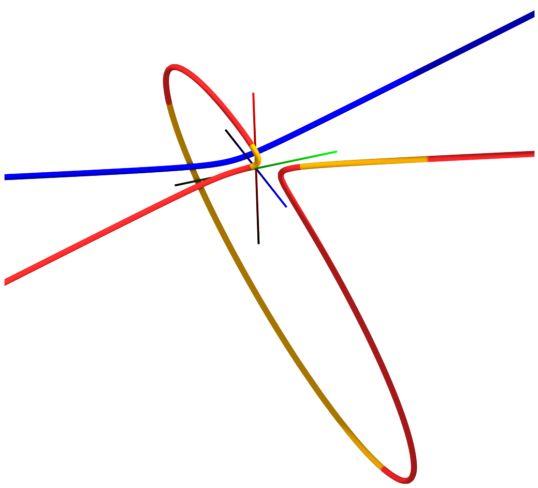
$$T_0 = \begin{bmatrix} 0.1000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.1000 \\ 0.0000 & 0.0000 & -0.1000 \end{bmatrix}$$
$$T_x = \begin{bmatrix} -1.2252 & -0.7677 & 0.5336 \\ -0.7677 & -0.2813 & 0.3178 \\ 0.5336 & -0.2813 & 1.5065 \end{bmatrix}$$
$$T_y = \begin{bmatrix} 1.0168 & -1.0791 & 0.1176 \\ -1.0791 & -1.2755 & -1.3055 \\ 0.1176 & -1.2755 & 0.2587 \end{bmatrix}$$
$$T_z = \begin{bmatrix} 0.6253 & -0.5554 & 0.9307 \\ -0.5554 & -0.8702 & 1.1838 \\ 0.9307 & -0.8702 & 0.2449 \end{bmatrix}$$

Figure 4.10: Four Transition points: Four transition points on one degenerate curve and zero transition point on the other degenerate curve in scenario 2.



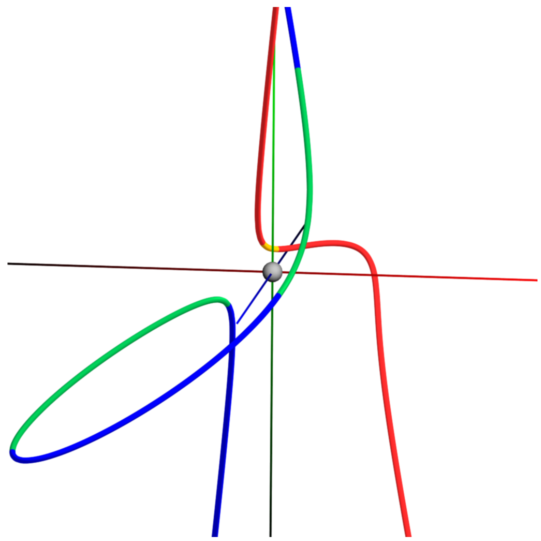
$$T_0 = \begin{bmatrix} 0.1000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.1000 \\ 0.0000 & 0.0000 & -0.1000 \end{bmatrix}$$
$$T_x = \begin{bmatrix} -1.2252 & -0.7677 & 0.5336 \\ -0.7677 & -0.2813 & 0.3178 \\ 0.5336 & -0.2813 & 1.5065 \end{bmatrix}$$
$$T_y = \begin{bmatrix} 1.0168 & -1.0791 & 0.1176 \\ -1.0791 & -1.2755 & -1.3055 \\ 0.1176 & -1.2755 & 0.2587 \end{bmatrix}$$
$$T_z = \begin{bmatrix} 0.6253 & -0.5554 & 0.9307 \\ -0.5554 & -0.8702 & 1.1838 \\ 0.9307 & -0.8702 & 0.2449 \end{bmatrix}$$

Figure 4.11: Four Transition points: Two transition points on each degenerate curves in scenario 2.



$$T_0 = \begin{bmatrix} 0.6000 & 0.2000 & 0.0000 \\ 0.2000 & 0.4000 & 0.0000 \\ 0.0000 & 0.4000 & -1.0000 \end{bmatrix}$$
$$T_x = \begin{bmatrix} 0.3447 & 1.7633 & 0.4000 \\ 1.7633 & -2.6369 & 0.5000 \\ 0.4000 & -2.6369 & 2.2922 \end{bmatrix}$$
$$T_y = \begin{bmatrix} 0.6909 & -0.3648 & 0.7771 \\ -0.3648 & 0.5383 & 1.4591 \\ 0.7771 & 0.5383 & -1.2292 \end{bmatrix}$$
$$T_z = \begin{bmatrix} 0.6107 & 0.0000 & 0.9000 \\ 0.0000 & 0.9452 & -1.1910 \\ 0.9000 & 0.9452 & -1.5559 \end{bmatrix}$$

Figure 4.12: Six transition points: Six transition points on one degenerate curve and zero transition on the other degenerate curve in scenario 2.



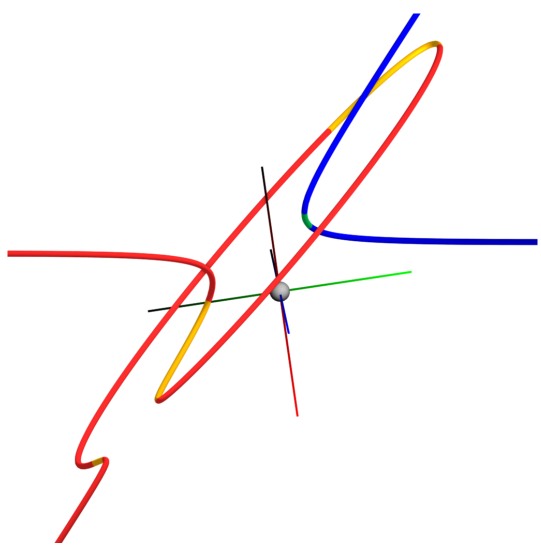
$$T_0 = \begin{bmatrix} 0.0000 & 0.1000 & 0.0000 \\ 0.1000 & 0.0000 & 0.1000 \\ 0.0000 & 0.0000 & -0.0000 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -1.2252 & -0.7677 & 0.5336 \\ -0.7677 & 0.3187 & 0.3178 \\ 0.5336 & 0.3187 & 0.9065 \end{bmatrix}$$

$$T_y = \begin{bmatrix} 1.0168 & -1.0791 & 0.1176 \\ -1.0791 & -1.2755 & 1.2945 \\ 0.1176 & -1.2755 & 0.2587 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.6253 & -0.5554 & 0.9307 \\ -0.5554 & -0.8702 & 1.1838 \\ 0.9307 & -0.8702 & 0.2449 \end{bmatrix}$$

Figure 4.13: Six transition points: Four transition points on one degenerate curve and two transition points on the other degenerate curve in scenario 2.



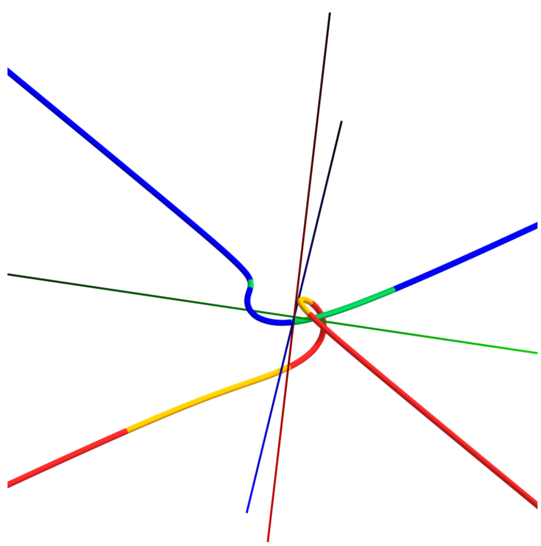
$$T_0 = \begin{bmatrix} 0.8046 & 0.6470 & 0.5960 \\ 0.6470 & -0.0900 & 1.1553 \\ 0.5960 & -0.0900 & -0.7146 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 2.0977 & 1.6935 & 0.1243 \\ 1.6935 & 0.1628 & 1.5706 \\ 0.1243 & 0.1628 & -2.2605 \end{bmatrix}$$

$$T_y = \begin{bmatrix} 2.4280 & 0.4236 & -1.3956 \\ 0.4236 & -1.5913 & 1.7224 \\ -1.3956 & -1.5913 & -0.8367 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 1.0004 & -0.2250 & -0.4504 \\ -0.2250 & -0.5826 & 0.8175 \\ -0.4504 & -0.5826 & -0.4178 \end{bmatrix}$$

Figure 4.14: Eight transition points: Six transition points on one degenerate curve and two transition points on the other degenerate curve in scenario 2.



$$T_0 = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.1000 \\ 0.0000 & 0.0000 & -0.0000 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -1.2252 & -0.7677 & 0.5336 \\ -0.7677 & 0.2187 & 0.3178 \\ 0.5336 & 0.2187 & 1.0065 \end{bmatrix}$$

$$T_y = \begin{bmatrix} 1.0168 & -1.0791 & 0.1176 \\ -1.0791 & -1.2755 & 1.2945 \\ 0.1176 & -1.2755 & 0.2587 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.6253 & -0.5554 & 0.9307 \\ -0.5554 & -0.8702 & 1.1838 \\ 0.9307 & -0.8702 & 0.2449 \end{bmatrix}$$

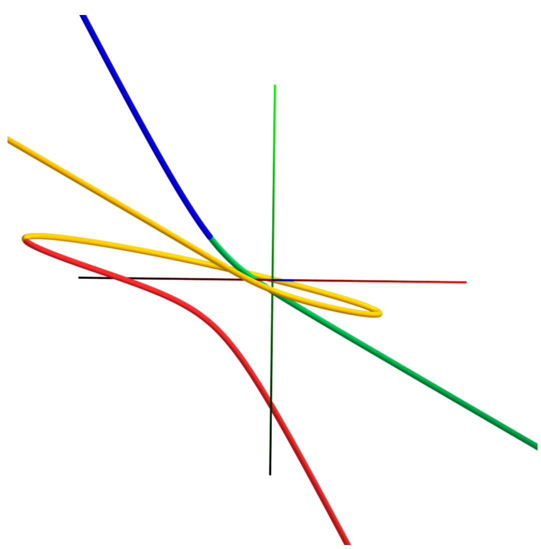
Figure 4.15: Eight transition points: Four transition points each on two degenerate curves in scenario 2.

4.2.3 Scenario 3: Two WT curves

In such case, there is at least two transition points in the tensor field. Also, there are odd number of transition points on each of the degenerate curve. In such tensor fields, one degenerate can have maximum of 7 transition points and minimum of 1 transition point. There cannot be zero transition point on a degenerate curve in such configuration because a WT curve always has an odd number of transition points. In this case too, we have upper bound of 8 transition points in all the observed linear tensor field. Table 4.4 shows different configurations possible under this scenario.

Number of transition points	**Two**	**Four**	**Six**	**Eight**	**Ten**
Possible Combinations	1,1 Figure 4.16	3,1 Figure 4.17	5,1 Figure 4.18	7,1 Figure 4.20	9,1 Not Found
			3,3 Figure 4.19	5,3 Figure 4.21	7,3 Not Found
					5,5 Not Found

Table 4.4: Combination of different configurations in scenario 3.



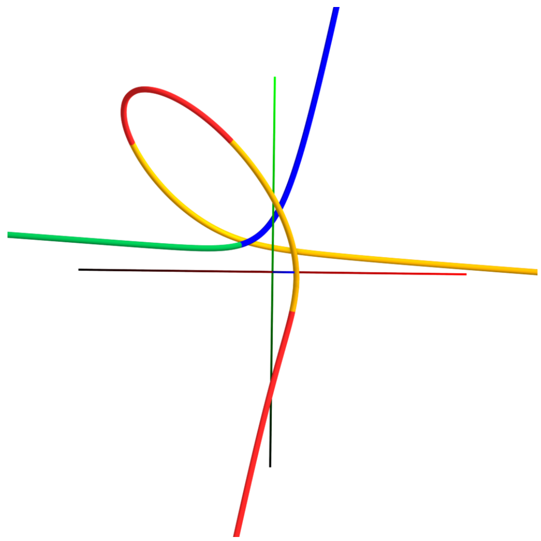
$$T_0 = \begin{bmatrix} -0.0116 & 0.0237 & 0.0960 \\ 0.0237 & 0.6919 & 0.0966 \\ 0.0960 & 0.6919 & -0.6803 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 1.3459 & 2.1036 & -1.0505 \\ 2.1036 & 0.4072 & -1.4391 \\ -1.0505 & 0.4072 & -1.7531 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -5.3075 & 5.1121 & -7.4489 \\ 5.1121 & 3.2839 & -8.1517 \\ -7.4489 & 3.2839 & 2.0236 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 2.5061 & 0.5772 & 2.5073 \\ 0.5772 & -0.1838 & 0.1619 \\ 2.5073 & -0.1838 & -2.3223 \end{bmatrix}$$

Figure 4.16: One transition point each on each of the two degenerate curves in scenario 3.



