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Title AN IMPULSE-RESPONSE MEASUREMENT APPARATUS FOR LINEAR SYSTEMS USING A RECTANGULAR INPUT WITH A POISSON DISTRIBUTION OF ZERO CROSSINGS

Abstract approved

For many applications it is desirable to be able to experimentally determine the impulse response of a linear system while it is in operation. Use of correlation techniques allows this to be done without noticeably disturbing the quiescent output value of the system.

An impulse response measurement apparatus is designed and constructed using correlation techniques. This device is based on the Input-Output Crosscorrelation Theorem. A Poisson rectangular wave is used as the input.

The Poisson rectangular wave is a waveform which alternates between $+E_m$ volts and $-E_m$ volts at random intervals of time. The zero crossings of the wave, where it shifts from one value to the other, follow a Poisson probability distribution.

An analytical method is developed which indicates the deviation of the apparatus output from the ideal impulse response. This error
is expressed as a function of the characteristics of the Poisson rectangular input and the system under investigation. Results are obtained for the generalized second-order system, although the approach will work for any linear system. These results indicate that the error can be made sufficiently small to allow the use of a Poisson rectangular input.

The Poisson rectangular input greatly simplifies the cross-correlation circuitry. The multiplier circuit in the crosscorrelator is required to multiply the system output by +Em or -Em only. This permits the use of a very simple multiplication and averaging circuit. A closed-loop tape transport was found to be very satisfactory in achieving the variable delay required in the crosscorrelator. Delays in the millisecond range are obtained by the use of two sets of write-read heads.
AN IMPULSE-RESPONSE MEASUREMENT APPARATUS FOR
LINEAR SYSTEMS USING A RECTANGULAR INPUT WITH
A POISSON DISTRIBUTION OF ZERO CROSSINGS

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AN IMPULSE-RESPONSE MEASUREMENT APPARATUS FOR LINEAR SYSTEMS USING A RECTANGULAR INPUT WITH A POISSON DISTRIBUTION OF ZERO CROSSINGS

INTRODUCTION

For many applications, especially in adaptive control systems, it is desirable to be able to experimentally determine the impulse response of a linear system while it is in operation. The application of test impulses, large enough to cause an output response, will disturb the quiescent value of the output. In cases where this quiescent output value cannot be disturbed, correlation techniques may be used to determine the impulsive response of the system.

A device which uses correlation techniques to determine impulse response is based on the Input - Output Crosscorrelation Theorem. This theorem states that the input - output crosscorrelation of a linear system under white noise excitation is proportional to the system unit-impulse response.

![Diagram](image)

Figure 1. Determination of the unit-impulse response of a linear system by crosscorrelation.
With reference to Figure 1, the theorem can be expressed as

\[ \phi_{io}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_i(t) f_o(t+\tau) dt \]

which can be shown to be equal to

\[ \int_{-\infty}^{\infty} h(\nu) \phi_{ii}(\tau-\nu) d\nu. \]

Therefore the input - output crosscorrelation of a linear system is the convolution of the unit-impulse response, \( h(\nu) \), and the input autocorrelation, \( \phi_{ii}(\tau) \).

It is found that the autocorrelation function of a noise with a flat power density spectrum, or white noise, is an impulse function. Consequently, \( \phi_{io}(\tau) \), the crosscorrelation function between the random input, \( f_i(t) \), applied to the system and the corresponding system output, \( f_o(t) \), will directly give the impulse response of the system. Therefore if a white noise input is applied to the system shown in Figure 1, the impulse response for any linear system can be determined.

The main drawback of this technique is to find a low cost device with a multiplication and averaging capability. This device must operate with inputs which vary rapidly in phase and amplitude. Also, the random noise input might possess instantaneous amplitudes which are either sufficiently large or small to cause saturation or undetectability respectfully in the system under investigation.

If a Poisson rectangular wave as shown in Figure 2 is used as the random input \( f_i(t) \), the above problems are eliminated.
The amplitude of Figure 2 is always kept at either $+ \text{Em}$ or $- \text{Em}$ volts while the occurrence of switchovers between these two amplitudes is random. Since the input amplitude varies between two fixed amplitudes, the multiplying and averaging circuitry is greatly simplified. The saturation - undetectability problem is also eliminated since $\text{Em}$ can be adjusted to fall within the operating range of the system.

The one undesirable characteristic of the Poisson rectangular wave when used in the system of Figure 1 is the fact that its autocorrelation function is not the ideal impulse function, but is of an exponential form. Therefore, the response obtained experimentally from the crosscorrelation plot will not be identical with the calculated impulse response.

The objectives of this thesis are:

1. Make an analysis of the Figure 1 system which will be valid when using the Poisson rectangular wave for $f_i(t)$. 

Figure 2. A Poisson rectangular wave.
2. Determine an analytical method by which \( \phi_{io}(\tau) \) for a linear system can be calculated as a function of the magnitude and the number of zero crossings per second of the Poisson rectangular wave input.

