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The theory and numerical calculation of physical fields that are governed by linear partial differential equations of the elliptic, parabolic and hyperbolic types is a topic of fundamental interest in the qualitative theory of various resources. Often, field situations require the solution of the basic equations for different types of domains with common boundaries. It will be shown that the introduction of fractional powers of the Laplacian can be a helpful device in such cases.

The general setting involves coupled systems of P.D.E.'s defined over two regions with a common boundary segment, Σ. These mixed boundary value problems are then reduced to a single surface equation on Σ with the dependency on one variable eliminated by the introduction of an operator H in the original boundary condition. This operator, which arises as a fractional power of the Laplacian on Σ, is our main mathematical tool.

Several models of physical fields that are well suited to a

reformulation in terms of H have been investigated by G. Bodvarsson. In this thesis we will consider the example of the linearized equations for internal water waves. This involves modeling of waves of infinitesimal amplitude of a surface interface, under the influence of gravity and surface tension, in a homogeneous incompressible and inviscid fluid.

While the use of H is highly motivated through specific physical examples the main analytical results of this work are obtained in considering its use in a more general format. A reformulation of the basic equations governing internal water waves leads to a new type of wave operator with the spacial part of the operator arising as a fractional power of the Laplacian. In considering a mixed problem for this wave equation, an existence and uniqueness theorem is obtained within two settings. A class of generalized solutions is defined, which is well suited for a formal eigenfunction expansion solution technique, allowing existence and uniqueness in an L^2 sense. Further, a result on eigenfunction expansions of distributions with compact support is given. This leads to existence, by the same solution technique, to the generalized wave equation in a distributional sense.

FRACTIONAL POWERS OF THE LAPLACIAN WITH APPLICATION TO SMALL AMPLITUDE WATER WAVES

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FRACTIONAL POWERS OF THE LAPLACIAN WITH APPLICATION TO SMALL AMPLITUDE WATER WAVES

Introduction

The theory and numerical calculation of physical fields that are governed by linear partial differential equations of the elliptic, parabolic and hyperbolic types is a topic of fundamental interest in the qualitative theory of various resources. For example, fluid pressure fields in petroleum, natural gas, water and geothermal reservoirs are governed by equations of the parabolic type. The propagation of seismic signals in subsurface formations involves equations of the hyperbolic type. Finally, the fields of potential type in exploration methods are governed by elliptic equations.

Often, field situations require the solution of the basic equations on domains consisting of different types of regions with common boundaries. It will be shown that the introduction of fractional powers of the Laplacian can be a helpful device in simplifying and clarifying some aspects of the underlying theory in such cases. The general setting involves mixed boundary and initial value problems consisting of coupled systems of P.D.E.'s defined over two regions with a common boundary segment, Σ. By the introduction of an operator H in the original boundary condition this type of problem is then reduced to a single surface equation on Σ such that the dependency on one variable is eliminated. This operator, that is referred to as a cross-surface differential operator and is our main mathematic tool arises as a fractional power of the Laplacian on Σ .

The main application to be considered in this paper involves linearized equations for surface and internal water waves of infinitesimal amplitude in homogeneous or layered incompressible and inviscid liquids. The case of gravity waves on deep water is considered as a simple special case that portrays the precise setting which motivates the definition of our cross-surface differential operator. Through the use of H, the basic equations governing such water waves are reformulated resulting in a single scalar equation for the amplitude of the wavelike motion off the horizontal surface Σ.

The theories of gravity and internal water waves has been of interest to mathematicians and physicists for a long time. Early work on the subject can be traced back to Cauchy and Poisson at the start of the nineteenth century and research in the field is still very active. A few well known books in this area giving an account of the classical theory as well as containing more recent results are, J.J. Stoker, Water Waves (1957), O.M. Phillips, The Dynamics of the Upper Ocean (1966) and B. Kinsman, Wind Waves (1965). Although it is clear that the theory is highly developed it is interesting to note that, until recently, no single equation has been set forth that could be referred to as the equation for gravity or internal water waves of infitesimal amplitude. Other oscillatory scalar fields are governed by well defined wave equations that provide the basis for theoretical investigations. An example is the central role played by d'Alembert's equation in the theory of sound . G. Bodvarsson has derived such an equation for the case of gravity waves (1977, [1]) and has more recently applied this technique to internal water waves under the influence of both gravity and surface tension.

In the first two chapters of the present thesis, the setting for the definition of the cross-surface differential operator H is developed along with an eigenfunction expansion representation. Subsequently, in Chapter III, we derive a new type of wave equation where the spatial part of the systems operator arises as a cross-surface operator that is a fractional power of the Laplacian. Several other models of physical fields which are well suited to a reformulation in terms of such operators have been investigated by Bodvarsson (1977 [2]) and this author leading to equations of the parabolic type. Further work on these examples, following the scope of this thesis, will be forthcoming.

The use of H is highly motivated through specific physical examples and, while one concern of this thesis is to develope a rigorous setting for its application to the water wave model, the main analytical results are obtained in a more general setting. A mixed initial and boundary value problem involving the wave operator $\partial_{tt} + c^2 H$ is investigated where an existence and uniqueness theorem is obtained within two settings. In Chapter IV we apply the classical method of energy integrals to mixed typed problems governed by equations of the hyperbolic type (Vladimirov, 1971). Upon defining an energy integral associated with a suitably smooth solution to the mixed problem a key result is obtained on the representation of this integral in terms of the data. This leads to L^2-norm estimates of solutions and first time derivatives in terms of the data which immediately yields uniqueness and continuous dependence theorems for this class of solutions. Existence, however, presents more difficulties. The main solution technique is by a formal eigenfunction expansion. That is, we assume an eigenfunction expansion for the solution and derive an ordinary differential equation for the undetermined coefficients by formally carrying the operations and initial conditions through the summation. A class of generalized solutions is defined which is well suited for this expansion technique. It is verified that the L^2-estimates obtained earlier also hold for this class of solutions allowing for the main theorem on existence, uniqueness and continuous dependence on the data to be proven.

The above results demonstrate that a formal eigenfunction expansion solution technique yields existence in an L^2-sense. The final section of Chapter IV is devoted to showing further properties of a generalized solution which will attest to the suitability of this class of solutions not only for the solution method used but also for the problem at hand. By making use of the distributional setting for H developed in Chapter II, it is shown in Chapter V that the same solution technique yields existence in a distributional sense. In fact, far weaker restrictions can be imposed on the input data. This leads to the formulation of a generalized wave equation where the initial data and the causal source term are distributions of compact support. An existence theorem for this problem is proven in the last section of Chapter V.

I. THE CROSS-SURFACE DIFFERENTIAL OPERATOR H

1.1 Preliminary Considerations on a Symbolic Calculus for Functions of an Operator

The main goal of this chapter is to develop, through a constructive procedure an operator H, which behaves like the square-root of the negative 2-dimensional Laplacian that we denote by

$$\Pi = -\Delta_2 = -(\partial_{xx} + \partial_{yy}) .$$

As mentioned in the introduction, this operator plays a central role in the reformulation of the equations for the water waves that are derived in chapter III. On a more theoretical slant, we want to emphasize the square-root nature of this operator and take up an analysis of the resulting new type of wave equation. Here, a square-root of an operator is with respect to the operation of composition. Hence, provided the domains are properly defined, B is a square-root of A if and only if

$$B^2 = B \circ B = A .$$

This first section will be devoted to a brief account of a symbolic calculus for functions of operators. More specifically, we will be concerned about the meaning of functions of an operator f(A), for what class of functions such operators exist and what are proper representations of these operators. This is a subject found in most texts on functional analysis or operator theory with the spectral theorems as one of the main results of interest. Here, the emphasis lies in obtaining a representation of f(A) that resembles the diagonalization of matrices in the finite dimensional case. Briefly, these results go as follows.

Consider the case when A is a bounded, self-adjoint operator on a Hilbert Space. Then a representation for A is obtained (Schecter, 1971) in terms of a family of orthogonal projection operators $\{E(\lambda)\}$ such that

$$A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda) \; .$$

Since A is bounded and self-adjoint its spectrum $\sigma(A)$ is contained in a closed interval [a,b] of the real line. Hence, the integral is over a finite length and is to be understood as the limit under the operator norm in the space of bounded linear operators of sums of the form

$$\sum_{k=1}^{n} \lambda_k' \, [E(\lambda_k) - E(\lambda_{k-1})] \quad \text{as} \max_k \, (\lambda_k - \lambda_{k-1}) \to 0 \quad \text{where}$$

$\{\lambda_k\}$ partions [a,b] and λ_k' is bounded by $\lambda_{k-1} \leq \lambda_k' \leq \lambda_k$. The family, $\{E(\lambda)\}$, is called the <u>Resolution of the Identity</u> corresponding to A. As a consequence of the spectral theorem we can define for any continuous function f on [a,b]

$$f(A) = \int_a^b f(\lambda)\, dE(\lambda),$$

satisfying

$$||f(A)|| = \max_{\lambda\varepsilon\sigma(A)} |f(\lambda)| \ .$$

These definitions can be generalized to the following two cases (Rudin, 1973).

case i) Take A to be a normal operator. Then we obtain the analogous representation

$$A = \int_{\sigma(A)} \lambda\, dE(\lambda) \ \text{, to be interpreted as; } (Ax,y) = \int_{\sigma(A)} \lambda\, d(E(\lambda)x,y)$$

for all $x,y \varepsilon H$ where $(.,.)$ is the inner product. Also, the definitions are generalized to include bounded Borel functions f on $\sigma(A)$ and denoted by

$$f(A) = \int_{\sigma(A)} f(\lambda)\, dE(\lambda) \ \text{, where } ||f(A)|| \leq \sup \{|f(\lambda)| : \lambda\varepsilon\ \sigma(A)\} \ ,$$

with equality holding for $f \varepsilon\ C(\sigma(A))$.

case ii) A is a linear, normal not necessarily bounded operator with a dense domain of definition. The above calculus now applies to all measurable functions on $\sigma(A)$.

