ARTIFICIAL SATELLITE LIFE DURATION
IN NEAR-EARTH ORBITS

by

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CONTENTS

SYMBOLS

a	semimajor axis
A	area projected to free stream
$\vec{A}$	satellite acceleration
b	constant (defined on page 27)
$\vec{B}$	vector constant (defined on page 27)
c	most probable molecular speed
C_d	drag coefficient
$\vec{C}$	vector constant (defined on page 28)
d	distance from focus to directrix of ellipse
$\vec{D}$	drag vector

D-D' directrix

e $|\hat{E}|$

$\hat{E}$ vector constant (defined on page 29)

f drag on satellite

F force on satellite

F focus of ellipse

g local acceleration of gravity

$\tilde{G}$ gravity force

h height measured from earth's surface (20,891,199.6 ft from earth's center)

H dissociated hydrogen

K universal gravitation constant ($K = 3.44032 \times 10^{-8}$ ft^3/slugs sec^2)

K_1 constant (defined on page 44)

°K degrees Kelvin

L characteristic dimension

m satellite mass

M mass of earth ($M = 4.0919 \times 10^{13}$ slugs)

N dissociated nitrogen

$\tilde{N}$ unit vector in the direction of the principal normal to the satellite flight path

N_2 molecular nitrogen

O dissociated oxygen

O_2 molecular oxygen

p pressure

P period of revolution

r magnitude of radius vector from earth center to satellite

r_1 radial distance (defined on page 24)

r_a radius to satellite at apogee

r_p radius to satellite at perigee

R universal gas constant (6.13 ft-lb/gm-cm°K)

$\bar{R}$ radius vector from earth center to satellite

$\bar{R}_1$ unit vector in direction of $\bar{R}$

s arc length along satellite flight path

t time

T temperature

$\bar{T}$ unit vector tangent to satellite flight path

U total energy of satellite

v speed of satellite

v_o initial perigee speed

v_1 second perigee passage speed

v_p speed at perigee

$\bar{V}$ velocity of satellite

$\bar{V}_r$ velocity of atmosphere due to atmospheric rotation

$\bar{V}_t$ thermal velocity of atmospheric molecules

β defined by $\beta = \frac{g}{RT}$ (slope of logarithmic density curve)

δ density of atmosphere

δ_o initial value of density (defined on page 9)

δ_r reference density (defined on page 38)

δ_P atmospheric density at perigee

α angle measured between $\bar{R}$ and the fixed direction of $\bar{E}$

ϵ eccentricity of satellite orbit

θ angle measured between ray from focus to perigee and focus to satellite (see Figure 4.3)

λ mean free path of gas molecule

μ molecular weight (atomic mass units per molecule, 1 AMU = 1.137 X 10^{-28} slugs)

ρ radius of curvature of satellite flight path

ϕ included angle between unit radius and unit tangent vector (see Figure 4.1)

ARTIFICIAL SATELLITE LIFE DURATION IN NEAR-EARTH ORBITS

INTRODUCTION

Up to the present time, the majority of artificial earth satellites have had orbits which could be considered to be, at least partially, within the earth's 1000 mile thick atmosphere. Future satellites may also occupy this region of space for reasons of desirability or necessity. For example, if no provision for shielding is provided for the manned observation satellite, rather restrictive limits are imposed on the selection of operating altitudes by the Van Allen radiation belts. The region of space, from sea level up to 400 miles in height above the surface of the earth and latitudes less than 60°, is relatively free of radiation. The next radiation free region is above 30,000 miles (17, p. 4). This region is much higher than would be desired for an observation satellite.

If consideration is given to the flight mechanics of an earth satellite in orbit within the earth's atmosphere, then account must be made for the small but important drag force which will be present. This force, when integrated over long periods of time, is sufficient to force the satellite to descend back toward the earth.

This thesis contains an analysis of satellite motion with drag and a solution to the corresponding differential equation of motion. The resulting solution allows prediction of expected satellite lifetimes. As pointed out above, the lifetimes at "low" altitudes are of interest. For this reason, results are presented for orbits with perigee heights less than 600 miles and eccentricities between 0 (circular orbits) and 0.2 (elliptical orbits).

An exact solution to the equations of satellite dynamical motion, with drag, is extremely difficult to obtain. The variables involved are not amenable to exact description. For example, the upper atmospheric density depends to an extent on the latitude, the time of year, the time of day, and the state of the atmospheric tide. Since an exact solution is virtually unobtainable (at lease in closed form), an approximate solution is practical and may be carried out by making a number of simplifying assumptions. This approach is used in this thesis.

A study is made of the variables involved with the problems of satellite engineering. There are many assumptions that can be made in connection with these variables and still allow a sufficiently accurate solution. A relation expressing drag in terms of velocity and density is derived for use in the differential equation of motion. A new model atmosphere is then developed to relate atmospheric density with height (up to 600 miles). This model is used in the drag relation.

The solution to the controlling differential equation of motion is carried out completely for the simplified case of a circular orbit. The result is an equation relating orbital lifetime to the other controlling variables. The solution is then generalized to include elliptical orbits and a comparison is carried out between predicted lifetimes and actual lifetimes of the present day satellites. This serves to confirm the accuracy of the solution.

SOME IMPORTANT CONSIDERATIONS

DRAG

It is a well established fact that the atmospheric density monotonically decreases with altitude above the earth's surface until some vague point often referred to as the edge or outer bound of the atmosphere is reached. Cis-lunar space extends beyond this altitude and remains at a more or less constant density of 1.42×10^7 particles/ ft^3.

Any satellite orbiting within this sphere of atmospheric gas will be influenced by it in the form of drag. The atmospheric density at high altitudes is extremely small compared to sea level values. Over a long period of time, the integrated effect of drag will be sufficient to gradually reduce the orbital altitudes to the more dense regions of the atmosphere and finally terminate the initially established orbit.

The atmospheric drag force depends upon the type of flow that a satellite will experience. Classical aerodynamics is based on the assumption of continuum flow. A body placed in this type of fluid flow will influence not only the molecules that actually impinge upon its surface, but will also influence the molecules in the vicinity of the body. This results from the numerous collisions between the molecules themselves. The aerodynamic forces and flow patterns result from a collective flow behavior of the gas. As might be expected, however, the continuum concept is not valid at satellite altitudes. This is simply because the extremely low density at these heights vastly decrease the number of collisions between the gas molecules themselves. The gas in the vicinity of a body is practically unaffected by its presence. The criteria for a continuum is that $L/\lambda >> 1$, where L is a characteristic dimension associated with the aerodynamic body, usually the boundary layer thickness, and λ is the mean free path of the gas molecule (15, p. 56-58). Noncontinuum flow ($L/\lambda < 1$) is usually referred to as free molecule flow. The dynamical aspects of free molecule flow are investigated using the methods of kinetic theory.

Kinetic theory treats the interactions of the gas molecules with the immersed body surface in accordance with the conservation laws of classical mechanics (momentum and energy). Statistical methods are used in the theory; this presupposes that the actual behavior of gas is equivalent to the average behavior. The model gas molecule is analyzed in this theory by giving it the characteristics of a small rigid elastic sphere. For free molecule flow, collisions between gas molecules are ignored since these are few when compared with collisions against the aerodynamic body.

Internal energy of the gas is usually accounted for by assuming that the gas has a Maxwellian distribution of thermal velocity. However, at satellite speeds the internal energy of the gas may be neglected. For molecular speed ratios greater than about six, the contribution of internal energy to drag force is very small. (The molecular speed ratio is the ratio of the relative speed between the gas and satellite to the most probable molecular speed). A satellite orbiting at a height of 400 miles has a molecular speed ratio greater than six.

In addition to the random thermal motion mentioned, the atmospheric gas molecule will have an ordered mass motion. This component of velocity arises because the viscous nature of the earth's atmosphere requires that, without external interaction, the entire atmosphere rotate with the same angular velocity as the earth. The presence of this velocity, theoretically, should be accounted for in the equation of motion.

Velocities expressed in the equation of motion are measured with respect to a Newtonian frame of reference. Thus, if the satellite is traveling at a velocity $\bar{V}$, it will have a relative velocity with respect to the gas molecules of $\bar{V} - (\bar{V}_r + \bar{V}_t)$ where $\bar{V}_r$ is the velocity of the gas molecule due to atmospheric rotation and $\bar{V}_t$ is the thermal velocity (negligible at satellite speeds). A comparison of calculated characteristics with the orbital characteristics of Explorer IV show that a 5 per cent error is introduced by not accounting for atmospheric motion due to the earth's rotation (14, p. 19). Rough estimates of this effect are in agreement with this percentage. Since this effect is small, it will be neglected in the analysis. Both $\bar{V}_r$ and $\bar{V}_t$, then, are set equal to zero and are not considered in the derivation of the equation of motion.

The drag relation for flow about a satellite can now be developed using the concepts and methods of kinetic theory. Consider a satellite traveling through the upper atmosphere. A drag force f will be imparted to the satellite by the gas molecules as they collide with it. The magnitude of this force is equal to the time rate of change of the gas momentum. Since both the thermal and atmospheric motion of the gas may be neglected, the satellite is essentially encountering molecules at rest. This is equivalent to having the dynamical situation of a uniform stream of molecules striking the satellite surface with a speed v. The total momentum imparted to the satellite will depend upon how the gas molecules are reflected from the surface.

It has been shown by Millikan (7, p. 226) that the fraction of molecular flow reflected specularly (mirror like) is extremely small, less than 3 to 10 per cent for most materials and surface conditions. This follows from the fact that structural surfaces, though highly polished, are considered rough on the molecular scale. The surface projections (roughness) are large compared to the characteristic diameter of the impinging molecules. The initial molecular contact is likened to throwing tennis balls into the mouth of a cave. All the translational energy is removed while the molecule temporarily remains in contact with the surface. During this contact previous directional history is erased and later the molecule is re-emitted randomly with a thermal energy corresponding to the surface temperature of the vehicle. This type of reflection is known as diffuse reflection.

The expression for drag force is (Newton's second law)

$$f = (\delta A v)\Delta v \qquad (1.1)$$

where (δAv) is the mass of gas that collides with the satellite per second and $\Delta v = v_{initial} - v_{final}$. A 100 per cent diffuse reflection is assumed by setting $\Delta v = v$. Substituting this quantity into equation (1.1) gives

$$f = \delta A v^2 \tag{1.2}$$

Equation (1.2) is identical with a relation derived by Sir Isaac Newton for drag on a flat plate oblique to the flow. It is interesting to note that years ago engineers had erroneously applied this relation to calculate wind loads on buildings and structures. Their answers were always somewhat high since more recent work has shown that equation (1.2) is not valid for continuum type flow.

The classical expression for drag force on a body immersed in a fluid is

$$f = \frac{1}{2} C_d \, \delta A v^2 \tag{1.3}$$

Equating equations (1.2) and (1.3) gives a drag coefficient of two for free molecule flow. This result is in agreement with experiment as shown in Figure 1.1 (11, p. 345). This figure indicates the thermal effects at lower molecular speed ratios by the rise of the curve. Corresponding curves for flat plates (13, p. 42) and cones (16, p. 40) give virtually the same results. These results imply that equation (1.2) is independent of body shape.