3. With the information obtained from 2, establish a criterion relating the deviation of \( \phi_{io}(\tau) \) from the calculated impulse response to the characteristics of the Poisson wave input and some characteristic of a given system. Such a characteristic could be \( \omega_n \) for a second-order system, where \( \omega_n \) is the undamped natural frequency of the system.

4. Design and construct the apparatus.

5. Experimentally verify the results obtained in 1, 2, and 3.
FOURIER TRANSFORM CONSIDERATIONS

Before proceeding any further, it might be well to include a brief discussion of the relationship between physical reality and the Fourier transform mathematics used in this thesis.

The impulse response will first be considered.

A unit-impulse may be defined as shown in Figure 3.

\[ f(t) = \lim_{a \to 0} \frac{1}{a} \left( U(t) - U(t-a) \right) \]

Figure 3. Definition of a unit-impulse \( \delta(t) \).

Assume a system with a transfer function \( H(s) \) and initially at rest. Since \( \mathcal{L}[\delta(t)] = 1 \), where \( \mathcal{L} \) is the Laplace transform, the output is \( W(s) = H(s) \mathcal{L}[\delta(t)] = H(s) \). Therefore, \( W\delta(t) = h(t) \). In other words, the response of an initially relaxed system to a unit-impulse is equal to the inverse Laplace transform of the transfer function.

The impulse function can also be synthesized using the Fourier transform approach. The Fourier transform pair for aperiodic functions is:

\[
\begin{align*}
  f(t) &= \int_{-\infty}^{\infty} F(\omega) e^{\text{i}\omega t} d\omega \\
  F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-\text{i}\omega t} dt
\end{align*}
\]
Given the aperiodic pulse shown in Figure 4a, application of the Fourier transform will result in the frequency spectrum shown in Figure 4b. Therefore, the rectangular pulse is resolvable into an infinity of infinitesimal sinusoids where relative amplitudes and phases are given in Figure 4b.

Note that $F(\omega)$ approaches a constant value of zero volt/ rad/sec in the limit as $a \to 0$. This is now the spectrum of the unit impulse. The unit-impulse can therefore be synthesized by an infinite series string of infinitesimal sinusoidal voltage sources all of equal relative magnitude. If this infinite number of voltage sources were all started at time $t = -\infty$, these sinusoidal components will have just the right relative magnitudes and phase angles for complete cancellation in the intervals $(-\infty, 0)$ and $(0, \infty)$ and addition into a unit impulse at $t = 0$. The energy density spectrum in Figure 4c will also become flat as $a \to 0$.

Consider now a function made up of these unit-impulses in an equally likely positive and negative Poisson distribution as shown in Figure 5a. The autocorrelation function of this wave form is an impulse function as seen in Figure 5b (5, p. 342). If the Wiener Theorem for autocorrelation is now applied, the flat power density spectrum as shown in Figure 5c results (5, p. 342). This function will be used for an input function in the derivation of the Input-Output Crosscorrelation Theorem. Note that the energy density
Figure 4. (a) An aperiodic pulse, (b) its frequency spectrum and (c) its energy density spectrum.
Figure 5. (a) Wave of equally likely positive and negative Poisson-distributed unit-impulses, (b) its autocorrelation function and (c) its power density function.
spectrum of a single unit-impulse is essentially changed into a power density spectrum where these unit-impulses occur repeatedly with time.
INPUT-OUTPUT CROSSCORRELATION THEOREM

Referring to Figure 1, an expression for $\phi_{io}(\tau)$ can be derived as follows:

$$\phi_{io}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_i(t) f_o(t+\tau) \, dt$$

Making use of the superposition integral $f_o(t) = \int_{-\infty}^{\infty} h(\nu) f_i(t-\nu) \, d\nu$

(5, p. 326):

$$f_o(t+\tau) = \int_{-\infty}^{\infty} h(\nu) f_i(t+\tau-\nu) \, d\nu$$

$$\phi_{io}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_i(t) \, dt \int_{-\infty}^{\infty} h(\nu) f_i(t+\tau-\nu) \, d\nu$$

Inverting the order of integration:

$$\phi_{io}(\tau) = \int_{-\infty}^{\infty} h(\nu) \, d\nu \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_i(t) f_i(t+\tau-\nu) \, dt$$

$$\phi_{ii}(\tau-\nu) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_i(t) f_i(t+\tau-\nu) \, dt$$

$$\phi_{io}(\tau) = \int_{-\infty}^{\infty} h(\nu) \, \phi_{ii}(\tau-\nu) \, d\nu$$

If the Figure 5a function is used as the input $f_i(t)$, $\phi_{ii}(\tau) = k \delta(\tau)$ and therefore $\phi_{io}(\tau) = \int_{-\infty}^{\infty} h(\nu) k \delta(\tau-\nu) \, d\nu = kh(\tau)$. This result states that for any $\tau$, $\phi_{io}(\tau)$ is of a magnitude proportional to the instantaneous value of the impulse response $h(t)$ at $t = \tau$.