When A is assumed to have a more simple structure, these spectral representations bear a simple relation to the eigenvalues of A. Suppose $A \varepsilon B(H)$ is normal and has a countable spectrum $\sigma(A) = \{\lambda_1, \lambda_2 \ldots\}$ consisting of eigenvalues of A (in this case, the isolated points of $\sigma(A)$ are necessarily eigenvalues) along with a complete orthonormal system (C.O.S.) of eigenfunctions $\{x_i\}$ to A, i.e., $Ax_i = \lambda_i x_i$ for all $i = 1,2,\ldots,$ where any two eigenfunctions corresponding to distinct eigenvalues are orthogonal and every $x \varepsilon H$ has a unique expansion of the form, $x = \sum_i \alpha_i x_i$. From this expansion we immediately obtain the representation,

$$Ax = \sum_i \lambda_i \alpha_i x_i \tag{1.1}$$

which can also be derived from a simplification of the spectral representation under the above conditions. Notice, the formal calculus now obeys the rule of carrying the function inside to the eigenvalues, i.e.,

$$f(A)x = \sum_i f(\lambda_i)\ \alpha_i x_i \ .$$

Another approach (Schecter, 1971) is via a Cauchy integral type formula. Let $A \varepsilon B(X)$, with X a complex Banach space. Then $(z-A)^{-1}$ is an analytic function of z on the resolvent of A, $\rho(A)$ (where $\lambda \varepsilon \rho(A)$ iff the null space $N(A-\lambda) = \{o\}$ and the range $R(A-\lambda) = X$, in which case $(A-\lambda)^{-1} \varepsilon B(X)$.). Then we obtain the representation,

$$A = \frac{1}{2\pi i} \int_c z\ (z-A)^{-1} dz\ ; \tag{1.2}$$

where c is any curve containing $\sigma(A)$ in its interior. Moreover, if $f(z)$ is analytic on an open set Ω with $\sigma(A) \subset \Omega$, then we can always find an open set ω with $\sigma(A) \subset \omega \subset \bar{\omega} \subset \Omega$, whose boundary $\partial\omega$ consists of a finite number of non-intersecting simple closed curves. Then

$$\frac{1}{2\pi i} \int_{\partial\omega} f(z)\ (z-A)^{-1} dz \tag{1.3}$$

defines an operator in $B(X)$ and is independent of the choice of ω. In this setting the formal calculus defines $f(A)$ as the integral in (1.3).

The representation given in (1.2) can, in many cases, be shown to reduce to an eigenfunction expansion such as in (1.1) presented earlier. In fact, when A is a differential operator arising within certain Sturm-Liouville problems, this approach is taken (Friedman, 1956). In such cases, the resolvent operator, which now plays the role of the integral operator induced by the Green's function for the differential equation, can be determined and the integration is explicitly carried out to obtain an eigenfunction expansion.

We will make frequent use of eigenfunction expansions for representations of operators and as a solution technique. The representation (1.1) therefore makes a good starting point for the problems to be considered below. However, our setting will involve a differential operator arising from a boundary value problem (BVP) context. This adds several complications to the general setting outlined above. In general, differential operators are unbounded and we must restrict the setting considerably to obtain the simple structure as above. Furthermore, within the context of a BVP, it is important to recognize the central role played by the domain of definition, that is, the specified boundary conditions and underlying function space. This point will be illustrated soon through an example.

In the next section, we will outline a setting, that is, a class of differential operators including regions of definition and boundary conditions, that allows the simple structure mentioned above. These operators have an eigenfunction expansion and the corresponding formal calculus will apply. In particular, the operator Π is included and consequently a square-root operator is obtained by carrying the square-root inside to the eigenvalues. However, some confusion can arise with the notion of a square-root. In general, if an operator has a square-root it need not be unique.

A well known result along these lines (Schecter, 1971) states that every bounded, positive operator on a Hilbert space has a unique bounded, positive square-root. This can be generalized to unbounded, non-negative, self-adjoint operators (Weidman, 1980) and, in all cases, the square-root is obtained via the spectral representation. Within the setting to be developed, Π will be a symmetric[1], positive operator of simple structure. To be precise, from now on a simple operator will refer to an operator whose spectrum consists of a countable set of eigenvalues of finite multiplicity (the multiplicity is the dimension of the eigenspace) and having a C.O.S. of eigenfunctions. We have in mind a preferred square-root of Π that also exhibits these same properties. Essentially, this square-root is obtained from the formal calculus.

Consider the following example with the differential operator, $-D^2$, where $D = {}^{d}/dx$. This is an easy enough operation and we can immediately recognize a square-root operation given by either $\pm iD$. But, we haven't as yet completely defined the operator to be con-

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(1) Some care is needed with this terminology. Many texts will refer to a differential operator as self-adjoint or formally self-adjoint when, on the domain specified, it is only symmetric. A is symmetric on a Hilbert space iff $(Au,v) = (u,Av)$ for all $u,v \in D_A$, in which case, the adjoint A^* is an extension of A but is not necessarily equal to A. It can be the case that by enlarging the domain of the operator, by closing the operator, it is then self-adjoint. Such an operator, whose closure is self-adjoint, is called essentially self-adjoint. Since, in most cases we are only interested in the symmetric nature of Π, the precise domain to obtain a closed, self-adjoint operator won't be specified.

sidered. A domain of definition has to be specified. Clearly, different choices of boundary conditions and underlying function spaces result in operators with entirely different characteristics. We will present an example where the expressions $\pm iD$ have no relation to the preferred notion of a square-root that is to be used.

In particular, lets take, $\Pi = -D^2$ defined on the interval, $(0,1)$, with the domain

$$D_{\Pi} = A \cap \{u(x) : u(o) = u(1) = o\} \text{ , where}$$

$$A = \{u \varepsilon L^2(0,1) : u' \text{ is absolutely continuous on } (0,1),\ \ u'' \varepsilon L^2(0,1)\}.$$

A is, in a classical sense, the largest class of functions we can apply $-D^2$ on and stay within an L^2-setting. The set A was chosen over a smooth class of functions such as $C^2[0,1]$ so that Π defines a closed, self-adjoint operator. In fact, if we define operators, Π_i for $i = 0,1,2$ by

$$D_{\Pi_0} = C_c^{\infty}(0,1) \text{ (infinitely differentiable functions with compact support)}$$

$$D_{\Pi_1} = \{u \varepsilon C^2(0,1)\ C[0,1] \text{ with } u'' \varepsilon L^2(0,1) \text{ and } u(0) = u(1) = 0\}$$

$$D_{\Pi_2} = A$$

where for all $i = 0,1,2$, Π_i is induced by $-D^2$; (Π_0 and Π_2 are known as the minimal and maximal operators, resp. induced by $-D^2$) then it can be shown (Weidman, 1980) that

$$\Pi_0^* = \Pi_2 \text{ , so } \Pi_2^* = \Pi_0^{**} = \bar{\Pi}_0 \text{ (the closure of } \Pi_0) \text{ ,}$$

$$D_{\bar{\Pi}_0} = A \quad \{u : u(0) = u'(0) = u(1) = u'(1) = 0\} \text{ and}$$

$$\bar{\Pi}_1 = \Pi \text{ .}$$

This shows that, although Π_0 is a symmetric operator, neither Π_0 nor Π_2 are self-adjoint or essentially self-adjoint. Also, this illustrates how, by enlarging the domain of Π_1, we obtain the closed operator Π. As to the domain of the adjoint, intuitively the situation is as follows. First, the adjoint is always closed and we must have the minimal homogeneous boundary conditions necessary to eliminate all boundary terms that arise upon integration by parts of, $(\Pi_i u,v)$, when transferring the differentiation to v. Since the boundary terms are,

$$-u'(x)\ v(x) + u(x)\ v'(x) \Big|_{x=0}^{x=1} \text{ ,}$$

it is only necessary to specify homogenous boundary conditions on the functions themselves (not, also, their derivatives). Hence, Π defines a self-adjoint operator (Weidman, 1980). For our purposes it is sufficient to deal with the symmetric and essentially self-adjoint operator, Π_1. The precise domain of its closure was given in this one dimensional case since it was relatively easy to come by.

In this setting, the normalized eigenfunctions and eigenvalues for Π are,

$$u_n(x) = \sqrt{2}\ \sin(n\pi x) \ , \ \lambda_n = n^2\pi^2 \ , \text{ for } n = 1,2,\ldots \ .$$

So, we do indeed have a simple operator with the C.O.S. of eigenfunctions giving rise to Fourier Sine expansions,

$$u(x) = \sum_{n=1}^{\infty} (u,u_n)\ u_n(x) \ ,$$

with convergence in $L^2(0,1)$ for all $u \varepsilon L^2(0,1)$. Of course, in this case quite a bit more can be said concerning convergence of Fourier series. For example, a result which will hold in the more general setting also, for every $u \varepsilon D_\Pi$, we have regular convergence (uniform convergence of the series of absolute values) on [0,1]. Also, the following is the eigenfunction expansion for Π,

$$\Pi\, u(x) = \sum_{n=1}^{\infty} n^2\pi^2 \,(u,u_n)\, u_n(x) = \sqrt{2} \sum_{n=1}^{\infty} n^2\pi^2\, \tilde{u}_n \sin(n\pi x) \ ,$$

where the notation $\tilde{u}_n$ refers to the expansion coefficient,

$$\tilde{u}_n = (u,u_n) = \sqrt{2} \int_0^1 u(x) \sin(n\pi x)\, dx \ .$$

Now, according to our formal calculus developed earlier, a square-root of Π is given by,

$$\Pi^{1/2} u(x) = \sqrt{2} \sum_{n=1}^{\infty} n\pi\, \tilde{u}_n \sin(n\pi x) \ . \tag{1.4}$$

What of its domain? In general, when dealing with a square-root of a differential operator the question of appropriate boundary conditions and underlying function space would appear to be a major problem. The above construction frees us of these considerations. Notice, we are not attempting to view $\Pi^{1/2}$ as induced by some differential expression (this may not be possible). Also, recall that one purpose in this example was to show that $\pm$ iD may have no bearing on the construction of a square-root of $-D^2$. In the setting chosen this is certainly the case since a general expansion for $\pm iu'(x)$ given by (1.4) doesn't hold. In fact, if the main concern is in obtaining an eigenfunction expansion representation for an operator that is to behave as a square-root of Π in an L^2-sense and maintains the same properties inherent to Π; then, (1.4) is the starting point for the definition of $\Pi^{1/2}$. The only necessary stipulation is to stay within an L^2-setting, i.e., that $\Pi^{1/2} u \in L^2(0,1)$. Hence, if we impose this restriction on its domain and set,

$$D_{\Pi^{1/2}} = \{u \in L^2(0,1) : \sum_{n=1}^{\infty} n^2\pi^2 \; |\tilde{u}_n|^2 < \infty\}$$

and define $\Pi^{1/2} u(x)$ by (1.4); then we can show directly that this gives a self-adjoint, simple operator where, as expected, its spectrum consists of the eigenvalues, $n\pi$, with the corresponding C.O.S. of eigenfunctions, $u_n(x) = \sqrt{2}\, \sin(n\pi x)$, for $n = 1,2 \ldots$. Moreover, by construction,

$$(\Pi^{1/2})^2 \; u(x) = \sqrt{2} \sum_{n=1}^{\infty} n^2\pi^2 \; \tilde{u}_n \; \sin(n\pi x) = \Pi \; u(x) \quad , \text{ in } L^2(0,1) \; .$$

To finish these preliminary considerations let's take a slight variation on the domain of Π . Set,

$$D_{\Pi} = A \cap \{u(x) : u(o) = u(1), \; u'(0) = u'(1)\} \; .$$

So, we now have periodic boundary conditions instead of the homogeneous Dirichlet condition of the last example. Again, it is known that Π is a self-adjoint, simple operator. The eigenvalues and eigenfunctions for Π are,

$$\lambda_k = (2k\pi)^2 \ , \ u_k(x) = e^{\pm i2k\Pi x} \ , \text{ for } k = 0,1,2,\ldots$$

