The diffusion theory of the uniform positive column of a glow discharge in electronegative gases is well known. The situation is complicated in electronegative gases due to the presence of negative ions and many more possible ionization processes, and the theory needs to be modified.

Here the diffusion theory for the uniform positive column in an electronegative gas (i.e. oxygen) is developed from the equations of diffusion and continuity of the charged particles. The ionization and deionization processes considered are ionization by primary electrons, electron attachment, collisional detachment of electrons from negative ions, and recombination at the walls of the discharge tube. Quasi-neutrality of charge is assumed and the ratio of negative ion to electron concentration is taken as constant throughout the positive column. The equations for ambipolar diffusion of ions and electrons are developed from the diffusion equations using the aforementioned assumptions and the simplifying assumption that the ion temperatures are equal to the temperature of the gas in the discharge tube. The equations of continuity are then solved for the concentrations of ions and electrons. It is found that the number densities of all charged particles vary radially as a zero order Bessel function. Their concentration at the axis of the discharge tube is found from an expression for the electric current through the discharge and the problem is completely determined and the constitution of the plasma is known at any point in the positive column.
DIFFUSION THEORY OF THE UNIFORM POSITIVE COLUMN OF A GLOW DISCHARGE IN AN ELECTRONEGATIVE GAS

by

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A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

June 1963
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Date thesis is presented September 12, 1962

Typed by Carol Baker
ACKNOWLEDGMENT

I wish to thank Dr. George L. Trigg, my original major professor, for suggesting this theoretical problem and for his willing assistance in certain technical matters by correspondence after he left Oregon State University to his new position with the Physical Review. I also thank Dr. David S. Burch for the benefit of his investigations into this problem and for assuming the duties of my major professor after Dr. Trigg left. I am grateful to Dr. Edwin A. Yunker, Chairman of the Department of Physics, for use of the facilities of the department. I also wish to express my appreciation to LCDR Robert C. Noll, USN, for helping me to obtain a leave of absence from the Navy in order that I could work on this thesis.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>FORMULATION OF THE PROBLEM</td>
<td>4</td>
</tr>
<tr>
<td>QUASI-NEUTRALITY</td>
<td>9</td>
</tr>
<tr>
<td>CONSTANT RATIO APPROXIMATION</td>
<td>10</td>
</tr>
<tr>
<td>AMBIPOLAR DIFFUSION</td>
<td>11</td>
</tr>
<tr>
<td>SOLUTION ASSUMING QUASI-NEUTRALITY AND CONSTANT RATIO</td>
<td>18</td>
</tr>
<tr>
<td>DISCUSSION</td>
<td>24</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>29</td>
</tr>
<tr>
<td>APPENDIX: NOTATION</td>
<td>31</td>
</tr>
</tbody>
</table>
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Ambipolar diffusion coefficient vs ratio of negative ion to electron concentration</td>
<td>16</td>
</tr>
</tbody>
</table>
DIFFUSION THEORY OF THE UNIFORM POSITIVE COLUMN OF A GLOW DISCHARGE IN AN ELECTRONEGATIVE GAS

INTRODUCTION

Electrical discharges in gases have been observed and carefully studied for well over a century. During the first fifty years or so many interesting and complex phenomena were noted, but knowledge of their nature was for the most part qualitative. In later years, with refined experimental techniques, quantitative results have been obtained, and theoretical explanations for many of these phenomena have been proposed. But there are still many aspects of electrical discharges, especially low pressure glow discharges, that lack satisfactory theoretical treatment. One of these is the prediction of the constitution of the plasma of the uniform positive column in electronegative gases such as oxygen.

Two forms of the positive column have been observed in oxygen (6) and certain other gases. The first is characterized by the presence of striations, which may be either stationary or moving, through which the optical and electrical properties of the plasma vary. The other form of the positive column is uniform in appearance, with no striations, and the optical and electrical properties of the plasma are regarded as constant along the length of the column and as having radial symmetry about the axis of a cylindrical
The voltage gradient along the length of the uniform positive column is higher than the average voltage gradient along the length of the striated column, at least in oxygen (6, p. 764-765). It is the uniform positive column that is treated in this paper.

The first satisfactory theoretical treatment of the positive column was proposed by Schottky in 1924 (17). Application of the Schottky theory is limited to electropositive gases, however, since it fails to include the possibility of the presence of negative ions in the discharge. Briefly, the theory assumes the presence of approximately equal numbers of positive ions and electrons forming a quasi-neutral plasma. When the discharge is first established, the more mobile electrons give the walls of the discharge tube a small negative charge and set up a small radial electric field. The positive ions and electrons migrate to the walls in equal numbers under the combined effects of this field and diffusion, and recombine there. The plasma is sustained by an ionization process due to the longitudinal electric field. The equations of continuity may be written for the positive ions and electrons, their solutions being zero-order Bessel functions of an argument which depends on the radius of the discharge tube. The concentration of positive ions and electrons at the axis is found from an expression for the current through the
discharge and the problem is then completely determined (4, p. 245-246).