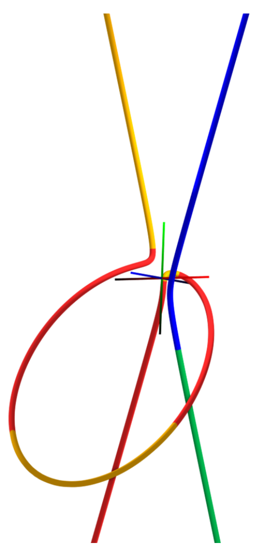
$$T_0 = \begin{bmatrix} -0.0116 & 0.0237 & 0.0960 \\ 0.0237 & 0.6919 & 0.0966 \\ 0.0960 & 0.6919 & -0.6803 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 1.2510 & 1.2321 & -0.6147 \\ 1.2321 & 0.4235 & -1.3982 \\ -0.6147 & 0.4235 & -1.6745 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -1.3341 & -0.0351 & -0.7498 \\ -0.0351 & 0.2000 & -2.0022 \\ -0.7498 & 0.2000 & 1.1341 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 2.5167 & -0.8126 & -2.4270 \\ -0.8126 & -0.1636 & -0.0210 \\ -2.4270 & -0.1636 & -2.3531 \end{bmatrix}$$

Figure 4.17: Three transition points on one degenerate curve and one transition point on the other degenerate curve in scenario 3.



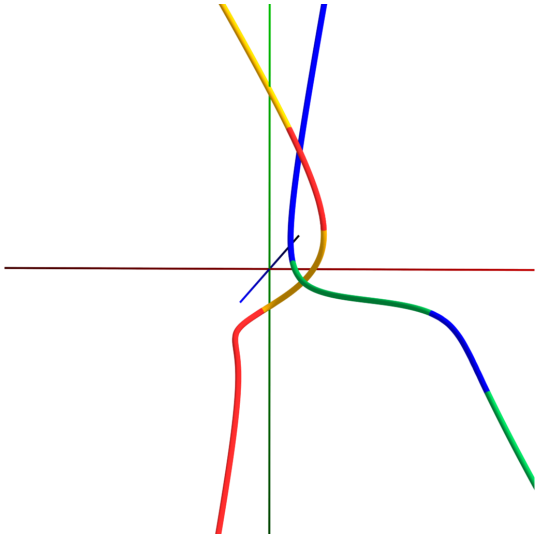
$$T_0 = \begin{bmatrix} 0.6000 & 0.2000 & 0.0000 \\ 0.2000 & 0.4000 & 0.0000 \\ 0.0000 & 0.4000 & -1.0000 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.3447 & 1.7633 & 0.4000 \\ 1.7633 & -2.6369 & 0.5000 \\ 0.4000 & -2.6369 & 2.2922 \end{bmatrix}$$

$$T_y = \begin{bmatrix} 0.6909 & -0.2648 & 0.7771 \\ -0.2648 & 0.5383 & 1.4591 \\ 0.7771 & 0.5383 & -1.2292 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.6107 & 0.0000 & 0.9000 \\ 0.0000 & 0.9452 & -1.1910 \\ 0.9000 & 0.9452 & -1.5559 \end{bmatrix}$$

Figure 4.18: Six transition points: Five transition points on one degenerate curve and one transition point on the other degenerate curve in scenario 3.



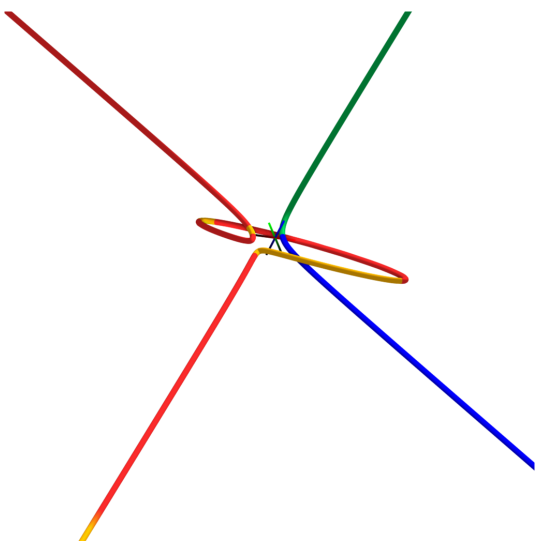
$$T_0 = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.1000 & 0.0000 \\ 0.0000 & 0.1000 & -0.1000 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -0.4552 & -0.0843 & -0.9109 \\ -0.0843 & -0.7403 & -0.2663 \\ -0.9109 & -0.7403 & 1.1955 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.7000 & 0.0000 & 1.0000 \\ 0.0000 & 0.4000 & 0.0000 \\ 1.0000 & 0.4000 & 0.3000 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.0000 & 0.0000 & 0.2000 \\ 0.0000 & 0.0000 & 0.2000 \\ 0.2000 & 0.0000 & -0.0000 \end{bmatrix}$$

Figure 4.19: Six transition points: Three transition points on each of the two degenerate curves in scenario 3.



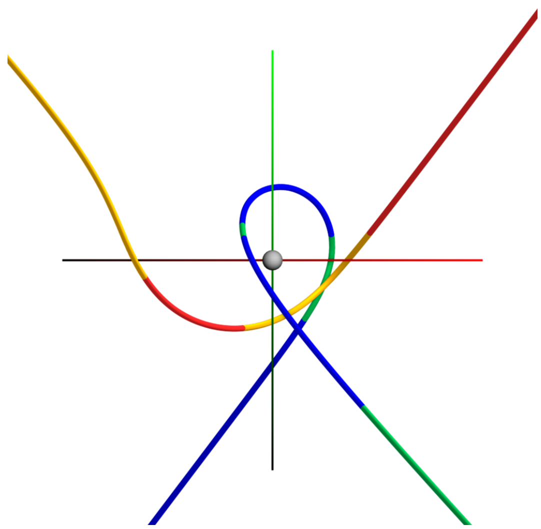
$$T_0 = \begin{bmatrix} 1.7209 & -0.3590 & 0.0865 \\ -0.3590 & -0.1476 & 0.0207 \\ 0.0865 & -0.1476 & -1.5733 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -0.3493 & -0.2422 & -0.0384 \\ -0.2422 & -1.7299 & -0.1965 \\ -0.0384 & -1.7299 & 2.0792 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.1407 & -0.0844 & 1.9154 \\ -0.0844 & 0.2167 & -0.2852 \\ 1.9154 & 0.2167 & -0.0760 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.1134 & -0.0840 & 2.9275 \\ -0.0840 & 0.1737 & 0.0740 \\ 2.9275 & 0.1737 & -0.0603 \end{bmatrix}$$

Figure 4.20: Eight transition points: Seven transition points on one degenerate curve and one transition point on the other degenerate curve in scenario 3.



$$T_0 = \begin{bmatrix} -0.5728 & -0.4436 & -0.0024 \\ -0.4436 & 0.5732 & 0.2141 \\ -0.0024 & 0.5732 & -0.0004 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.0170 & -0.9867 & 2.4017 \\ -0.9867 & -0.7061 & 2.2959 \\ 2.4017 & -0.7061 & 0.6891 \end{bmatrix}$$

$$T_y = \begin{bmatrix} 1.8383 & -1.7101 & 0.8759 \\ -1.7101 & 0.0000 & 0.0000 \\ 0.8759 & 0.0000 & -1.8383 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \\ 0.0000 & 0.0000 & -0.0000 \end{bmatrix}$$

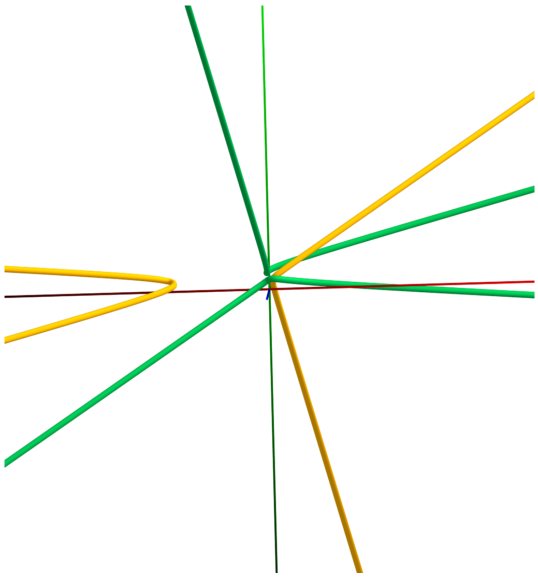
Figure 4.21: Eight transition points: Five transition points on one degenerate curve and three transition points on the other degenerate curve in scenario 3.

4.2.4 Scenario 4: Four WW curves

When there are four WW curves, there are always either none or even number of transition points on a degenerate curve. The observed maximum and minimum number of transition points in such field are 8 and 0 respectively. Following are the configuration (Table 4.5) which are possible to occur:

Number of transition points	**Zero**	**Two**	**Four**	**Six**	**Eight**	**Ten**
Possible Combinations	0,0,0,0 Figure 4.22	2,0,0,0 Figure 4.23	4,0,0,0 Not Found	6,0,0,0 Not Found	8,0,0,0 Not Found	10,0,0,0 Not Found
			2,2,0,0 Figure 4.24	4,2,0,0 Figure 4.25	6,2,0,0 Not Found	8,2,0,0 Not Found
				2,2,2,0 Figure 4.26	4,2,2,0 Not Found	6,4,0,0 Not Found
					2,2,2,2 Figure 4.27	6,2,2,0 Not Found
						4,4,2,0 Not Found
						4,2,2,2 Not Found

Table 4.5: Combination of different configurations in scenario 4.



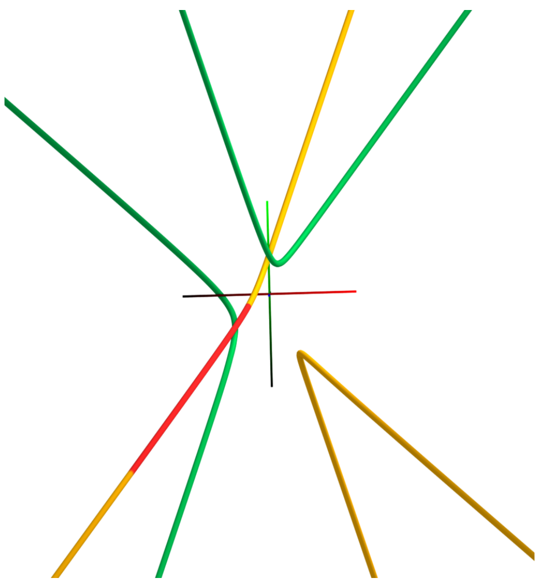
$$T_0 = \begin{bmatrix} 0.1000 & 0.0100 & 0.0000 \\ 0.0100 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.1000 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -0.7208 & 1.4853 & -0.4368 \\ 1.4853 & 0.9841 & -1.0347 \\ -0.4368 & 0.9841 & -0.2633 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -3.1768 & -0.7047 & -0.6298 \\ -0.7047 & 0.1884 & 1.4414 \\ -0.6298 & 0.1884 & 2.9884 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.1166 & 0.3200 & -0.2222 \\ 0.3200 & -0.5140 & -0.8002 \\ -0.2222 & -0.5140 & 0.3974 \end{bmatrix}$$

Figure 4.22: Zero transition point in scenario 4.



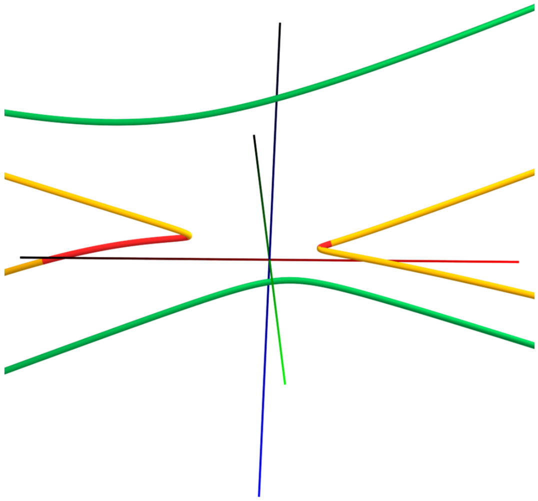
$$T_0 = \begin{bmatrix} -0.6180 & -0.1320 & 0.2179 \\ -0.1320 & 0.6083 & 0.3606 \\ 0.2179 & 0.6083 & 0.0097 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.0170 & -0.9867 & 2.4017 \\ -0.9867 & -0.7061 & 2.2959 \\ 2.4017 & -0.7061 & 0.6891 \end{bmatrix}$$

$$T_y = \begin{bmatrix} 0.1788 & 1.2135 & -1.4717 \\ 1.2135 & -0.8925 & -0.5553 \\ -1.4717 & -0.8925 & 0.7137 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.3460 & -0.4611 & -0.4414 \\ -0.4611 & -0.9169 & -0.0763 \\ -0.4414 & -0.9169 & 0.5709 \end{bmatrix}$$

Figure 4.23: Two transition points on one degenerate curve and zero transition point on all other degenerate curves in scenario 4.