(we now have multiplicity 2). Hence, the eigenfunction representations for Π and the square-root $\Pi^{1/2}$ are,

$$\Pi u(x) = \sum_{k=-\infty}^{\infty} (2k\pi)^2 \ \tilde{u}_k \ e^{i2k\pi x} \ ,$$

$$\Pi^{1/2} u(x) = \sum_{k=-\infty}^{\infty} (2k\pi) \ \tilde{u}_k \ e^{i2k\pi x} \tag{1.5}$$

with domain consisting of all $L^2(0,1)$ functions, $u(x)$, such that

$$\sum_{k=-\infty}^{\infty} (2k\pi)^2 \ |\tilde{u}_k|^2 < \infty \ .$$

Next, define an operator T, induced by iD, with domain

$$D_T = \{u \varepsilon L^2(0,1) : u \text{ is absolutely continuous on } (0,1),\ u' \varepsilon L^2(0,1),\ u(0) = u(1)\} \ .$$

It can be shown (Rudin, 1973) that this operator is also self-adjoint and simple. Moreover, the eigenvalues and eigenfunctions for T are precisely $2k\pi$ and $e^{i2k\pi x}$ for $k = 0,\pm1,\pm2,\ldots$, as for $\Pi^{1/2}$. Hence, T has the same representation given in (1.5) and, so, defines a satisfactory square-root of Π. This example is enlightening in illustrating the domain of the preferred square-root of Π when it is induced by a differential expression. Essentially, the same boundary conditions are carried over to the square-root operator (still periodic B.C.'s but involving one less derivative, however, the eigenfunctions for Π are determined by the one condition, $u(0) = u(1)$). Implicit in the formal calculus rule of construction is the use of the same boundary conditions. This is necessarily the case since we maintain the same eigenfunctions.

These simple examples demonstrate the close connection between the domain of definition of a differential operator and basic properties of closure, symmetry and self-adjointness. Also, they serve to illustrate a construction of a square-root of a differential operator that will be tied in with the construction of the cross-surface differential operator H to be developed in the next section.

1.2 Construction of H

The construction of H takes place through a process of embedding the problem into a higher dimensional setting. We will be concerned with the two dimensional Laplacian defined on a region $\Sigma \subset R^2$, and we construct H by embedding Σ as a boundary face in a region $B \subset R^3$. Then, we perform operations in B and take a limit back to Σ. This is a natural scheme for the applications of H, since they involve harmonic functions in a region such as B with specified boundary values on the surface Σ. The square-root nature of the operator arises through the technique of factoring out the 2-dimensional Laplacian from the equation for a harmonic function on B. That is, if u(P) is harmonic for $P \in B$ then,

$$\Delta_3 u(P) = (\partial_{zz} + \Delta_2)\, u(P) = 0 \quad \text{implies that} \quad \frac{\partial^2 u}{\partial z^2}(P) = \Pi_2 u(P) \ .$$

Hence H is constructed by extending a boundary value function harmonically off Σ, performing one derivative w.r.t. z in B and then taking a limit as P approaches Σ. This will be explained in detail later. First, we want to outline a general setting within which Π is a simple operator and allows eigenfunction expansion methods as presented in the first section.

Let $\Sigma \subset R^2$ be a bounded region in the plane (in many cases this setting can be generalized to include unbounded regions and, in fact, several examples where this is the case will be presented). We will consider the elliptic Sturm-Liouville type operator,

$$L = -\text{div}(p \text{ grad } . \,) + q$$

defined on Σ, where we associate to L the domain D_L, as all functions, u(x), of class $C^2(\Sigma) \cap C^1(\bar{\Sigma})$ satisfying the boundary condition,

$$\alpha u + \varepsilon \frac{\partial u}{\partial \tilde{n}} \bigg|_{\gamma} = 0 \ , \text{ where } \gamma = \partial\Sigma \ ,$$

plus the condition that $\Delta u \varepsilon\ L^2(\Sigma)$. Also, the following restrictions are imposed on all coefficients;

$$p \varepsilon C^1(\bar{\Sigma}),\ q \varepsilon C(\bar{\Sigma}) \text{ with } p > 0,\ q \geqslant 0 \text{ on } \Sigma$$

and for the boundary condition, we consider the cases where

$$\varepsilon \equiv 0 \text{ or } \varepsilon \equiv 1,\ \alpha \varepsilon C(\gamma),\ \alpha \geqslant 0,\ \alpha + \varepsilon > 0 \text{ on } \gamma \ .$$

We wish to have a setting where L is a symmetric, simple operator. This will depend on the smoothness of the boundary γ and coefficients, p, q and α. We will call γ a <u>sufficiently smooth curve</u> if, for the above setting, we have the following properties;

<u>property i</u> All Green's formulas hold for functions $u, v \varepsilon C^2(\Sigma) \cap C(\bar{\Sigma})$ with $\Delta u, \Delta v \varepsilon\ L^2(\Sigma)$ and having a correct normal derivative over γ.

By a <u>correct normal derivative</u> we mean that for every $u \varepsilon\ C^1(\Sigma)$, with γ a surface of class C^1, there exists a uniform limit w.r.t. $S \varepsilon \gamma$ of

$$\frac{\partial u}{\partial \tilde{n}_S}(S') \text{ as } S' \to S \text{ with } S' \varepsilon - \tilde{n}_S$$

where, $\tilde{n}_S$ is the outward unit normal to γ at S. The value of this limit will be denoted in the usual fashion as the normal derivative, $\partial u/\partial \tilde{n}\,(S)$, evaluated at $S \varepsilon \gamma$. Under these conditions this defines a continuous function over γ. In particular, Green's 2nd formula, which is hypothesised, states that

$$\int_\Sigma v\, Lu - u\, Lv = \int_\gamma u \frac{\partial v}{\partial \tilde{n}} - v \frac{\partial u}{\partial \tilde{n}} .$$

With the above boundary conditions, this gives the symmetric nature of L.

property ii L is a simple operator. That is, the eigenvalues of L are countable, no finite limit points, with finite multiplicity and a corresponding C.O.S. of eigenfunctions that can be chosen real-valued. Moreover, for every $u \varepsilon D_L$, the eigenfunction expansion converges regularly over $\bar{\Sigma}$ (uniform convergence of the series of absolute values) and allows term-by-term partial differentiation, once, w.r.t. each variable with the result converging in $L^2(\Sigma)$.

On the basis of this property, we can order the eigenvalues by magnitude $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, $\lambda_k \to \infty$ as $k \to \infty$. We have repeated the eigenvalues according to multiplicity and such that each λ_k corresponds to exactly one eigenfunction $u_k(s)$ satisfying $Lu_k = \lambda_k u_k$ with $u_k \varepsilon D_L$. Also, we have that for every $u \varepsilon D_L$,

$$u(S) = \sum_{k=1}^{\infty} \tilde{u}_k u_k(S) \text{ (converging regularly over } \bar{\Sigma})$$

and (1.6)

$$\text{grad } u(S) = \sum_{k=1}^{\infty} \tilde{u}_k \text{ grad } u_k(S) \text{ (converging in } L^2(\Sigma))$$

<u>property iii</u> There exists a <u>Neumann function</u> $N(S,R)$ defined on $\Sigma x \Sigma$ of the form,

$$N(S,R) = \frac{1}{2\pi} \ln \frac{1}{|S-R|} + \eta(S,R) \text{ , where}$$

$$\Delta_S \eta(S,R) = \frac{1}{|\Sigma|} \text{ , for } S,R \varepsilon \Sigma$$

and satisfies the appropriate boundary condition such that

$$\frac{\partial N}{\partial \tilde{n}_S} (S,R) = 0 \text{ , for all } S \varepsilon \gamma,\ R \varepsilon \Sigma \text{ .}$$

Moreover, $N(S,R)$ is of class, $C^2(\Sigma \setminus \{R\} \cap C(\bar{\Sigma} \setminus \{R\})$ w.r.t. S and uniformly of order,

$$|N(S,R)| = 0 \left(\ln \frac{2D}{|S-R|}\right) \text{ , (where } D = \text{diam } (\Sigma)). \qquad (1.7)$$

The general approach in developing this theory is to seek the Neumann function (some authors refer to all such kernels as Green's functions or, in the case of non-uniqueness, as a Green's function in the broad sense) via the method of potentials. This entails that the potentials satisfy certain criterion and that we can solve a certain type of Neumann boundary value problem. To meet these conditions, the necessary restrictions and smoothness of γ can be spotted. Vladimirov (1971) develops this setting with the Dirichlet boundary condition for a class of sufficiently smooth Liapunov surfaces or lines. A much more general setting including Neumann boundary conditions is considered by Miranda (1970). Once the Neumann function is obtained, the eigenvalue problem, $Lu = \lambda u$, can be expressed as an integral equation with this Neumann function as the kernel. Property ii is then obtained as a consequence of the Fredholm and Hilbert-Schmidt theorems.

We will also note some consequences of this setting that will be referred to below. From the order of N(S,R) in (1.7) and by Green's 2nd formula, we can show that N is a symmetric kernel, i.e., $N(S,R) = N(R,S)$ for all $S,R \varepsilon\ \Sigma$ and has the bilinear expansion,

$$N(S,R) = \sum_k \frac{u_k(S) \quad u_k(R)}{\lambda_k} \tag{1.8}$$

where the sum is k=1,2,... since $\lambda_0=0$ and which converges, uniformly for $R\varepsilon\bar{\Sigma}$, in $L^2(\Sigma)$ w.r.t. S. As a result of (1.8) and by Dini's lemma, we have

$$\sum_k \frac{|u_k(S)|^2}{\lambda_k{}^2} = \int_\Sigma |N(S,R)|^2 d\Sigma_R \tag{1.9}$$

which shows that the series in the L.H.S. of (1.9) converges uniformly over $\bar{\Sigma}$ (since $\int_\Sigma |N(S,R)|^2 d\Sigma_R$ defines a continuous function of $S\varepsilon\bar{\Sigma}$). Next, we have the following representation theorem for solutions to the Poisson equation,

$$-\Delta u = f \text{ on } \Sigma$$

(problem ψ)

$$\frac{\partial u}{\partial \tilde{n}} = \phi \text{ on } \gamma$$

where $f\varepsilon\ C(\Sigma) \cap L^2(\Sigma)$, $\phi\varepsilon C(\gamma)$ and satisfies the solvability condition of

$$\int_\Sigma f = \int_\gamma \phi \ .$$

Then, if u(s) is a solutiion to (problem ψ) with $u\varepsilon\ C^2(\Sigma) \cap C(\bar{\Sigma})$ and having a correct normal derivative over γ, we have that

$$u(s) = \int_\Sigma N(S,R)\ f(R)d\Sigma_R + \int_\gamma N(S,R)\phi(R)d\gamma_R + \bar{u} \tag{1.10}$$

where $\bar{u}$ is an arbitrary constant equal to the average value of u (Duff and Naylor, 1966). This assumes that solution to problem ψ with the above properties exists (for weaker hypotheses see Miranda, 1970). For existence, the following result holds. By further restricting $f \varepsilon\ C^1(\Sigma) \cap C(\bar{\Sigma})$ with average value $\bar{f} = 0$ the unique solution to

$$\text{with} \quad \begin{array}{l} -\Delta u = f \text{ on } \Sigma \\ \left.\frac{\partial u}{\partial \tilde{n}}\right|_\gamma = 0 \end{array} \quad \text{is} \quad u(S) = \int_\Sigma N(S,R)f(R)d\Sigma_R .$$

(if $\bar{f} \neq 0$ we can't have a solution but we can replace f with $f - \bar{f}$) . Equation (1.10) can be used to transform the eigenvalue problem $Lu = \lambda u$, $\partial u/\partial \tilde{n}|_\gamma = 0$ into an equivalent integral equation.