In electronegative gases the situation is more complicated and the theory must be extended to take into account the effects of the negative ions. Thompson (14, 15, 16) has done quite a bit of experimental work with the striated positive column of an oxygen discharge. His results seem to indicate that in that form of discharge, the concentration of negative ions is many times that of the electrons (14). There is no reason not to suspect a similar situation in the uniform positive column. These ions may contribute significantly to the current and other discharge characteristics, indicating the need for a more comprehensive theory. Although many workers have remarked on the qualitative aspects of such a treatment, the theory itself is far from complete.

In 1949, Seeliger (18) investigated the one-dimensional case of a discharge between parallel walls in an electronegative gas. Assuming that the ratio of negative ion to electron concentration is constant and much smaller than the ratio of electron to ion mobilities, he solved the diffusion equation for the rates of production and loss of electrons and ions. His assumption of the ratio of negative ion to electron concentration being small is disputed by the work of Thompson mentioned above.
Konyukov (11, 12, 13) has recently investigated the effects of negative ions on the positive column under many different conditions. His work was concerned mainly with the manner in which the ratio of negative ion to electron concentration varied when various ionization and deionization processes were considered. He did not attempt to determine completely the constitution of the plasma in terms of the discharge parameters as is done in this paper.

**FORMULATION OF THE PROBLEM**

The flux of charged particles in the positive column of a glow discharge in an electronegative gas in a cylindrical discharge tube is determined by the electric fields present and by diffusion, according to the equations

\[
\Gamma_+ = -D_+ \nabla n_+ + n_+ \mu_+ E, \\
\Gamma_- = -D_- \nabla n_- - n_- \mu_- E, \\
\Gamma_e = -D_e \nabla n_e - n_e \mu_e E,
\]

Here \( \Gamma_i \) is the particle flux density, \( D_i \) the diffusion coefficient, \( n_i \) the particle number density, \( \mu_i \) the electrical mobility, and \( E \) the electric field strength. (A complete list of the notational symbols used throughout this paper will be found in the Appendix.)

Thompson (15, p. 515-516) has identified the main ions present
in the striated positive column of an oxygen glow discharge to be 
\( O_2^+ \) and \( 0^- \). We will assume these to be the ions present in the uni-
form positive column also. If \( Z_{n_e} \) is the rate of ionization of gas 
molecules by primary electrons, \( Y_{n_e} \) is the rate of negative ion 
production by electron attachment to neutral gas molecules, and \( X_{n_} \) 
is the rate of negative ion destruction by collisional detachment, the 
equation of continuity may be written for each type of particle:

\[
\begin{align*}
\text{div} \vec{\Gamma}_+ &= Z_{n_e}, \quad (4) \\
\text{div} \vec{\Gamma}_- &= Y_{n_e} - X_{n_-}, \quad (5) \\
\text{div} \vec{\Gamma}_e &= Z_{n_e} - Y_{n_e} + X_{n_-}. \quad (6)
\end{align*}
\]

The ionization processes considered above are probably the most 
likely to occur in the positive column of oxygen glow discharges 
(16, p. 520-522).

Poisson's equation for this system is

\[
\text{div} \vec{E} = 4\pi e (n_+ - n_- - n_e). \quad (7)
\]

It is desired to obtain solutions \( n_i \) of the above equations in 
cylindrical coordinates \((r, \phi, z)\) coaxial with the discharge tube.

Since the wall of the tube acts as a sink for all current carri-
ers, we expect one boundary condition to be

\[
n_i = 0 \quad (8)
\]
at \( r = R \). Consideration of symmetry demands that a second condition be
\[
\frac{\partial n_1}{\partial r} = 0 \tag{9}
\]
at \( r = 0 \). Finally, because we are investigating the case of a uniform positive column, we may require that
\[
\frac{\partial n_1}{\partial \phi} = 0, \tag{10}
\]
\[
\frac{\partial n_1}{\partial z} = 0, \tag{11}
\]
and that
\[
\frac{\partial \vec{E}}{\partial \phi} = 0, \tag{12}
\]
\[
\frac{\partial \vec{E}}{\partial z} = 0, \tag{13}
\]
everywhere.

These conditions for uniformity together with equations for the particle flux densities show that the electric current density
\[
\vec{J} = e(\vec{\Gamma}_+ - \vec{\Gamma}_- - \vec{\Gamma}_e) \tag{14}
\]
is everywhere independent of the \( \phi \) and \( z \) coordinates. From the previous equation and the equations of continuity it is evident that
\[
\text{div } \vec{J} = 0. \tag{15}
\]
Thus, the net current entering any volume \( V \).
is zero. If $V$ is taken as a cylinder of radius $r$ coaxial with the discharge tube, the divergence theorem

$$\int_V (\text{div } \mathbf{J}) \, dV = \int_A (\mathbf{J} \cdot \hat{n}) \, dA,$$  \tag{17}$$

(where $A$ is the surface enclosing volume $V$ and $\hat{n}$ is the unit normal to that surface) along with the axial and azimuthal independence of $\mathbf{J}$ requires that

$$J_r = 0.$$  \tag{18}$$

When combined with equation (14) this requires that

$$\Gamma_+ - \Gamma_- - \Gamma_e = 0,$$  \tag{19}$$

where $\Gamma_i$ is the radial component of $\mathbf{J}_i$. Equation (19) merely says that the flux of positive ions through our cylindrical volume $V$ must be balanced by an equal flux of negative ions and electrons.