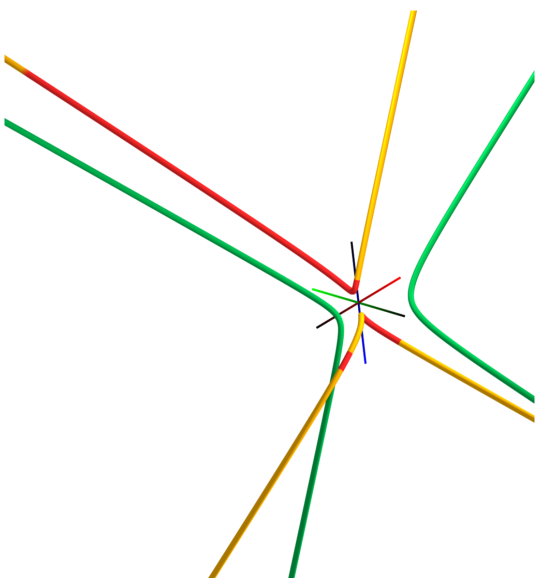
$$T_0 = \begin{bmatrix} 0.0514 & 0.0333 & -0.0778 \\ 0.0333 & 0.0053 & 0.0011 \\ -0.0778 & 0.0053 & -0.0566 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.0463 & 0.0657 & 0.0708 \\ 0.0657 & -0.0437 & 0.2232 \\ 0.0708 & -0.0437 & -0.0026 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -2.8003 & 1.0429 & -1.3473 \\ 1.0429 & -0.3228 & -0.8642 \\ -1.3473 & -0.3228 & 3.1232 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.1166 & 0.3200 & -0.2222 \\ 0.3200 & -0.5140 & -0.8002 \\ -0.2222 & -0.5140 & 0.3974 \end{bmatrix}$$

Figure 4.24: Two transition points on two degenerate curves and zero transition point on other two degenerate curves in scenario 4.



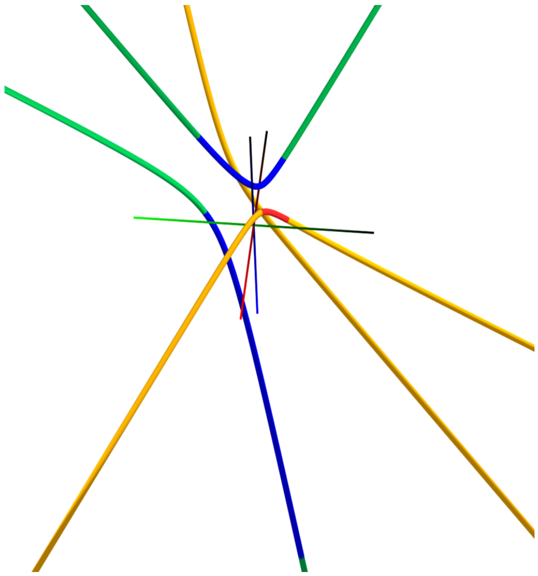
$$T_0 = \begin{bmatrix} -0.1935 & 0.3977 & -0.0206 \\ 0.3977 & -0.0732 & 0.0714 \\ -0.0206 & -0.0732 & 0.2667 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -0.2132 & 0.7479 & -0.6931 \\ 0.7479 & 0.3928 & 1.2670 \\ -0.6931 & 0.3928 & -0.1796 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -1.2350 & 0.0565 & -3.1983 \\ 0.0565 & 0.1481 & 0.9768 \\ -3.1983 & 0.1481 & 1.0870 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.8562 & -0.7682 & -0.1183 \\ -0.7682 & 0.9399 & -0.3332 \\ -0.1183 & 0.9399 & -0.0837 \end{bmatrix}$$

Figure 4.25: Six transition points: Four transition points on one degenerate curve, two transition points on one degenerate curve and zero transition point on one degenerate curve in scenario 4.



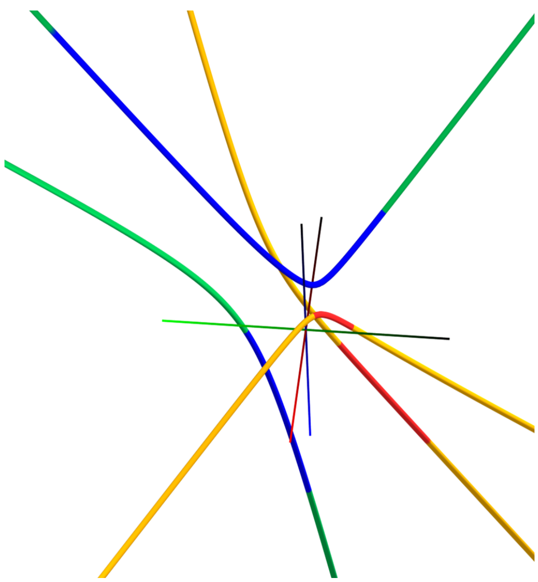
$$T_0 = \begin{bmatrix} 0.3595 & -0.1305 & 0.0404 \\ -0.1305 & 0.2716 & 0.1598 \\ 0.0404 & 0.2716 & -0.6311 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 2.0977 & 0.6955 & 1.3215 \\ 0.6955 & 0.1578 & 1.5729 \\ 1.3215 & 0.1578 & -2.2555 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.6349 & -0.5709 & -0.0637 \\ -0.5709 & 1.1157 & 2.0381 \\ -0.0637 & 1.1157 & -0.4808 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.9947 & -0.9103 & -0.0900 \\ -0.9103 & -0.2301 & 0.2761 \\ -0.0900 & -0.2301 & -0.7646 \end{bmatrix}$$

Figure 4.26: Six transition points: Two transition points on three degenerate curves and zero transition point on one degenerate curve in scenario 4.



$$T_0 = \begin{bmatrix} 0.3595 & -0.1305 & 0.0404 \\ -0.1305 & 0.2716 & 0.1598 \\ 0.0404 & 0.2716 & -0.6311 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 2.1977 & 0.6955 & 1.3215 \\ 0.6955 & 0.1578 & 1.5729 \\ 1.3215 & 0.1578 & -2.3555 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.4385 & -0.5842 & -0.0540 \\ -0.5842 & 1.0463 & 1.7287 \\ -0.0540 & 1.0463 & -0.6078 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.9947 & -0.9103 & -0.0900 \\ -0.9103 & -0.2301 & 0.2761 \\ -0.0900 & -0.2301 & -0.7646 \end{bmatrix}$$

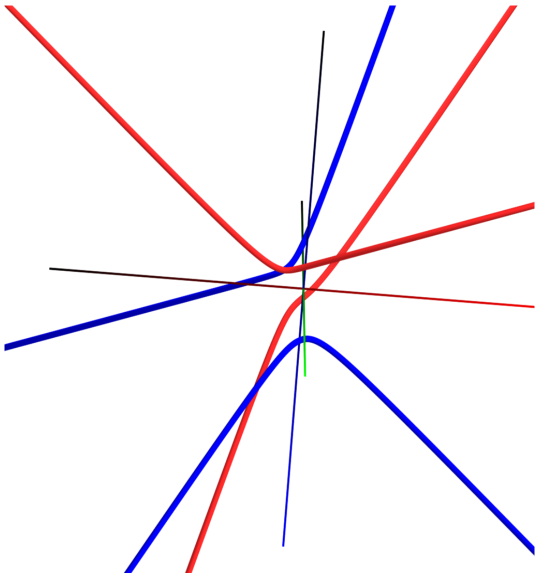
Figure 4.27: Eight transition points: Two transition points on all four degenerate curves in scenario 4.

4.2.5 Scenario 5: Four TT curves

When there are four TT curves, there are always either none or even number of transition points on a degenerate curve similar to scenario 4. In this case too, the observed maximum and minimum number of transition points are 8 and 0 respectively. Table 4.6 provides all the possible configurations under this scenario.

Number of transition points	**Zero**	**Two**	**Four**	**Six**	**Eight**	**Ten**
Possible Combinations	0,0,0,0 Figure 4.28	2,0,0,0 Figure 4.29	4,0,0,0 Not Found	6,0,0,0 Not Found	8,0,0,0 Not Found	10,0,0,0 Not Found
			2,2,0,0 Figure 4.30	4,2,0,0 Figure 4.31	6,2,0,0 Not Found	8,2,0,0 Not Found
				2,2,2,0 Figure 4.32	4,2,2,0 Not Found	6,4,0,0 Not Found
					2,2,2,2 Figure 4.33	6,2,2,0 Not Found
						4,4,2,0 Not Found
						4,2,2,2 Not Found

Table 4.6: Combination of different configurations in scenario 5.



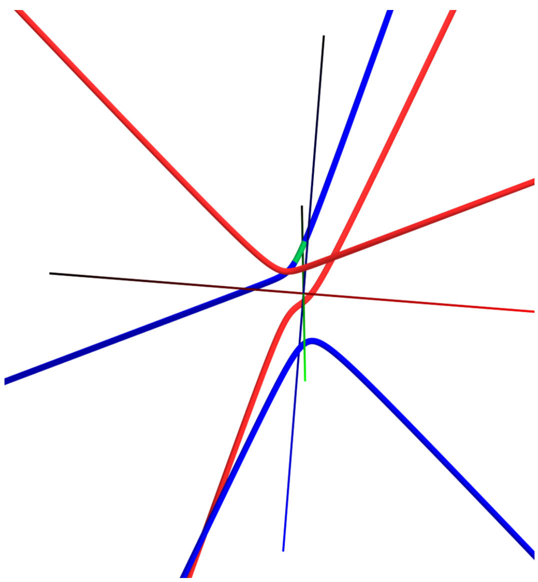
$$T_0 = \begin{bmatrix} -0.0116 & 0.0237 & 0.0960 \\ 0.0237 & 0.6919 & 0.0966 \\ 0.0960 & 0.6919 & -0.6803 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -2.7407 & -0.9808 & -4.3788 \\ -0.9808 & 8.3682 & -5.1143 \\ -4.3788 & 8.3682 & -5.6275 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.8359 & -5.3236 & -10.5531 \\ -5.3236 & 3.4134 & 3.9742 \\ -10.5531 & 3.4134 & -2.5775 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -1.0416 & -3.9385 & 9.0661 \\ -3.9385 & -0.9106 & 0.6563 \\ 9.0661 & -0.9106 & 1.9522 \end{bmatrix}$$

Figure 4.28: Zero transition point in scenario 5.



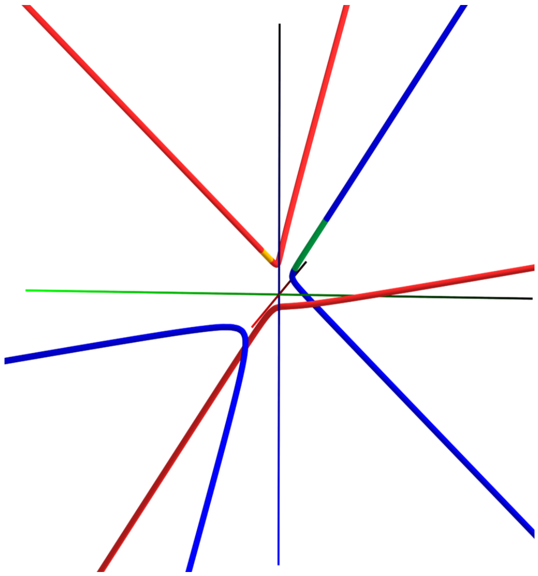
$$T_0 = \begin{bmatrix} -0.0116 & 0.0237 & 0.0960 \\ 0.0237 & 0.6919 & 0.0966 \\ 0.0960 & 0.6919 & -0.6803 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -2.7407 & -0.9808 & -4.3788 \\ -0.9808 & 8.3682 & -5.1143 \\ -4.3788 & 8.3682 & -5.6275 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.1979 & -6.2851 & -9.8869 \\ -6.2851 & 2.1413 & 5.3048 \\ -9.8869 & 2.1413 & -1.9434 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.6756 & -3.7720 & 9.1113 \\ -3.7720 & -1.4930 & 0.2499 \\ 9.1113 & -1.4930 & 2.1686 \end{bmatrix}$$

Figure 4.29: Two transition points on one degenerate curve and zero transition point on all other degenerate curves in scenario 5.



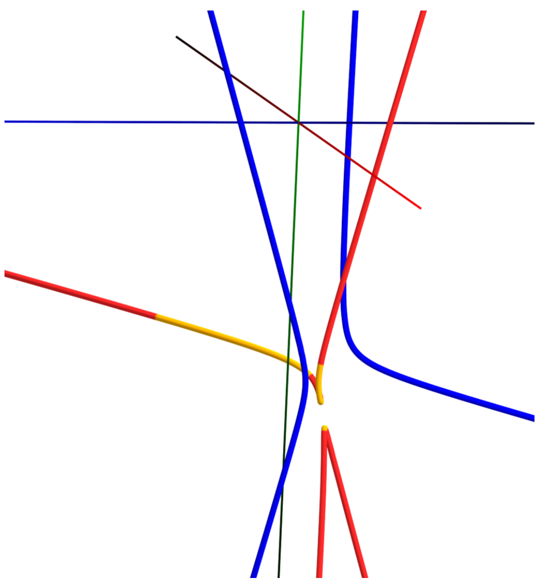
$$T_0 = \begin{bmatrix} -0.0116 & 0.0237 & 0.0960 \\ 0.0237 & 0.6919 & 0.0966 \\ 0.0960 & 0.6919 & -0.6803 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -2.7407 & -0.9808 & -4.3788 \\ -0.9808 & 8.3682 & -5.1143 \\ -4.3788 & 8.3682 & -5.6275 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.1979 & -6.2851 & -9.8869 \\ -6.2851 & 2.1413 & 5.3048 \\ -9.8869 & 2.1413 & -1.9434 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.2904 & -3.5801 & 9.1146 \\ -3.5801 & -2.0936 & -0.1753 \\ 9.1146 & -2.0936 & 2.3840 \end{bmatrix}$$

Figure 4.30: Two transition points on two degenerate curves and zero transition point on all other degenerate curves in scenario 5.