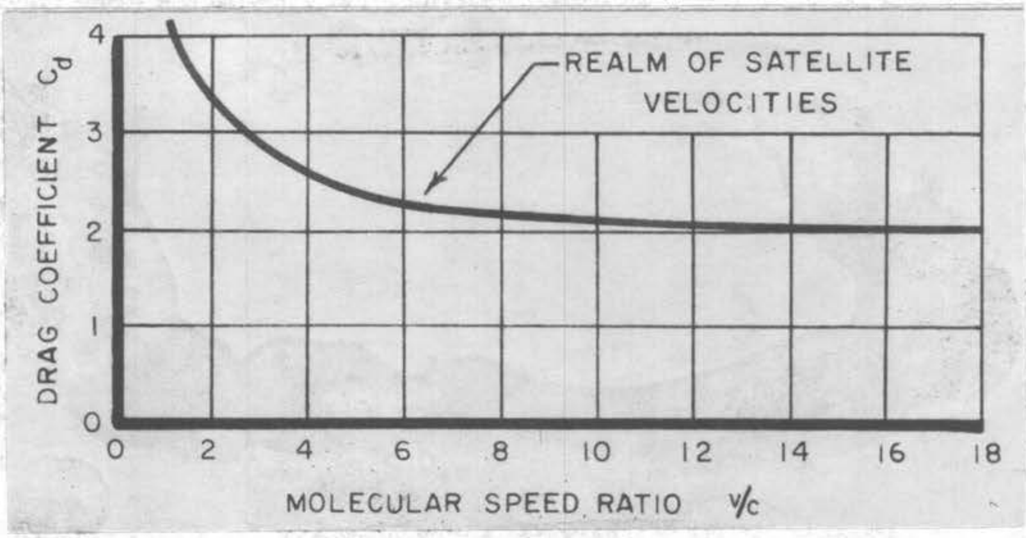


Figure 1.1

ATMOSPHERIC DENSITY VARIATION

In order to successfully use equation (1.2) to compute drag, it is necessary to have a relation for the density δ in terms of the altitude h. This type of relationship may be theoretically derived and is known as the barometric formula. The derivation of this formula assumes that the earth's gravitational field, the atmospheric temperature, and atmospheric composition to be uniform with height. Consider a vertical column of gas of unit cross-sectional area extending from $h = 0$ at the earth's surface to an indefinite height. If this gas is a continuous fluid of mass density δ in hydrostatic equilibrium, then the layer of fluid between h and $h + dh$ is subject to a pressure force $p(h)$ on its lower surface and $p(h + dh) = p(h) + dp$ on its upper surface. Let g be the local acceleration of gravity, then,

$$-dp = \delta g dh \tag{1.4}$$

Assuming the perfect gas law by inserting $dp = d\delta \; RT$ gives

$$-\frac{d\delta}{\delta} = \frac{gdh}{RT} \tag{1.5}$$

Integration of the density in equation (1.5) is carried out between the initial density δ_o (h = 0) and the density at height h.

$$\delta = \delta_o e^{-\frac{gh}{RT}} \tag{1.6}$$

Equation (1.6) will represent the earth's density variation only to the extent in which the assumptions are valid. Obviously neither the gravitational acceleration g nor the temperature T remains constant with height. The gas constant R will change as the composition of the atmosphere changes. Despite these variations, however, equation (1.6) is still useful in that it gives the proper relationship between the various parameters.

The following table illustrates the variation of average atmospheric composition and temperature with altitude as given in reference (1, p. 114).

From sea level to 60 miles, the composition of the atmosphere and the gravitational acceleration remain essentially constant. Equation (1.6) is applied to this region of space by assuming an average value for the temperature.

(Composition by % Volume)

Height Miles	O_2	O	N_2	N	H	Molecular Weight μ	Temp. °K
0	21	0	79	0	0	28.8	288
45	21	0	79	0	0	28.8	219
60	20	1	79	0	0	28.7	230
70	12	13	75	0	0	26.9	260
80	5	26	69	0	0	25.1	300
95	4	26	65	5	0	24.4	450
125	2	46	41	11	0	21.0	700
155	1	59	24	16	0	18.6	800
190	0.5	66	13	20	0.5	17.3	900
250	0	69	3	27	1	15.6	1000
Above	0	69	0	30	1	15.32	1000+

From 60 to 250 miles, the atmosphere is partially dissociated. This results in a variable molecular weight. The change in this quantity and the temperature variation renders equation (1.6) useless for this region. Fortunately, sufficient data have been compiled during the International Geophysical Year to establish the density variation of the atmosphere in this region. Figure 1.2 is representative of the current IGY data (18, p. 3).

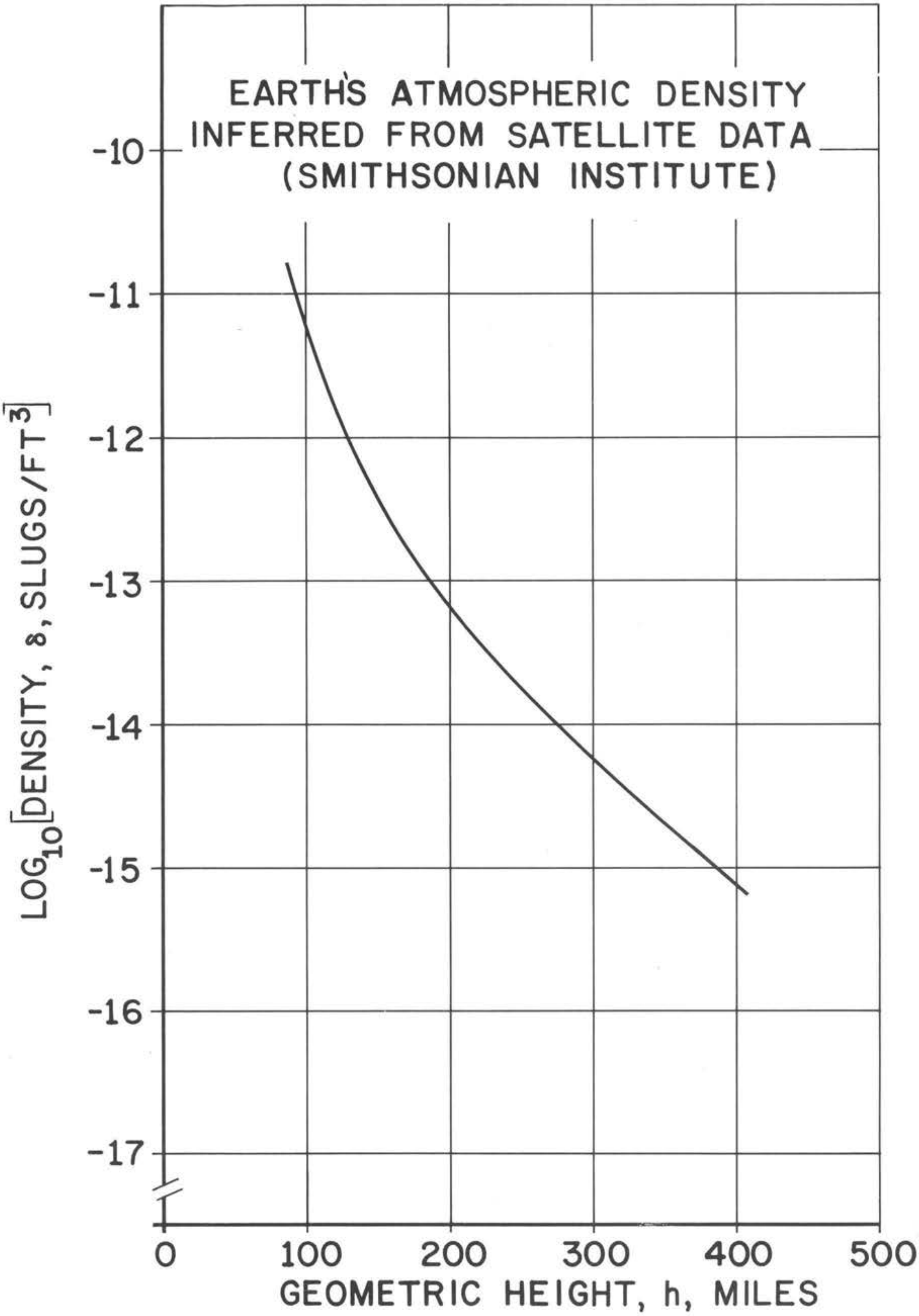

EARTH'S ATMOSPHERIC DENSITY
INFERRED FROM SATELLITE DATA
(SMITHSONIAN INSTITUTE)
-10
-11
-12
-13
-14
-15
-16
-17
LOG10[DENSITY, δ, SLUGS/FT3]
0
100
200
300
400
500
GEOMETRIC HEIGHT, h, MILES

FIGURE 1.2

Beyond 250 miles the atmospheric composition is completely dissociated and the molecular weight once again becomes constant. The temperature continues to increase but not as rapidly as before. Equation (1.6) is used to determine the density variation from 250 to 600 miles. The gravitational acceleration is averaged between these altitudes and a constant average temperature of 1140°K is used. The atmospheric density variation with altitude is constructed by dividing the atmosphere into three space regions and analyzing each region separately.

Region 1: From sea level to 60 miles the atmosphere is assumed to consist of air ($\mu = 28.8$) at an average temperature of 249°K with a sea level density of 2.378×10^{-3} slugs/ft^3. The average gravitational acceleration for this region is 31.7 ft/sec^2. Using these values in equation (1.6) gives:

$$\delta = 2.378 \times 10^{-3} \, e^{-4.093 \times 10^{-5} h} \tag{1.7}$$

This relation generates densities that are very close in value to the well known ARDC model atmosphere which has been used up to altitudes of 80 miles successfully for many years.

Region 2: Between the heights of 60 and 250 miles, the atmospheric density is assumed to correspond to the IGY data given in Figure 1.2.

Region 3: From 250 to 600 miles, the atmosphere is assumed to be composed of completely dissociated oxygen, nitrogen, and hydrogen

($\mu = 15.32$) at an average temperature of 1140°K. Using an initial density, δ_o, of 3.17×10^{-12} slugs/ft^3 matches equation (1.6) to the region two data. The average gravitational constant is equal to 26.2 ft/sec^2. Using the above values in equation (1.6) gives:

$$\delta = 3.17 \times 10^{-12} e^{-3.93 \times 10^{-6} h} \tag{1.8}$$

Extrapolated calculations using equation (1.8) show that the density of cis-lunar space is reached at 1000 miles. This altitude then corresponds to the "edge" of the atmosphere.

Figure 1.3 depicts the complete atmospheric density variation.

SATELLITE MOTION

In addition to the drag force just described, an earth satellite will be influenced by the earth's gravitational field and to a lesser extent by the gravitational fields of the moon, sun, and planets. The earth's gravitational equipotential surfaces are not spherical, but assume the shape of a geoid very closely. Any perturbations the non-spherical character of this field induces are referred to as oblateness effects. These effects decrease as the orbital altitude is increased.

The gravity and drag forces, assumed to be the only ones acting on the satellite, combine to determine the motion of the satellite according to Newton's laws. This motion can generally be thought to consist of three phases.