Now, getting back to the problem of defining the operator, H, let $\Sigma \subset R^2$ be a bounded region whose boundary $\partial\Sigma = \gamma$ is a sufficiently smooth curve as outlined above. Then, a complete definition of Π is given by

$$\Pi = -\Delta_2 \text{ on } \Sigma \text{ , with } D_\pi = C^2(\Sigma) \cap C^1(\bar{\Sigma}) \cap \{u : \Delta_2 u \varepsilon L^2(\Sigma) \text{ , } \left.\frac{\partial u}{\partial \tilde{u}}\right|_\gamma = 0\}$$

which certainly falls within the previous setting. So, there exists a C.O.S. of eigenfunctions $u_k(S)$, with corresponding eigenvalues λ_k such that

$$\Pi\, u_k(S) = \lambda_k u_k(S) \text{ for } S\varepsilon\, \Sigma\ , \quad \frac{\partial u_k}{\partial \tilde{n}}\Big|_\gamma = 0\ , \ u_k\ \varepsilon\ D_\Pi$$

and with all the properties in i) - iii) and results in (1.8) - (1.10) holding.

Now, form the cylindrical region $B \subset R^3$ by $B = \Sigma x(0,\infty)$ with $\partial B = \Sigma \cup \Gamma$ (so Γ represents the sides, $\gamma \times [0,\infty)$).

Definition 1.1

Given a function $\phi(S)$ defined for $S\varepsilon\Sigma$, solve the BVP

$$-\Delta u(P) = 0\ , \ P\varepsilon B$$

(BVP*) with $\dfrac{\partial u}{\partial \tilde{n}}\Big|_\Gamma = 0$ and $u\big|_\Sigma = \phi$.

Then, we define the Cross-Surface Differential Operator H by

$$H\phi(S) = -\frac{\partial}{\partial z} u(P)\Big|_{z\downarrow 0}$$

Provided the limit as $z\downarrow 0$ exists.

Of course, a class of boundary values for $\phi(S)$ allowing a solution to BVP*, in what sense it is a solution and ensuring that the limiting process back to Σ exists has to be specified. However, given a suitable function $\phi(S)$ defined on Σ, there will exist a unique harmonic extension u to B with those boundary values and satisfying the prescribed boundary condition on Γ. This definition explains the use of the terminology that H is a cross-surface differential operator since, we then perform a differentiation w.r.t. z off Σ and take a limit as $z \downarrow 0$, back to Σ.

Formally, it is easy to see that H will behave like a square-root of Π. To apply H twice to $\phi(S)$ we need to extend the boundary values $H\phi(S)$ to a harmonic function $v(P)$ on B with these values on Σ. If ϕ is sufficiently smooth, this unique harmonic extension will be given by,

$$v(P) \stackrel{*}{=} - \partial_{zz} u(P) \ .$$

So,

$$H^2\phi(S) = H(H\phi)\ (S) = - \frac{\partial}{\partial z} v(P)\big|_{z\downarrow 0} = \partial_{zz}\ u(P)\big|_{\ z\downarrow 0}$$

$$= -\Delta_2 u(P)\big|_{z\downarrow 0} \stackrel{*}{=} \Pi\phi(S) \ .$$

This illustrates how we seek a square-root of Π by factoring it from the 3-dimensional Laplacian. However, it must be stressed that these manipulations are purely formal in nature and considerable work is needed to make this rigorous. In particular, the necessary conditions to justify the starred (*) steps above must be examined.

Since one of the main interests is to obtain a square-root of Π in an L^2-sense, a good approach to this problem is through an eigenfunction expansion method. Several results will come from this approach. First, a result (see theorem 1.2) giving a class of functions which allows a classical solution to B.V.P.* is given. This gives a suitable setting for H for the application to the water wave model in chapter III. It must be emphasized, however, that this is a formal definition for H in the sense that Definition (1.1) gives the prescription for calculating $H\phi$ when the limit as $z \downarrow 0$ exists. Lemmas 1.3 and 1.4 in the next section give domains of definition for H guaranteeing that this limit exists. Then, H is related back to the formal calculus methods presented in section 1.1, where we ignore the previous background with the boundary value problem BVP*, and begin the definition of H by an eigenfunction expansion representation. This yields an operator which is a square-root of Π in an L^2-sense. Finally, in chapter II, this eigenfunction expansion method is extended to a distributional setting.

By a <u>classical solution to BVP*</u> we will refer to a solution $u(p)$ satisfying, $u \in C^2(B) \cap C^1(\bar{B} \setminus \Gamma) \cap C(\bar{B})$ with $\partial u/\partial \tilde{n} = 0$ on Γ as a correct normal derivative. This entails the following compatibility conditions on ϕ for a solution; namely, $\phi \in C^1(\Sigma) \cap C(\bar{\Sigma})$ with $\partial\phi/\partial\tilde{n} = 0$ on γ as a correct normal derivative.

A solution to BVP* is sought through a partial expansion in terms of the eigenfunctions $u_k(S)$. That is, we try a solution of the form,

$$u(P) = \sum_k a_k(z)u_k(S) \text{ , where } a_k(z) = (u,u_k) = \int_\Sigma u(P)u_k(S)d\Sigma_s \text{ .}$$

Then, formally plugging this into BVP*, a differential equation for the undetermined coefficients $a_k(z)$ is obtained as

$$a_k''(z) = \lambda_k a_k(z) \text{ , for } z > 0 \text{ , and we want } a_k(z) \text{ bounded as } z \to \infty.$$

This implies that,

$$a_k(z) = a_k e^{-\lambda_k^{1/2} z}$$

where the boundary condition $u = \phi$ on Σ (for $z = 0$) is satisfied by setting

$$a_k = \tilde{\phi}_k = (\phi, u_k) \text{ .}$$

Hence, we have the following expansion for u(P),

$$u(P) = \sum_k \tilde{\phi}_k \, e^{-\lambda_k^{1/2} z} u_k(S) \tag{1.11}$$

(here, $S \varepsilon \Sigma$ and $P = (S,z) \varepsilon B$) .

From here, an analysis of this expansion to determine the smoothness of u in terms of various restrictions on ϕ involves the repeated use of the Cauchy-Buniakowski inequality. This allows the expansion to be factored into two series, one containing the eigenfunctions or their derivatives, scaled down by dividing by the appropriate power of the eigenvalues so it will converge. But, then, the other series, which contains the expansion coefficients of ϕ, must pick up those powers of λ_k. This allows the necessary restrictions on ϕ, to obtain a solution, to be spotted. This is illustrated through the following estimate.

Let β be a non-negative integer and $\alpha = (\alpha_1, \alpha_2)$ a multi-index with $D^\alpha = \partial^{\alpha_1}/\partial x^{\alpha_2} \, \partial^{\alpha_2}/\partial y^{\alpha_2}$ (here, $S = (x,y) \varepsilon \Sigma$) . Then, for any real r we have

$$\left| \sum_{k=n}^{m} \tilde{\phi}_k \frac{\partial^\beta}{\partial z^\beta} (e^{-\lambda_k^{1/2} z}) D^\alpha u_k(S) \right| < \left(\sum_{k=n}^{m} \lambda_k^{r+\beta} |\tilde{\phi}_k|^2 e^{-2\lambda_k^{1/2} z} \right)^{1/2} \left(\sum_{k=n}^{m} \frac{|D^\alpha u_k(S)|^2}{\lambda_k^r} \right)^{1/2} \tag{1.12}$$

In particular, for $\beta = \alpha_1 = \alpha_2 = 0$, with $r=2$, by (1.9) there exists a constant M such that

$$\left|\sum_{k=n}^{m} \tilde{\phi}_k e^{-\lambda_k^{1/2} z} u_k(s)\right| < M\left(\sum_{k=n}^{m} \lambda_k^2 |\tilde{\phi}_k|^2\right)^{1/2} \to 0 \text{ as } n,m \to \infty,$$

if we assume that $\Delta\phi \varepsilon L^2(\Sigma)$ (since then, $\sum_{k=1}^{\infty} \lambda_k^2 |\tilde{\phi}_k|^2 = ||\Delta\phi||_2^2 < \infty$). In this case, $u \varepsilon C(\bar{B})$. In fact, if we only assume $\phi \varepsilon L^2(\Sigma)$, we have uniform convergence of these partial sums on any region of the form $\bar{\Sigma} \times [\varepsilon,\infty)$, for any $\varepsilon>0$. This holds, since, by keeping the exponential terms, for k sufficiently large

$$\max_{z \varepsilon [\varepsilon,\infty)} \lambda_k^2 e^{-2\lambda_k^{1/2} z} < 1 \text{ and } \sum_k |\tilde{\phi}_k|^2 = ||\phi||_2^2 < \infty.$$

It seems clear that for $z>0$, $u(P)$ given by (1.11) should define an infinitely continuously differentiable function due to the rapid convergence of the exponential term. In fact, we will show that $u(P)$ defines a generalized harmonic function and hence (Vladimirov, 1971) for $\Delta\phi \varepsilon L^2(\Sigma)$, when $u \varepsilon C(\bar{B})$, it is harmonic on B.

Let Ψ be a test function, i.e., of class $C^\infty(B)$ with compact support in B. Then,

$$(u,\Delta\psi) = \sum_k \tilde{\phi}_k \int_\Sigma \int_0^\infty e^{-\lambda_k^{1/2} z} u_k(S)\, \Delta\psi(P)\, dz d\Sigma_S$$

$$= \sum_k \tilde{\phi}_k \left\{ \int_\Sigma u_k(S)d\Sigma_S \int_0^\infty e^{-\lambda_k^{1/2} z} \partial_{zz}\psi(P)dz - \int_0^\infty e^{-\lambda_k^{1/2} z} dz \int_\Sigma u_k(S)\Pi\psi(P)d\Sigma_S \right\}.$$

Here, by integration by parts

$$\int_0^\infty e^{-\lambda_k^{1/2} z} \partial_{zz}\psi(P)\, dz = \lambda_k \int_0^\infty e^{-\lambda_k^{1/2} z} \psi(P)\, dz \text{ , and}$$

$$\int_\Sigma u_k(S)\Pi\psi(P)d\Sigma_S = \lambda_k \int_\Sigma u_k(S)\ \psi(p)d\Sigma_S.$$

So, both terms cancel in the above series and $(u,\Delta\psi) = 0$.