Equations (1) - (3) may therefore be written,

$$\Gamma_+ = -D_+ n_+ + n_+ \mu_r E_r,$$  \tag{20}$$

$$\Gamma_- = -D_- n_- - n_- \mu_r E_r,$$  \tag{21}$$

$$\Gamma_e = -D_e n_e - n_e \mu_e E_e,$$  \tag{22}$$

where $n_i' = \frac{dn_i}{dr} = \frac{\partial n_i}{\partial r}$ in view of equations (10) and (11).
Substituting equations (20)-(22) into equation (19) and solving for \( E_r \) leaves

\[
E_r = \frac{(D_+n_1^1 - D_-n_1^- - D_e n_e^1)}{(n_+\mu_+ + n_-\mu_- + n_e\mu_e)}.
\]  

By use of equations (7), (20)-(22), and (23), the equations of continuity may be written

\[
-D_+n_+'' - D_+n_+'/r + \mu_+ n_+ (D_+n_1^1 - D_-n_1^- - D_e n_e^1)/(n_+\mu_+ + n_-\mu_- + n_e\mu_e) + 4\pi e (n_+ - n_ - n_e) \mu_+ n_+ = Zn_e,
\]  

\[
-D_-n_-'' - D_-n_-'/r - \mu_ - n_ - (D_+n_1^1 - D_-n_1^- - D_e n_e^1)/(n_+\mu_+ + n_-\mu_- + n_e\mu_e) - 4\pi e (n_+ - n_ - n_e) \mu_- n_- = Yn_e - Xn_.,
\]  

\[
-D_e n_e'' - D_e n_e'/r - \mu_e n_e (D_+n_1^1 - D_-n_1^- - D_e n_e^1)/(n_+\mu_+ + n_-\mu_- + n_e\mu_e) + 4\pi e (n_+ - n_ - n_e) \mu_e n_e = Zn_e - Yn_e + Xn_.,
\]

where \( n_1^n = d^2 n_1 / dr^2 = \delta^2 n_1 / \delta r^2 \).

Equations (24)-(26) are strongly coupled and non-linear. A solution would be very difficult to obtain and we are not even sure of the existence of a unique solution to the equations as they stand. Our procedure will be to remove the nonlinearities and interdependencies by making certain simplifying assumptions concerning the nature of
the discharge and solve the equations in the simplified form (2).

QUASI-NEUTRALITY

Most theories of the positive column (17, 18, 11, 12, 13, 14) assume quasi-neutrality of charge. That is, that the number densities of charged particles are such that the net charge density in the column is approximately zero, enough being reserved to provide the radial field. The charge density of one sign can deviate appreciably from that of the other only over distances of the order of or smaller than the Debye shielding length. Under the conditions of pressure and electric field strength for most glow discharges, the Debye length is small compared to the linear dimensions of the discharge tube and quasi-neutrality can be assumed (1, p. 399).

It was shown in the previous section that the flux of charged particles out of a cylindrical region coaxial with the discharge tube must be such that the flux of positive ions is balanced by an equal flux of negative ions and electrons. If the radius of this region is large compared to the Debye length, quasi-neutrality may be assumed in this region as well as in the whole positive column. The condition for quasi-neutrality may be written

\[ n_+ = n_- + n_e. \] (27)
The assumption of quasi-neutrality simplifies the equations of the previous section considerably, but in order to determine completely the problem and remove all interdependencies one more assumption concerning the relationship between the electron and ion concentrations must be made.

**CONSTANT RATIO APPROXIMATION**

One such assumption requires that the number densities of both types of negatively charged particles vary in the same way with \( r \). That is, that the ratio of negative ion to electron concentration is constant throughout the positive column. This approximation is made by Seeliger (18, p. 95) and by Konyukov (13, p. 972) in their theories of the positive column where the ionization processes are as we have assumed. It is also made by Thompson (14, p. 818) for striated columns. The condition for constant ratio of negative ion to electron concentration may be written

\[
n_\text{e} = \frac{n_\text{e}}{n_\text{e}}
\]

for an arbitrary region within the positive column.

Once both quasi-neutrality and constant ratio are assumed, the problem is completely determined. Before solving the equations of continuity for the number densities of charged particles it will be
interesting to investigate the equations for the radial flux of ions and electrons under these assumptions.