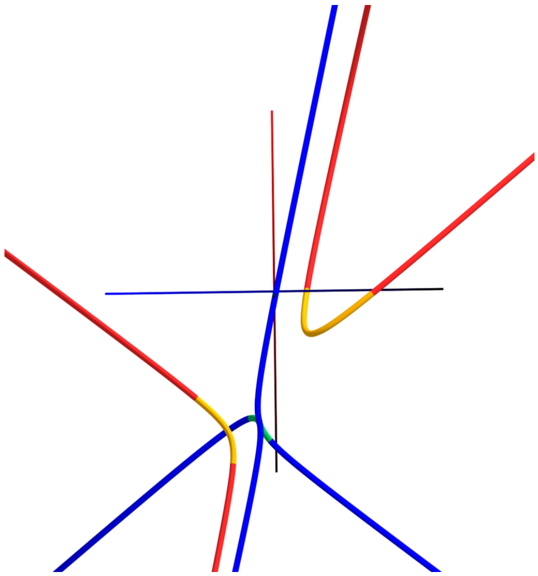
$$T_0 = \begin{bmatrix} -0.1871 & 0.0410 & -0.6872 \\ 0.0410 & -0.2922 & -0.0559 \\ -0.6872 & -0.2922 & 0.4793 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -0.7152 & 1.6814 & -0.4420 \\ 1.6814 & 0.9881 & -1.0377 \\ -0.4420 & 0.9881 & -0.2729 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.1135 & 0.0821 & -0.4112 \\ 0.0821 & -0.0645 & 0.0276 \\ -0.4112 & -0.0645 & 0.1780 \end{bmatrix}$$

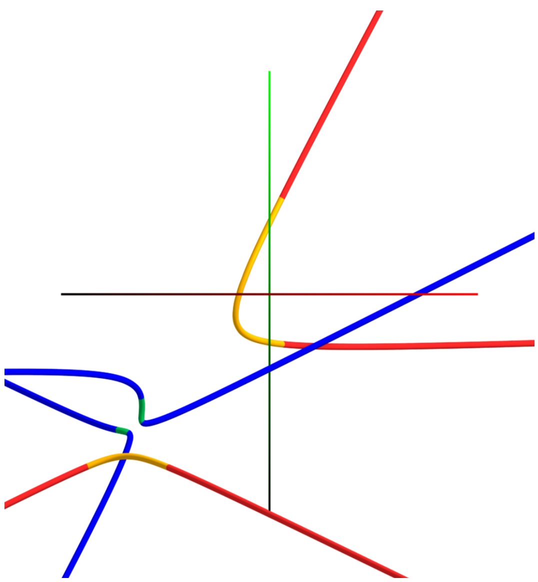
$$T_z = \begin{bmatrix} 0.0779 & 0.4396 & -0.3575 \\ 0.4396 & -0.6411 & -1.0759 \\ -0.3575 & -0.6411 & 0.5632 \end{bmatrix}$$

Figure 4.31: Six transition points: Four transition points on one degenerate curve, two transition points on one degenerate curve and zero transition point on one degenerate curve in scenario 5.



$$T_0 = \begin{bmatrix} -0.0088 & 0.2469 & -0.1972 \\ 0.2469 & 1.0442 & -0.1593 \\ -0.1972 & 1.0442 & -1.0354 \end{bmatrix}$$
$$T_x = \begin{bmatrix} -1.5613 & 0.1249 & 0.8366 \\ 0.1249 & 2.0668 & -0.9217 \\ 0.8366 & 2.0668 & -0.5055 \end{bmatrix}$$
$$T_y = \begin{bmatrix} 1.7777 & 0.1847 & -0.4755 \\ 0.1847 & -0.4547 & 0.9872 \\ -0.4755 & -0.4547 & -1.3230 \end{bmatrix}$$
$$T_z = \begin{bmatrix} 0.8519 & 1.1166 & 2.5128 \\ 1.1166 & 0.2589 & -0.1784 \\ 2.5128 & 0.2589 & -1.1108 \end{bmatrix}$$

Figure 4.32: Six transition points: Two transition points each on three degenerate curves and zero transition point on one degenerate curve in scenario 5.

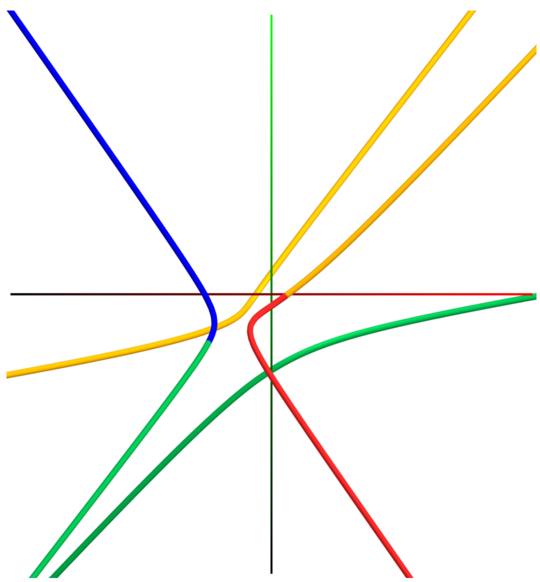


$$T_0 = \begin{bmatrix} -0.0088 & 0.2469 & -0.1972 \\ 0.2469 & 0.9442 & -0.1593 \\ -0.1972 & 0.9442 & -0.9354 \end{bmatrix}$$
$$T_x = \begin{bmatrix} -1.5613 & 0.1249 & 0.8366 \\ 0.1249 & 2.0668 & -0.9217 \\ 0.8366 & 2.0668 & -0.5055 \end{bmatrix}$$
$$T_y = \begin{bmatrix} 1.7777 & 0.1847 & -0.4755 \\ 0.1847 & -0.4547 & 0.9872 \\ -0.4755 & -0.4547 & -1.3230 \end{bmatrix}$$
$$T_z = \begin{bmatrix} 0.8519 & 1.1166 & 2.5128 \\ 1.1166 & 0.2589 & -0.1784 \\ 2.5128 & 0.2589 & -1.1108 \end{bmatrix}$$

Figure 4.33: Eight transition points: Two transition points each on all four degenerate curves in scenario 5.

4.2.6 Scenario 6: Two WW curves and two WT curves

When there are two WW curves and two WT curve, there are minimum of two transition points. None of the observed cases had zero transition point which clearly make sense as there are two WT curves and each WT curve have at least one transition point. Similar to our previous analysis, a degenerate curve can only have their ends in wedge and trisector if there is at least one switch along the curve. Two WW curve either have zero transition point or even number of transition points. Consequently, we have even number of transition points in this scenario. Table 4.7 show all possible configuration under this scenario.



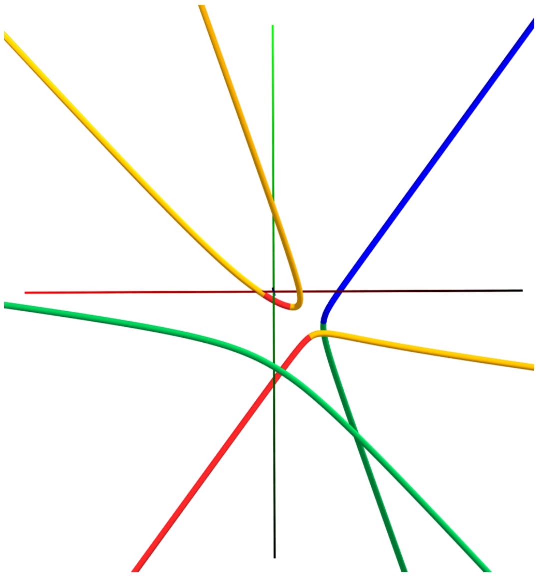
$$T_0 = \begin{bmatrix} 0.1259 & 0.0591 & -0.0846 \\ 0.0591 & 0.1465 & 0.1165 \\ -0.0846 & 0.1465 & -0.2724 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.2587 & 0.0861 & -0.9337 \\ 0.0861 & -0.9061 & 0.9366 \\ -0.9337 & -0.9061 & 0.6474 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.7781 & 1.0794 & -0.8078 \\ 1.0794 & 1.8484 & -0.3140 \\ -0.8078 & 1.8484 & -1.0703 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -1.7610 & 1.0224 & -1.0675 \\ 1.0224 & -0.0609 & 1.4906 \\ -1.0675 & -0.0609 & 1.8219 \end{bmatrix}$$

Figure 4.34: One transition point each on two degenerate curves and zero transition point on other two degenerate curves in scenario 6.



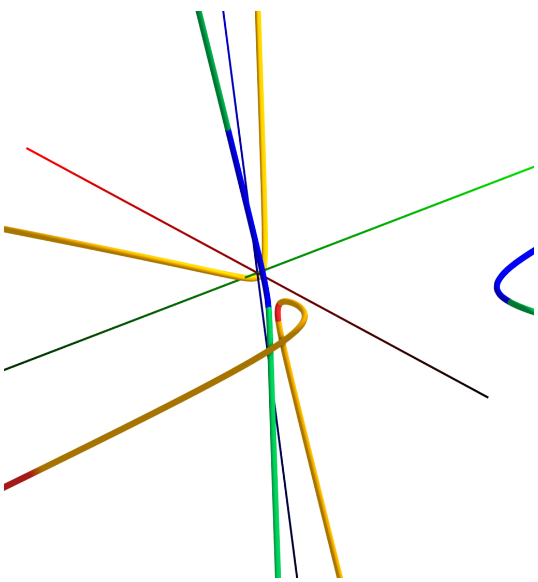
$$T_0 = \begin{bmatrix} 0.1259 & 0.0591 & -0.0846 \\ 0.0591 & 0.1465 & 0.1165 \\ -0.0846 & 0.1465 & -0.2724 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.2587 & 0.0861 & -0.9337 \\ 0.0861 & -0.9061 & 0.9366 \\ -0.9337 & -0.9061 & 0.6474 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.7781 & 1.0794 & -0.8078 \\ 1.0794 & 1.8484 & -0.3140 \\ -0.8078 & 1.8484 & -1.0703 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -1.7610 & 0.4224 & -1.0675 \\ 0.4224 & -0.0609 & 1.4906 \\ -1.0675 & -0.0609 & 1.8219 \end{bmatrix}$$

Figure 4.35: Two transition points on one degenerate curve, one transition point each on two degenerate curves and zero transition point on one degenerate curve in scenario 6.



$$T_0 = \begin{bmatrix} -0.0904 & -0.0700 & -0.0004 \\ -0.0700 & 0.0904 & 0.0338 \\ -0.0004 & 0.0904 & -0.0001 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -0.9648 & -0.2299 & -0.7033 \\ -0.2299 & 0.3605 & -0.0693 \\ -0.7033 & 0.3605 & 0.6043 \end{bmatrix}$$

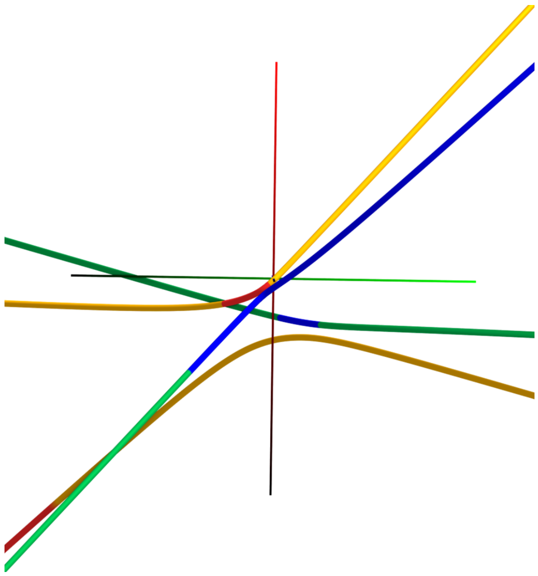
$$T_y = \begin{bmatrix} 2.1181 & -0.4236 & 0.4121 \\ -0.4236 & 0.2050 & 0.9737 \\ 0.4121 & 0.2050 & -2.3231 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.1649 & -1.0237 & 0.1652 \\ -1.0237 & 0.0070 & 0.6842 \\ 0.1652 & 0.0070 & 0.1579 \end{bmatrix}$$

Figure 4.36: Six transition points: Three, two, one and zero transition point on each of the degenerate curves respectively in scenario 6.

Number of transition points	**Two**	**Four**	**Six**	**Eight**	**Ten**
Possible Combinations	1,1,0,0 Figure 4.34	4,0,0,0 Not Found	6,0,0,0 Not Found	8,0,0,0 Not Found	10,0,0,0 Not Found
		3,1,0,0 Not Found Asymmetry with scenario 7	5,1,0,0 Not Found	7,1,0,0 Not Found	9,1,0,0 Not Found
		2,2,0,0 Not Found	4,2,0,0 Not Found	6,2,0,0 Not Found	8,2,0,0 Not Found
		2,1,1,0 Figure 4.35	4,1,1,0 Not Found Asymmetry with scenario 7	6,1,1,0 Not Found	8,1,1,0 Not Found
			3,2,1,0 Figure 4.36	5,3,0,0 Not Found	7,3,0,0 Not Found
			2,2,1,1 Figure 4.37	5,2,1,0 Not Found	7,2,1,0 Not Found
				4,2,1,1 Figure 4.38 Asymmetry with scenario 7	6,4,0,0 Not Found
				3,3,2,0 Not Found	6,3,1,0 Not Found
				3,2,2,1 Not Found Asymmetry with scenario 7	6,2,1,1 Not Found
					5,5,0,0 Not Found
					5,4,1,0 Not Found
					5,3,2,0 Not Found
					5,2,2,1 Not Found
					4,4,1,1 Not Found
					4,3,2,1 Not Found
					3,3,2,2 Not Found

Table 4.7: Combination of different configurations in scenario 6.