With the help of the information presented in connection with
the Fourier Transforms the fact that $\phi_{10}(\tau)$ corresponds to the impulse response $h(t)$ can be seen on a purely intuitive basis. First, the Figure 5c power density spectrum is divided into a series of discrete frequencies - each with a power equal to the integral of the power density curve as shown in Figure 6. The white noise source can now be approximated by a very large series array of voltage sources, with each source possessing an equal and finite voltage magnitude.

The Figure 1 diagram can be redrawn as shown in Figure 7, since it can be shown that

$$\phi_{10}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_i(t) f_o(t+\tau) dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_i(t-\tau) f_o(t) dt$$

(5, p. 75-80). When the series array voltage source is applied to the system as shown in Figure 7, each discrete $\omega_i$ will be modified in amplitude and phase according to the amplitude and phase characteristics of the system $h(t)$. Each $\omega_i$ will be delayed by the same amount when passing through the delay $\tau$.

Since the system is linear, superposition holds, and $f_o(t)$ can be considered to be the sum of the instantaneous outputs due to each $\omega_i$. These two outputs, $f_o(t)$ and $f_i(t-\tau)$, are now multiplied at each instant of time and the average taken. This will give the magnitude of the impulse response plot at time $t = \tau$. 
Figure 6. Figure 5 power density function incremented into a series of discrete frequencies.

Figure 7. Impulse response system with idealized input.
INPUT-OUTPUT CROSSCORRELATION THEOREM WITH POISSON RECTANGULAR WAVE INPUT

The Poisson rectangular wave and its autocorrelation function are shown in Figure 8 (5, p. 53). This wave has the probability distribution function $P_{\xi}(n; \tau) = \frac{(k\tau)^n}{n!} e^{-k\tau}$, where $\xi$ is the number of zero crossings in a given interval $\tau$ if the average number of zero crossings per unit of $\tau$ is $k$. The complete set of possible values of $\xi$ is $n = 1, 2, 3, \ldots$ Since $\phi_{11}(\tau)$ is not an impulse function, when the Poisson rectangular wave is used as the input to the Figure 7 system, the equation $\phi_{10}(\tau) = \int_{-\infty}^{\infty} h(\nu) \phi_{11}(\tau-\nu) d\nu$ will not reduce to the impulse response of the system.

At this point, it would be desirable to develop an analytical method by which the error between $\phi_{10}(\tau)$ and the $h(\tau)$ of a given system as a function of $k$, the average number of zero crossings per second of the Poisson rectangular wave, could be determined. With this method available, the theoretical value of $k$ required for $\phi_{10}(\tau) - h(\tau) < \epsilon$, where $\epsilon$ is any desired maximum error, can be determined. This process will now be developed for the generalized second-order system $G(s) = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$ although the approach is equally valid for any linear transfer function.

The impulse response of $G(s)$ is:

$$h(t) = \omega_n \sqrt{1 - \delta^2} e^{-\delta\omega_n t} \sin(\omega_n \sqrt{1 - \delta^2} t)$$
\[ \phi_{11}(\tau) = E_m^2 e^{-2k|\tau|} \]

\[ k = \text{average number of zero crossings per second} \]

Figure 8. (a) Poisson rectangular wave and (b) its autocorrelation function.
With $\omega_n = 1$ and $\delta = 0.4$, the impulse response is shown in Figure 9a.

The process of the convolution of $h(t)$ and $\phi_{11} (\tau)$ to obtain $\phi_{i0} (\tau)$ in $\phi_{i0} (\tau) = \int_{-\infty}^{\infty} h(\nu) \phi_{11} (\tau-\nu) d\nu$ is shown graphically in Figure 9. To obtain $\phi_{11} (\tau_1 -\nu)$, $\phi_{11} (\nu)$ is first shifted to the right by the amount $\tau_1$ as shown in Figure 9c. This new function is $\phi_{11} (\nu-\tau_1)$. If $\phi_{11} (\nu-\tau_1)$ is reflected about the $\tau_1$ axis, $\phi_{11} (\tau_1 + \nu)$ results as shown in Figure 9d. Note that in this case $\phi_{11} (\nu-\tau_1)$ is symmetrical about the $\tau_1$ axis, so the reflection of $\phi_{11} (\nu-\tau_1)$ to obtain $\phi_{11} (\tau_1 -\nu)$ is not necessary. The product of the amplitudes of the two functions is now taken for all $\nu$ as shown in Figure 9e. The area under the product curve in Figure 9e corresponds to the point at $\tau = \tau_1$ on the curve in Figure 3f, which represents the convolution of the two functions.

