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Title: CHARACTERISTICS OF RC DIGITAL FILTERS AS BANDPASS CIRCUITS

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One of the most promising methods of constructing bandpass filters without inductors is the RC digital filter. It is simply a number of parallel RC branches which are sequentially switched between the source and the load. If the source signal is at the same frequency as the switching rate of the RC branches, it will pass through to the load, virtually unattenuated. As the frequencies begin to differ, there is a rapid increase in the amount of attenuation.

The design relationships are very simple compared to LC circuits with comparable values of Q. The RC branches are designed for a low pass response in the conventional manner and it is then transposed to center around the switching frequency. As this switching frequency increases, the bandwidth remains constant, and large values of Q and thus obtained.

Low pass and broad bandpass filters are normally required at

the input and output of the digital filter, but their design is usually very simple and easy to construct. They are necessary to eliminate the undesirable harmonic properties of the digital filter.

Digital filters will probably be constructed entirely by integrated circuit techniques in the future; at present, other techniques are required since the values of capacitance now obtainable by this process are limited.

This paper will follow a derivation for the transfer function of a digital filter, show results of some laboratory tests, outline a design procedure, and briefly describe one presently in commercial use.

Characteristics of RC Digital Filters as Bandpass Circuits

by

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CHARACTERISTICS OF RC DIGITAL FILTERS AS BANDPASS CIRCUITS

I. INTRODUCTION

There are two methods for designing bandpass circuits without the use of inductors (7). One is active RC circuits with negative feedback. The second method is the digital filter, the method to be described in this paper. In the simplest terms, it is a number of sequentially switched RC circuits. If these circuits are switched at the same frequency as an input signal, the overall circuit acts as a bandpass filter. As the two frequencies begin to differ, the signal is attenuated according to certain relationships among the resistance, capacitance, switching frequency, and number of RC branches. Both types of circuits are useful in designing bandpass filters with integrated circuits. Especially at low audio frequencies, where large inductors are normally required, these circuits offer obvious advantages.

The active RC networks have an inherent problem of component sensitivity which can make them quite unreliable. This means the bandpass characteristic changes appreciably with small changes in component values (resistance, for example). The digital filter, on the other hand, is no more sensitive than normal LC bandpass filters (7).

Another desirable feature of digital filters is the ability to "tune" the filter center frequency by merely changing the rate at which the RC circuits are switched. A clock can be used to set this switching rate and can be designed to change quite easily. One possibility for utilizing this feature is in frequency hopping communications systems (5). They require a transmitter and receiver which will operate simultaneously at changing frequencies according to a predetermined pattern and rate. The digital filter could be made to change its switching rate according to this pattern, and would provide a bandpass response centered at the desired frequency.

The digital filter can achieve a very high Q by proper selection of the R and C values. These values are easier to realize than the inductance necessary in LC filters. The R and C determine the bandwidth of the filter, and when using a high center frequency, a high value of Q is thus obtained.

Probably the greatest promise for digital filters lies in the area of integrated circuits, since inductors cannot be made by this process. The limit on the values of capacitance attainable is the limiting factor at the present time.

II. PHYSICAL DESCRIPTION OF DIGITAL FILTER OPERATION

This section describes the operation of the digital filter using a physical, non-mathematical approach.

Types of Digital Filters

There are two types of digital filters--shunt and series switched (5). This paper investigated only the shunt type; however, both have similar response characteristics. A schematic of each type is shown in Figures 1 and 2, where mechanical rotary switches are used. The two switches in the series type have to rotate at the same frequency and be synchronized to the same RC branches. The discussion for the remainder of this paper covers the shunt filter, although most results can be assumed for both types.

Theory of Operation

In Figure 1, as the switch is connected across any one of the RC branches, an RC low pass circuit appears between V_s and V_o. If three branches are used, each capacitor is connected for a full one-third cycle (where there is no appreciable gap between switch contacts); and if the frequency of the input signal is the same as f_o, then any one capacitor will see the same 120° of input signal each time it is connected. This is shown in Figure 3 for two possible

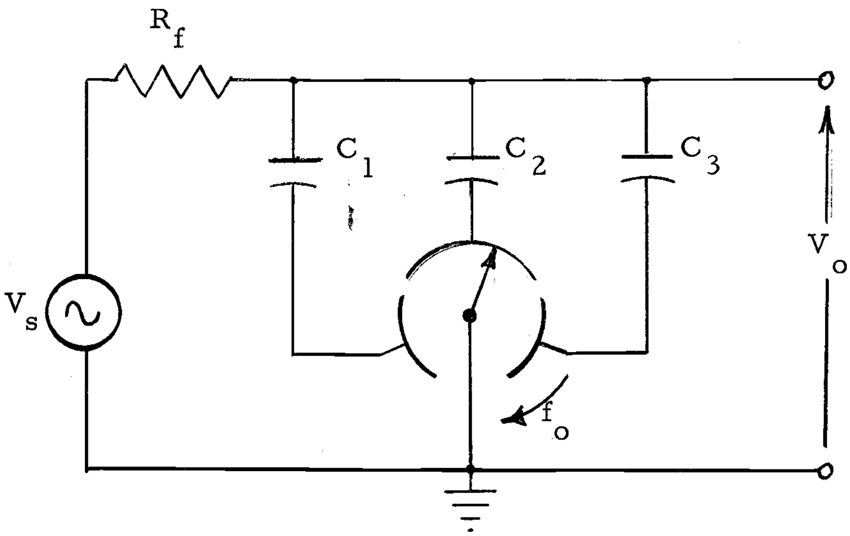


Figure 1. Shunt-switched digital filter.

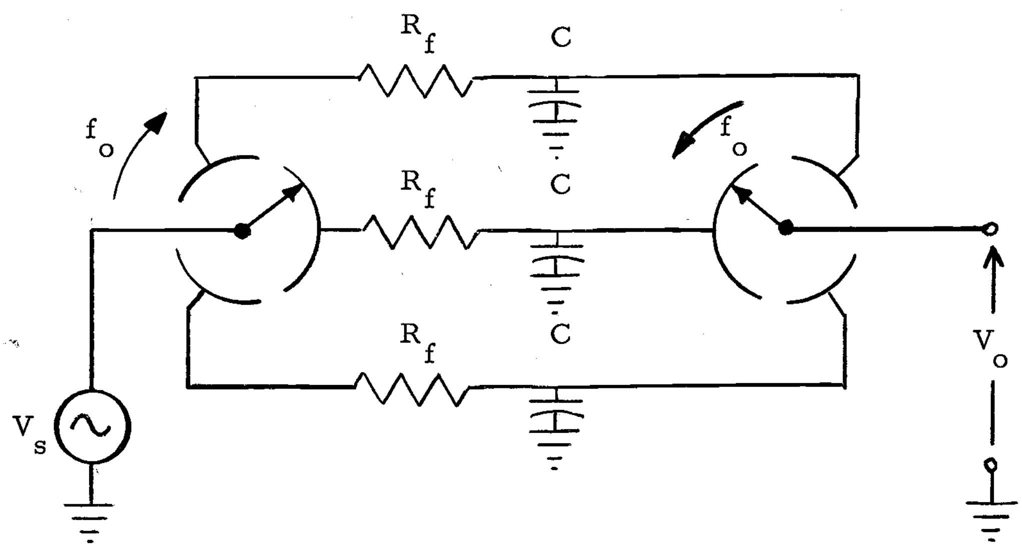
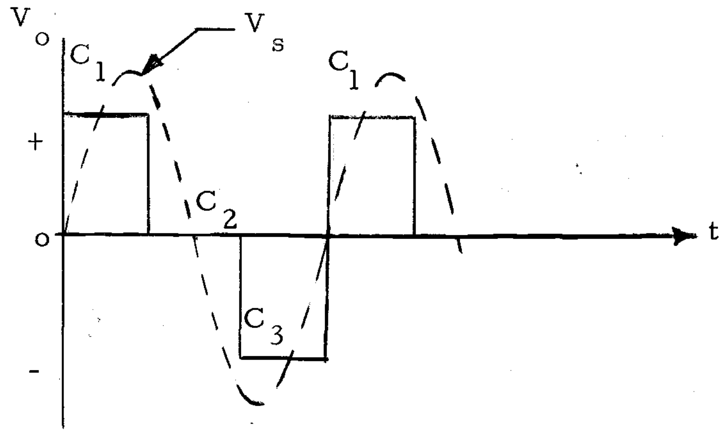
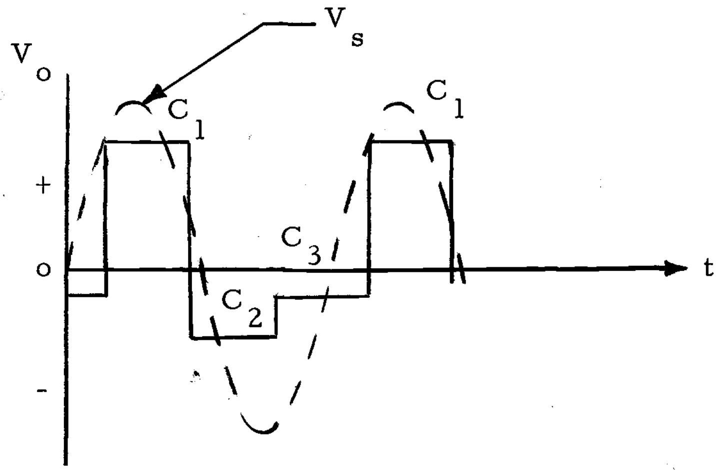


Figure 2. Series-switched digital filter.



(A)



(V)

Figure 3. Capacitors acquiring average voltage of input signal (V_s), for two different phase relationships.

phase differences between input and switching frequencies. For the amount of time that each capacitor is switched into the circuit, it accumulates voltage until after a number of repetitions, this voltage reaches the average value of input voltage over the interval.

A capacitor can charge or discharge only when it is grounded by the rotary switch, and all other capacitors are floating (one side being disconnected) so that they store their charge. The output at any instant is the voltage on the capacitor that is switched in at that moment. Thus, the overall output is a sampled version of the input wave, where the sampled values equal the average of the input value over each sampling interval.

When the input frequency starts to differ from the commutating frequency, a capacitor does not see exactly the same value of input voltage each time it is connected into the circuit. If the two frequencies are only slightly different, the values across each capacitor change only slightly during each switching interval. The capacitors will be able to follow these changes, provided the RC charging time constant is not too large. As the difference between the two frequencies increases, the rate of variation across each capacitor also increases. This effect will be the same whether the input frequency is getting larger or smaller then the switching frequency. It is only the amount of difference between the two that is important. Eventually this rate will be faster than the RC time constant can follow, and the average value of each capacitor will approach zero. The frequency response plot of the filter then becomes similar to an LC bandpass filter that attenuates signals at a 6 db per octave rate (or equivalently, 20 db per decade). There has thus been a transformation from a number of low pass filters to one bandpass filter.

As is now evident, the RC time constant must be large compared to the amount of time each capacitor is switched on, or else instead of "averaging" the input voltage, the capacitor voltage will follow the input at all times, no matter how large the difference between the two frequencies.

Electronic Models

The next step is to replace the mechanical switches with electronic equivalents. This is shown in Figure 4 for the shunt type filter. An electronic model for a series type digital filter is presented in one of the references (5, p. 93). In Figure 4, the transistors are providing the switching function. A sequence of delayed pulses is required, as shown, in order to turn each branch on at its appropriate time. The output voltage at any time is the voltage across the branch that is turned ON. The switching frequency (f_o) is defined by the pulse period (T) for any one branch. All pulses are assumed to be of equal duration, T/N, where N is the number of branches. The capacitors must also be of equal value.

