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Title A SYNTHESIS OF SIGNIFICANT DEVELOPMENTS IN THE HISTORY, CALCULATION, AND PROPERTIES OF THE NUMBER e

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This thesis brings together under one cover a survey of the history of the real number e along with a study of the present state of its theory and calculation.

# A Synthesis of Significant Developments in the History, Calculation, and Properties of the Number e

by

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# A SYNTHESIS OF SIGNIFICANT DEVELOPMENTS IN THE HISTORY, CALCULATION AND PROPERTIES OF THE NUMBER e

## INTRODUCTION

Interest in the number represented by the letter e is relatively recent. It first came into notice in the 16th century. This interest grew and developed under such capable minds as Napier, Louiville, Leibniz, Newton, Euler, and Hermite to mention only a few. The history of e is roughly broken into three periods. The first is the 17th century when the mathematicians were using approximation methods to find the value of e. The first half of the 18th century is almost completely dominated by the work of Euler. The third part is in the latter half of the 18th century when the actual nature of e was investigated and the irrationality and transcendence of e was discovered.

This thesis is divided into three chapters. The first deals with the actual historical development of e. The second deals with the development of e from the definition of the logarithm. It includes the proof of the irrationality and transcendence of e. Also included is a report on the normality of e which, although not truely a mathematical concept, sheds a more knowledgeable outlook on e. The third has to do with the approximations that have been made of $e$, and takes us to the electronic computer and its amazing ability to give a huge number of places in the decimal expansion.

# CHAPTER 1

# HISTORY

## INTRODUCTION

This chapter is devoted to the historical development of our knowledge of e , (see Appendix I and II). The number e is defined and used in several ways by mathematicians. We here define it terms of the calculus as:

$$e = \lim_{n \to \infty} (1 + 1/n)^n$$

A fifteen place approximation is 2.718281828459045.

## John Napier's Logarithms

The ancient Greek mathematicians considered many problems, the solutions of which completely escaped their grasp. One such problem was the quadrature of the hyperbola (see Appendix III). Pi is involved in the formulas of the circumference and area of a circle. In similar manner e is involved in the area between the arm of the hyperbola and a chord perpendicular to the transverse axis. Unknown to the Greeks they were dealing with a number similar to pi, but which would have to wait nineteen centuries to have its mystery revealed.

The discovery of e , as with many discoveries, came about accidently. John Napier, a Scotish nobleman, had two burning interests in life. One was religion and the other was mathematics. He was very active in both disciplines and is reported to have said, "If I am ever famous, it will be due to my religious publications." History has shown him to be remembered and famous for his accomplishments in mathematics.

To Napier we owe the invention of the decimal point, the logarithms, two trigonometry formuli known as "Napier's Analogies", and a mechanical device for multiplying, dividing and taking square roots known as "Napier's Rods". As C. G. Knott observed, "Perhaps no other mathematician has as clear a title to his invention as Napier has to logarithms."

In 1614, Napier publish <u>Mirifici Logarithmorum Canonis Descriptio</u>, being interpreted "A Description of the Admirable Table of Logarithms". In the <u>Descriptio</u> appears the first table of Logarithms (Appendix VI). It is explained that these tables are to be used to shorten multiplication and division, operations so fundamental that to shorten them seemed impossible. (Kasner, 1963)

M. J. Cajori in his <u>History of Mathematics</u>, 1919, recounts that Henry Briggs (1556 - 1631) a professor of geometry at Oxford University

> "was so struck with admiration of Napier's book that he left his studies in London to

> do homage to the Scottish philosopher. Briggs was delayed in his journey, and Napier complained to a common friend, 'Ah, John, Mr. Briggs will not come.' At that moment knocks were heard at the gate, and Briggs was brought into the lord's chamber. Almost one quarter of an hour was spent, each beholding the other without speaking a word. At last Briggs began, 'My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy, viz, the logarithms; but my lord, being by you found out, I wonder nobody found it out before, when now it is so easy." (p. 151)

At this conference between these two great men, Briggs suggested the possibility of making the logarithm of one, or some power of ten, equal to zero. Napier agreed the idea had great merit and encouraged Briggs to continue his inquiry on that point. Thus, in 1624 when Briggs published his work <u>Arithmetica logarithmic</u>, we find him using the base ten in his logarithms, and making the logarithms of one equal to zero.

Napier defined his logarithms as follows. Consider a line segment AB and an infinite ray DE, as shown in figure 1. Let points C and F start moving simultaneously from A and D, respectively, along these lines with the same initial rate. Suppose C moves with a velocity always numerically equal to the distance CB, that is, a decreasing velocity. At the same time, F moves with a uniform velocity. Then, Napier defined DF to be the logarithm of CB. That is, setting DF = x and

CB = y , x = Nap log y.

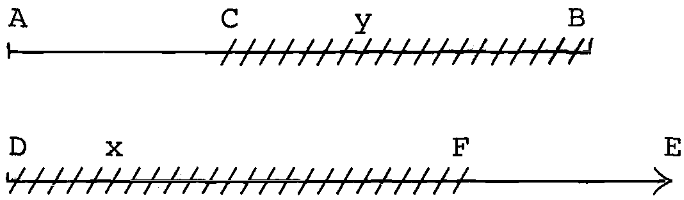


Figure 1.

An association of this type between an arithmetic progression and a geometric progession was suggested as early as 1544 by Michael Stifel. But not until Napier did anyone come to a reasonable process of actually putting the association to good use.

Our present day natural logarithms, also called Napierian logarithms, are claimed by many people to be the logarithms that John Napier invented. Such a claim is not substantiated. The table in Appendix VI is a copy of part of the logarithm Napier presented to the public in 1614 in the <u>Descriptio</u>. One observes that the logarithms decrease as the number increases, contrary to what happens with natural logarithms.

The question arises as just what is the connection between Napier's logarithms and the number e that we are discussing. The idea of base, or base number, never suggested itself to Napier. Yet we find the relationship between the natural logarithms and Napierian logarithms only a difference of the bases used. Napier, unknowingly, used

1/e as the base of his logarithms. We wish to show this using the calculus.

Referring to figure 1, let $AB = 10^7$. Using $CB = y$, then $AC = 10^7 - y$. Then the velocity of C is $-\frac{dy}{dt} = +y$, or $\frac{dy}{y} = -dt$. Integrating, we have $\log_e y = -t + k$. To evaluate the constant of integration we set $t = 0$. Thus, at $t = 0$, $y = 10^7$, and $\log_e 10^7 = -0 + k$ or $k = \log_e 10^7$. Therefore, $\log_e y = -t + \log_e 10^7$. The velocity of $F = dx/dt$ so that $x = 10^7 t$. From this we see that:

$$\begin{aligned} \text{Nap log } y = x = 10^7 t &= 10^7 \ (\log_e 10^7 - \log_e y) \\ &= 10^7 \log_e (10^7/y) \\ &= 10^7 \log_{1/e}(y/10^7) \end{aligned}$$

We see that Napier used 1/e as the base of his logarithms.

Two years after Napier first published his logarithms, there appeared on the market a translation of his book into the English language by Edward Wright. In the second edition published in 1618, there appeared an appendix compiled by William Oughtred (Appendix IV) in which appears the first table of what we recognize as natural logarithms. Thus, four years after logarithms were introduced, logarithms to the base number e were produced.

Discovery of e

After Napier's publication of the Descriptio logarithms became popular as a help in solving problems involving lengthy multiplication and division as well as helping with calculations involving the extremely large numbers that come in astronomy. But, knowledge of the number $e$ had not actually emerged to its full stature. In fact, its being a real number had been only hinted at.

The quadrature of a hyperbola was still an unsolved puzzle, and many great men were involved in its solution. In 1647 a Belgian Jesuit, Gregory St. Vincent, published a work in geometry in which, among other things, he considered the quadrature of a hyperbola. A theorem that he proved is:

> "If the equation of the hyperbola be referred to its asymptotes as axes and the abscissae be taken in geometrical progression, the hyperbolic trapezia standing on the abscissae are equal." (Mitchell and Strain, 1938. p. 481)

Vincent's pupil, Alfons de Sarasa, made additions to this theorem and was the first to state it in terms of logarithms. In modern terminology, this theorem states that the area between the hyperbola and the corresponding asymptote is divided into equal areas if we use a geometric progression as the points on the asymptote. As an example consider the following: let $xy = 4$ be the hyperbola, and use as the geometric progression the sequence $1 , 2 , 4 , 8 , 16 , \ldots$ The area 'a' in figure 2 is

$$\int_1^2 \frac{4}{x}\, dx = 4 \ln 2$$

While the area in 'b' of figure 1 is

$$\int_2^4 \frac{4}{x} dx = 4 \ (\ln x)\Big|_2^4$$

$$= 4 \ (\ln 4 - \ln 2)$$

$$= 4 \ (\ln 2^2 - \ln 2)$$

$$= 4 \ (2 \ln 2 - \ln 2)$$

$$= 4 \ln 2$$

All areas thus constructed using the geometric series are equal.

St. Vincent had thus reduced the problem of the quadrature of the hyperbola to one of infinite series.

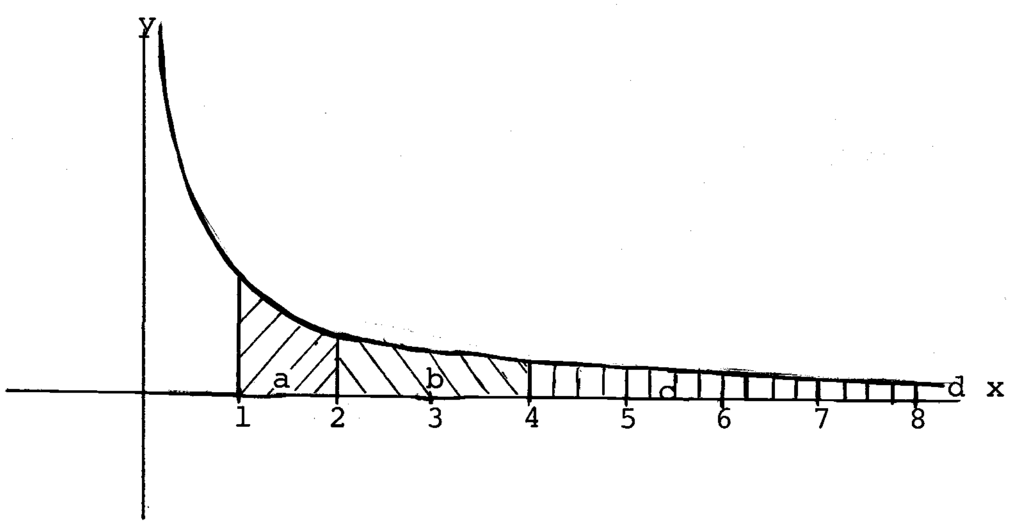


Figure 2.