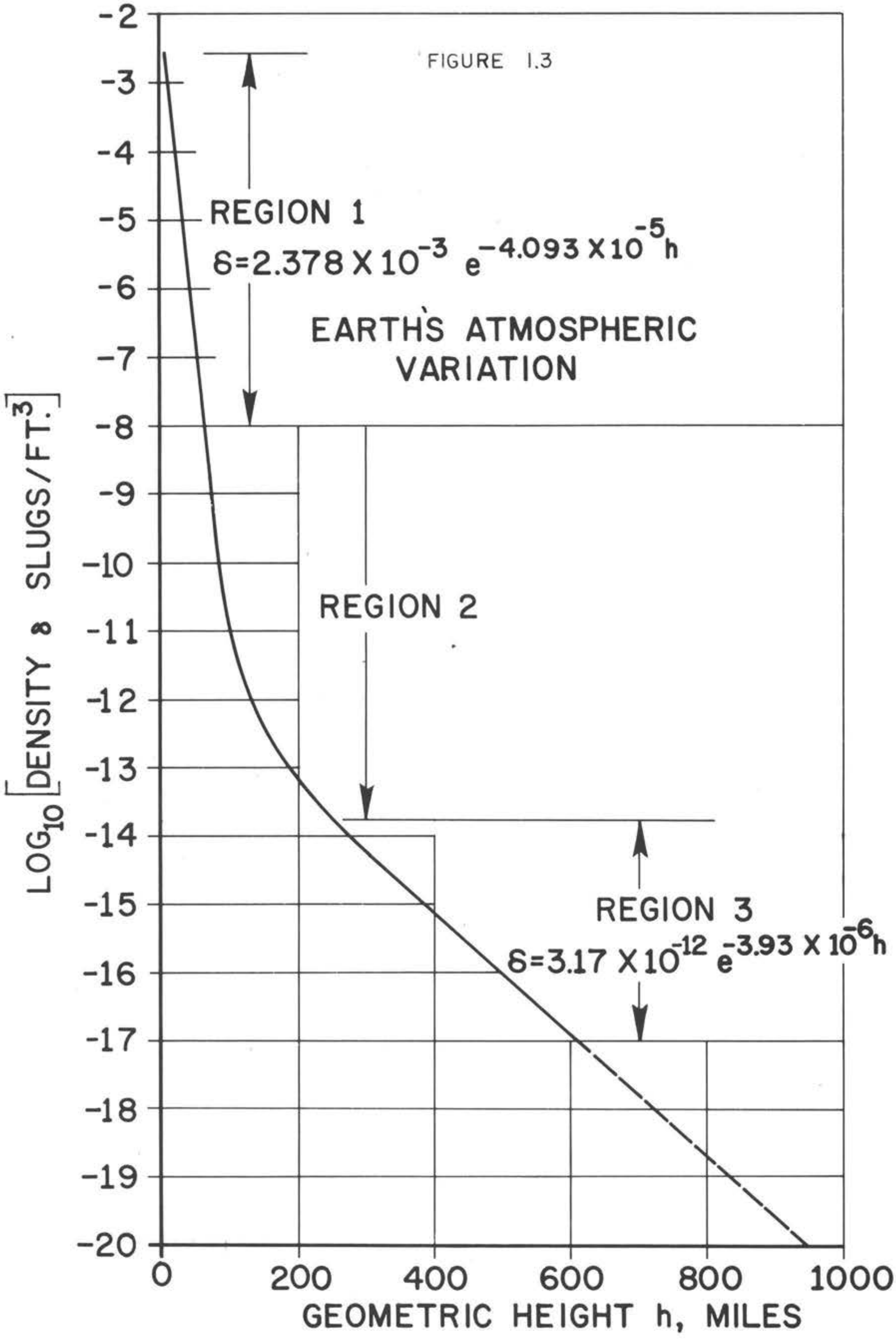


FIGURE 1.3
REGION 1
δ=2.378 X 10^-3 e^-4.093 X 10^-5 h
EARTH'S ATMOSPHERIC
VARIATION
REGION 2
REGION 3
δ=3.17 X 10^-12 e^-3.93 X 10^-6 h
LOG10 [DENSITY δ SLUGS/FT.^3]
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12
-13
-14
-15
-16
-17
-18
-19
-20
0
200
400
600
800
1000
GEOMETRIC HEIGHT h, MILES

According to Nielsen (9, p. 6-7), a satellite originally moves in an almost elliptical path depending on the initial orbit eccentricity and height. However, in the first phase called circularization of the ellipse, the perigee altitude decreases slowly while the apogee altitude decreases more rapidly. The circularization of elliptical orbits has been studied by Henry (4, p. 21-24) and others using energy methods. These studies show that the altitude at perigee decreases less rapidly than at apogee until a nearly circular orbit is achieved.

The next phase of the trajectory is termed spiral decay. In this phase the distance of the satellite from the center of the earth is a monotonically decreasing function of time. During this phase the satellite is traveling at nearly circular orbital speed, but with its angular momentum decreasing slowly because of drag.

In the final or terminal phase, the air drag becomes so important that the satellite turns downward into the lower atmosphere and reaches the earth in a small part of a revolution. During this phase aerodynamic heating is important and the path is not even approximately elliptical.

EARTH SATELLITE LIFETIMES

EQUATION OF MOTION

The fundamental relation underlying the behavior of earth satellites is given by Newton's second law. The resultant force acting on a body equals the time rate of change of its momentum. In symbols,

$$\sum \bar{F} = \frac{d}{dt} (m\bar{V}) \tag{2.1}$$

The mass of an earth satellite will remain essentially constant so that equation (2.1) may be rewritten as

$$\sum \bar{F} = m \frac{d\bar{V}}{dt} \tag{2.2}$$

Only two forces will be assumed to act on the satellite. They are the atmospheric drag force and the earth's gravitational pull. The small perturbing forces due to the gravitational fields of the sun and planets are neglected, as well as the perturbation forces caused by the oblateness effect of the earth. The satellites are reckoned to be close enough to the earth to neglect external gravitational fields, yet far enough away so that the earth's gravitational field "looks" spherical.

Under these conditions, the gravitational force $\bar{G}$ will act along the radius vector $\bar{R}$ and the drag force will act along the tangent to the flight path. These forces are depicted in Figure 2.1.

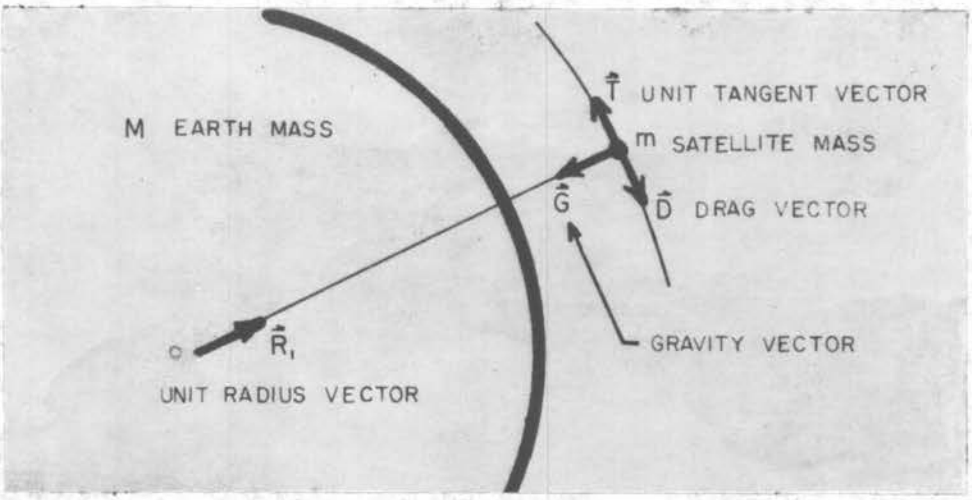


Figure 2.1

The gravitational force $\vec{G}$ is given by Newton's Law of Gravitation. It is equal in magnitude to $\frac{KMm}{r^2}$, where K is the universal gravitation constant, M the mass of the earth, m the mass of the satellite, and r the magnitude of the radius vector $\vec{R}$. The drag force is given by the relation, $\frac{1}{2} C_d \delta A v^2$, where C_d is the drag coefficient previously computed by methods of kinetic theory to be equal to two. The atmospheric density δ is a function of height shown in Figure 1.3. The area A is the cross-sectional area of the satellite presented to the free stream, and v is the relative scalar speed between the satellite and the oncoming molecules.

The sum of the forces acting on the satellite is given as

$$\sum \vec{F} = - \frac{KMm}{r^2} \vec{R}_1 - \delta A v^2 \vec{T} \tag{2.3}$$

The positive unit radius vector $\vec{R}_1$ is directed outward from the center of the earth and the positive unit tangent is directed along the flight path in the direction of the velocity vector.

The radius vector $\bar{R}$ has magnitude r and is in the direction of the unit vector $\bar{R}_1$. In vector notation,

$$\bar{R} = r\bar{R}_1 \tag{2.4}$$

The radius vector $\bar{R}$ is a function of arc length along the path so that

$$\dot{\bar{R}} = \frac{d\bar{R}}{dt} = \frac{d\bar{R}}{ds}\frac{ds}{dt} \tag{2.5}$$

The scalar velocity v equals $\frac{ds}{dt}$ and the unit tangent vector $\bar{T}$ equals $\frac{d\bar{R}}{ds}$ (2, p. 90). Thus,

$$\dot{\bar{R}} = v\bar{T} = \bar{V} \tag{2.6}$$

The first time derivative of $\bar{R}$ is the vector velocity $\bar{V}$. It is tangent to the path and has magnitude equal to the speed. The second time derivative of $\bar{R}$ gives

$$\ddot{\bar{R}} = v\,\frac{d\bar{T}}{dt} + \frac{dv}{dt}\,\bar{T} \tag{2.7}$$

The unit tangent is a function of arc length along the path. Thus,

$$\frac{d\bar{T}}{dt} = \frac{d\bar{T}}{ds}\frac{ds}{dt} \tag{2.8}$$

The scalar velocity v equals $\frac{ds}{dt}$ and the unit vector $\bar{N}$ in the direction of the principal normal to the path is equal to $\rho \frac{d\hat{T}}{ds}$ where ρ is the radius of curvature of the flight path (2, p. 92). Equation (2.7) can be rewritten to give

$$\ddot{\hat{R}} = v\left(\frac{v}{\rho}\hat{N}\right) + \frac{dv}{dt}\hat{T} \tag{2.9}$$

$$\ddot{\hat{R}} = \frac{v^2}{\rho}\hat{N} + \frac{dv}{dt}\hat{T} = \hat{A} \tag{2.10}$$

The second time derivative of $\bar{R}$ is the vector acceleration. The acceleration $\hat{A}$ of the satellite is a vector lying in the plane of the tangent and the principal normal to the flight path (equation 2.10). The tangential component is the time derivative of the speed. The normal component is the square of the speed divided by the instantaneous radius of curvature ρ .