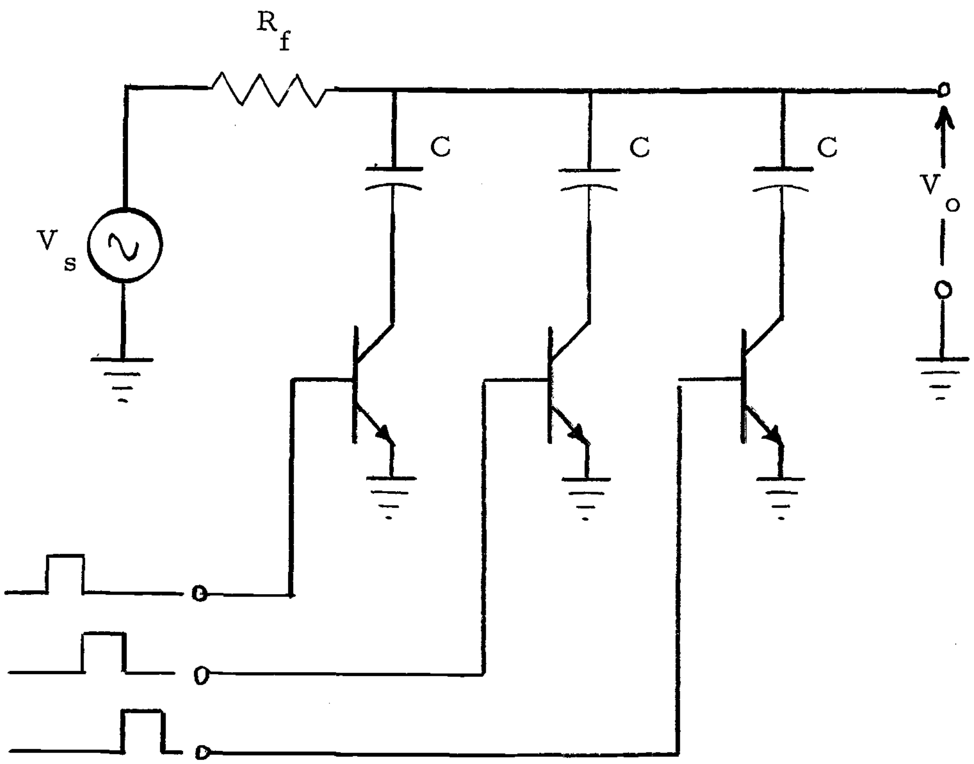
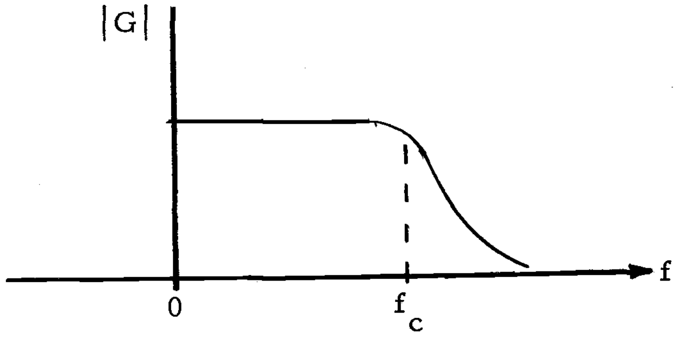


Figure 4. Digital filter with electronic switches.

Response Characteristics

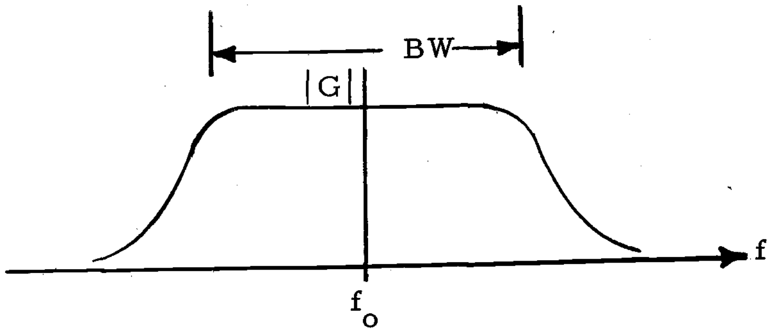
The output characteristics are dependent upon f_0, R, C, and N. If R and C are picked to provide a low pass filter with a passband of f_c hertz, the digital filter will transpose this to a passband of $2f_c/N$ hertz centered about f_o. This is shown in Figure 5. Since this bandwidth does not depend upon the center frequency (f_o), the Q can become very large as the switching frequency increases (since $Q = f_o/BW$). A value of $Q = 2500$ was obtained using $R = 10,000$ ohms, $C = 0.1$ microfarads, $N = 9$, and $f_o = 100$ KHz. Since the bandwidth $(BW) = 2/2\pi NRC = 40$ hertz, $Q = 100$ KHz/40 hz $= 2500$.

When showing responses on log paper, the low pass filter response is shown with a straight line passing through the 3 db frequency, and sloping at 20 db per decade. When the filter is transformed around f_o by the digital filter, its bandpass slope will increase as f_o increases. The relationship is not a linear one, and the bandpass slope cannot be shown by a straight line. For instance, if the low pass filter has a 3 db frequency of 40 hertz, at 400 hertz it will be down 20 db, and at 4 KHz it will be down 40 db. When transposed to center about $f_o = 20$ KHz, and using a filter with $N = 4$ branches, the bandpass response will be flat at 20 KHz $\pm$ 10 hz. It will then drop 20 db for another $\pm$ hz, or will be down 20 db at 20.110 KHz and



$f_c = 1/2\pi RC$

(A)



$BW = 2[1/2\pi NRC]$

(B)

Figure 5. Digital filter transposes low pass filter (A) to bandpass filter (B), centered about f_o. ($|G|$ = gain)

19.890 KHz. It will drop another 20 db for a 1 KHz change, so that the response is down 40 db at 21.110 KHz and 18.890 KHz. Thus, it is the amount that the input frequency differs from f_o which follows the 20 db per decade slope, and not the overall characteristic. These relationships are illustrated in Figures 6 and 7.

An understanding of how N affects the response can be had by considering the amount of time any one capacitor is actually in the circuit. If an RC branch normally has a 3 db response at 40 hz, but is only "there" one-fourth of the time, it would then have a 3 db response equal to (1/4) times 40 = 10 hz. This will be discussed later in more detail.

The Averaging Process

To understand why the capacitors charge to the "average" value of input voltage, it is necessary to look at the integral relationships for an RC low pass circuit. The output voltage is given by

$$V_o = 1/C \int idt,$$

where i is the current through both R and C, with no output load. The integration is over one switching period for this example. With no initial charge on the capacitor, the first few switching periods place an intermediate voltage on the capacitor. A condition will

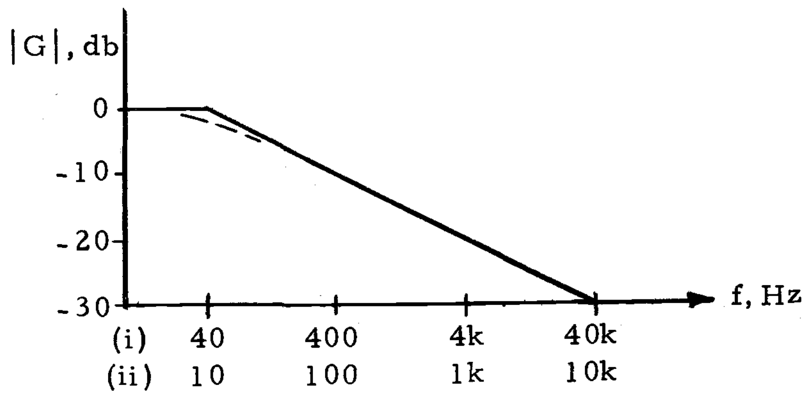


Figure 6. Filter characteristic for: (i) low pass RC filter (f_c = 40 Hz), (ii) frequency difference from f_o in a digital filter with N = 4.

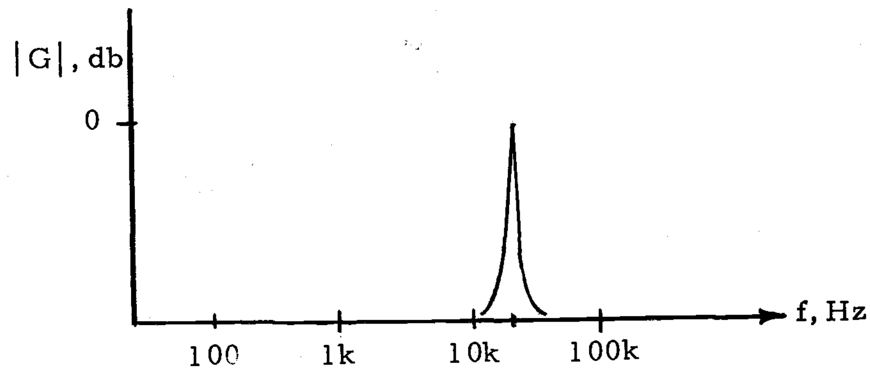


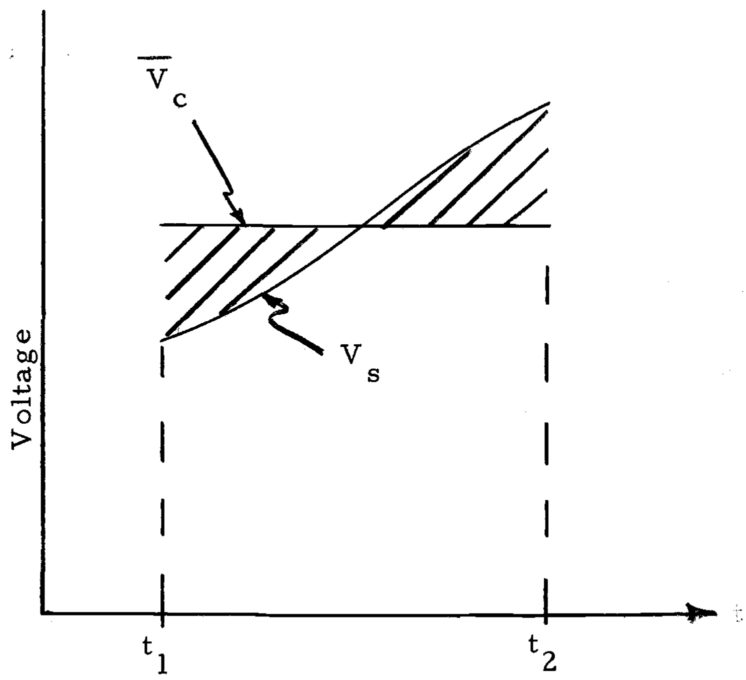
Figure 7. Actual bandpass response for digital filter with f_o = 20 KHz.

be reached similar to the one shown in Figure 8. At time t_1, the input voltage is less than that already existing on C, and hence some charge is lost. V_c will approach a final level such that there is an equal amount of area (shown by diagonal lines) above and below $\overline{V}_c$. Although current (and charge) are lost and gained during each switching period, the voltage does not change appreciably due to the relatively large RC time constant (large compared to the switching ON time). Further mathematical justification for the averaging process is given in the Appendix.

Harmonic Response

The digital filter is very sensitive to harmonics of the switching frequency, f_o. This would be expected since all harmonics of f_o are also synchronized to the filter. If $2f_o$ is the input frequency, each RC network in a four branch circuit sees 180° of input signal, instead of 90°, during each switching interval. But when the input is $4f_o$, each RC branch sees 360° of input signal, the average value of which is zero. There is therefore no response at Nf_o (N = 4).

A response is also present between zero hertz and the original 3 db cutoff frequency of the RC circuits. This would be expected since the branches will pass frequencies below the cutoff frequency whenever they are switched into the circuit. But each branch is still



$t_2 - t_1$ = switching interval

Figure 8. The process of acquiring the average voltage on a capacitor.

only ON one-fourth of the time, so that the cutoff frequency is still divided by 4 (when $N = 4$). Alternately, since each RC branch is only ON one-fourth of the time, it can only follow input changes one-fourth as fast. This is effectively increasing the RC time constant by four, or decreasing its low pass cutoff frequency to one-fourth (3, p. 33). This will be further explained by Equation 22.

III. MATHEMATICAL THEORY OF OPERATION

Two articles have derived transfer functions describing the operation of a digital filter. The work described here was done by James Thompson of Motorola Semiconductor Products, Inc. (7).

