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A theory of higher order connections is developed as a reduction of the structure group of the jet bundle of sections of a smooth vector bundle. This definition leads naturally to higher order curvatures and covariant derivatives.

Curvature conditions for a second order partial differential operator to have constant coefficients in the top order part are given. Also obtained are results on the reduction of real analytic C-R structures to the Cauchy-Riemann equations on a complex manifold via complexification and the study of the convexity of tubular neighborhoods in it.

HIGHER ORDER DIFFERENTIAL GEOMETRY
AND SOME RELATED QUESTIONS

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TABLE OF CONTENTS

HIGHER ORDER DIFFERENTIAL GEOMETRY AND SOME RELATED QUESTIONS

INTRODUCTION

We have developed a theory of higher order differential geometry to study differentiable manifolds endowed with a system of partial differential operators. The classical example is given by the complex structure of a manifold and many results are known. A generalization of this example is the Cauchy-Riemann (C-R) structure which is a system of first order partial differential operators in one unknown function. The theory of higher order connections (to be used in the study of higher order systems) is applied to a single second order partial differential operator. The results obtained in this situation can be considered dual to the classical theory of Riemannian geometry or to the conformal theory of Herman Weyl and Schouten if we allow a positive multiplicator. Differential operators of order larger than two generally produce a system of infinitely many algebraic invariants and therefore depend on "moduli". We are hopeful, however, that results similar to second order can be derived for "generic" systems. Finally we have initiated the global study of C-R structures and obtained for analytic C-R structures two theorems which should enable us to reduce this study to the study of ℓ-convexity of complex manifolds. The first theorem is a generalization of the theorem of Bruhat-Whitney on the existence of complexifications. The complexification of a C-R structure can be reduced in dimension in a functional way so that the theorem of Bruhat-Whitney is a special case of our theorem for a totally real structure. The second theorem deals with the convexity of the complexification and can be considered as a generalization of a theorem of Grauert in the case of a totally real structure. Although the second theorem has only been proved for compact manifolds, we hope to remove this restriction later.

Our program, of which this is only the beginning, is to connect the theory of higher order differential geometry with a higher order invariant introduced as the curvature of successive jet bundles. This invariant is much more subtle than the topological invariants and we have not as yet determined if its vanishing is completely determined by topological conditions. We have also initiated the study of higher order Grassmann manifolds and the generalized Gauss map, but have not yet clarified the connection (if any) between the higher order invariant of the tangent structure and the generalized Gauss map.

1. VECTOR BUNDLES

A paracompact real manifold with C^∞ differentiable structure will be simply called a <u>manifold</u>. If M and N are manifolds of dimensions k and n, respectively, and $f: N \to M$ is a smooth map, for each $q \in N$ we have the linear map

$$df_q : TN_q \to TM_{f(q)} .$$

The map f is a <u>submersion</u> if df_q is surjective for every $q \in N$, i.e. rank $df_q = k$. It follows by the implicit function theorem that $f^{-1}(p)$ is an $(n-k)$-dimensional submanifold of N (or empty).

<u>Definition (1.1)</u> <u>An</u> <u>n-dimensional</u> <u>vector</u> <u>bundle</u> is a triple (E,π,M), where $\pi: E \to M$ is a submersion and $\pi^{-1}(p)$ is an n-dimensional real vector space for every $p \in M$ such that:

(i) the function

$$E\times_M E = \{(x,y) \in E\times E \mid \pi(x) = \pi(y)\} \to E$$

defined by $(x,y) \to x+y$ is smooth, and

(ii) the function $R \times E \to E$ defined by $(c,x) \to cx$ is also smooth.

E is called the total space, π the projection, and $\pi^{-1}(p)$ the fibre over $p \in M$. When there is no ambiguity we will use E to denote the vector bundle (E,π,M).

From (1.1) and the implicit function theorem we see that every vector bundle is locally trivial, i.e. there exists an open cover $\{U_i\}$ of M and diffeomorphisms

(1.2) $$h_i: \pi^{-1}(U_i) \to U_i \times R^n$$

such that $\pi \circ h_i^{-1}$ is the projection on U_i for every i. The trivializations h_i and h_j are both defined on $\pi^{-1}(U_i \cap U_j)$ and hence

$$h_i \circ h_j^{-1}(p,x) = (p, g_{ij}(p)x),$$

with $g_{ij}: U_i \cap U_j \to GL(n,R) = GL(n)$ smooth and satisfying the cocycle condition

(1.3) $$g_{ij}\, g_{jk} = g_{ik} \ \underline{\text{on}} \ U_i \cap U_j \cap U_k.$$

The collection of smooth maps $\{g_{ij}: U_i \cap U_j \to GL(n)\}$ are called the <u>transition functions</u> of the vector bundle (E, π, M) (relative to the local trivializations h_i).

If (E, π, M) is a vector bundle, then $\pi^{-1}(p)$ is a vector space which generally has no canonical coordinatization. As a result the trivializations and the transition functions are not unique, even if one uses the same trivializing cover. This necessitates a notion of equivalence of vector bundles.

<u>Definition (1.4)</u> Two n-dimensional vector bundles (E, π, M) and (F, θ, M) are <u>isomorphic</u> if there exists a diffeomorphism $f: E \to F$ such that

(i) $\pi = \theta \circ f$, and

(ii) for every $p \in M$, the induced map

$f_p: \pi^{-1}(p) \to \theta^{-1}(p)$ is a vector space isomorphism.

One can show that two vector bundles with transition functions $\{g_{ij}: U_i \cap U_j \to GL(n)\}$ and $\{h_{ij}: U_i \cap U_j \to GL(n)\}$ on the same trivializing cover $\{U_i\}$ are isomorphic if and only if there exist smooth maps $\lambda_i: U_i \to GL(n)$ such that

(1.5) $$h_{ij} = \lambda_i \, g_{ij} \, \lambda_j^{-1} \quad \underline{\text{on}} \quad U_i \cap U_j .$$

We let $\mathbb{1}$ denote the vector bundle $(M \times R^n$, projection on $M, M)$ and call a vector bundle isomorphic to is *trivial*. Steenrod [1] p. 53 shows that every vector bundle with smoothly contractible base space is trivial.

A *section* of the vector bundle (E,π,M) is a smooth map $s: M \to E$ such that

$$\pi \circ s(p) = p \quad \textit{for every} \quad p \in M.$$

In terms of the local trivializations (1.2) a section is a collection of smooth maps $\{s_i: U_i \to R^n\}$ such that

(1.6) $$s_i = g_{ij} \, s_j \quad \textit{on} \quad U_i \cap U_j.$$

The vector space of sections of (E,π,M) will be denoted by $\Gamma(E,\pi,M)$ or more simply by $\Gamma(E)$.

Since the fibre over $p \in M$ is an n-dimensional vector space in (1.1), we chose our trivializations (1.2) to map onto $U_i \times R^n$. One may, however, replace R^n by any n-dimensional topological vector space V, thus obtaining transition functions which are smooth maps $g_{ij}: U_i \cap U_j \to GL(V)$ satisfying (1.3). Here $GL(V)$ denotes the space of linear isomorphisms of V, which is a Lie group diffeomorphic to $GL(n)$. We will say that the vector bundle (E,π,M) has *(standard) fibre* V if the trivializations are given as diffeomorphisms of the form

$$h_i: \pi^{-1}(U_i) \to U_i \times V.$$

If the vector bundle (E,π,M) with fibre V has transition functions $\{g_{ij}: U_i \cap U_j \to G\}$, where G is a closed subgroup of $GL(V)$, we say that (E,π,M) has structure group G.

Definition (1.7) Two vector bundles (E,π,M) and (F,θ,M) with the same fibre V and the same structure group G, say with transition functions $\{g_{ij}: U_i \cap U_j \to G\}$ and $\{h_{ij}: U_i \cap U_j \to G\}$ respectively, are G-isomorphic if there exist smooth maps $\lambda_i: U_i \to G$ such that

$$h_{ij} = \lambda_i \, g_{ij} \, \lambda_j^{-1} \ \underline{\text{on}} \ U_i \cap U_j.$$

Note that isomorphic as defined in (1.5) is the same as $GL(n)$-isomorphic for vector bundles with fibre R^n.

Suppose now that (E,π,M) is a vector bundle as in (1.7) and that H is a closed subgroup of G. A collection of smooth maps $\{\lambda_i: U_i \to G\}$ such that

$$(1.8) \qquad \lambda_i \, g_{ij} \, \lambda_j^{-1} \in H \ \underline{\text{on}} \ U_i \cap U_j$$

is called a reduction of the structure group of (E,π,M) from G to H. The following well-known result is probably due to Ehresmann [2] in 1942.

Theorem (1.9) *In the setting of the preceeding paragraph, there exists a reduction of the structure group of* (E,π,M) *from* G *to* H *if the homogeneous space* G/H *is smoothly contractible.*

The remainder of this section will be devoted to some basic constructions involving vector bundles. We will assume henceforth that (E,π,M) and (F,θ,M) are vector bundles with fibres R^n and R^m respectively, and that they have transition functions $\{g_{ij}: U_i \cap U_j \to GL(n)\}$ and $\{h_{ij}: U_i \cap U_j \to GL(m)\}$ respectively, on the open cover $\{U_i\}$ of M.

Suppose that $f: N \to M$ is a smooth map. The set $f^*E = \dot{\bigcup}_{q \in N} \pi^{-1}(f(q))$ (disjoint union) has an obvious projection onto N and a unique structure as an n-dimensional vector bundle such that $s \circ f \in \Gamma(f^*E)$ if $s \in \Gamma(E)$. This vector bundle is called the *bundle induced from* E *by* f. The transition functions of f^*E with fibre R^n are given on the trivializing cover $\{f^{-1}(U_i)\}$ of N in terms of the transition functions of E on $\{U_i\}$ by

(1.10) $\{g_{ij} \circ f: f^{-1}(U_i) \cap f^{-1}(U_j) \to GL(n)\}$.

The set $E \oplus F = \bigcup_{p \in M} \pi^{-1}(p) \oplus \theta^{-1}(p)$ has an obvious projection onto M and a unique structure as an $(n+m)$-dimensional vector bundle such that $(s,t) \in \Gamma(E \oplus F)$ if $s \in \Gamma(E)$ and $t \in \Gamma(F)$. This vector bundle is called the Whitney sum of E and F. The transition functions of $E \oplus F$ with fibre R^{n+m} are given in terms of the transition functions of E and F on $\{U_i\}$ by

(1.11) $$\{g_{ij} \oplus h_{ij} = \begin{pmatrix} g_{ij} & 0 \\ 0 & h_{ij} \end{pmatrix} : U_i \cap U_j \to GL(n+m)\}.$$

The set $E \otimes F = \bigcup_{p \in M} \pi^{-1}(p) \otimes_R \theta^{-1}(p)$ has an obvious projection onto M and a unique structure as an (nm)-dimensional vector bundle such that $s \otimes t \in \Gamma(E \otimes F)$ if $s \in \Gamma(E)$ and $t \in \Gamma(F)$. This vector bundle is called the tensor product of E and F. The transition functions of $E \otimes F$ with fibre R^{nm} are given in terms of the transition functions of E and F on $\{U_i\}$ by

(1.12) $$\{g_{ij} \otimes h_{ij} : U_i \cap U_j \to GL(nm)\},$$

where $\otimes$ denotes the Kronecker product of matrices.

The set $E^* = \bigcup_{p \in M} \pi^{-1}(p)^*$, with $*$ denoting the dual vector space, has an obvious projection onto M and a unique structure as an n-dimensional vector bundle such that the following property holds on U_i for every i. If $s_1, \cdots, s_n$ are linearly independent sections of E over U_i and $t_1(p), \cdots, t_n(p)$ is the basis of $\pi^{-1}(p)^*$ dual to $s_1(p), \cdots, s_n(p)$, then $t_1, \cdots, t_n$ are sections of E^* over U_i. This vector bundle is called the <u>dual bundle</u> of E. The transition functions of E^* with fibre R^n are given in terms of the transition functions of E on $\{U_i\}$ by

(1.13) $$\{{}^t g_{ij}^{-1} \colon U_i \cap U_j \to GL(n)\}.$$

2. JETS

Suppose that M and N are manifolds of dimensions k and n respectively, and that $p \in M$ and $q \in N$. By a <u>local map</u> from (M,p) to (N,q) we mean a smooth map $f: M \to N$ defined in an arbitrary neighborhood of $p \in M$ such that $f(p) = q$. Let $F(M,p; N,q)$ denote the set of all local maps from (M,p) to (N,q). If (x,U) and (y,V) are charts of M and N respectively, with $p \in U$ and $q \in V$, we define the <u>r-jet equivalence</u> on $F(M,p; N,q)$ by $f \sim g$ if

$$\frac{\partial^{|\alpha|}(y^i \circ f)}{\partial x^\alpha}(p) = \frac{\partial^{|\alpha|}(y^i \circ g)}{\partial x^\alpha}(p)$$

for every $i = 1, 2, \cdots, n$ and every multi-index $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{N}^k$ of non-negative integers with

$$|\alpha| = \alpha_1 + \cdots + \alpha_k \leq r, \quad \text{where} \quad \frac{\partial^{|\alpha|} F}{\partial x^\alpha} = \frac{\partial^{|\alpha|} F}{\partial x_1^{\alpha_1} \cdots \partial x_k^{\alpha_k}}.$$

This equivalence relation is independent of the choice of charts (x,U) and (y,V) by the chain rule. The equivalence class of $f \in F(M,p; N,q)$ is denoted by $j^r f(p)$

and the set of all such equivalence classes is denoted by $J^r(M,p;\ N,q)$. By allowing the <u>source points</u> $p \in M$ and/or the <u>target points</u> $q \in N$ to vary, we obtain the sets

$$J^r(M,p;\ N) = \bigcup_{q\in N} J^r(M,p;\ N,q)$$

$$J^r(M;\ N,q) = \bigcup_{p\in M} J^r(M,p;\ N,q)$$

$$J^r(M;\ N) = \bigcup_{p\in M} J^r(M,p;\ N)$$

These sets all inherit natural structures as manifolds. The manifold structure of $J^r(M;N)$ for instance, is given by the charts $\Psi\colon J^r(U;V) \to R^{k+n\binom{k+r}{r}}$ (U and V as above) defined by

$$(2.1)\qquad \Psi(j^r f(p)) = (x(p),\ \frac{1}{\alpha!}\frac{\partial^{|\alpha|}(y^i \circ f)}{\partial x^\alpha})$$

for $|\alpha| \leq r$. The other manifold structures are similarly defined and hence

$$\dim J^r(M;N) = k+n\binom{k+r}{r}$$

$$\dim J^r(M,p;N) = n\binom{k+r}{r}$$

$$\dim J^r(M;N,q) = k+n[\binom{k+r}{r}-1]$$

$$\dim J^r(M,p;N,q) = n[\binom{k+r}{r}-1].$$

The <u>source map</u> $\alpha\colon J^r(M;N) \to M$ and the <u>target map</u> $\beta\colon J^r(M;N) \to N$ are both smooth by (2.1).

A specific example of such a manifold is $J^r(R^k,0;GL(n))$ which is diffeomorphic to the open subset $GL(n) \times R^{n^2[\binom{k+r}{r}-1]}$ of $R^{n^2\binom{k+r}{r}}$. The diffeomorphism is given by

(2.2) $$j^r g(0) \to (\frac{1}{\alpha!} D^\alpha g^{ij}(0))$$

with $i,j = 1, \cdots, n$ and $|\alpha| \leq r$, where D^α denotes the partial with respect to the usual coordinates in R^k and g^{ij} are the components of g with respect to the usual coordinates on $GL(n)$ as an open subset of R^{n^2}.

$J^r(R^k,0;GL(n))$ inherits a group structure from the group structure of $GL(n)$ by

(2.3) $$j^r g(0)\ j^r h(0) = j^r(gh)(0),$$

where $gh \in F(R^k,0;GL(n))$ is defined by $gh(x) = g(x)h(x)$. The identity element of $J^r(R^k,0;GL(n))$ is $j^r I(0)$ where $I \in F(R^k,0;GL(n))$ denotes the constant map with image δ^{ij}

(Kronecker delta). The inverse of $j^r g(0)$ is $j^r(g^{-1})(0)$ where $g^{-1} \in F(R^k,0;GL(n))$ is defined by $g^{-1}(x) = (g(x))^{-1}$. Since the group operation (2.3) and the inverse operation are smooth, $J^r(R^k,0;GL(n))$ is a Lie group.

A local map in $F(R^k,0;GL(n))$ is called <u>flat</u> if it is constant on a neighborhood of 0 in R^k. Since g and h flat imply gh is flat, the set of r-jets of flat maps, denoted by $FJ^r(R^k,0;GL(n))$, is a closed subgroup of $J^r(R^k,0;GL(n))$.

For each $g \in GL(n)$ let g denote the constant map from R^k to $GL(n)$ with image g. The smooth map

$$g \in GL(n) \to j^r g(0) \in FJ^r(R^k,0;GL(n))$$

defines an isomorphism of Lie groups.

As a consequence we will use $GL(n)$ to denote $FJ^r(R^k,0;GL(n))$ and write g for $j^r h(0)$, where $h \in F(R^k,0;GL(n))$ is flat with $h(0) = g$. With this convention the identity element of $J^r(R^k,0;GL(n))$ is denoted by I.

The target map $\beta\colon J^r(R^k,0;GL(n)) \to GL(n)$ is a homomorphism of Lie groups as

(2.4) $\beta(j^r(gh)(0)) = g(0)h(0) = \beta(j^r g(0))\beta(j^r h(0))$.

The kernel of β consists of all elements of $J^r(R^k,0;GL(n))$ with target (δ^{ij}), i.e. the kernel of β is $J^r(R^k,0;GL(n),I)$. $J^r(R^k,0;GL(n),I)$ is diffeomorphic by (2.2) to $\delta^{ij} \times R^{n^2[\binom{k+r}{r}-1]}$ and hence we have

<u>Proposition (2.5)</u> $J^r(R^k,0;GL(n),I)$ <u>is a closed normal subgroup of</u> $J^r(R^k,0;GL(n))$ <u>and it is smoothly contractible</u>.

Another example of a manifold of the form $J^r(M,p;N)$ is the $n\binom{k+r}{r}$-dimensional manifold $J^r(R^k,0;R^n)$ which is diffeomorphic to $R^{n\binom{k+r}{r}}$. $J^r(R^k,0;R^n)$ inherits a vector space structure from R^n by

(2.6)
$$j^r s(0) + j^r t(0) = j^r(s+t)(0)$$
$$c\, j^r s(0) = j^r(cs)(0),$$

where $s+t,\ cs \in F(R^k,0;R^n)$ are defined for $s,\ t \in F(R^k,0;R^n)$ and $c \in R$ by $s+t\ (x) = s(x) + t(x)$

and $(cs)(x) = c\,s\,(x)$.

Definition (2.7) If G is a Lie group and M is a manifold an action of G on M on the left is a smooth map $A\colon G \times M \to M$ such that

(i) $A(gh,p) = A(g,A(h,p))$ and

(ii) $A(e,p) = p$,

for every $g, h \in G$ and $p \in M$. If M has a linear structure and $A(g, p+q) = A(g,p) + A(g,q)$, the action is said to be linear.

An action of G on M on the right is a smooth map $B\colon M \times G \to M$ satisfying the corresponding conditions. We have a smooth map

$$*\colon J^r(R^k,0;GL(n)) \times J^r(R^k,0;R^n) \to J^r(R^k,0;R^n)$$

defined by

$$j^r g(0) * j^r s(0) = j^r(gs)(0), \tag{2.8}$$

where $gs \in F(R^k,0;R^n)$ is defined by the usual matrix multiplication, i.e. $gs(x) = g(x)s(x)$. (2.8) defines a linear action of $J^r(R^k,0;GL(n))$ on $J^r(R^k,0;R^n)$ on the left. Because of the similarity of the action (2.8) and operation (2.3), we will simply write $j^r g(0)\ j^r s(0)$ for

$j^r g(0) * j^r s(0)$.

An important Lie group arising in any study of jets is the <u>group of invertible</u> (<u>or regular</u>) <u>r-jets on R^k</u>, denoted by $L^r(k)$. $L^r(k)$ consists of those $j^r f(0) \in J^r(R^k,0;R^k,0)$ for which $f \in F(R^k,0;R^k,0)$ is a local diffeomorphism and has operation given by composition, i.e.

(2.9) $$j^r f(0) \circ j^r g(0) = j^r(f \circ g)(0).$$

The identity element of $L^r(k)$ is given by $1 = j^r f(0)$ where $f \in F(R^k,0;R^k,0)$ is defined by $f(x) = x$. The inverse of $j^r f(0)$ is given by $j^r(f^{-1})(0)$ where $f^{-1} \in F(R^k,0;R^k,0)$ is the inverse local diffeomorphism of f, i.e. $f^{-1}(x) = y$ if $y = f(x)$.