Equation (2.2) may be rewritten as

$$\sum\hat{F} = m\ddot{\hat{R}} \tag{2.11}$$

Equations (2.3) and (2.11) are combined to give

$$-\frac{KMm}{r^2}\hat{R}_1 - \delta Av^2\hat{T} = m\ddot{\hat{R}} \tag{2.12}$$

$$\ddot{\hat{R}} = -\frac{KM}{r^2}\hat{R}_1 - \delta\frac{A}{m}v^2\hat{T} \tag{2.13}$$

Equations (2.10) and (2.13) are combined to give

$$\frac{v^2}{\rho}\hat{N} + \frac{dv}{dt}\hat{T} = -\frac{KM}{r^2}\hat{R}_1 - \delta v^2\frac{A}{m}\hat{T} \tag{2.14}$$

This vector equation completely describes the motion of earth satellites under the assumptions given so far. The equation is nonlinear, and a general solution in closed form has not yet been devised.

CIRCULAR ORBITS

Equation (2.14) may be used to describe the motion of an initially circular orbiting satellite quite easily. A satellite in this initial configuration will fall back to the earth under the influence of drag and gravity in a spiral orbit of small pitch. Operating on equation (2.14) by dot product multiplication first with the unit normal vector and second with the unit tangent vector results in two scalar equations. These may be solved simultaneously for time in terms of radial distance r and density δ. Figure 3.1 aids in visualizing these equations.

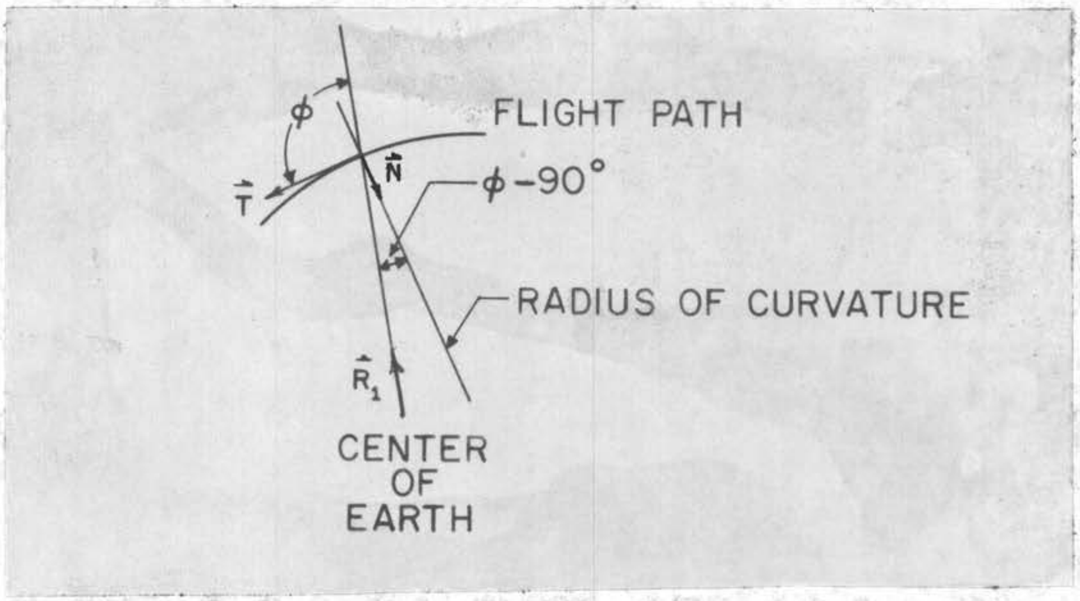


Figure 3.1

Dot product multiplication with the unit normal vector gives

$$\frac{v^2}{\rho} = \frac{KM}{r^2} \sin\phi \tag{3.1}$$

For the near circular orbit, the radius of curvature of the flight path is given approximately by

$$\rho = \frac{r}{\sin\phi} \tag{3.2}$$

Using this allows equation (3.1) to be written as

$$v^2 = \frac{KM}{r} \tag{3.3}$$

Dot product multiplication with the unit tangent vector gives

$$\frac{dv}{dt} = - \frac{KM}{r^2} \cos\phi \ - \delta\, v^2 \frac{A}{m} \tag{3.4}$$

This equation is simplified by using equation (2.13). Multiplying both members of equation (2.13) by $2\dot{\vec{R}}\cdot$ gives

$$2\dot{\vec{R}}\cdot\ddot{\vec{R}} = - 2 \frac{KM}{r^2} \vec{R}_1\cdot\dot{\vec{R}} - 2\delta\frac{A}{m} v^2 \vec{T}\cdot\dot{\vec{R}} \tag{3.5}$$

Use is now made of the fact that $\vec{R}_1\cdot\dot{\vec{R}} = \vec{R}_1\cdot(r\dot{\vec{R}}_1 + \dot{r}\vec{R}_1) = \dot{r}$ and that $2\dot{\vec{R}}\cdot\ddot{\vec{R}} = \frac{d}{dt}(\dot{\vec{R}}\,\dot{\vec{R}}) = \frac{d}{dt}(v^2)$, and $\vec{T}\cdot\dot{\vec{R}} = v$, to give

$$\frac{d(v^2)}{dt} = - 2 \frac{KM}{r^2}\frac{dr}{dt} - 2\,\delta\, v^3 \frac{A}{m} \tag{3.6}$$

The integral of equation (3.6) is the energy equation. If equation (3.3) and equation (1.2) are now substituted for v^2 and $\delta A v^2$, equation (3.6) reduces to

$$\frac{d}{dt}\left(\frac{KM}{r}\right) = -\frac{2KM}{r^2}\frac{dr}{dt} - 2\frac{fv}{m} \tag{3.7}$$

$$fv + \frac{1}{2}\frac{KMm}{r^2}\frac{dr}{dt} = 0 \tag{3.8}$$

The change in radial distance with time is $\frac{dr}{dt} = \vec{R}_1 \cdot \dot{\vec{R}} = + v \cos\phi$.

$$fv = -\frac{1}{2}\frac{KMm}{r^2} v \cos\phi \tag{3.9}$$

or

$$-\frac{KM}{r^2}\cos\phi = +\frac{2f}{m} \tag{3.10}$$

Substituting for $-\frac{KM}{r^2}\cos\phi$ in equation (3.4) allows this equation to be written as

$$\frac{dv}{dt} = \delta v^2 \frac{A}{m} \tag{3.11}$$

In equations (3.3) and (3.11) the time t is the independent variable. Velocity v is the function of radial distance r and r in turn is a function of time t. Thus the derivative of velocity with respect to time is given by

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} \tag{3.12}$$

Equation (3.3) can be used to find $\frac{dv}{dr}$.

$$v = \frac{\sqrt{KM}}{r^{1/2}} \tag{3.13}$$

$$\frac{dv}{dr} = -\frac{1}{2}\frac{1}{r}\frac{\sqrt{KM}}{r^{1/2}} \tag{3.14}$$

$$\frac{dv}{dr} = -\frac{1}{2}\frac{v}{r} \tag{3.15}$$

Therefore from equation (3.12)

$$\frac{dv}{dt} = -\frac{1}{2}\frac{v}{r}\frac{dr}{dt} \tag{3.16}$$

Equations (3.11) and (3.16) combine to give

$$-\frac{1}{2}\frac{v}{r}\frac{dr}{dt} = -\delta v^2 \frac{A}{m} \tag{3.17}$$

$$-\frac{1}{2}\frac{dr}{dt} = \delta vr \frac{A}{m} \tag{3.18}$$

Substituting for v from equation (3.13) gives

$$-\frac{1}{2}\frac{dr}{dt} = \delta \sqrt{KM}\, r^{1/2} \frac{A}{m} \tag{3.19}$$

$$-\frac{1}{2}\int_{r}^{r_1} \frac{dr}{r^{1/2}} = \sqrt{KM}\,\frac{A}{m}\int_{o}^{t} \delta\, dt \qquad (3.20)$$

where r_1 is a radial distance less than r. For small increments $r-r_1$ the density is approximately constant and may be taken outside the integral sign. Using this to integrate equation (3.20) gives

$$\left(r^{1/2} - r_1^{\ 1/2}\right) = \delta\sqrt{KM}\,\frac{A}{m}\,t \qquad (3.21)$$

Since the time to descend from a given radial height r to a slightly lower radial height r_1 is of interest, equation (3.21) is expressed in terms of the independent variable t.

$$t = \frac{2\left(r^{1/2} - r_1^{\ 1/2}\right)}{\sqrt{KM}\ \frac{A}{m}\ \delta} \qquad (3.22)$$

The total time spent in orbit can be determined by summing up the time spent in each segment $(r-r_1)$. The smaller the choice of segments, the more accurate the final answer will be.

Calculations using equation (3.22) were carried out using $KM = 1.40775 \times 10^6\ ft^3/sec^2$ (3, p. 636). The value of the density was determined by using the average density between r and r_1 from Figure 1.3.

The initial altitude of the satellite was broken up into 100,000 foot intervals and equation (3.22) applied to each interval.

Approximately a one per cent error is introduced by using segments of this size. Even though equation (3.22) is not strictly applicable for very low altitudes, the computations were carried out down to sea level. Very little error was introduced by doing this since the time spent in the low altitude regions is only a small fraction of the total time spent in orbit.

Satellite lifetimes for a range of altitudes up to 600 miles is shown in Figure 3.2. This figure gives the total lifetime expected from an initially established circular orbit.

ELLIPTICAL ORBITS

If there is no drag, the last term in equation (2.14) is set equal to zero and this equation reduces to the well known equation of particle motion under an inverse square force of attraction.

$$\ddot{\vec{R}} = -\frac{KM}{r^2}\vec{R}_1 \qquad (4.1)$$

Needless to say, there is a vast amount of published literature on the solution of this equation. These solutions are a great aid in arriving at an approximate solution to equation (2.14). The pertinent information which is used from these solutions will be