We can show directly that (1.11) defines a $C^\infty(B)$ function and allows term-by-term partial differentiation of all orders by making use of an estimate on the partial derivatives of the eigenfunctions in terms of a power of the eigenvalues. This result, which will be given later in Section 2.2 (see lemma 2.5), where it is used to extend the setting for H to include distributions, states that for k sufficiently large,

$$|D^\alpha u_k(S)| < C\ \lambda_k^m \text{ , for all S in a compact subset of } \Sigma$$

(here, m depends on $|\alpha|$).

Then, choosing $r = m^2 + 2$, the 2nd series in (1.12) is dominated by

$$\sum_{k=n}^{m} \frac{1}{\lambda_k^{\,2}} \to 0 \text{ as } n,m \to \infty \quad (\text{since, in 2-dimensions, } \lambda_k \sim k$$
$$\text{see Courant and Hilbert (1953)})$$

and, no matter what r is, the 1st series converges if z is bounded away from zero. In particular, this shows that

$$\frac{\partial u}{\partial \tilde{n}_P}(P') = \sum_k \tilde{\phi}_k \, e^{-\lambda_k^{1/2} z} \, \frac{\partial u_k}{\partial \tilde{n}_S}(S')$$

where $S\varepsilon\gamma$, $S'\varepsilon-\tilde{n}_S$ and $P = (S,z)$, $P' = (S',z)$ with $z>0$. Then, since $u_k \varepsilon C^1(\bar{\Sigma})$ with $\frac{\partial u_k}{\partial \tilde{n}} = 0$ on γ, given any $\varepsilon>0$ there exists a $\delta>0$ such that

$$|S'-S| < \delta,\ S\varepsilon\gamma,\ S'\varepsilon\Sigma \text{ implies that } \left|\frac{\partial u_k}{\partial \tilde{n}_S}(S')\right| < \varepsilon.$$

So, we obtain

$$\left|\frac{\partial u}{\partial \tilde{n}_P}(P')\right| \leq \Big(\sum_k \lambda_k^2 \, e^{-\lambda_k^{1/2} z} |\tilde{\phi}_k|^2\Big)^{1/2} \Big(\sum_k \frac{\left|\frac{\partial u_k}{\partial \tilde{n}_S}(S')\right|^2}{\lambda_k^{\,2}}\Big)^{1/2}$$

$$\leq \varepsilon \Big(\sum_k \lambda_k^2 \, |\tilde{\phi}_k|^2\Big)^{1/2} \Big(\sum_k \frac{1}{\lambda_k^2}\Big)^{1/2}.$$

f $\Delta\phi\varepsilon L^2(\Sigma)$, then these last two series will converge which shows

t

$$\left.\frac{\partial u}{\partial \tilde{n}}\right|_{\Gamma\setminus\gamma} = 0 \quad \text{as a correct normal derivative.}$$

Hence, we have seen that if $\phi\varepsilon D_{\Pi}$, then (1.11) defines a harmonic function $u(P)$ on B with $u\varepsilon C^{\infty}(B)\cap C(\bar{B})$. However, we only have a correct normal derivative over $\Gamma\setminus\gamma$. The corner to the domain B, at γ, poses some problems for obtaining a classical solution. The sense in which $\partial u/\partial\tilde{n} = 0$ on γ is not as a correct normal derivative with uniform convergence as we approach the boundary but with convergence in $L^2(\Sigma)$. This follows from property ii where, if $\phi\varepsilon D_{\Pi}$, then for $z=0$

$$\frac{\partial u}{\partial \tilde{n}_S}(S') = \sum_k \tilde{\phi}_k \frac{\partial u_k}{\partial \tilde{n}_S}(S') \to 0 \text{ as } S' \to S, \text{ in } L^2(\Sigma).$$

Also, to obtain $u\varepsilon C^1(\bar{B}\setminus\Gamma)$ or equivalently that the eigenfunction expansion for $\phi(S)$,

$$\phi(S) = \sum_k \tilde{\phi}_k\, u_k(S) \tag{1.13}$$

defines a function in $C^1(\Sigma)$, a more stringent growth condition on the expansion coefficients $\tilde{\phi}_k$ must be imposed. By assuming $\phi\varepsilon D_{\Pi}$ we have $\Delta\phi\varepsilon L^2(\Sigma)$ which is equivalent to $\sum_k \lambda_k|\tilde{\phi}_k|^2 < \infty$. By imposing the more stringent growth condition, namely,

$$\sum_k \lambda_k^6 \, |\tilde{\phi}_k|^2 < \infty \, ,^{(2)} \tag{1.14}$$

the series in (1.13) now allows term-by-term partial differentiation once w.r.t. all variables with the partial sums converging uniformly on any compact subset of Σ. Hence, ϕ given by (1.13) is of the class $C^1(\Sigma)$.

In this case, the argument used to show

$$\left.\frac{\partial u}{\partial \tilde{n}}\right|_{\Gamma\setminus\gamma} = 0 \quad \text{as a correct normal derivative,}$$

can be extended to include all of Γ since we no longer have to require z to be greater than zero.

So, in summary the following theorem has been proven.

Theorem 1.2

If $\phi \varepsilon D_\Pi$ satisfies the growth condition of (1.14); then u(P) given by (1.11) is a classical solution to BVP* (i.e. it is a solution with $u \varepsilon C^\infty(B) \cap C^1(\bar{B}\setminus\Gamma) \cap C(\bar{B})$ and $\partial u/\partial \tilde{n} = 0$ on Γ as a correct normal derivative). However, by only assuming $\phi \varepsilon D_\Pi$, u(P) is still a

(2) It is possible this can be weakened. Lemma 2.5 mentioned earlier, gives the following estimate for a single partial differentiation; $|\partial_{x_i} u_k(S)| < C \lambda_k^2$, for k sufficiently large and for all S in a compact subset of Σ. Then a similar argument as given on page 31 using (1.12) where we now choose r=6 gives the condition in (1.14).

harmonic function on B with $u \varepsilon C(\bar{B})$, where $u = \phi$ on Σ and $\partial u / \partial \tilde{n} = 0$ on $\Gamma \setminus \gamma$ as a correct normal derivative.

Now we obtain an eigenfunction representation for H. For $z>0$, $\partial_z u\ (P)$ can be calculated by term-by-term differentiation of (1.11) to yield

$$H\phi(S) = \lim_{z \downarrow 0} \sum_k \lambda_k^{1/2} e^{-\lambda_k^{1/2} z} \tilde{\phi}_k u_k(S). \tag{1.15}$$

Recall that, for $\phi \varepsilon D_\Pi$, an eigenfunction representation for Π is given by

$$\Pi\phi(S) = \sum_k \lambda_k \phi_k u_k(S), \text{ with convergence in } L^2(\Sigma).$$

Notice the similarity in the expansion given in (1.15) and that of a square-root of Π as dictated by the formal calculus. The exponential factor and limit as $z \downarrow 0$ reflects the scheme of embedding the surface problem into a 3-dimensional region, performing a differentiation there and then coming back down to the surface. If ϕ is smooth enough, for example, satisfying the first set of conditions of Theorem 1.2 ($\phi \varepsilon D_\Pi$ and condition (1.14) holds), then as we will see in the next section the limit as $z \downarrow 0$ can be brought inside the series yielding an expansion which obeys the formal calculus rule of bringing the square-root inside to the eigenvalues. In general, we can't expect this to be the case. For example, if we take $\phi \varepsilon D_{\Pi}$ or even $\phi \varepsilon L^2(\Sigma)$, u(P) given in (1.11) still defines a harmonic funciton on B so we can perform the differentiation w.r.t. z for $z>0$. However, the behavior as $z \downarrow 0$ is unknown and so the limit and exponential factor must remain in (1.15).

In the next section a weaker growth condition than (1.14) will be given which allows the limit as $z \downarrow 0$ to be brought inside the series. Also, a domain for H will be specified which yields exactly the same square-root operator as the symolic calculus methods outlined in the first section.

1.3 Domains of Definition

In considering an appropriate domain for H, just as was the case when defining a square-root operator by the formal calculus rule in the 1-dimensional example of Section 1.1, the previous background involving BVP* can be ignored. If the concern is to obtain an operator that behaves as a square-root of Π in $L^2(\Sigma)$; then the representation in (1.15) can be used as a starting point for the definition of H. Later, in Chapter III, we will return to the full construction of H, since by design, it fits the models we wish to apply H to.

If we define the square-root operator,

$$\Pi^{1/2}\,\phi(S) = \sum_k \lambda_k^{1/2}\,\tilde{\phi}_k\, u_k(S) \tag{1.16}$$

defined on all functions, $\phi(S)$, that satisfy

$$\sum_k \lambda_k\,|\tilde{\phi}_k|^2 < \infty\,; \qquad \underline{\text{condition (A)}}$$

then, it might seem that the exponential factor and limit as $z\downarrow 0$, in (1.15), should give an extension of this operator, since these features would seem to help the convergence of the series. Two classes of functions will be considered as a domain of definition for H. In one case, the series

$$\sum_k \lambda_k^{1/2}\, e^{-\lambda_k^{1/2} z}\,\tilde{\phi}_k\, u_k(S) \tag{1.17}$$

will converge in $L^2(\Sigma)$, uniformly w.r.t. $z\varepsilon[0,\infty)$. So, $H\phi\varepsilon L^2(\Sigma)$, and, somewhat surprisingly, is precisely the operator $\Pi^{1/2}$. In the other case, $H\phi(S)$, will define a continuous function on $\bar{\Sigma}$ with the series in (1.17) converging uniformly on $\bar{\Sigma}\times[0,\infty)$.

<u>Lemma 1.3</u>

$H\phi\varepsilon L^2(\Sigma)$ iff ϕ satisfies condition (A), in which case $H\phi(S)$ in given (1.15) equals $\Pi^{1/2}\,\phi(S)$ in $L^2(\Sigma)$. Hence, taking

$$D_H = \{\phi : \phi \text{ satisfies condition (A)}\} = D_{\Pi^{1/2}},$$

$H = \Pi^{1/2}$ and this notation is justified, since H is a square-root of Π in $L^2(\Sigma)$.