One can show as in (2.2) that $L^r(k)$ is a $k[\binom{k+r}{r}-1]$-dimensional manifold which is diffeomorphic to $GL(k) \times R^{k[\binom{k+r}{r}-(k+1)]}$.

The smooth map

$$\circ : J^r(R^k,0;R^n) \times L^r(k) \to J^k(R^k,0;R^n)$$

defined by

(2.10) $$j^r s(0) \circ j^r f(0) = j^r(s \circ f)(0)$$

is a linear action of $L^r(k)$ on $J^r(R^k,0;R^n)$ on the right.

We also have a smooth map

$$\circ : J^r(R^k,0;GL(n)) \times L^r(k) \to J^r(R^k,0;GL(n))$$

defined by

(2.11) $$j^r g(0) \circ j^r f(0) = j^r(g \circ f)(0),$$

which is an action of $L^r(k)$ on $J^r(R^k,0;GL(n))$ on the right. This action preserves the group structure of $J^r(R^k,0;GL(n))$ and the action of $J^r(R^k,0;GL(n))$ on $J^r(R^k,0;R^n)$, i.e.

(2.12) $$(GH) \circ X = (G \circ X)\ (H \circ X)$$

(2.13) $$(GS) \circ X = (G \circ X)\ (S \circ X)$$

for every $G, H \in J^r(R^k,0;GL(n))$, $S \in J^r(R^k,0;R^n)$ and $X \in L^r(k)$.

The action (2.11) also preserves flatness. In fact

(2.14) $$g \circ X = g \in GL(n),$$

whenever $g \in GL(n)$ and $F \in L^r(k)$.

3. VECTOR BUNDLE CONSTRUCTIONS INVOLVING JETS

Let M denote a fixed k-dimensional manifold throughout this chapter. In chapter 2 we saw that $J^r(M;R^n)$ is a $[k+n\binom{k+r}{r}]$-dimensional manifold and that the source map $\alpha\colon J^r(M;R^n) \to M$ is smooth. Since $\alpha^{-1}(p)$ is the $n\binom{k+r}{r}$-dimensional vector space $J^r(M,p;R^n)$ with operations similar to (2.6), we see from (2.1) that $(J^r(M;R^n), \alpha, M)$ is an $n\binom{k+r}{r}$-dimensional vector bundle.

If (x,U_i) is a chart of M, the maps $h_i\colon \alpha^{-1}(U_i) \to U_i \times J^r(R^k,0;R^n)$ defined by

$$h_i(j^r f(p)) = (p, j^r(f\circ(x-x(p))^{-1})(0))$$

are trivialization of the vector bundle $J^r(M;R^n)$ with fibre $J^r(R^k,0;R^n)$. If (y,U_j) is another chart of M with $p \in U_i \cap U_j$, then

$$\begin{aligned} & j^r(f\circ(x-x(p))^{-1})(0) \\ &= j^r(f\circ(y-y(p))^{-1})(0)\circ j^r((y-y(p))\circ(x-x(p))^{-1})(0). \end{aligned}$$

The transition functions of $J^r(M;R^n)$ are therefore given by the smooth maps $D^r_{ij}(p) = j^r((y-y(p))\circ(x-x(p))^{-1})(0)$ for each $p \in U_i \cap U_j$. Note that $D^r_{ij}(p) \in L^r(k)$ for every $p \in U_i \cap U_j$ and that the collection $\{D^r_{ij}\}$ satisfy the cocycle condition (1.3) if the group operation on $L^r(k)$ is defined by $(X,Y) \to Y\circ X$. Denoting this Lie group by $L^r(k)'$, it follows from (2.10) that $L^r(k)'$ acts linearly on $J^r(R^k,0;R^n)$ on the left and hence can be identified with a subgroup of $GL(J^r(R^k,0;R^n))$. We have established

Proposition (3.1) $J^r(M;R^n)$ *is an* $n\binom{k+r}{r}$*-dimensional vector bundle with fibre* $J^r(R^k,0;R^n)$, *structure group* $L^r(k)'$ *and transition functions* $\{D^r_{ij}: U_i \cap U_j \to L^r(k)'\}$ *defined by*

$$D^r_{ij}(p) = j^r((y-y(p)) \circ (x-x(p))^{-1})(0).$$

Of primary interest is the $\binom{k+r}{r}$-dimensional vector bundle $J^r(M;R)$. Note (by the first paragraph of this chapter) that $\alpha: J^r(M;R,0) \to M$ is also a vector bundle. Since

$$J^r(M,p;R) = R \oplus J^r(M,p;R,0)$$

as vector spaces, we see from (2.1) that

$$J^r(M;R) = \mathbb{1} \oplus J^r(M;R,0)$$

as vector bundles.

The k-dimensional vector bundle $J^1(M;R,0)$ has trivializations $h_i: \alpha^{-1}(U_i) \to U_i \times R^k$ defined by

$$h_i(j^1 f(p)) = (p, (\frac{\partial f}{\partial x_1}(p), \cdots, \frac{\partial f}{\partial x_k}(p)))$$

for every $f \in F(M,p;R,0)$, where (x,U_i) is a chart of M with $p \in U_i$. If (y,U_j) is another chart of M with $p \in U_i \cap U_j$, then (by the chain rule)

$$h_i \circ h_j^{-1}(p,u) = (p, {}^t\frac{\partial(x)}{\partial(y)}^{-1}(p)\, u),$$

where $\frac{\partial(x)}{\partial(y)}$ is the usual Jacobian of the change of variable used to define the transition functions of the tangent bundle of M. Therefore

(3.2) $$J^1(M;R,0) = T^*(M),$$

the cotangent bundle of M.

The vector bundle $J^r(M;R,0)$ is therefore called the <u>rth order cotangent bundle of M</u> and is denoted by $T_r^*(M)$. These vector bundles and their dual bundles have been studied by Pohl [3] and Feldman [4]. The dual vector bundle of $T_r^*(M)$ is called the rth order tangent bundle of M and is denoted by $T_r(M)$. Pohl shows that the fibre of $T_r(M)$ over $p \in M$ is the vector space spanned by the linear functionals

$$\{ \frac{\partial^{|\alpha|}}{\partial x^\alpha}(p) \mid \alpha \in \mathbb{N}^k \text{ and } 0 < |\alpha| \leq r \},$$

where (x,U_i) is a chart of M with $p \in U_i$. A section of $T_r(M)$ is therefore an rth order partial differential operator on M without constant term. From these definitions we have

$$J^r(M;R) = 1\!\!1 \oplus T_r^*(M)$$

(3.3)

$$J^r(M;R)^* = 1\!\!1 \oplus T_r(M).$$

A section of $J^r(M;R)^*$ is therefore an rth order partial differential operator on M.

The remainder of this chapter is devoted to constructing vector bundles from a given vector bundle by taking jets of sections as in Palais [5] p. 59.

Suppose that (E,π,M) is an n-dimensional vector bundle and r is a positive integer. For each $p \in M$, let

(3.4) $$J^r(E)_p = \{j^r s(p) \mid s \in \Gamma(E)\}.$$

Note that $J^r(E)_p$ is a vector space (as $\Gamma(E)$ is itself a vector space) with operations

$$j^r s(p) + j^r t(p) = j^r(s+t)(p)$$
$$cj^r s(p) = j^r(cs)(p)$$

for every $s,t \in \Gamma(E)$ and $c \in R$. Letting

(3.5) $$J^r(E) = \bigcup_{p\in M} J^r(E)_p$$

we have a natural projection $\alpha\colon J^r(E) \to M$ (given by the source map) and $J^r(E)$ has a unique structure as a vector bundle such that

(3.6) $$j^r s\colon M \to J^r(E)$$

is a section of $J^r(E)$ whenever $s \in \Gamma(E)$.

A section of $J^r(E)$ of the form (3.6) is called holonomous. The linear map

(3.7) $$j^r\colon \Gamma(E) \to \Gamma(J^r(E))$$

is called the r-jet extension map.

Since a section of the vector bundle $\mathbb{1}$ is a smooth map from M to R, we see that $J^r(\mathbb{1}) = J^r(M;R)$ and hence by (3.3) that

$$J^r(\mathbb{1}) = \mathbb{1} \oplus T_r^*(M). \tag{3.8}$$

Assume now that the vector bundle (E,π,M) has fibre R^n, trivializations $h_i\colon \pi^{-1}(U_i) \to U_i \times R^n$ on a cover $\{U_i\}$ of charts and transition functions $g_{ij}\colon U_i \cap U_j \to GL(n)$. We define trivializations $H_i\colon \alpha^{-1}(U_i) \to U_i \times J^r(R^k,0;R^n)$ of $J^r(E)$ by

$$H_i(j^r s(p)) = (p, j^r(s_i \circ (x-x(p))^{-1})(0)), \tag{3.9}$$

where $s_i = (\text{projection on } R^n) \circ h_i \circ s\colon U_i \to R^n$ and (x,U_i) is a chart of M. If (y,U_j) is another chart of M with $p \in U_i \cap U_j$, then by (1.6) and (2.13) we have

$$\begin{aligned} & j^r(s_i \circ (x-x(p))^{-1})(0) \\ (3.10) \quad & = j^r(g_{ij} \circ (x-x(p))^{-1})(0)\, j^r(s_j \circ (x-x(p))^{-1})(0) \\ & = [j^r(g_{ij} \circ (y-y(p))^{-1})(0)\, j^r(s_j \circ (y-y(p))^{-1})(0)] \circ D^r_{ij}(p), \end{aligned}$$

where D^r_{ij} is defined in (3.1).

The transition functions of $J^r(E)$ with fibre $J^r(R^k,0;R^n)$ are therefore given on $U_i \cap U_j$ by

$$(3.11)\quad J^r g_{ij}(p) = (j^r(g_{ij}\circ(y-y(p))^{-1})(0),\ D^r_{ij}(p))$$

as elements of $J^r(R^k,0;GL(n)) \times L^r(k)$. To discover the group operation on $J^r(R^k,0;GL(n)) \times L^r(k)$ abbreviate $J^r g_{ij}(p)$ to $(j^r(g_{ij}\circ y^{-1}),\ j^r(y\circ x^{-1}))$ and note that

$$J^r g_{ij}(p) J^r g_{jk}(p) = (j^r(g_{ij}\circ y^{-1}),\ j^r(y\circ x^{-1}))(j^r(g_{jk}\circ z^{-1}),\ j^r(z\circ y^{-1})),$$

while

$$J^r g_{ik}(p) = (j^r(g_{ik}\circ z^{-1}),\ j^r(z\circ x^{-1}))$$

$$= (j^r((g_{ij}\circ z^{-1})(g_{jk}\circ z^{-1})),\ j^r(z\circ x^{-1}))$$

by (1.3). The operation on $J^r(R^k,0;GL(n)) \times L^r(k)$ is therefore defined by

$$(3.12)\qquad (G,X)\ (H,Y) = ((G\circ Y^{-1})H, Y\circ X).$$

$J^r(R^k,0;GL(n)) \times L^r(k)$ is a Lie group under the above operation with identity $(I,1)$ and inverse given by

$$(3.13)\qquad (G,X)^{-1} = (G^{-1}\circ X,\ X^{-1}).$$

Returning to (3.10) we define the smooth map

$$*: J^r(R^k,0;GL(n)) \times L^r(k) \times J^r(R^k,0;R^n) \to J^r(R^k,0;R^n)$$

by

$$(3.14) \qquad (G,X) * S = (G\circ X)\ (S\circ X).$$

A straightforward calculation shows that (3.14) defines a linear action of $J^r(R^k,0;GL(n)) \times L^r(k)$ on $J^r(R^k,0;R^n)$ on the left. As a result we can consider $J^r(R^k,0;GL(n)) \times L^r(k)$ to be a subgroup of $GL(J^r(R^k,0;R^n))$. Note that

$\dim GL(J^r(R^k,0;R^n)) = n^2\binom{k+r}{r}^2$ and

$\dim J^r(R^k,0;GL(n)) \times L^r(k) = n^2\binom{k+r}{r} + k[\binom{k+r}{r}-1]$

so that it may be a relatively "small" subgroup. To summarize, we state

Theorem (3.15) *$J^r(E)$ is an $n\binom{k+r}{r}$-dimensional vector bundle with fibre $J^r(R^k,0;R^n)$ and structure group $J^r(R^k,0;GL(n)) \times L^r(k)$. If $\{g_{ij}: U_i \cap U_j \to GL(n)\}$ are the transition functions of E, then $\{J^r g_{ij}: U_i \cap U_j \to J^r(R^k,0;GL(n)) \times L^r(k)\}$ defined by (3.11) are the transition functions of $J^r(E)$.*

4. HIGHER ORDER CONNECTIONS

Recall from chapter 2 that $GL(n)$ is a closed subgroup of $J^r(R^k,0;GL(n))$. If $g,h \in GL(n)$ and $X \in L^r(k)$, then $(g \circ X^{-1})f = gf \in GL(n)$ by (2.14) and hence $GL(n) \times L^r(k)$ is a closed subgroup of $J^r(R^k,0;GL(n)) \times L^r(k)$ by (3.12).

An <u>rth</u> <u>order</u> <u>connection</u> on the vector bundle (E,π,M) with transition functions $\{g_{ij}: U_i \cap U_j \to GL(n)\}$ is defined to be a reduction of the structure group of $J^r(E)$ from $J^r(R^k,0;GL(n)) \times L^r(k)$ to $GL(n) \times L^r(k)$. Recall from (1.8) and (3.11) that such a reduction is given by a collection of smooth maps $\Theta_i: U_i \to J^r(R^k,0;GL(n)) \times L^r(k)$ such that

(4.1) $$\Theta_i J^r g_{ij} \, \Theta_j^{-1} \in GL(n) \times L^r(k) \quad \underline{\text{on}} \quad U_i \cap U_j.$$

Letting $\Theta_i = (A_i,B_i)$, $\Theta_j = (A_j,B_j)$, $G_{ij}(p) = j^r(g_{ij} \circ (y-y(p))^{-1})(0)$ and $D_{ij} = D^r_{ij}$, (4.1) becomes

(4.2) $$(A_i,B_i)(G_{ij},D_{ij})(A_j,B_j)^{-1} = (h_{ij},H_{ij}) \quad \underline{\text{on}} \quad U_i \cap U_j,$$

with $h_{ij} \in GL(n)$ and $H_{ij} \in L^r(k)$. Using (3.13) and (2.12) in performing the multiplication in (4.2) we obtain

$$h_{ij} = (A_i \circ D_{ij}^{-1} \circ B_j)(G_{ij} \circ B_j)(A_j^{-1} \circ B_j) \tag{4.3}$$

$$H_{ij} = B_j^{-1} \circ D_{ij} \circ B_i .$$

Since $h_{ij} \in GL(n)$ it is the target of the right hand side of (4.3) and hence

$$h_{ij} = \beta(A_i) g_{ij} \beta(A_j)^{-1} \quad \underline{\text{on}} \quad U_i \cap U_j . \tag{4.4}$$

As a result (4.2) becomes

$$(A_i, B_i)(G_{ij}, D_{ij}) = (\beta(A_i) g_{ij} \beta(A_j)^{-1}, B_j^{-1} \circ D_{ij} \circ B_i)(A_j, B_j)$$

and multiplication yields

$$(4.5) \quad \begin{aligned} (A_i \circ D_{ij}^{-1}) G_{ij} &= (\beta(A_i) g_{ij} \beta(A_j)^{-1} \circ B_j^{-1}) A_j \\ D_{ij} \circ B_i &= B_j \circ B_j^{-1} \circ D_{ij} \circ B_i . \end{aligned}$$

Since the second line of (4.5) always holds we see by (2.14) that an rth order connection on E is given by a collection of smooth maps $A_i : U_i \to J^r(R^k, 0; GL(n))$ such that

$$(A_i \circ D_{ij}^{-1}) G_{ij} = \beta(A_i) g_{ij} \beta(A_j)^{-1} A_j \quad \underline{\text{on}} \quad U_i \cap U_j . \tag{4.6}$$

The reduction in (4.2) is therefore independent of B_i and

B_j. Note that if $\Theta_i = (A_i, 1)$ with A_i as in (4.6), then

(4.7) $\Theta_i J^r g_{ij} \Theta_j^{-1} = (\beta(A_i) g_{ij} \beta(A_j)^{-1}, D_{ij})$ <u>on</u> $U_i \cap U_j$.

Fixing $p \in U_i \cap U_j$ and recalling the definitions of G_{ij} and D_{ij}, we see from (4.6) that

(4.8)
$$j^r(a_i \circ (x - x(p)) \circ (y - y(p))^{-1})(0) j^r(g_{ij} \circ (y - y(p))^{-1})(0)$$
$$= a_i(0) g_{ij}(p) a_j^{-1}(0) j^r a_j(0),$$

where $A_i(p) = j^r a_i(0)$ and $A_j(p) = j^r a_j(0)$.

To "lift" (4.8) from R^k to M, note that $J^r(M,p;GL(n))$ is a Lie group for each $p \in M$ and if $f \in F(M,p;R^k,0)$ is a local diffeomorphism then the map

$$j^r g(0) \rightarrow j^r(g \circ f)(p)$$

from $J^r(R^k,0;GL(n))$ to $J^r(M,p;GL(n))$ is an isomorphism of Lie groups. Letting $f = y - y(p)$ we conclude from (4.8) that

$$j^r(a_i \circ (x - x(p)))(p) j^r g_{ij}(p)$$
$$= a_i(0) g_{ij}(p) a_j^{-1}(0) j^r(a_j \circ (y - y(p)))(p)$$

or

$$\Lambda_i(p)j^r g_{ij}(p) = \beta(\Lambda_i(p))g_{ij}(p)\beta(\Lambda_j(p))^{-1}\Lambda_j(p),$$

where $\Lambda_i(p) = j^r(a_i \circ (x-x(p)))(p)$ and $\Lambda_j(p) = j^r(a_j \circ (y-y(p)))(p)$. An rth order connection on E is therefore equivalent to a collection of smooth maps $\{\Lambda_i\colon U_i \to J^r(U_i;GL(n))\}$ such that $\Lambda_i(p) \in J^r(U_i,p;GL(n))$ and

$$(4.9) \qquad g_{ij}\beta(\Lambda_j)^{-1}\Lambda_j = \beta(\Lambda_i)^{-1}\Lambda_i j^r g_{ij} \quad \underline{\text{on}} \quad U_i \cap U_j.$$

To normalize the connection, write

$$(4.10) \qquad \Omega_i = \beta(\Lambda_i)^{-1}\Lambda_i,$$

so that $\beta(\Omega_i) = I$ on U_i.

Definition (4.11) An r-th order connection on E (with transition functions $\{g_{ij}\colon U_i \cap U_j \to GL(n)\}$) is a collection $\Omega = \{\Omega_i\colon U_i \to J^r(U_i;GL(n))\}$ of smooth maps such that $\Omega_i(p) \in J^r(U_i,p;GL(n),I)$ and

$$(4.12) \qquad g_{ij}\Omega_j = \Omega_i j^r g_{ij} \quad \underline{\text{on}} \quad U_i \cap U_j.$$

The condition in (4.12) is compatible in $U_i \cap U_j \cap U_k$, as if $g_{ij}\Omega_j = \Omega_i j^r g_{ij}$ on $U_i \cap U_j$ and $g_{jk}\Omega_k = \Omega_j j^r g_{jk}$ on $U_j \cap U_k$, then

$$g_{ij}\Omega_k = g_{ij}g_{jk}\Omega_k = g_{ij}\Omega_j j^r g_{jk} = \Omega_i j^r g_{ij} j^r g_{jk}$$
$$= \Omega_i j^r (g_{ij}g_{jk}) = \Omega_i j^r g_{ik}$$

on $U_i \cap U_j \cap U_k$.

Remark (4.13) A first order connection on E is the classical definition of a Cartan connection. To see this, note by (3.2) that we can write $\Omega_i = (I, \omega_i)$ and $\Omega_j = (I, \omega_j)$, where ω_i and ω_j are $n \times n$ matrices of 1-forms on U_i and U_j respectively. Since $j^1 g_{ij} = (g_{ij}, dg_{ij})$ on $U_i \cap U_j$ (4.12) becomes

$$(g_{ij}, 0)(I, \omega_j) = (I, \omega_i)(g_{ij}, dg_{ij}),$$

or

$$(g_{ij}, g_{ij}\omega_j) = (g_{ij}, dg_{ij} + \omega_i g_{ij})$$

by Leibnitz' rule. We conclude that

$$g_{ij}\omega_j = dg_{ij} + \omega_i g_{ij} \ \underline{\text{on}} \ U_i \cap U_j,$$

which is the definition of a Cartan connection.

Theorem (4.14) Connections of all orders exist on every vector bundle.