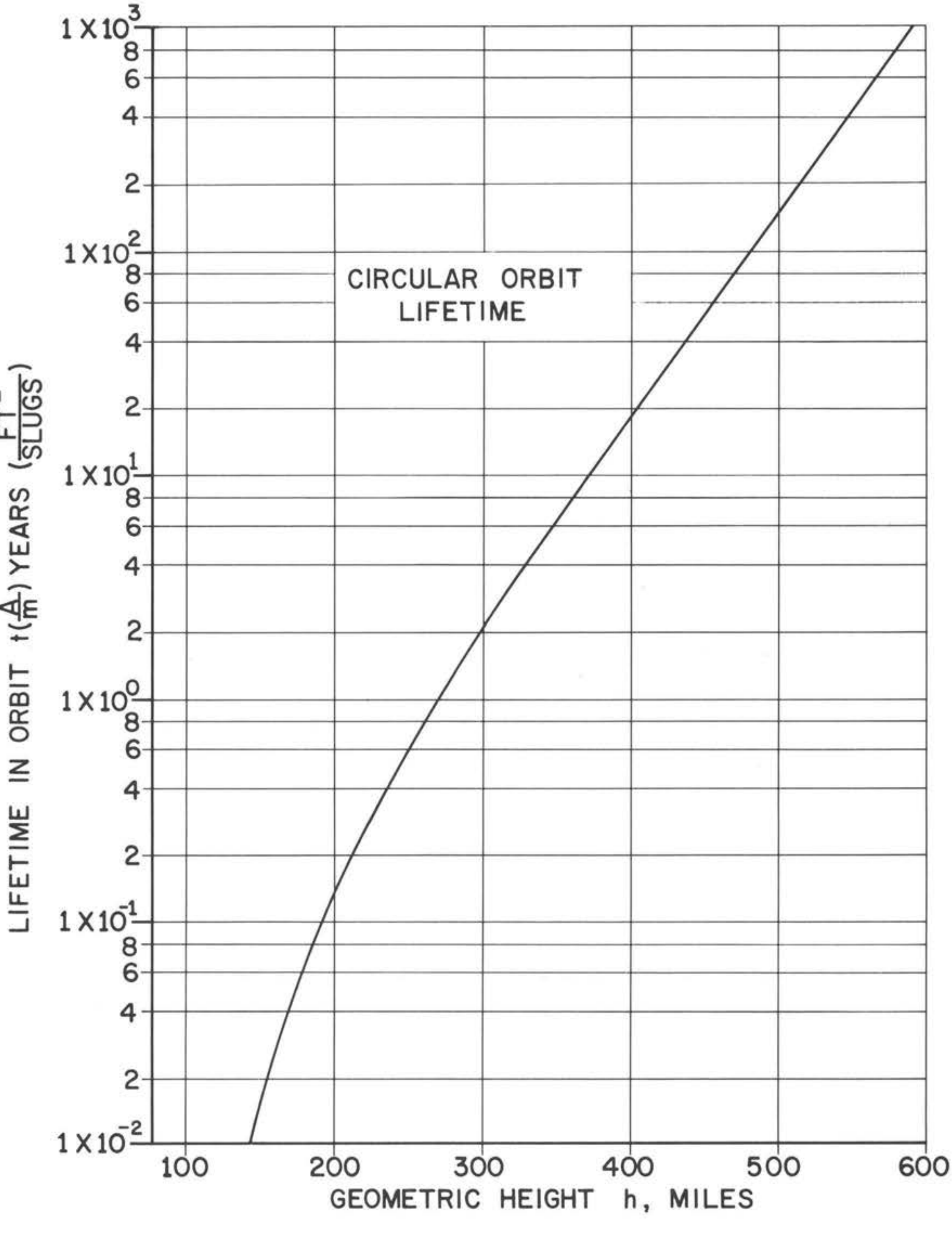

CIRCULAR ORBIT
LIFETIME
LIFETIME IN ORBIT $t\left(\frac{A}{m}\right)$ YEARS $\left(\frac{FT^2}{SLUGS}\right)$
GEOMETRIC HEIGHT h, MILES
1×10^{3}
1×10^{2}
1×10^{1}
1×10^{0}
1×10^{-1}
1×10^{-2}
100
200
300
400
500
600

FIGURE 3.2

derived here for the sake of completeness. The following material through page 29 is condensed from work by Hostetter (5, p. IV15-IV20).

Taking the cross product with the radial vector $\bar{R}$ in equation (4.1) gives

$$\bar{R} \times \ddot{\bar{R}} = - \frac{KM}{r^2} \bar{R} \times \bar{R}_1 = 0 \tag{4.2}$$

or

$$\frac{d}{dt} (\bar{R} \times \dot{\bar{R}}) = 0 \tag{4.3}$$

thus

$$\bar{R} \times \dot{\bar{R}} = \bar{B} \tag{4.4}$$

where $\bar{B}$ is a vector constant. The equation

$$\bar{R} \times m\dot{\bar{R}} = m\bar{B} \tag{4.5}$$

states that the angular momentum of the satellite remains constant during the motion. The equivalent scalar equation is, (see Figure 4.2 for definition of θ)

$$r^2 \frac{d\theta}{dt} = b \tag{4.6}$$

As the area of the triangle described by the radius vector in time dt is $\frac{1}{2}$ r (r$d\theta$), the left hand side of equation (4.6) is twice the area described by the radius vector per unit time. Therefore the equation states that the radius vector describes equal areas in equal times. This is Kepler's second law of planetary motion.

To find the equation of the path, the cross product of equation (4.1) with the vector constant $\vec{B}$ is applied. Use of equation (4.4) is then made to give equation (4.7). Note that $\vec{R} = r\vec{R}_1$.

$$\ddot{\vec{R}} \times \vec{B} = \frac{KM}{r^3} (\vec{R} \times \dot{\vec{R}}) \times \vec{R} \qquad (4.7)$$

The vector triple product in equation (4.7) is expanded in the usual way. Note that $\vec{R} \cdot \dot{\vec{R}} = r\dot{r}$.

$$\ddot{\vec{R}} \times \vec{B} = \frac{KM}{r^3} (r^2\dot{\vec{R}} - r\dot{r}\,\vec{R}) \qquad (4.8)$$

Equation (4.8) is now simplified by using the identity $\dot{\vec{R}} = (\dot{r}\vec{R}_1 + r\dot{\vec{R}}_1)$ and noting that $\ddot{\vec{R}} \times \vec{B} = \frac{d}{dt} (\dot{\vec{R}} \times \vec{B})$

$$\frac{d}{dt} (\dot{\vec{R}} \times \vec{B}) = KM \frac{d\vec{R}_1}{dt} \qquad (4.9)$$

$$\dot{\vec{R}} \times \vec{B} = \frac{KM\vec{R}}{r} + \vec{C} \qquad (4.10)$$

where $\vec{C}$ is a vector constant of integration. The time is eliminated from equation (4.10) by multiplying both members by $\vec{R}\cdot$, interchanging the dot and cross in the resulting left member, and applying equation (4.4) to get

$$\vec{B} \cdot \vec{B} = KM \frac{\vec{R} \cdot \vec{R}}{r} + \vec{R} \cdot \vec{C} \qquad (4.11)$$

$$b^2 = KMr + \bar{R}\cdot\bar{C} \tag{4.12}$$

$$r = \frac{b^2}{KM} - \bar{R}\cdot\bar{E} \tag{4.13}$$

where $\bar{E} = \frac{\bar{C}}{KM}$. Let $e = |\bar{E}|$ and α be the angle between $\bar{R}$ and the fixed direction of $\bar{E}$. Then $\bar{R}\cdot\bar{E} = re\cos\alpha$. Using this to rewrite equation (4.13) gives

$$r = \frac{b^2/KM}{1 + e\cos\alpha} \tag{4.14}$$

Equation (4.14) may be shown to be an equation of an ellipse by comparing this equation to the equation of an ellipse obtained from geometry. By definition, an ellipse is the curve traced by a point j (satellite) which moves so that the ratio of its distance from a fixed point F (center of earth) to its distance from a straight line D-D' is a constant less than unity, (Figure 4.1).

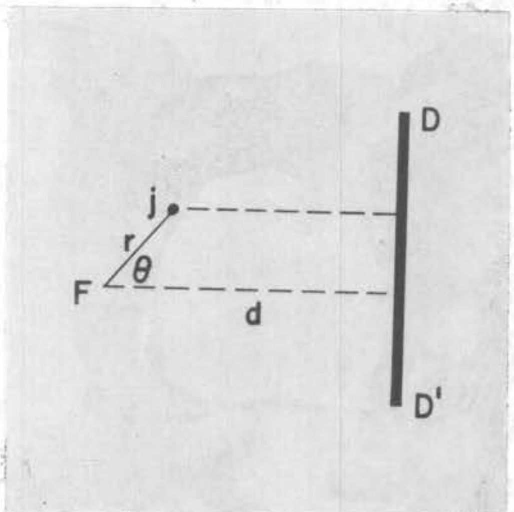


Figure 4.1

The point F is known as a focus, D-D' a directrix, and the ratio of the distance of j from the focus to its distance from the directrix is known as the eccentricity of the ellipse ($\epsilon < 1$). If d is the distance of the focus from the directrix, then,

$$\epsilon = \frac{r}{d - r \cos \theta} \tag{4.15}$$

Equation (4.15) is the equation of an ellipse

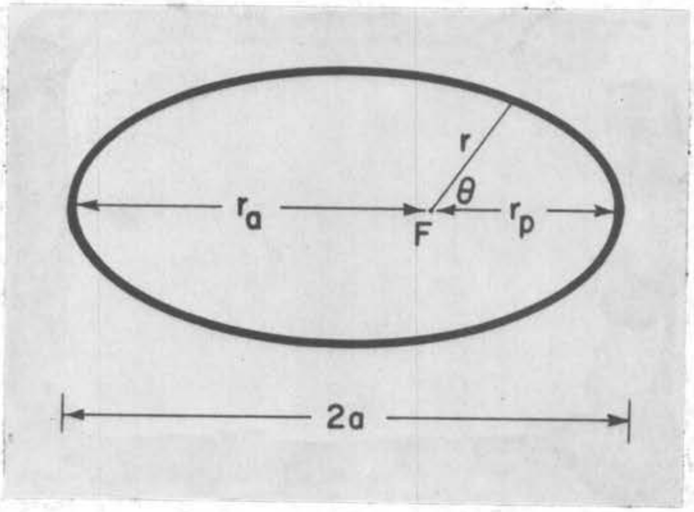


Figure 4.2

The major axis 2a is the sum of the minimum value r_p (perigee for F at the center of the earth) of the radius vector corresponding to $\theta = 0$ and the maximum value r_a (apogee for F at the center of the earth) corresponding to $\theta = \pi$. Since

$$r_p = \frac{\epsilon d}{1 + \epsilon} \qquad r_a = \frac{\epsilon d}{1 - \epsilon} \tag{4.16}$$

it follows that

$$2a = r_p + r_a = \frac{2\,d}{1 - \epsilon^2} \tag{4.17}$$

$$d = \frac{a}{\epsilon}\left(1 - \epsilon^2\right) \tag{4.18}$$

Replacing d in equation (4.15) with equation (4.18) gives

$$r = \frac{a\left(1 - \epsilon^2\right)}{1 + \epsilon\cos\theta} \tag{4.19}$$

By comparing equation (4.19) with equation (4.14) it is seen that equation (4.14) is also an equation of an ellipse if $e < 1$. In which case $e = \epsilon$, $\alpha = \theta$, and $b^2/KM = a(1 - \epsilon^2)$. Equation (4.14) can be rewritten as

$$r = \frac{b^2/KM}{1 + \epsilon\cos\theta} \tag{4.20}$$

Equation (4.19) states that the satellite motion (with no drag) is completely determined by its initial eccentricity and semimajor axis a.

The energy equation is found by multiplying both members of equation (4.1) by $2\dot{\bar{R}}\cdot$ and noting that $\bar{R}\cdot\dot{\bar{R}} = r\dot{r}$.

$$2\dot{\bar{R}}\cdot\ddot{\bar{R}} = -\frac{2KM}{r^3}\,\bar{R}\cdot\dot{\bar{R}} \tag{4.21}$$

$$\frac{d}{dt}\,(\dot{\bar{R}}\cdot\dot{\bar{R}}) = -\,\frac{2KM}{r^2}\,\frac{dr}{dt} \tag{4.22}$$

$$\dot{\bar{R}}\cdot\dot{\bar{R}} = -\,2\,KM\int\frac{dr}{r^2} \tag{4.23}$$

$$v^2 = \frac{2KM}{r} + \frac{2U}{m} \tag{4.24}$$

The quantity $\frac{2U}{m}$ is the constant of integration. The total energy U given by

$$U = \frac{1}{2}\,mv^2 - \frac{KMm}{r} \tag{4.25}$$

The first term on the right is the kinetic energy, the second the potential energy.