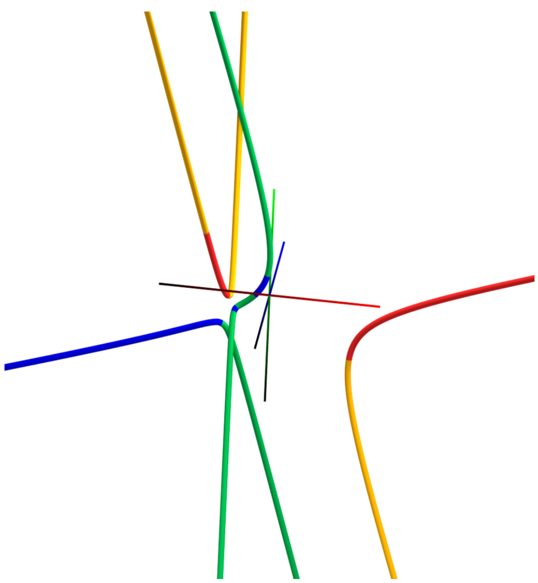
$$T_0 = \begin{bmatrix} 0.2265 & -0.0192 & 0.0255 \\ -0.0192 & 0.1711 & 0.1007 \\ 0.0255 & 0.1711 & -0.3976 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 2.1977 & 0.6955 & 1.3215 \\ 0.6955 & 0.1578 & 1.4729 \\ 1.3215 & 0.1578 & -2.3555 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.4385 & -0.5842 & -0.0540 \\ -0.5842 & 1.0463 & 1.7287 \\ -0.0540 & 1.0463 & -0.6078 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.9947 & -0.9103 & -0.0900 \\ -0.9103 & -0.2301 & 0.2761 \\ -0.0900 & -0.2301 & -0.7646 \end{bmatrix}$$

Figure 4.37: Six transition points: Two transition points on two degenerate curves and one transition point each on two degenerate curves in scenario 6.



$$T_0 = \begin{bmatrix} -0.055630 & 0.121152 & -0.144964 \\ 0.121152 & 0.030503 & -0.073200 \\ -0.144964 & -0.073200 & 0.025127 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -0.539884 & 0.435815 & -0.387068 \\ 0.435815 & 1.407228 & -0.618981 \\ -0.387068 & -0.618981 & -0.867344 \end{bmatrix}$$

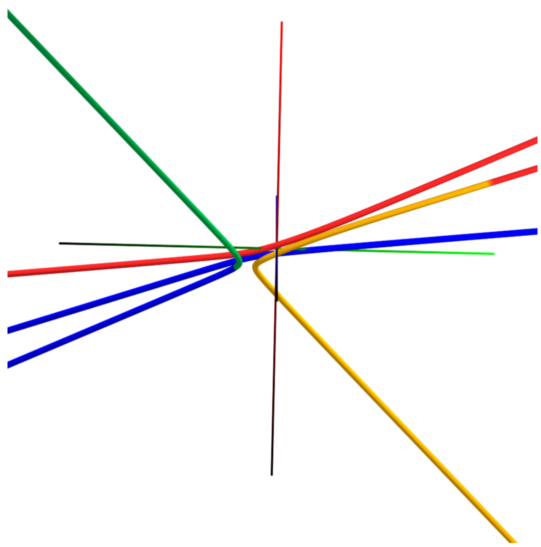
$$T_y = \begin{bmatrix} 1.277507 & -0.065352 & -0.207895 \\ -0.065352 & 0.942943 & -1.103464 \\ -0.207895 & -1.103464 & -2.22045 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.629354 & -0.006736 & -0.012501 \\ -0.006736 & -1.862845 & 1.245910 \\ -0.012501 & 1.245910 & 1.233491 \end{bmatrix}$$

Figure 4.38: Eight transition points: Four transition points on one degenerate curve, two transition points on one degenerate curve and one transition point each on two degenerate curves in scenario 6.

4.2.7 Scenario 7: Two TT curves and two WT curves

When there are two TT curves and two WT curves, the same analysis follows as scenario 6. Table 4.8 shows all possible configurations under this scenario.



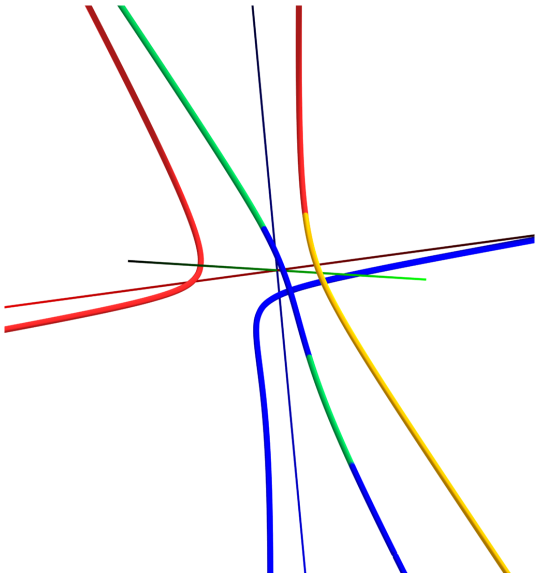
$$T_0 = \begin{bmatrix} 0.0344 & 0.0096 & -0.0154 \\ 0.0096 & 0.0440 & 0.0060 \\ -0.0154 & 0.0440 & -0.0784 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.2587 & 0.0861 & -1.0600 \\ 0.0861 & 0.4939 & 1.3366 \\ -1.0600 & 0.4939 & -0.7526 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.3769 & 0.3177 & -0.0512 \\ 0.3177 & 0.3068 & -0.4099 \\ -0.0512 & 0.3068 & 0.0701 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -1.7922 & 1.4648 & -0.7536 \\ 1.4648 & 0.8485 & 0.9022 \\ -0.7536 & 0.8485 & 0.9437 \end{bmatrix}$$

Figure 4.39: One transition point each on two degenerate curves and zero transition point on two degenerate curves in scenario 7.



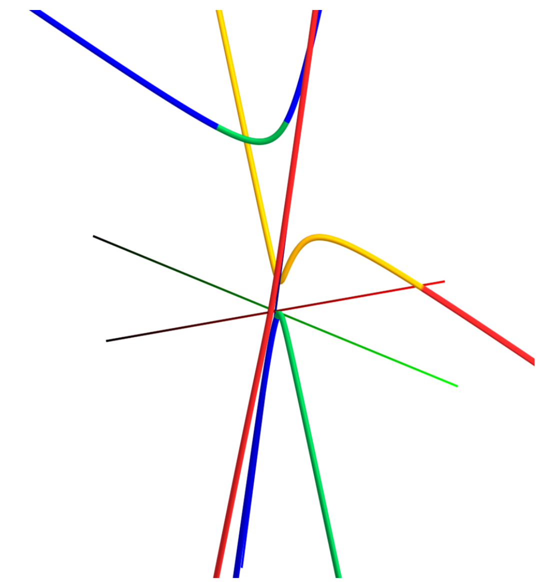
$$T_0 = \begin{bmatrix} 0.059004 & 0.147258 & -0.035296 \\ 0.147258 & 0.195146 & 0.006435 \\ -0.035296 & 0.195146 & -0.254150 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.258700 & 0.086100 & -1.060000 \\ 0.086100 & 0.493900 & 1.336600 \\ -1.060000 & 0.493900 & -0.752600 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.376900 & 0.317700 & -0.051200 \\ 0.317700 & 0.306800 & -0.409900 \\ -0.051200 & 0.306800 & 0.070100 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -1.792200 & 1.464800 & -0.753600 \\ 1.464800 & 0.848500 & 0.902200 \\ -0.753600 & 0.848500 & 0.943700 \end{bmatrix}$$

Figure 4.40: Three transition points on one degenerate curve, one transition point on one degenerate curve and zero transition point on two degenerate curves in scenario 7.



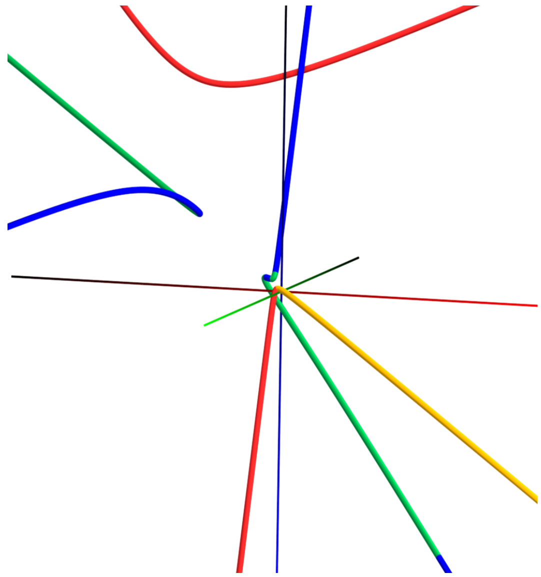
$$T_0 = \begin{bmatrix} -0.0075 & -0.0030 & 0.0134 \\ -0.0030 & 0.0350 & 0.0338 \\ 0.0134 & 0.0350 & -0.0275 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -2.6455 & -0.1284 & 1.4676 \\ -0.1284 & -1.2834 & 0.6693 \\ 1.4676 & -1.2834 & 3.9289 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.5925 & 2.9152 & -0.2388 \\ 2.9152 & 1.3966 & -1.0381 \\ -0.2388 & 1.3966 & -0.8041 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.2953 & 0.1556 & 0.0102 \\ 0.1556 & -0.1746 & 0.2295 \\ 0.0102 & -0.1746 & 0.4699 \end{bmatrix}$$

Figure 4.41: Two transition points on one degenerate curve, one transition point on two degenerate curves and zero transition point on one degenerate curve in scenario 7.



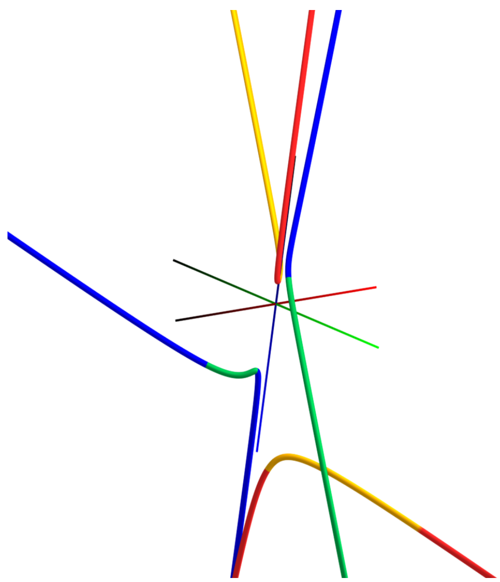
$$T_0 = \begin{bmatrix} -0.0199 & -0.0015 & -0.0191 \\ -0.0015 & 0.0017 & -0.0046 \\ -0.0191 & 0.0017 & 0.0183 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -0.2661 & 0.1807 & -0.4794 \\ 0.1807 & 0.8413 & -0.4275 \\ -0.4794 & 0.8413 & -0.5752 \end{bmatrix}$$

$$T_y = \begin{bmatrix} 1.1745 & -0.4655 & 0.4963 \\ -0.4655 & -0.5951 & -0.1226 \\ 0.4963 & -0.5951 & -0.5794 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.2799 & -0.0822 & -0.1720 \\ -0.0822 & -0.5679 & 0.6355 \\ -0.1720 & -0.5679 & 0.8478 \end{bmatrix}$$

Figure 4.42: Four transition points on one degenerate curve, one transition point on two degenerate curves and zero transition point on one degenerate curve in scenario 7.



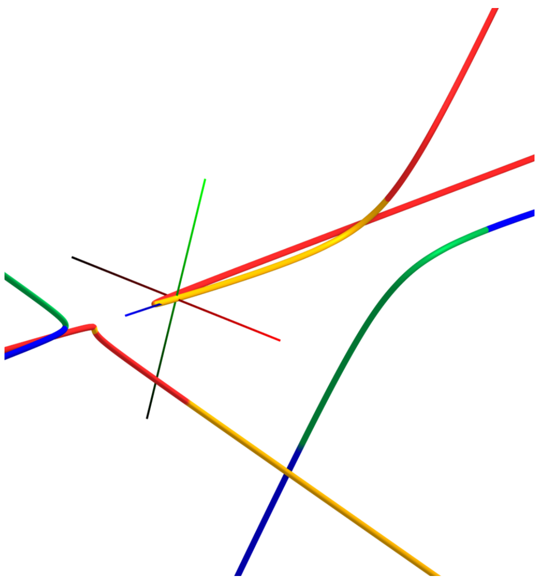
$$T_0 = \begin{bmatrix} -0.0712 & -0.0053 & -0.0683 \\ -0.0053 & 0.0060 & -0.0165 \\ -0.0683 & 0.0060 & 0.0652 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -2.6455 & -0.1284 & 1.4676 \\ -0.1284 & -1.2834 & 0.6693 \\ 1.4676 & -1.2834 & 3.9289 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.5925 & 2.9152 & -0.2388 \\ 2.9152 & 1.3966 & -1.0381 \\ -0.2388 & 1.3966 & -0.8041 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.2953 & 0.1556 & 0.0102 \\ 0.1556 & -0.1746 & 0.2295 \\ 0.0102 & -0.1746 & 0.4699 \end{bmatrix}$$

Figure 4.43: Two transition points each on two degenerate curves, one transition point each on two degenerate curves in scenario 7.



$$T_0 = \begin{bmatrix} 0.039117 & -0.006686 & 0.137009 \\ -0.006686 & 0.107261 & 0.005092 \\ 0.137009 & 0.005092 & -0.146378 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -2.394048 & -0.116195 & 1.328106 \\ -0.116195 & -1.161414 & 0.605600 \\ 1.328106 & 0.605600 & 3.555462 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.442778 & 2.827740 & -0.393000 \\ 2.827740 & 1.542000 & -1.032207 \\ -0.393000 & -1.032207 & -1.099222 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -0.084154 & 0.044343 & 0.002900 \\ 0.044343 & -0.049800 & 0.065421 \\ 0.002900 & 0.065421 & 0.133954 \end{bmatrix}$$

Figure 4.44: Three transition points on one degenerate curve, two transition points each on two degenerate curves and one transition point in one degenerate curve in scenario 7.