**AMBIPOLAR DIFFUSION**

Assuming quasi-neutrality and constant ratio, it is possible to eliminate the electric field strength from the expressions for the radial flux of ions and electrons as given by equations (20)-(22). The equations are further simplified if we assume that the positive and negative ion temperatures are equal,

\[ T_+ = T_- = T, \]  

approximately the temperature of the gas in the positive column. The ratio of electron to ion temperature is thus written

\[ K = \frac{T_e}{T_i}. \]  

These conditions along with the Einstein relation (1, p. 414)

\[ \frac{D_i}{\mu_i} = \frac{kT_i}{e} \]  

determine the diffusion coefficients for the negative ions and electrons in terms of the diffusion coefficient for the positive ions, the mobilities of the ions and electrons, and the ratio of electron to ion temperature:
Because we have assumed quasi-neutrality and constant ratio, we may write the positive and negative ion concentrations in terms of that of the electrons and the ratio of negative ion to electron concentration,

\[ n_+ = (a + 1)n_e, \]
\[ n_- = an_e. \]

Substituting equations (19) and (32)-(35) into equations (20)-(22) results in

\[ r_+ + r_e = - (a + 1)D_+ n_e' + (a + 1)\mu_+ n_e E, \]
\[ r_- = - (a\mu_-/\mu_+)D_+ n_e' - a\mu_- n_e E, \]
\[ r_e = - (K\mu_-/\mu_+)D_+ n_e' - \mu_e n_e E. \]

Solving equation (38) for \( n_e E \) gives

\[ n_e E = - \frac{r_e}{\mu_e} - KD_+ n_e'/\mu_+. \]

and substituting the result into equations (36) and (37) leaves

\[ r_+ + [1 + (a + 1)\mu_+\mu_e]r_e = - (a + 1)(K + 1)D_+ n_e', \]
\[ r_- = (a\mu_-/\mu_+)r_e = (a\mu_-/\mu_+)(K - 1)D_+ n_e'. \]
The desired equation is obtained by subtracting equation (41) from (40) and solving the difference for \( \Gamma_e \):

\[
\Gamma_e = - \left\{ \frac{(K + 1) + \left[ K \left( \frac{\mu_-}{\mu_+} + 1 \right) - \left( \frac{\mu_-}{\mu_+} - 1 \right) \right]}{(1 + \frac{\mu_+}{\mu_+ e}) + \left( \frac{\mu_+}{\mu_+ e} + \frac{\mu_-}{\mu_+ e} \right) a} \right\} D_e n'_e, \tag{42}
\]

which might be written

\[
\Gamma_e = - D_e \frac{n'_e}{n'_e}, \tag{43}
\]

where the ambipolar diffusion coefficient for electrons is

\[
D_e = \left\{ \frac{(K + 1) + \left[ K \left( \frac{\mu_-}{\mu_+} + 1 \right) - \left( \frac{\mu_-}{\mu_+} - 1 \right) \right]}{(1 + \frac{\mu_+}{\mu_+ e}) + \left( \frac{\mu_+}{\mu_+ e} + \frac{\mu_-}{\mu_+ e} \right) a} \right\} \tag{44}
\]

Similarly, we may substitute \( n'_i/a \) for \( n'_e \) everywhere in equations (40) and (41), multiply equation (40) by \( (a \mu_-/\mu_+) \) and equation (41) by \( [1 + (a + 1) \mu_+/\mu_e] \), add the two equations and solve the sum for \( \Gamma_- \):

\[
\Gamma_- = - \left\{ \frac{2\mu_-/\mu_e - (K - 1)\mu_-/\mu_+}{(1 + \frac{\mu_+}{\mu_+ e}) + \left( \frac{\mu_+}{\mu_+ e} + \frac{\mu_-}{\mu_+ e} \right) a} \right\} D_+, \tag{45}
\]

This equation may also be written in terms of an ambipolar diffusion coefficient,

\[
\Gamma_- = - D_+ - \frac{a}{n'_i}, \tag{46}
\]

where the ambipolar diffusion coefficient for negative ions is
\[ D_+^a = \frac{\left[ \frac{2\mu_-}{\mu_e} - (K - 1)\mu_- + \frac{1}{\mu_e} \right] + (2\mu_-/\mu_e) a}{(1 + \mu_+/(\mu_e) + (\mu_+/\mu_e + \mu_-/\mu_e) a} \]  

(47)

For the sake of being complete, we may write the equation for the radial flux of positive ions by adding equations (44) and (45), and substituting \( \Gamma_+ + \Gamma_e = \Gamma^+ \), \( n_+ = n_e/(a + 1) \), and \( n_+ = an_+/(a + 1) \) into their sum,

\[ \Gamma_+ = -\frac{(K + 1) + (2\mu_-/\mu_e) a}{(1 + \mu_+)/(\mu_e) + (\mu_+/\mu_e + \mu_-/\mu_e) a} D_+ n_+^a \]  

(48)

This may be written

\[ \Gamma_+ = -D_+^a n_+^a \]  

(49)

where the ambipolar diffusion coefficient for positive ions is

\[ D_+^a = \frac{(K + 1) + (2\mu_-/\mu_e) a}{(1 + \mu_+)/(\mu_e) + (\mu_+/\mu_e + \mu_-/\mu_e) a} D_+^a \]  

(50)

Equations (43), (46), and (49) are the desired equations for the radial diffusion of electrons and ions. Only two of the three equations are really independent, the third is given only for the sake of physical completeness.