Eight years later, in 1655, John Wallis published his <u>Arithmetica Infinitorum</u>, in which he explains the method of effecting the quadratures. His method was that of interpolation which gave him the value of $\pi$ as

$$\pi = 2\,\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdots\cdots}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdots\cdots}$$

For a detailed account of the method of interpolation, the reader is referred to W. W. R. Ball (1889). This difficult and elaborate method used by Wallis was used by many mathematicians of the seventeenth century. Yet, this very difficulty led Isaac Newton to the discovery of the binomial theorem, and was the means of interesting Lord Brouncher in obtaining the area bounded by the equalateral hyperbola, $xy = 1$, one of its asymptotes, and the lines $x = 1$ and $x = 2$.

In 1667 James Greogory showed how to compute logarithms by approximating to the hyperbolic asymptotic spaces, that is, the area between the hyperbola and the asymptotes, by means of a series of inscribed and circumscribed polygons. Thus, the quadrature of the hyperbola became equivalent to the computation of lograthims. The logarithms were called hyperbolic logarithms.

The calculus had its beginning in finding areas, volumes and tangents to curves. The means of expressing the hyperbolic logarithm of numbers also came from determining areas connected with the hyperbola, as explained above.

Isaac Barrow was a mathematician and theologian as well as the gifted teacher of Isaac Newton. Barrow was interested in the quadrature of the hyperbola and seems to have improved on Gregory St. Vincents' work in that field. To Newton was intrusted the care and publication of Barrow's Lectiones Opticae. Newton took the liberty of writing two appendices in the 1704 edition. The second appendix was titled De quadratura curvarum and was written in response to a letter from Leibniz asking for the method used in deriving the binomial theorem. This appendix contains three methods for obtaining the value of a number by use of a series: first, interpolation, the method used to obtain the binomial theorem; second, the binomial theorem; and third, fluxions. In the appendix the series

$$y = z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \ldots.$$

is derived and Newton explains that the use of this series greatly simplifies the work in the computation of logarithms. We know this series as the exponential series, and observe that Newton was the first to derive it.

A professor of geometry at Oxford University, Dr. Edmund Halley, published an essay in 1695 dealing with the construction of logarithms and antilogarithms without regard to the hyperbola. In the closing paragraph of the essay, Dr. Halley writes,

> "Thus, I hope I have cleared up the doctrine of logarithms, and shewn their construction and use independent from the hyperbola, whose affection have hitherto been made use of for this purpose, tho' this be a matter purely Arithmetical, nor properly demonstrable from the Principles of Geometry. Nor have I been obliged to have recourse to the Method of Indivisible or the Arithmetick of Infinites; the whole being no other than an easie Corolary to Mr. Newton's General Theorem for forming Roots and Powers." (p. 67)

For the first time we have logarithms being constructed by using exponents.

In 1772 Roger Cotes suggested that a relation existed between the exponential and the trigonometric functions. He invented the terms modulus and modular ratio. The modulus is the ratio of $1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + - - -$ to one. The modular ratio was the ratio of one to

$$1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} - - -$$

The modulus we recognize as the ratio of 2.71828.... to 1 and the modular ratio as the ratio of 1 to the number 0.36789794411171.... Probably due to Cotes' early death, he did not fully recognize the relation between the trigonometric and the exponential function. His discovery becomes $i\theta = \log_e (\cos\theta + i \sin\theta)$. This relationship was left for Euler to bring forth.

One of the first men to have a clear understanding of the nature of logarithms as exponents was William Jones. He realized that almost any number might be taken as the base of a system of logarithms. Quite possible he was the first to recognize $e$ as belonging to the real numbers. William Jones wrote a tract in 1771 in which he explains that "any number may be expressed by some single power of the same radical number." By radical number Mr. Jones means "base."

In the first half of the eighteenth century, the contribution to the concept of $e$ is almost soley centered upon one man, Leonard Euler. One of the most notable treatises dealing with $e$ was published by Euler (1748) entitled, Introductio In Analysin Infinitorum. In this he derives the exponential series from the binomial series, the same as Halley had done. He further explained that the sum of this series, when $z = 1$, would be denoted by the letter "c". Later he changed the notation to $e$. From papers and letters written by Euler, it appears that he used $e$ for this number as early as 1727. The use of the symbol is original with Euler. He recognized the existence of a real number as the sum of a series and also as the base of the system of hyperbolic logarithms.

Euler did a great amount of work with continued fractions and developed many continued fractions that involved

e. The connections between the trigonometric and exponential function which he discovered are

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \ , \ \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

Perhaps, as Kasmer and Newman (1963) report, the most beautiful discovery accredited to Euler is the formula $e^{i\pi} + 1 = 0$ which combines in one simple formula the two most popularly known transcentental numbers, the base of the complex number system and the two identities of the arithmetic operations on real numbers.

> "Elegant, concise and full of meaning, we can only reproduce it and not stop to inquire into its implications. It appeals equally to the mystic, the scientist, the philosopher, the mathematician. For each it has its own meaning." (Kasner and Newman. 1940. p. 103)

## The Nature of e

Leonard Euler may have been the first mathematician to infer that e is irrational. In his work with continued fractions he suggested that to every rational number there corresponds a finite continued fraction and an infinite continued fraction can have only an irrational value.

The first published proof that e is irrational was by J. H. Lambert in 1761. Lambert used Euler's continued fraction for $\frac{e - 1}{2}$ and developed a continued fraction whose value was for $\frac{e^x - 1}{e^x + 1}$.

But $\dfrac{e^x - 1}{e^x + 1} = \dfrac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} = \tanh \dfrac{x}{2} = \dfrac{1}{i} \tan \dfrac{ix}{2}$. By letting $\dfrac{ix}{2} = z$, the continued fraction for $\tan z$ is obtained.

From this continued fraction, Lambert was able to prove that $e$ could not be rational.

In 1815 J. B. Fourier proved $e$ to be irrational (Tropfke, 1902) by using the series $e = \sum_{0}^{\infty} \dfrac{1}{K!}$. By assuming $e$ was rational he was able to contradict the number property that there is no integer between zero and one.

Attempts to prove $e$ was transcendental were made as soon as the difference between an algebraic number and a transcendental number was clearly defined. In 1844 Lioville proved the existence of transcendental numbers. Charles Hermite, a French mathematician, in 1873, proved in two distinct ways that $e$ was transcendental. Since then the proof has been much simplified, but the simplified proofs still depend largely upon a function that Hermite used. That $e$ is a transcendental number is still referred to as Hermite's theorem.

# CHAPTER 2

## MATHEMATICAL PROPERTIES OF e

### INTRODUCTION

In this chapter we propose to develop the exponential function, $e^x$, by beginning with the definition of the logarithm of $x$. This closely follows the historical approach. In many calculus texts, to save time and get directly to the differentiation and integration of the exponential function, $e^x$ is defined as a limit or as the sum of an infinite series. We propose to develop both of these formulas and thus find several ways of expressing $e$. In this approach we define the exponential function as the inverse of the logarithmic function and derive a limit and series expression for $e^x$. This development follows closely that found in Calculus: An Introductory Approach by Ivan Niven (1961). The development is more detailed than given by Niven.

### Development of $e^x$

Our notation for the logarithm of $x$ will be $L(x)$ throughout this chapter. This is to caution us against using intuitive properties that we may already know about the logarithmic function.

We define $L(x)$ as follows:

$$L(x) = \int_1^x \frac{1}{t}\,dt = \int_1^x t^{-1}\,dt \quad \text{for any} \quad x > 0$$

We note that $L(1) = \int_1^1 \frac{1}{t}\,dt = 0$ and $L(x) < 0$ for $0 < x < 1$. These follow from the general definition of an integral.

<u>Theorem 2.1.</u> The derivative of $L(x)$ is $1/x$. ie $L'(x) = 1/x$.

Proof: By definition of a derivative

$$L'(x) = \lim_{\Delta x \to 0} \frac{L(x + \Delta x) - L(x)}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_1^{x + \Delta x} \frac{1}{t}\,dt - \int_1^x \frac{1}{t}\,dt\right]$$

These integrals may be combined to give

$$L'(x) = \lim_{\Delta x \to 0}\left[\frac{1}{\Delta x}\int_x^{x + \Delta x} \frac{1}{t}\,dt\right]$$

Consider the limit as $\Delta x$ tends to zero through positive values. The integrand has maximum value $1/x$ and minimum value $1/(x + \Delta x)$.

Thus

$$\frac{\Delta x}{x + \Delta x} \leq \int_{x}^{x + \Delta x} \frac{1}{t}\, dt \leq \frac{\Delta x}{x}$$

Dividing by $\Delta x$ we have

$$\frac{1}{x + \Delta x} \leq \frac{1}{\Delta x} \int_{x}^{x + \Delta x} \frac{1}{t}\, dt \leq \frac{1}{x}$$

As $\Delta x$ tends to zero we see that the center term is caught between $1/x$ and a fraction tending to $1/x$. It follows that $L'(x) = 1/x$.

Now as $\Delta x$ tends to zero through negative values the minimum value of the integrand is $1/x$ and the maximum value is $1/(x + \Delta x)$.

Then we have

$$\frac{-\Delta x}{x} \leq \int_{x + \Delta x}^{x} \frac{1}{t}\, dt \leq \frac{-\Delta x}{x + \Delta x}$$

Multiplying by $-1/\Delta x > 0$, and interchanging the upper and lower limits on the integral we have

$$\frac{1}{x} \leq \frac{1}{\Delta x} \int_{x}^{x + \Delta x} \frac{1}{t}\, dt \leq \frac{1}{x + \Delta x}$$

As $\Delta x$ tends to zero it follows that $L'(x) = 1/x$.

Therefore, $L'(x) = 1/x$ for all $x > 0$.

Many theorems could be developed, but only those that will be of value in establishing our goal will be considered.

Theorem 2.2. For any positive numbers $a$ and $x$,

$$L(ax) = L(a) + L(x)$$

Proof: In this proof we shall use theorems from the calculus that are proved in most calculus texts. Let us regard $a$ as a constant and $x$ as the variable. Let $u = ax$. Then $y = L(u)$ , $u = ax$

$$dy/du = L'(u) = 1/u \quad , \quad du/dx = a$$

$$\begin{aligned} d/dx\ [L(ax)] = dy/dx &= dy/du \cdot du/dx \\ &= 1/u \cdot a \\ &= 1/ax \cdot a \\ &= 1/x \end{aligned}$$

Thus $L(ax)$ and $L(x)$ have the same derivative and differ by at most a constant. $L(ax) = L(x) + c$.