Number of transition points	**Two**	**Four**	**Six**	**Eight**	**Ten**
Possible Combinations	1,1,0,0 Figure 4.39	4,0,0,0 Not Found	6,0,0,0 Not Found	8,0,0,0 Not Found	10,0,0,0 Not Found
		3,1,0,0 Figure 4.40 Asymmetry with scenario 6	5,1,0,0 Not Found	7,1,0,0 Not Found	9,1,0,0 Not Found
		2,2,0,0 Not Found	4,2,0,0 Not Found	6,2,0,0 Not Found	8,2,0,0 Not Found
		2,1,1,0 Figure 4.41	4,1,1,0 Figure 4.42 Asymmetry with scenario 6	6,1,1,0 Not Found	8,1,1,0 Not Found
			3,2,1,0 Not Found	5,3,0,0 Not Found	7,3,0,0 Not Found
			2,2,1,1 Figure 4.43	5,2,1,0 Not Found	7,2,1,0 Not Found
				4,2,1,1 Not Found Asymmetry with scenario 6	6,4,0,0 Not Found
				3,3,2,0 Not Found	6,3,1,0 Not Found
				3,2,2,1 Figure 4.44 Asymmetry with scenario 6	6,2,1,1 Not Found
					5,5,0,0 Not Found
					5,4,1,0 Not Found
					5,3,2,0 Not Found
					5,2,2,1 Not Found
					4,4,1,1 Not Found
					4,3,2,1 Not Found
					3,3,2,2 Not Found

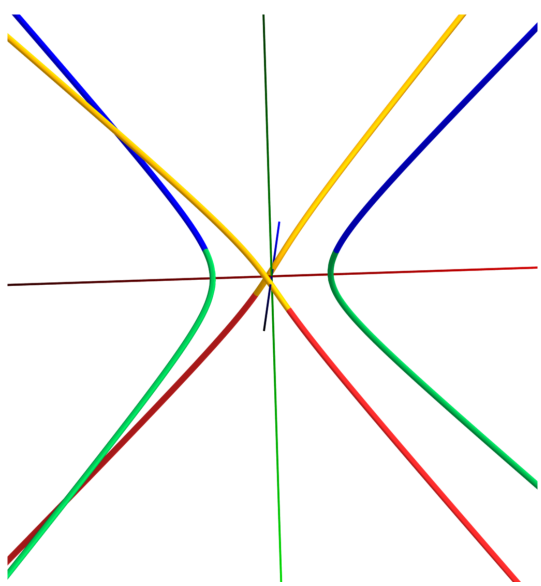
Table 4.8: Combination of different configurations in scenario 7.

4.2.8 Scenario 8: Four WT curves

In this scenario, there are four WT curves and each of the WT curve have at least one or odd number of transition point(s). Since there are even number of degenerate curves, there are even number of transition points and each curve will have at least one transition point on them (Figure 4.45). So, the minimum number of transition points in such cases are always four. Table 4.9 shows different configurations possible under this scenario.

Number of transition points	**Four**	**Six**	**Eight**	**Ten**
Possible Combinations	1,1,1,1 Figure 4.45	3,1,1,1 Figure 4.46	5,1,1,1 Not Found	7,1,1,1 Not Found
			3,3,1,1 Figure 4.47	5,3,1,1 Not Found
				3,3,3,1 Not Found

Table 4.9: Combination of different configurations in scenario 8.



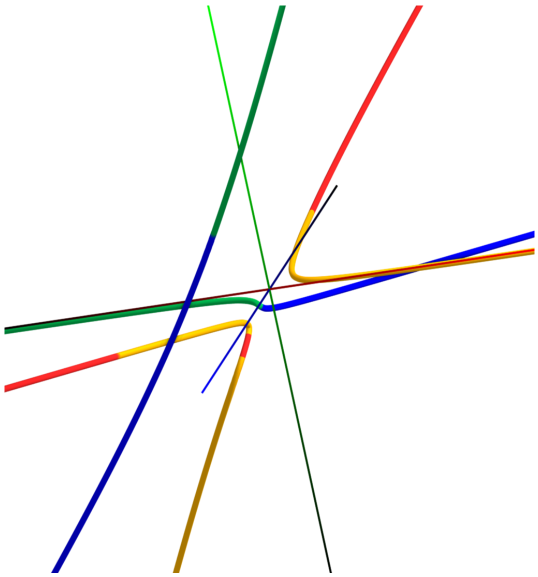
$$T_0 = \begin{bmatrix} 0.1000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.1000 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 \\ 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 \end{bmatrix}$$

$$T_y = \begin{bmatrix} 0.0000 & 0.0000 & 1.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 0.0000 & -0.0000 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \\ 0.0000 & 0.0000 & -0.0000 \end{bmatrix}$$

Figure 4.45: One transition point each on all four degenerate curves in scenario 8.



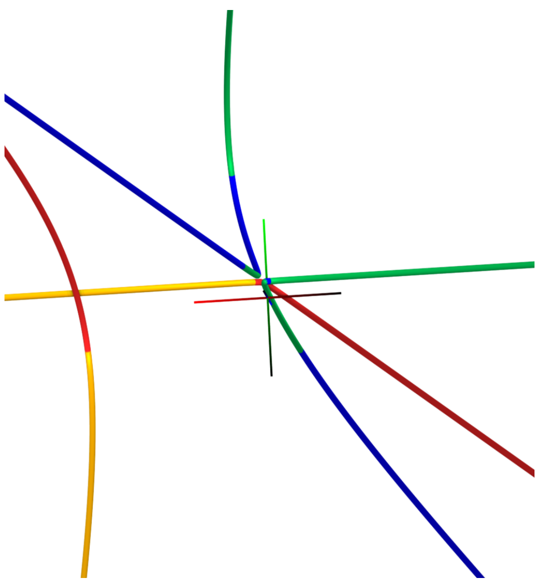
$$T_0 = \begin{bmatrix} -0.0637 & -0.0962 & -0.1312 \\ -0.0962 & -0.0483 & 0.0111 \\ -0.1312 & -0.0483 & 0.1121 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -0.3433 & -0.5896 & -3.0368 \\ -0.5896 & 1.9754 & -0.7279 \\ -3.0368 & 1.9754 & -1.6321 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.1180 & -0.8664 & 0.0275 \\ -0.8664 & -1.8729 & 1.9143 \\ 0.0275 & -1.8729 & 1.9909 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.5255 & 0.0557 & 0.1163 \\ 0.0557 & -0.3915 & -0.0771 \\ 0.1163 & -0.3915 & -0.1340 \end{bmatrix}$$

Figure 4.46: Three transition points on one degenerate curve and one transition point each on the other three degenerate curves in scenario 8.



$$T_0 = \begin{bmatrix} 0.275800 & 0.411167 & 0.338027 \\ 0.411167 & 0.018058 & 0.111983 \\ 0.338027 & 0.111983 & -0.293858 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -2.986146 & -5.128552 & -26.415173 \\ -0.5896 & 17.182735 & -6.331500 \\ -26.415173 & -6.331500 & -14.196589 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.609759 & -0.795132 & -0.132900 \\ -0.795132 & -1.857500 & 2.178074 \\ -0.132900 & 2.178074 & 2.467259 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.148799 & 0.016169 & 0.032596 \\ 0.016169 & -0.113324 & -0.023562 \\ 0.032596 & -0.023562 & -0.035475 \end{bmatrix}$$

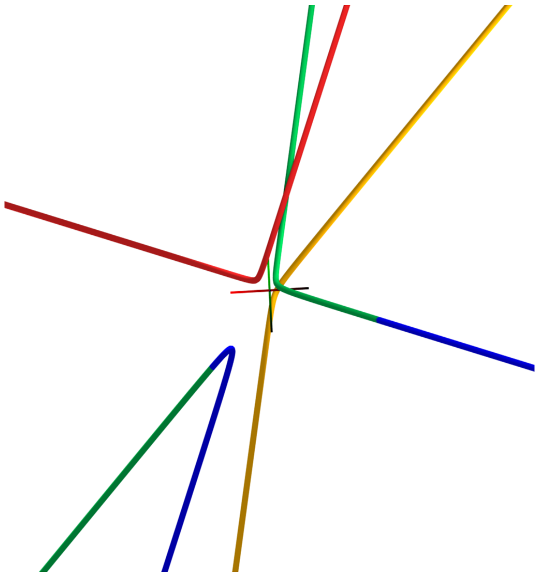
Figure 4.47: Eight transition points: Three transition points each on two degenerate curves and one transition point each of the other two degenerate curves in scenario 8.

4.2.9 Scenario 9: One WW curve, one TT curve and two WT curves

In this scenario, there are four degenerate curves: one WW curve, one TT curve and two WT curves. The two WW curves have zero or even number of transition point(s). There are even number of WT curves and each of these WT curve have odd number of transition points. The product of even number with odd number is always even. So two WT curve contributes to even number of transition points. As a result, in this scenario, we always have even number of transition points. Figure 4.51) shows the power of our interactive interface. With lack of navigability and cluttering in visualization, it is always difficult to see all transition points at once. The zoom and pan-in/pan out functionalities in the interface helps in visualizing cases when the degenerate curve does not fit in given viewport size. Table 4.10 shows all the possible configurations under this scenario.

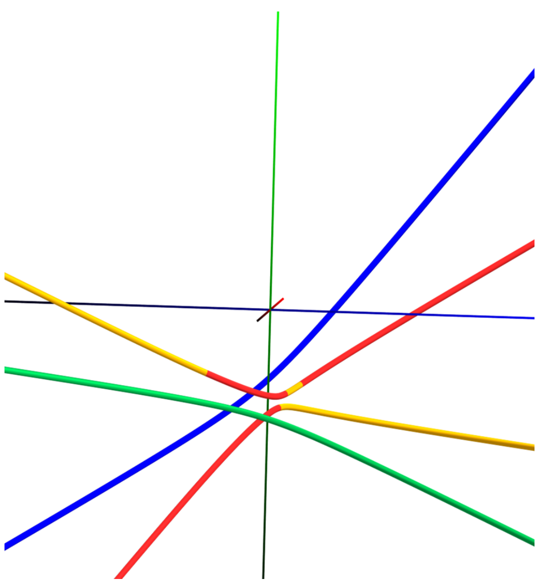
Number of transition points	**Two**	**Four**	**Six**	**Eight**	**Ten**
Possible Combinations	1,1,0,0 Figure 4.48	3,1,0,0 Figure 4.49	5,1,0,0 Not Found	7,1,0,0 Not Found	9,1,0,0 Not Found
		2,1,1,0 Figure 4.50	4,1,1,0 Figure 4.51	6,1,1,0 Not Found	8,1,1,0 Not Found
			3,3,0,0 Not Found	5,3,0,0 Not Found	7,3,0,0 Not Found
			3,2,1,0 Figure 4.52	5,2,1,0 Not Found	7,2,1,0 Not Found
			2,2,1,1 Not Found	4,2,1,1 Figure 4.53	6,3,1,0 Not Found
				3,3,2,0 Not Found	6,2,1,1 Not Found
				3,2,2,1 Figure 4.54	5,4,1,0 Not Found
					5,3,2,0 Not Found
					5,2,2,1 Not Found
					4,3,3,0 Not Found
					4,3,2,1 Not Found
					3,3,2,2 Not Found

Table 4.10: Combination of different configurations in scenario 9.