A computer program to perform the above series of steps was developed for the IBM 1410/7010 Operation System. The Fortran program is shown in the Appendix.

The functions convolved in this program are:

$$h(t) = \omega_n / \sqrt{1 - \delta^2} e^{-\delta \omega_n t} \sin(\omega_n \sqrt{1 - \delta^2} t)$$

$$\phi_{11} (t) = E m^2 e^{-2k|\tau|}$$
Figure 9. Steps in the convolution of the functions $h(\nu)$ and $\phi_{11}(\nu)$ where $\phi_{11}(\nu) = e^{-0.8|\nu|}$. 
Where \( \omega_n = 1 \text{ rad/sec} \), \( \delta = 0.4 \), \( E_m = 1.0 \text{ volt} \) and \( k = K \omega_n \) zero crossings/sec. Note that with the choice of constants above, the average number of zero crossings per second, \( k \), is expressed in terms of the \( \omega_n \) of the second order system.

A description of this convolution program follows:

1. The function \( h(\nu) = 1.09e^{-0.4\nu}\sin 0.916\nu \) is computed at intervals \( \Delta\nu = 0.05 \text{ sec} \) from \( \nu = 0.05 \text{ sec} \) to \( \nu = 12.5 \text{ sec} \) (5 time constants). These values are stored in a subscript table.

2. The function \( \phi_{11}(\nu) = e^{-2k\nu} \) is computed at intervals \( \Delta\nu = 0.05 \text{ sec} \) to \( \nu = 25 \text{ sec} \) for a given value of \( K \). These values are stored in a subscript table.

3. \( \phi_{11}(\nu) \) is now displaced by an amount \( T_1 \) relative to \( h(\nu) \) in order to obtain \( \phi_{11}(T_1-\nu) \).

4. The product of the two functions is now taken increment by increment and these products are summed from \( \nu = 0 \text{ sec} \) to \( \nu = 12.5 \text{ sec} \).

5. Steps three and four result in one point on the convolution curve.

6. To obtain the value of the convolution at \( T = 0 \text{ sec} \), \( T_1 \) is set equal to zero as shown in Figure 10.
Figure 10. $h(\nu)$ and $\phi_{11}(\nu)$ ready to be multiplied and summed increment by increment by computer with $\tau_1 = 0$.

Figure 11. $h(\nu)$ and $\phi_{11}(\nu)$ ready to be multiplied and summed increment by increment with $\tau_1 = 0.5$. 
7. Steps three and four are repeated with $\tau_1$ increasing by 0.5 sec each time until $\tau_1 = 7.5$ sec (3 time constants of $h(\nu)$). Figure 11 shows the two functions ready to be multiplied and summed with $\tau_1 = 0.5$ sec.

8. The process is repeated starting with step two and a different value for $K$.

The results of the convolution can be generalized to apply to any value of $\omega_n$ and any Poisson rectangular wave magnitude $E_m$ as shown in Figure 12. Referring to Figure 12, when $k = 0$ crossings/sec, $\phi_{11}(T) = E_m^2$ volt$^2$ for all $\nu$. With $k = 0$, the convolution is simply the area under the $h(\nu)$ curve.

$$\phi_{10}(T) = \int_{-\infty}^{\infty} h(\nu) \phi_{11}(T-\nu) d\nu = E_m^2 \int_{-\infty}^{\infty} 1.09e^{-0.4\nu} \sin0.916\nu d\nu$$

$$= E_m^2 \text{volt}^2$$

The error introduced in the computer computation of the above integral is 0.1%. This is due to the fact that both $h(\nu)$ and $\phi_{11}(\nu)$ are divided into a finite number of increments and the integral is carried only to five time constants of $h(\nu)$. This error will increase as $K$ increases.

Assume $k = 8.0 \omega_n$, and $h(\nu)$ to have a constant value of 0.59 at $\omega_n T = 1.5$ radians and over the interval in which $\phi_{11}(\nu) = e^{-16|\nu|}$ is of noticeable magnitude as shown in Figure 13. The convolution $\phi_{10}(1.5)$, can be closely approximated as shown below:
Figure 12. Computed $\phi_{10}(\tau)$ as a function of $K$. 
Figure 13. $\phi_{11}(1.5)$ for $k = 8.0\omega_n$ superimposed on $h(\nu)$ plot.

Figure 14. $\int_{-\infty}^{\infty} h(\nu) \phi_{ii}(\tau-\nu) \, d\nu = 0.05 \, E m^2 h(\tau) \, \text{volt}^2$ when $k = \infty$. 
\[
\phi_{10}(1.5) = 0.59 \int_{-\infty}^{\infty} \phi_{11}(\nu) d\nu = 1.18 \int_{-\infty}^{\infty} e^{-1.6\nu} d\nu = 0.074
\]

The computer data lists \( \phi_{10}(1.5) = 0.077 \)

\% error = \( \frac{0.077 - 0.074}{0.074} < 5\% \)

Further error computations reveal that the points on the computed curves of Figure 12 for all \( k = 8.0 \) are accurate within 10\%.