By setting $x = 1$ we have $L(a) = c$ , where $L(1) = 0$.

Therefore $L(ax) = L(x) + L(a)$

Lemma 1. $L(2) < 1$ and $L(3) > 1$.

$$L(2) = \int_1^2 \frac{1}{t}\, dt = \int_1^{3/2} \frac{1}{t}\, dt + \int_{3/2}^2 \frac{1}{t}\, dt = I_1 + I_2$$

as $1/t < 1$ when $t > 1$ we may replace $1/t$ by its maximum of one in $I_1$ and $2/3$ in $I_2$.

$$I_1 = \int_1^{3/2} dt = t\Big|_1^{3/2} = 3/2 - 1 = 1/2$$

$$I_2 = \int_{3/2}^{2} \frac{2}{3}\, dt = \frac{2}{3}\, t\Big|_{3/2}^{2} = 4/3 - 1 = 1/3$$

$L(2) = I_1 + I_2 = 1/2 + 1/3 = 5/6 < 1$

$L(3) = \int_1^3 \frac{1}{t}\, dt$ which we shall break down into the following six integrals:

$$\int_1^3 \frac{1}{t}\, dt = \int_1^{5/4} \frac{1}{t}\, dt + \int_{5/4}^{3/2} \frac{1}{t}\, dt + \int_{3/2}^{7/4} \frac{1}{t}\, dt + \int_{7/4}^{2} \frac{1}{t}\, dt + \int_{2}^{5/2} \frac{1}{t}\, dt + \int_{5/2}^{3} \frac{1}{t}\, dt$$

and let the respective integrals equal

$$I_3 + I_4 + I_5 + I_6 + I_7 + I_8$$

As $1/t < 1$ when $t > 1$ we replace $1/t$ by the minimum values.

Thus;

$$L(3) = \int_1^{5/4} \frac{4}{5}\, dt + \int_{5/4}^{3/2} \frac{2}{3}\, dt + \int_{3/2}^{7/4} \frac{4}{7}\, dt + \int_{7/4}^{2} \frac{1}{2}\, dt + \int_{2}^{5/2} \frac{2}{5}\, dt + \int_{5/2}^{3} \frac{1}{3}\, dt$$

$$= \frac{4}{5}\left(\frac{1}{4}\right) + \frac{2}{3}\left(\frac{1}{4}\right) + \frac{4}{7}\left(\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4}\right) + \frac{2}{5}\left(\frac{1}{2}\right) + \frac{1}{3}\left(\frac{1}{2}\right)$$

$$= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{5} + \frac{1}{6}$$

$$= \frac{841}{840} > 1$$

Therefore: $L(3) > 1$

From the definition of $L(x)$ we can see that since the integrand $1/t$ is positive, the value of $L(x)$ increases as $x$ increases and decreases as $x$ decreases. That is, $L(x)$ is a monotonic increasing function. We prove that as $x$ tends to infinity $L(x)$ is unbounded, that is for every positive number $M$ (no matter how large) there exist values of $x$ such that $L(x) > M$. When $y = x$ we know from Theorem 2.2 that $L(x^2) = 2L(x)$. Again with $y = x^2$ we have $L(x^3) = 3L(x)$, and by induction, $L(x^n) = nL(x)$ for every integer $n \geq 1$. When $x = 2$ this becomes $L(2^n) = nL(2)$ and hence we have $L(2^n) > M$ when $n > M/L(2)$. Thus $L(x)$ is unbounded.

Since the function $y = L(x)$ is a monotonic increasing differentiable function, with $L'(x) \neq 0$ for all positive values of $x$, the calculus reveals that the inverse function is also differentiable. (Apostol, 1961. p. 197). We denote the inverse function by $x = E(y)$. Then $E'(y) = dx/dy$. We know from calculus (Apostol, 1961. p. 197) that $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$, with $dy/dx = L'(x)$ and $dx/dy = E'(y)$ we have $L'(x) \cdot E'(y) = 1$. Since $L'(x) = 1/x$ and $E'(y) = 1/L'(x) = 1/(1/x)$ then $E'(y) = x$. Also $x = E(y)$. Writing this using usual variable we have $E(x) = E'(x)$.

By using Lemma 1 above, we can obtain a useful condition on $E(x)$. Let $L(a) = c$ , $L(x) = y$ , $L(ax) = w$ then $w = c + y$. In terms of the inverse function we get

$$a = E(c) \;,\; x = E(y) \;,\; ax = E(w) \text{ and } ax = E(c + y)$$

(1) $\quad$ Hence $E(c + y) = ax = E(c) \cdot E(y)$

This property shall now be used to analyze the function $E(x)$. By substitution:

(2) $\quad y = 0 \qquad E(0) = 1$

$\qquad y = 1,\ c = 1,\ E(2) = [E(1)]^2$

$\qquad y = 1,\ c = 2,\ E(3) = [E(1)]^3$

$\qquad y = 1,\ c = 3,\ E(4) = [E(1)]^4$

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(3) $\quad y = 1,\ c = n - 1 \;,\; E(n) = [E(1)]^n$

Now, for any positive rational number $p/q$ , $p,q$ positive integers, form the product with $q$ factors;

$E(p/q) \cdot E(p/q) \;............E(p/q) = [E(p/q)]^q$.

By successively applying (1) to the left side we find the total product to equal $E(p)$. From (3) we obtain

(4) $\quad [E(p/q)]^q = E(p) = [E(1)]^p,\ E(p/q) = [E(1)]^{p/q}$

By replacing $c$ for $p/q$ and $y$ for $-p/q$ in (1) and then using (2) and (4) we obtain

(5) $\quad E(p/q) \cdot E(-p/q) = 1$ or $E(-p/q) = [E(1)]^{-p/q}$

With $E(1)$ playing such an important role in the above equation, we name $E(1) = e$. Then the equations (3), (4)

(5) become

$$E(n) = e^{n} \,,\; E(p/q) = e^{p/q} \quad\text{and}\quad E(-p/q) = e^{-p/q}$$

We conclude that

(6) $E(x) = e^{x}$ for all rational values of $x$.

Now we show that (6) is true for irrational values of $x$. Consider one specific irrational number $\alpha$. Let the sequence of rational numbers

(7) $a_1,\, a_2,\, a_3.....a_n,....$

have $\alpha$ as a limit. Consider the sequence of numbers

(8) $e^{a_1},\, e^{a_2},\, e^{a_3},......e^{a_n},...$

We wish to show that this sequence has a well defined limit. Sequence (8) can be written

(9) $E(a_1),\, E(a_2),........,\, E(a_n),......$

Since $E(x)$ is differentiable it is continuous and $\lim\limits_{n \to \infty} E(a_n)$ exists and is equal to $E(\alpha)$. But (8) is the same sequence as (9) then $\lim\limits_{n \to \infty} e^{a_n}$ exists. This limit we shall call $e^{\alpha}$. Thus $E(x) = e^{x}$ is true for all real $x$.

Having established that $E(x)$ is a differentiable continuous function for all real values of $x$, we turn our attention to finding an infinite series converging to $E(x)$. The normal development is through power series, using the Maclaurin and Taylor series. We choose to continue the development through the limit concepts and some basic theorems in calculus.

The following theorems are used in the development of the infinite series for $e^x$.

II.1. If $y$ is a differentiable function of $u$, and $u$ is a differentiable function of $x$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} .$$

II.2. If $a$ and $b$ are positive, then $a > b$ and $\frac{1}{a} < \frac{1}{b}$ are equivalent.

II.3. If two integrable function $F(x)$ and $G(x)$ satisfy inequality $F(x) \leq G(x)$ for all values of $x$, $a \leq x \leq b$, then $\int_a^b F(x)\, dx \leq \int_a^b G(x)\, dx.$

II.4. Let $c$ be a constant and let $\{a_n\}$ be a sequence of numbers with no term less than $c$, and $\{b_n\}$ be a sequence of numbers with no term greater than c.

ie. $a_n \geq c$, $b_n \leq c$ for all values of $n$.

If $\lim_{n \to \infty} (a_n - b_n) = 0$ then $\lim_{n \to \infty} a_n = c$ and $\lim_{n \to \infty} b_n = c$.

II.5. For any constant $c$, $\lim_{n \to \infty} \frac{c^n}{n!} = 0$.

We first find the derivative of $e^{-x}$. With $y = e^u$ and $u = -x$, $dy/du = e^u$, $du/dx = -1$, and $dy/dx = -e^{-x}$.

Then by integration, $\int_0^b e^{-x}\, dx = -e^{-x} \Big|_0^b = 1 - e^{-b}$.

When $x > 0$ we know $e^x > 1$, so that by II.2,

$e^{-x} = \frac{1}{e^x} \leq 1$. Thus for $x \geq 0$, $e^{-x} \leq 1$. Using II.3 with $F(x) = e^{-x}$ and $G(x) = 1$, and when $b > 0$

$\int_0^b e^{-x}\, dx \leq \int_0^b 1dx$ giving us $1 - e^{-b} \leq b$ or $1 - b \leq e^{-b}$

This holds for any positive $b$. With $x$ also positive, we may replace $b$ by $x$ and obtain $1 - x \leq e^{-x}$. Again applying II.3 with $F(x) = 1 - x$ and $G(x) = e^{-x}$, $b > 0$

$\int_0^b (1 - x)\, dx \leq \int_0^b e^{-x}\, dx$ giving $b - \frac{b^2}{2} \leq -e^{-b} + 1$ or

$1 - b + \frac{b^2}{2} \geq e^{-b}$. As this holds for any $b > 0$, it holds for $x > 0$. Thus $1 - x + x^2/2 \geq e^{-x}$. Integrating this inequality we obtain

$$\int_0^b (1 - x + x^2/2)\, dx \geq \int_0^b e^{-x}\, dx \quad \text{where} \quad b > 0$$

$$b - b^2/2 + b^3/2\cdot 3 \geq - e^{-b} + 1 \text{ or}$$

$$1 - b + b^2/2 - b^3/3! \leq e^{-b} \quad \text{for all } b > 0.$$

For $x > 0$ we have $1 - x + x^2/2 - x^3/3! \leq e^{-x}$.

By iteration of this process we obtain the following chain of inequalities:

$e^{-x} \leq 1$ $\qquad$ $e^{-x} \geq 1 - x$

$e^{-x} \leq 1 - x + x^2/2$ $\qquad$ $e^{-x} \geq 1 - x + x^2/2 - x^3/3$

$e^{-x} \leq 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!}$ $\qquad$ $e^{-x} \geq 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

etc....