Developing the Transfer Function

It must first be assumed that no two RC branches are on at the same time, such that the pulse length, τ, is less than T/N. Then each branch can be looked at separately, as represented by Figure 9. The switches are represented by the function $a_n(t)$, shown in part B. They have zero impedance when ON and infinite impedance at all other times. There is no load current. The current through C is therefore

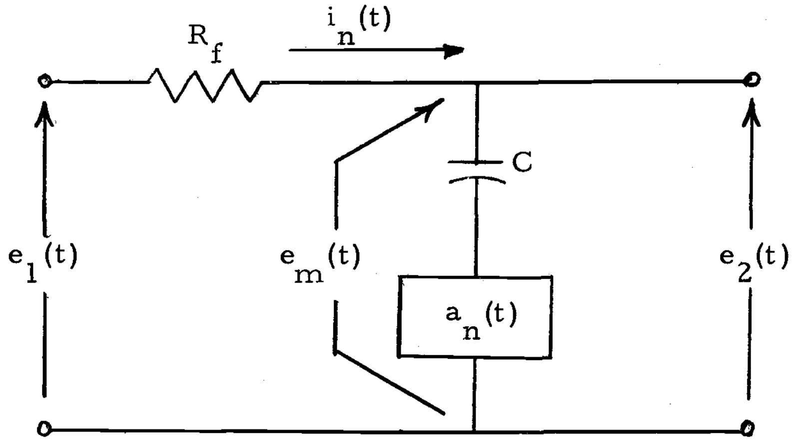
[1] $$i_n(t) = a_n(t) \cdot \frac{e_1(t) - e_m(t)}{R},$$

where $a_n(t) = 1$ when ON.

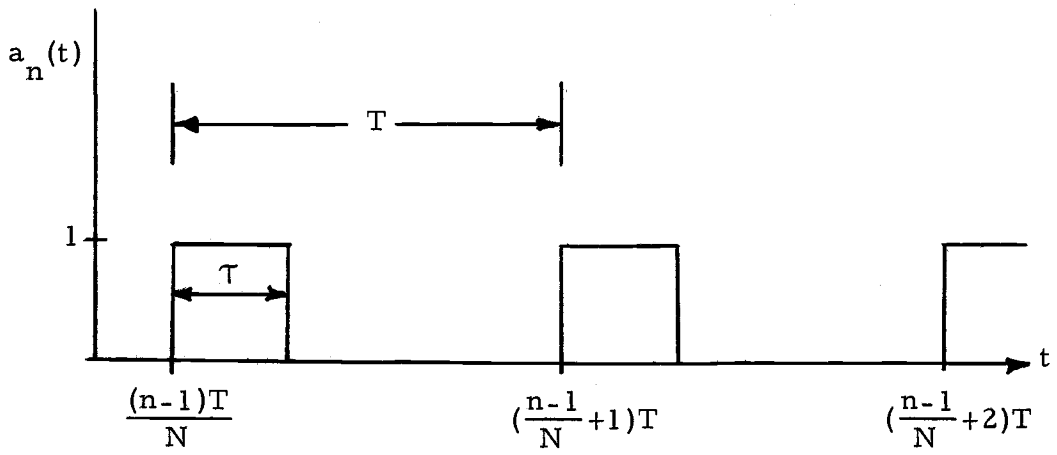
Multiplication in the time domain is equivalent to a convolution of transforms in the frequency domain, so that

[2] $$I_n(s) = A_n(s) * \frac{E_1(s) - E_m(s)}{R},$$

where $*$ signifies convolution. For a capacitor,



(A)



(B)

Figure 9. Developing a transfer function for the digital filter. (A) represents a single RC branch, and (B) shows characteristics of $a_n(t)$.

$$I_n(s) = sCE_m(s),$$

or

[3] $$RCsE_m(s) = A_n(s) * E_1(s) - A_n(s) * E_m(s).$$

The Laplace transform of $a_n(t)$ can be written (6, p. 217)

[4] $$A_n(s) = [\exp(-\frac{(n-1)Ts}{N})][\frac{(1-\exp(-\tau s))}{s(1-\exp(-Ts))}] .$$

The first term in Equation 4 describes the phasing of the pulses, while the second part defines the pulse period and duration.

$A_n(s)$ is here in a form of $N(s)/D(s)$, which can be transformed back into the time domain by use of the Laplace Expansion Theorem (2, p. 202-205). Thus, when

$$F(s) = N(s)/D(s),$$

[5] $$f(t) = \sum_k N(s_k)/D'(s_k)[\exp(s_k t)],$$

where

$$D'(s_k) = \frac{d}{ds}[D(s)]\Big|_{s=s_k}$$

and s_k are the poles of $F(s)$.

The poles (s_k) of $A_n(s)$ are given when

$$1 - \exp(-Ts_k) = 0,$$

or

$$s_k = \frac{j2\pi k}{T}, \qquad (-\infty < k < +\infty).$$

Then

$$[6] \qquad \frac{N(s_k)}{D'(s_k)} = \exp\left[-j2\pi k\left(\frac{n-1}{N}\right)\right] \frac{1-\exp(-j2\pi k\tau/T)}{j2\pi k}$$

$$= \frac{\tau}{T} \exp\left\{-j\pi k\left[\frac{2(n-1)}{N} + \frac{\tau}{T}\right]\right\} \cdot \left[\frac{\sin \pi k\tau/T}{\pi k\tau/T}\right]$$

$$= \frac{\tau}{T} \exp(-j\beta_k) M_k,$$

where

$$M_k = \frac{\sin \pi k\tau/T}{\pi k\tau/T}$$

and

$$\beta_k = \pi k\left[\frac{2(n-1)}{N} + \frac{\tau}{T}\right].$$

Then, by Equation [5]

$$[7] \qquad a_n(t) = \sum_k \frac{\tau}{T} \exp(-j\beta_k) M_k \exp(j2\pi k\tau/T).$$

By the definition of convolution,

$$F(s) * G(s) = L[f(t) \cdot g(t)] = \sum_k \frac{N(s_k)}{D'(s_k)} L[\exp(s_k t)\, g(t)].$$

Using the complex shifting theorem,

[8] $$L[\exp(s_k t)\, g(t)] = G(s - s_k).$$

Then

[9] $$F(s) * G(s) = \sum_k \frac{N(s_k)}{D'(s_k)}\, G(s - s_k).$$

Applying this to Equation [3] and [7],

[10] $$RCsE_m(s) = \frac{\tau}{T} \sum_k M_k \exp(-j\beta_k) E_1(s - j2\pi k/T)$$

$$- \frac{\tau}{T} \sum_k M_k \exp(-j\beta_k) E_m(s - j2\pi k/T).$$

It was earlier stated that RC must be much larger than τ, and that $RC \gg T$ (see Appendix). Therefore, πRC will be much larger than T, or equivalently,

$$\frac{1}{RC} << \frac{\pi}{T} = \frac{\omega_o}{2}.$$

This implies that the capacitive reactance is negligible compared to R for input frequencies greater than half the switching frequency, or

$$x_c = 0 \quad \text{for} \quad \omega > \frac{\pi}{T}.$$

The voltage across the capacitor would then be small compared to the voltage across R, or

$$|E_m(\omega)| = 0 \quad \text{for} \quad \omega \geq \frac{\pi}{T}.$$

Therefore the only non-zero term in $E_m(s-j2\pi k/T)$, in Equation [10], is when $k = 0$. Since $M_o = 1$, Equation [10] can be written

$$RCsE_m(s) = \frac{\tau}{T} \sum_k M_k \exp(-j\beta_k) E_1(s-j2\pi k/T) - \frac{\tau}{T} E_m(s),$$

or

[11] $$E_m(s) = \frac{1}{\frac{T}{\tau} RCs+1} \sum_k M_k \exp(-j\beta_k) E_1(s-j2\pi k/T).$$

The output terminal can be considered a summing junction of the N branches over a period of the switching frequency. Then

$$e_2(t) = \sum_{n=1}^{N} a_n(t) \cdot e_m(t),$$

or

[12] $$E_2(s) = \sum_{n=1}^{N} A_n(s) * E_m(s).$$

and using the relations for $A_n(s)$ and Equation [9], and a new summing index p.

[13] $$E_2(s) = \frac{\tau}{T} \sum_{n=1}^{N} \sum_p \exp(-j\beta_p) M_p E_m(s-j2\pi p/T),$$

where

$$\beta_p = \pi p \left[\frac{2(n-1)}{N} + \frac{\tau}{T} \right].$$

Using Equation [11] for E_m, $E_2(s)$ becomes

$$[14] \qquad E_2(s) = \frac{\tau}{T} \sum_{n=1}^{N} \sum_{p} M_p \exp(-j\beta_p) \frac{1}{\frac{\tau}{T} RC\left(s - \frac{j2\pi p}{T}\right) + 1}$$

$$\cdot \sum_{k} M_k \exp(-j\beta_k) E_1\left(s - \frac{j2\pi p}{T} - \frac{j2\pi k}{T}\right)$$

$$= \frac{\tau}{T} \sum_{n=1}^{N} \sum_{p} \sum_{k} M_k M_p \exp - j(\beta_p + \beta_k) \frac{E_1(s - j2\pi(k+p)/T)}{\frac{\tau}{T} RC\left(s - \frac{j2\pi p}{T}\right) + 1} .$$

Only the exponential term contains values of n, and can be summed first (over n):

$$[15] \qquad \sum_{n=1}^{N} \exp \left\{ -j\pi(k+p)\left[\frac{2(n-1)}{N} \right] \right\} .$$

Now assume $k+p = mN$, with m an integer. This says that the switching function $a_n(t)$ is some integer multiple of N. Equation [15] then becomes

$$\sum_{n=1}^{N} \exp\left[-j2\pi m(n-1) \right] = N.$$

Since m and n are integers, the exponential is always equal to one, and sums out to N. Using this and $k + p = mN$, Equation [14] becomes

$$[17] \quad E_2(s) = \frac{N\tau}{T}\sum_p \frac{M_p}{\frac{T}{\tau}RC\left(s-\frac{j2\pi p}{T}\right)+1}\sum_m M_{mN-p}\exp\left(-\frac{j\pi mN\tau}{T}\right)E_1\left(s-\frac{j2\pi mN}{T}\right)$$

If τ is allowed to approach T/N, then

$$[18] \qquad \exp\left(-\frac{j\pi mN\tau}{T}\right) \rightarrow \exp(-j\pi m) = (-1)^m .$$

It will now be necessary to band limit the input, so that

$$[19] \qquad |E_1(\omega)| = 0 \quad \text{for} \quad \omega > \frac{\pi N}{T} .$$

This states that there are no input frequencies beyond the Nth multiple of half the switching frequency. This could be easily accomplished using only a low pass RC filter before the digital filter. Since $E_2(s)$ is a function of $E_1(s-\frac{j2\pi mN}{T})$, the spectrum of $E_2(\omega)$ can be shown as in Figure 10, where all other terms in Equation [17] are neglected. This shows that there are no overlapping spectrums when Equation [19] is satisfied.

If this condition were not satisfied, then certain input frequencies that were multiples of the switching frequency would appear in the output as f_o. Even with the limiting conditions, $E_2(s)$ still has harmonic responses of both $2\pi p/T$ and $2\pi mN/T$ (see Equation [17]). Therefore the filtering properties are still somewhat less than ideal at this point.

If the output, $E_2(s)$, is now filtered by a broad bandpass filter at $f_o = 2\pi/T$, the only non-zero terms in Equation [17] will be with $m = 0$ and $p = \pm 1$ (that is, harmonics of the switching frequency are removed from the output):

$$E_2(s) = \frac{N\tau}{T} E_1(s) \left[\frac{M_1 M_{-1}}{\frac{T}{\tau} RC(s - \frac{j2\pi}{T}) + 1} + \frac{M_1 M_{-1}}{\frac{T}{\tau} RC(s + \frac{j2\pi}{T}) + 1} \right] . \tag{20}$$

But

$$M_1 = M_{-1} = \frac{\sin \pi\tau/T}{\pi\tau/T} = \frac{\sin \pi/N}{\pi/N} ,$$

and since $N\tau/T$ approaches one,

$$\frac{E_2(s)}{E_1(s)} = \left[\frac{\sin \pi/N}{\pi/N}\right]^2 \left[\frac{1}{NRC(s - \frac{j2\pi}{T}) + 1} + \frac{1}{NRC(s + \frac{j2\pi}{T}) + 1} \right] . \tag{21}$$

Considering that the transfer characteristic of a low pass RC filter can be written

$$\frac{E_2(s)}{E_1(s)} = \frac{1}{sRC + 1} , \tag{22}$$

Equation [21] is in a similar form, except that it is translated in frequency to center about $1/T$ hertz instead of zero, as shown in Figure 11. Thus the RC low pass characteristic has become a bandpass characteristic centered about $2\pi/T$, with a bandwidth $1/N$ times as large.