Proof:

It suffices, by the above construction, to show that there exists a reduction of the structure group from $J^r(R^k,0;GL(n)) \times L^r(k)$ to $GL(n) \times L^r(k)$. By (1.9) it suffices, in turn, to show that the homogeneous space

$$J^r(R^k,0;GL(n)) \times L^r(k)/GL(n) \times L^r(k)$$

is smoothly contractible. Note that each element (G,X) of $J^r(R^k,0;GL(n)) \times L^r(k)$ can be written uniquely in the form

$$((G\circ X)\beta(G)^{-1},1)(\beta(G),X). \tag{4.15}$$

Recall that G is a semidirect product of K by H if K is a normal subgroup of G, $G = KH$ and $K \cap H = \{e\}$. It follows from (4.15) that $J^r(R^k,0;GL(n)) \times L^r(k)$ is a semidirect product of $J^r(R^k,0;GL(n),I) \times 1$ by $GL(n) \times L^r(k)$ and hence (4.14) is established by (2.5) and the following lemma.

Lemma (4.16) If the Lie group G is a semidirect product of K by H (with K and H closed), then the homogeneous space G/H is diffeomorphic to K.

Proof

Since $K \cap H = \{e\}$ the function $\phi: G/H \to K$ given by

$\phi(gH) = K$ if $g = kh$ with $k \in K$ and $h \in H$ is well-defined and one-to-one. If $\pi: G \to G/H$ denotes the natural projection, then $\phi^{-1} = \pi|K$ and hence G/H is diffeomorphic to K by Warner [6] p. 120.

We will now exhibit a few properties of higher order connections. The first proposition relates higher order connections on $GL(n)$-isomorphic vector bundles.

Proposition (4.17) If the vector bundle E has transition functions $\{g_{ij}: U_i \cap U_j \to GL(n)\}$ and $\Omega = \{\Omega_i\}$ is an r-th order connection on E, then $\tilde{\Omega} = \{\lambda_i \Omega_i j^r \lambda_i^{-1}\}$ is an r-th order connection on the vector bundle with transition functions $\{\lambda_i g_{ij} \lambda_j^{-1}\}$.

Proof

On $U_i \cap U_j$ we have

$$g_{ij}\, \Omega_j = \Omega_i\, j^r g_{ij}$$

and hence

$$(\lambda_i g_{ij} \lambda_j^{-1})(\lambda_j \Omega_j j^r \lambda_j^{-1}) = (\lambda_i \Omega_i j^r \lambda_i^{-1})(j^r \lambda_i j^r g_{ij} j^r \lambda_j^{-1})$$

$$= (\lambda_i \Omega_i j^r \lambda_i^{-1}) j^r (\lambda_i g_{ij} \lambda_j^{-1}).$$

Proposition (4.18) $\Omega = I$, i.e. $\Omega = \{\Omega_i \equiv I$ for every $i\}$, is an r-th order connection on E if and only if the transition functions of E are locally constant.

Proof

$\Omega = I$ is an r-th order connection on E if and only if

$$g_{ij} = j^r g_{ij} \quad \text{on} \quad U_i \cap U_j$$

and this holds if and only if the transition functions are locally constant.

Proposition (4.19) If Ω_1, Ω_2 and Ω_3 are r-th order connections on E, then so is

$$\Omega_1 \, \Omega_2^{-1} \, \Omega_3.$$

Proof

On $U_i \cap U_j$ we have

$$g_{ij}\Omega_{1j}\Omega_{2j}^{-1}\Omega_{3j} = \Omega_{1i}j^r g_{ij}\Omega_{2j}^{-1}L_{3j} = \Omega_{1i}\Omega_{2i}^{-1}g_{ij}\Omega_{3j}$$

$$= \Omega_{1i}\Omega_{2i}^{-1}\Omega_{3i}j^r g_{ij}.$$

Corollary (*4.20*) *If* E *has locally constant transition functions, then the r-th order connections on* E *form a group*.

Proof

I is an r-th order connection on E by (4.18). Letting $\Omega_2 = I$ and then letting $\Omega_1, \Omega_3 = I$ in (4.19) we see that the product of r-th order connections is an r-th order connection and the inverse of an r-th order connection is an r-th order connection.

Remark (*4.21*) The group of r-th order connections on $1\!\!1$ consists of the sections of the "group bundle" $(J^r(M;R^*,1),\alpha,M)$, since the condition to be a connection on $1\!\!1$ is $\Omega_i = \Omega_j$ on $U_i \cap U_j$.

Proposition (*4.22*) *If* Ω_1 *and* Ω_2 *are r-th order connections on* E, *then so is*

$$\mu_1 \cdot \Omega_1 + \mu_2 \cdot \Omega_2,$$

where $\mu_i: M \to R$ *are smooth maps satisfying* $\mu_1 + \mu_2 = 1$, *and $\cdot$ denotes the action of R on $g\ell(n) = R^{n^2}$ on the left given by scalar multiplication.*

Proof

On $U_i \cap U_j$ we have

$$g_{ij}(\mu_1 \cdot \Omega_{1j} + \mu_2 \cdot \Omega_{2j}) = \mu_1 \cdot g_{ij}\Omega_{1j} + \mu_2 \cdot g_{ij}\Omega_{2j}$$

$$= \mu_1 \cdot \Omega_{1i} j^r g_{ij} + \mu_2 \cdot \Omega_{2i} j^r g_{ij}$$

$$= (\mu_1 \cdot \Omega_{1i} + \mu_2 \cdot \Omega_{2i}) j^r g_{ij}.$$

Proposition (*4.23*) *If* Ω *is an r-th order connection on* E, *then* $\rho(\Omega)$ *is an m-th order connection on* E, *where* $\rho: J^r(M;GL(n)) \to J^m(M;GL(n))$ *is the natural projection with* $1 \leq m < r$.

Proof

Since $\rho: J^r(M,p;GL(n)) \to J^m(M,p;GL(n))$ is a homomorphism of Lie groups, we see that $g_{ij}\Omega_j = \Omega_i j^r g_{ij}$ on $U_i \cap U_j$ implies

$$g_{ij}\rho(\Omega_j) = \rho(g_{ij}\Omega_j) = \rho(\Omega_i j^r g_{ij}) = \rho(\Omega_i) j^m g_{ij}$$

on $U_i \cap U_j$.

Higher order connections also have nice properties relating to vector bundle constructions.

Proposition (4.24) *If* $f\colon N \to M$ *is a smooth map and* Ω *is an r-th order connection on* E, *then* $f^*\Omega$ *is an r-th order connection on* f^*E, *where*

$$f^*j^r g(p) = j^r(g\circ f)(q) \in J^r(N,q;GL(n))$$

if $g \in F(M,p;GL(n))$ *and* $f(q) = p$.

Proof

Fix $p \in U_i \cap U_j$ and let $\Omega_i(p) = j^r h_i(p)$ and $\Omega_j(p) = j^r h_j(p)$ so that $g_{ij}(p)\Omega_j(p) = \Omega_i(p) j^r g_{ij}(p)$ is

$$j^r(g_{ij}(p)h_j)(p) = j^r(h_i g_{ij})(p).$$

If $f(q) = p$, then

$$j^r((g_{ij}(p)h_i)\circ f)(q) = j^r((h_i g_{ij})\circ f)(q)$$

$$(g_{ij}\circ f)(q) j^r(h_j\circ f)(q) = j^r(h_i\circ f)(q) j^r(g_{ij}\circ f)(q)$$

and hence

$$(g_{ij}\circ f) f^*\Omega_j = f^*\Omega_i j^r(g_{ij}\circ f) \quad \underline{\text{on}} \quad f^{-1}(U_i)\cap f^{-1}(U_j).$$

This gives the desired result by (1.10).

In the next two propositions suppose that F is a vector bundle with transition functions $\{h_{ij}: U_i \cap U_j \to GL(m)\}$.

Proposition (4.25) If Ω^E and Ω^F are r-th order connections on E and F respectively, then $\Omega^E \oplus \Omega^F$ is an r-th order connection on $E \oplus F$, where

$$j^r g(p) \oplus j^r h(p) = j^r(g \oplus h)(p) \in J^r(M,p;GL(n+m))$$

if $g \in F(M,p;GL(n))$ *and* $h \in F(M,p;GL(m))$.

Proof

On $U_i \cap U_j$ we have

$$(g_{ij} \oplus h_{ij})(\Omega_j^E \oplus \Omega_j^F) = (g_{ij}\Omega_j^E) \oplus (h_{ij}\Omega_j^F)$$

$$= (\Omega_i^E j^r g_{ij}) \oplus (\Omega_i^F j^r h_{ij}) = (\Omega_i^E \oplus \Omega_i^F)(j^r g_{ij} \oplus j^r h_{ij})$$

$$= (\Omega_i^E \oplus \Omega_i^F) j^r (g_{ij} \oplus h_{ij}),$$

which gives the result by (1.11).

Proposition (4.26) If Ω^E and Ω^F are r-th order connections on E and F respectively, then $\Omega^E \otimes \Omega^F$ is an r-th order connection on $E \otimes F$, where

$$j^r g(p) \otimes j^r h(p) = j^r(g \otimes h)(p) \in J^r(M,p;GL(nm))$$

if $g \in F(M,p;GL(n))$ *and* $h \in F(M,p;GL(m))$.

Proof

The proof is the same as (4.25) except we replace $\oplus$ by $\otimes$ and cite (1.12) instead of (1.11).

Proposition (4.27) *If* Ω *is an r-th order connection on* E, *then* ${}^t\Omega^{-1}$ *is an r-th order connection on* E^*, *where*

$$ {}^t(j^r g(p))^{-1} = j^r({}^t g^{-1})(p) \in J^r(M,p;GL(n)) $$

if $g \in F(M,p;GL(n))$.

Proof

On $U_i \cap U_j$ we have $g_{ij}\Omega_j = \Omega_i j^r g_{ij}$ and hence (as with matrices)

$$ {}^t g_{ij}^{-1}\ {}^t\Omega_j^{-1} = {}^t\Omega_i^{-1}\ {}^t j^r g_{ij}^{-1} = {}^t\Omega_i^{-1}\ j^r({}^t g_{ij}^{-1}) $$

on $U_i \cap U_j$. The conclusion follows from (1.13).

The existence of r-th order connections on E also gives information about $J^r(E)$ as a vector bundle with fibre $R^{n\binom{k+r}{r}}$.

Theorem (4.28) $J^r(E)$ *is* $GL(n\binom{k+r}{r})$-*isomorphic to* $E \otimes J^r(\mathbb{1})$.

Proof

Let $\phi\colon J^r(R^k,0;R^n) \to R^{n\binom{k+r}{r}}$ denote the vector space isomorphism defined by

$$\phi(j^r f(0)) = (D^\alpha f^1(0),\cdots,D^\alpha f^n(0)) \quad \underline{\text{for}} \quad |\alpha| \le r$$

from the usual coordinates on R^k and R^n. The map $\rho\colon J^r(R^k,0;GL(n)) \times L^r(k) \to GL(n\binom{k+r}{r})$ defined by

$$\rho(G)\phi(S) = \phi(T) \quad \underline{\text{if}} \quad G * S = T,$$

where $S,T \in J^r(R^k,0;R^n)$, is a group representation. Note from (4.7) that

$$(\beta(A_i)^{-1}A_i,1)\,J^r g_{ij}(\beta(A_j)^{-1}A_j,1)^{-1} = (g_{ij},D_{ij}) \quad \underline{\text{on}} \quad U_i \cap U_j.$$

The collection $\{\rho(\beta(A_i)^{-1}A_i,1)\colon U_i \to GL(n\binom{k+r}{r})\}$ is a $GL(n\binom{k+r}{r})$-isomorphism from $J^r(E)$ to the vector bundle with transition functions $\{\rho(g_{ij},D_{ij})\}$. A section of the vector bundle with transition functions $\{(g_{ij},D_{ij})\}$ is a collection of smooth maps $\{S_i\colon U_i \to J^r(U_i;R^n)\}$ such that $S_i(p) \in J^r(U_i,p;R^n)$ and

$$S_i = g_{ij}\,S_j \quad \underline{\text{on}} \quad U_i \cap U_j.$$

Since S_j is a section of $J^r(M;R^n)$ over U_j its components transform like sections of $J^r(\mathbb{1})$. From the definition of ϕ and (3.1) one can show that $\{\rho(g_{ij},D_{ij})\}$ are the transition functions of the vector bundle $E \otimes J^r(\mathbb{1})$ with fibre $R^{n\binom{k+r}{r}}$.

5. INTEGRABLE CONNECTIONS AND CURVATURE

Let (E,π,M) be a vector bundle with fibre R^n and transition functions $\{g_{ij}: U_i \cap U_j \to GL(n)\}$ as in chapter 4. An r-th order connection on E is called integrable if the reduction of the structure group of $J^r(E)$ is achieved by an isomorphism of E, i.e.

Definition (5.1) An r-th order connection $\Omega = \{\Omega_i\}$ on E is integrable (or flat) if there exist smooth maps $\lambda_i: U_i \to GL(n)$ such that

$$\Omega_i = \lambda_i^{-1}\, j^r\lambda_i \quad \underline{\text{on}} \quad U_i$$

for every i.

Although Chern [7] p. 34 gives a necessary and sufficient condition (involving frame bundles and universal covering bundles) for a Cartan connection to be integrable, we have been unable to find the following easy result in the literature.

Theorem (5.2) *There exists an integrable r-th order connection on E if and only if E is $GL(n)$-isomorphic to a vector bundle with locally constant transition functions.*

<u>Proof</u>

If $\Omega = \{\Omega_i = \lambda_i^{-1} j^r \lambda_i\}$ is an r-th order integrable connection on E, then by the definition of a connection

$$g_{ij}(\lambda_j^{-1} j^r \lambda_j) = (\lambda_i^{-1} j^r \lambda_i) j^r g_{ij}$$

$$\lambda_i g_{ij} \lambda_j^{-1} = j^r \lambda_i j^r g_{ij} j^r \lambda_j^{-1} = j^r(\lambda_i g_{ij} \lambda_j^{-1})$$

on $U_i \cap U_j$. Since $j^r(\lambda_i g_{ij} \lambda_j^{-1}) \in GL(n)$ <u>on</u> $U_i \cap U_j$ the collection $\{\lambda_i g_{ij} \lambda_j^{-1}\}$ are locally constant transition functions of a vector bundle GL(n)-isomorphic to E. The converse is obvious from this argument.

Note that $\Omega = I$ is an integrable r-th order connection on any vector bundle with locally constant transition functions.

<u>Remark</u> <u>(5.3)</u> If $\lambda: M \to R^*$ is any smooth map, then $\lambda^{-1} j^r \lambda$ defines an integrable r-th order connection on $\mathbb{1}$ as in (4.21).

<u>Corollary</u> <u>(5.4)</u> <u>If there exists an integrable r-th order connection on</u> E <u>and</u> M <u>is simply connected,</u> E <u>is trivial</u>.

Proof

Koszul [8] p. 58 shows that a vector bundle over a simply connected space with locally constant transition functions is trivial, so the result follows from (5.2).

Since the condition in (5.2) that E is $GL(n)$-isomorphic to a vector bundle with locally constant transition functions is independent of r, one expects a relationship between integrable connections of various orders on E.

Proposition (5.5) $\Omega = \{\lambda_i^{-1} j^1 \lambda_i\}$ is an integrable first order connection on E if and only if $\hat{\Omega} = \{\lambda_i^{-1} j^r \lambda_i\}$ is an integrable r-th order connection on E.

Proof

From the proof of (5.2) we see that Ω and $\hat{\Omega}$ are integrable if and only if $\lambda_i g_{ij} \lambda_j^{-1}$ is locally constant.

Letting ρ denote the natural projection to first order in (4.23), we have

Proposition (5.6) If Ω and Ω' are integrable r-th order connections on E with $\rho(\Omega) = \rho(\Omega')$, then $\Omega = \Omega'$.

Proof

If $\Omega = \{\lambda_i^{-1} j^r \lambda_i\}$ and $\Omega' = \{\mu_i^{-1} j^r \mu_i\}$, then the condition $\rho(\Omega) = \rho(\Omega')$ is

$$\lambda_i^{-1} j^1 \lambda_i = \mu_i^{-1} j^1 \mu_i \quad \underline{\text{on}} \quad U_i$$

for every i. Now $\mu_i \lambda_i^{-1} = j^1(\mu_i \lambda_i^{-1})$ on U_i, so $\mu_i \lambda_i^{-1} = g_i$ is locally constant and

$$\Omega_i' = \mu_i^{-1} j^r \mu_i = (g_i \lambda_i)^{-1} j^r (g_i \lambda_i) = \lambda_i^{-1} g_i^{-1} g_i j^r \lambda_i = \Omega_i$$

on U_i for every i.

The results (5.5) and (5.6) establish

Theorem (5.7) *There is a one-to-one correspondence between integrable first order and integrable r-th order connections on* E.

Many of the constructions of higher order connections in chapter 4 preserve integrability in one way or another.

Proposition (5.8) *In the setting of* (4.17), $\tilde{\Omega}$ *is integrable if and only if* Ω *is integrable.*

Proof

$$\Omega_i = \mu_i^{-1} j^r \mu_i \quad \text{on} \quad U_i \quad \text{if and only if}$$

$$\tilde{\Omega}_i = \lambda_i \mu_i^{-1} j^r \mu_i j^r \lambda_i^{-1} = (\mu_i \lambda_i^{-1})^{-1} j^r (\mu_i \lambda_i^{-1}) \quad \underline{\text{on}} \quad U_i .$$

Proposition (5.9) In the setting of (4.24), $f^*\Omega$ is an integrable connection on f^*E if Ω is an integrable connection on E.

Proof

If $\Omega_i = \lambda_i^{-1} j^r \lambda_i$ on U_i, then

$$f^*\Omega_i = (\lambda_i^{-1} \circ f) j^r (\lambda_i \circ f) = (\lambda_i \circ f)^{-1} j^r (\lambda_i \circ f) \quad \underline{\text{on}} \quad f^{-1}(U_i).$$

Proposition (5.10) In the setting of (4.25), $\Omega^E \oplus \Omega^F$ is integrable if and only if Ω^E and Ω^F are integrable.

Proof

If $\Omega_i^E = \lambda_i^{-1} j^r \lambda_i$ and $\Omega_i^F = \mu_i^{-1} j^r \mu_i$ on U_i, then

$$\Omega_i^E \oplus \Omega_i^F = \lambda_i^{-1} j^r \lambda_i \oplus \mu_i^{-1} j^r \mu_i = (\lambda_i^{-1} \oplus \mu_i^{-1})(j^r \lambda_i \oplus j^r \mu_i)$$

$$= (\lambda_i \oplus \mu_i)^{-1} j^r (\lambda_i \oplus \mu_i) \quad \underline{\text{on}} \quad U_i .$$

Conversely, if

$$\begin{bmatrix} \lambda_i & \sigma_i \\ \eta_i & \mu_i \end{bmatrix} \begin{bmatrix} \Omega_i^E & 0 \\ 0 & \Omega_i^F \end{bmatrix} = \begin{bmatrix} j^r \lambda_i & j^r \sigma_i \\ j^r \eta_i & j^r \mu_i \end{bmatrix} \quad \underline{\text{on}} \quad U_i ,$$

then $\Omega_i^E = \lambda_i^{-1} j^r \lambda_i$ and $\Omega_i^F = \mu_i^{-1} j^r \mu_i$ on U_i.

Replacing $\oplus$ by $\otimes$ in the first half of the above proof we get

<u>Proposition</u> (<u>5.11</u>) <u>In the setting of</u> (<u>4.26</u>), $\Omega^E \otimes \Omega^F$ <u>is integrable if</u> Ω^E <u>and</u> Ω^F <u>are integrable</u>.

<u>Proposition</u> (<u>5.12</u>) <u>In the setting of</u> (<u>4.27</u>), ${}^t\Omega^{-1}$ <u>is integrable if and only if</u> Ω <u>is integrable</u>.

<u>Proof</u>

$$\Omega_i = \lambda_i^{-1} j^r \lambda_i \quad \text{on} \quad U_i \quad \text{if and only if}$$

$$ {}^t\Omega_i^{-1} = {}^t\lambda_i \; j^r \; ({}^t\lambda_i^{-1}) \quad \underline{\text{on}} \quad U_i .$$

To find necessary and sufficient conditions for a connection to be integrable we consider the first order case.