Thus for $x \geq 0$ , $e^{-x}$ is always between any two successive partial sums of the series

$1 - x/1! + x^2/2! - x^3/3! + x^4/4! - x^5/5 + x^6/6 - \ldots$

Take the partial sums $S_n$ as follows:

$S_0 = 1$

$S_1 = 1 - x$

$S^2 = 1 - x + x^2/2!$

.
.
.

$S_n = 1 - x + x^2/2! - x^3/3! + \ldots + (-1)^{2n+1}x^n/n!$

Every member of the sequence $S_1, S_3, S_5 \ldots$ is less than $e^{-x}$ , and every member of the sequence $S_0, S_2, S_4, \ldots$ is greater than $e^{-x}$. By fixing $x$ , we also fix $e^{-x}$. By II.4 if the $\lim\limits_{n \to \infty} (S_{2n-1} - S_{2n}) = 0$ then both sequences tend to some constant c.

$$\lim_{n \to \infty} (S_{2n-1} - S_{2n}) = \lim_{n \to \infty} \left[ \frac{x^{2n-1}}{(2n-1)!} - \frac{x^{2n}}{(2n)!} \right]$$

With $x$ fixed and by II.5 this limit tends to zero as $n$ tends to infinity. We conclude by II.4 that both sequences have the same limit, namely $e^{-x}$ . As $S_{2n-1}$ and $S_{2n}$ are complementary subsequences of $S_n$ , we conclude that

$1 - x/1! + x^2/2! - x^3/3! + \ldots + (-1)^{2n-1}\ x^n/n! + \ldots = e^{-x}$

for all $x \geq 0$. Replacing $(-x)$ by $x$ we obtain

$e^x = 1 + x/1! + x^2/2! + x^3/3! + x^4/4! + \ldots + x^n/n! + \ldots.$

for all $x \leq 0$.

We need to prove that the above series is true for values of $x > 0$. Consider the function $f(x) = 1 + xe^c - e^x$ where $c \geq 0$ and fixed. We see that $f(0)=0$ and $f'(x) = e^c - e^x$. Since $e^x$ is a monotonic increasing function then $f'(x) > 0$ when $0 < x < c$. Hence $f(x)$ is a monotonic increasing function for $0 \leq x \leq c$. It follows that with $f(x) \geq 0$, $1 + xe^c - e^x \geq 0$ or $1 + xe^c \geq e^x \geq 1$. Integrating this inequality we obtain:

$$\int_0^b (1 + xe^c)\ dx \geq \int_0^b e^x\ dx \geq \int_0^b 1\ dx \qquad 0 \leq b \leq c$$

$b + b^2/2.e^c \geq e^b - 1 \geq b$. Replacing $b$ by $x$ we get

$$x + x^2/2 \cdot e^c \geq e^x - 1 \geq x$$

Integrating this inaquality

$$\int_0^b (x + x^2/2 \cdot e^c)\ dx \geq \int_0^b (e^x - 1)\ dx \geq \int_0^b x\ dx \qquad 0 \leq b \leq c$$

$b^2/2 + e^c b^3/2\cdot 3 \geq e^b - 1 - b \geq b^2/2$

Again replacing by $x$ and integrating

$$\int_0^b (x^2/2 + x^3/3!\ \cdot e^c)\ dx \geq \int_0^b (e^x - 1 - x)\ dx \geq \int_0^b x^2/2\ dx$$

$b^3/3! + b^4/4! \cdot e^c \geq e^b - 1 - b - b^2/2 \geq b^3/3! \quad 0 \leq b \leq c$

Repetition of this process leads to

$$\frac{b^n}{n!} + \frac{b^{n+1}}{(n+1)!} e^c \geq e^b - 1 - b - b^2/2 - \ldots$$

$$\ldots - b^{n-1}/(n-1)! \geq b^n/n$$

But the $\lim_{n \to \infty} b^n/n! = 0$ by II.5 and

$$\lim_{n \to \infty} \left[ \frac{b^n}{n!} + \frac{b^{n+1}}{(n+1)!} e^c \right] = \lim_{n \to \infty} \frac{b^n}{n!} + e^c \lim_{n \to \infty} \frac{b^{n+1}}{(n+1)!}$$

$$= 0 + e^c \cdot 0 = 0$$

Thus, as $n$ tends to infinity,

$$e^b - 1 - b - b^2/2! - b^3/3! - \ldots - b^{n-1}/(n-1)! \ldots = 0$$

Replacing $b$ by $x$ where $0 \leq x \leq c$ where $c$ is any positive number we obtain

$$e^x - 1 - x - x^2/2! - x^3/3! - \ldots - x^{n-1}/(n-1)! - \ldots = 0$$

or

$$e^x = 1 + x + x^2/2! + x^3/3! + \ldots + x^n/n! + \ldots\ldots$$

for <u>all</u> values of $x$.

By substituting $x = 1$ in the above series, we find a convenient series representing $e$.

$$e = 1 + 1/1! + 1/2! + 1/3! + \ldots + 1/n! + \ldots$$

We wish to establish another representation for the number $e$, only this time in the form of a limit.

Previously it was shown that $L(2) < 1$ and $L(3) > 1$. Using the inverse function of $L(x)$, we have $E(1) > 2$ and $E(1) < 3$. Thus $E(1)$ is bounded above by the real number three. We now prove $\lim_{n \to \infty} (1 + 1/n)^n = e$.

With $L'(x) = 1/x$ then by definition of a derivative

$$\lim_{\Delta x \to 0} \frac{L(x + \Delta x) - L(x)}{\Delta x} = \frac{1}{x}.$$ When $x = 1$, $L(x) = 0$ and we obtain

$$\lim_{\Delta x \to 0} \frac{L(1 + \Delta x) - 0}{\Delta x} = 1 \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{1}{\Delta x} L(1 + \Delta x) = 1.$$

Let $\Delta x \to 0$ through the positive sequence $1, 1/2, 1/3, \ldots$ so that the above limit becomes a limit of a sequence. With $1/n = \Delta x$, $\Delta x \to 0$ implies $n \to \infty$.

Thus $\lim_{n \to \infty} [n \cdot L(1 + 1/n)] = 1$

By a property of $L(x)$, $n \cdot L(1 + 1/n) = L(1 + 1/n)^n$ thus

$$\lim_{n \to \infty} [n \cdot L(1 + 1/n)] = \lim_{n \to \infty} [L(1 + 1/n)^n] = 1$$

For brevity let $L(1 + 1/n)^n = a_n$. Then the sequence $a_1, a_2, a_3, \ldots$ has limit 1. Since $E(x)$ is continuous and differentiable, we can conclude that the sequence

$\{E(a_n)\} = e^{a_1}, e^{a_2}, e^{a_3}, \ldots\ldots$ has limit $e^1$ or $e$.

But $E(x)$ and $L(x)$ are inverse function. Then

$$E(a_n) = E[L(1 + 1/n)^n] = (1 + 1/n)^n$$

Therefore $e = \lim_{n \to \infty} (1 + 1/n)^n$.

Irrationality of e

Many authors now include in their analysis books the topic of the irrationality of $e$. Lambert (Tropfke, 1902) in 1761 first proved the irrationality of $e$ by using two of Euler's continued fractions. The following is essentially the proof given by J. B. Fourier in 1815. The proof that $e$ is irrational is accomplished by assuming $e$ to be rational and arriving at a contradiction. This is the basis of the following proof.

Let us assume that $e$ is rational and therefore may be expressed in the form $e = p/q$ where $p,q$ are positive integers. The infinite series for $e$ is:

$$e = \sum_{n=0}^{\infty} 1/n! = 1 + 1/1! + 1/2! + 1/3! + \ldots + 1/n! + \ldots$$

If we let $$S_q = \sum_{n=0}^{q} 1/n! = 1 + 1/1! + 1/2! + \ldots + 1/q!$$

then $$e - S_q = \sum_{n=q+1}^{\infty} 1/n! = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \ldots.$$

$$= \frac{1}{(q+1)!}\left[1 + \frac{1}{q+2} + \frac{1}{(q+2)(q+3)} + \ldots\right]$$

thus $$e - S_q < \frac{1}{(q+1)!}\left[1 + \frac{1}{q+1} + \frac{1}{(q+1)^2} + \ldots.\right]$$

The second factor on the right is a geometric infinite series with the ratio being $1/(q + 1)$. We know the geometric series converges with the ratio $|r| < 1$. With $q \geq 1$ $\quad 0 < \frac{1}{q + 1} < 1.$ Thus the geometric series converges to

$$\frac{1}{1 - \frac{1}{q + 1}} = \frac{q + 1}{q}$$

Therefore $e - S_q < \frac{1}{(q + 1)!} \cdot \frac{q + 1}{q} = \frac{1}{q!q}$

that is, $0 < e - S_q < 1/(q!q)$

Multiplying the inequality by the integer $q!$ we have

$$0 < q!(e - S_q) < 1/q$$

But by assumption $q!e$ is an integer and

$q!S_q = q!\ [1 + 1/1! + 1/2! + 1/3! + \ldots + 1/q!]$

is also an integer.

This means that $q!(e - S_q)$ is an integer. Since $q \geq 1$, $1/q \leq 1$, and $0 < q!(e - S_q) < 1/q$ our conclusion is that there is an integer between 0 and 1. Thus we have reached a contradiction by assuming $e$ is rational. Thus, $e \neq p/q$. Therefore $e$ is an irrational number.

Transcendence

The number $e$ was first proved transcendental by Charles Hermite in 1873. This proof is relatively complicated; however once the breakthrough was made, several simpler proofs were found. Subsequent mathematicians who were to offer proofs of a less demanding nature, though still substantial, are Stieltjes, Hilbert, Gordan, Mertens, Klein and Vahlen. Translation of Hermite's original proof is found in Smith's Source Book in Mathematics, 1929, pp. 99 to 106. Proofs other than those mentioned above may be found in the following sources, Klein (1932), Beman (1897), Young (1911), Hobson (1913) and Niven (1957). The following proof of the transcendentality of $e$ is that of Herstein (1965).

This proof is expanded extensively beyond what can be found in the literature. The original source omitted several items which we have entered to make the proof more readable. This proof should be comprehensible to the bright student of elementary calculus.

We prove the transcendence of $e$ by the indirect method. It is assumed, to the contrary, that $e$ is an algebraic number; then $e$ is a root of a polynomial equation with integral coefficients.