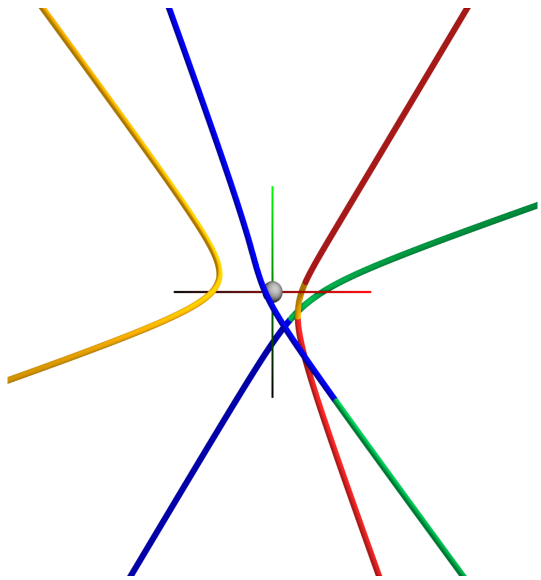
$$T_0 = \begin{bmatrix} -0.5728 & -0.4436 & -0.0024 \\ -0.4436 & 0.5732 & 0.2141 \\ -0.0024 & 0.5732 & -0.0004 \end{bmatrix}$$
$$T_x = \begin{bmatrix} 0.0170 & -0.9867 & 2.4017 \\ -0.9867 & -0.7061 & 2.2959 \\ 2.4017 & -0.7061 & 0.6891 \end{bmatrix}$$
$$T_y = \begin{bmatrix} 1.8382 & -1.7101 & 0.8759 \\ -1.7101 & -1.0792 & 0.0928 \\ 0.8759 & -1.0792 & -0.7590 \end{bmatrix}$$
$$T_z = \begin{bmatrix} 0.3460 & -0.4611 & -0.8414 \\ -0.4611 & -0.9169 & -0.0763 \\ -0.8414 & -0.9169 & 0.5709 \end{bmatrix}$$

Figure 4.48: One transition point each on two degenerate curves and zero transition point on the other two degenerate curves in scenario 9.



$$T_0 = \begin{bmatrix} 0.2118 & 0.5286 & -0.1267 \\ 0.5286 & -0.3995 & -0.2769 \\ -0.1267 & -0.3995 & 0.1877 \end{bmatrix}$$
$$T_x = \begin{bmatrix} 1.3756 & -0.0904 & 0.8231 \\ -0.0904 & 0.4304 & 0.0281 \\ 0.8231 & 0.4304 & -1.8060 \end{bmatrix}$$
$$T_y = \begin{bmatrix} -0.1087 & 1.8888 & -0.6382 \\ 1.8888 & -0.4513 & -1.3544 \\ -0.6382 & -0.4513 & 0.5600 \end{bmatrix}$$
$$T_z = \begin{bmatrix} -1.3906 & 0.2273 & -1.0665 \\ 0.2273 & -0.0239 & 1.4640 \\ -1.0665 & -0.0239 & 1.4145 \end{bmatrix}$$

Figure 4.49: Three transition points on one degenerate curve, one transition point on one degenerate curve and zero transition point on the other two degenerate curves in scenario 9.



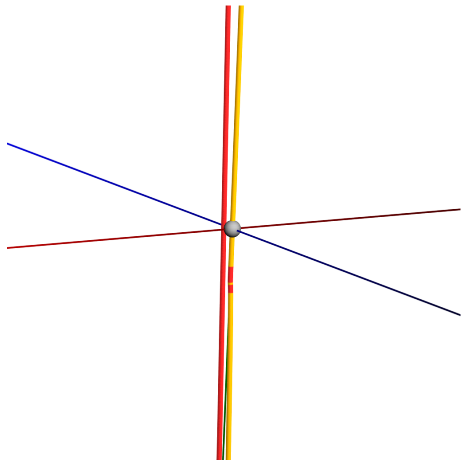
$$T_0 = \begin{bmatrix} -0.5728 & -0.4436 & -0.0024 \\ -0.4436 & 0.5732 & 0.2141 \\ -0.0024 & 0.5732 & -0.0004 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 0.0170 & -0.9867 & 2.4017 \\ -0.9867 & -0.7061 & 2.2959 \\ 2.4017 & -0.7061 & 0.6891 \end{bmatrix}$$

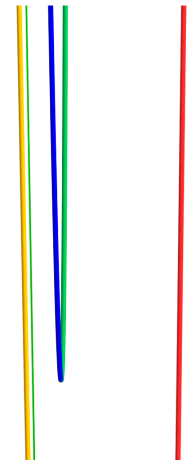
$$T_y = \begin{bmatrix} 1.8383 & -1.7101 & 0.8759 \\ -1.7101 & -1.0792 & 0.0928 \\ 0.8759 & -1.0792 & -0.7591 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.3460 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \\ 0.0000 & 0.0000 & -0.3460 \end{bmatrix}$$

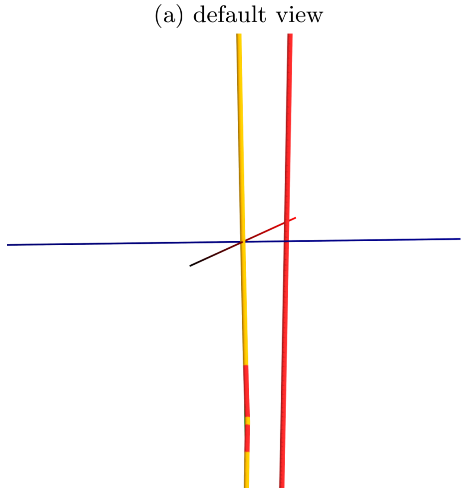
Figure 4.50: Two transition points on one degenerate curve, one transition point on two degenerate curves and zero transition point on one degenerate curve in scenario 9.



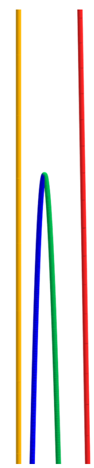
(a) default view



(b) North view



(c) center view



(d) South view

$$T_0 = \begin{bmatrix} 0.0132 & -0.0030 & 0.0361 \\ -0.0030 & 0.0303 & 0.0062 \\ 0.0361 & 0.0303 & -0.0435 \end{bmatrix}$$

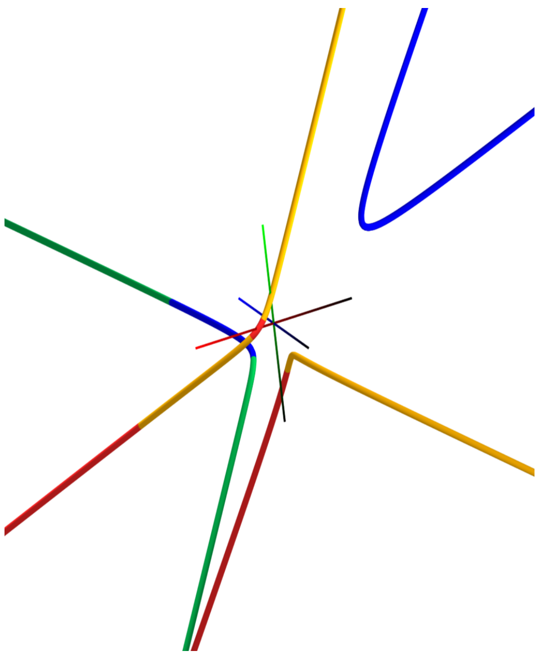
$$T_x = \begin{bmatrix} 25.7044 & -0.1015 & -1.2306 \\ -0.1015 & -2.5293 & -2.9795 \\ -1.2306 & -2.5293 & -23.1751 \end{bmatrix}$$

$$T_y = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.1000 \\ 0.0000 & 0.0000 & -0.0000 \end{bmatrix}$$

$$T_z = \begin{bmatrix} -2.0914 & 5.9905 & 0.4267 \\ 5.9905 & -6.8082 & -3.4781 \\ 0.4267 & -6.8082 & 8.8996 \end{bmatrix}$$

(e) Tensor values

Figure 4.51: Six transition points: Four transition point on one degenerate curve, one transition point each on two degenerate curves and zero transition point on one degenerate curve in scenario 9.



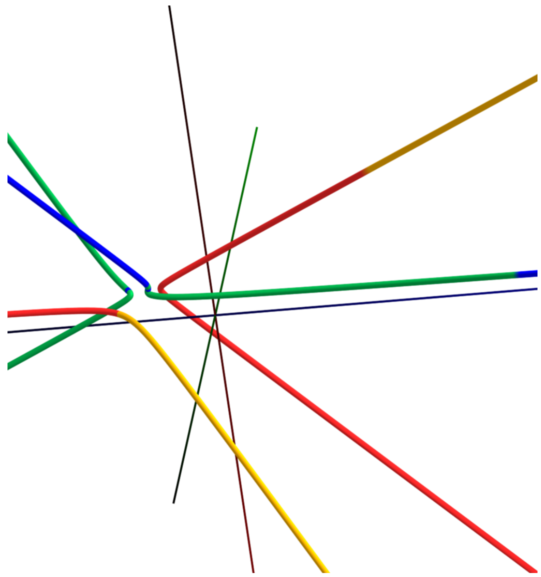
$$T_0 = \begin{bmatrix} 0.3595 & -0.0305 & 0.4040 \\ -0.0305 & 0.0716 & 0.0598 \\ 0.4040 & 0.0716 & -0.4311 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -2.2608 & -1.2583 & 0.3668 \\ -1.2583 & 0.7141 & 1.4100 \\ 0.3668 & 0.7141 & 1.5467 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.1393 & -0.5707 & 0.0859 \\ -0.5707 & 0.8933 & 1.7705 \\ 0.0859 & 0.8933 & -0.7540 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.9947 & -0.9103 & -0.1900 \\ -0.9103 & -0.2301 & 0.2761 \\ -0.1900 & -0.2301 & -0.7646 \end{bmatrix}$$

Figure 4.52: Six transition points: Three transition points on one degenerate curve, two transition points on one degenerate curve, one transition point on one degenerate curve and zero transition point on one degenerate curve in scenario 9.



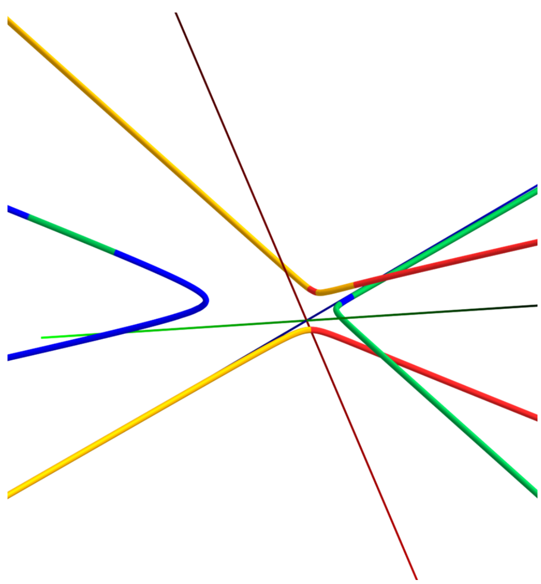
$$T_0 = \begin{bmatrix} 0.3595 & -0.1305 & 0.1404 \\ -0.1305 & -0.0284 & 0.3598 \\ 0.1404 & -0.0284 & -0.3311 \end{bmatrix}$$

$$T_x = \begin{bmatrix} 2.0977 & 0.6935 & 1.4243 \\ 0.6935 & 0.1628 & 1.5706 \\ 1.4243 & 0.1628 & -2.2605 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.3666 & -1.0240 & -0.6439 \\ -1.0240 & 0.2514 & 1.8586 \\ -0.6439 & 0.2514 & 0.1152 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.9947 & -0.9103 & -0.0900 \\ -0.9103 & -0.2301 & 0.2761 \\ -0.0900 & -0.2301 & -0.7646 \end{bmatrix}$$

Figure 4.53: Eight transition points: Four transition points on one degenerate curve, two transition points on one degenerate curve, one transition point each on two degenerate curves in scenario 9.



$$T_0 = \begin{bmatrix} -0.0405 & -0.0305 & 0.0404 \\ -0.0305 & 0.0716 & 0.0598 \\ 0.0404 & 0.0716 & -0.0311 \end{bmatrix}$$

$$T_x = \begin{bmatrix} -1.3608 & -1.2583 & 0.8668 \\ -1.2583 & 0.6141 & 1.6100 \\ 0.8668 & 0.6141 & 0.7467 \end{bmatrix}$$

$$T_y = \begin{bmatrix} -0.0803 & -0.5831 & 0.1212 \\ -0.5831 & 0.9727 & 1.7662 \\ 0.1212 & 0.9727 & -0.8924 \end{bmatrix}$$

$$T_z = \begin{bmatrix} 0.8947 & -0.9103 & -0.4214 \\ -0.9103 & -0.2301 & 0.2761 \\ -0.4214 & -0.2301 & -0.6646 \end{bmatrix}$$

Figure 4.54: Eight transition points: Three transition points on one degenerate curve, two transition points each on two degenerate curves, one transition point on one degenerate curve in scenario 9.

4.3 Conjectures

Based on our observation from nine scenarios covering all the possible configurations under structurally stable conditions, we have following results.

Conjecture 1. *Under structurally stable conditions, a linear tensor field has at most eight transition points.*

In all the scenarios presented above, we found the maximum number of degenerate points to be eight. Figure 4.55 shows all the scenarios having eight transition points.

On the other hand, the lower-bound on the number of transition points is same as the number of WT curves in a 3D linear tensor field. In all the cases shown in figure 4.56, we have zero number of WT curves and hence we have zero as the lower bound.

We also observed the maximum transition points on each of WW, TT and WT curves and we have following conjectures for them based on observation.