A first order connection $\Omega = \{\Omega_i\}$ on E is integrable (by definition) if and only if there exist smooth maps $\lambda_i \colon U_i \to GL(n)$ such that

$$\lambda_i \Omega_i = j^1 \lambda_i \quad \underline{\text{on}} \quad U_i \tag{5.13}$$

for every i. By (3.2) we can write $\Omega_i = (I, \omega_i)$, where ω_i is an $n \times n$ matrix of 1-forms on U_i, $j^1\lambda_i = (\lambda_i, d\lambda_i)$ and $\lambda_i = (\lambda_i, 0)$ so that (5.13) becomes

$$(\lambda_i,0)(I,\omega_i) = (\lambda_i d\lambda_i) \quad \underline{\text{on}} \quad U_i.$$

Using Leibnitz' rule as in (4.13) we conclude that (5.13) is equivalent to

(5.14) $$\lambda_i\omega_i = d\lambda_i \qquad \underline{\text{on}} \quad U_i.$$

If (5.14) is assumed and d denotes the usual exterior derivative, then

$$d\lambda_i \wedge \omega_i + \lambda_i d\omega_i = d^2\lambda_i = 0$$

$$\lambda_i \omega_i \wedge \omega_i + \lambda_i d\omega_i = 0$$

$$\omega_i \wedge \omega_i + d\omega_i = 0 \qquad \underline{\text{on}} \quad U_i.$$

Hence a necessary condition for the existence of a solution λ_i to (5.14) is that the $n\times n$ matrix of 2-forms

(5.15) $$\kappa_i = d\omega_i + \omega_i \wedge \omega_i$$

vanish on U_i. One can show by a standard theorem in first order partial differential equations in Spivak [9] p. 6-11 that this condition is also sufficient. The collection $\kappa = \{\kappa_i = d\omega_i + \omega_i\wedge\omega_i\}$ is called the <u>curvature</u> of the first order connection $\Omega = \{\Omega_i = (I,\omega_i)\}$. This is the classical definition of the curvature of a Cartan connection.

Theorem (5.16) A first order connection on E is integrable if and only if its curvature κ is zero, i.e. $\kappa_i = 0$ on U_i for every i. The curvature $\kappa = \{\kappa_i\}$ of any first order connection on E satisfies

$$g_{ij}\kappa_j = \kappa_i g_{ij} \quad \text{on} \quad U_i \cap U_j .$$

The second part of (5.16) is well-known and proved by a straightforward calculation.

Suppose now that Ω is an r-th order connection on E. The usual curvature of Ω is defined to be the curvature of the first order connection $\rho(\Omega)$ on E. This curvature is denoted by κ as before. If $\kappa = 0$ then from (5.6) there is a unique integrable r-th order connection Λ on E such that $\rho(\Lambda) = \rho(\Omega)$. The higher order curvature of Ω is defined to be the collection

$$K = \{K_i = \Lambda_i - \Omega_i\}. \tag{5.17}$$

Note that $\Lambda_i - \Omega_i$ is a smooth map from U_i to $J^r(U_i; g\ell(n), 0)$ with trivial first order derivatives as $\rho(\Lambda_i) = \rho(\Omega_i)$.

From (5.5) and (5.17) we have

Theorem (5.18) An r-th order connection on E is *integrable if and only if its usual curvature* κ *and its higher order curvature* K *are zero, i.e.* $\kappa_i = 0$ *and* $K_i = 0$ *on* U_i *for every* i.

Proposition (5.19) The higher order curvature $K = \{K_i\}$ *of an r-th order connection on* E *with* $\kappa = 0$ *(so* K *is defined) satisfies*

$$g_{ij}K_j = K_i j^r g_{ij} \quad \text{on} \quad U_i \cap U_j .$$

Proof

K is the difference of two r-th order connections on E by (5.17).

The definition of the higher order curvature is somewhat of a disappointment, in spite of (5.18) and (5.19), as we had hoped to obtain "higher order characteristic classes".

To give a summary of the construction of the real Pontryagin classes of the vector bundle E, suppose that Ω is a first order connection on E with curvature κ. For every $\lambda \varepsilon R$

$$\det (\lambda I - \kappa_i) = \det (\lambda I - g_{ij}\kappa_j g_{ij}^{-1})$$

$$= \det (g_{ij}(\lambda I - \kappa_j) g_{ij}^{-1}) = \det (\lambda I - \kappa_j)$$

on $U_i \cap U_j$ by (5.16) and hence we define the global 2i-forms $p_i(\kappa)$ on M by

$$\det (\lambda I-\kappa) = \lambda^n + p_1(\kappa)\lambda^{n-1} + \cdots + p_{n-1}(\kappa)\lambda + p_n(\kappa).$$

As in Bott [10] p. 27, one can show without too much difficulty that the differential forms $p_i(\kappa)$ are closed and independent of the connection and the local trivialization. Also the deRham cohomology classes $\{p_i(\kappa)\} \in H^{2i}(M)$ are trivial if i is odd, so we define the <u>i-th Pontryagin class of</u> E by

$$p_i(E) = \{p_{2i}(\kappa)\} \in H^{4i}(M),$$

and the <u>total Pontryagin class of</u> E by

$$p(E) = 1 + p_1(E) + \cdots + p_{[\frac{n}{2}]}(E) \in H^*(M).$$

The problem with the higher order curvature on E is that it is only defined when $\kappa = 0$. In this case E can be retrivialized to have locally constant transition functions and then $\Omega = I$ is an r-th order connection on E with $K = 0$. Hence any theory of "higher order characteristic classes" from K, which is independent of the connection and the trivialization, would be trivial.

There is, however, another definition of a higher order curvature which could possibly lead to results in this area. A manifold M is called flat if there is an integrable first order connection on the tangent bundle TM. $J^1(TM)$ is also a vector bundle which may or may not have integrable first order connections. This leads to

Definition (5.20) A manifold M is r-flat for $r = 1,2,\cdots$ if there is an integrable first order connection on the vector bundle

$$J^{(r)}(TM) = J^1(J^1 \cdots J^1(TM)\cdots) \qquad (\underline{r} \text{ times}).$$

Theorem (5.21) If M is flat it is r-flat for every r.

Proof

Consider the case $r = 1$. If Ω is an integrable first order connection on TM, then $\Omega \otimes (I \oplus {}^t\Omega^{-1})$ is an integrable first order connection on $TM \otimes (\mathbb{1} \oplus T^*M)$ by (5.10), (5.11) and (5.12). $J^1(TM)$ is $GL(k(k+1))$-isomorphic to $TM \otimes (\mathbb{1} \oplus T^*M)$ by (3.8) and (4.28), and hence integrable first order connections exist on $J^1(TM)$ by (4.17) and (5.8).

The proof of the general case is now obvious from

Lemma (5.22) $J^{(r)}(TM)$ is GL-isomorphic to a vector *bundle formed from Whitney sums and tensor products of* TM, T*M *and* $\mathbb{1}$.

Proof

The proof by induction is clear from (4.28) and the definition of $J^{(r)}$.

In view of (5.21) we state

Conjecture (5.23) *There exist r-flat manifolds which are not flat.*

Note that S^2 is not flat, as S^2 flat implies by (5.4) that TS^2 is trivial, which contradicts the well-known result that there are no non-vanishing vector fields on S^2.

In the case $r = 1$, (5.23) is equivalent by (5.2) to the existence of a manifold M such that:

(i) TM *is not* GL(k)-isomorphic to a vector bundle with locally constant transition functions, and

(ii) $TM \otimes (\mathbb{1} \oplus T^*M)$ *is* GL(k(k+1))-isomorphic to a vector bundle with locally constant transition functions.

If E is GL(n)-isomorphic to a vector bundle with locally constant transition functions, there are first order connections on E with usual curvature zero, and therefore the total Pontryagin class of E is 1. Hence a candidate for a manifold M satisfying (i) and (ii) above would be one for which $p(TM \otimes (\mathbb{1} \oplus T^*M)) = 1$ and yet $p(TM) \neq 1$. Using the standard formulas of Hirzebruch [11] p. 64 for calculating characteristic classes of tensor products and Whitney sums and using the fact that T^*M is $GL(k)$-isomorphic to TM (so $TM \otimes (1 \oplus T^*M) \cong TM \oplus (TM \otimes TM))$, we can show that this is impossible. In fact one can show in this way that

$$p_1(TM \otimes (\mathbb{1} \oplus T^*M)) = (2k+1)\, p_1\ (TM)$$

and hence $p(TM \otimes (\mathbb{1} \oplus T^*M)) = 1$ implies $p_1(TM) = 0$. A calculation under the inductive assumption $p_m(TM) = 0$ for $m = 1, \cdots, i-1$ with $i \leq [\frac{k}{2}]$ shows that

$$p_i\ (TM \otimes (\mathbb{1} \oplus T^*M)) = \alpha\, p_i\ (TM),$$

where α is a positive integer which depends on i and the dimension of M. Again $p(TM \otimes (\mathbb{1} \oplus T^*M)) = 1$ implies $p_i(TM) = 0$ and hence $p(TM) = 1$.

There are other conjectures which also arise.

<u>Conjecture</u> (<u>5.24</u>) <u>Every manifold is r-flat for large enough</u> r.

Conjecture (_5.25_) _If_ M _is r-flat for some_ r, _then_ p(TM) = 1.

If these conjectures are true, the minimum r for which M is r-flat would be a new _differential invariant_.

6. COVARIANT DERIVATIVES

Suppose that $\Omega = \{\Omega_i\}$ is an r-th order connection on the vector bundle (E,π,M) with transition functions $\{g_{ij}: U_i \cap U_j \to GL(n)\}$. The covariant derivative is the composition of the r-jet extension map (3.7) and the bundle isomorphism Ω.

Definition (6.1) The covariant derivative of $s \in \Gamma(E)$ with respect to Ω is the collection

$$\nabla_\Omega s = \{\nabla_\Omega s_i = \Omega_i j^r s_i : U_i \to J^r(U_i;R^n)\}.$$

On $U_i \cap U_j$ the covariant derivative satisfies

$$g_{ij}\nabla_\Omega s_j = g_{ij}\Omega_j j^r s_j = \Omega_i j^r g_{ij} j^r s_j$$

$$= \Omega_i j^r (g_{ij}s_j) = \Omega_i j^r s_i = \nabla_\Omega s_i$$

and hence $\nabla_\Omega s \in \Gamma(J^r(E)_{red})$, where $J^r(E)_{red}$ denotes r-th jet bundle of E with reduced structure group. Since $\nabla_\Omega(s+t) = \nabla_\Omega s + \nabla_\Omega t$ for $s,t \in \Gamma(E)$ we have

Proposition (6.2) $\nabla_\Omega : \Gamma(E) \to \Gamma(J^r(E)_{red})$ is linear.

If $s \in \Gamma(E)$ with $j^r s(p) = 0$, then

$$\nabla_\Omega s(p) = \Omega_i(p) j^r s_i(p) = 0$$

and hence ∇_Ω is an <u>r-th order differential operator</u> from E to $J^r(E)_{red}$ as defined by Palais [5] p. 61. Note also that if $f\colon M \to R$ is a smooth map, then

(6.3) $$\nabla_\Omega(fs) = j^r f \; \nabla_\Omega s$$

for every $s \in \Gamma(E)$ as

$$\nabla_\Omega(fs_i) = \Omega_i j^r(fs_i) = \Omega_i j^r f \; j^r s_i$$

$$= j^r f \; \Omega_i j^r s_i = j^r f \; \nabla_\Omega s_i \quad \underline{\text{on}} \quad U_i.$$

<u>Remark</u> (<u>6.4</u>) The first order part of the covariant derivative of a first order connection is the classical definition of the covariant derivative of a Cartan connection. As in (4.13) we write $\Omega_i = (I, \omega_i)$ where ω_i is an $n \times n$ matrix of 1-forms on U_i. Since $j^1 s_i = (s_i, ds_i)$ on U_i we have

$$\nabla_\Omega s_i = \Omega_i j^1 s_i = (I, \omega_i)(s_i, ds_i)$$

$$= (s_i, \; ds_i + \omega_i s_i) \quad \underline{\text{on}} \quad U_i.$$

The first order part $\{ds_i + \omega_i s_i\}$ is the classical definition of the covariant derivative of the Cartan connection $\{\omega_i\}$.

Since s_i is also carried along in (6.1) some higher order results appear to be different from the classical results. This appearance, however, is superficial. For example, we have

Proposition (6.5) *If* $\Omega = \Omega^E \otimes \Omega^F$ *in the setting of* (4.26) *then*

$$\nabla_\Omega(s\otimes t) = \nabla_{\Omega^E} s \otimes \nabla_{\Omega^F} t$$

for every $s \in \Gamma(E)$ *and* $t \in \Gamma(F)$.

Proof

On U_i we have

$$(\Omega_i^E \otimes \Omega_i^F)\ j^r\ (s_i \otimes t_i) = (\Omega_i^E\ j^r\ s_i) \otimes (\Omega_i^F\ j^r\ t_i).$$

In a similar fashion one shows

Proposition (6.6) *If* $\Omega = \Omega^E \oplus \Omega^F$ *in the setting of* (4.25) *then*

$$\nabla_\Omega(s\oplus t) = \nabla_{\Omega^E} s \oplus \nabla_{\Omega^F} t$$

for every $s \in \Gamma(E)$ *and* $t \in \Gamma(F)$.

Theorem (6.7) Suppose that $\Omega = \{\Omega_i = \lambda_i^{-1} j^r \lambda_i\}$ is an integrable r-th order connection on E and that F is the GL(n)-isomorphic vector bundle with locally constant transition functions $\{\lambda_i g_{ij} \lambda_j^{-1}\}$ from (5.2). The collection $\{\lambda_i \nabla_\Omega s_i\}$ is then a holonomous section of $J^r(F) = J^r(F)_{red}$.

Proof:

From (6.2) we have

$$\lambda_i^{-1} j^r \lambda_i \; j^r s_i = \nabla_\Omega s_i = g_{ij} \nabla_\Omega s_j = g_{ij} \lambda_j^{-1} j^r \lambda_j j^r s_j$$

on $U_i \cap U_j$ and hence $\lambda_i \nabla_\Omega s_i = j^r(\lambda_i s_i)$ and

$$j^r(\lambda_i s_i) = \lambda_i g_{ij} \lambda_j^{-1} \; j^r(\lambda_j s_j) \quad \text{on} \quad U_i \cap U_j .$$

7. CONNECTIONS IN THE BASE AND DIFFERENTIAL OPERATORS

In this chapter we will consider a further reduction of the structure group of the jet bundle.

Note that $GL(k)$ can be considered to be a subgroup of $L^r(k)$ by identifying $g \in GL(k) \subset R^{k^2}$ with $j^r\overline{g}(0) \in L^r(k)$, where $\overline{g} \in F(R^k,0;R^k,0)$ is the linear transformation determined by g in the usual coordinates on R^k. Therefore

$$GL(k) = \{j^r f(0) \in L^r(k) \,|\, D^\alpha f^i(0)=0 \ \underline{\text{for}} \ 1 \leq i \leq k \ \underline{\text{and}} \ |\alpha| \geq 2\}$$

and hence $GL(k) = L^1(k)$ in the case $r = 1$. The smooth map $L\colon L^r(k) \to GL(k)$ defined by

$$L(j^r f(0)) = j^r(\tau f)(0),$$

where τf denotes the linear part of the Taylor series expansion of f at 0 in the usual coordinates on R^k, is a homomorphism of Lie groups. Note that L is the identity in the case $r = 1$ and also that the kernel of L is given by

(7.1) $K^r(k)=\{j^r f(0) \in L^r(k) \mid D^{e_j} f^i(0)=\delta^{ij}$ _for_ $1 \leq i,j \leq k\}$.

Suppose now that (E,π,M) is a vector bundle with transition functions $\{g_{ij}: U_i \cap U_j \to GL(n)\}$. A _strong r-th order connection on_ E is defined to be a reduction of the structure group of $J^r(E)$ from $J^r(R^k,0;GL(n)) \times L^r(k)$ to $GL(n) \times GL(k)$.

Theorem (7.2) Strong connections of all orders exist on every vector bundle.

Proof

The Lie group $J^r(R^k,0;GL(n)) \times L^r(k)$ is a semi-direct product of $J^r(R^k,0;GL(n),I) \times K^r(k)$ by $GL(n) \times GL(k)$ as

$$(G,X) = ((G \circ L(x))\beta(G)^{-1}, L(X)^{-1} \circ X)(\beta(G),L(X)).$$

$J^r(R^k,0;GL(n),I) \times K^r(k)$ is contractible by (2.5) and (7.1) and hence strong r-th order connections exist by (1.9) and (4.16).

We proceed now to find conditions for a collection of smooth maps $(A_i,B_i): U_i \to J^r(R^k,0;GL(n)) \times L^r(k)$ to reduce the structure group of $J^r(E)$ to $GL(n) \times GL(k)$.

As in chapter 4

$$(A_i,B_i)(G_{ij},D_{ij})(A_j,B_j)^{-1} = (h_{ij},L_{ij}) \quad \underline{\text{on}} \quad U_i \cap U_j$$

with $(h_{ij},L_{ij}) \in GL(n) \times GL(k)$ implies

$$h_{ij} = \beta(A_i) g_{ij} \beta(A_j)^{-1}$$

$$L_{ij} = L(B_j)^{-1} \circ L(D_{ij}) \circ L(B_i).$$

The equation

$$(A_i,B_i)(G_{ij},D_{ij})$$

$$= (\beta(A_i) g_{ij} \beta(A_j)^{-1}, L(B_j)^{-1} \circ L(D_{ij}) \circ L(B_i))(A_j,B_j)$$

implies

$$(A_i \circ D_{ij}^{-1}) G_{ij} = \beta(A_i) g_{ij} \beta(A_j^{-1}) A_j$$

(7.3)

$$D_{ij} \circ B_i = B_j \circ L(B_j)^{-1} \circ L(D_{ij}) \circ L(B_i) \text{ on } U_i \cap U_j.$$

From (7.3) one can show that

$$(\beta(A_i)^{-1} A_i, B_i \circ L(B_i)^{-1})(G_{ij},D_{ij})(\beta(A_j)^{-1} A_j, B_j \circ L(B_j)^{-1})^{-1}$$

(7.4)

$$= (g_{ij}, L(D_{ij})) \quad \text{on} \quad U_i \cap U_j.$$

Since the first line of (7.3) is independent of B and the second line of (7.3) is independent of A, we conclude that the collection (A_i, B_i) above is a strong r-th order connection on E if and only if the A_i form an r-th order connection on E as discussed in chapter 4 and the smooth maps $B_i\colon U_i \to L^r(k)$ satisfy

$$(7.5) \quad B_j^{-1} \circ D_{ij} \circ B_i = L(B_j)^{-1} \circ L(D_{ij}) \circ L(B_i) \quad \underline{\text{on}} \quad U_i \cap U_j .$$

Since $D_{ij}(p) = D_{ij}^r(p) = j^r((y-y(p))\circ(x-x(p))^{-1})(0)$ define the transition functions of $J^r(\mathbb{1})$ with fibre $J^r(R^k,0;R)$ by (3.1), the condition (7.5) does not depend on E and therefore leads to

<u>Definition</u> <u>(7.6)</u> <u>An r-th order connection in the base space</u> M (relative to a cover $\{U_i\}$) is a collection of smooth maps $B_i\colon U_i \to L^r(k)$ satisfying (7.5).

<u>Remark</u> <u>(7.7)</u> Since L is the identity in the case $r = 1$, condition (7.5) is automatically satisfied and hence a first order connection in the base space M is any collection of smooth maps $B_i\colon U_i \to L^1(k)$.

One can <u>normalize</u> the r-th order connection in the base by letting $\Omega_i = B_i \circ L(B_i)^{-1}$ and $\Omega_j = B_j \circ L(B_j)^{-1}$ so that (7.5) becomes

$$(7.8) \qquad \Omega_j^{-1} \circ D_{ij} \circ \Omega_i = L(D_{ij}) \quad \underline{\text{on}} \quad U_i \cap U_j.$$

As a result, $\Omega_i = 1$ on U_i for every i is the only first order normalized connection in the base.

One can also "lift" the r-th order connection $\{B_i\colon U_i \to L^r(k)\}$ to the manifold by defining

$$(7.9) \qquad \Theta_i(p) = j^r(b \circ (x - x(p)))(p)$$

if $B_i(p) = j^r b(0)$. In this way we obtain smooth maps $\{\Theta_i\colon U_i \to RJ^r(U_i; R^k)\}$ such that $\Theta_i(p) \in RJ^r(U_i, p; R^k, 0)$, where $RJ^r(U_i, p; R^k, 0)$ denotes the regular jets, i.e. the jets $j^r g(p)$ for which there exists $f \in F(R^k, 0; U_i, p)$ satisfying $j^r(g \circ f)(0) = 1 \in L^r(k)$. The condition (7.5) to be an r-th order connection, however, does not "lift".

A connection in the base is integrable if it arises from a change of coordinates.