Let $f(x)$ be a polynomial of degree $r$ with real coefficients. Let $F(x) = f(x) + f'(x) + f^2(x) + \ldots\ldots + f^r(x)$ where $f^i(x)$ stands for the ith derivative of $f(x)$ with respect to $x$.

$$e^{-x}F(x) = e^{-x}f(x) + e^{-x}f'(x) + \dots + e^{-x}f^{r}(x)$$

$$\frac{d}{dx}\,[e^{-x}F(x)] = - e^{-x}f(x) + e^{-x}f'(x) - e^{-x}f'(x) + e^{-x}f^{2}(x)$$
$$+ \dots + e^{-x}f^{r}(x) - e^{-x}f^{r}(x)$$

Since $f^{r+1}(x) = 0$.

Thus $\frac{d}{dx}\,[e^{-x}F(x)] = -e^{-x}f(x)$

The mean value theorem of calculus asserts that if $g(x)$ is a continuously differentiable single valued function on the closed interval $[a,b]$ then

$$\frac{g(b) - g(a)}{b - a} = g'[a + \emptyset\cdot(b - a)]\ ,\ 0 < \emptyset < 1$$

The function $e^{-x}F(x)$ is a continuously differentiable single valued function on the closed interval $[0,k]$, where $k$ is any positive integer. Then by the mean value theorem

$$\frac{e^{-k}F(k) - F(0)}{k - 0} = - e^{-(0+\emptyset_k\cdot k)}\ f(0 + \emptyset\cdot k)$$

or

$$e^{-k}F(k) - F(0) = -e^{-\emptyset_k\cdot k}\ f(k\cdot\emptyset_k)k$$

where $\emptyset_k$ depends on $k$ and is a real number between 0 and 1.

Multiplying this equation by $e^{k}$ we obtain

$$F(k) - e^{k}F(0) = -e^{(1 - \emptyset_k)k}\ f(k\cdot\emptyset_k)k$$

By substituting successive values of k into the above equation we get

$$
\begin{aligned}
&F(1) - e^{1}F(0) = -e^{(1-\phi_1)} f(\phi_1) &&= s_1\\
&F(2) - e^{2}F(0) = -2e^{2(1-\phi_2)} f(2\phi_2) &&= s_2\\
(1)\quad &F(3) - e^{3}F(0) = -3e^{3(1-\phi_3)} f(3\phi_3) &&= s_3\\
&\quad\vdots\\
&F(n) - e^{n}F(0) = -ne^{n(1-\phi_n)} f(n\phi_n) &&= s_n
\end{aligned}
$$

To show $e$ is transcendental, we assume it is not a transcendental number. That is, assume $e$ to be an algebraic number that satisfies the relation

$$(2)\quad c_n e^n + c_{n-1}e^{n-1} + c_{n-2}e^{n-2} + \dots + c_1 e + c_0 = 0$$

where $c_0$ , $c_1$ , $c_2$ , .... are integers and $c_0 > 0$.

In equations (1) we will multiply the first one by $c_1$ , the second by $c_2$ and in general the nth one by $c_n$.

$$
\begin{aligned}
&c_1F(1) - c_1eF(0) = -c_1e^{(1-\phi_1)} f(\phi_1) &&= c_1s_1\\
&\quad\vdots\\
&c_nF(n) - c_ne^{n}F(0) = -c_nne^{n(1-\phi_n)} f(n\phi_n) &&= c_ns_n
\end{aligned}
$$

Adding these together we get

$$c_1F(1) + c_2F(2) +\ldots+ c_nF(n) - F(0)(c_1e + c_2e^2 +\ldots+ c_ne^n)$$

$$= c_1s_1 + c_2s_2 + c_3s_3 + \ldots + c_ns_n$$

This simplifies by (2) into

(3) $c_0F(0) + c_1F(1) +\ldots+ c_nF(n) = c_1s_1 + c_2s_2 +\ldots+ c_ns_n$

Recall that $F(x)$ was constructed from an arbitrary polynomial $f(x)$. Hermite, the French mathematician who was the first to prove $e$ transcendental, composed a very specific polynomial to test this condition. It was

$$f(x) = \frac{1}{(p-1)!} x^{p-1}(1-x)^p(2-x)^p \ldots (n-x)^p$$

where $p$ is any prime number chosen so that $p > n$ and $p > c_0$. By expanding $f(x)$ we get the form

$$f(x) = \frac{(n!)^p}{(p-1)!} x^{p-1} + \frac{a_ox^p}{(p-1)!} + \frac{a_1x^{p+1}}{(p-1)!} + \ldots$$

where Hermite proved that $a_0$, $a_1$, .... are integers.

The reader may demonstrate to himself that when $i \geq p$, $f^i(x)$ is a polynomial, with coefficients which are integers all of which are multiples of $p$. Thus for any integer $j$, $f^i(j)$, for $i \geq p$ is an integer and is a multiple of $p$.

From its very definition $f(x)$ has a root of multiplicity $p$ at $x = 1, 2, \ldots, n$. Thus for $j = 1, 2, 3, \ldots\ldots, n$, $f(j) = 0$, $f'(j) = 0$, ...., $f^{p-1}(j) = 0$.

However, from the discussion above,

$F(j) = f(j) + f'(j) + \ldots + f^{p-1}(j) + f^{p}(j) + \ldots + f^{r}(j)$

for $j = 1, 2, 3, \ldots, n$ is an integer and is a multiple of $p$.

Let us turn our attention to $F(0)$. Since $f(x)$ has a root of multiplicity $p - 1$ at $x = 0$, $f(0) = f'(0) = \ldots = f^{p-2}(0) = 0$. For $i \geq p$, $f^{i}(0)$ is an integer which is a multiple of $p$. But $f^{p-1}(0) = (n!)^{p}$ and since $p > n$ and is a prime number $p$ does not divide $(n!)^{p}$ so that $f^{p-1}(0)$ is an integer not divisible by $p$. Since

$F(0) = f(0) + f'(0) + \ldots + f^{p-2}(0) + f^{p-1}(0) + f^{p}(0) + \ldots + f^{r}(0)$ and

$f(0) + f'(0) + \ldots + f^{p-2}(0) = 0$ and

$f^{p-1}(0)$ is not divisible by $p$ and

$f^{p}(0) + \ldots + f^{r}(0)$ is divisible by $p$, we conclude that $F(0)$ is an integer not divisible by $p$. Since $c_0 > 0$, $p > c_0$ and because $p$ does not divide $F(0)$ but $p$ divides $F(1)$, and $p$ divides $F(2)$, and ...., and $p$ divides $F(n)$, we can assert that $c_0F(0) + c_1F(1) + \ldots + c_nF(n)$ is an integer and is not divisible by $p$.

By (3) we know

$c_oF(0) + c_1F(1) + \ldots + c_nF(n) = c_1s_1 + c_2s_2 + \ldots + c_ns_n$

But $s_i = -ie^{i(1 - \emptyset_i)} f(i\emptyset_i)i$

$$= \frac{-ie^{i(1 - \emptyset_i)}(i\emptyset_i)^{p-1}(1 - i\emptyset_i)^p \ldots(n - i\emptyset_i)^p}{(p - 1)!}$$

with $0 < \emptyset_i < 1$ and $i < n$ we make these substitution and obtain the following inequality.

$$|s_i| \leq \frac{e^n n^p (n!)^p}{(p - 1)!}$$

holding $n$ fixed, $\lim\limits_{p \to \infty} \frac{e^n n^p (n!)^p}{(p - 1)!}$

$$= e^n \lim_{p \to \infty} \frac{n^p (n!)^p}{(p - 1)!}$$

$$= e^n \cdot 0$$

$$= 0$$

Thus $|s_i|$ tends to zero as $p$ tends to infinity. Then a prime number can be found that is larger than both $c_0$ and $n$ and large enough to force

$|c_1 s_1 + c_2 s_2 + \ldots + c_n s_n| < 1$

But $c_1 s_1 + c_2 s_2 + \ldots + c_n s_n = c_0 F(0) + \ldots + c_n F(n)$ and must be an integer. Since it is smaller in value than one, and it is an integer, $c_1 s_1 + c_2 s_2 + \ldots + c_n s_n$ must be zero. Thus,

$c_0F(0) + c_1F(1) + \ldots + c_nF(n) = 0$. This cannot be true since $p$ does not divide $c_0F(0) + c_1F(1) + \ldots + c_nF(n)$, whereas $p$ does divide zero. This contradiction, coming from our assumption that $e$ is algebraic, proves $e$ must be transcendental.

Normality

One of the reasons that $e$ has been computed to 100,256 decimal places is to examine this decimal expansion for normality. A normal number has been defined by Borel in 1909 (Niven, 1957) as "a real number $x$ which has all possible blocks of $j$ digits appearing with the relative frequency of $1/10^j$ (or $1/g^j$, where $g$ is the base of the representation of $x$)." Borel also defined an 'absolute' normal number as one which is normal to every base. He proved that "almost all" real numbers are normal to every base.

An example of a normal number is the positive integers listed in their natural order as a decimal expansion.

0.12345678910111213141516171819202122232425262728293031..

This number has all digits in their proper proportions and all blocks of digits in their proper proportions. (Davis, 1961).

Mathematicians have not been lax in developing the theory of normal numbers. There have been several theorems established concerning the properties of normal numbers. Schmidt (1960) proved that there exist numbers that are normal to one base but not absolutely normal. These theorems are based on measure theory and describe the properties of a normal number. That is, this number is normal, therefore the following theorems apply. Unfortunately, the theorems give us no way to tell if a given number is normal.

The first study of $e$ for normality was done by John von Newman (1950). Further studies have been done in the hope of establishing the random characteristics of the decimal places of $e$. If this could be done it would make the transcendental numbers available as a computer source of internally programmed pseudo-random numbers for Monte Carlo methods.

H. Geiringer, in 1954 stated that since $e$, and other transcendentals as $\pi$ and Euler's constant, are formed by mathematical laws, they do not form a 'random' sequence or a 'collective' even though we are not yet able to establish any prevailing regularities. She continues by saying that such numbers do exhibit 'local' or 'restricted' randomness.

Many mathematicians have tried to show the random and normal properties of $e$, John von Neumann (1950), R. K. Pathria (1961), and Fisher and Yates (1938) have great interest in this field.

To test e for normality many statistical tests have been carried out. These were done using a 60,000 digit decimal expansion of e. The tests include (1) poker, (2) serial, (3) frequency, (4) gap test, (5) chi-square measure for each of the above tests, (6) the chi-square for the accumulated frequency of the counts of single digits computed every 500 digits, (7) the chi-square values for 60 blocks of 1000 digits (8) a goodness-of-fit test using the chi-squares for 120 blocks of 500 digits, (9) a study of these 120 blocks as Bernoulli trials, (10) plots of the deviation of each separate digit from zero to nine from expectation for the accumulated counts of 600 blocks of 100 digits, (11) the values of $S_n^i/n$ and $S_n^{ij}/n$, where ratios represent relative frequencies of single and pairs of digits for normality studies (12) and finally a test of randomness based on a criteria for randomness in a 'collective' proposed by von Mises wherein one chooses a subset by some choice of place selection.