It is interesting to see how the values of the ambipolar diffusion coefficients compare with each other at different values of \( a \). We will assume that the positive column is made up of electrons.
and $O_2^+$ and $O^-$ ions (15, p. 515-516) and will plot $D_1^a$ against $a$ for the uniform positive column of an oxygen discharge under typical operating conditions. These are taken from Güntherschulze (6, p. 764) who noted that for a 25 ma discharge in a 1.5 cm radius tube: $E = 12.3 \text{ v/cm}$, $p_o = 0.437 \text{ torr}$, and $T = 347^\circ \text{K}$. The electron temperature and the drift velocity of the electrons at $E/p_o = 28.2$ v/cm-torr are taken from the data of Healey and Kirkpatrick (9, p. 94) and are assumed to be the same as in a Townsend discharge in oxygen: $T_e = 19950^\circ \text{K}$, and $v_e = 9.55 \times 10^6 \text{ cm/sec}$. The data of Healey and Kirkpatrick are small by a factor of 2 at lower values of $E/p_o$ when compared to those measured by various others, but are the only data available for $E/p_o$ large enough to correspond to Güntherschulze's data. The drift velocities of the $O_2^+$ and $O^-$ ions are assumed to be the same as those in a Townsend discharge at this value of $E/p_o$ and are taken from Burch and Geballe (3, p. 42): $v_+ = 4.1 \times 10^4 \text{ cm/sec}$ and $v_- = 8.65 \times 10^4 \text{ cm/sec}$. Using these data it is found that: $K = T_e/T = 57.5$, $\mu_+/\mu_e = v_+/v_e = 4.3 \times 10^{-3}$, $\mu_-/\mu_e = v_-/v_e = 9.06 \times 10^{-3}$, $\mu_-/\mu_+ = v_-/v_+ = 2.11$, and $D_+ = v_+ kT/eE = 10^2 \text{ cm}^2/\text{sec}$. These numbers are substituted into equations (44), (47) and (50) and $D_1^a$ is plotted against $a$ in Figure 1.

It can be seen from Figure 1 how the degree of electronegativity of the gas affects the radial motions of the ions and electrons.
FIG. I
Ambipolar diffusion coefficients vs. ratio of negative ion to electron concentration
For very low values of $a$ the gas is essentially electropositive and the negative ions can be neglected. Here, we see that the ambipolar diffusion coefficients of the positive ions and the electrons are equal as in the Schottky theory (17, p. 637). This, along with quasi-neutrality, requires that the positive ions and the electrons migrate to the walls of the tube in the same manner and in equal numbers. Similarly, for very highly electronegative gases, corresponding to very high values of $a$, the electrons are so few as to be negligible when considering radial diffusion, and the ambipolar diffusion coefficients of the positive and negative ions are seen to be equal, which along with quasi-neutrality requires that they also migrate to the walls in the same manner and in equal numbers. We expect the situation for a moderately electronegative gas like oxygen to be somewhere between these limits.

It is also noticed from Figure 1 that for $a$ less than $6.58 \times 10^3$ the ambipolar diffusion coefficient for negative ions is negative. This would require the radial motion of the negative ions to be toward the axis. The concentration of negative ions would build up infinitely at the axis in this case had we not included a negative ion destruction mechanism, collisional detachment, in our theory.

The numerical value for the situation depicted by the example of this section will next be determined, and the problem completely
SOLUTION ASSUMING QUASI-NEUTRALITY AND CONSTANT RATIO

If both quasi-neutrality and constant ratio are assumed, the problem is completely determined and equations (49), (46), and (43) may be used for the radial flux of ions and electrons. The number densities of the negative ions and electrons may also be written

\[ n_- = a n_e, \quad (51) \]
\[ n_e = n_- / a, \quad (52) \]
\[ n_e = n_+ / (a + 1). \quad (53) \]

Thus, the equations of continuity may be written entirely in terms of the number density of each particle:

\[ -D_+^a n_+'' - D_+^a n_+'/r = Zn_+ / (a + 1), \quad (54) \]
\[ -D_-^a n_-'' - D_-^a n_-'/r = Yn_- / a - Xn_-, \quad (55) \]
\[ -D_e^a n_e'' - D_e^a n_e'/r = Zn_e - Yn_e + aXn_e. \quad (56) \]

These equations may be rearranged and are recognized as Bessel's equations of order zero,

\[ n_+'' + n_+'/r + \left[ Z / (a + 1) D_+^a \right] n_+ = 0, \quad (57) \]
\[ n_-'' + n_-'/r + \left[ Y - aX / a D_-^a \right] n_- = 0, \quad (58) \]
\[ n''_e + n'_e /r + \left[ (Z - Y + aX)/D_e^a \right] n_e = 0, \]  
with solutions (10, p. 131)

\[ n_+ = n^+_0 J_0 \left( \frac{Z}{(a + 1)D_+^a} \right)^{\frac{1}{2}r}, \]  
\[ n_- = n^-_0 J_0 \left( \frac{(Y - aX)/aD_-^a}{2r} \right)^{\frac{1}{2}r}, \]  
\[ n_e = n^e_0 J_0 \left( \frac{(Z - Y + aX)/D_e^a}{2r} \right)^{\frac{1}{2}r}. \]

The boundary condition at the wall demands that \( n_i = 0 \) at \( r = R \).