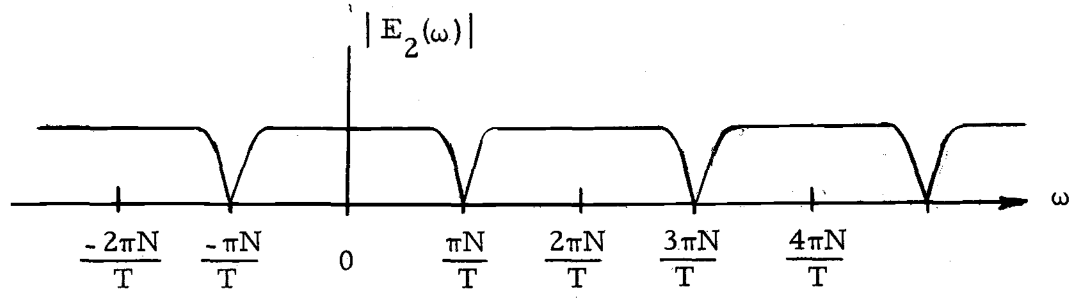


Figure 10. Output spectrum with input band limited:
$E_2(\omega) = f[E_1(s - \frac{j2\pi mN}{T})]$, $E_1(\omega) = 0$ for $\omega \geq \frac{\pi N}{T}$

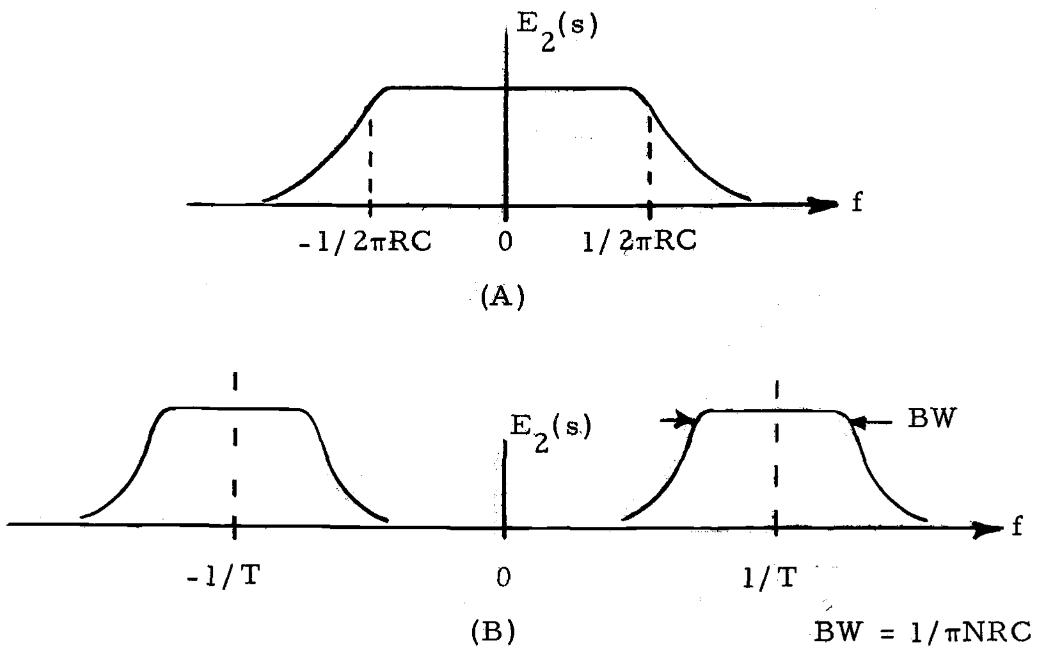


Figure 11. Transforming low pass characteristic of RC filter (A) to bandpass characteristic of digital filter (B), where output has been further filtered by a broad bandpass filter.

As the input frequency approaches $2\pi/T$, the first term in brackets in Equation [21] approaches one, while the second term becomes very small. Thus the input voltage is attenuated by a constant, $[\frac{\sin \pi/N}{\pi/N}]^2$, with no phase shift. This represents an insertion loss at the center frequency of the passband, f_o. As the input frequency begins to increase, it becomes attenuated and undergoes a phase shift similar to a low pass filter, ranging between 0 and $-\pi/2$ radians. The frequency difference can also become negative and cause positive phase shifts. The overall phase shift thus varies from $-\pi/2$ to $\pi/2$, with zero phase shift at f_o.

To show how the insertion loss would change as the number of branches were varied, refer to Table 1.

Table 1. Digital filter attenuation of input frequency.

N	$[\frac{\sin \pi/N}{\pi/N}]^2$	-db
3	0.75	-2.48
4	0.81	-1.86
5	0.89	-1.00
6	0.91	-0.82
7	0.93	-0.62
8	0.94	-0.54

If the output had been filtered by only a low pass RC filter which passed $1/T$ hertz, then $p = 0$ would also have to be considered in Equation [17], giving an additional term

[23] $$\frac{NT}{T}\left[\frac{1}{\frac{T}{T}RC(s-0)+1}\right]=\frac{1}{NRC\,s+1}$$

as T/T approaches N. This is the same form as Equation [22], except that the RC constant has been increased by N. The output spectrum, $E_2(s)$, will now contain an RC low pass characteristic around zero hertz, giving an overall spectrum as shown in Figure 12.

Since the bandwidth of the output is

[24] $$BW = 2\left[\frac{1}{2\pi NRC}\right] = \frac{1}{\pi NRC}$$

the circuit Q will be

[25] $$Q = \frac{f_o}{BW} = \pi f_o NRC = \frac{\omega_o NRC}{2}.$$

If the output of the digital filter is not filtered (although the input still is), the transfer function contains all values of p and m. There are then many harmonics present in this output, and knowing the results of Fourier series expansion, some type of square wave might be expected. A squared-off version of the input is what actually results, as can be seen in Figure 13. This figure shows the input wave, the digital filter output, and the output component at f_o (which has been filtered out with the 50 hz bandpass filter in the GR Wave Analyzer). Since four branches were used (as part B shows), the attenuation can be found from Table 1. Part C reads about 0.8 volts

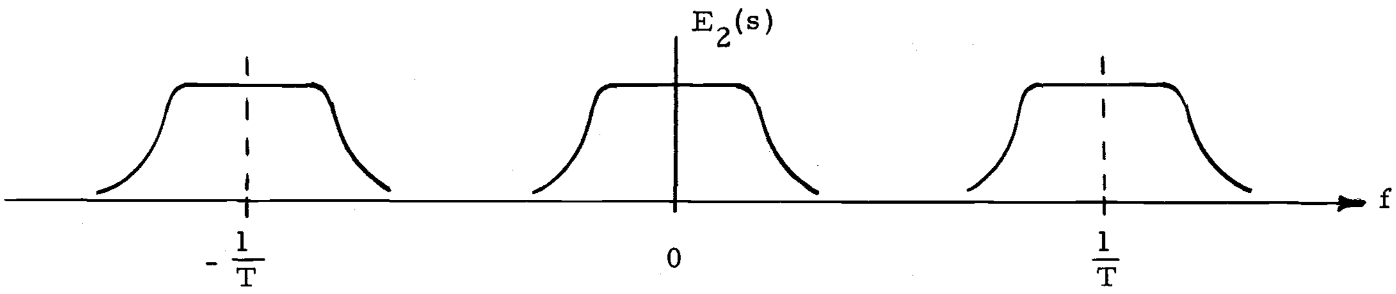
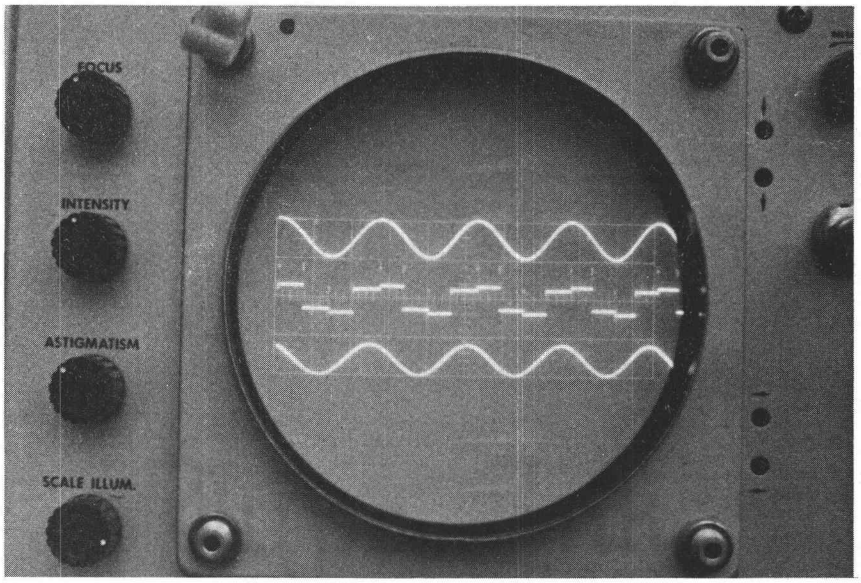


Figure 12. Output spectrum of digital filter, where output was further filtered by a low pass filter. All three bandwidths equal $1/\pi NRC$.



First waveform: Input waveform to digital filter.
Second waveform: Unfiltered output of digital filter.
Third waveform: Output after bandpass filtering.

Figure 13. Digital filter waveforms.

peak to peak, as expected. There is a slight phase shift, but this was due to the 50 hz bandpass filter, and not the digital filter.

Loading Effects

Also affecting the output is any load resistance (R_L) present. Looking at only one RC branch, this would have the effect of limiting the gain of the transfer function when the capacitive reactance is large, or when the input frequency is near f_o. As the input frequency moves far away from f_o, the reactance becomes small and the capacitor shorts out R_L. Using a filter with N sections, the load resistance effectively increases to NR_L. If $N = 4$, R_L is connected to one RC branch only one-fourth of the time, and can therefore affect its output only one-fourth of the time. In "real" time, this appears as a resistor four times as large. This effect is proven more rigorously in a second article describing digital filter theory (4).

Other Topics

Also included in the article just mentioned is a discussion describing the digital filter using a more general approach. In particular, the switching function, $a_n(t)$, is allowed to be sinusoidal in form, or a general type of jump modulation. The filter was also used to realize an all-pass constant delay characteristic over a limited bandwidth. However, the main points in both articles were similar to the subjects brought out in this paper.

IV. EXPERIMENTAL RESULTS

A digital filter was built in the lab and data was taken to test some of the theories covered in the previous section. Also, some of the more practical aspects were looked into which could cause results to differ from the expected.

Basic Set-Up

On hand was an Adar Associates SQ-260 Multi-channel Programmable Pulse Generator which was capable of supplying pulses on as many as 12 different output channels, and could be adjusted as to frequency, pulse length and amplitude, and programmed into any position in the basic cycle. Using $N = 3$, the separate output channel pulses would appear as shown previously in Figure 4. Also used was a Hewlett-Packard Electronic Counter (HP 5245L), a Tektronix Operational Amplifier plug-in unit (Type 0), a Wavetek voltage controlled generator (VCG111), which allowed the signal voltage to be synchronized to the switching pulse rate, a General Radio Wave Analyzer (GR 1900-A), and other voltmeters and oscilloscopes, as needed.

The operational amplifiers were used to prevent loading effects. The complete circuit is shown in Figure 14.

A variation of this circuit was also used, the silicon switching transistors being replaced by Fairchild 2N4360 field-effect transistors.