At the point of nearest approach to the earth (perigee) the velocity is entirely transverse, so that

$$v_p = r_p\,\frac{d\theta}{dt} \tag{4.26}$$

where v_p is the velocity at perigee. Using equation (4.6) with this expression gives

$$v_p = \frac{b}{r_p} \tag{4.27}$$

From equation (4.20)

$$\frac{1}{r_p} = \frac{KM}{b^2} + \frac{KM}{b^2}\epsilon \qquad (\cos\theta = 1) \tag{4.28}$$

With the aid of (4.27) this reduces to

$$v_p = \frac{KM}{b} + \frac{\epsilon KM}{b} \tag{4.29}$$

Substituting these values for r_p and v_p in the energy equation gives

$$\left(\frac{KM}{b} + \frac{\epsilon KM}{b}\right)^2 - 2KM\left(\frac{KM}{b^2} + \frac{\epsilon KM}{b^2}\right) = \frac{2U}{m} \tag{4.30}$$

which reduces to

$$\frac{2U}{m} = -\left(\frac{KM}{b}\right)^2 \left(1 - \epsilon^2\right) \tag{4.31}$$

$$\epsilon = \sqrt{1 + \frac{2U}{m}\left(\frac{b}{KM}\right)^2} \tag{4.32}$$

For elliptical orbits, the eccentricity ϵ is less than one. Equation (4.32) requires that for the elliptic orbit the total energy be negative.

The greatest radial distance a satellite obtains from the center of the earth is the apogee distance. This is given by equation (4.20) to be

$$r_a = \frac{b^2/KM}{1 - \epsilon} \tag{4.33}$$

The smallest radial distance a satellite obtains from the center of the earth is the perigee position. This is obtained from equation (4.20) to be

$$r_p = \frac{b^2/KM}{1 + \epsilon} \tag{4.34}$$

The major axis is given by $2a = r_a + r_p$ (see Figure 4.2)

$$2a = \frac{b^2/KM \ (1 + \epsilon + 1 - \epsilon)}{1 - \epsilon^2} \tag{4.35}$$

$$\frac{1}{a} = \frac{KM}{b^2} \left(1 - \epsilon^2\right) \tag{4.36}$$

Comparing equations (4.31) and (4.36) shows that

$$\frac{2U}{m} = - \frac{KM}{a} \tag{4.37}$$

$$U = - \frac{KMm}{2a} \tag{4.38}$$

from which is concluded that all elliptic orbits of the same major axis have the same energies irrespective of their eccentricities. The energy equation is rewritten as

$$\frac{1}{2} mv^2 - \frac{KMm}{r} = - \frac{KMm}{2a} \tag{4.39}$$

The magnitude of the potential energy is then always larger than the kinetic energy.

The period of revolution P for elliptical orbits can be determined by using equation (4.6) and geometry. Multiplying both sides of this equation gives

$$\frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} b \tag{4.40}$$

The left hand side expresses the area swept out by the radius vector per unit time. During one revolution the total area of the ellipse must be given by $\frac{1}{2}$ bP. This can be set equal to the area as determined from geometry, $\pi a^2 \sqrt{1-\epsilon^2}$.

$$bP = 2\pi a^2 \sqrt{1-\epsilon^2} \tag{4.41}$$

From equation (4.36)

$$\sqrt{1-\epsilon^2} = \frac{b}{\sqrt{aKM}} \tag{4.42}$$

thus

$$P = \frac{2\pi a^{3/2}}{\sqrt{KM}} \tag{4.43}$$

Equations (4.1) - (4.43) were derived assuming the drag term in equation (2.14) to be equal to zero. The above results will, however, approximate the actual motion predicted by equation (2.14) if the motion is confined to one or two revolutions. The drag term becomes important when long time periods are involved.

The following procedure is used to arrive at an approximate solution for elliptic orbit lifetime.

The fundamental assumption that the perigee height r_p remains fixed until the elliptical orbit is circularized is used. From equation (4.43), the time required for the first revolution is found knowing the initial magnitude of the semimajor axis a. This same equation is used to find the period of the second revolution after calculating the change in a, as determined from the energy equation. This presupposes knowledge of the change in velocity at perigee due to drag. The change in velocity at perigee due to drag during one revolution is formulated by integrating the drag term in equation (3.4). Naturally, once the period of the second revolution is determined, the period for the third revolution may be determined in the same way and so on until the elliptic orbit is circularized.

The time required for the circularization of the ellipse is equal to the sum of the periods for each revolution, in going from the initially established elliptical orbit, to the circular orbit at perigee height r_p. The period is expressed in equation (4.43) as

$$P = \frac{2\pi a^{3/2}}{\sqrt{KM}} \qquad (4.43)$$

The velocity change at perigee due to change in orbital eccentricity can be determined from the derivative of the energy equation (4.39) written for conditions at perigee.

$$\frac{1}{2} m v_p^{\,2} - \frac{KMm}{r_p} = - \frac{KMm}{2a} \qquad (4.44)$$

Differentiation of equation (4.44), assuming r_p constant, gives

$$2v_p dv_p = \frac{KM}{a^2} da \qquad (4.45)$$

Solving for v_p in (4.44) and substituting into equation (4.45) gives

$$dv_p = \frac{\sqrt{KMr_p}}{\sqrt{8}\ a^2\sqrt{1 - r_p/2a}}\, da \qquad (4.46)$$

The change in velocity at perigee due to drag during one revolution is obtained from equation (3.4) which is repeated here. Note that once the change in velocity due to drag is obtained, the corresponding change in major axis, 2a, is given by equation (4.46).

$$\frac{dv}{dt} = -\frac{KM}{r^2}\cos\phi - \delta v^2 \frac{A}{m} \qquad (4.47)$$

The first term on the right hand side of this equation describes the periodic variation in acceleration as the satellite moves in orbit. The second term expresses the acceleration resulting from atmospheric drag. The acceleration resulting from drag only is given by

$$\frac{dv}{dt} = -\ \delta v^2 \frac{A}{m} \qquad (4.48)$$

$$dv = -\ \delta v^2 \frac{A}{m}\, dt \qquad (4.49)$$

The velocity is obtained from energy equation (4.39).

$$v^2 = 2KM\left(\frac{1}{r} - \frac{1}{2a}\right) \qquad (4.50)$$

Equation (4.6) can be rearranged to give

$$dt = \frac{r^2}{b}\, d\theta \tag{4.51}$$

Using these last two equations to express v^2 and dt in equation (4.49) gives

$$dv = -\,\delta\, \frac{A}{m}\, 2KM \left(\frac{1}{r} - \frac{1}{2a}\right) \frac{r^2}{b}\, d\theta \tag{4.52}$$

The radial distance r is expressed in terms of θ by using the equation of an ellipse repeated here.

$$r = \frac{a\left(1 - \epsilon^2\right)}{1 + \epsilon \cos\theta} \tag{4.19}$$

The density δ is given by equation (1.6), $\delta = \delta_o e^{-\beta h}$. $\beta = -\frac{g}{RT}$.
This may be rewritten as

$$\delta = \delta_r\, e^{-\beta r} \tag{4.53}$$

where δ_r includes the constant obtained in changing variables from h to r. Substituting the last two expressions into equation (4.52) gives

$$dv = -\,\delta_r\, \frac{A}{m}\, \frac{2KM}{b} \left[\frac{1 + \epsilon \cos\theta}{a(1 - \epsilon^2)} - \frac{1}{2a}\right]$$

$$\frac{a^2(1 - \epsilon^2)^2}{(1 + \epsilon \cos\theta)^2} \exp\left[-\frac{\beta\, a(1 - \epsilon^2)}{1 + \epsilon \cos\theta}\right] d\theta \tag{4.54}$$

Equation (4.54) may be simplified by noting that dv is directly proportional to density (4.52). Since density varies very rapidly with altitude, the greatest change in velocity will occur near perigee. It is assumed that the total change in velocity for one orbit may be approximated by the velocity change in the vicinity of perigee (small values of Θ). Cos Θ is expanded by using $\cos \Theta = 1 - \frac{\Theta^2}{2}$

$$dv = -\delta_r \frac{A}{m} \frac{2KM}{b} \left(\frac{1 + \epsilon - 1/2\, \epsilon\, \Theta^2}{a(1 - \epsilon^2)} - \frac{1}{2a} \right)$$

$$\left[\frac{a(1 - \epsilon^2)}{(1 + \epsilon - 1/2 \epsilon \Theta^2)} \right]^2 \exp \left[- \frac{\beta\, a(1 - \epsilon^2)}{1 + \epsilon - 1/2 \epsilon\, \Theta^2} \right] d\Theta \qquad (4.55)$$

The ellipse equation (4.19) written for $\Theta = 0$ gives $r_p = \frac{a(1 - \epsilon^2)}{1 + \epsilon} = a(1 - \epsilon)$ where r_p = perigee radius. Using this in equation (4.55) gives

$$dv = -\delta_r \frac{A}{m} \frac{2KM}{b} r_p \left[1 - \frac{\epsilon\, \Theta^2}{2(1 + \epsilon)} - \frac{r_p}{2a} \right]$$

$$\left[1 - \frac{\epsilon\, \Theta^2}{2(1 + \epsilon)} \right]^{-2} \exp \left[- \frac{\beta\, r_p}{1 - \frac{\epsilon\, \Theta^2}{2(1 + \epsilon)}} \right] d\Theta \qquad (4.56)$$

The expansion for $\left[1 - \frac{\epsilon\,\theta^2}{2(1+\epsilon)}\right]^{-1}$ is given as

$$\left[1 - \frac{\epsilon\,\theta^2}{2(1+\epsilon)}\right]^{-1} = \left[1 + \frac{\epsilon\,\theta^2}{2(1+\epsilon)} + \frac{\epsilon\,\theta^4}{4(1+\epsilon)^2} + \cdots\right] \qquad (4.57)$$

Using the first two terms of this expansion in equation (4.56) and rearranging $\left[\delta_p = \delta_r\, e^{-\beta r_p}\right]$ gives

$$dv = -\,\delta_p\, r_p\, \frac{A}{m}\, \frac{2KM}{b} \left[1 + \frac{\epsilon\,\theta^2}{2(1+\epsilon)} - \frac{r_p}{2a} - \frac{r_p \epsilon\,\theta^2}{2a(1+\epsilon)}\right] \exp\left[-\frac{\beta r_p \epsilon\,\theta^2}{2(1+\epsilon)}\right] d\theta \qquad (4.58)$$

The above equation should be integrated between the small angles (in the vicinity of perigee) previously assumed. Integration is simplified, however, if advantage is taken of the fact that the integrand of equation (4.58) is a function of density. At $\theta = 0$ the density assumes its perigee value δ_p. As θ is increased, the corresponding value of density r pidly becomes vanishingly small when compared with δ_p (See Figure 4.3). Negligible error is introduced if integration is carried out between the limits $-\infty$ and $+\infty$ instead of the "small" values of θ. Neglecting higher order terms in the expansion of cos θ eliminated any periodicity in equation (4.58).