The computed \( \phi_{10}(\tau) \) will approach the value \( \phi_{10}(\tau) = 0.05 \, E m^2 h(\tau) \) as \( K \to \infty \) as shown in Figure 14. From the convolution integral

\[
\phi_{10}(\tau) = \int_{-\infty}^{\infty} h(\nu) \phi_{11}(\tau - \nu)
\]

it is evident that, in the physical system, \( \phi_{10}(\tau) \to 0 \) as \( k \to \infty \). It can be concluded from Figure 15 that if \( k > \omega_n \), \( \phi_{10}(\tau) \) will be an adequate representation of \( h(\nu) \).
Figure 15. Curves of Figure 12 normalized about $h(1.5\text{ sec})$ on the unit-impulse response plot.
POISSON RECTANGULAR WAVE GENERATOR

A Schmitt trigger configuration as shown in Figure 16 was used to generate the Poisson rectangular waveform. A General Radio type 1390-B Random Noise Generator is used to trigger the Schmitt circuit. Potentiometer $P_1$ adjusts the bias on the base of $T_1$ to approximately $-3.2\, \text{v}$. The hysteresis of this particular Schmitt circuit is $1.2\, \text{v}$, so any noise generator voltage excursion less than $-0.6\, \text{v}$ will turn $T_2$ on, and any excursion greater than $+0.6\, \text{v}$ will turn $T_2$ off.

The output of the noise generator is Gaussian, as shown in Figure 17, and the average number of zero crossings per second which will occur at the output of the Schmitt circuit can be expressed as a function of the RMSV value of the noise. The output of the type 1390-B noise generator has the second probability density (5, p. 144):

$$P_{\xi_1 \xi_2}(V_1, V_2; \tau_1) = \frac{1}{2\pi \sqrt{2\phi_{11}(0) - \phi_{11}(\tau_1)^2}} e^{-\frac{V_1^2 + V_2^2}{2(\phi_{11}(0) - \phi_{11}(\tau_1))}}$$

Assume that the noise can be considered white noise, and therefore $\phi_{11}(\tau_1) = 0$.

$$P_{\xi_1 \xi_2}(V_1, V_2; \tau_1) = \frac{1}{2\pi \phi_{11}(0)} e^{-\frac{V_1^2 + V_2^2}{2\phi_{11}(0)}}$$
Figure 16. Poisson rectangular wave generator.

Figure 17. Variable-variance, normal density output of Type 1390-B Noise Generator.
Let the triggering level of the Schmitt circuit be ± a.

\[ P(-b<\xi_1<-a, a<\xi_2<b; t_2-t_1 = \tau_1) = P_{joint} \]

\[ = \int_{-b}^{b} \int_{-a}^{a} P_{\xi_1 \xi_2}(V_1, V_2; \tau_1) d\xi_1 d\xi_2 \]

\[ = \frac{1}{2\pi \phi_{11}(0)} \int_{-a}^{a} \int_{-b}^{b} e^{-\frac{(V_1^2 + V_2^2)}{2\phi_{11}(0)^2}} dV_1 dV_2 \]

Solving the expression and letting \( b \to \infty \):

\[ P_{joint} = \frac{\phi_{11}(0)}{2\pi a^2} e^{-\frac{a^2}{\phi_{11}(0)^2}} \]

\( \phi_{11}(0) \) of a Gaussian density is actually the variance, and the variance is equal to the \((\text{RMSV})^2\) of the density. The noise generator output is indicated in RMSV, so referring to the above equation, the relative number of zero crossings as a function of the noise RMSV value can be calculated.

In the experimental circuit, the noise generator was made a non-ideal voltage source by placing a resistor in series with it. This allows the rate of increase of the number of zero crossings for a given change in generator RMSV to be less than that specified by the above equation. This is desirable for ease of calibration.
CROSSCORRELATOR CIRCUITRY

The Figure 7 diagram is redrawn in Figure 18. The time delay \( T \) must be of the type which delays all the frequency components of \( f_i(t) \) by the same amount.

![Diagram](image)

Figure 18. Impulse response measurement system.

A continuous loop tape transport was constructed to achieve this delay. The general configuration of the delay circuit is shown in Figure 19. Note that the delay between a read head and a write head is determined by the distance between the two heads and velocity of the tape. The particular transport used had a tape speed of 7.5 inches/sec, therefore, the delay is approximately 135ms per inch of head spacing.

To achieve delays of the order of 0.5ms, two sets of read-write heads were used as seen in Figure 19. The input \( f_i(t) \) is also delayed before it is applied to the linear system. Therefore, the delay \( T_2 \) is the net delay between the two sets of heads. By careful
Figure 19. Impulse response measurement system with continuous tape loop delay device.
adjustment of the variable delay arm, net delays of the order of 0.5ms can be obtained.