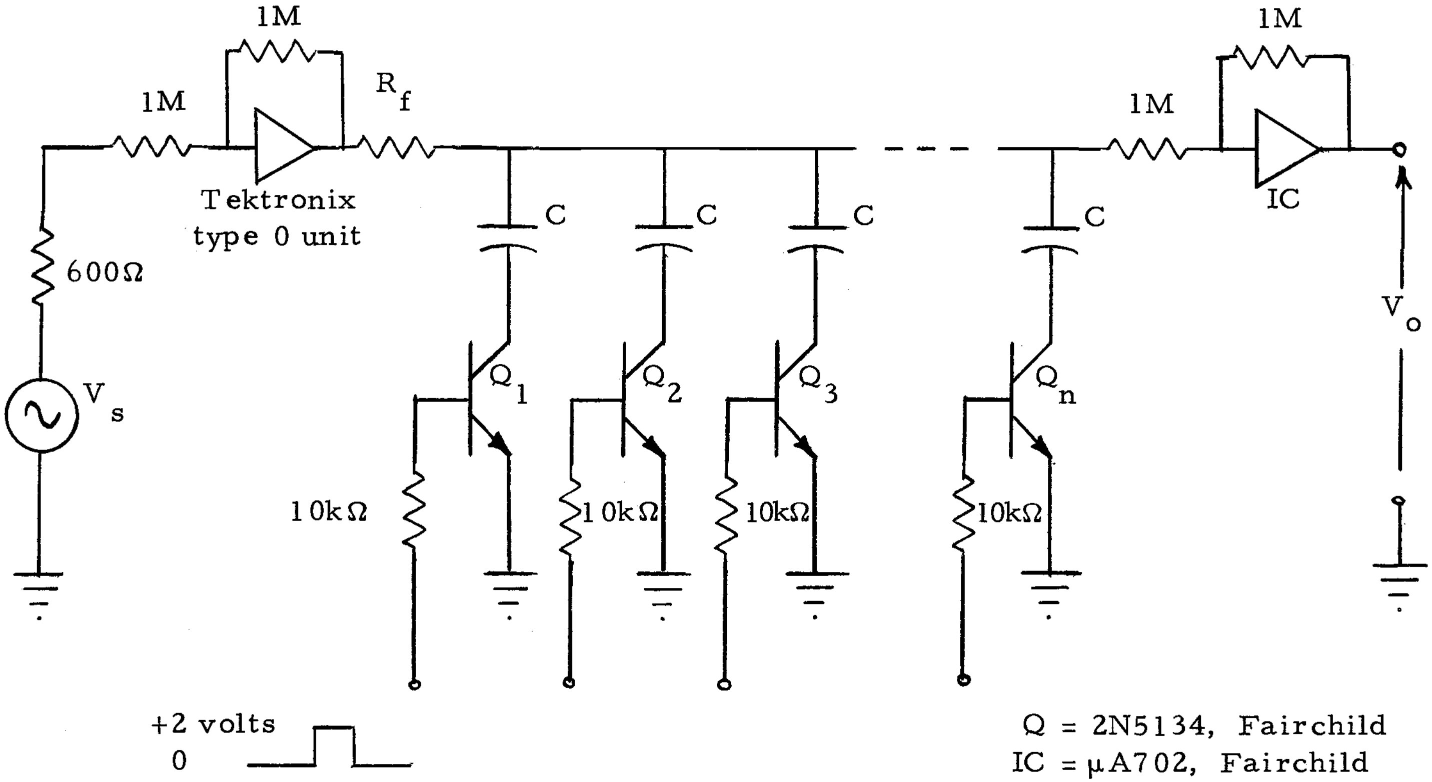


Figure 14. Digital filter used for testing.

There was considerable difference in the data obtained from these switches, as can be seen in some of the graphs on the next few pages. The reason was that field-effect transistors (FET's) make better AC switches than junction transistors. This will be further discussed after Figure 19. The circuit for an FET switching branch is shown in Figure 15. The FET was placed on a Tektronix Transistor Curve-Tracer to determine what gate to source voltage would be necessary to completely turn them off. The resulting trace is shown in Figure 16. It can be seen that it is almost off at four volts. Six volts was used in the lab.

Bandwidth

It was observed when using a very large R_f, it was necessary to keep stray output capacitance and instrumentation capacitance to a minimum. If the $R_f C_{out}$ constant should be such as to cause it to short out the V_s frequency, the performance of the digital filter will be severely degraded.

One area of investigation was the effects of the different parameters on the bandwidth of the digital filter passband. The RC time constant was first varied, and results are shown in Figure 17. Next, the number of RC branches (N) was varied, with results shown in Figure 18. Although there is appreciable difference when using the two types of switches, the important point to note at present is

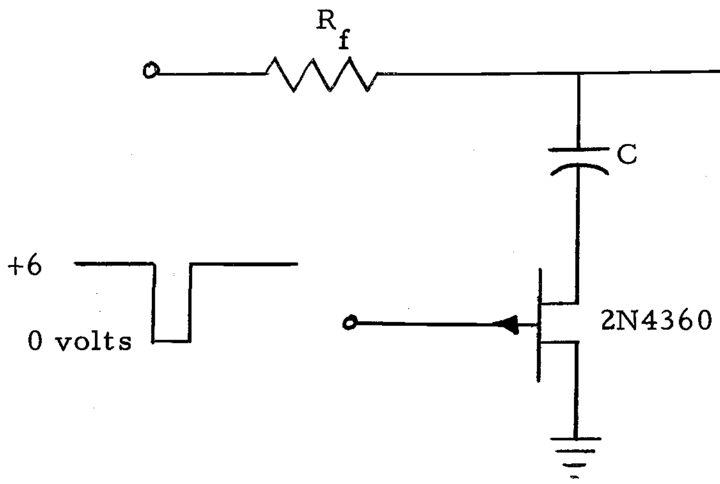


Figure 15. Switching branch using a field-effect transistor.

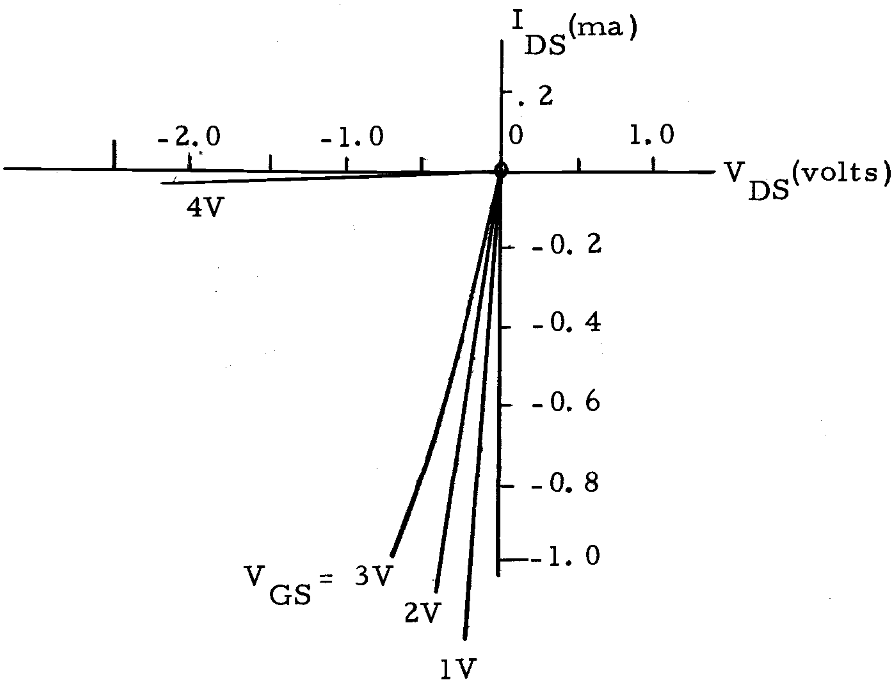


Figure 16. 2N4360 characteristics. (Results were similar when positive drain voltage was applied.)

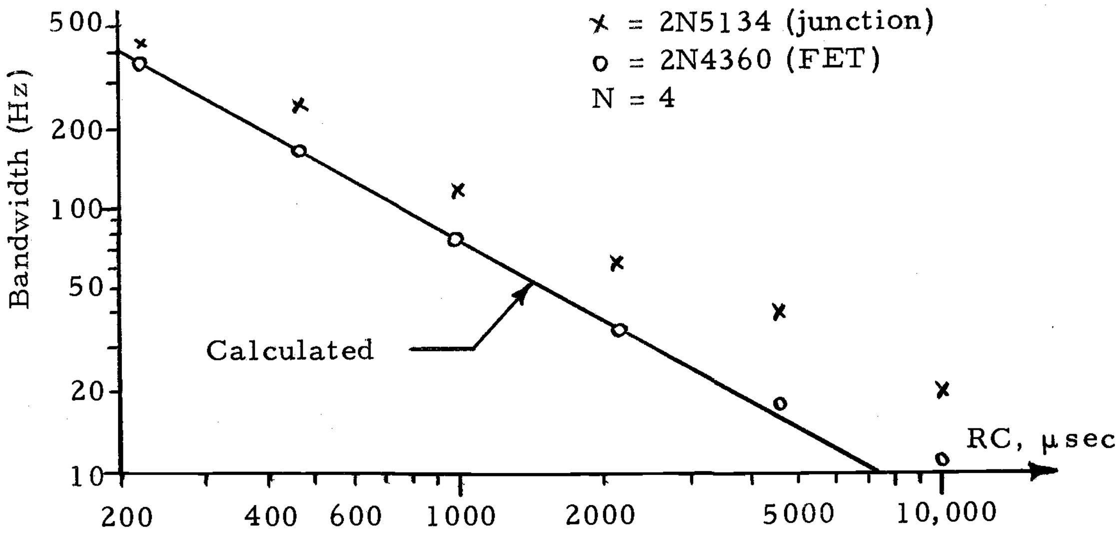


Figure 17. Data points and calculated values of bandwidth for various RC constants.

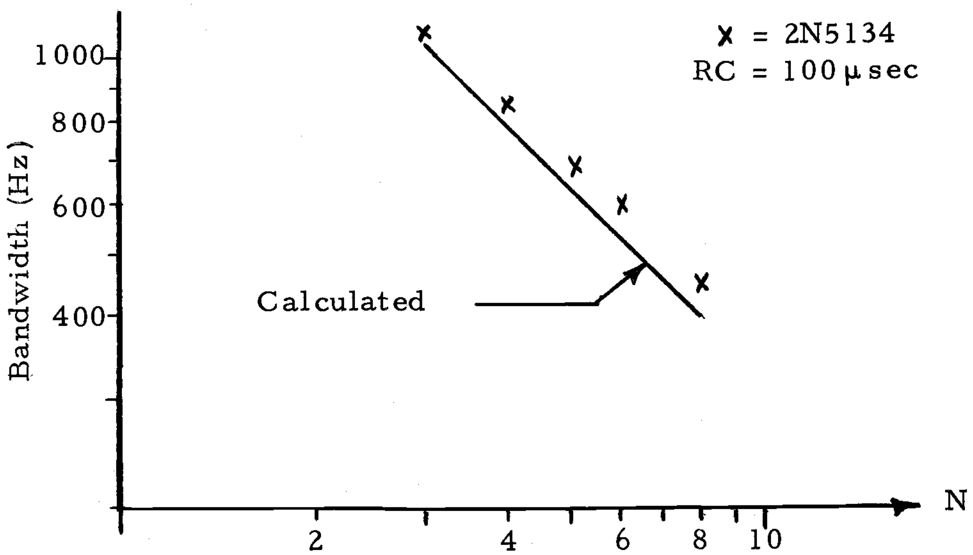


Figure 18. Data points and calculated values of bandwidth for various values of N.

bandwidth being inversely proportional to N and to the RC constant. This is as expected from Equation [24].

The effect on bandwidth as the switching frequency varied was next tested. The results are shown in Table 2. As expected, there is little variation since bandwidth was previously shown to depend on only N, R and C. It was not expected that the bandwidth would change.

Table 2. Effect of switching frequency on bandwidth. (RC = 4.7 msec, N = 4, Calculated BW = 17 hz).

f_o	BW
10 KHz	16 hz
20 KHz	18 hz
50 KHz	18 hz

Harmonic Properties

Harmonic properties of the filter were also examined. It was decided to feed harmonics of the switching frequency into the input (thus violating the condition in Equation [19]) to see what effects this would have on the output. For example, using f_o = 20 KHz and V_s = 40 KHz, and looking at the output with a highly selective voltmeter (bandwidth = 50 hz) set for 20 KHz, a strong 20 KHz signal was observed. This test was repeated using various input frequencies and different values of N, although f_o was kept at 20 KHz. The results are tabulated in Table 3.