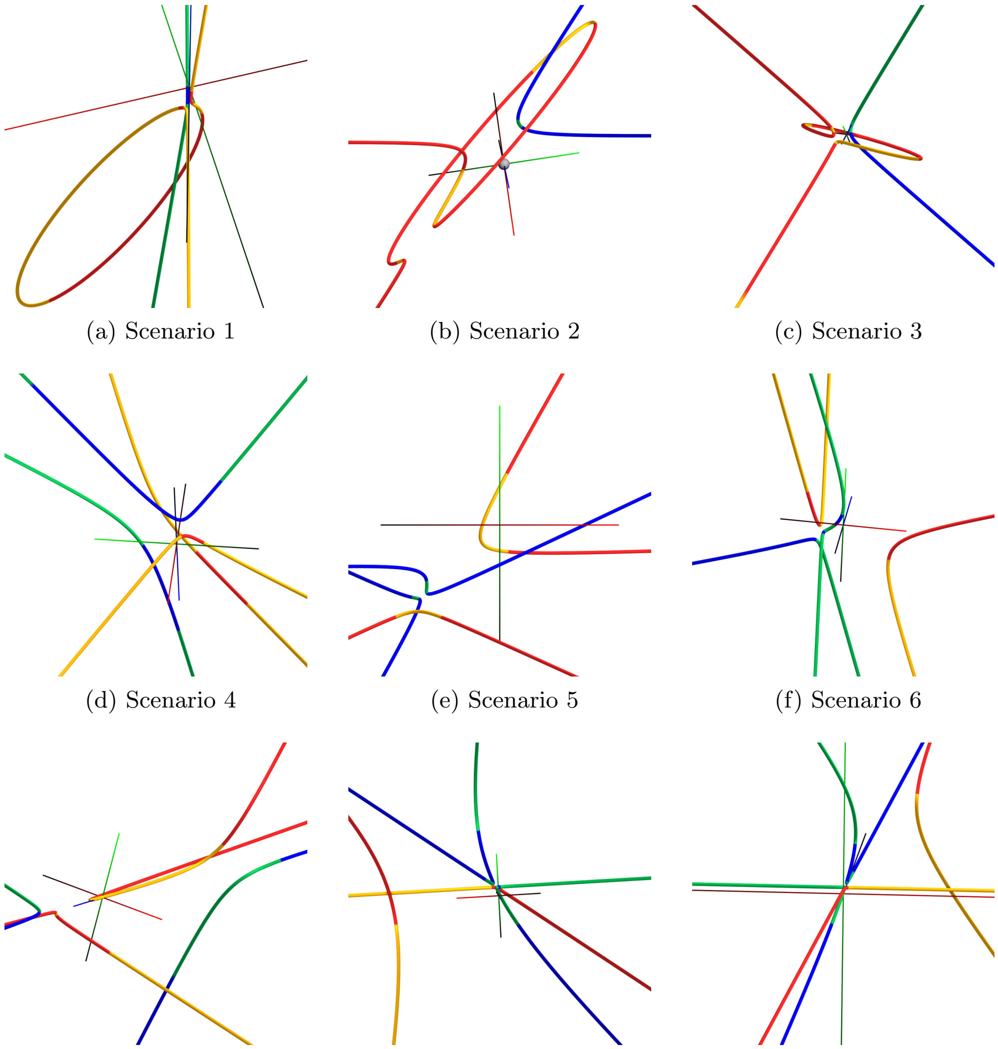
Conjecture 2. *Under structurally stable conditions, there are maximum of 6 transition points on a WW curve and 4 transition points on a WW curve, when there are two and four curves in a linear tensor field, respectively.*

The figure 4.57 shows the maximum number of transition points on WW curve.

Conjecture 3. *Under structurally stable conditions, there are maximum of 6 transition points on a TT curve and 4 transition points on a TT curve, when there are two and four curves in a linear tensor field, respectively.*

The figure 4.58 shows the maximum number of transition points on TT curve.

Conjecture 4. *Under structurally stable conditions, there are maximum of 7 transition points on a WT curve and 3 transition points on a WT curve, when there are two and four curves in a linear tensor field, respectively.*



(a) Scenario 1 (b) Scenario 2 (c) Scenario 3

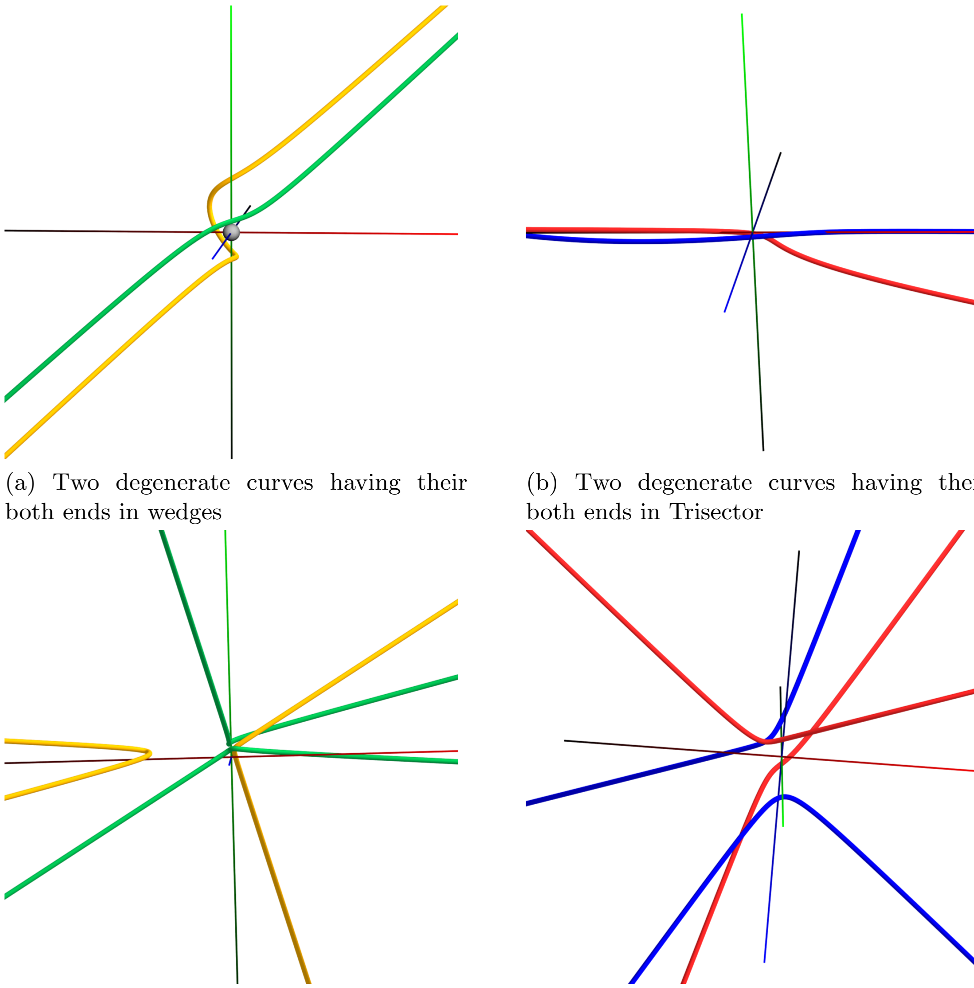
(d) Scenario 4 (e) Scenario 5 (f) Scenario 6

(g) Scenario 7 (h) Scenario 8 (i) Scenario 9

Figure 4.55: Eight transition.

The figure 4.59 shows the maximum number of transition points on WT curve.

Our analysis further reveals that a linear tensor field can have at most 7 transition point on a single degenerate curve when there are only two degenerate curves and 4



(a) Two degenerate curves having their both ends in wedges

(b) Two degenerate curves having their both ends in Trisector

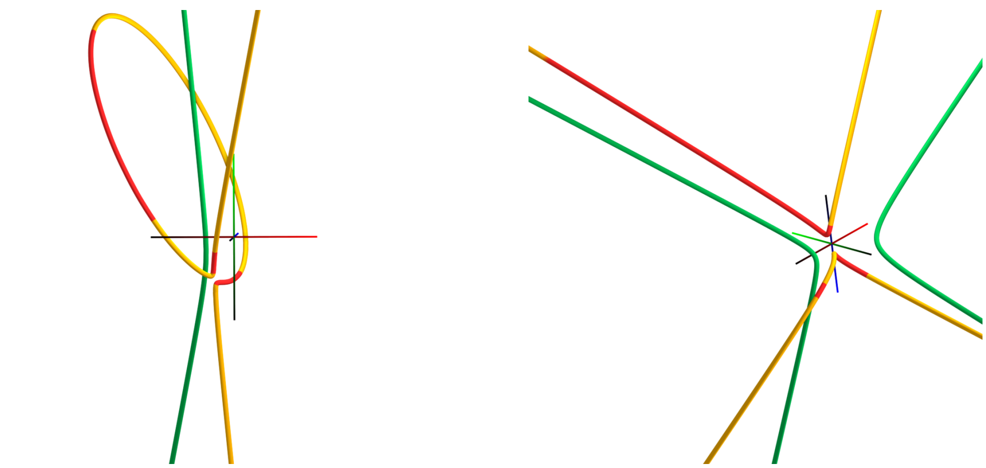
(c) Four degenerate curves having their both ends in wedges

(d) Four degenerate curves having their both ends in wedges

Figure 4.56: Zero transition.

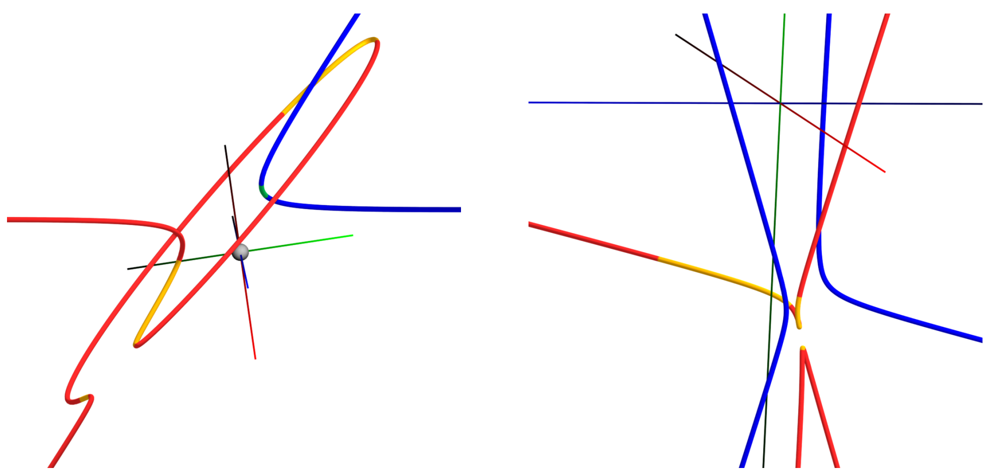
transition points on a single degenerate curve when there are four degenerate curves.

Conjecture 5. *Under structurally stable conditions, a linear tensor field can have at most 7 transition points on a single degenerate curve only when it contains only two degenerate curves and at most 4 transition points on a single degenerate curve when it contains four degenerate curves.*



(a) Two degenerate curves (b) Four degenerate curves

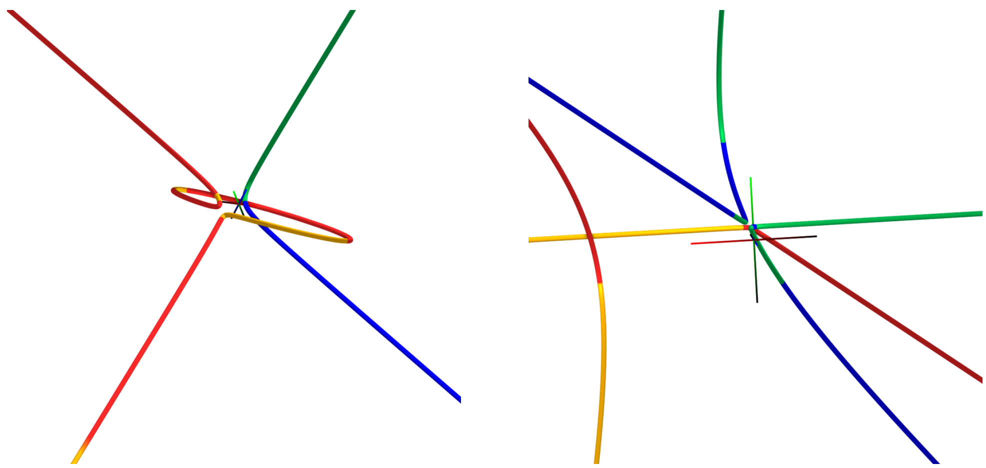
Figure 4.57: Maximum number of transition points on a WW curve.



(a) Two degenerate curves (b) Four degenerate curves

Figure 4.58: Maximum number of transition points on a TT curve.

Figure 4.20 and figure 4.27 are the basis of our above conjecture.



(a) Two degenerate curves (b) Four degenerate curves

Figure 4.59: Maximum number of transition points on a WT curve.

Chapter 5: Conclusions and Future Work

In this thesis, we provided an interactive interface to change different design parameters and generate any 3D linear symmetric tensor fields. The interface provides both geometric as well as algebraic approach to design 3D linear tensor fields. This helped us in studying the number of transition points both on a degenerate curve and in a 3D linear tensor field. We first showed that there are 20 transition points in a linear tensor field under structurally stable conditions. Moreover, we observe that there are at most 8 transition points in a linear tensor field. There are maximum of 6 transition points on a WW curve when there are two degenerate curves and maximum of 4 transition points on a WW curve when there are four degenerate curves. Also, we found that the maximum number of transition points on a TT curve, is 6 when there are two degenerate curves and, is 4 when there are four degenerate curves. When there are two degenerate curves in a linear tensor field, there are maximum 7 transition points on WT curve and there are maximum of 3 transition points on a WT curve when there are four degenerate curves in a linear tensor field. The lower bound on the number of transition points is same as the number of WT curves in a linear tensor field. Also, there are an odd number of transition points on a WT curve and an even number of transition points on WW or TT curves.

In future, we plan to find a tighter upper bound on the number of transition points in a 3D linear symmetric tensor field and also visualize all of them if there exist such an upper bound. Also, we plan to study bifurcation using our interactive interface in a 3D linear tensor field. Also we would like to automate these testing to get all the stable configurations which we were unable to find during our analysis.

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