<u>**Proof**</u> ($\Rightarrow$) This direction is easy. If $H\phi \in L^2(\Sigma)$, then we can calculate its expansion coefficients, namely

$$(H\phi, u_j) = \lim_{z \downarrow 0} \sum_k \lambda_k^{1/2} e^{-\lambda_k^{1/2} z} \tilde{\phi}_k (u_k, u_j) = \lim_{z \downarrow 0} \lambda_j^{1/2} e^{-\lambda_j^{1/2} z} \tilde{\phi}_j = \lambda_j^{1/2} \tilde{\phi}_j$$

Hence,

$$H\phi(S) = \sum_k \lambda_k^{1/2} \tilde{\phi}_k u_k(S) = \Pi^{1/2} \phi(S) \quad \text{, in } L^2(\Sigma) \text{ and}$$

$$||H\phi||_2^2 = \sum_k \lambda_k |\tilde{\phi}_k|^2 < \infty.$$

($\Leftarrow$) Next, suppose condition (A) is satisfied. It will be shown that

$$\lim_{z \downarrow 0} \sum_k \lambda_k^{1/2} e^{-\lambda_k^{1/2} z} \tilde{\phi}_k u_k(S) \text{ converges in } L^2(\Sigma) \text{ to } \Pi^{1/2} \phi(S).$$

For this it is sufficient to show that

$$\Big\|\sum_{K}\lambda_k^{1/2}\tilde{\phi}_k\,(1-e^{-\lambda_k^{1/2}z})\,u_k(S)\Big\|_2^2=\sum_k\lambda_k|\tilde{\phi}_k|^2\,(1-e^{-\lambda_k^{1/2}z})^2\to 0 \quad\text{as } z\downarrow 0.$$

Let $\varepsilon>0$ be given. By assumption, there exists N such that

$$\sum_{k=N+1}^{\infty}\lambda_k\,|\tilde{\phi}_k|^2<\frac{\varepsilon}{2}\,.$$

Set,

$$M=\sum_{k=1}^{N}\lambda_k\,|\tilde{\phi}_k|^2 \quad\text{and take,}\quad \delta=\frac{-\ln\,(1-\tilde{\varepsilon})}{\lambda_N^{1/2}}\,,\ \text{where}$$

$$\tilde{\varepsilon}=\left(\frac{\varepsilon}{2M}\right)^{1/2}.$$

Now, for every

$0<z<\delta$, $1-e^{-\lambda_N^{1/2}z}<\tilde{\varepsilon}$ which implies that $(1-e^{-\lambda_k^{1/2}z})^2<\frac{\varepsilon}{2M}$ for every $k=1,2,\ldots,N$

Hence, we have that

$$\sum_{k=1}^{\infty}\lambda_k\,|\tilde{\phi}_k|^2\,(1-e^{-\lambda_k^{1/2}z})^2=\sum_{k=1}^{N}(\text{same})+\sum_{k=N+1}^{\infty}(\text{same})$$

$$<\frac{\varepsilon}{2}+\sum_{k=N+1}^{\infty}\lambda_k\,|\tilde{\phi}_k|^2<\varepsilon\,.$$

Hence, $H\phi(S) = \Pi^{1/2}\phi(S) \in L^2(\Sigma)$. It is an easy matter to show that in this case, just as in the examples of Section 1.1, H (or $\Pi^{1/2}$) is a square-root of Π. If $\phi \in D_\Pi$, then $\Delta\phi \in L^2(\Sigma)$ which holds iff

$$\sum_k \lambda_k^2 \, |\tilde{\phi}_k|^2 < \infty ,$$

in particular, ϕ satisfies condition (A). So, the first part of this Lemma shows that,

$H\phi(S) = \sum_k \lambda_k^{1/2} \, \tilde{\phi}_k u_k(S)$ which implies that Hϕ also satisfies condition (A).

Then,

$$H^2\phi(S) = \sum_k \lambda_k^{1/2} \, (\widetilde{H\phi})_k \, u_k(S) = \sum_k \lambda_k \, \tilde{\phi}_k \, u_k(S) = \Pi\phi(S)$$

(all equalities are interpreted as in $L^2(\Sigma)$) ∎

Lemma 1.4

If ϕ satisfies,

$$\sum_k \lambda_k^3 \, |\tilde{\phi}_k|^2 < \infty , \qquad \text{condition (B)}$$

then $H\phi \in C(\bar{\Sigma})$. Moreover, in this case the limit as $z \downarrow 0$ can be brought inside the series with,

$$H\phi(S) = \sum_k \lambda_k^{1/2} \tilde{\phi}_k u_k(S).$$

Proof First, we will show that

$$\sum_k \lambda_k^{1/2} \tilde{\phi}_k u_k(S) \in C(\bar{\Sigma}), \text{ if } \phi \text{ satisfies condition (B).}$$

Here, using the estimate in (1.12) ($\beta = \alpha_1 = \alpha_2 = 0$, $z = 0$, $r = 3$) and (1.9)

$$\left|\sum_{k=n}^{m} \lambda_k^{1/2} \tilde{\phi}_k u_k(S)\right| \leq M \left(\sum_{k=n}^{m} \lambda_k^3 |\tilde{\phi}_k|^2\right)^{1/2} \to 0 \text{ as } n,m \to \infty.$$

(where, $M = \sup_{S\in\bar{\Sigma}} \int_{\Sigma} |N(S,R)|^2 d\Sigma_R$, which is finite due to the order of N from property iii).

Hence, the partial sums of the above series forms a uniform Cauchy sequence on $\bar{\Sigma}$, and converges uniformly to a continuous function on $\bar{\Sigma}$. Now, using the same argument as in the proof of Lemma 1.3 we can show that

$$\sum_k \lambda_k^{1/2} e^{-\lambda_k^{1/2} z} \tilde{\phi}_k u_k(S) \underset{\text{as } z\downarrow 0}{\to} \sum_k \lambda_k^{1/2} \tilde{\phi}_k u_k(S), \text{ uniformly on } \bar{\Sigma}.$$

Thus, $H\phi = \Pi^{1/2} \phi \in C(\bar{\Sigma})$. That is,

$$\left| \pi^{1/2} \phi(S) - \sum_k \lambda_k^{1/2} e^{-\lambda_k^{1/2} z} \tilde{\phi}_k u_k(S) \right| = \left| \sum_k \lambda_k^{1/2} \tilde{\phi}_k (1 - e^{-\lambda_k^{1/2} z}) u_k(S) \right|$$

$$< \left(\sum_k \lambda_k^3 |\tilde{\phi}_k|^2 (1 - e^{-\lambda_k^{1/2} z})^2 \right)^{1/2} \left(\sum_k \frac{|u_k(s)|^2}{\lambda_k^2} \right)^{1/2} ,$$

and by condition (B) and the same argument of Lemma 1.3 this converges uniformly on $\bar{\Sigma}$ to zero as $z \downarrow 0$ ∎

Notice that both condition (A) and (B) involve growth conditions on the expansion coefficients of ϕ. As more stringent growth conditions are imposed more smoothness of $H\phi$ is obtained. In fact, if we assume that

$$\sum_k \lambda_k^m |\tilde{\phi}_k|^2 < \infty \text{ , for all } m$$

then we can show that $H\phi \epsilon C^{\infty}(\Sigma)$ (this is immediate from the estimate in (1.12) and Lemma 2.5 to come in the next chapter).

II. A DISTRIBUTIONAL SETTING FOR H

2.1 Preliminary Results

The restrictions on $\phi(S)$ from Theorem 1.2 and conditions (A) and (B) give possible domains for H. Condition (B) allows the limit as $z \downarrow 0$ in (1.15) to be brought inside the series with $H\phi$ defining a continuous function over $\bar{\Sigma}$; while, condition (A) also allows this simplification of the representation in (1.15) but in an L^2-sense. The more stringent restrictions of Theorem 1.2 guarantees that H fits the construction of Definition 1.1 which is designed for the application to the water wave model. In this chapter, the definition of H is extended to a class of distributions. In the applications we have the occasion to apply H to a Dirac or delta distribution to obtain impulse response solutions. The bilinear expansion of this distribution, used there, points the way to a more general theory. In Section 2.3 a theorem, which is quite interesting on its own, demonstrating an eigenfunction expansion representation for distributions with compact support in Σ is given. This will allow an extension of the definition of H to this class of distributions by using a formula analogous to the expansion in (1.15). In this section some notation and preliminary results will be given that are needed for the remainder of the chapter.

Given a region $\Omega \subset R^n$, $\underline{D(\Omega)}$ will denote the space of test functions on Ω of class $c^{\infty}(\Omega)$ with compact support in Ω. The topology for $D(\Omega)$ yields convergence as follows;

$$\phi_n \to \phi \text{ in } D(\Omega) \quad \text{iff} \quad D^{\alpha}\phi_n \to D^{\alpha}\phi$$

uniformly on Ω for all multi-indices $\alpha = (\alpha_1,\ldots,\alpha_n)$, where

$$D^{\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

and there exists a compact set K such that $\operatorname{supp}\phi_n \subset K$ for all n sufficiently large. Often, the notation, <u>$E(\Omega)$</u>, is used to denote the space of functions of class $c^{\infty}(\Omega)$ where convergence is given by uniform convergence on Ω for derivatives of all order. The dual spaces of $D(\Omega)$ and $E(\Omega)$, consisting of all continuous linear functionals, are denoted by <u>$D'(\Omega)$</u> and <u>$E'(\Omega)$</u>, resp.. The elements of these dual spaces are called distributions and we will denote evaluation with the symbol $\langle .,. \rangle$, i.e., for $T \varepsilon D'(\Omega)$, $\phi \varepsilon D(\Omega)$ the evaluation of T at ϕ is denoted by, $\langle T,\phi \rangle$. The convergence used in these spaces is known as <u>weak* convergence</u> and is defined by

$$T_n \to T, \text{ weak* or in } D'(\Omega) \ \left(\text{or in } E'(\Omega)\right)$$

$$\text{iff } \langle T_n,\phi \rangle \to \langle T,\phi \rangle \text{ , in } \mathbb{C} \text{ for all } \phi \ \varepsilon\ D(\Omega) \ \left(\text{or } E(\Omega)\right).$$

A <u>regular distribution</u> will refer to a distribution induced by a locally integrable function on Ω, denoted by $L^1_{LOC}(\Omega)$. That is, if $f \varepsilon L^1_{LOC}(\Omega)$, then f defines a distribution on $D(\Omega)$ by

$$<f,\phi> = \int_\Omega f(x)\ \phi(x)dx.$$

If f has compact support in Ω, then, this definition defines an element of $E'(\Omega)$.

We will often refer to a sequence or series of locally integrable functions as converging in a distributional sense. This will refer to the corresponding sequence or series of regular distributions converging weak*. For example, given a series

$$\sum_{k=1}^{\infty} f_k(x)\ ,\ f_k \ \varepsilon\ L^1_{LOC}(\Omega)\ ,$$

This converges in $D'(\Omega)$ iff

$$\lim_{n\to\infty} < \sum_{k=1}^{n} f_k,\ \phi> = \sum_{k=1}^{\infty} <f_k,\phi> = \sum_{k=1}^{\infty} \int_\Omega f_k(x)\ d(x)\ dx$$

converges in $\mathbb{C}$, for all $\phi \varepsilon D(\Omega)$. It may be the case that this series doesn't converge to a regular distribution but still converges in $D'(\Omega)$.

Lemma 2.1

It is a well-known fact that the elements of the dual space, $E'(\Omega)$, are exactly the distributions in $D'(\Omega)$ with compact support.

Lemma 2.2

Define the norms,

$$||\phi||_{m,\Omega} = \sup_{\substack{|\alpha|\leq m \\ x\in\Omega}} |D^{\alpha}\phi(x)|$$

where, $|\alpha| = \alpha_1+\ldots+\alpha_n$, is the order of α. Then, given $T\in D'(\Omega)$, for every compact set $K\subset\Omega$, there exists a non-negative integer m and constant $c<\infty$ such that

a) $|<T,\phi>| \leq c\ ||\phi||_{m,\Omega}$, for all $\phi \in D(\Omega)$ with $\text{supp}\phi \subset K$.