The above tests were reported to the American Mathematical Monthly by R. G. Stoneham. (1965). He summarizes the statistical findings by writing

> "In general, all the statistical tests support the hypothesis of pure chance selection in the sequence of digits in e with but a few anomalies and also the relative frequencies of the single and pairs of digits in e are approx-

> imately 1/10 and 1/100, respectively, thus empirically supporting normality in the sense of Borel." (p. 484)

Although these statistical tests seem to suggest that e is a normal number, most mathematicians are not ready to dismiss the possibility that if carried out to a billion decimal places, e might begin to favor some one digit, or a sequence of digits might show up more frequently than others. As of yet, there is no mathematical proof that e is normal.

# CHAPTER 3

## DECIMAL APPROXIMATIONS

George Mallory was asked in 1922 why he attempted Mount Everest. "It was there", he answered (Davis, 1961). The number e is there and has been for some time, and although its computation to 100,000 decimal places is comparable to the conquering of Mount Everest by helicopter, there are enough difficulties present to make it interesting. We may learn something along the journey, if not at the destination.

An American astonomer and mathematician, Simon Newcomb, who lived in the last half of the nineteenth century, said,

> "Ten decimal places are sufficient to give the circumference of the earth to the fraction of an inch, and thirty decimals would give the circumference of the whole visible universe to a quantity imperceptible with the most powerful microscope." (Kasner and Newman, 1963, p. 78)

Yet men continue to look for greater and greater decimal expansions in the number e. Perhaps it is the sportsmanlike interest in making a record that attracts them to this labor. Yet, man's insatiable curiosity and drive to know and do everything is reason enough in itself for these approximations.

The demand for lists of random digits has increased considerably in the last quarter century, particularly in connection with the application of Monte Carlo methods to various problems in mathematical physics and the drawing of random samples in statistics. Test for randomness in the sequence of digits in the decimal approximation for e have been referred to in chapter 2. These tests give every evidence that e is a random sequence of digits.

Perhaps if e had failed the test for randomness, it would have been more significant than the success. There are no known properties of e that would predict such a failure, and hence the failure would give new information about the nature of e. Another reason for obtaining more decimal places is the possibility of new properties of e being revealed.

At one time it was believed that the decimal approximations would reveal the repetitive or terminating property of e. Since Lambert showed that e was irrational, all hope of e being a repetitive or terminating decimal was dispelled.

Use of continued fractions

In a discussion of the history of mathematics, E. T. Bell (1945) gives the following on continued fractions:

> "The first systematic discussion of continued fractions was Euler in 1737. Apart from the sporadic appearence in the arithematic of the Greeks, the Hindus, and the Moslems that can now be interpreted as results of continued

fractions." (p. 476)

Euler not only named the number e and calculated it to 23 decimal places, but he also developed several continued fractions converging to e. These continued fractions play an important part in the history of e as well as in its decimal expansion. The following seven continued fractions are credited to Euler.

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + 5 \ldots}}}}}$$

$$e^{\frac{1}{2}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + 1 \ldots}}}}}}}$$

$$\frac{1}{e - 1} = \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + 6 \ldots\ldots}}}}}$$

In 1875 Glaisher used this continued fraction to find $1/(e - 1) = .581976706869....$

$$\frac{e - 1}{2} = \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \cfrac{1}{\ldots}}}}}}$$

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ldots}}}}}}}$$

$$\frac{e + 1}{e - 1} = 2 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \cfrac{1}{\ldots}}}}}$$

$$\frac{e^2 - 1}{2} = 3 + \cfrac{1}{5 + \cfrac{1}{7 + \cfrac{1}{9 + \cfrac{1}{11 + \cfrac{1}{\ldots}}}}}$$

The mathematically prolific Euler also found the following two equations. From these $e^x$ and $e^{-x}$ can readily be expressed in terms of infinite products.

$$\frac{e^x + e^{-x}}{2} = (1 + z)(1 + z/9)(1 + z/25)(1 + z/49)\ldots.$$

$$\frac{e^x - e^{-x}}{2} = (1 + z)(1 + z/4)(1 + z/9)(1 + z/16)\ \ldots.$$

J. H. Lambert was an admirer of Euler and had great interest in the above continued fractions. From these fractions Lambert developed and used the following continued fraction for the first proof of the irrationality of e.

$$\frac{e^x - 1}{e^x + 1} = \cfrac{1}{\frac{2}{x} + \cfrac{1}{\frac{6}{x} + \cfrac{1}{\frac{10}{x} + \cfrac{1}{\frac{14}{x} + \cfrac{1}{\ldots.}}}}}$$

<u>Fractions approximating e</u>

After men found that e was a decimal fraction that could not be expressed as a ratio of two integers, the search began for a close approximation using the quotient of two integers.

It is well known (Trigg, 1962) that

$\frac{2721}{1001} \approx 2.7182817$ approximates the value of e, being accurate to six decimal places. This is equivalent to

$$e \approx \frac{4}{7} + \frac{16}{11} + \frac{9}{13} = \frac{11}{7} + \frac{5}{11} + \frac{9}{13} = \frac{4}{7} + \frac{5}{11} + \frac{22}{13}$$

In closely examining the above sums, we find in all three cases the denominators are consecutive primes. The third sum has the sum of the numerator equaling the sum of the denominators, ie. 37.

The approximating fraction may also be written as

$$\frac{877 + 907 + 937}{7 \cdot 11 \cdot 13}$$

which the denominator is the product of three consecutive primes, and the numerator is the sum of three primes in arithmetic progression with the common difference being 30. The numerator may be written with primes in at least 23 other ways in which 907 is the mean. The smaller terms of the arithmetic progression's are 3,13,31,61,67,73,151, 157,193,271,283,331,367,433,487,523,601,613,643,661,727, 751, and 823.

Of the fractions with denominators less than 100, the one most closely approximating $e$ by defect is $\frac{106}{39} \approx$ 2.717949 being accurate to three decimal places. The fraction most closely approximating $e$ by excess is $\frac{193}{71} \approx$ 2.718310 which is accurate to four decimal places.

When Hermite's proof of the transcendental nature of $e$ was published, he also gave the following fraction as approximating $e$.

$$e = \frac{58291}{21444} = 2.718289,$$ correct to six decimal places. Hermite also approximated $e^2 = \frac{158452}{21444}$.

Electronic computer calculations of e

After Euler computed e to 23 decimal places, other men joined in the task of the decimal computation of e. With curiosity prodding them ahead, the mathematicians reached further and further into that non-terminating expansion. In 1849 F. J. Studnicke had e expanded to 113 decimal places. After many years of work, J. W. Boorman had the decimal expansion to a staggering 346 places. With the Boorman expansion in 1884, the computation of e diminished as it required years of a man's life to come up with more decimal places. Any further expansion was to wait until the development of statistics and the invention of electronic computer.

The advent of the electronic computers brought man an amazing ability to do calculation in a fractional part of the time previously taken. It was inevitable that computers would be brought to the task of the decimal expansion of e. With the formula

$$e = 1 + 1 + 1/2! + 1/3! + 1/4! + \ldots.$$

as the guide, the computers were able to extend the expansion beyond previously dreamed of limits. In 1949 the ENIAC (Electronic Numerical Integrator and Calculator) at the Army's Ballistic Reasearch Laboratories in Aberdeen, Maryland, was used to compute an approximation of e to 2,036 places. As the electronic computers increased in reliability and speed, the accuracy of decimal approximations of e further increased. In 1952 an electronic computer at the University of Illinois, under the guiding eye of D. J. Wheeler, carried e to the staggering total of 60,000 decimal places! In the United Kingdom, H. G. Simon used an IBM computer in 1961 to obtain e to 20,000 decimal places with the corresponding digits being the same as Wheeler's expansion.

August 19, 1961 was the date that Daniel Shanks and J. Wrench had reserved to use the IBM 7090 for the great test. To have a decimal expansion of e to 100,000 places was their goal. After just 2.5 hours, the computer had computed and printed the expansion of e to 100,256 decimal places. (Shanks & Wrench, 1961).

In Appendix III is printed e to 2,450 decimal places.

It has been predicted by Shanks and Wrench that to compute e to one million decimal places

> "It would take months and then the memory would be too small. One would really want a computer 100 times as fast, 100 times as reliable and 10 times as large." (p. 78)

With the rapid advance of technology, computers are faster, more reliable and larger. We can expect e to one million decimal places to be computed soon.

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# APPENDICES

## APPENDIX I

Conspectus of the historical development of the concept of 'e'.

Hyperbola - from antiquity

Natural logarithms - 1618

Hyperbolic Logs - 1647

Exponential Series - 1665

Logarithms as Exponents - 1695

Exponential and Trigonometric Relation - 1706

The number 'e' - 1728

Limit of Series

Limit of continued fraction

$\lim_{n \to \infty} \left[1 + \frac{1}{n}\right]^n$

Approximations

Irrationality of 'e' - 1761

Transcendence of 'e' - 1873

APPENDIX II

Significant Contributions of men in the history of e

| DATE | NAME | CONTRIBUTION |
| --- | --- | --- |
| 1614 | Napier, John | Invented logarithms |
| 1618 | Oughtred, William | Table of logarithms to base e |
| 1624 | Briggs, Henry | Table of logarithms to base 10 |
| 1647 | Vincent, Gregory St. | Quadrature of hyperbola. |
| 1655 | Wallis, John | Method of quadrature of hyperbola in the invention of interpolation. |
| 1667 | Gregory, James | Computed logarithms by means of a series of inscribed and circumscribed polygons. |
| 1666 | Newton, Isaac | Expansion of series by the binomial theorem. |
| 1676 | Leibnez, G. W. | Developed series of vers $x$, $\sin x$, $\cos x$, $e^x$, $e^{-x}$. |
| 1676 | Newton, Isaac | Developed exponential series |
| 1694 | Leibniz, G. W. | Relationship between calculus and exponential function. |
| 1695 | Bernoulli, John | Determined an expansion for $x \log x$ in a series. |
| 1695 | Halley, Edmund | Logarithms independently of hyperbola. |
| 1722 | Cotes, Roger | Relation between the exponential and trigonometric function. |
| 1727 | Euler, Leonhard | Symbol of e |
| 1740 | Euler, Leonhard | Cos x and Sin x in terms of $e^x$. |

| | | |
|---|---|---|
| 1740 | Euler, Leonhard | Computed e to 23 decimal places. |
| 1748 | Euler, Leonhard | Developed series for e. |
| 1749 | Jones, William | Logarithms are exponents. |
| 1761 | Lambert, J. H. | Proved e irrational. |
| 1849 | Studnicke, F. J. | e to 113 decimal places. |
| 1873 | Hermite, Charles | Proved e transcendental. |
| 1875 | Glaisher, | $1/(e - 1) = .581976706869...$ |
| 1884 | Boorman, J. W. | e to 345 decimal places. |
| 1952 | Gruenberger and Marlowe | e to 3000 decimal places. |
| 1953 | Page and Pfeil | e to 3333 decimal places. |
| 1953 | Wheeler, David | e to 60,000 decimal places. |
| 1961 | Simon, H. G. | e to 20,000 decimal places. |
| 1961 | Wrench and Shanks | e to 100,256 decimal places. |

## APPENDIX III

### Quadrature of a hyperbola

Due to the profound effect that the quadrature of a hyperbola had on the development of the logarithms, it was felt that a brief discussion on the subject should appear in this work.