Since the first root of \( J_0(M) \) is at \( M = 2.40 \), the arguments of each of the Bessel functions in equations (60)-(62) must be equal to 2.40 \( r/R \). Thus,

\[ \left[ \frac{Z}{(a + 1)D_+^a} \right]^{\frac{1}{2}} = 2.40/R, \]  
\[ \left[ \frac{(Y - aX)/aD_-^a}{2} \right]^{\frac{1}{2}} = 2.40/R, \]  
\[ \left[ \frac{(Z - Y + aX)/D_e^a}{2} \right]^{\frac{1}{2}} = 2.40/R. \]

As in the previous section, only two of the above equations are completely independent as \( D_e^a = (a + 1)D_+^a - a D_-^a \) as can be seen from equations (50), (47), and (44). For this reason equation (65) will not be considered in what follows.
The ionization, attachment, and detachment rates may be written

\[ Z = \alpha n_e = \alpha n_e E, \quad (66) \]
\[ Y = \eta n_e = \eta n_e E, \quad (67) \]
\[ X = \delta n_e = \delta n_e E. \quad (68) \]

Substituting equations (47), (50), and (66)-(68) into equations (64) and (65), writing \( D_+ = \mu_+ kT/e \) by the Einstein relation, squaring the results and rearranging leaves

\[
\frac{(\alpha/p_0)(E/p_0)(p_0 R)^2}{(2.40)^2(\mu_+ / \mu_e)(kT/e)} = \frac{(a + 1)[(K + 1) + (2\mu_- / \mu_e)a]}{(1 + \mu_+ / \mu_e) + (\mu_+ / \mu_e + \mu_- / \mu_e)a}
\]

\[
\frac{[\eta/p_0] - a(\mu_- / \mu_e)(\delta/p_0)](E/p_0)(p_0 R)^2}{(2.40)^2(\mu_+ / \mu_e)(kT/e)} = \frac{a[(2\mu_- / \mu_e) - (K - 1)\mu_- / \mu_+ + (2\mu_- / \mu_e)a]}{(1 + \mu_+ / \mu_e) + (\mu_+ / \mu_e + \mu_- / \mu_e)a}
\]

If the coefficients for the ionization, attachment, and detachment processes were all known as functions of \( E/p_0 \), then equations (69) and (70) could be solved for \( a \) and used to determine the relation between \( E/p_0 \) and \( p_0 R \). Information on \( a/p_0 \) and \( \eta/p_0 \) at various
values for of $E/p_0$ is available for a Townsend discharge in oxygen from the work of Harrison and Geballe (7, p. 372) and will be assumed to be valid in the positive column of our glow discharge. However, there are not sufficient data available on the detachment process to determine $\delta/p_0$ even in the Townsend discharge. Because of this unavailability of data we will use Güntherschulze's experimental determination of $E/p_0$ vs $p_0 R$ for the specific example cited in the previous section and will solve equation (69) for $a$ and equation (70) for $\delta/p_0$. Recalling that for this example $E/p_0 = 28.2 \text{v/cm-torr}$, we find from Harrison and Geballe (7, p. 372) that $a/p_0 = 3.85 \times 10^{-2} /\text{cm-torr}$, and $\eta/p_0 = 8.8 \times 10^{-2} /\text{cm-torr}$.

Equation (69) may be written as a quadratic equation in $a$,

$$ (2\mu_-/\mu_e) a^2 + [(K + 1) + 2\mu_-/\mu_e - (\mu_+ /\mu_e + \mu_- /\mu_e) f] a + [(K + 1) - (1 + \mu_+ /\mu_e) f] = 0, $$

where $f$ is the term on the left side of equation (69). Denoting the coefficient of $a^2$ as $A$, that of $a$ as $B$, and the constant term as $C$, equation (74) may be written

$$ A a^2 + B a + C = 0. $$

The solution of this quadratic equation takes the standard form
\[ a = -\frac{B}{2A} \pm \sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}} \frac{1}{2}. \]  

(73)

The plus sign is used in the above equation since \( a \) must be positive.