The smallest such m that works for all sets K is called the Order of T. Also,

b) If $T \in E'(\Omega)$, then it has finite order.

For proofs of these results see Rudin (1973).

Lemma 2.3

Within the setting outlined at the beginning of Section 1.2, the eigenfunctions $u_k(S)$, for Π, satisfy

$$u_k \in C^\infty(\Sigma) \cap C'(\bar{\Sigma}) \quad \text{(see Vladimivov (1971))}.$$

Lemma 2.4

If $\phi \in D(\Sigma)$ then

$$\sum_k \lambda_k^m |\tilde{\phi}_k|^2 < \infty, \quad \text{for all real } m \geq 0$$

Hence, by the final remark of Section 1.3, pg. 44, $H\phi \in C^\infty(\Sigma)$.

Proof Since ϕ has compact support in Σ and is infinitely differentiable, for any multi-index $\alpha = (\alpha_1, \alpha_2)$, $D^\alpha \phi \in L^2(\Sigma)$. In particular, $\Delta^n \phi \in L^2(\Sigma)$ for every non-negative integer n. But,

$$\Delta^n \phi(S) = \sum_k \lambda_k^n \tilde{\phi}_k u_k(S) \in L^2(\Sigma) \text{ iff } \sum_k \lambda_k^{2n} |\tilde{\phi}_k|^2 < \infty.$$

This shows the result we are after holds for all non-negative, even integers. But since, $\lambda_k \to \infty$ as $k \to \infty$, it is easy to see that this also holds for all real non-negative numbers ∎

2.2 An Estimate on the Derivatives of the Eigenfunctions

In this section, we will prove a result which gives an asymptotic estimate on the derivatives of $u_k(S)$ in terms of a power of λ_k which holds uniformly on compact subsets of Σ. Previous references to this result have been made within the proof of Theorem 1.2, for its use in connection with the estimate in (1.12), and in the remark after Lemma 2.4. It will also play a major role in the next section where an eigenfunction representation for distributions with compact support is obtained.

Lemma 2.5

For every compact set $K \subset \Sigma$ and multi-index α

$$|D^\alpha u_k(S)| \sim O(\lambda_k^{|\alpha|+1}) \text{ as } k \to \infty, \text{ uniformly w.r.t. } S \in K.$$

That is, there exists constants C, N (depending on α and K) such that

$$\sup_{S \in K} |D^\alpha u_k(S)| \leq C\, \lambda_k^{|\alpha|+1} \quad \text{for all } k \geq N.$$

Proof The estimates on $u_k(S)$ and their first partials are straightforward. From (1.9) we know that

$$\sum_k \frac{|u_k(S)|^2}{\lambda_k^2}, \text{ converges uniformly on } \bar{\Sigma}.$$

This implies that, $|u_k(S)| \sim O(\lambda_k)$ as $k \to \infty$, uniformly for $S \in \bar{\Sigma}$ (where this is to be interpreted as given in the statement of Lemma 2.5). Also, from (1.10), this eigenvalue problem is equivalent to the integral equation,

$$u_k(S) = \lambda_k \int_{\Sigma} N(S,R)\, u_k(R)\, d\Sigma_R.$$

Now, from property iii), we have

$$\left| \frac{\partial}{\partial x_i} N(S,R) \right| = O\left(\frac{1}{|S-R|}\right), \quad \left| \frac{\partial^2}{\partial x_i \partial x_j} N(S,R) \right| = O\left(\frac{1}{|S-R|^2}\right). \qquad (3)$$

Hence, the 1st partials of $u_k(S)$ can be obtained by bringing the differentiation inside the integral sign, yielding

$$\left|\frac{\partial}{\partial x_i} u_k(S)\right| \sim (\lambda_k^2) \text{ as } k \to \infty, \text{ uniformly on } \bar{\Sigma}.$$

However, the 2nd partials and higher order derivatives can't be handled so easily as indicated by the order of the 2nd partial of N(S,R).

---

(3) These orders on the Neumann function N(S,R) and its 1st and 2nd partials are not as easy to obtain as when Dirichlet B.C. is imposed. Also, it is not immediate from the sketch of the general setting in property iii) how they are derived. For a more in depth account of this situation see Miranda, 1970.

We will demonstrate the technique to circumvent this problem and obtain the appropriate estimates for higher order derivatives by first considering the 2nd partial,

$$\frac{\partial^2}{\partial x_i \partial x_j} u_k(S).$$

Given any compact set $K \subset \Sigma$, let Σ' be a subregion of Σ with $\partial\Sigma' = \gamma'$ a "parallel" boundary to γ such that $K \subset \Sigma' \subset \Sigma$. Then, we split up the integral equation for the eigenvalue problem as,

$$u_k(S)=\lambda_k\{\int_{\Sigma\setminus\Sigma'} N(S,R)u_k(R)d\Sigma_R+\int_{\Sigma'} \eta(S,R)u_k(R)d\Sigma_R+ \frac{1}{2\pi}\int_{\Sigma'} \ln\frac{1}{|S-R|}\, u_k(R)d\Sigma_R\}$$

Denote the first two integrals by I_1 and the last by I_2. Since, N is of class $C^\infty(K \times \Sigma\setminus\Sigma')$ and η of class $C^\infty(K \times \Sigma')$, further derivatives of I_1, over K, can be carried inside the integral sign and yields terms $\sim O(\lambda_k)$ as $k \to \infty$, uniformly for $S \epsilon K$.

The last term is written as,

$$I_2 = \frac{1}{2\pi}\ln\frac{1}{|S|} * \tilde{u}_k,\ \text{where}\ \tilde{u}_k(S) = \begin{cases} u_k(S) & , S \in \bar{\Sigma}' \\ 0 & , \text{on } R^2\setminus\bar{\Sigma}' \end{cases}$$

I_2 is of class of $C^1(R^2)$ and allows first order partial differentiation to be brought inside the integral sign. Moreover,

$$\frac{1}{2\pi}\ln\frac{1}{|S|}*\frac{\partial}{\partial x_i}\tilde{u}_k=\frac{1}{2\pi}\int_{\Sigma'}\ln\frac{1}{|S-R|}\frac{\partial u_k}{\partial x_i}(R)d\Sigma_R-\frac{1}{2\pi}\int_{\gamma'}\ln\frac{1}{|S-R|}u_k(R)\cos(\tilde{n},x_i)d\gamma_R$$

For some examples on the calculation of derivatives of functions with discontinuities and convolutions see Vladimirov (1971).

This last expression, along with $\frac{\partial}{\partial x_i}(I_2)$, define continuous functions on R^2, hence, since we can differentiate I_2 on either convolute (this holds as distributions but both expressions are continuous functions) we have,

$$\frac{\partial}{\partial x_i}u_k(S) = \lambda_k\{\int_{\Sigma\setminus\Sigma'}\frac{\partial}{\partial x_i}N(S,R)u_k(R)d\Sigma_R + \int_{\Sigma'}\frac{\partial}{\partial x_i}\eta(S,R)\, u_k(R)\, d\Sigma_R$$

$$+\frac{1}{2\pi}\int_{\Sigma'}\ln\frac{1}{|S-R|}\frac{\partial}{\partial x_i}u_k(R)d\Sigma_R-\frac{1}{2\pi}\int_{\gamma'}\ln\frac{1}{|S-R|}u_k(R)\cos(\tilde{n},x_i)d\gamma_R\}$$

This representation transfers the derivative from the singular part of the Kernel, $N(S,R)$, to $u_k(S)$. Now, we are allowed to bring another partial differentiation inside the integral sign in all terms. Carrying this out, all terms but the 3rd integral are asymptotically of order λ_k (since they only contain u_k) and the 3rd integral, involving $\frac{\partial}{\partial x_i}u_k$, is of order $\lambda_k{}^2$. Therefore, including the original coefficient λ_k,

$$\left|\frac{\partial^2}{\partial x_i\partial x_j}u_k(S)\right| \sim O(\lambda_k^3) \text{ as } k\to\infty, \text{ uniformly on } K.$$

And since, K, i and j were arbitrary, we have shown that for every compact set $K \subset \Sigma$

$$|D^{\alpha}u_k(S)| \sim O(\lambda_k^{|\alpha|+1}) \text{ as } k \to \infty, \text{ uniformly on } K \text{ for all } \alpha \text{ with } |\alpha| \leq 2.$$

It is clear this process can be continued to obtain estimates on

$$\frac{\partial^r}{\partial x_{i_1} \cdot \cdot \cdot \partial x_{i_r}} u_k(S) \quad , \text{ for all } r$$

by induction on r; where we will transfer all but the last partial derivative from the singular part $\frac{1}{2\pi} \ln \frac{1}{|S-R|}$, of N(S,R) to $u_k(S)$. That is, in exactly the same manner, we can write

$$\frac{\partial^{r-1}}{\partial x_{i_1} \cdot \cdot \cdot \partial x_{i_{r-1}}} u_k(S) = \lambda_k \{ \frac{\partial^{r-1}}{\partial x_{i_1} \cdot \cdot \cdot \partial x_{i_{r-1}}} (I_1) +$$

$$+ \frac{1}{2\pi} \int_{\Sigma'} \ln \frac{1}{|S-R|} \frac{\partial^{r-1}}{\partial x_{i_1} \cdot \cdot \cdot \partial x_{i_{r-1}}} u_k(R) d\Sigma_R + g(S) \},$$

Where I_1, as before, defines a function of class $C^{\infty}(K)$ allowing all differentiation to be carried inside the integral signs to the kernels N and η. Hence, this term will always be asymptotically of order λ_k. Here, g(S), represents the sum of boundary integrals (we pick up one more integral after each iteration) which only involves differentiation of u_k of order $\leq$ r-2 (in fact, there is an integral containing a derivative of u_k of order r', for $r' = 0,1,2,\ldots, r-2$). Thus, by the inductive hypothesis that

$$\left|\frac{\partial^m}{\partial x_{i_1} \cdots \partial x_{i_m}} u_k(S)\right| \sim O(\lambda_k^{m+1}) \text{ as } k \to \infty, \text{ uniformly on } K$$

and holding for all $m \leq r-1$, we see that the 2nd term is of the highest order of λ_k, namely λ_k^r. Moreover, all terms allow another partial differentiation to be taken inside the integrals to the kernels keeping the same order of differentiation on u_k. Therefore,

$$\left|\frac{\partial^r}{\partial x_{i_1} \cdots \partial x_{i_r}} u_k(S)\right| \sim O(\lambda_k^{r+1}) \text{ as } k \to \infty, \text{ uniformly on } K.$$

This inductive step verifies that for every multi-index α and compact set $K \subset \Sigma$ since,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_{i_1} \cdots \partial x_{i_{|\alpha|}}}, \text{ with } x_{i_j} = x_1 \text{ or } x_2$$

Then in fact

$$|D^\alpha u_k(S)| \sim O(\lambda_k^{|\alpha|+1}) \text{ as } k \to \infty, \text{ uniformly on } K \;\blacksquare$$

2.3 Eigenfunction Expansions of Distributions with Compact Support and an Extension of the Definition of H

The goal of this section is to obtain an eigenfunction representation for distributions, $T \varepsilon D'(\Sigma)$, with compact support. We restrict ourselves to compactly supported distributions, since, by Lemma 2.1 these distributions in fact make up $E'(\Sigma)$ and can be extended to functions of class $C^{\infty}(\Sigma)$. In particular, by Lemma 2.3, we can define the evaluation of T on the eigenfunctions of Π. However, the expansion that will be obtained will actually refer to the restriction of T to $D(\Sigma)$. This is necessary since we must have a class of functions in $L^2(\Sigma)$ having eigenfunction expansions[(4)]. In fact, it will be shown that these expansions converge in $E(\Sigma)$.