Table 3. Input harmonics that appear in the output at the switching frequency. (R = 10 Kohms, C = 0.1 microfarads)

Input Frequency	Output @ 20 KHz (mvolts) N = 3	N = 4	N = 5	N = 6	N = 8
20 KHz	130	140	145	160	155
40 KHz	48	1	10	5 1/2	10
60 KHz	2	80	8	6	8
80 KHz	17	1	30	3 1/2	3
100 KHz	12	50	4	22	2
120 KHz	2	6	17	2 1/2	2
140 Khz	7	30	3	10	10
160 KHz	1	1	4	3 1/2	2
180 KHz	0	25	9	2	5
200 KHz	0	2	1	0	2

If design conditions were such that input harmonics were inherently present, it might be best to use a large number of RC branches. With 8 branches, there was a strong 20 KHz output only when 20 KHz was at the input. But with three branches, there was an output with 20 KHz, 40 KHz, 80 KHz or 100 KHz at the input. In each case, it is to be noted the very small output when the input is Nf_o. This was discussed earlier, being the case where each branch sees one complete cycle of input voltage, which has an "average" voltage of zero.

Due to limitations of the test equipment, it was difficult to make extensive measurements of output harmonics of an input frequency. The pulse generator would not operate properly below 9 KHz, and the selective voltmeter would not measure above 60 KHz. Harmonics in the output would correspond to the case where the output of the digital filter is not further filtered by a broad bandpass filter as discussed before Equation [20]. The results are shown in Table 4.

Table 4. Output harmonics of the switching frequency. ($f_s = f_o$ = 9 KHz, R = 47 Kohms, N = 3, C = 0.02 microfarads)

Output frequency (KHz)	Output voltage (mv)
9 (f_o)	145
18 ($2f_o$)	55
27 ($3f_o$)	10
36 ($4f_o$)	20
45 ($5f_o$)	10
54 ($6f_o$)	1

From these results, and the fact that there is also an output response near zero hertz (up to $1/2\,\pi NRC$), it is obvious why bandpass filtering around the switching frequency is required. Since the harmonics are generated within the digital filter itself, they can only be eliminated by using an additional filter. However, the digital filter is providing the high Q, while the other filter has only to remove harmonics, which is relatively easy.

Bandpass Response

In order to adequately show the overall bandpass response of the digital filter around f_o, it is necessary to plot the response as a function of the difference in frequency between the input and f_o. This will be easy to interpret on logarithmic paper, and the slope can be easily checked. In plotting the "actual" response as a function of frequency (instead of the difference), a very sharp spike occurs which is difficult to plot for large values of Q. An example of this response as a function of frequency difference is shown in Figure 19.

There are three regions to this response, all of which deviate slightly from the expected. The insertion loss should vary as the number of RC branches (N) vary, as described in Equation [21], but not as the RC constant varies. The reason for this is that transistors do not make ideal AC switches. They are unidirectional devices, and when they are switched off by removing their base current, it is still possible for the capacitor in the collector circuit to be placing a negative potential at the collector, thereby leaving the collector-base junction forward biased. Current is drawn across this junction until the forward bias is removed, causing the capacitor to lose some of its stored charge. This then causes the response for all N capacitors to be somewhat less than expected. However, in both this case, and in Figures 17 and 18, the calculations provide fairly close approximations, and digital filters could still be easily built with these inherent discrepancies.

To further investigate this matter, one of the junction transistors was placed in the curve tracer, and biasing the base-collector

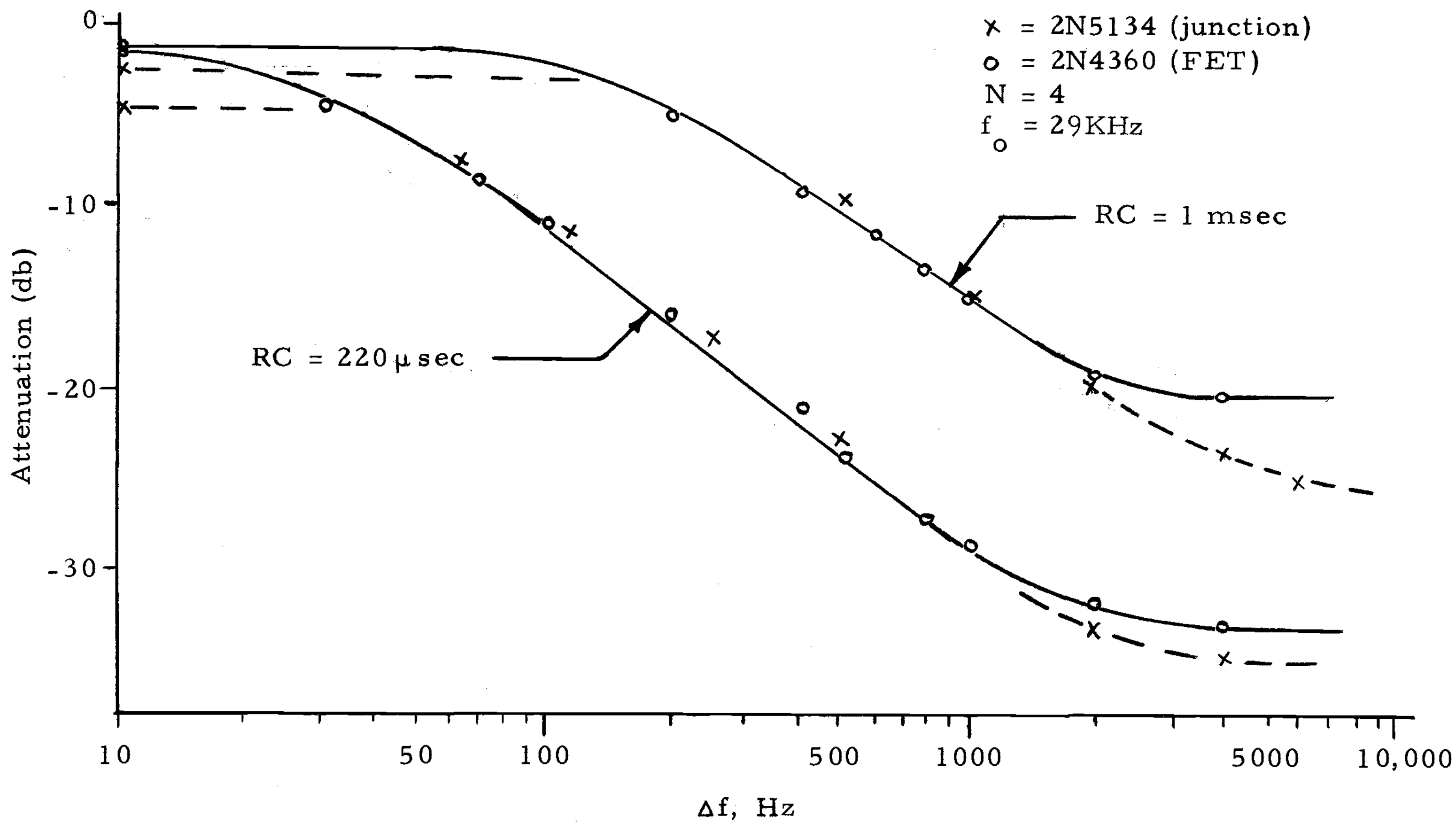


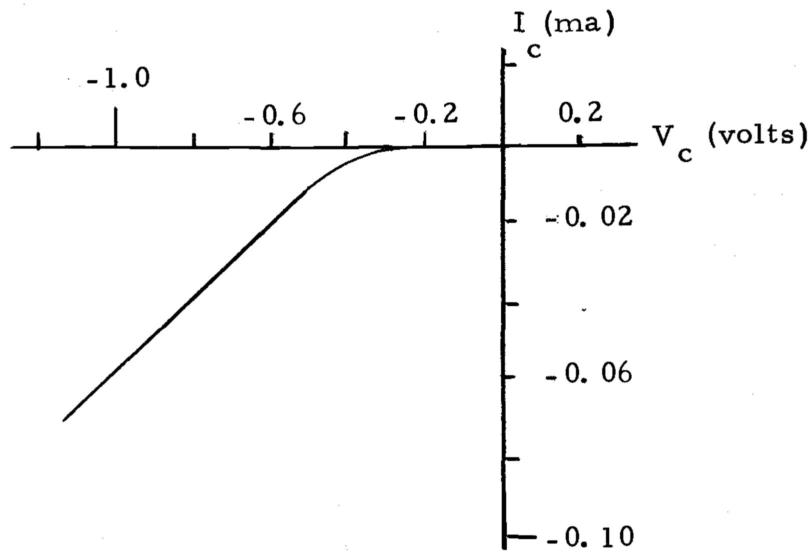
Figure 19. **Frequency response curves for digital filter.**

junction in the forward direction (collector negative for an NPN transistor), it was possible to obtain an operating curve with the base grounded through a 10 Kohm resistor. This trace is shown in Figure 20. As long as there was more than 0.4 volts forward bias, the junction had a resistance of about 700 ohms. Below 0.4 volts, the junction was cut off. Thus, in a branch which was supposed to be off, the capacitor may have been actually discharging through the 10 Kohm base resistance, the 700 ohm junction resistance, and whichever RC branch was ON at the time, until the junction bias fell below 0.4 volts. The result was therefore an increase in insertion loss and a widening of the passband beyond calculated values.

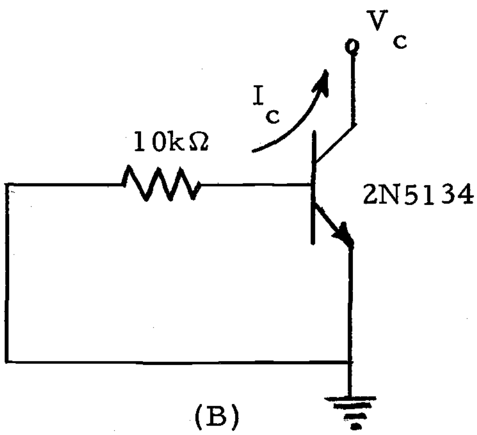
The problem was remedied when using the field-effect transistors. When an FET is biased off from gate to source, it is completely off for up to large values of drain voltage. When the bias is removed, it is free to conduct in either direction through its saturation resistance. It is therefore bidirectional.

The second region of Figure 19 is the slope, which is very close to the expected 20 db per decade. This slope was identical on both sides of f_o, so that there is arithmetic symmetry. Since Equation [21] involves $s - j2\pi/T$, this is to be expected.

There is finally the flat region to the right. This is a measure of the maximum attenuation possible, "out-of-band," with the digital filter. It is limited by imperfections of the transistors and is caused



(A)



(B)

Figure 20. Operating curve (A) of circuit shown (B). Base-emitter is off, but base-collector is still forward biased. $R_{CB} = 700\Omega$ when $V_{CB} > 0.4$ volts.

by the saturation resistance from collector to emitter (or source to drain)(5). Taking this resistance into account, the equivalent circuit of one branch is as shown in Figure 21. The resistance causes a second break point in the response curve at

$$f_{c_2} = 1/2\pi C R_{ce}(sat).$$

For frequencies above this, the capacitive reactance is small compared to $R_{ce}(sat)$, and hence the output voltage will not decrease beyond

$$\frac{V_o}{V_s} = \frac{R_{ce}(sat)}{R_f} \quad . \qquad [26]$$

Thus by proper choice of components, the maximum attenuation desired can be designed into the final circuit. The fact that the FET's were unable to attenuate the out-of-band signals as well is an indication of their higher saturation resistance. From the data in Figure 19, the saturation resistance of the FET's was calculated to be 200 to 250 ohms, while the junction transistors had values from 100 to 200 ohms.

One final aspect of Figure 19 is the response curve for $RC = 220$ microseconds, which for $f_o = 29$ KHz does not really satisfy the previously stated condition that $RC \gg T$. Thus, although this condition is generally desirable, it is not absolutely required. If the

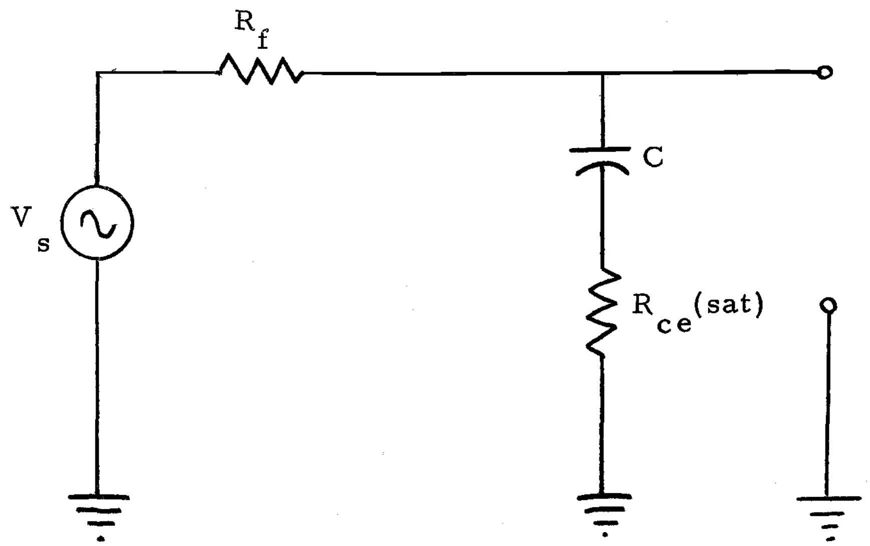


Figure 21. Equivalent circuit for digital filter branch which includes saturation resistance of transistor.