Definition (7.10) An r-th order connection $\{B_i\colon U_i \to L^r(k)\}$ in the base space M is integrable if there exist diffeomorphisms $g_i: U_i \to R^k$ such that

$$\Theta_i(p) = j^r(g_i - g_i(p))(p) \quad \underline{\text{for every}} \quad p \in U_i,$$

i.e. if

$$B_i(p) = j^r((g_i - g_i(p)) \circ (x - x(p))^{-1})(0) \quad \underline{\text{for every}} \quad p \in U_i.$$

Note that $\{B_i = 1$ on $U_i\}$ is an integrable first order connection in the base space M by taking $g_i = x$. For the higher order case we make

Definition (7.11) A manifold M has a linear structure (relative to the cover $\{U_i\}$) if there are local coordinates on U_i for which the corresponding transition functions of TM are locally constant.

Since the Jacobian of the change of variable is locally constant, the local coordinates are related by linear equations. For some results on manifolds with linear structures see Chern [5] p. 39 ff.

Theorem (7.12) *There exist r-th order integrable connections in the base space M for $r \geq 2$ if and only if M has a linear structure.*

Proof

Suppose that $\{B_i : U_i \to L^r(k)\}$ is an integrable r-th order connection for $r \geq 2$ with $g_i : U_i \to R^k$ as in (7.10). From (7.5) we have

$$j^r((y-y(p))\circ(g_j-g_j(p))^{-1}\circ(y-y(p))\circ(x-x(p))^{-1}$$

$$\circ(g_i-g_i(p))\circ(x-x(p))^{-1})(0)$$

$$\varepsilon\ GL(k)\ \underline{\text{for}}\ \underline{\text{every}}\ p\ \varepsilon\ U_i \cap U_j.$$

Since $r \geq 2$ the local coordinates $x' = x\circ g_i^{-1}\circ x$ and $y' = y\circ g_j^{-1}\circ y$ on U_i and U_j respectively are related by linear equations and hence M has a linear structure.

Conversely, if M has a linear structure given by x' and y' coordinates on U_i and U_j respectively, then $B_i(p) = j^r((g_i-g_i(p))\circ(x-x(p))^{-1})(0)$ with

$$g_i = x\circ(x')^{-1}\circ x$$

is an integrable r-th order connection in the base space M.

Remark (7.13) The condition for integrability in (7.12) is independent of r. Note indeed that

$$B_i(p) = j^2((g_i - g_i(p)) \circ (x - x(p))^{-1})(0)$$

is an integrable second order connection in the base if and only if

$$\hat{B}_i(p) = j^r((g_i - g_i(p)) \circ (x - x(p))^{-1})(0)$$

is an integrable r-th order connection in the base.

The existence of r-th order connections in the base space M gives results about vector bundle isomorphisms just as r-th order connections on E did in chapter 4. From (7.8) we see that an r-th order connection in the base is an $L^r(k)'$-isomorphism from $J^r(\mathbb{1})$ to the vector bundle with transition functions $\{L(D_{ij})\}$ and fibre $J^r(R^k,0;R)$. One can show as in (4.28) that this gives a $GL(\binom{k+r}{r})$-isomorphism of the vector bundles $J^r(\mathbb{1})$ and

(7.14) $$\mathbb{1} \oplus T^*M \oplus S^2(T^*M) \oplus \cdots \oplus S^r(T^*M),$$

where $S^m(T^*M)$ denotes the <u>m-th symmetric tensor product of</u> T^*M as in Palais [5] p. 52. A strong r-th order connection on E therefore gives a $GL(n\binom{k+r}{r})$-isomorphism between $J^r(E)$ and

(7.15) $$E \otimes (\mathbb{1} \oplus T^*M \oplus \cdots \oplus S^r(T^*M)).$$

By duality we also conclude from (7.14) that $T_r(M)$ is $GL(\binom{k+r}{r}-1)$-isomorphic to the vector bundle

(7.16) $$TM \oplus S^2(TM) \oplus \cdots \oplus S^r(TM).$$

An r-th order partial differential operator on M without constant term is therefore a section of the vector bundle (7.16) after applying a connection in the base space M. Note that if $\rho: L^r(k)' \to GL(N)$ is a group representation then $\rho': L^r(k)' \to GL(N)$ defined by

(7.17) $$\rho'(F) = {}^t\rho(F)^{-1}$$

is also a group representation. This is the method of applying a connection in the base to the vector bundle $J^r(\mathbb{1})^*$.

The rest of this chapter is devoted to an examination of the local effect of applying a connection in the base space M to the vector bundle $T_r(M)$ as described in the preceeding paragraph.

To examine duality in general, suppose that (E,π,M) is an N-dimensional vector bundle with transition functions $\{g_{ij}: U_i \cap U_j \to GL(N)\}$ relative to the trivializations

$\{h_i: \pi^{-1}(U_i) \to U_i \times R^N\}$. Defining $\{s_m^{(i)}\}$ for $1 \leq m \leq N$ by

$$(7.18) \qquad s_m^{(i)}(p) = h_i^{-1}(p, e_m),$$

where e_m is the usual m-th unit vector in R^N, we obtain N linearly independent sections of E over U_i, i.e. a <u>local frame</u> of E over U_i. Letting $g_{ij} = g = (g_{mn})_{(m,n)}$ we have

$$(7.19) \qquad s_n^{(i)} = \sum_m s_m^{(i)} g_{mn} \quad \underline{\text{on}} \quad U_i \cap U_j \quad \underline{\text{for}} \quad 1 \leq n \leq N.$$

Letting $\{t_m^{(i)}\}$ denote the dual coframe as in (1.13) we also get

$$(7.20) \qquad t_m^{(i)} = \sum_n g_{mn} t_n^{(j)} \quad \underline{\text{on}} \quad U_i \cap U_j \quad \underline{\text{for}} \quad 1 \leq m \leq N.$$

Returning to $T_r^*(M)$ and $T_r(M)$ we let $N = \binom{k+r}{r} - 1$ and define the vector space isomorphism

$\phi: J^r(R^k, 0; R, 0) \to R^N$ by

$$\phi(j^r f(0)) = \left(\frac{1}{\alpha!} D^\alpha f(0)\right) \quad \underline{\text{for}} \quad 0 < |\alpha| \leq r.$$

Recall from chapter 3 that the vector bundle

$T_r^*(M) = J^r(M;R,0)$ has trivializations

$H_i:\alpha^{-1}(U_i) \to U_i \times J^r(R^k,0;R,0)$ defined by

$$H_i(j^r f(p)) = (p, j^r(f\circ(x-x(p))^{-1})(0)).$$

By composing with ϕ we also obtain trivializations

$h_i: \alpha^{-1}(U_i) \to U_i \times R^N$ of $T_r^*(M)$ defined by

$$h_i(j^r f(p)) = (p, \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(p)) \quad \underline{\text{for}} \quad 0 < |\alpha| \leq r.$$

For each $f \varepsilon F(U_i,p;R,0)$ define $\tau^r f \varepsilon F(U_i,p;R,0)$ by

$$\tau^r f = \sum_{0<|\alpha|\leq r} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(p) \quad (x-x(p))^\alpha,$$

where $g^\alpha = g_1^{\alpha_1} \cdots g_k^{\alpha_k}$ for each $\alpha \varepsilon \mathbf{N}^k$. Since

(7.21) $$j^r f(p) = j^r(\tau^r f)(p) = \sum_{0<|\alpha|\leq r} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(p) j^r((x-x(p))^\alpha)(p)$$

by Taylor's theorem, we conclude that

(7.22) $$\{X^\alpha(p) = j^r((x-x(p))^\alpha)(p)\} \quad \underline{\text{for}} \quad 0 < |\alpha| \leq r$$

is the local frame of $T_r^*(M)$ over U_i determined by h_i as in (7.18).

The group representation $\rho: L^r(k)' \to GL(N)$ defined from ϕ by

$$S \circ G = T \quad \underline{\text{if}} \quad \rho(G)\phi(S) = \phi(T),$$

where $S,T \in J^r(R^k,0;R,0)$, can now be determined from $\rho(D^r_{ij}(p))$ by finding the relationship between the collections $\{X^\alpha(p)\}$ and

$$\{Y^\beta(p) = j^r((y-y(p))^\beta)(p)\} \quad \underline{\text{for}} \quad 0 < |\beta| \leq r$$

as in (7.19). Letting $f = (y-y(p))^\beta$ in (7.21) we obtain

$$(7.23) \qquad Y^\beta(p) = \sum_{0<|\alpha|\leq r} \frac{1}{\alpha!} \frac{\partial^{|\alpha|}(y-y(p))^\beta}{\partial x^\alpha}(p)\; X^\alpha(p).$$

Since $D^r_{ij}(p) = j^r((y-y(p))\circ(x-x(p))^{-1})(0) \in L^r(k)'$ we see that $\rho: L^r(k)' \to GL(N)$ is defined by

$$(7.24) \qquad \rho(j^r g(0)) = \left(\frac{1}{\alpha!} D^\alpha(g^\beta)(0)\right)_{(\alpha,\beta)}.$$

Note that $D^\alpha(g^\beta)(0) = 0$ if $|\alpha| < |\beta|$ as $g(0) = 0$.

For the local theory we assume that $U_i = U_j = U$, so that $\{X^\alpha\}$ is a local frame of $T^*_r(M)$ over U. The collection $\{Y^{\beta}\}$ is called an <u>integrable frame</u> of $T_r^*(M)$ over U as it arises from the smooth map $Y: U \to RJ^r(U;R^k)$ defined by $Y(p) = j^r(y-y(p))(p)$ and hence the $\{Y^{\beta}\}$ are related to the $\{X^{\alpha}\}$ by applying ρ to an integrable isomorphism of $T_*^r(U)$ with structure group $L^r(k)'$ as in (7.10). A "non-integrable" frame of $T_r^*(M)$ over U arises from an arbitrary smooth map $\theta: U \to RJ^r(U;R^k)$ with $\theta(p) = j^r g(p) \in RJ^r(U,p;R^k,0)$ by defining

(7.25) $$\theta^{\beta}(p) = j^r(g^{\beta})(0) \quad \underline{\text{for}} \quad 0 < |\beta| \leq r.$$

The collection $\{\theta^{\beta}: U \to T_r^*(M)\}$ is then a local frame with $\{\theta^{\beta}(p)\}$ related to $\{X^{\alpha}(p)\}$ by

$$\rho(j^r(g\circ(x-x(p)^{-1})(0)) \in \rho(L^r(k)').$$

In fact, letting $f = g^{\beta}$ in (7.21) we obtain

(7.26) $$\theta^{\beta}(p) = \sum_{0<|\alpha|\leq r} \frac{1}{\alpha!} \frac{\partial^{|\alpha|}(g^{\beta})}{\partial x^{\alpha}}(p)\ X^{\alpha}(p)$$

for $0 < |\beta| \leq r$.

Definition (7.27) A jet frame is a local frame $\{\theta^\beta: U \to T_r^*(M)\}$ for $0 < |\alpha| \leq r$ defined from a smooth map $\theta: U \to RJ^r(U;R^k)$ as above by (7.25).

There are, of course, many other local frames for the vector bundle $T_r^*(M)$ if $r > 1$. In fact, the dimension of the jet frames of the fibre of $T_r^*(M)$ over $p \in M$ is kN, while the dimension of the arbitrary frames is N^2. Since $N = \binom{k+r}{r}-1$ these dimensions are the same in the case $r = 1$.

To study the local effect of applying a connection in the base space M to $T_r(M)$ let $\{\frac{1}{\alpha!}\frac{\partial}{\partial x^\alpha} : U \to T_r(M)\}$ and $\{\frac{1}{\beta!}\frac{\partial}{\partial \theta^\beta}: U \to T_r(M)\}$ denote the dual coframes to $\{x^\alpha\}$ and $\{\theta^\beta\}$ respectively. The constant $\frac{1}{\alpha!}$ is inserted because $\frac{\partial}{\partial x^\alpha}(p) \in J^r(U,p;R,0)^*$ is naturally defined by

$$\langle \frac{\partial}{\partial x^\alpha}(p),\ j^r f(p)\rangle = \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(p),$$

and hence

$$\left\langle \frac{\partial}{\partial x^\alpha}(p),\ j^r((x-x(p))^\beta)(p) \right\rangle$$

$$= \frac{\partial^{|\alpha|}(x-x(p))^\beta}{\partial x^\alpha}(p)$$

$$= \alpha!\ \delta^{\alpha\beta}.$$

The desired result on the local effect of reframing $T_r(M)$ with a jet frame follows from (7.19), (7.20) and (7.26).

Theorem (7.28) *If* $\{\theta^\beta : U \to T_r^*(M)\}$ *is a jet frame and* $p \in U$, *then*

$$\frac{\partial}{\partial x^\alpha}(p) = \sum_{0<|\beta|\leq r} \frac{1}{\beta!} \frac{\partial^{|\alpha|}(g^\beta)}{\partial x^\alpha}(p) \frac{\partial}{\partial \theta^\beta}(p),$$

where $\theta(p) = j^r g(p) \in J^r(U,p;R^k,0)$.

Suppose now that D is a section of $T_r(M)$ over U so that it can be written in the form

$$\text{(7.29)} \qquad D = \sum_{0<|\alpha|\leq r} a_\alpha \frac{\partial}{\partial x^\alpha},$$

with $a_\alpha : U \to R$ smooth. In terms of the jet frame $\{\theta^\beta : U \to T_r^*(M)\}$ we have

(7.30) $$D = \sum_{0<|\beta|\leq r} \left(\sum_{0<|\alpha|\leq r} a_\alpha \, g_{\alpha\beta} \right) \frac{\partial}{\partial\theta^\beta},$$

where $g_{\alpha\beta}(p) = \frac{1}{\beta!} \frac{\partial^{|\alpha|}(g^\beta)}{\partial x^\alpha}(p)$ as in (7.28).

This leads to

<u>Question (7.31)</u> <u>What are necessary conditions on</u> D <u>in</u> (7.29) <u>for the existence of a jet frame</u> $\{\theta^\beta : U \to T_r^*(M)\}$ <u>such that</u> D <u>has constant coefficients in the</u> $\{\frac{\partial}{\partial\theta^\beta} : U \to T_r(M)\}$ <u>frame?</u>

We will now examine the above question in the case $r = 2$. A calculation from (7.28) shows that

$$\frac{\partial}{\partial x_i}(p) = \sum_\ell \frac{\partial g_\ell}{\partial x_i}(p) \frac{\partial}{\partial\theta_\ell}(p)$$

$$\frac{\partial}{\partial x_i \partial x_j}(p) = \sum_\ell \frac{\partial^2 g_\ell}{\partial x_i \partial x_j}(p) \frac{\partial}{\partial\theta_\ell}(p) + \sum_{\ell,m} \frac{\partial g_\ell}{\partial x_i}(p) \frac{\partial g_m}{\partial x_j}(p) \frac{\partial}{\partial\theta_\ell \partial\theta_m}(p)$$

as $g(p) = 0$. If D is a section of $T_2(M)$ over U given by

$$D = \sum_i a_i \frac{\partial}{\partial x_i} + \sum_{i,j} a_{ij} \frac{\partial}{\partial x_i \partial x_j}$$

$$= \sum_\ell b_\ell \frac{\partial}{\partial\theta_\ell} + \sum_{\ell,m} b_{\ell m} \frac{\partial}{\partial\theta_\ell \partial\theta_m},$$

where $a_{ij} = a_{ji}$ and $b_{\ell m} = b_{m\ell}$, then

$$b_\ell(p) = \sum_i a_i(p) \frac{\partial g_\ell}{\partial x_i}(p) + \sum_{i,j} a_{ij}(p) \frac{\partial^2 g_\ell}{\partial x_i \partial x_j}(p)$$

$$b_{\ell m}(p) = \sum_{i,j} a_{ij}(p) \frac{\partial g_\ell}{\partial x_i}(p) \frac{\partial g_m}{\partial x_j}(p).$$

Since the smooth map $\theta: U \to RJ^2(U;R^k)$ is subject only to the condition $\theta(p) \in RJ^2(U,p;R^k,0)$ we see that (7.31) in the case $r = 2$ becomes

Question (7.32) *When do there exist smooth maps* $g_i^\ell, g_{ij}^\ell: U \to R$ *with* $\det(g_i^\ell) \neq 0$ *and* $g_{ij}^\ell = g_{ji}^\ell$ *such that*

(7.33) $$b_\ell = \sum_i a_i g_i^\ell + \sum_{i,j} a_{ij} g_{ij}^\ell \quad \text{for} \quad 1 \leq \ell \leq k$$

(7.34) $$b_{\ell m} = \sum_{i,j} a_{ij}\, g_i^\ell\, g_j^m \quad \text{for} \quad 1 \leq \ell,\ m \leq k$$

are constant on U?

Note that (7.34) at $p \in U$ gives the relationship between the coefficients of the symmetric quadratic forms $\sum_{i,j} a_{ij}(p)\xi_i\xi_j$ and $\sum_{\ell,m} b_{\ell,m}\,\eta_\ell\eta_m$ under the linear change of variable

$$\xi_i = \sum_\ell g_i^\ell(p)\eta_\ell \quad \text{for} \quad 1 \leq i \leq k.$$

More generally note that

$$(7.35)\qquad \left(\sum_{i,j} a_{ij}\xi_i\xi_j\right) \circ (g_m^\ell)_{(\ell,m)} = \sum_{\ell,m}\left(\sum_{i,j} a_{ij}g_i^\ell g_j^m\right)\xi_\ell\xi_m$$

is an action of $GL(k)$ on the space of symmetric quadratic forms in k indeterminants on the right. Since this action partitions the space of symmetric quadratic forms into orbits, we see that a necessary condition for D to have constant coefficients as in (7.31) is that the symmetric quadratic forms

$$A(p,\xi) = \sum_{i,j} a_{ij}(p)\xi_i\xi_j$$

belong to the same orbit for each $p \in U$.

The orbits of the action (7.35) are studied in matrix theory by associating the $k \times k$ symmetric matrix $A = (a_{ij})_{(i,j)}$ to the symmetric quadratic form $\sum_{i,j} a_{ij}\xi_i\xi_j$ and defining A and B to be <u>congruent</u> if there exists $G \in GL(k)$ such that $B = GA{}^tG$. Perlis [12] p. 91 shows that every $k \times k$ symmetric matrix is congruent to a matrix of the form

$$(7.36)\qquad \begin{bmatrix} I_\iota & 0 & 0 \\ 0 & -I_{\rho-\iota} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where the *rank* ρ and the *index* ι are uniquely determined by A. Moreover, two $k \times k$ symmetric matrices are congruent if and only if they have the same rank and index.

If $\rho(p)$ and $\iota(p)$ denote the rank and index respectively of the symmetric quadratic form $A(p,\xi)$, then the necessary condition for D to have constant coefficients as in (7.31) is

(7.37) $\rho(p)$ *and* $\iota(p)$ *are constant on* U.

We will assume for the remainder of this discussion that D satisfies (7.37). Let ρ and ι denote the constant rank and index respectively and assume that $\rho > 0$ so that D is not a first order operator. The matrix (7.36) leads us to the following stronger version of (7.34).

Question (7.38) Does there exist a smooth map $G: U \to GL(k)$ *such that*

$$A(p,\xi) \circ G(p) = \sum_{j=1}^{\iota} \xi_j^2 - \sum_{j=\iota+1}^{\rho} \xi_j^2 \quad \text{for every } p \in U?$$

To pose this question in matrix form let $S(k,\rho)$ denote the submanifold of $g\ell(k)$ consisting of all $k \times k$ symmetric matrices of rank ρ and then let

$$d = \text{diag } (1,\cdots,1,-1,\cdots,-1,0,\cdots,0) \in S(k,\rho)$$

with ι ones and $\rho-\iota$ negative ones. If $A(p)$ denotes the $k\times k$ symmetric matrix of rank ρ associated to the symmetric quadratic form $A(p,\xi)$, then
$F: U \times GL(k) \to S(k,\rho)$ defined by

$$F(p,g) = g\ A(p)\,{}^t g$$

is smooth. A careful examination of the components of F,

$$F(p,(g^i_j))_{(\ell,m)} = \sum_{i,j} a_{ij}(p) g^\ell_i g^m_j,$$

in the usual coordinates from $g\ell(k)$ shows that

$$\text{rank } \frac{\partial(F)}{\partial(g)}(p,g) = k + (k-1)+\cdots+(k-\rho+1)$$

$$= \dim S(k,\rho) \quad \underline{\text{for every}} \quad (p,g) \in F^{-1}(d).$$

We therefore conclude that d is a regular value of F and hence by Spivak [9] p. 2-30 that

$$E = F^{-1}(d) = \{(p,g) \in U\times GL(k) \mid gA(p)\,{}^t g = d\}$$

is a submanifold of $U \times GL(k)$. If $\pi: E \to U$ is the projection to U, then the matrix version of (7.38) asks for the existence of a smooth map $s: U \to E$ such that

$\pi \circ s(p) = p$ *for every* $p \in U$.