An equation for an ellipse is $x^2/a^2 + y^2/b^2 = 1$. The area of the ellipse derived by the calculus is $\pi ab$. With the circle being a special ellipse, the irrational and trandcendental number $\pi$ was discussed in great depth by the Greeks. Not as popular, but certainly as important was the area between the hyperbola and a chord perpendicular to the transverse axis. An equation for a hyperbola is $x^2/a^2 - y^2/b^2 = 1$. The area between the curve and a chord found by calculus involves the number $e$ in its inverse form, that of a natural logarithm. Following is the quadrature of a hyperbola, using the calculus.

The graph of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is given below. The area to be found is shaded.

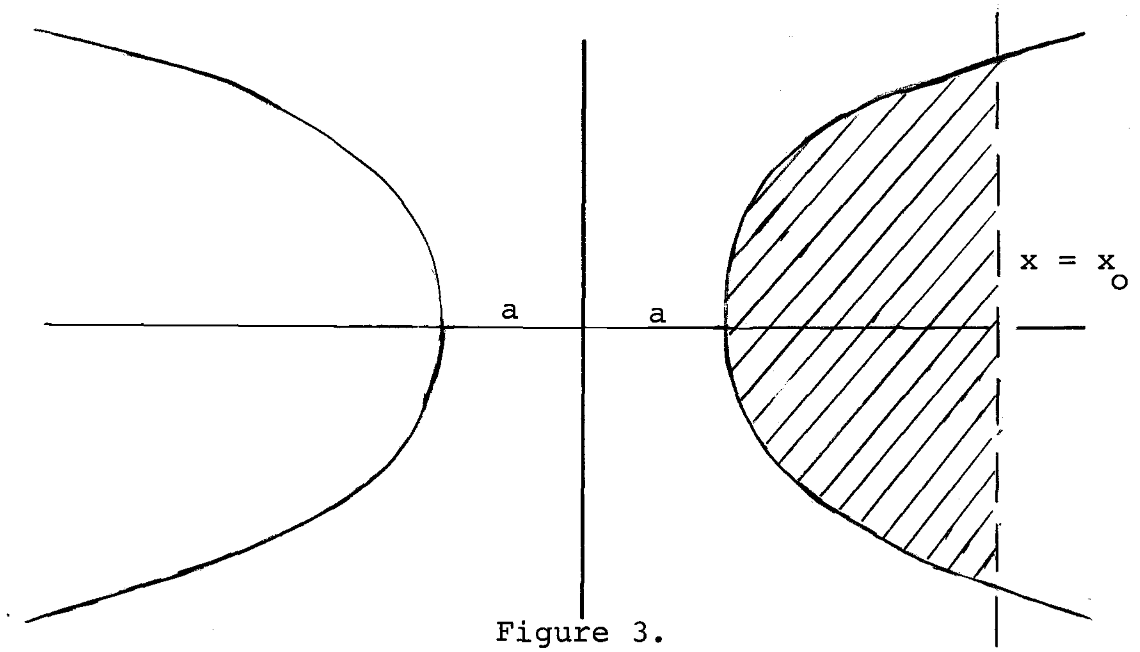


Figure 3.

$$A = \frac{b}{a}\int_a^{x_o} (x^2 - a^2)^{\frac{1}{2}}\, dx$$

$$= \frac{b}{a}\left[\tfrac{1}{2}\, x(x^2 - a^2)^{\frac{1}{2}} - \tfrac{1}{2}a^2 \ln\left(x + (x^2 - a^2)^{\frac{1}{2}}\right)\right]_a^{x_o}$$

$$= \frac{b}{a}\left[\tfrac{1}{2} x_o(x_o^2 - a^2)^{\frac{1}{2}} - \tfrac{1}{2}a^2 \ln(x_o + (x_o^2 - a^2)^{\frac{1}{2}}) + \frac{b}{a}(\tfrac{1}{2}a^2 \ln a)\right]$$

$$= \frac{x_o b}{2a}(x_o^2 - a^2)^{\frac{1}{2}} - \tfrac{1}{2}ab \ln\left(x_o + (x_o^2 - a^2)^{\frac{1}{2}}\right) + \tfrac{1}{2}ab \ln a$$

since $y_o = \frac{b}{a}(x_o^2 - a^2)^{\frac{1}{2}}$ , by substitution

$$= \tfrac{1}{2}\left[x_o y_o - ab \ln\left(x_0 + \frac{ay_o}{b}\right) + ab \ln a\right]$$

$$= \tfrac{1}{2}\left[x_o y_0 - ab \ln a\left(\frac{x_o}{a} + \frac{y_o}{b}\right) + ab \ln a\right]$$

$$= \tfrac{1}{2}\left[x_o y_o - ab \ln\left(\frac{x_o}{a} + \frac{y_o}{b}\right) - ab \ln a + ab \ln a\right]$$

$$= \frac{1}{2}\left[x_o y_o - ab \ln\left(\frac{x_o}{a} + \frac{y_o}{b}\right)\right]$$

Since we want the area in the complete arc of the hyperbola, that is, the area between the hyperbola and the chord $x = x_o$ , we must double the above to obtain, in addition, the lower half of the arc. The area becomes

$$= x_o y_o - ab \ln \frac{x_o}{a} + \frac{y_o}{b}$$

The area is in terms of the natural logarithms, inverse of the exponential function.

## APPENDIX IV

### DECIMAL REPRESENTATION OF e TO 2450 DECIMAL PLACES

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 2. | 71828 | 18284 | 59045 | 23536 | 02874 | 71352 | 66249 | 77572 | 47093 | 69995 |
| | 95749 | 66967 | 62772 | 40766 | 30353 | 54759 | 45713 | 82178 | 52516 | 64274 |
| | 27466 | 39193 | 20030 | 59921 | 81741 | 35966 | 29043 | 57290 | 03342 | 95260 |
| | 59563 | 07381 | 32328 | 62794 | 34907 | 63233 | 82988 | 07531 | 95251 | 01901 |
| | 15738 | 34187 | 93070 | 21540 | 89149 | 93488 | 41675 | 09244 | 76146 | 06680 |
| | 82264 | 80016 | 84774 | 11853 | 74234 | 54424 | 37107 | 53907 | 77449 | 92069 |
| | 55170 | 27618 | 38606 | 26133 | 13845 | 83000 | 75204 | 49338 | 26560 | 29760 |
| | 67371 | 13200 | 70932 | 87091 | 27443 | 74704 | 72306 | 96977 | 20931 | 01416 |
| | 92836 | 81902 | 55151 | 08657 | 46377 | 21112 | 52389 | 78442 | 50569 | 53696 |
| | 77078 | 54499 | 69967 | 94686 | 44549 | 05987 | 93163 | 68892 | 30098 | 79312 |
| | 77361 | 78215 | 42499 | 92295 | 76351 | 48220 | 82698 | 95193 | 66803 | 31825 |
| | 28869 | 39849 | 64651 | 05820 | 93923 | 98294 | 88793 | 32036 | 25094 | 43117 |
| | 30123 | 81970 | 68416 | 14039 | 70198 | 37679 | 32068 | 32823 | 76464 | 80429 |
| | 53118 | 02328 | 78250 | 98194 | 55815 | 30175 | 67173 | 61332 | 06981 | 12509 |
| | 96181 | 88159 | 30416 | 90351 | 59888 | 85193 | 45807 | 27386 | 67385 | 98422 |
| | 87922 | 84998 | 92086 | 80582 | 57492 | 79610 | 48419 | 84443 | 63463 | 24496 |
| | 84875 | 60233 | 62482 | 70419 | 78623 | 20900 | 21609 | 90235 | 30436 | 99418 |
| | 49146 | 31409 | 34317 | 38143 | 64054 | 62531 | 52096 | 18369 | 08887 | 07016 |
| | 76839 | 64243 | 78140 | 59271 | 45635 | 49061 | 30310 | 72085 | 10383 | 75051 |
| | 01157 | 47704 | 17189 | 86106 | 87396 | 96552 | 12671 | 54688 | 95703 | 50354 |
| | 02123 | 40784 | 98193 | 34321 | 06817 | 01210 | 05627 | 88023 | 51930 | 33224 |
| | 74501 | 58539 | 04730 | 41995 | 77770 | 93503 | 66041 | 69973 | 29725 | 08868 |
| | 76966 | 40355 | 57071 | 62268 | 44716 | 25607 | 98826 | 51787 | 13419 | 51246 |
| | 65201 | 03059 | 21236 | 67719 | 43252 | 78675 | 39855 | 89448 | 96970 | 96409 |
| | 75459 | 18569 | 56380 | 23637 | 01621 | 12047 | 74272 | 28364 | 89613 | 42251 |
| | 64450 | 78182 | 44235 | 29486 | 36372 | 14174 | 02388 | 93441 | 24796 | 35743 |
| | 70263 | 75529 | 44483 | 37998 | 01612 | 54922 | 78509 | 25778 | 25620 | 92622 |
| | 64832 | 62779 | 33386 | 56648 | 16277 | 25164 | 01910 | 59004 | 91644 | 99828 |
| | 93150 | 56604 | 72580 | 27786 | 31864 | 15519 | 56532 | 44258 | 69829 | 46959 |
| | 30801 | 91529 | 87211 | 72556 | 34754 | 63964 | 47910 | 14590 | 40905 | 86298 |
| | 49679 | 12874 | 06870 | 50489 | 58586 | 71747 | 98546 | 67757 | 57320 | 56812 |
| | 88459 | 20541 | 33405 | 39220 | 00113 | 78630 | 09455 | 60688 | 16674 | 00169 |
| | 84205 | 58040 | 33637 | 95376 | 45203 | 04024 | 32256 | 61352 | 78369 | 51177 |
| | 88386 | 38744 | 39662 | 53224 | 98506 | 54995 | 88623 | 42818 | 99707 | 73327 |
| | 61717 | 83928 | 03494 | 65014 | 34558 | 89707 | 19425 | 86398 | 77275 | 47109 |
| | 62953 | 74152 | 11151 | 36835 | 06275 | 26023 | 26484 | 72870 | 39207 | 64310 |
| | 05958 | 41166 | 12054 | 52970 | 30236 | 47254 | 92966 | 69381 | 15137 | 32275 |
| | 36450 | 98889 | 03136 | 02057 | 24817 | 65851 | 18063 | 03644 | 28123 | 14965 |
| | 50704 | 75102 | 54465 | 01172 | 72115 | 55194 | 86685 | 08003 | 68532 | 28183 |
| | 15219 | 60037 | 35625 | 27944 | 95158 | 28418 | 82947 | 87610 | 85263 | 98139 |
| | 55990 | 06737 | 64829 | 22443 | 75287 | 18462 | 45780 | 36192 | 98197 | 13991 |
| | 47564 | 48826 | 26039 | 03381 | 44182 | 32625 | 15097 | 48279 | 87779 | 96437 |
| | 30899 | 70388 | 86778 | 22713 | 83605 | 77297 | 88241 | 25611 | 90717 | 66394 |
| | 65070 | 63304 | 52795 | 46618 | 55096 | 66618 | 56647 | 09711 | 34447 | 40160 |
| | 70462 | 62156 | 80717 | 48187 | 78443 | 71436 | 98821 | 85596 | 70959 | 10259 |
| | 68620 | 02353 | 71858 | 87485 | 69652 | 20005 | 03117 | 34392 | 07321 | 13908 |
| | 03293 | 63447 | 97273 | 55955 | 27734 | 90717 | 83793 | 42163 | 70120 | 50054 |
| | 51326 | 38354 | 40001 | 86323 | 99149 | 07054 | 79778 | 05669 | 78533 | 58048 |
| | 96690 | 62951 | 19432 | 47309 | 95876 | 55236 | 81285 | 90413 | 83241 | 16072 |