RC constant were too small however, the capacitor would be able to follow the input voltage variations, the bandpass characteristics of the harmonics would start to overlap, and there would result a total loss of bandpass characteristics.

Zero Signal Noise

When Equation [26] has been employed to provide very low response to out-of-band signals, a limit is approached due to the inherent noise of the digital filter, the zero signal noise (5). This noise is caused by unequal saturation voltages in the transistors, and is coupled out through the capacitors. If this noise is larger than the designed out-of-band response, then it will be the dominant feature at those frequencies. For the test set-up, approximately 0. 6 mv rms of zero-signal noise was observed at the output at 20 KHz, using the wave analyzer. It has been observed that this value rises rapidly as the switching frequency increases, approaching rather large values (20 mv) above 500 KHz (5).

Overlapping Switching Pulses

One method attempted for decreasing the zero signal noise was to cause the switching pulses to overlap slightly. Whenever two pulses are ON simultaneously, the respective capacitors will tend to discharge into each other through the transistor saturation resistances, until they are equal. When $V_s = 0$, all N capacitors would eventually equalize out so that no variations are coupled to the output. In the circuit utilizing the FET's, by balancing the amplitudes of the switching pulses from the pulse generator, the output noise was about 0.6 mv in the 20 KHz to 50 KHz range. Then when setting each switching pulse to overlap 100 nsec. on both sides of the pulse, this noise could not be balanced down to any lower value. This method was therefore unable to reduce the noise level beyond that possible in a well designed filter with non-overlapping pulses. By deliberately unbalancing them, the worst amount of noise was 10 mv, whereas by deliberately unbalancing the non-overlapping pulses, noise rose as high as 100 mv. The overlapping pulses were therefore less sensitive to variations in switching pulse amplitudes. If the switches were all integrated, there would be no noticeable variation among their characteristics, and probably would cause no problem. Using discrete transistors, there was some variation, and therefore overlapping pulses might be desirable. There was a definite disadvantage to this process, however: an increase of insertion loss. This would be expected since it is a condition similar to the case where the collector-base junction was still forward biased after the transistor was switched off, causing the capacitor to partially discharge. With overlapping pulses, two transistors are fully ON for the overlap time, and the capacitors will try to equalize through the two transistor saturation resistances. Since these resistances are quite low, the discharge constant is usually much smaller than the charging constant. However, it discharges for a much shorter time. It will be the relative ratios of these RC constants and charging times that will determine the final voltage on the capacitors, and hence at the output of the filter. An exact mathematical expression for the loss as a function of pulse overlap would be difficult, especially since it is not known to which values the capacitors are charging or discharging. It is only known that they will attempt to neutralize each other when both are ON. Some experiments were run to test the effects on gain as the pulses are overlapped. This is shown in Figure 22. It would be expected that for equal overlap times, gain would decrease as the frequency increases, since the pulse period must decrease, and hence the charging time on the capacitors decreases.

The horizontal scale is the relative amount of overlap per pulse length. Thus a 50 microsecond pulse with 100 nsec. of overlap (at each edge of the pulse) would have a value $100 \times 10^{-9}/50 \times 10^{-6} = 0.002$. In general, it can be seen that the gain falls very rapidly as the overlap time is increased. Overlapping pulses would therefore not normally be a desirable feature, unless there was an excessive zero signal noise problem.

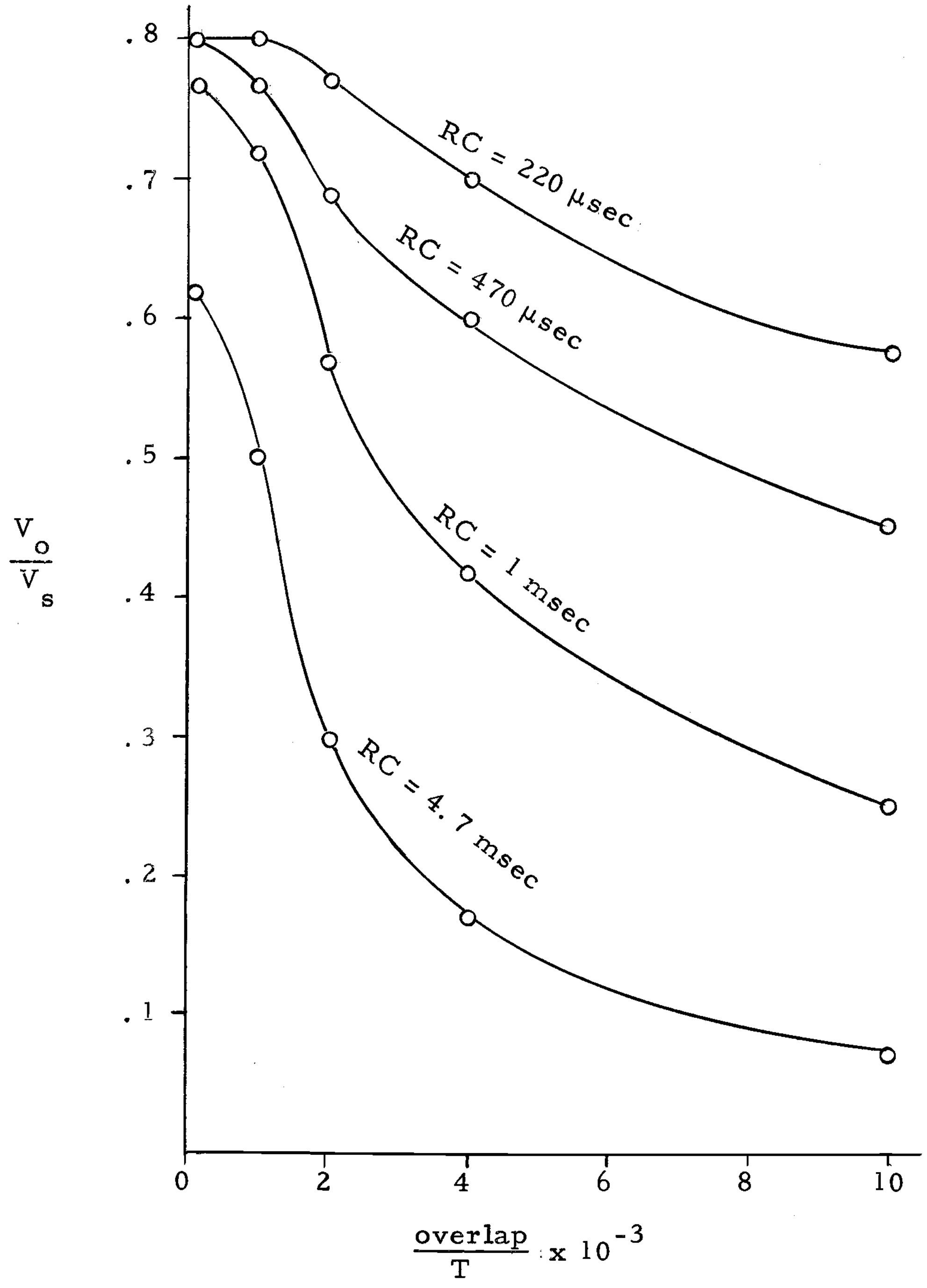


Figure 22. Digital filter gain as a function of relative amount of overlap of switching pulses, and for various RC constants (N=4).

Voltage Spikes in the Output

The non-overlapping pulses are more desirable, but they can cause an effect which should be noted. During the time of the gap, all switches are OFF, thereby placing the input signal directly on the output load. With no input signal, the saturation voltages of the transistors are AC coupled to the output. During the gap, zero voltage is coupled out. There is thus an effective spike equal to the saturation voltage which is coupled out every time a transistor is switched. Using the junction type transistors, output spikes were similar to those in Figure 23. The spike width was approximately the same as the gap width of the switching pulses.

These spikes are also noticeable when there is an input signal, as can be seen in Figure 13. When the digital filter output is followed by a bandpass filter, the spikes can be eliminated, and should normally cause no problem.

When there was a large gap between pulses, the output was the same as the input signal during the gap. Thus, when an out-of-band signal was applied, it would pass through during the gaps, but would not when a branch was ON. Therefore large gaps are not desirable.

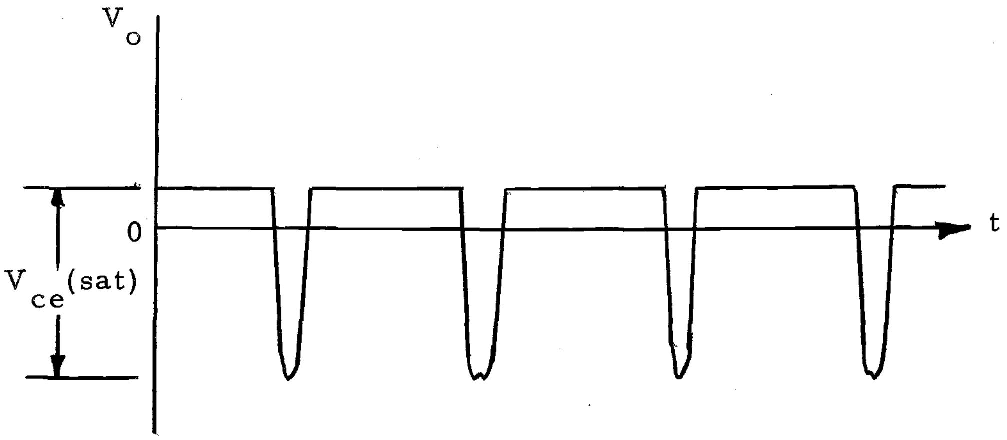


Figure 23. Output voltage spikes of digital filter with no input signal. They are caused by the saturation voltages of the transistors.

V. A COMPLETE CIRCUIT

All tests were run on the circuits previously described. However, a more practical circuit was built using integrated circuits in a ring counter. It is shown in Figure 24. The clock pulsed at 4 KHz, which meant the digital filter switching rate was 1 KHz. The bandwidth was 17 hz, giving $Q = 1000/17 = 59$.

The logic circuits in the ring counter were such that when the clock pulse was applied, a ONE would be forced to move to the following stage. However, this switching would occur after the clock was applied, thereby causing about 100 nsec. overlap in switching pulses. Therefore the results previously described in Figure 22 occurred. The effect was slight at 1 KHz, but when the switching rate was changed to 10 KHz, the insertion loss rose to about 10 db.