It follows from (7.37) that $\pi^{-1}(p)$ is nonempty for every $p \in U$ and hence that π is a submersion. If

$$Z = \{g \in GL(k) \mid g\, d^t g = d\},$$

then $\pi^{-1}(p) = (p, Zg)$ where (p,g) is any element of $\pi^{-1}(p)$ and hence the triple (E, π, U) is a *fibre bundle* with fibre Z. Smooth sections of E over U exist by Steenrod [1] p. 53 if we assume that U is smoothly contractible. We have now established

Theorem (7.39) Suppose that D *is a second order partial differential operator on* M *given on a smoothly contractible chart* (x,U). *If the associated symmetric quadratic forms* $A(p,\xi)$ *have constant rank* $\rho > 0$ *and constant index* ι *on* U, *then there exists a jet frame* $\{\theta^\beta : U \to T_*^2(M)\}$ *such that the second order terms of* D *on* U *are*

$$\sum_{j=1}^{\iota} \frac{\partial}{\partial \theta_j^2} - \sum_{j=\iota+1}^{\rho} \frac{\partial}{\partial \theta_j^2} . \tag{7.40}$$

Note that we have only used the first order part of the jet frame θ in reducing the second order part of D to constant coefficients. Since (7.33) with $b_\ell = 0$ amounts to solving k equations in the $k\binom{k+2}{2}$ unknowns g_{ij}^{ℓ} for every $p \in U$, we expect that D can be completely reduced to the form (7.40).

In the first order case we see that if D is a section of $T(M)$ over U given by

$$D = \sum_i a_i \frac{\partial}{\partial x_i} = \sum_\ell b_\ell \frac{\partial}{\partial \theta_\ell} ,$$

then

$$b_\ell(p) = \sum_i a_i(p) \frac{\partial g_\ell}{\partial x_i}(p) \quad \underline{\text{for every}} \quad p \in U.$$

Letting $A(p,\xi) = \sum_i a_i(p)\xi_i$ we assume that $A(p,\xi) \not\equiv 0$ for every $p \in U$ so that D is a nonvanishing vector field on U. The action of $GL(k)$ on the space of linear forms is given by matrix multiplication and leads to the definition of the smooth map $F: U \times GL(k) \to R^k$ by

$$F(p,g) = gA(p),$$

where $A(p) = (a_1(p),\cdots,a_k(p)) \in R^k$. One now checks that $d = (1,0,\cdots,0) \in R^k$ is a regular value of F and hence concludes that $E = F^{-1}(d)$ is a submanifold of $U \times GL(k)$. As in the second order case we see that $\pi\colon E \to U$ is a fibre bundle with fibre

$$Z = \{g \in GL(k) \mid gd = d\}$$

and hence obtain

Theorem (7.40) *If* D *is a nonvanishing vector field on a smoothly contractible chart* (x,U), *then there exists a jet frame* $\{\theta^\beta\colon U \to T^*(M)\}$ *such that*

$$D = \frac{\partial}{\partial\theta_1} \quad \text{on} \quad U.$$

The higher order cases are much more difficult, especially in the top order. Already for the third order case we see that if D is given on the chart (x,U) by

$$D = \sum_{i,j,q} a_{ijq} \frac{\partial}{\partial x_i \partial x_j \partial x_q} + \sum_{ij} a_{ij} \frac{\partial}{\partial x_i \partial x_j} + \sum_i a_i \frac{\partial}{\partial x_i}$$

$$= \sum_{\ell,m,n} b_{\ell mn} \frac{\partial}{\partial\theta_\ell \partial\theta_m \partial\theta_n} + \sum_{\ell,m} b_{\ell,m} \frac{\partial}{\partial\theta_\ell \partial\theta_m} + \sum_\ell b_\ell \frac{\partial}{\partial\theta_\ell},$$

where the a's and b's are symmetric, then

$$b_{\ell mn}(p) = \sum_{i,j,q} a_{ijq}(p) \frac{\partial g_\ell}{\partial x_i}(p) \frac{\partial g_m}{\partial x_j}(p) \frac{\partial g_n}{\partial x_q}(p).$$

A necessary condition for D to have constant coefficients as in (7.31) is that the symmetric cubic forms

$$A(p,\xi) = \sum_{i,j,q} a_{ijq}(p)\xi_i\xi_j\xi_q$$

belong to the same orbit of the action of $GL(k)$ on the space of symmetric cubic forms. Since there are an infinite number of orbits for $k \geq 3$ this condition is very strong.

We suspect that these algebraic conditions on the r-th order coefficients of a section of $T_r(M)$ are not sufficient to reduce to constant coefficients as in (7.31) because we must then solve $\binom{k+r-2}{r-1}$ equations in $k\binom{k+2}{2}$ unknowns at the $(r-1)$-st level.

A question similar to (7.31) which arises is

<u>Question</u> (<u>7.41</u>) <u>What are necessary conditions on</u> D <u>in</u> (<u>7.29</u>) <u>for the existence of an integrable jet frame</u> $\{Y^\beta: U \to T_r^*(M)\}$ <u>such that</u> D <u>has constant coefficients in the</u> $\{\frac{\partial}{\partial y^\beta}: U \to T_r(M)\}$ <u>frame?</u>

In the case $r = 2$, this question becomes

<u>Question</u> (<u>7.42</u>) <u>When do there exist smooth maps</u> $g_i^\ell: U \to R$

with $\det(g_i^\ell) \neq 0$ *and* $\frac{\partial g_i^\ell}{\partial x_j} = \frac{\partial g_j^\ell}{\partial x_i}$ *such that*

$$b_\ell = \sum_i a_i g_i^\ell + \sum_{i,j} a_{ij} \frac{\partial g_i^\ell}{\partial x_i} \quad \text{for} \quad 1 \leq \ell \leq k$$

$$b_{\ell m} = \sum_{i,j} a_{ij}\, g_i^\ell\, g_j^m \quad \text{for} \quad 1 \leq \ell, m \leq k$$

are constant on U?

The last equation leads to the study of the equivalence of symmetric quadratic differential forms under change of coordinates on U. Eisenhart [13] p. 25 shows that a symmetric quadratic differential form $\sum_{i,j} a_{ij}\, dx_i dx_j$ (of maximal rank k) is equivalent to a symmetric quadratic differential form with constant coefficients if and only if the components of the Riemann tensor vanish. Recall that the components of the Riemann tensor are defined by

$$R_{hij\ell} = \frac{1}{2}\left(\frac{\partial^2 a_{h\ell}}{\partial x_i \partial x_j} + \frac{\partial^2 a_{ij}}{\partial x_h \partial x_\ell} - \frac{\partial^2 a_{hj}}{\partial x_i \partial x_\ell} - \frac{\partial^2 a_{i\ell}}{\partial x_h \partial x_j}\right)$$

$$+ \sum_{m,n} a^{mn}([ij,n][h\ell,m]-[i\ell,n][hj,m]),$$

where

$$[ij,\ell] = \frac{1}{2}\left(\frac{\partial a_{i\ell}}{\partial x_j} + \frac{\partial a_{j\ell}}{\partial x_i} - \frac{\partial a_{ij}}{\partial x_\ell}\right)$$

$$(a^{\ell m})_{(\ell,m)} = (a_{ij})_{(i,j)}^{-1} .$$

Moreover, $\sum_{i,j} a_{ij}\, dx_i dx_j$ is equivalent to

$$\sum_{j=1}^{\iota} (dx_j)^2 - \sum_{j=\iota+1}^{k} (dx_j)^2$$

if the components of the Riemann tensor vanish and $A(p,\xi)$ has index ι. Hence we have

Theorem (7.43) *Suppose that D is a second order partial differential operator on M given on a chart (x,U). If the symmetric matrix $A(p)$ is nonsingular for every $p \in U$ and the components of the Riemann tensor of the symmetric quadratic differential form $\sum_{i,j} a_{ij}\, dx_i dx_j$ vanish on U, then there exists an integrable jet frame $\{Y^\beta : U \to T_2^*(M)\}$ such that the second order terms of D on U are*

$$\sum_{j=1}^{\iota} \frac{\partial}{\partial y_j^2} - \sum_{j=\iota+1}^{k} \frac{\partial}{\partial y_j^2} .$$

In analysis one is usually interested in operators which are conformally equivalent to constant coefficients, so we ask

<u>Question</u> (<u>7.44</u>) <u>What are necessary conditions on</u> D <u>in</u> (<u>7.29</u>) <u>for the existence of an integrable jet frame</u> $\{\theta^\beta : U \to T_r^*(M)\}$ <u>such that</u> D <u>in the</u> $\{\frac{\partial}{\partial y^\beta} : U \to T_r(M)\}$ <u>frame is of the form</u>

$$\lambda\Big(\sum_{0<|\beta|\leq r} b_\beta \frac{\partial}{\partial y^\beta}\Big),$$

<u>where</u> $\lambda\colon U \to R^+$ <u>is smooth and</u> b_β <u>is constant for every</u> β?

In the second order case we see from Eisenhart [13] p. 89 as before that a symmetric quadratic differential form $\sum_{i,j} a_{ij}\, dx_i dx_j$ (of maximal rank k) is conformally equivalent to a symmetric quadratic differential form with constant coefficients if and only if the components of the conformal Riemann tensor vanish and hence we have

<u>Theorem</u> (<u>7.45</u>) <u>Suppose that</u> D <u>is a second order partial differential operator on</u> M <u>given on a chart</u> (x,U). <u>If the symmetric matrix</u> A(p) <u>is nonsingular for every</u> $p \in U$ <u>and the components of the conformal Riemann tensor of the symmetric quadratic differential form</u> $\sum_{i,j} a_{ij}\, dx_i dx_j$

vanish on U, *then there exists an integrable jet frame* $\{Y^\beta : U \to T_2^*(M)\}$ *and a smooth map* $\lambda: U \to R^+$ *such that the second order terms of* D *on* U *are*

$$\lambda\Big(\sum_{j=1}^{l} \frac{\partial}{\partial y_j^2} - \sum_{j=l+1}^{k} \frac{\partial}{\partial y_j^2}\Big).$$

Remark *(7.46)* The condition that the conformal Riemann tensor vanish on U in (7.45) is automatic by Eisenhart in the cases

(i) $k = 2$

(ii) $k = 3$ *and* $a_{ij} = 0$ *for* $i \neq j$.

Finally we can ask to globalize the questions (7.31), (7.41) and (7.44) by a connection in the base. For example, we have

Question *(7.47)* *What are necessary conditions on* $D \in \Gamma(T_r(M))$ *for the existence of an r-th order connection* $\theta = \{\theta_i\}$ *in the base space* M *such that* D *has constant coefficients in the* $\{\frac{\partial}{\partial \theta_i^\beta} : U_i \to T_r(M)\}$ *frame for every* i?

8. SOME REMARKS ON ANALYTIC CAUCHY-RIEMANN STRUCTURES

In the preceeding chapter we have seen how to reduce the top order part of first and second order partial differential operators of general type to canonical form. The theory for a system of first order partial differential operators is due to Jacobi. If we consider a first order system with complex variable coefficients, however, the situation is not so simple. The study of systems of this type leads to the introduction of Cauchy-Riemann (C-R) structures which must be studied first locally and then globally. The local study is rather complicated because the system, although assumed involutive, may not have a sufficient number of functional independent solutions. We therefore restrict ourselves to the consideration of real analytic C-R structures where a local embedding theorem is available and hence a global study can be undertaken.

Let M be a submanifold of real dimension m of some open set in $\mathbb{C}^q$, i.e. a <u>locally closed submanifold</u> of $\mathbb{C}^q$. For each $p \in M$ there exists a neighborhood U of p in $\mathbb{C}^q$ and smooth maps $f_1,\dots,f_{2q-m}: U \to R$ such that

i) $M \cap U = \{z \varepsilon U | f_1(z,\bar{z}) = \cdots = f_{2q-m}(z,\bar{z}) = 0\}$

ii) at every point $y \varepsilon U$ we have

$$(df_1 \wedge \cdots \wedge df_{2q-m})_y \neq 0.$$

If U is sufficiently small one can also find smooth maps $\phi_1,\cdots,\phi_q$: $D \to \mathbb{C}$, where D is an open set in R^m, such that

i) $M \cap U = \phi(D)$, where ϕ: $D \to \mathbb{C}^q$ is the smooth map given by the equations

ii)
$$z_j = \phi_j(t_1,\cdots,t_m) \quad \underline{\text{for}} \quad 1 \leq j \leq q$$

$$\text{rank } \frac{\partial(\phi_1,\cdots,\phi_q,\bar{\phi}_1,\cdots,\bar{\phi}_q)}{\partial(t_1,\cdots,t_m)} = m \quad \underline{\text{on}} \quad D.$$

Note that if M is a real analytic locally closed submanifold of $\mathbb{C}^q$, then the above remarks hold with the maps f_α and ϕ_j being real analytic.

Suppose now that M is a closed submanifold of an open set Ω in $\mathbb{C}^q$ and let $\underset{\sim}{\mathcal{J}}(M)$ denote the sheaf on Ω of germs of C^∞ functions which vanish on M. If $U = U(p)$ is defined as above, then

(α) $\underset{\sim}{\mathcal{J}}(M)(U) = \underset{\sim}{C}^\infty(U)(f_1,\cdots,f_{2q-m})$, where $\underset{\sim}{C}^{\infty}$ is the sheaf of germs of C^{∞} functions on U, and

$$(\beta) \quad \underset{\sim}{\mathcal{J}}(M)(U) = \{g \varepsilon \underset{\sim}{C}^{\infty}(U) \mid g(\phi_1, \cdots, \phi_q, \overline{\phi}_1, \cdots, \overline{\phi}_q) \equiv 0\}.$$

A complex tangent vector to $\mathbb{C}^q$ is an expression of the form

$$X = \sum_{i=1}^{q} a_i \frac{\partial}{\partial z_i} + \sum_{i=1}^{q} b_i \frac{\partial}{\partial \overline{z}_i} \quad \underline{\text{with}} \quad a_i, b_j \ \varepsilon \ \mathbb{C}$$

This vector is <u>tangent to</u> M <u>at</u> $p \varepsilon M$ (by definition) if

$$X(f)(p) = 0 \quad \underline{\text{for every}} \quad f \varepsilon \underset{\sim}{\mathcal{J}}(M)_p.$$

The vector X is called <u>holomorphic</u> (or of type (1,0)) if the b_i are all zero. Similarly X is <u>antiholomorphic</u> (or of type (0,1)) if the a_i are all zero.

Let $\ell(p)$ denote the complex dimension of the vector space of holomorphic tangent vectors to M at $p \varepsilon M$, i.e.

$$\ell(p) = \dim_{\mathbb{C}} HT(M)_p,$$

where

$$HT(M)_p = \{X = \sum_{i=1}^{q} a_i \frac{\partial}{\partial z_i} \mid X(f)(p) = 0 \quad \underline{\text{for every}} \quad f \varepsilon \underset{\sim}{\mathcal{J}}(M)_p\}.$$

Lemma (8.1) *The function* $\ell(p)$ *is an upper semicontinuous function of* p *along* M. *Moreover,*

$$m - q \leq \ell(p) \leq \frac{m}{2}.$$

Proof

Since $\underset{\sim}{\mathcal{I}}(M)_p = \underset{\sim}{C}^{\infty}_p(f_1,\cdots,f_{2q-m})$, a vector $X = \sum_{i=1}^{q} a_i \frac{\partial}{\partial z_i}$ is tangent to M at p if and only if

$$X(f_\alpha)(p) = 0 \quad \text{for} \quad 1 \leq \alpha \leq 2q-m.$$

Thus $\ell(p)$ is the dimension of the subspace of vectors $a = (a_1,\cdots,a_q) \in \mathbb{C}^q$ such that

$$\sum_{i=1}^{q} a_i \frac{\partial f_\alpha}{\partial z_i}(p) = 0 \quad \text{for} \quad 1 \leq \alpha \leq 2q-m.$$

Hence

$$\ell(p) = q\text{-rank}\,\frac{\partial(f_1,\cdots,f_{2q-m})}{\partial(z_1,\cdots,z_q)}(p).$$

Since the rank of a matrix of C^∞ functions is lower semicontinuous, it follows that $\ell(p)$ is upper semicontinuous. Moreover, the rank of the matrix in question is less than or equal to $2q-m$ and thus

$$\ell(p) \geq q - (2q-m) = m - q.$$

Since f is real valued we also have that

$$\text{rank}\ \frac{\partial(f_1,\cdots,f_{2q-m})}{\partial(z_1,\cdots,z_q)} = \text{rank}\ \frac{\partial(f_1,\cdots,f_{2q-m})}{\partial(\overline{z}_1,\cdots,\overline{z}_q)},$$

while

$$\text{rank}\ \frac{\partial(f)}{\partial(z,\overline{z})} = 2q - m.$$

Therefore

$$\text{rank}\ \frac{\partial(f)}{\partial(z)} \geq \frac{1}{2}(2q-m)$$

and consequently

$$\ell(p) \leq q - \frac{1}{2}(2q-m) = \frac{m}{2}.$$

<u>Definition</u> (<u>8.2</u>) A locally closed submanifold M of $\mathbb{C}^q$ is <u>generic</u> if the following conditions are satisfied.

i) $\dim_{\mathbb{R}} M = m \geq q$

ii) $\dim_{\mathbb{C}} HT(M)_p = \ell(p)$ is constant along M

iii) $\ell(p)$ is minimal, i.e. $\ell(p) = m-q$.

<u>Proposition</u> (<u>8.3</u>) M <u>is generic in a neighborhood of</u> $p \in M$ <u>if and only if one of the following conditions is satisfied</u>.

$$(\alpha) \quad \text{rank} \ \frac{\partial(f_1,\cdots,f_{2q-m})}{\partial(z_1,\cdots,z_q)}(p) = 2q-m$$

$$(\beta) \quad \text{rank} \ \frac{\partial(\phi_1,\cdots,\phi_q)}{\partial(t_1,\cdots,t_m)}(t_\circ) = q \ \underline{\text{if}} \ p = \phi(t_\circ), \ t_\circ \in D.$$

Proof

The first statement follows from the proof of (8.1).

To prove (β), note that there is a neighborhood W of $t_\circ$ in D such that

$$f_\alpha(\phi(t),\bar{\phi}(t)) \equiv 0 \ \underline{\text{on}} \ W \ \underline{\text{for}} \ 1 \le \alpha \le 2q-m.$$

Therefore

$$\sum_{j=1}^{q} \frac{\partial\phi_j}{\partial t_k}\frac{\partial f_\alpha}{\partial z_j} + \sum_{j=1}^{q} \frac{\partial\bar{\phi}_j}{\partial t_k}\frac{\partial f_\alpha}{\partial \bar{z}_j} \equiv 0$$

for $1 \le \alpha \le 2q-m$ and $1 \le k \le m$ and hence for $t \in W$

$$X_k = \frac{\partial}{\partial t_k} = \sum_{j=1}^{q} \frac{\partial\phi_j}{\partial t_k}\frac{\partial}{\partial z_j} + \sum_{j=1}^{q} \frac{\partial\bar{\phi}_j}{\partial t_k}\frac{\partial}{\partial \bar{z}_j} \quad \underline{\text{for}} \quad 1 \le k \le m$$

are m linearly independent complex tangent vectors to M at $\phi(t) \in M$. These span the full space of complex tangent vectors to M at $\phi(t) \in M$. Note that the holomorphic tangent vectors are those vectors $Y = \sum_{m=1}^{m} \theta_k X_k$ with

$(\theta_1,\cdots,\theta_m) \in \mathbb{C}^m$ such that

$$(8.4) \qquad \sum_{k=1}^{m} \theta_k \frac{\partial\overline{\phi}_j}{\partial t_k} = 0 \quad \underline{\text{for}} \quad 1 \leq j \leq q.$$

Consequently

$$(8.5) \qquad \ell(\phi(t)) = m\text{-rank}\, \frac{\partial(\overline{\phi}_1,\cdots,\overline{\phi}_q)}{\partial(t_1,\cdots,t_m)}(t)$$

and thus $\ell(\phi(t)) = m-q$ is equivalent to

$$\text{rank}\, \frac{\partial(\overline{\phi}_1,\cdots,\overline{\phi}_q)}{\partial(t_1,\cdots,t_m)}(t) = q.$$

If this condition is satisfied at $t = t_o$ it is automatically satisfied in a neighborhood W of t_o in D, so (8.3) is established.

Note from (8.5) that if we write $\ell(t)$ for $\ell(\phi(t))$ and if $\ell(t) = \ell$ is constant in a neighborhood W of t_o in D, i.e. in a neighborhood of $p = \phi(t_o)$ on M, then we can choose C^∞ functions $\theta_k^{(j)}: W \to \mathbb{C}$ for $1 \leq j \leq \ell$ and $1 \leq k \leq m$ such that the vector fields

$$Y^{(j)} = \sum_{k=1}^{m} \theta_k^{(j)}(t) \frac{\partial}{\partial t_k} \quad \underline{\text{for}} \quad 1 \leq j \leq \ell$$

span $HT(M)_{p'}$ for each $p' \in M$ near p. The functions

$\theta_k^{(j)}$ can be chosen to be real analytic if the ϕ_j's are real analytic.