APPENDIX V

This table appears from the second edition (1618) of Edward Wright's English translation of Napier's Descripto. According to Glasier (1915) it was probably written by William Oughtred. This is the first logarithm table to the base e. In modern notation it is $10^6 \ln_e N$.

| sin | logarithm | sin | logarithm | sin | logarithm |
|---|---|---|---|---|---|
| 1 | 000000 | 100 | 4605168 | 10000 | 9210337 |
| 2 | 693146 | 200 | 5298314 | 20000 | 9803483 |
| 3 | 1098612* | 300 | 5703780 | 30000 | 10308949 |
| 4 | 1386294 | 400 | 5991462 | 40000 | 10596631 |
| 5 | 1609437 | 500 | 6214605 | 50000 | 10819774 |
| 6 | 1791758 | 600 | 6396925 | 60000 | 11002095 |
| 7 | 1045905 | 700 | 6551077 | 70000 | 11156246 |
| 8 | 2079441 | 800 | 6684609 | 80000 | 11289778 |
| 9 | 2197223 | 900 | 6802391 | 90000 | 11407560 |
| 10 | 2302584 | 1000 | 6907753 | 100000 | 11512921 |
| 20 | 2995730 | 2000 | 7600899 | 200000 | 12206067 |
| 30 | 3401196 | 3000 | 8006365 | 300000 | 12611533 |
| 40 | 3688878 | 4000 | 8294047 | 400000 | 12899215 |
| 50 | 3911021 | 5000 | 8517190 | 500000 | 13122358 |
| 60 | 4094342 | 6000 | 8699511 | 600000 | 13304679 |
| 70 | 4248493 | 7000 | 8853662 | 700000 | 13458830 |
| 80 | 4382025 | 8000 | 8987194 | 800000 | 13592362 |
| 90 | 4499807 | 9000 | 9104976 | 900000 | 13710114 |

The supplement of the table for tenth and hundreths parts

| | | | | | |
|---|---|---|---|---|---|
| 11 | 095311 | 17 | 530628 | 104 | 39222 |
| 12 | 182321 | 18 | 587786 | 105** | 48790 |
| 13 | 262364 | 19 | 641953 | 106 | 58269 |
| 14 | 336473 | 101 | 9951 | 107 | 67659 |
| 15 | 405465 | 102 | 19803 | 108 | 76962 |
| 16 | 470004 | 103 | 29560 | 109 | 86177 |

* This logarithm is printed 1096612 in the _Appendix_.

** Printed 126 in the _Appendix_.

APPENDIX VI

Napier's Original (1614) Logarithms

Gr. 0

| min | Sinus | Logarithmi | Differentia | Logarithmi | Sinus | | min | Sinus | Logarithmi | Differentia | Logarithmi | Sinus | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | infinitum | infinitum | 0 | 10000000 | 60 | 30 | 87265 | 47413852 | 47413471 | 381 | 9999619 | 30 |
| 1 | 2909 | 81425681 | 81425680 | 1 | 10000000 | 59 | 31 | 90174 | 47085961 | 47085554 | 407 | 9999593 | 29 |
| 2 | 5818 | 74494213 | 74494211 | 2 | 9999998 | 58 | 32 | 93083 | 46768483 | 46768049 | 434 | 9999566 | 28 |
| 3 | 8727 | 70439564 | 70439560 | 4 | 9999996 | 57 | 33 | 95992 | 46460773 | 46460312 | 461 | 9999539 | 27 |
| 4 | 11636 | 67562746 | 67562739 | 7 | 9999993 | 56 | 34 | 98901 | 46162254 | 46161765 | 489 | 9999511 | 26 |
| 5 | 14544 | 65331315 | 65331304 | 11 | 9999989 | 55 | 35 | 101809 | 45872392 | 45871874 | 518 | 9999482 | 25 |
| 6 | 17453 | 63508099 | 63508083 | 16 | 9999986 | 54 | 36 | 104718 | 45590688 | 45590140 | 548 | 9999452 | 24 |
| 7 | 20362 | 61966595 | 61966573 | 22 | 9999980 | 53 | 37 | 107627 | 45316714 | 45316135 | 579 | 9999421 | 23 |
| 8 | 23271 | 60631284 | 60631256 | 28 | 9999974 | 52 | 38 | 110536 | 45050041 | 45049430 | 611 | 9999389 | 22 |
| 9 | 26180 | 59453453 | 59453418 | 35 | 9999967 | 51 | 39 | 113445 | 44790296 | 44789652 | 644 | 9999357 | 21 |
| 10 | 29088 | 58399857 | 58399814 | 43 | 9999959 | 50 | 40 | 116353 | 44537132 | 44536455 | 677 | 9999323 | 20 |
| 11 | 31997 | 57446759 | 57446707 | 52 | 9999950 | 49 | 41 | 119262 | 44290216 | 44289505 | 711 | 9999289 | 19 |
| 12 | 34906 | 56576646 | 56576584 | 62 | 9999940 | 48 | 42 | 122171 | 44049255 | 44048509 | 746 | 9999254 | 18 |
| 13 | 37815 | 55776222 | 55776149 | 73 | 9999928 | 47 | 43 | 125079 | 43813959 | 43813177 | 782 | 9999218 | 17 |
| 14 | 40724 | 55035148 | 55035064 | 84 | 9999917 | 46 | 44 | 127988 | 43584078 | 43583259 | 819 | 9999181 | 16 |
| 15 | 43632 | 54345225 | 54345129 | 96 | 9999905 | 45 | 45 | 130896 | 43359360 | 43358503 | 857 | 9999143 | 15 |
| 16 | 46541 | 53699843 | 53699734 | 109 | 9999892 | 44 | 46 | 133805 | 43139582 | 43138686 | 896 | 9999105 | 14 |
| 17 | 49450 | 53093600 | 53093577 | 123 | 9999878 | 43 | 47 | 136714 | 42924534 | 42923599 | 935 | 9999065 | 13 |
| 18 | 52359 | 52522019 | 52521881 | 138 | 9999863 | 42 | 48 | 139622 | 42714014 | 42713039 | 975 | 9999025 | 12 |
| 19 | 55268 | 51981356 | 51981202 | 154 | 9999847 | 41 | 49 | 142531 | 42507833 | 42506817 | 1016 | 9998984 | 11 |
| 20 | 58177 | 51468431 | 51468361 | 170 | 9999831 | 40 | 50 | 145439 | 42305826 | 42304768 | 1058 | 9998942 | 10 |
| 21 | 61086 | 50980537 | 50980450 | 187 | 9999813 | 39 | 51 | 148348 | 42107812 | 42106711 | 1101 | 9998900 | 9 |
| 22 | 63995 | 50515342 | 50515137 | 205 | 9999795 | 38 | 52 | 151257 | 41913644 | 41912499 | 1145 | 9998856 | 8 |
| 23 | 66904 | 50070827 | 50070603 | 224 | 9999776 | 37 | 53 | 154165 | 41723175 | 41721986 | 1189 | 9998811 | 7 |
| 24 | 69813 | 49645239 | 49644995 | 244 | 9999756 | 36 | 54 | 157074 | 41536271 | 41535037 | 1234 | 9998766 | 6 |
| 25 | 72721 | 49237030 | 49236765 | 265 | 9999736 | 35 | 55 | 159982 | 41352795 | 41351515 | 1280 | 9998720 | 5 |
| 26 | 75630 | 48844826 | 48844539 | 287 | 9999714 | 34 | 56 | 162891 | 41172626 | 41171299 | 1327 | 9998673 | 4 |
| 27 | 78539 | 48467431 | 48467122 | 309 | 9999692 | 33 | 57 | 165799 | 41006643 | 41005268 | 1375 | 9998625 | 3 |
| 28 | 81448 | 48103763 | 48103431 | 332 | 9999668 | 32 | 58 | 168708 | 40821746 | 40820322 | 1424 | 9998577 | 2 |
| 29 | 84357 | 47752859 | 47752503 | 356 | 9999644 | 31 | 59 | 171616 | 40650816 | 40649343 | 1473 | 9998527 | 1 |
| 30 | 87265 | 47413852 | 47413471 | 381 | 9999619 | 30 | 60 | 174524 | 40482764 | 40481241 | 1523 | 9998477 | 0 |

Gr. 89 min