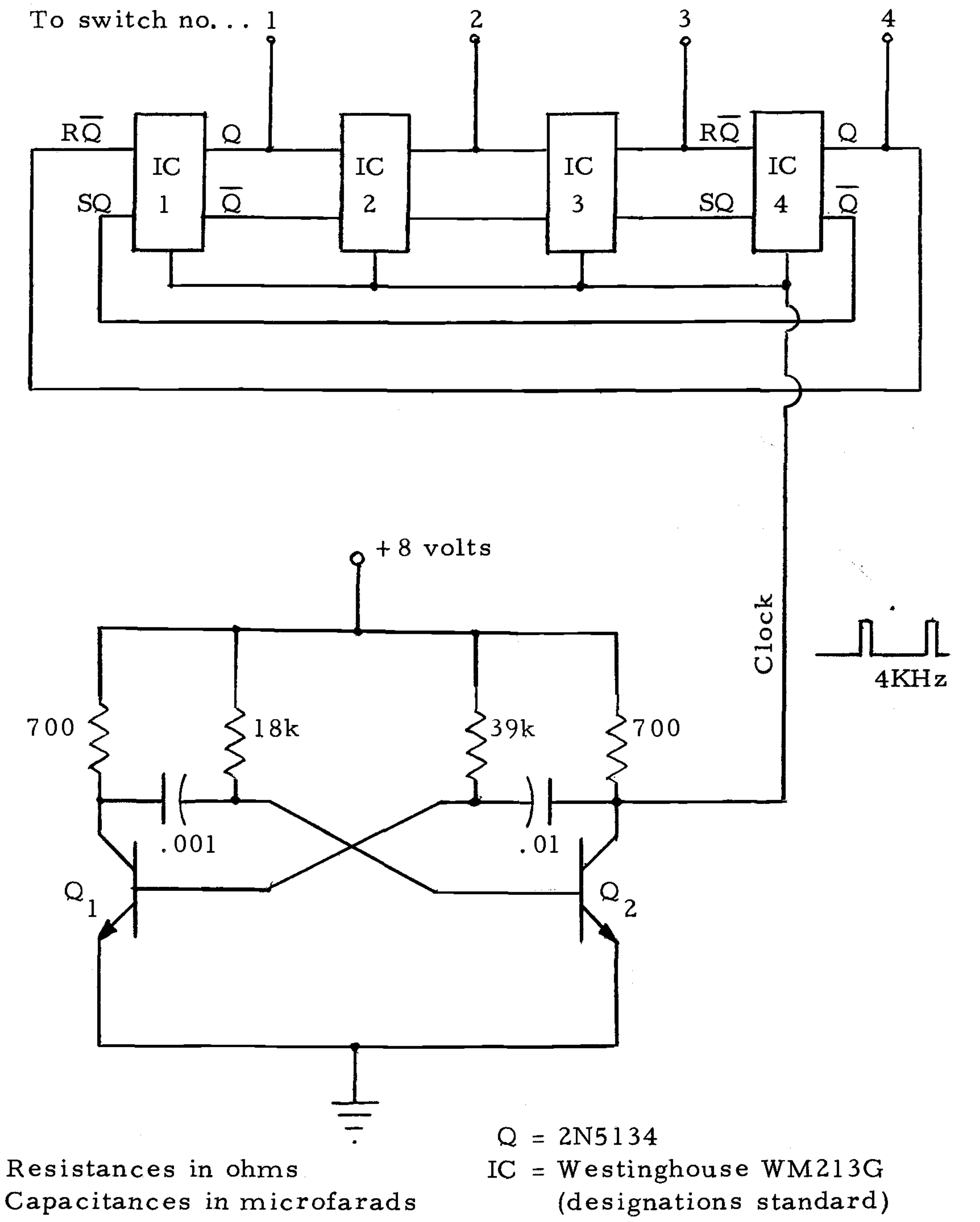


Figure 24. A pulse supply for the digital filter shown in Figure 14.

VI. PROCEDURE FOR DESIGNING DIGITAL FILTERS

The following procedure provides a step by step approach to designing digital filters. It is intended for shunt type filters, although series types would be similar.

1. Decide what center frequency, f_o, is desired.
2. Determine the value of Q required.
3. Determine the amount of out-of-band attenuation required.
4. Decide what type of devices are going to be used for the switches. Field-effect transistors give results more consistent with theory. Junction transistors will work, but they will increase insertion loss and bandwidth, possibly by a factor of two.
5. By testing, or referring to specification sheets, determine the saturation resistance of the switches. Values between 100 and 200 ohms could be assumed.
6. Determine the value of R_f by use of:

   $R_f = R_{ce}(\text{sat})\ V_s/V_o$, where the voltage ratio is the value in the out-of-band region, determined in step 3.
7. Pick the number N of RC branches. This may depend upon the method of supplying pulses. If a ring counter is used, any number is feasible, although 3, 4, or 5 is usually sufficient. If a combination counter and logic circuit is

used, it will probably be much easier to pick $N = 2^m$, where m is an integer greater than or equal to two. Thus $N = 4$ would be a logical choice for such a pulse supply. Also, the clock rate can now be determined, being equal to Nf_o. Thus, for a digital filter with four branches, and operating at 1 KHz, the clock would have to be 4 KHz.

7. Determine the value of C necessary to achieve the Q desired. This will be

$$C = Q/\pi f_o NR,$$

or using bandwidth (BW),

$$C = 1/\pi NR(BW).$$

8. Examine the pulse supply again to see if the switching pulses are overlapping, and if they are, by how much. If there is too much overlap, it may be necessary to pick another type of pulse supply, or else design a special circuit which will prevent the pulses from being applied to the switches for a short duration, during which the pulse supply is changing state. See for example, Thompson (6, p. 48). However, a small amount of overlap may help to reduce zero signal noise.

9. If there is any possibility of frequencies greater than

$Nf_o/2$ in the input, place a low pass filter at this point to reduce these frequencies to zero.

10. Place either a low pass or broad bandpass filter on the output, which will pass f_o. The low pass filter is to remove all harmonics of f_o, but will allow frequencies close to zero hertz to pass. If this is undesirable, use the bandpass filter.
11. Use isolation buffers whenever loading effects may occur. By using operational amplifiers, both isolation and filtering (step 10) can be accomplished at once.

VII. A DIGITAL FILTER PRESENTLY IN COMMERCIAL USE

This author was fortunate in being able to talk with a company which is currently utilizing digital filters in some of their equipment. Collins Radio Company at Newport Beach uses them in some of their data communications equipment. They are constructed using hybrid circuit techniques (a combination of integrated and discrete components). Two innovations in their design were noteworthy (7). One was the use of a current driver at the input of the filter instead of the resistor R_f. The current source effectively appeared as an infinite impedance, thereby giving extremely high Q. Second was their concept of "quenching". When the input signal is removed, it takes a few switching cycles to remove the stored charge on the capacitors. In their particular application this was undesirable. By turning all N branches ON at once (quenching), the charges were quickly neutralized, reducing the output voltage to zero. The filter was then ready for the next input signal.

Another aspect of their design was an operational amplifier with LC feedback to obtain the broad bandpass output filter. It was buffered from the digital filter by a compound emitter follower to prevent loading (8).

Finally, to prevent switching pulses from overlapping, an integrated circuit pulse supply was used wherein the logic required one stage to shift states <u>before</u> the following stage would shift. The rise time of the shifting pulse was longer than the fall time due to the inherent nature of the particular integrated circuits used. The result was an effective gap between pulses, as shown in Figure 25 (7).

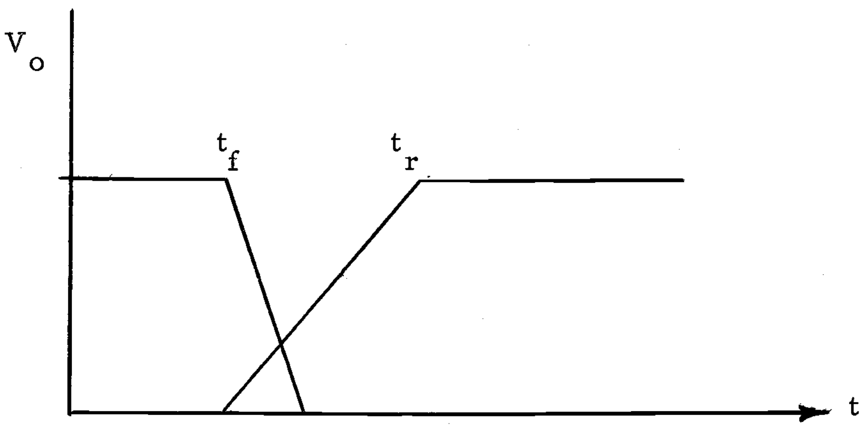


Figure 25. Switching pulse gap in digital filter designed by Collins Radio Company.

VIII. CONCLUSION

This paper has presented a general study of the characteristics of RC digital filter operation. Their basic design and explanations of their operation have been presented, using both a mathematical and physical approach. It was shown how the bandpass characteristic varies with the component values and the number of switching branches. Some of the problems encountered in the physical realization are shown and the extent of their degradation are noted. The use of digital filters at low frequencies could be an obvious advantage over conventional LC filters which normally require large and sometimes unreliable inductors. The Q of digital filters can be made very large with a minimum of design effort and physical components.

It is usually desirable to band-limit both the input and the output of the digital filter about the switching frequency, and to isolate it with buffer amplifiers.

Complete integration of the filter would be desirable, although there are limits on the Q available since capacitors and resistors are usually limited to values less than 300 pf and 20 Kohms (1). This would give a bandpass of approximately 50, 000/N hertz, which would be much too large for most applications. At present, the hybrid circuit appears to offer the most attractive possibilities in miniaturization.

Bidirectional switches give results most consistent with theory since unidirectional devices do not work well as AC switches. However, useful digital filters could be designed with either type of switch.

Finally, note must be taken as to whether the switching pulses overlap or not, and if they do, how much effect this will have on the output of the digital filter.

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APPENDIX

APPENDIX

A mathematical explanation is given here for the "averaging" process on a switched RC circuit, as described in Section II (3, p. 27-31). The input voltage is a cosinusoid, whose form is

$$V_{in} = V \cos 2\pi t/T_o,$$

where

$$T_o = 1/f_o.$$

For an RC low pass filter whose time constant is large compared to T_o ($RC \gg T_o$), the capacitive reactance is small compared to the series resistance, and the input current can be given by

$$i_{in} = \frac{V}{R} \cos 2\pi t/T_o.$$

With an initially uncharged capacitor, the amount of voltage added during the first ON period is

$$\Delta V_1 = \frac{1}{C}\int_{-\tau/2}^{\tau/2} i_{in}(t)dt \qquad = \frac{1}{RC}\int_{-\tau/2}^{\tau/2} v_{in}(t)dt$$

$$= \frac{V}{RC}\int_{-\tau/2}^{\tau/2} \cos\frac{2\pi t}{T_o}\, dt \;= \frac{VT_o}{2\pi RC}\left[\sin\frac{2\pi t}{T_o}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{VT_o}{2\pi RC}\left[2\sin\frac{\pi\tau}{T_o}\right] \qquad = \frac{VT_o}{\pi RC}\sin\frac{\pi\tau}{T_o},$$

where τ is the time of the ON period.

When the gate is turned off, this voltage remains on the capacitor until the next ON period. Then two processes take place: (1) the charge on C from the first period partially leaks back through the source, and (2) the source adds more charge to the capacitor. The net charge during the second ON period is the algebraic sum of the charging and discharging currents. The net charging current is

$$i_{in_2} = \frac{v_{in}(t) - \Delta V_1}{R} .$$

Then the net voltage accumulated during the second period is

$$\Delta V_2 = \frac{1}{RC} \int_{-\tau/2}^{\tau/2} v_{in}(t)dt - \frac{1}{RC} \int_{-\tau/2}^{\tau/2} \Delta V_1 dt$$

$$= \Delta V_1 - \frac{\tau}{RC} \Delta V_1 = \Delta V_1 (1-\tau/RC).$$

Repeating this process gives an expression for the net voltage increment acquired during the n^{th} ON period:

$$V_n = \Delta V_1 (1-\tau/RC)^{n-1}.$$

The total voltage accumulated on the capacitor after n ON periods is

$$v_n = \Delta V_1 \sum_{p=1}^{n} (1-\tau/RC)^{p-1}.$$

This is a geometric progression summed through n terms, and can be written

$$v_n = \Delta V_1 \frac{[1-(1-\tau/RC)^n]}{\tau/RC} .$$

Since $(1-\tau/RC) < 1$,

$$V_{final} = \lim_{n \to \infty} v_n = \frac{RC}{\tau} \Delta V_1 = V \frac{\sin \pi\tau/T_o}{\pi\tau/T_o} .$$

Now we merely have to show that the average value of $v_{in}(t)$ is equal to V_{final}. This is:

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} V \cos \frac{2\pi t}{T_o} dt = \frac{VT_o}{2\pi\tau} [\sin \frac{2\pi t}{T_o}]_{-\tau/2}^{\tau/2} = \frac{VT_o}{2\pi\tau}(2 \sin \frac{\pi\tau}{T_o}) = \frac{V(\sin \pi\tau/T_o)}{\pi\tau/T_o} .$$

The final voltage value of the switched capacitor is then equal to the average value of input voltage over the switching interval. This result is dependent only on the fact that $RC \gg T_o$.