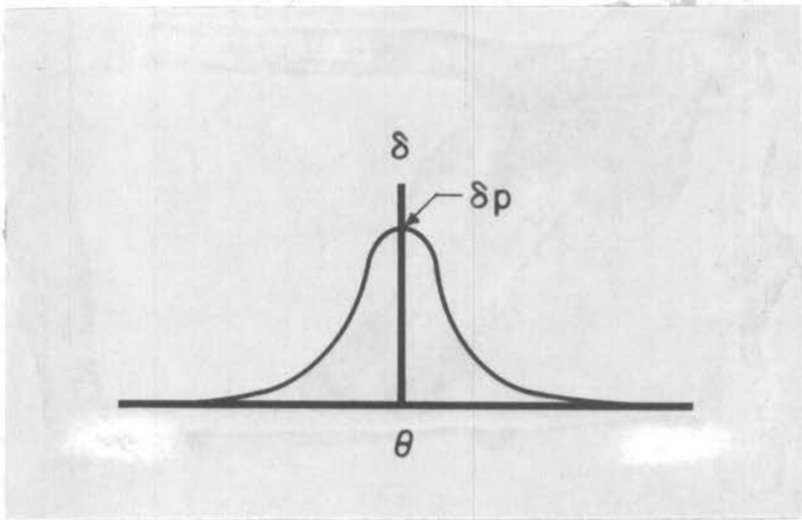


Figure 4.3

Integration of equation (4.58) gives

$$(\Delta v)_p = -\delta_p r_p \frac{A}{m} \frac{2KM}{b} \sqrt{\frac{2\pi(1+\epsilon)}{\beta\, r_p \epsilon}} \left[\left(1 - \frac{r_p}{2a}\right) + \left(1 - \frac{r_p}{a}\right) \frac{1}{2\beta r_p} \right] \qquad (4.59)$$

Where $(\Delta v)_p$ represents the velocity change, due to drag, in going between the two "small" angles. This velocity change is precisely the same magnitude as the difference between the initial and second perigee passage velocities $v_1 - v_0$ and will be used in this sense. The quantity $(\Delta v)_p$ is negative since drag decreases the perigee velocity. Note that $\frac{1}{\beta r_p} = 10^{-2}$. This makes the very last term in equation (4.59) two orders of magnitude smaller than $1 - \frac{r_p}{2a}$ and may be neglected. Thus,

$$(\Delta v)_p = -\delta_p \, r_p \frac{A}{m} \frac{2KM}{b} \left(1 - \frac{r_p}{2a}\right) \sqrt{\frac{2\pi(1+\epsilon)}{\beta \, r_p \, \epsilon}} \qquad (4.60)$$

The incremental velocity change at perigee due to drag $(\Delta v)_p$ per period of revolution may be set equal to the time rate of change of velocity at perigee.

$$\frac{(\Delta v)_p}{P} = \frac{dv_p}{dt} \qquad (4.61)$$

$$dt = \frac{P}{(\Delta v)_p} \, dv_p \qquad (4.62)$$

It would appear that this equation could be integrated for time by a step by step summation process. (A procedure that would require as many steps as satellite revolutions about the earth.) This laborsome summation process is avoided by substituting equations (4.43), (4.46), and (4.60) for P, dv_p, and $(\Delta v)_p$. Equation (4.19) is used to relate ϵ to a and integration is performed in the usual way.

$$t = -\frac{\sqrt{\pi \beta}}{4 \; 2 \, \delta_p \frac{A}{m} KM} \int_{\frac{r_a + r_p}{2}}^{r_p} \frac{b\sqrt{1 - \frac{r_p}{a}}}{\left(1 - \frac{r_p}{2a}\right)^2} \, da \qquad (4.63)$$

Integration is carried out for the variable, the semimajor axis a, from its initial value $\frac{r_a + r_p}{2}$ (elliptic orbit) to its final value r_p (circular orbit). The "constant" b may be expressed as $b = r_p v_p$ (equation 4.27). From equation (4.50)

$$v_p = \sqrt{2KM\left(\frac{1}{r_p} - \frac{1}{2a}\right)} \tag{4.64}$$

Thus

$$b = r_p\sqrt{2KM\left(\frac{1}{r_p} - \frac{1}{2a}\right)} \tag{4.65}$$

Substituting equation (4.65) into equation (4.63) gives

$$t = \frac{\sqrt{\pi}\,\beta\, r_p}{4\,\delta_p \frac{A}{m}\sqrt{KM}} \int_{r_p}^{\frac{r_a + r_p}{2}} \frac{\sqrt{1 - \frac{r_p}{a}}}{\sqrt{a\left(1 - \frac{r_p}{2a}\right)^3}}\, da \tag{4.66}$$

Expanding $\sqrt{\left(1 - \frac{r_p}{2a}\right)^3}$ by means of the binomial expansion gives

$$t = K_1 \int_{r_p}^{\frac{r_a + r_p}{2}} \sqrt{a - r_p} \left[\frac{1}{a} + \frac{3}{4} \frac{r_p}{a^2} + \frac{15}{32} \frac{r_p^2}{a^3} + \frac{35}{128} \frac{r_p^3}{a^4} + \frac{315}{2048} \frac{r_p^4}{a^5} + \ldots \right] da \tag{4.67}$$

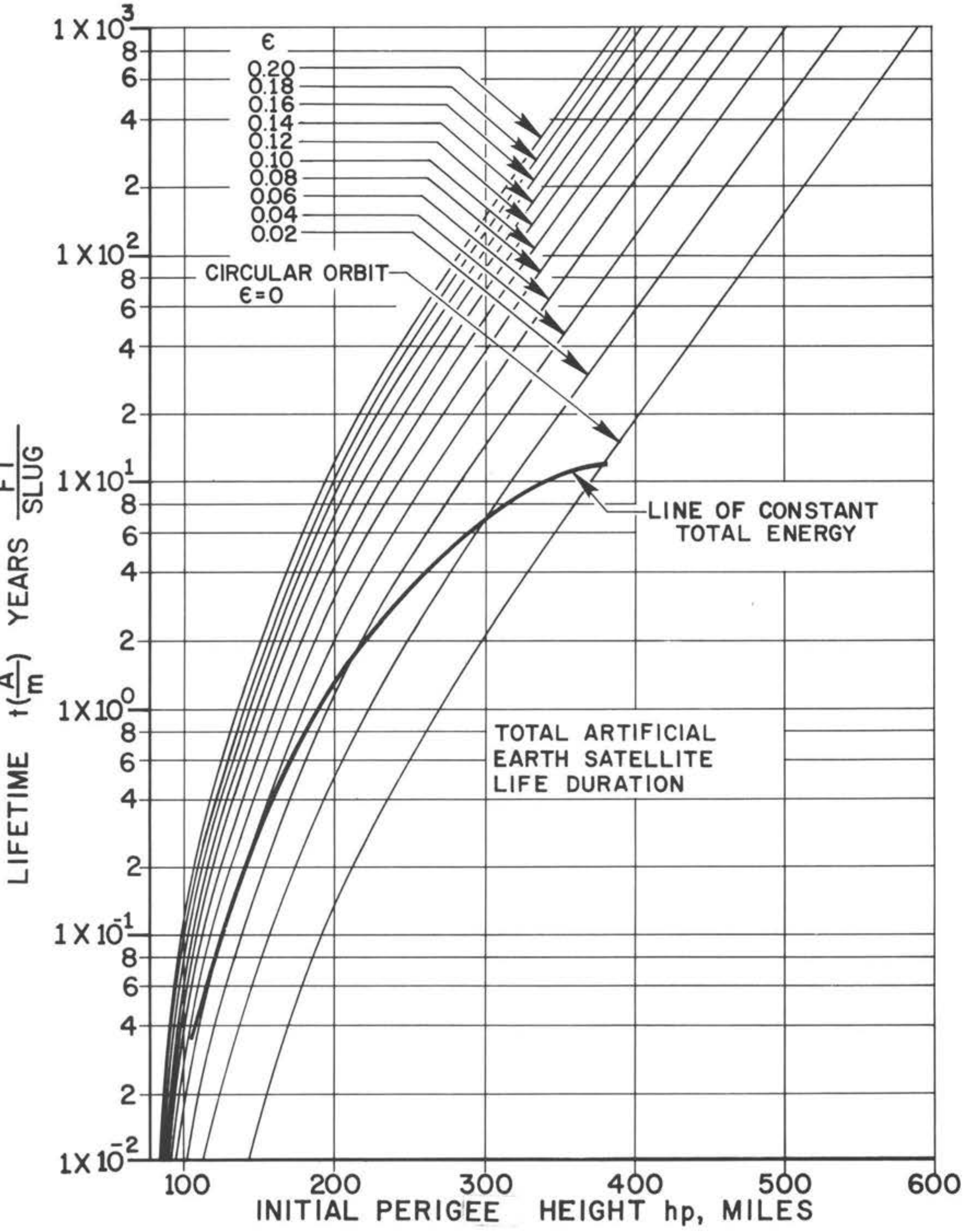
where $K_1 = \dfrac{\sqrt{\pi}\,\beta\, r_p}{4\,\delta_p \frac{A}{m}\sqrt{KM}}$. After integrating equation (4.67) and summing up the infinite series with due regard to convergence, equation (4.67) reduces to

$$t = K_1 \left\{ 1.086\left(\sqrt{\frac{r_a - r_p}{2}} - \sqrt{r_p}\ \tan^{-1}\sqrt{\frac{r_a - r_p}{2r_p}}\right) + \frac{\sqrt{(r_a - r_p)^3}}{\sqrt{8}\ a} \left[.902 + .311\left(\frac{r_p}{a}\right) + .123\left(\frac{r_p}{a}\right)^2 + .0385\left(\frac{r_p}{a}\right)^3 \right] \right\} \tag{4.68}$$

Using the identity $\epsilon = \dfrac{r_a - r_p}{r_a + r_p}$ and the definition $a = \frac{1}{2}(r_a + r_p)$, equation (4.68) is expressible in terms of the initial perigee distance r_p, and initial eccentricity ϵ .

$$t = K_1 \left\{ 1.086\ \sqrt{r_p}\left(\sqrt{\frac{\epsilon}{1-\epsilon}} - \tan^{-1}\sqrt{\frac{\epsilon}{1-\epsilon}}\right) + \sqrt{\epsilon^3\left(\frac{r_p}{1-\epsilon}\right)}\ \left[.902 + .311(1-\epsilon) + .123(1-\epsilon)^2 + .0385(1-\epsilon)^3\right]\right\} \tag{4.69}$$

Equation (4.69) expresses the time required for a satellite to circularize from an initially established elliptical orbit of eccentricity ϵ and perigee distance r_p to a circular orbit of radius r_p. The time used in decaying from the circular orbit to the earth's surface is obtained from Figure 3.2. When this time is added to that obtained with equation (4.69) the total satellite lifetime is determined. This procedure was used to compute the ordinates of Figure 4.4.