For the example of the last section mentioned above we find that \( f = 630, \ A = 1.81 \times 10^{-2} \), \( B = 50.1 \), and \( C = -574 \). Substituting these numbers into equation (75) gives \( a = 8.4 \); there are 8.4 times as many negative ions as there are electrons in the positive column for this example. Substituting this value of \( a \) and the data mentioned previously into equation (70) and solving for \( \delta/p_0 \) results in a value of \( 1.88/\text{cm-torr} \). For the particular example we are describing, the rate of negative ion destruction by collisional detachment is \( X_n^- = (\delta/p_0)(p_0)(v^-)(n_e) = 59.7 \times 10^4 \ \text{n}_e/\text{sec} \), the rate of electron attachment is \( Y_n^+ = (\eta/p_0)(p_0)(v_e^+)(n_e) = 36.5 \times 10^4 \ \text{n}_e^+/\text{sec} \), and the rate of ionization by primary electrons is \( Z_n^+ = (\alpha/p_0)(p_0)(v_e^+)(n_e^+) = 16.1 \times 10^4 \ \text{n}_e^+/\text{sec} \). Using these numbers and the equations of continuity, we see that the total rate of production of positive ions by ionization and deionization processes is \( Z_n^+ = 16.1 \times 10^4 \ \text{n}_e^+/\text{sec} \), that for negative ions is \( Y_n^- - X_n^- = -23.2 \times 10^4 \ \text{n}_e^-/\text{sec} \), and that for electrons is \( Z_n^- - Y_n^- + X_n^- = 39.3 \times 10^4 \ \text{n}_e^-/\text{sec} \). The balance of the plasma is maintained by ambipolar diffusion of positive ions and electrons to the walls of the discharge.
tube and of negative ions to the axis. Notice that the net rate of production of negative ions by the attachment and detachment processes is negative. This means that the ambipolar diffusion of negative ions has to be radially inward to balance this effect. This was mentioned in a different light in the section on ambipolar diffusion.

The concentration of electrons at the axis of the tube may be determined from consideration of the current passing axially through the column. The current density is principally axial and will be given by

$$J = (n_+ \mu_+ + n_- \mu_- + n_e \mu_e) e E.$$  \(\text{(74)}\)

Using quasi-neutrality and constant ratio to write the positive and negative ion concentrations in terms of that of the electrons and using the Bessel function solution for the electron concentration, the expression for the current density becomes,

$$J = [\mu_e + a \mu_- + (a + 1) \mu_+] e E n_e^o \int_0^R 2 \pi r J_o(2.40 \frac{r}{R}) dr.$$  \(\text{(75)}\)

The above equation is integrated to get an expression for the electric current passing through the discharge,

$$I = [\mu_e + a \mu_- + (a + 1) \mu_+] e E n_e^o \int_0^R 2 \pi r J_o(2.40 \frac{r}{R}) dr.$$  \(\text{(76)}\)
On performing the integration indicated in equation (76), the expression for the current becomes:

\[ I = \left[ \mu_e + a \mu_i + (a + 1) \mu_{+i} \right] eE \ n_e^0 \ \left( \frac{2 \pi R^2}{2} \right) J_1 (2, 40) \]  

(10, p. 155). The constant terms in the above equation may be combined, the value \( J_1 (2, 40) = 0.520 \) (10, p. 158) substituted, and the equation written

\[ I = 1.36 \ R^2 \ eE [\mu_e + a \mu_i + (a + 1) \mu_{+i}] n_e^0. \]  

Equation (78) is then solved for the concentration of electrons at the axis,

\[ n_e^0 = \frac{1}{1.36 \ R^2 e [\mu_e + a \mu_i + (a + 1) \mu_{+i}]} \]  

where \( \mu_i E \) is written \( v_i \). For our example \( I = 25 \) ma, so \( n_e^0 = 9.59 \times 10^9 \) \( /cm^3 \), \( n_i^0 = 8.05 \times 10^{10} \) \( /cm^3 \), and \( n_{+i}^0 = 9.00 \times 10^{10} \) \( /cm^3 \). These numbers along with equations (63)-(65) substituted into equations (60)-(62) completely determine the problem, and the constitution of the plasma at any point in the positive column is known.

**DISCUSSION**

Because of the difficulty in obtaining good data from the literature, only one example of the theory presented here was
worked out completely. There is a real need for better measurements of $E/p_o$ vs $p_o R$ in the uniform positive column of oxygen discharges. Many of the earlier workers did not measure the gas temperature along with their other data as Güntherschulze did (6, p. 764), making it impossible to calculate a diffusion coefficient for the ions and to correct the pressure to standard temperature. The gas density, not the pressure, is the fundamental quantity involved, and unless the pressure can be corrected to standard temperature the data are not useful for our purposes. The data of Güntherschulze mentioned above are available only for a range of $E/p_o$ of about 20 - 30 volts/cm-torr. We could use data for a more complete range of $E/p_o$, say 12-50 volts/cm-torr, complete with gas temperatures and at a given constant current to calculate $a$, $b/p_o$, and $n_e^o$ for different values of $E/p_o$.

The data on the ion and electron drift velocities, the electron temperature, and the ionization and attachment coefficients were all taken from experiments in Townsend discharges. The energy distribution of the charged particles in a Townsend discharge may differ very much from that in the uniform positive column of a glow discharge making the data used unacceptable for this application. In the absence of information on this question, the Townsend discharge results were used to illustrate the theory.
Thompson has been able to determine the ratio of electron to negative ion temperature in the striated positive column of an oxygen glow discharge from an experimental determination of the energy distribution of the negatively charged particles (14, p. 819). It would be desirable, if possible, to repeat these experiments in the uniform positive column in oxygen under various conditions.