Referring to Figure 20, T₄ and T₅ form a polarity reversing circuit. The base of T₅ is adjusted to -7.25v. When the T₃ collector is in the -11.5v state, approximately 5mA will flow from right to left through each write head. When the T₃ collector is in the -3v state, the same current will flow in the opposite direction. This current will cause saturation of the tape either in one direction or the other - depending on the state of the Schmitt - Poisson generator circuit.

The output of read head No. 1 is a series of approximately 40mv pulses of the polarity shown in Figure 21. This pulse is amplified by T₆ sufficiently to allow triggering of the Schmitt circuit made up of T₇ and T₈. The output of this Schmitt circuit is a reconstruction of the waveform at the collector of T₃, but delayed by the desired amount as seen in Figure 21. The read head No. 2 input is delayed and reconstructed in the same manner. Note that this technique of tape saturation eliminates the need of an erase head between the two sets of heads.

Due to frequency response limitations of the tape heads and the tape, the delay circuit will not restore the portions of the Poisson rectangular wave in which more than one zero crossing occurs in 0.17ms. At this point, the arbitrary assumption is made that the
Figure 20. Complete
(All tracing...
Figure 21. Poisson wave delay process.

Figure 22. (a) Impulse response measurement system with filter $H_1(\omega)$ included in delay circuit, (b) same configuration but different grouping of elements.
system operation will not be noticeably affected if crossover times less than 0.17ms do not occur greater than 10% of the time. The maximum average number of zero crossings per second the Poisson generator waveform can have and still satisfy the above assumption is:

\[
P_{\xi}(n; T) = \frac{(kT)^n}{n!} e^{-kT}
\]

\[n = \frac{1}{0.17\text{ms/zero crossing}} = 6 \text{ zero crossings/ms}.
\]

\[T = 1\text{ms}
\]

\[P_{\xi}(6; 1) = 0.1 = \frac{k^6}{6!} e^{-k}
\]

\[k_{\text{max}} = 4000 \text{ zero crossings/sec}.
\]

An area of possible further investigation is concerned with the use of a filter having a variable cutoff frequency (and hence a variable time delay). Referring to Figure 22, an expression can be derived relating the characteristics of a filter and \(\phi_{10}(T)\).

\[
\phi_{ab}(T) = \lim_{T \to \infty} \frac{1}{2T} \int_{T}^{T} f_a(t-T)f_b(t)dt = \lim_{T \to \infty} \frac{1}{2T} \int_{T}^{T} f_a(t)f_b(t+T)dt
\]

\[f_a(t) = \int_{-\infty}^{\infty} h_1(\nu) f_i(t-\nu) d\nu
\]

\[f_b(t) = \int_{-\infty}^{\infty} h_2(\sigma) f_i(t-\sigma) d\sigma
\]
\[
\phi_{ab}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{\infty} h_1(v) f_i(t-v) \, dv \int_{-\infty}^{\infty} h_2(\sigma) f_i(t+T-\sigma) \, d\sigma
\]

\[
\phi_{ab}(\tau) = \int_{-\infty}^{\infty} h_1(v) \, dv \int_{-\infty}^{\infty} h_2(\sigma) \, d\sigma \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_i(t-v) f_i(t+T-\sigma) \, dt
\]

\[
\phi_{ii}(\tau+\sigma) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_i(t-v) f_i(t+T-\sigma) \, dt
\]

\[
\phi_{ab}(\tau) = \int_{-\infty}^{\infty} h_1(v) \, dv \int_{-\infty}^{\infty} h_2(\sigma) \, d\sigma \phi_{ii}(\tau+\sigma)
\]

Let \( t = \sigma - \nu \), \( \sigma = t + \nu \)

\[
\phi_{ab}(\tau) = \int_{-\infty}^{\infty} h_1(v) \, dv \int_{-\infty}^{\infty} h_2(\nu+t) \, dt \phi_{ii}(\tau-t)
\]

\[
\phi_{ab}(\tau) = \int_{-\infty}^{\infty} \phi_{ii}(\tau-t) \, dt \int_{-\infty}^{\infty} h_1(v) h_2(\nu+t) \, dv
\]

\[
\phi_{h_1 h_2}(t) = \int_{-\infty}^{\infty} h_1(v) h_2(\nu+t) \, dv
\]

\[
\phi_{ab}(\tau) = \int_{-\infty}^{\infty} \phi_{h_1 h_2}(t) \phi_{ii}(\tau-t) \, dt
\]

Therefore, the output crosscorrelation \( \phi_{ab}(\tau) \) of the linear system is the convolution of the input autocorrelation and the unit-impulse response crosscorrelation.