For an open set Ω in R^m we have

Definition (8.6) A Cauchy-Riemann structure (C-R structure) on Ω of type ℓ is a system of complex valued C^∞ vector fields on Ω

$$P_k = \sum_{j=1}^{m} c_k^j(x) \frac{\partial}{\partial x_j} \quad \underline{\text{for}} \quad 1 \leq k \leq \ell$$

such that

(i) the system is involutive, i.e. there exist $k_{\mu\sigma}^{\sigma} \in C^\infty(\Omega)$ such that

$$[P_\mu, P_\nu] = \sum_{\sigma=1}^{\ell} k_{\mu\nu}^{\sigma}(x) P_\sigma \quad \underline{\text{for}} \quad 1 \leq \mu < \nu \leq \ell,$$

(ii) the vector fields $P_1, \cdots, P_\ell, \overline{P}_1, \cdots, \overline{P}_\ell$ are linearly independent at each point of Ω.

In particular the ℓ vector fields $P_1, \cdots, P_\ell$ are linearly independent at each point of Ω.

Let $(\Omega, P_1, \cdots, P_\ell)$ and $(\Omega', P_1', \cdots, P_\ell')$ be two C-R structures of type ℓ on the open sets Ω and Ω' of R^m respectively. A diffeomorphism $\tau: \Omega \to \Omega'$ is called an

<u>isomorphism for the C-R structures</u> if $\tau_* P_j$ is in the span of $P'_1, \cdots, P'_\ell$ for $1 \leq j \leq \ell$ and $(\tau^{-1})_* P'_j$ is in the span of $P_1, \cdots, P_\ell$ for $1 \leq j \leq \ell$. A C-R structure is <u>real analytic</u> if the vector fields P_k are real analytic. An isomorphism of real analytic C-R structures is defined by a real analytic diffeomorphism.

For an m-dimensional locally closed submanifold M of $\mathbb{C}^q$, we have

<u>Proposition (8.7)</u> <u>If the dimension $\ell(p)$ of the holomorphic tangent space to M at $p \in M$ is constant and equal to ℓ, then the holomorphic tangent vector fields give a C-R structure of type ℓ to M. Moreover, if M is real analytic the C-R structure is also real analytic</u>.

<u>Proof</u>

We can choose a local C^∞ (or real analytic) basis for the holomorphic tangent vectors by (8.4). Also if $X_1, \cdots, X_\ell$ is a basis of holomorphic vector fields, then these and their conjugates are linearly independent as

$$HT(M)_p \cap \overline{HT(M)}_p = 0.$$

Since every holomorphic vector field is characterized as a vector field $X = \Sigma\, a_i(x) \frac{\partial}{\partial z_i}$, with a_i C^∞ (or real analytic) in a neighborhood of $p \in M$ in $\mathbb{C}^q$, satisfying the property

$$X\, \underset{\sim}{\mathcal{J}}(M) \subset \underset{\sim}{\mathcal{J}}(M),$$

it follows that if X_1 and X_2 are two such vector fields then so is $[X_1, X_2]$. The system of holomorphic vector fields is therefore involutive and the proof is complete.

Remark (8.8) It is customary to take the C-R structure on M defined by the antiholomorphic vector fields.

Proposition (8.9) *Every locally closed submanifold of* $\mathbb{C}^q$ *satisfying the assumptions of (8.7) is locally isomorphic to a generic locally closed submanifold of* $\mathbb{C}^{m-\ell}$.

Proof

Let $p \in M$ and let $\mathbb{C}^\ell = HT(M)_p$ be the holomorphic tangent space to M at p. Also let $R^m \supset \mathbb{C}^\ell$ be the real tangent space to M at p. We may assume that p is the origin in $\mathbb{C}^q$ and then $R^m + i\, R^m = \mathbb{C}^{m-\ell}$.

Consider the holomorphic projection λ of $\mathbb{C}^q = \mathbb{C}^{m-\ell} \times \mathbb{C}^{q-m+\ell}$ onto $\mathbb{C}^{m-\ell}$ near $0 \in \mathbb{C}^q$ and note that

$\tau = \lambda|_M$ is a diffeomorphism (analytic if M is analytic) of a neighborhood W of p in M onto its image. This map preserves the C-R structure since it is given by the restriction to M of holomorphic functions. Let $z = (z_1, \cdots, z_q)$ be holomorphic coordinates in $\mathbb{C}^q$, let $\xi = (\xi_1, \cdots, \xi_{m-\ell})$ be holomorphic coordinates in $\mathbb{C}^{m-\ell}$ and let

$$\xi_j = g_j(z_1, \cdots, z_q) \quad \underline{\text{for}} \quad 1 \le j \le m-\ell$$

be the equations of the projection map of $\mathbb{C}^q$ onto $\mathbb{C}^{m-\ell}$. If M is given by parametric equations $z_j = \phi_j(t_1, \cdots, t_m)$ then $\lambda(M)$ is given by parametric equations

$$\xi_j = g_j(\phi_1(t), \cdots, \phi_q(t)) \quad \underline{\text{for}} \quad 1 \le j \le m-\ell$$

and hence

$$\frac{\partial}{\partial t_k} = \sum_{j=1}^{m-\ell} \sum_{\alpha=1}^{q} \frac{\partial g_j}{\partial z_\alpha} \frac{\partial \phi_\alpha}{\partial t_k} \frac{\partial}{\partial \xi_j} \quad \underline{\text{for}} \quad 1 \le k \le m.$$

The antiholomorphic tangent vectors are given by

$Y = \sum_{k=1}^{m} \theta_k \frac{\partial}{\partial t_k}$ with

(8.10) $$\sum_{k=1}^{m} \theta_k \frac{\partial \phi_\alpha}{\partial t_k} = 0 \quad \underline{\text{for}} \quad 1 \le \alpha \le q,$$

and these equations have ℓ linearly independent solutions $\theta = (\theta_1, \cdots, \theta_m) \in \mathbb{C}^m$. Since (8.10) implies

(8.11) $$\sum_{k,\alpha} \theta_k \frac{\partial g_j}{\partial z_\alpha} \frac{\partial \phi_\alpha}{\partial t_k} = 0 \quad \underline{\text{for}} \quad 1 \le j \le m-\ell,$$

we see that the antiholomorphic tangent vectors to M are mapped into antiholomorphic tangent vectors to $\lambda(M)$. Also note from (8.11) that

$$\text{rank} \frac{\partial(g)}{\partial(t)} (t_\circ) = m - \ell \quad \underline{\text{for}} \quad p = \phi(t_\circ),\ t_\circ \in D,$$

and hence $\lambda(M)$ is generic and λ is an isomorphism in a neighborhood of $\lambda(p)$.

Finally let us recall

Theorem (8.12) *If $(\Omega, P_1, \cdots, P_\ell)$ is an analytic C-R structure of type ℓ defined on an open set $\Omega \subset R^m$, then for each $p \in \Omega$ there exists a neighborhood $\omega(p)$ and an analytic locally closed imbedding $\tau: \omega(p) \to \mathbb{C}^q$ $(q=m-\ell)$ such that the locally closed submanifold $M = \tau(\omega(p))$ of $\mathbb{C}^q$ is generic and the C-R structure on $\omega(p)$ coincides with the one M inherits as a locally closed submanifold of $\mathbb{C}^q$.*

Remark An example of Nirenberg shows that (8.12) fails to hold if the C-R structure is not analytic.

We will assume henceforth that M is a real analytic manifold of dimension m with a real analytic C-R structure of type ℓ. Since M can be locally realized as a generic, real analytic locally closed submanifold of $\mathbb{C}^{m-\ell}$, we let

$$z_\alpha = \phi_\alpha(t_1,\cdots,t_m) \quad \underline{\text{for}} \quad 1 \leq \alpha \leq m-\ell \quad \underline{\text{and}} \quad t \in D \subset R^m$$

be a set of parametric equations of M near p in $\mathbb{C}^{m-\ell}$ so that

$$\text{rank}\ \frac{\partial(\phi_1,\cdots,\phi_{m-\ell})}{\partial(t_1,\cdots,t_m)} = m - \ell.$$

M can also be given in a sufficiently small neighborhood U of p in $\mathbb{C}^{m-\ell}$ by equations

$$f_\beta(z_1,\cdots,z_{m-\ell},\bar{z}_1,\cdots,\bar{z}_{m-\ell}) = 0$$

for $1 \leq \beta \leq 2(m-\ell)-m = m-2\ell$ with

$$\text{rank}\ \frac{\partial(f_1,\cdots,f_{m-2\ell})}{\partial(z_1,\cdots,z_{m-\ell})} = m-2\ell.$$

The functions ϕ and f are real analytic with ϕ complex valued and f real valued.

If the analytic tangent space to M at p, taking p at the origin, is $\{z_{\ell+1} = \cdots = z_{m-\ell} = 0\}$ and if the real tangent space to M at p is $\{y_{\ell+1} = \cdots = y_{m-\ell} = 0\}$ $(z_j = x_j+iy_j)$, then the equations of M near the origin are of the form

$$y_{\ell+j} = g_{\ell+j}(z_1,\cdots,z_\ell,x_{\ell+1},\cdots,x_{m-\ell}) \quad \underline{\text{for}} \quad 1\leq j\leq m-\ell$$

with the functions $g_{\ell+j}$ real analytic and real valued.

As a short digression we prove the following proposition for a real analytic m-dimensional submanifold M of an open set Ω in R^s.

Proposition (8.13) *For every real analytic function* $g: M \to \mathbb{C}$ *there exists a real analytic function* $G: \Omega \to \mathbb{C}$ *such that* $G|_M = g$.

Proof

Let $\mathcal{A}$ be the sheaf of germs of real analytic complex valued functions in Ω, let $\mathcal{A}_M$ be the sheaf of real analytic complex valued functions on M and let $\mathcal{J}_M$ be the subsheaf of $\mathcal{A}$ of germs of functions vanishing on M, so that we have an exact sequence of sheaves:

(8.14) $$0 \to \mathcal{J}_M \to \mathcal{A} \to \mathcal{A}_M \to 0.$$

Fixing $p \varepsilon M$ we may assume, by a convenient choice of local real analytic coordinates $(x_1,\cdots,x_s)$ in R^s vanishing at p, that in a neighborhood U of p in R^s we have

$$M \cap U = \{x_{m+1} = \cdots = x_s = 0\}$$

so that $\mathcal{J}(M) = \mathcal{O}(x_{m+1},\cdots,x_s)$.

This shows that $\mathcal{J}(M)$ is an $\mathcal{O}$-coherent sheaf. As a result we have $H^1(\Omega,\mathcal{J}(M)) = 0$ which gives a surjective map

$$(8.15) \qquad H^\circ(\Omega,\mathcal{O}) \to H^\circ(M,\mathcal{O}_M) \to 0$$

from the exact cohomology sequence of (8.14) as desired.

Returning to the situation described prior to the above digression we have

Proposition (8.16) *If M is a generic locally closed submanifold of $\mathbb{C}^{m-\ell}$ as described earlier, then there exists a neighborhood $\tilde{D}$ of D in $\mathbb{C}^m$ (with holomorphic coordinates $\eta_j = t_j + is_j$ for $1 \leq j \leq m$) and a neighborhood W of M in $\mathbb{C}^{m-\ell}$ such that the map ϕ extends to a holomorphic open surjective map $\tilde{\phi}: \tilde{D} \to W$ with the property* that

$$\text{rank}\ \frac{\partial(\tilde{\phi}_1,\cdots,\tilde{\phi}_{m-\ell})}{\partial(\eta_1,\cdots,\eta_m)} = m-\ell \quad \text{on} \quad \tilde{D}.$$

Proof

The functions $\tilde{\phi}_\alpha = \phi_\alpha(t_1+is_1,\cdots,t_m+is_m)$ are holomorphic in a certain neighborhood Ω of D in $\mathbb{C}^m$ and

$$\text{rank}\ \frac{\partial(\tilde{\phi}_1,\cdots,\tilde{\phi}_{m-\ell})}{\partial(\eta_1,\cdots,\eta_m)} = m-\ell \quad \text{on} \quad D.$$

Thus there exists an open set $\tilde{D}$ in Ω with $D \subset \tilde{D} \subset \Omega$ where the rank is equal to $m-\ell$ which is maximal. The map $\tilde{D} \to \mathbb{C}^{m-\ell}$ defined by the function $\tilde{\phi}$ is therefore holomorphic and of maximal rank. Thus it is open and $W = \tilde{\phi}(\tilde{D})$ is the desired neighborhood of M in $\mathbb{C}^{m-\ell}$.

Proposition (8.17) *If $f: M \to \mathbb{C}$ is a real analytic function on M with $\bar{\partial}_M f = 0$, then*

(α) *there exists a holomorphic function F defined in some neighborhood W_F of M in $\mathbb{C}^{m-\ell}$ such that $F|_M = f$, and*

(β) *if F_1 and F_2 satisfy (α) and $W \subset W_{F_1} \cap W_{F_2}$*

is a neighborhood of M *connected with* M, *then*

$$F_1|_W = F_2|_W.$$

Proof

By the previous proposition we can select $\tilde{D}$ so small that $\xi_1 = \tilde{\phi}_1, \cdots, \xi_{m-\ell} = \tilde{\phi}_{m-\ell}$ are part of a system of holomorphic coordinates $(\xi_1, \cdots, \xi_m)$ in $\tilde{D}$. Since $\phi: D \to W$ is a real analytic isomorphism, $g = \phi^* f$ is real analytic in D. Letting $g = g(t_1, \cdots, t_m)$ and assuming $\tilde{D}$ is sufficiently small (depending on f) we can find $G = G(\xi_1, \cdots, \xi_m)$ holomorphic in $\tilde{D}$ such that $G|_D = g$. The condition $\bar{\partial}_M f = 0$ states that

$$\Sigma \theta_k \frac{\partial f}{\partial t_k} = 0 \quad \underline{\text{if}} \quad \theta \in \mathbb{C}^m \quad \underline{\text{satisfies}} \quad \Sigma \theta_k \frac{\partial \phi_\alpha}{\partial t_k} = 0$$

i.e.

$$d\phi_1 \wedge \cdots \wedge d\phi_{m-\ell} \wedge dg = 0 \quad \underline{\text{on}} \quad D$$

or

$$d\xi_1 \wedge \cdots \wedge d\xi_{m-\ell} \wedge \Sigma \frac{\partial G}{\partial \xi_j} d\xi_j \Big|_D = 0$$

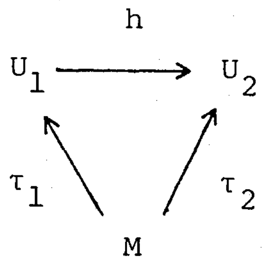
i.e.

$$\frac{\partial G}{\partial \xi_j} \Big|_D = 0 \quad \underline{\text{for}} \quad m-\ell < j \leq m.$$

This means that the holomorphic function $\frac{\partial G}{\partial \xi_j}$ vanishes in D and thus in $\tilde{D}$ (assuming $\tilde{D}$ is connected with D). Hence G is independent of $\xi_{m-\ell+1}, \cdots, \xi_m$, i.e. $G = G(\xi_1, \cdots, \xi_{m-\ell})$. Thus setting $F = G(z_1, \cdots, z_{m-\ell})$ we get a holomorphic function on $W = W(f)$ such that $F|_M = f$ and (α) is proved. The unicity of F also follows from the above considerations.

Corollary (8.18) If $\tau_1, \tau_2 \colon M \to \mathbb{C}^{m-\ell}$ are two real analytic locally closed imbeddings of M into $\mathbb{C}^{m-\ell}$ which induce the natural C-R structure on M and which are therefore generic, then there exist neighborhoods U_1 and U_2 of $\tau_1(M)$ and $\tau_2(M)$ respectively in $\mathbb{C}^{m-\ell}$ and a biholomorphic map $h\colon U_1 \to U_2$ such that

(α)



is a commutative diagram.

Moreover, if U_1 and U_2 are connected with $\tau_1(M)$ and

$\tau_2(M)$ *respectively, then the biholomorphic map* h *satisfying* (α) *is unique.*

We are now ready to consider global complexifications of a real analytic m-dimensional manifold M with a real analytic C-R structure of type ℓ.

Definition (*8.19*) A *complexification* of M is a complex manifold X of complex dimension $q = m-\ell$ and a real analytic closed imbedding $\tau: M \to X$ such that τ respects the C-R structure of M, i.e. τ imbedds M as a generic submanifold of X so that the C-R structure inherited by $\tau(M)$ from X coincides with the given C-R structure on M.

Theorem (*8.20*) (a.) *If* M *is a real analytic m-dimensional manifold with a real analytic C-R structure of type* ℓ, *then there exists a complexification* (M,τ,X) *of* M.

(b.) *If* (M,τ,X) *and* (M,σ,Y) *are two complexifications of* M, *then there exist neighborhoods* U *of* $\tau(M)$ *in* X *and* V *of* $\sigma(M)$ *in* Y *and a biholomorphic map* $h: U \to V$ *such that the diagram*

$$\begin{array}{ccc} U & \xrightarrow{\;h\;} & V \\ & {}_{\tau}\nwarrow \quad \nearrow_{\sigma} & \\ & M & \end{array}$$

commutes.

Proof

The proof is broken up into parts due to its length.

Part 1 We consider (b.), i.e. the unicity of the germ of the complexification of M along M. Note that the map $\sigma \circ \tau^{-1}: \tau(M) \to \sigma(M)$ extends by (8.18) to a neighborhood U of $\tau(M)$ in X. The extension $h: U \to Y$ is holomorphic and one to one from U to $h(U)$ if U is sufficiently small. In fact if k is a holomorphic extension of $\tau \circ \sigma^{-1}: \sigma(M) \to \tau(M)$ then $k \circ h$ is the identity in a neighborhood of $\tau(M)$ and $h \circ k$ is the identity in a neighborhood of $\sigma(M)$. Thus if U is sufficiently small and connected with $\tau(M)$, h is a uniquely defined biholomorphic map onto the open neighborhood $V = h(U)$ of $\sigma(M)$ in Y which is connected with $\sigma(M)$.

Part 2 To establish (a.) we proceed as in the proof that Bruhat-Whitney [14] gave for the case $\ell = 0$.

We can find three locally finite open covers of M with the same index set I, say $\{V_i'\}$, $\{U_i'\}$ and $\{T_i'\}$, such that

$$V_i' \subset\subset U_i' \subset\subset T_i' \quad \underline{\text{for every}} \quad i \in I.$$

For each $i \in I$ we can find local complexified models

$T_i \subset \tilde{T}_i$ for T_i', where $\tilde{T}_i$ is an open set in $\mathbb{C}^q (q=m-\ell)$, T_i is a generic real analytic submanifold of $\tilde{T}_i$ and with a real analytic isomorphism $\phi_i : T_i' \to T_i$ compatible with the C-R structure of T_i' and the one induced by $\tilde{T}_i$ on T_i. We now set

$$(8.21) \qquad U_i = \phi_i U_i', \qquad V_i = \phi_i V_i'$$

$$(8.22) \qquad U_{ij} = \phi_i (U_i' \cap U_j'), \qquad V_{ij} = \phi_i (V_i' \cap V_j')$$

$$T_{ij} = \phi_i (T_i' \cap T_j').$$

The isomorphism

$$(8.23) \qquad \phi_j \circ \phi_i^{-1} : T_{ij} \to T_{ji}$$

extends by (8.18) to a biholomorphic map $\psi_{ji} : \tilde{T}_{ij} \to \tilde{T}_{ji}$ of a neighborhood $\tilde{T}_{ij}$ of T_{ij} in $\tilde{T}_i$ to a neighborhood $\tilde{T}_{ji}$ of T_{ji} in $\tilde{T}_j$. We can assume that $\tilde{T}_{ij}$ is empty if T_{ij} is empty and that $\psi_{ij} = \psi_{ji}^{-1}$. For every ordered pair (i,j) we can select open sets $\tilde{U}_{ij}$ in $\tilde{T}_{ij}$ such that $\tilde{U}_{ij} \subset\subset \tilde{T}_{ij}$, $\psi_{ji} \tilde{U}_{ij} = \tilde{U}_{ji}$ and

$$(8.24) \qquad \tilde{U}_{ij} \cap T_i = U_{ij} \ \underline{\text{and}} \ \bar{\tilde{U}}_{ij} \cap T_i = \bar{U}_{ij},$$

where the bar denotes the closure of the set. Since $\overline{V}_i \cap \psi_{ij}(\overline{V}_j \cap \overline{U}_{ji})$ is a compact subset of U_{ij} we can choose open sets $\tilde{W}_{ij} \subset \tilde{T}_{ij}$ such that $\tilde{W}_{ij} \subset\subset \tilde{U}_{ij}$, $\tilde{W}_{ij} = \psi_{ij}(\tilde{W}_{ji})$ and

(8.25) $$\overline{V}_i \cap \psi_{ij}(\overline{V}_j \cap \overline{U}_{ji}) \subset \tilde{W}_{ij}.$$

The subsets $\overline{V}_i - \tilde{W}_{ij}$ and $\psi_{ij}(\overline{V}_j \cap \overline{U}_{ji}) - \tilde{W}_{ij}$ of T_i are compact and disjoint, and therefore contained in disjoint open sets $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$ of $\tilde{T}_i$ respectively, so we have

$$\overline{V}_i \subset \tilde{A}_{ij} \cup \tilde{W}_{ij}$$

(8.26)

$$\psi_{ij}(\overline{V}_j \cap \overline{U}_{ji}) \subset \tilde{B}_{ij} \cup \tilde{W}_{ij}.$$

We now choose $\tilde{A}_i$ open in $\tilde{T}_i$ such that

(8.27) $$\tilde{A}_i \cap T_i = V_i, \qquad \overline{\tilde{A}}_i \cap T_i = \overline{V}_i$$

and

(8.28) $\tilde{A}_i \subset \tilde{A}_{ij} \cup \tilde{W}_{ij}$ <u>for all</u> $j \varepsilon I$ <u>such that</u> $T_{ij} \neq \phi$.