LIFETIME $t(\frac{A}{m})$ YEARS $\frac{FT^2}{SLUG}$
INITIAL PERIGEE HEIGHT hp, MILES
ϵ
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
CIRCULAR ORBIT $\epsilon = 0$
LINE OF CONSTANT TOTAL ENERGY
TOTAL ARTIFICIAL EARTH SATELLITE LIFE DURATION

FIGURE 4.4

DISCUSSION OF RESULTS

Figure (4.4) represents the main objective of this thesis. From this figure, the total life duration of an earth satellite may be determined. This prediction requires knowledge of the initial eccentricity, perigee height, and the frontal area to mass ratio of the satellite. The lifetime in years is evaluated by dividing the ordinate value as read from the figure by the area to mass ratio $\frac{A}{m}$. Note that life duration is inversely proportional to the area to mass ratio $\frac{A}{m}$. A 100 year lifetime is reduced to a one year lifetime by increasing the area to mass ratio by a factor of 100, other factors being held constant.

It is interesting to speculate on the lifetime of the proposed NASA 100-foot sphere communication satellite. This vehicle is composed of a .0005-inch thick Mylar plastic coated structure with vapor-deposited aluminum on the outer surface. It will have an area to mass ratio of about 1700 ft^2/slug. This means, if it were put into a circular orbit at 600 miles, it would have a lifetime of only 270 days.

The lowest lifetime shown in this figure is 3.65 days for $\frac{A}{m} = 1$. Even with eccentricities as large as 0.20, a satellite can not maintain a lifetime greater than this value for perigee altitudes less than 80 miles. The per cent change in satellite lifetime is very sensitive to perigee height, especially in lower altitude regions. For example, the circular orbit lifetime may be increased by a factor of ten by increasing the initial orbital altitude from 142 miles to

189 miles. The slope of the curves ("slope" with respect to the logarithmic vertical scale) decreases as the perigee altitude increases up to about 250 miles, beyond this height the slope remains nearly constant. This reflects the trend of the density variations given in Figure 1.3.

The highest altitude considered in Figure 4.4 is 600 miles which just lies within the range of the model atmosphere derived earlier. Satellite lifetimes at this perigee height are of the order of thousands of years (for $\frac{A}{m} = 1$).

At any given perigee height, the lifetime as well as the total energy of a satellite increases See equation (4.38) with an increase in eccentricity. The constant energy line illustrated shows that the circular orbit maximizes total lifetime for a given value of total energy. In following a line of constant energy, the lifetime in orbit decreases with an increase in eccentricity.

The effects of the many assumptions used in deriving equation (4.69) may be summarized below.

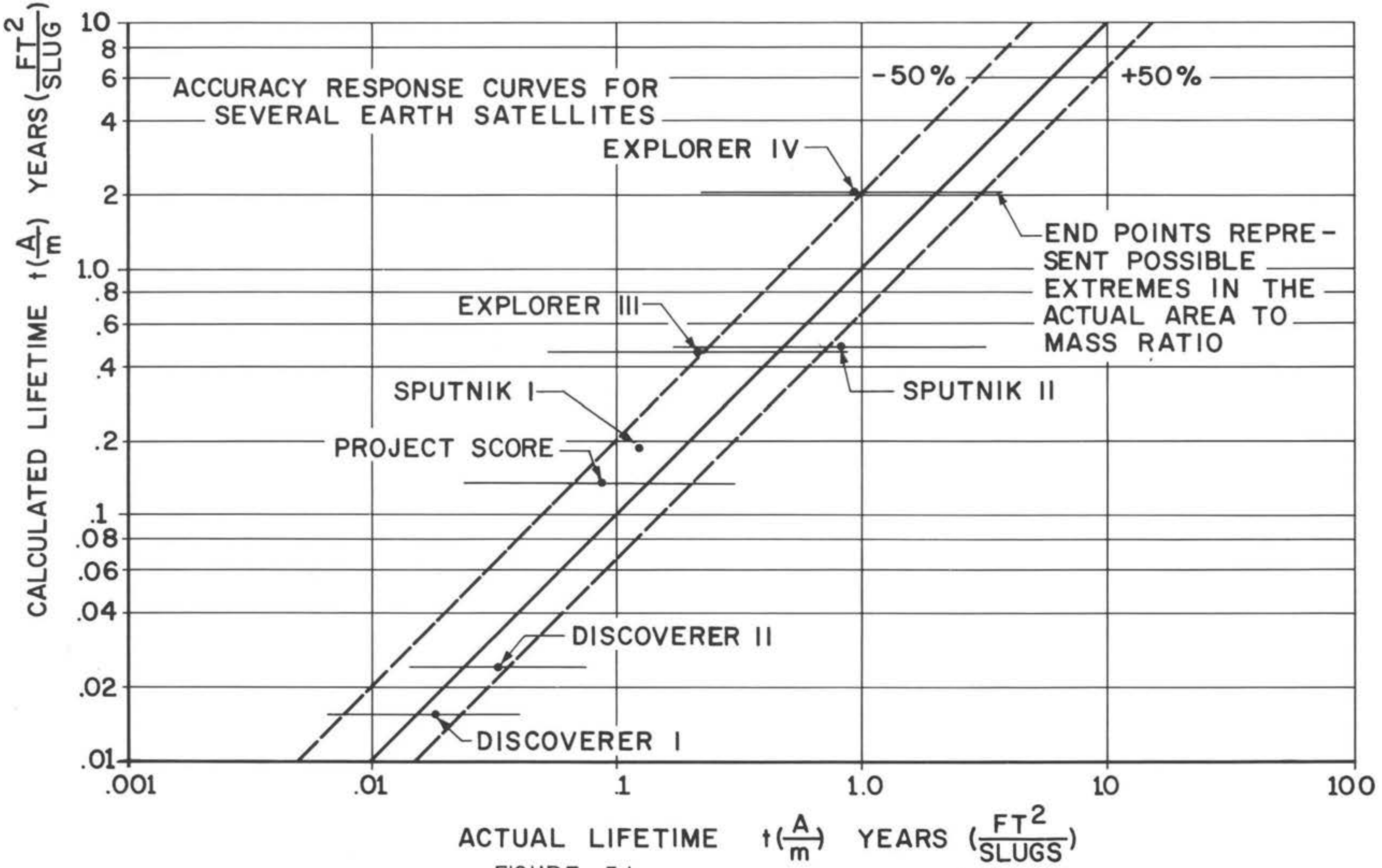
1. It was assumed that the density curve slope (Fig. 1.3) was constant. Actually, the value of β varies over a small portion of the range of altitudes given in Figure 4.4. At 60 miles, $\beta = 4.093 \times 10^{-5}\ ft^{-1}$ (equation 1.7) not the constant value of $\beta = 3.93 \times 10^{-6}\ ft^{-1}$ (equation 1.8) used. As a consequence, the predicted lifetimes in the lower altitude regions are low.
2. For posigrade orbits, neglecting the atmospheric rotation means

satellite lifetime predictions are low.

3. Neglecting oblateness means satellite lifetime predictions should be high (9, p. 40).
4. Assuming a constant perigee height during the circularization of the ellipse may mean that lifetimes are predicted low or high depending on the interrelationship between the additional time needed to go to a lower perigee height and the effect of the resulting greater density in reducing the circularization time. In addition, higher lifetime predictions will result during the circular decay.

The analysis neglects all of these effects as well as other smaller ones not mentioned. It is assumed that these combined effects are small when compared with the gross effects of the other parameters (for example consider the effect of a small error in density determination).

Figure 5.1 aids in estimating the over-all accuracy of the results. Lifetimes are calculated for several actual earth satellites and compared with their measured lifetimes. Unfortunately, even though the satellite lifetimes are known with a good deal of accuracy, the quantity $\frac{A}{m}$ is not. Sputnik I is the only spherical satellite that has returned to the earth so far. (Being spherical there is no question to the value of A). The other data points are plotted assuming an average value of $\frac{A}{m}$ (tumbling) with the possible extremes indicated. Notice that lifetime predictions using an average value of $\frac{A}{m}$ are within 50 per cent of the diagonal line representing absolute accuracy. The data points lie on either side of the diagonal. This indicates that there are no consistent errors in the analysis. Indeed, the data points corresponding to Sputnik II and Explorer III would tend to indicate that lifetime prediction errors are entirely due to uncertainties in the actual value of the area-to-mass ratio. For these seven data points, the lifetimes are predicted on the average 5 per cent low, with maximum deviations amounting to ± 50 per cent of lifetime.



ACCURACY RESPONSE CURVES FOR SEVERAL EARTH SATELLITES
-50%
+50%
EXPLORER IV
END POINTS REPRESENT POSSIBLE EXTREMES IN THE ACTUAL AREA TO MASS RATIO
EXPLORER III
SPUTNIK I
SPUTNIK II
PROJECT SCORE
DISCOVERER II
DISCOVERER I
CALCULATED LIFETIME $t(\frac{A}{m})$ YEARS $(\frac{FT^2}{SLUG})$
ACTUAL LIFETIME $t(\frac{A}{m})$ YEARS $(\frac{FT^2}{SLUGS})$
10
8
6
4
2
1.0
.8
.6
.4
.2
.1
.08
.06
.04
.02
.01
.001
.01
.1
1.0
10
100

FIGURE 5.1

CONCLUSIONS

1. The dynamical model of satellite motion presented allows solution to the resulting equation of motion in terms of lifetime.
2. The accuracy of the solution may not be reliably determined from satellite data due to uncertainties in the magnitude of area to mass ratio of the satellites.
3. Accurate predictions of satellite data require that the orientation of the satellite (if other than spherical) be known.
4. For a given input energy, the circular orbit allows maximum lifetime.
5. For lifetimes greater than 10 to 20 years, the unshielded manned satellite must have an $\frac{A}{m}$ (ft^2/slug) ratio less than one.

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APPENDIX

Record of Successful Earth Satellite Launches
(to Sept. 1959)

Name	Maximum Possible Area to Mass Ratio	Minimum Possible Area to Mass Ratio	Perigee Height Miles	Eccentricity	Lifetime	Mass Slugs
Sputnik I	0.50	0.50	142	0.0517	92 days	5.72
Sputnik II	8.04 (?)	0.4 (?)	140	0.0986	161 days	34.8
Explorer I	3.48	0.205	224	0.1394	in orbit	0.956
Vanguard I	2.03	2.03	409	0.1893	in orbit	0.1009
Explorer III	3.48	0.205	121	0.1661	94 days	0.964
Sputnik III	0.359	0.278	135	0.1120	in orbit	90.9
Explorer IV	2.90	0.178	163	0.1288	453 days	1.192
Project Score	3.13	0.289	110	0.0909	34 days	272
Vanguard II	3.39	3.39	347	0.1679	in orbit	0.644
Discoverer I	2.33	0.486	99	0.0588	6 days	40.4
Discoverer II	1.92	0.392	142	0.00938	13 days	50.0
Explorer VI	1.33	0.829	157	0.762	in orbit	4.42
Discoverer V	2.18	0.451	136	0.037	in orbit	52.8
Discoverer VI	(?)	(?)	138	0.0466	67 days	(?)
Explorer VII	(?)	(?)	316.6	0.0433	in orbit	2.84