If the time delay \( \tau \) is made equal to zero, \( \phi_{ab}(0) \) could be
plotted as a function of the impulse response of the filter. \( \phi_{ab}^{(0)} \) could then be compared with \( h_2(\tau) \). The impulse response of the filter could be calculated from its frequency characteristics.

The multiplication and averaging circuitry is shown in Figure 23. In the expression \( \phi_{io}(T) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_i(t-T) f_o(t) \, dt \), for the system under consideration, \( f_i(t-T) \) has only two possible values - either \( +Em \) or \( -Em \). This greatly simplifies the multiplier circuitry.

To aid in the development of this circuitry, assume that it is desired to autocorrelate the square wave shown in Figure 24a. In Figure 23, during the time points 1 and 2 are both \( +Em \) or \( -Em \), the \( V_{be} \) of \( T_1 \) and \( T_2 \) are zero, and therefore \( I_c = 0 \). When points 1 and 2 are of opposite polarity, \( I_c = I_{c \, \text{max}} \). Assuming circuit linearity, it is evident that \( I_c \) will vary as the inverse of the autocorrelation function as shown in Figure 24c. Note that the zero axis in Figure 24c corresponds to \( I_c = Em/Ze \) where \( Ze \) is the equivalent emitter impedance. \( -Em^2 \) corresponds to \( I_c = 2Em/Ze \) and \( +Em^2 \) corresponds to \( I_c = 0 \). The Figure 23 circuit will effectively linearly multiply and average any input magnitude by \( +Em \) or \( -Em \). This circuit can, therefore, be used to multiply and average the two functions \( f_i(t-T) \) and \( f_o(t) \).

The final version of the multiplication and averaging circuit is shown in Figure 25. The input impedance of \( T_2 \) and \( T_3 \) is small.
Figure 23. Multiplication and averaging circuit.

(a) Function to be autocorrelated.

(b) Autocorrelation function of (a).

(c) $I_c$ as a function of $\tau$.

Figure 24. Autocorrelation of squarewave with the Figure 23 circuit.
Figure 25. Final multiplication and averaging circuit.
enough to require the emitter followers $T_1$ and $T_4$. The input impedance to $T_1$ and $T_4$ is large enough to prevent excessive loading on the outputs of the Schmitt restoring circuits.

$R$ and $C$ make up a simple R-C filter which reduces the fluctuations in the $I_c$ meter reading. Note that the effective resistance in the R-C network is the resistor $R$ in parallel with the output impedance of $T_5$. The output impedance of $T_5$ is as high as possible since it is in the common base configuration. Since $\phi_{10}(\tau)$ is ideally a steady state value when $\tau$ is held fixed, the $R$ and $C$ of filter should be made as large as possible. The one drawback of doing this, however, is the fact that the rise-time of the filter also increases with increase in $R$ and $C$ values. With an increase in rise-time, the settling time required between readings of $\phi_{10}(\tau)$, when $\tau$ is varied, is greater. Therefore, a compromise must be used, where the meter fluctuation is tolerable, and the rise-time is sufficiently small.

Referring to Figure 20, if a square wave generator is substituted for the noise generator, and the second order system is bypassed, the multiplication and averaging circuit may be adjusted. Figure 20 is now essentially an autocorrelator, so when $\tau$ is set equal to zero, the square wave autocorrelation function is maximum, and the $I_c$ meter reading should be zero. $I_c$ can be reduced to zero through the adjustment of potentiometers $P_1$ and $P_2$. When $\tau$ is
adjusted to a $180^\circ$ delay with respect to the square wave, the auto-
correlation function is minimum, so the $I_c$ meter reading will be
maximum. The range of $I_c$ has now been determined, and the $I_c$
meter scale can be calibrated as shown in Figure 24c.

Any noise generated by the system under test can be neglected
if its average value of the noise is zero and the noise voltage ex-
cursions remain within the linear range of the multiplication and
averaging circuit.
EXPERIMENTAL RESULTS

A series of ensembles of the Poisson wave generator output was obtained by photographing the display on a Tektronics Type 545 Oscilloscope. The approximate average number of zero crossings per second as a function of the output of the noise generator can be determined from these ensembles.

The second-order system shown in Figure 27 has an
\[ \omega_n = 146 \text{ rad/sec.} \] Therefore, \( K = \frac{240}{146} = 1.65. \) Figure 26 shows the experimental \( \phi_{10}(\tau) \) obtained when the Poisson wave input has an average of 240 zero crossings per sec. The time constant for the R-C filter in Figure 25 used to obtain the Figure 26 results was \( RC = 4.4 \text{ sec.} \)

Referring to Figure 12, the maximum amplitude of the impulse should be approximately 0.35 \( E_m \) at \( \omega_n \tau = 1.3 \) radians. In Figure 26, \( E_m \) is equivalent to \( I_c = 0 \), and \( E_m = 0 \) is equivalent to \( I_c = 2 \text{ma.} \) Therefore, the minimum \( I_c \) should be \( I_{c \text{ min}} = 2 \text{ma} - 0.35(2 \text{m}) = 1.3 \text{ma.} \) This minimum should occur at
\[ \tau = \frac{1.3 \text{ radians}}{146 \text{ radians/sec}} = 9 \text{ ms.} \] Note that \( I_{c \text{ min}} \) in Figure 26 is approximately 1.22\( \text{ma} \) at \( \tau = 9 \text{ ms.} \) This experimental value is, therefore, within 10% of the calculated value.