It would also be desirable to determine experimentally which kinds of positive and negative ions are present in the uniform positive column of an oxygen discharge. This might possibly be done by the use of a radio frequency mass spectrometer probe such as Thompson utilized for striated oxygen discharges (15, p. 515-516). If it were found that the ion species were different from those Thompson found in the striated column, the mobilities for the proper ions should be substituted into our theory in place of the values used. The theory itself would not have to be altered unless it were found that there were two or more kinds of positive or negative ions present. In this case another diffusion equation and another equation of continuity could be written for each new ion species and the theory developed in the same manner.

There are many possible ionization and deionization processes that could take place in the positive column. The ones we assumed for the purposes of our theory are perhaps the most likely, but it
would be desirable to know exactly which processes take place and their rates. These could be determined from the variations of their cross-sections with the energies of the particles, together with the energy distributions of these same particles in the uniform positive column. There are insufficient data available at present to allow such determinations.

Although it is made by many other workers (11, 13, 14, 18), the assumption that the ratio of negative ion to electron concentration is constant throughout the positive column remains hypothetical. Its only vindication is the fact that its incorporation into our theory allows us to obtain physically reasonable solutions to the diffusion and continuity equations. The situation might actually be quite different. The work of Thompson (14, p. 820) in striated oxygen discharges, although far from conclusive, seems to indicate that an assumption of constant electron density might be better. If possible, experiments should be undertaken to determine the validity of this assumption and the theory modified if necessary.

The experiments mentioned above are desirable to test out hypotheses and to provide numbers for insertion into our theory. Although these are very difficult projects in themselves, the real test of our theory lies in the accuracy of its predictions. Experiments should be undertaken to determine whether the radial diffusion of
negative ions in the uniform positive column of an oxygen glow discharge operating under the conditions of our example is toward the axis as we have predicted. We also need a more direct experimental determination of the constitution of the plasma at different points in the positive column with which to compare our results. Thompson (14, p. 820) has determined the radial variation of the charged particle concentration in the striated positive column of an oxygen discharge, but many more experiments need to be done before we will have sufficiently accurate results with which to compare those of our theory. In conclusion we can only say that we have presented a very likely theory based on the best experimental evidence available at present. We can only wait until new and more sophisticated experiments are performed for final verification or modification.
BIBLIOGRAPHY


APPENDIX
Vector quantities are represented by a horizontal bar over the literal symbol used to represent that quantity, i.e. \( \overline{E}, \overline{I}, \) and \( \overline{J} \).

The gradient of a scalar quantity \( n_1 \) is written \( \text{grad} \ n_1 \) and the divergence of a vector quantity \( \overline{E} \) is denoted \( \text{div} \ \overline{E} \). The components of vector quantities are referred to in cylindrical coordinates \((r, \phi, z)\) coaxial with the cylindrical discharge tube, the \( r \)-component of \( \overline{E} \) being written \( E_r \) unless otherwise noted.

The subscript \( i \) on a literal symbol stands for the type of particle involved: + for positive ions, - for negative ions, and \( e \) for electrons. The value of a quantity at the axis of the discharge tube is denoted by the superscript \( o \).

The notational symbols used in this paper are the following:

- \( n_i \) = particle number density \((\text{number of particles/cm}^3)\)
- \( \mu_i \) = electrical mobility \((\text{cm}^2/\text{v-sec})\)
- \( D_i \) = diffusion coefficient \((\text{cm}^2/\text{sec})\)
- \( D_i^a \) = ambipolar diffusion coefficient \((\text{cm}^2/\text{sec})\)
- \( T_i \) = particle kinetic temperature \((^\circ\text{K})\)
- \( T \) = gas temperature \((^\circ\text{K})\)
- \( e \) = magnitude of electron charge \((1.60 \times 10^{-19} \text{ coulomb})\)
- \( k \) = Boltzmann constant \((8.62 \times 10^{-5} \text{ ev/}^\circ\text{K})\)
\( p_0 \) = pressure corrected to standard temperature (torr)

\( R \) = radius of discharge tube (cm)

\( v_i \) = drift velocity of a particle in the direction of the electric field (cm/sec)

\( I \) = electric current passing longitudinally through the discharge tube (amp)

\( J_0 \) = zero-order Bessel function

\( J_1 \) = first-order Bessel function

\( \Gamma_i \) = particle flux density (number of particles passing through 1 cm\(^2\) per sec)

\( E \) = electric field strength (v/cm)

\( J \) = electric current density (amp/cm\(^2\))

\( Z \) = rate of electron and positive ion production by primary electron collisions with gas molecules (number ion pairs produced per electron per sec)

\( Y \) = rate of negative ion formation by attachment of electrons to gas molecules (number of attachments per electron per sec)

\( X \) = rate of negative ion destruction by collisional detachment of electrons from collisions with gas molecules (number of detachments per negative ion per sec)

\( \alpha \) = ionization coefficient (mean number of ionizations per cm of electron drift in the direction of the electric field)

\( \eta \) = attachment coefficient (mean number of attachments per cm of electron drift in the direction of the electric field)

\( \delta \) = detachment coefficient (mean number of detachments per cm of negative ion drift in the direction of the electric field)