The last condition can be satisfied since there are only a finite number of $j \in I$ such that T_{ij} is nonempty.

Since $\bar{\tilde{A}}_i \cap \bar{\tilde{U}}_{ij}$ is compact and contained in $\tilde{T}_{ij}$ we have $\overline{\psi_{ji}(\tilde{A}_i \cap \tilde{U}_{ij})} \subset \psi_{ji}(\bar{\tilde{A}}_i \cap \bar{\tilde{U}}_{ij})$. By (8.24) and (8.27) we have

$$\psi_{ji}(\bar{\tilde{A}}_i \cap \bar{\tilde{U}}_{ij}) \cap T_j = \psi_{ji}(\bar{\tilde{A}}_i \cap \bar{\tilde{U}}_{ij} \cap T_i)$$

$$= \psi_{ji}(\bar{V}_i \cap \bar{U}_{ij})$$

and hence

(8.29) $$\overline{\psi_{ji}(\tilde{A}_i \cap \tilde{U}_{ij})} \cap T_j \subset \psi_{ji}(\bar{V}_i \cap \bar{U}_{ij}).$$

Part 3 For any point $x \in U_i$ there exists an open set $\tilde{U}_i(x)$ in $\tilde{T}_i$ containing x and satisfying the following five conditions:

(1) $\tilde{U}_i(x) \subset \tilde{U}_{ij}$ for every index j such that $x \in U_{ij}$.

(2) $\tilde{U}_i(x) \subset \tilde{B}_{ij} \cup \tilde{W}_{ij}$ for every index j such that $x \in \psi_{ij}(\bar{V}_j \cap \bar{U}_{ji})$. (Compare (8.26).)

(3) $\tilde{U}_i(x) \cap \psi_{ij}(\tilde{A}_j \cap \tilde{U}_{ji})$ is empty for every index j such that $\phi_i^{-1}(x) \notin \overline{V_j'}$.

(4) $\tilde{U}_i(x) \subset \psi_{ij}(\tilde{U}_{ji} \cap \tilde{U}_{jk}) \cap \psi_{ik}(\tilde{U}_{ki} \cap \tilde{U}_{kj})$ for every

pair of indices (j,k) such that $x \in U_{ij} \cap U_{ik}$ (i.e. $\phi_i^{-1}(x) \in U_i' \cap U_j' \cap U_k'$).

(5) $\psi_{ji} = \psi_{jk} \circ \psi_{ki}$ on $\tilde{U}_i(x)$ for all (j,k) as in (4).

The conditions (1), (2) and (4) are satisfied because the number of indices involved is finite. Condition (5) is satisfied on $U_{ij} \cap U_{ik}$ and hence in a neighborhood. Condition (3) is trivially satisfied if $\tilde{U}_{ji}$ is empty. Otherwise there are at most finitely many j's such that $\tilde{U}_{ji}$ is nonempty and $\phi_i^{-1}(x) \notin \overline{V_j'}$ implies $x \notin \psi_{ij}(\bar{V}_j \cap \bar{U}_{ji})$ so $x \notin \psi_{ij}(\tilde{A}_j \cap \tilde{U}_{ji})$ by (8.29).

Part 4 Set

$$\tilde{U}_i = \bigcup_{x \in U_i} \tilde{U}_i(x)$$

and let $\tilde{V}_i$ be a neighborhood of V_i in $\tilde{T}_i$ which is contained in $\tilde{A}_i$ and relatively compact in $\tilde{U}_i$. Note from (8.27) that $\tilde{V}_i \cap T_i = V_i$ and $\bar{\tilde{V}}_i \cap T_i = \bar{V}_i$. Setting

$$\tilde{V}_{ij} = \tilde{V}_i \cap \psi_{ij}(\tilde{V}_j \cap \tilde{U}_{ji})$$

(8.30)

$$\tilde{V}_{ijk} = \tilde{V}_{ij} \cap \tilde{V}_{ik}$$

we see that $\tilde{V}_{ij} \subset \tilde{U}_{ij}$ and $\psi_{ij}\colon \tilde{V}_{ji} \tilde{\to} \tilde{V}_{ij}$.

Let $y \in \tilde{V}_{ijk}$, so $y \in \tilde{U}_i(x)$ for some $x \in U_i$. Since $\tilde{V}_{ijk}$ intersects $\psi_{ij}(\tilde{V}_j \cap \tilde{U}_{ji})$ and $\psi_{ik}(\tilde{V}_k \cap \tilde{U}_{ki})$ it also meets $\psi_{ij}(\tilde{A}_j \cap \tilde{U}_{ji})$ and $\psi_{ik}(\tilde{A}_k \cap \tilde{U}_{ki})$ and hence $x \in \tilde{U}_{ij} \cap \tilde{U}_{ik}$ by (3). Therefore $\psi_{ki}(y) \in \tilde{V}_k \cap \tilde{U}_{kj}$ and $\psi_{jk} \circ \psi_{ki}(y) = \psi_{ji}(y)$ by (2), so

$$\begin{aligned} z = \psi_{ji}(y) &\in \tilde{V}_j \\ &\in \psi_{ji}(\tilde{V}_i \cap \tilde{U}_{ij}) \\ &\in \psi_{jk}(\tilde{V}_k \cap \tilde{U}_{kj}) \\ &\in \tilde{V}_{jik}. \end{aligned}$$

We therefore conclude that $\psi_{ji}(\tilde{V}_{ijk}) \subset \tilde{V}_{jik}$ and by symmetry $\psi_{ij}(\tilde{V}_{jik}) \subset \tilde{V}_{ijk}$, so

$$\psi_{ji}\colon \tilde{V}_{ijk} \tilde{\to} \tilde{V}_{jik}.$$

We also have $\psi_{ij} = \psi_{ji}^{-1}$ and $\psi_{ji} = \psi_{jk} \circ \psi_{ki}$ on $\tilde{V}_{ijk}$

so $\{\tilde{V}_i, \psi_{ij}\}$ is an amalgamation system and we can construct the amalgamated sum

$$\{\tilde{\Omega}, \psi_i\} = \lim \{\tilde{V}_i, \psi_{ij}\}.$$

This sum is obtained from the disjoint union $\dot{\bigcup}_{i \varepsilon I} \tilde{V}_i$ by dividing out by the equivalence relation

$$x \varepsilon \tilde{V}_i \sim y \varepsilon \tilde{V}_j \ \underline{\text{if}} \ x \varepsilon \tilde{V}_{ij},\ y \varepsilon \tilde{V}_{ji} \ \underline{\text{and}}$$
$$y = \psi_{ji}(x).$$

The space $\tilde{\Omega}$ is a complex manifold if it is Hausdorff. The natural maps $\psi_i : \tilde{V}_i \to \tilde{\Omega}$ are given by the canonical open imbeddings

$$\tilde{V}_i \to \dot{\bigcup} \tilde{V}_i \to \tilde{\Omega},$$

and the isomorphism ϕ_i merge into an isomorphism ϕ of M onto a generically imbedded real analytic submanifold of $\tilde{\Omega}$.

<u>Part</u> <u>5</u> The proof will be complete if we show that $\tilde{\Omega}$ has a Hausdorff topology.

We will first show that $\tilde{V}_{ij} \subset \tilde{U}_{ij}$ and more precisely that $\tilde{V}_{ij} \subset \tilde{W}_{ij}$ (when T_{ij} is nonempty). If

$y \in \tilde{V}_{ij}$ then there exists $x \in U_i$ such that $y \in \tilde{U}_i(x)$ since $\tilde{V}_{ij} \subset \tilde{V}_i \subset \tilde{U}_i$.

If $x \notin \psi_{ij}(\bar{V}_j \cap \bar{U}_{ji})$ then $\phi_i^{-1}(x) \notin V_j'$ and hence $y \notin \psi_{ij}(\tilde{A}_j \cap \tilde{U}_{ji})$ by (3). Since this contradicts the fact that $y \in \tilde{V}_{ij}$ in (8.30), we have that $x \in \psi_{ij}(\bar{V}_j \cap \bar{U}_{ji})$ and hence by (2) that $y \in \tilde{B}_{ij} \cup \tilde{W}_{ij}$. Since $\tilde{V}_{ij} \subset \tilde{V}_i \subset \tilde{A}_i$ we see from (8.28) that $y \in \tilde{A}_{ij} \cup \tilde{W}_{ij}$ and hence that $y \in \tilde{W}_{ij}$ as $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$ are disjoint. In conclusion we have

$$\tilde{V}_{ij} \subset \tilde{W}_{ij} \subset \tilde{U}_{ij}.$$

Suppose now that x' and y' are two points of $\tilde{\Omega}$ with $x' \neq y'$. Let $x \in \tilde{V}_i$ and $y \in \tilde{V}_j$ such that $\psi_i(x) = x'$ and $\psi_j(y) = y'$. It suffices to show that there exist neighborhoods A of x in $\tilde{V}_i$ and B of y in $\tilde{V}_j$ such that no point of A is equivalent to a point of B.

If this was not the case we could find sequences $\{x_k\}$ in $\tilde{T}_i$ and $\{y_k\}$ in $\tilde{T}_j$ converging to x and y respectively with $x_k \in \tilde{V}_{ij}$, $y_k \in \tilde{V}_{ji}$ and $x_k = \psi_{ij}(y_k)$ for every k. Since $\tilde{V}_{ij} \subset \tilde{U}_{ij}$ we see that $x \in \tilde{U}_{ij}$ and by symmetry $y \varepsilon \tilde{U}_{ji}$, so $x = \psi_{ij}(y)$ by continuity. Thus

$$y \varepsilon \tilde{V}_j \cap \tilde{U}_{ji} \subset \tilde{V}_{ji},$$

$$x \varepsilon \tilde{V}_i \cap \psi_{ij}(\tilde{V}_j \cap \tilde{U}_{ji}) = \tilde{V}_{ij}$$

and $y = \psi_{ji}(x)$, so x is equivalent to y which is contrary to our assumption.

This completes the proof of (8.20).

Suppose now that M is a real analytic m-dimensional manifold with two real analytic C-R structures α and β of types $\ell(\alpha)$ and $\ell(\beta)$ respectively.

Definition (8.31) The C-R structure α dominates the C-R structure β and we write $\alpha > \beta$ if for every $p \varepsilon M$

$$\text{span}(Y_1(p),\cdots,Y_{\ell(\beta)}(p)) \subset \text{span}(X_1(p),\cdots,X_{\ell(\alpha)}(p)),$$

where $X_1,\cdots,X_{\ell(\alpha)}$ and $Y_1,\cdots,Y_{\ell(\beta)}$ are sets of linearly independent vector fields on a neighborhood U of p in M which define α and β respectively.

The C-R structure with $\ell(\beta) = 0$ is called the totally real structure of M and we write $\beta = 0$. Note that every C-R structure α on M dominates 0.

As an immediate consequence of (8.16) we have

Theorem (8.32) *If* $\tau_o\colon M \to X^m$ *and* $\tau_\alpha\colon M \to X^{m-\ell(\alpha)}$ *are complexifications of* M *with respect to the C-R structures* 0 *and* α *respectively, then there exist neighborhoods* U *of* $\tau_o(M)$ *in* X^m *and* V *of* $\tau_\alpha(M)$ *in* $X^{m-\ell(\alpha)}$ *and a surjective holomorphic map* $\phi\colon U \to V$ *of maximal rank such that the diagram*

$$\begin{array}{ccccc} & \overset{\tau_o}{\nearrow} & U & \subset & X^m \\ M & & \downarrow \phi & & \\ & \underset{\tau_\alpha}{\searrow} & V & \subset & X^{m-\ell(\alpha)} \end{array}$$

is commutative. Moreover, ϕ *is uniquely defined if* U *is connected with* $\tau_o(M)$ *and sufficiently small.*

Corollary (8.33) *If* $\tau_\beta\colon M \to X^{m-\ell(\beta)}$ *and* $\tau_\alpha\colon M \to X^{m-\ell(\alpha)}$ *are complexifications of* M *with respect to the C-R structures* β *and* α *respectively and if* $\alpha > \beta$, *then there exist neighborhoods* U *of* $\tau_\beta(M)$ *in* $X^{m-\ell(\beta)}$ *and* V *of* $\tau_\alpha(M)$ *in* $X^{m-\ell(\alpha)}$ *and a surjective holomorphic map* $\phi\colon U \to V$ *of maximal rank such that the diagram*

$$\begin{array}{ccccc} & \overset{\tau_\beta}{\nearrow} & U & \subset & X^{m-\ell(\beta)} \\ M & & \downarrow \phi & & \\ & \underset{\tau_\alpha}{\searrow} & V & \subset & X^{m-\ell(\alpha)} \end{array}$$

<u>commutes</u>. <u>Moreover</u>, ϕ <u>is uniquely defined if</u> U <u>is connected with</u> $\tau_\beta(M)$ <u>and sufficiently small</u>.

<u>Proof</u>

Let

$$z_j = \phi_j(t_1,\cdots,t_m) \quad \underline{\text{for}} \quad 1\leq j\leq m-\ell(\alpha),\ t \in D \subset R^m$$

and

$$\xi_k = \psi_k(t_1,\cdots,t_m) \quad \underline{\text{for}} \quad 1\leq k\leq m-\ell(\beta),\ t \in D \subset R^m$$

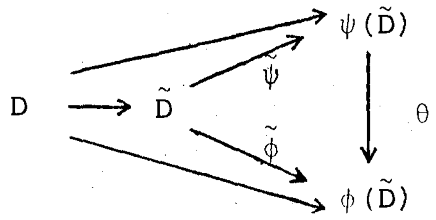
be local parametric equations of M in $\mathbb{C}^{m-\ell(\alpha)}$ and $\mathbb{C}^{m-\ell(\beta)}$ respectively which realize the C-R structures of α and β respectively. If $\theta = (\theta_1,\cdots,\theta_m) \in \mathbb{C}^m$, then $\Sigma\theta_k \frac{\partial\psi_j}{\partial t_k} = 0$ for $1 \leq j \leq m-\ell(\beta)$ implies $\Sigma\theta_k \frac{\partial\phi_s}{\partial t_k} = 0$ for $1 \leq s \leq m-\ell(\alpha)$ since $\alpha>\beta$. Thus

$$\frac{\partial\phi_s}{\partial t_k} = \Sigma\ c_j^s \frac{\partial\psi_j}{\partial t_k} \quad \underline{\text{on}} \quad D,$$

so $d\phi \wedge d\psi_1 \wedge \cdots \wedge d\psi_{m-\ell(\beta)} = 0$ on D and hence there are holomorphic extensions $\tilde{\phi}$ and $\tilde{\psi}$ of ϕ and ψ respectively in a neighborhood $\tilde{D}$ of D in $\mathbb{C}^m$ such that

$$d\tilde{\phi} \wedge d\tilde{\psi}_1 \wedge \cdots \wedge d\tilde{\psi}_{m-\ell(\beta)} = 0 \quad \underline{\text{on}} \quad \tilde{D}.$$

It now follows that the map $\tilde{\phi}$ factors through the map $\tilde{\psi}$ and hence we have the commutative diagram:



Since θ is uniquely defined and $\tilde{D}$ is locally the complexification of M with the totally real structure, the conclusion follows from this local statement.

The final theorem of this chapter is a result about the convexity of the tubular neighborhoods in the complexification. It is a theorem of Grauert [15] when $\ell = 0$.

If X is an n-dimensional complex manifold we say that X is ℓ-<u>complete</u> if there exists a C^∞ function $\phi: X \to R$ such that

i) $B_c = \{x \varepsilon X | \phi(x) < c\}$ is relatively compact in X for every $c \varepsilon R$, and

ii) at each point $x_o \varepsilon X$ the Levi form of ϕ

$$\mathcal{L}(\phi)_{x_o}(u) = \Sigma \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta}(x_o) u_\alpha \bar{u}_\beta$$

has at least $n-\ell$ positive eigenvalues.

Theorem (8.34) *If M is a compact m-dimensional real analytic manifold with a real analytic C-R structure of type ℓ and $\tau: M \to X^{m-\ell}$ is a complexification of M, then there exists an open neighborhood U of $\tau(M)$ in X which is an ℓ-complete manifold.*

Remark Since every open neighborhood U of $\tau(M)$ in X is a complexification of M we see that $\tau(M)$ has a fundamental system of neighborhoods in X which are ℓ-complete.

Proof

Let $\{U_i\}$ be a locally finite covering of $\tau(M)$ in X by open charts $U_i \subset C^{m-\ell}$ such that

$$\tau(M) \cap U_i = \{z \in U_i \mid f^i_\alpha(z,\bar{z}) = 0 \ \underline{\text{for}} \ 1 \leq \alpha \leq m-2\ell\},$$

where the functions $f^i_\alpha: U_i \to R$ are real analytic with

$$(8.35) \qquad \operatorname{rank} \frac{\partial(f^i_1,\cdots,f^i_{m-2\ell})}{\partial(z_1,\cdots,z_{m-\ell})} = m-2\ell \ \underline{\text{on}} \ U_i.$$

For each index i we set

$$v^i = \sum_{\alpha=1}^{m-2\ell} |f^i_\alpha|^2$$

so that

$$\partial\bar{\partial}v^i \,|_{M\cap U_i} = \Sigma|\partial f^i_\alpha|^2 \geq 0.$$

From (8.35) we see that

$$\partial\bar{\partial}v^i|_{\mathbb{C}^{m-2\ell}(y)} > 0 \quad \underline{\text{for every}} \quad y \in M \cap U_i,$$

where $\mathbb{C}^{m-2\ell}(y)$ denotes the linear space spanned by $\partial f_1^i(y), \cdots, \partial f_{m-2\ell}^i(y)$.

Let $\{\rho_i : U_i \to R\}$ be a partition of unity subordinate to the cover $\{U_i\}$ with $\rho_i \geq 0$ and set

$$\theta = \sum_i \rho_i v^i.$$

There exists a sufficiently small open neighborhood W of $\tau(M)$ in X such that θ is smoothly defined in W with $\theta \geq 0$, $\theta(x) = 0$ if and only if $x \in \tau(M)$, and with $\mathfrak{L}(\theta)$ having at least $(m-\ell)-\ell = m-2\ell$ positive eigenvalues.

Since $\tau(M)$ is a compact subset of W there exists an open neighborhood U of $\tau(M)$ in W with U relatively compact in W. Hence $\partial U = \bar{U} - U$ is a compact set which does not meet $\tau(M)$ and we set

$$\delta = \min_{x \in \partial U} \theta(x) > 0.$$

If $V = \{x \in U | \theta(x) < \delta\}$ then $\bar{V} \subset \bar{U}$ and $g(x) = \frac{1}{\delta}\theta(x)$ has the following properties on V:

i) $0 \leq g(x) < 1$ and $g(x) = 0$ if and only if $x \varepsilon \tau(M)$.

ii) $\mathcal{L}(g)_x$ has at least $m-2\ell$ positive eigenvalues for every $x \varepsilon V$ (as $\mathcal{L}(g)_x > 0$ on a subspace $\mathbb{C}^{m-2\ell}(x)$ of the complex tangent space at x).

We now define $\phi: V \to R$ by

$$\phi(x) = \frac{1}{1-g(x)}$$

so that $B_c = \{x \varepsilon V | \phi(x) < c\}$ is relatively compact in V for every $c \varepsilon R$. Also $\mathcal{L}(\phi)$ has at least $m-2\ell$ positive eigenvalues on V and hence V is ℓ-complete and (8.34) is established.

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