The calculated and experimental curves of \( \phi_{10}(0) \) as a function of \( K \) are shown in Figure 28. The calculated values were taken
Figure 26. $I_c$ of $T_{13}$ in Figure 20 as a function of time delay $\tau$ for system shown in Figure 27.

![Figure 26](attachment:figure26.png)

Figure 27. Unit-impulse response of second-order system shown above and curve (a) normalized about point $\frac{h(t)}{\omega_n} = 0.602$ on impulse response curve.

![Figure 27](attachment:figure27.png)
Figure 28. (a) Calculated $\phi_{10}(0)$ as a function of $K$, 
(b) experimental $\phi_{10}(0)$ as a function of $K$. 

$$k = K\omega_n \ln \phi_{11}(\tau) = E_m^2 e^{-2k|\tau|}$$
from Figure 12. The curves are in agreement at the points where
the experimental number of zero crossings per second is known.
SUMMARY AND CONCLUSIONS

An impulse response measurement system has been designed and constructed which will experimentally determine the impulse response of any linear system. This device is based on the Input-Output Crosscorrelation Theorem which states that the unit-impulse response may be obtained by a crosscorrelation measurement of the input and output when the input is white noise. A noise may be considered white if the autocorrelation function of the noise appears as an impulse function to the system under investigation.

The use of a Poisson rectangular wave input instead of a Gaussian input greatly simplifies the crosscorrelation circuitry. The linear system output is multiplied by either $+Em$ or $-Em$ only. This permits the use of a very simple multiplication and averaging circuit.

Use of a closed-loop tape transport for obtaining the delay $\tau$ was found to be very satisfactory. Delays in the millisecond range are obtained by the use of two sets of write-read heads.

The one undesirable characteristic of the Poisson rectangular wave input is the fact that its autocorrelation function is not an impulse function, but of the exponential form. A computer program was developed to determine the error in the crosscorrelator output due to this non-impulse autocorrelation function. Results are
obtained for the generalized second-order system, although the approach will work for any linear system. As seen in Figure 15, when $k > \omega_n$, $\phi_{i0}(T)$ will be a close approximation of the impulse response of the second-order system. The experimental results give strong support to these theoretical derivations.
BIBLIOGRAPHY


APPENDIX
FORTRAN LISTING

```
DIMENSIONFUN1(1), FUN2(1), FUN3(1, 250), PHI(500), CROS1(250), CROS2(251), HL(5)

C ANALYSIS OF IMPULSE RESPONSE ERROR ROBERT R. BEUTLER

00101 FORMAT(1H , 5F10.6)
00103 FORMAT(32H1INPUT - OUTPUT CROSSCORRELATION)
DELTA=0.4
I=1
J=0
TIME=0.
FUN1(I)=1. - DELTA**2
FUN2(I)=SQRT(FUN1(I))

00002 J=J+1
TIME=TIME+0.05
FUN3(I, J)=1. / (FUN2(I)*EXP(Delta*TIME))*SIN(FUN2(I)*TIME)
IF (TIME-12.5) 2, 1, 1

00001 WRITE(3, 103)
CROS2(1)=0.
M=0

00006 M=M+1
GOTO(201, 202, 203, 204), M

00201 HK=0.001
GOTO3
00202 HK=0.
GOTO3
00203 HK=1.0
GOTO3
00204 HK=8.0

00003 TIME=0.
I=0

00004 TIME=TIME+0.05
I=I+1
T1=TIME-12.5
T2=2.*HK*ABS(T1)
PHI(I)=1./EXP(T2)
IF (TIME-25.4) 15, 15

00015 L=0
DO30J = 1, 250
CROS1(J)=FUN3(1, J)*PHI(J+249)

00030 CROS2(J+1)=CROS2(J)+CROS1(J)
WRITE(3, 101) CROS2(251)
DO11I=10, 150, 10
DO5J = 1, 250
K=J+I-250
CROS1(J)=FUN3(1, J)*PHI(K)

00005 CROS2(J+1)=CROS2(J)+CROS1(J)
L=L+1
HL(L)=CROS2(251)
IF (L-5) 40, 50, 50

00040 GOTO11
00050 WRITE (3, 101)HL(1), HL(2), HL(3), HL(4), HL(5)
I=0

0011 CONTINUE
0021 IF (M-4) 6, 7, 7
```