Before building up to this result, we will illustrate the situation with the Dirac delta distribution. Let $\delta_R \varepsilon D'(\Sigma)$, often denoted by $\delta(S-R)$, be defined by

$$<\delta_R, \phi> = \phi(R).$$

(4) A treatment of expansions for tempered distributions is given by B. Simon, 1970. There, the results are somewhat more complete in that they allow an iff statement in the types of theorems to come. This is because the eigenfunctions considered there, called harmonic oscillator wave-functions, are in the space of rapidly decreasing test functions, hence, there is no need to restrict the domain of the tempered distributions.

It is easy to show that δ_R has the bilinear expansion

$$\delta_R = \sum_k u_k(R)\ u_k \tag{2.1}$$

Where we regard u_k as a regular distribution and convergence is in $D'(\Sigma)$ (i.e. weak*). In fact, for every $\phi \in D(\Sigma)$,

$$\lim_{n\to\infty} < \sum_{k=1}^{n} u_k(R)\ u_k, \phi> = \sum_{k=1}^{\infty} u_k(R)\ < u_k, \phi>$$

$$= \sum_{k=1}^{\infty} \tilde{\phi}_k\ u_k(R) = \phi(R).$$

So, the series in the R.H.S. of (2.1) does indeed converge weak* defining a distribution in $D'(\Sigma)$ and as seen above, equals δ_R.

If we were to formally treat δ_R as a regular distribution induced by the symbolic function $\delta(S-R)$ then

$$u_k(R) = <\delta_R, u_k> = \int_\Sigma \delta(S-R)\ u_k(S)\ d\Sigma_S = \big(\delta(S-R),\ u_k(S)\big).$$

So, we see that the terms $u_k(R)$ in (2.1) play the role of the expansion coefficients in an eigenfunction expansion for the symbolic function $\delta(S-R)$. Given the expansion coefficients of a function, the representation in (1.15), shows how to define the evaluation of H of this function. This suggests we define $H(\delta_R)$ by

$$H(\delta_R) = \lim_{z\downarrow 0} \sum_k \lambda_k^{1/2}\ e^{-\lambda_k^{1/2} z}\ u_k(R)\ u_k \tag{2.2}$$

where we have calculated the expansion coefficients $(\tilde{\delta}_R)_k$, as $u_k(R)$.

It must be verified that the R.H.S. of (2.2) does indeed define a distribution on $D(\Sigma)$. Also, it will be shown that, as expected, the limit as $z\downarrow 0$ can be brought inside the series.

Lemma 2.6

$H(\delta_R)$ defined by (2.2) is in $D'(\Sigma)$ with

$$H(\delta_R) = \sum_k \lambda_k^{1/2} u_k(R)\ u_k, \text{ and}$$

$$\langle H(\delta_R), \phi\rangle = \langle \delta_R, H\phi\rangle = H\phi(R).$$

Proof Let $\phi \epsilon D(\Sigma)$. Then

$$\lim_{z\downarrow 0} \lim_{n\to\infty} \langle \sum_{k=1}^{n} \lambda_k^{1/2} e^{-\lambda_k^{1/2} z} u_k(R)\ u_k,\ \phi\rangle = \lim_{z\downarrow 0} \sum_{k=1}^{\infty} \lambda_k^{1/2} e^{-\lambda_k^{1/2} z}\ \tilde{\phi}_k u_k(R).$$

By, Lemma 1.4 and Lemma 2.4, the limit as $z\downarrow 0$ of this last series converges to

$$\sum_{k=1}^{\infty} \lambda_k^{1/2} \tilde{\phi}_k\ u_k(R) = H\phi(R) \blacksquare$$

The usual topology on $E(\Sigma)$, which yields convergence as defined earlier, is induced by the family of semi-norms

$$||\phi||_{M,N} = \sup_{\substack{|\alpha| \leq M \\ s\varepsilon K_N}} |D^\alpha \phi(S)| \ ,$$

where we decompose $\Sigma = \bigcup_{N=1}^{\infty} K_N$, $K_N \subset \Sigma$, compact. Clearly, an equivalent family of semi-norms w.r.t. convergence is given by

$$||\phi||_{\alpha,K} = \sup_{s\varepsilon K} |D^\alpha \phi(S)| \text{ , for any multi-index } \alpha \text{ and compact, } K\subset\Sigma.$$

We also introduce a family of norms on the space of complex sequences, $\phi = (\phi_k)_{k=1}^{\infty}$, by

$$||\phi||_m = (\sum_{k=1}^{\infty} \lambda_k^m \ |\phi_k|^2)^{1/2} .$$

If $\phi\varepsilon L^2(\Sigma)$, we will identify ϕ with its sequence of expansion coefficients, $(\tilde{\phi}_k)_{k=1}^{\infty}$.

Theorem 2.7

If $\phi\varepsilon D(\Sigma)$, then $||\phi||_m < \infty$ for all real $m \geq 0$, and

$$\phi(S) = \sum_k \tilde{\phi}_k \, u_k(S), \text{ with convergence in } E(\Sigma).$$

<u>**Proof**</u> The first part of this is Lemma 2.4. For the 2nd part, we will show that the partial sums, $\sum_{k=n}^{m} \tilde{\phi}_k \, u_k(S)$, form a Cauchy sequence in $E(\Sigma)$.Then, by completeness of $E(\Sigma)$ the series converges in $E(\Sigma)$ to some function $\psi(S)$. But both $\psi(S)$ and $\phi(S)$ will have the same expansion coefficients; which implies they agree in $L^2(\Sigma)$ and, hence, by continuity, $\psi = \phi$ on Σ. Here,

$$\Big|\Big| \sum_{k=n}^{m} \tilde{\phi}_k \, u_k(S) \Big|\Big|_{\alpha,K} = \sup_{S \varepsilon K} \Big| \sum_{k=n}^{m} \tilde{\phi}_k \, D^{\alpha} u_k(S) \Big|$$

$$< \Big(\sum_{k=n}^{m} \lambda_k^{2(|\alpha|+2)} |\tilde{\phi}_k|^2 \Big)^{1/2} \Big(\sum_{k=n}^{m} \sup_{S \varepsilon K} \frac{|D^{\alpha} u_k(S)|^2}{\lambda_k^{2(|\alpha|+2)}} \Big)^{1/2}$$

$$< C \Big(\sum_{k=n}^{m} \frac{1}{\lambda_k^2} \Big)^{1/2} ||\phi||_{2|\alpha|+4} \, ,$$

for some constant C and n,m sufficiently large by Lemma 2.5. By the first part of this theorem, this last expression converges to zero as $n,m \to \infty$ (recall, $\lambda_k \sim k$ as $k \to \infty$) ∎

<u>**Remark:**</u> This shows that the norms, $||\cdot||_{\alpha,K}$, are weaker w.r.t. convergence than the norms, $||\cdot||_m$. That is, since D^{α} is continuous on $E(\Sigma)$, for every $\phi \varepsilon E(\Sigma)$,

$$||\phi||_{\alpha,K} = \sup_{S\varepsilon K} \left|\sum_{k=1}^{\infty} \tilde{\phi}_k \, D^{\alpha}u_k(S)\right| < M \, ||\phi||_{2|\alpha|+4}$$

(using the same steps as in the above proof). However, the other direction, an estimate for $||\phi||_m$ in terms of a finite combination of the norms, $||\phi||_{\alpha,K}$, doesn't hold on $E(\Sigma)$. So, these two families of norms aren't equivalent. Also, in our setting (see footnote (4) page 56), a converse to Theorem 2.7 isn't true. Given a sequence, $\phi = (a_k)$, such that $||\phi||_m < \infty$ for all real $m \geq 0$; we can identify ϕ with the function

$$\phi(S) = \sum_k a_k \, u_k(S)$$

where, this series converges in $E(\Sigma)$. However, ϕ isn't necessarily in $D(\Sigma)$.

Theorem 2.8

Let $T\varepsilon D'(\Sigma)$ with compact support and set $\tilde{T}_k = \langle T, u_k \rangle$. Then, T has the eigenfunction expansion,

$$T = \sum_{k=1}^{\infty} \tilde{T}_k \, u_k \; ,$$

where we treat u_k as a regular distribution. Moreover,

$$|\tilde{T}_k| = O(\lambda_k^r) \text{ for some positive integer } r.$$

Note: This shows that distributions with compact support have finite order w.r.t. λ_k in their expansions. In other words, since $\lambda_k \sim k$ as $k\to\infty$, we can say that the identification of ϕ with its sequence of expansion coefficients $\tilde{\phi}_k$ represents $D(\Sigma)$ as sequences of rapid decrease; while, the identification of T with the coefficients $\tilde{T}_k$ represents $E'(\Sigma)$ as sequences of polynomial growth.

Proof The first part is immediate from Theorem 2.7. If $\phi\varepsilon D(\Sigma)$, then the expansion coefficients $\tilde{\phi}_k$ are exactly the evaluation $<u_k,\phi>$ and since,

$$\sum_k \tilde{\phi}_k \, u_k(S) = \sum_k u_k(S) <u_k,\phi>$$

converges in $E(\Sigma)$ to $\phi(S)$, by the continuity of T on $E(\Sigma)$ we have,

$$<T,\phi> = \sum_k <T,u_k> <u_k,\phi> = \sum_k \tilde{T}_k <u_k,\phi> .$$

Thus, $T = \sum_k \tilde{T}_k \, u_k$ on $D'(\Sigma)$.

For the next part, by Lemma 2.2, there exists a constant $C<\infty$ and a positive integer m such that

$|<T,\phi>| < C \, ||\phi||_{m,K}$, for every compact $K\subset\Sigma$, $\phi\varepsilon D(\Sigma)$ with $\text{supp}\,\phi\subset K$.

This implies, that in fact,

$$|\langle T,\phi\rangle| \leq C \sup_{\substack{|\alpha|\leq M \\ S\varepsilon\Sigma}} |D^{\alpha}\phi(S)| = C \sup_{S\varepsilon\Sigma} |D \;\phi(S)|$$

for some multi-index β with $|\beta|\leq m$ and for all $\phi\varepsilon D(\Sigma)$. Then, by the remark after Theorem 2.7, this shows that

$$|\langle T,\phi\rangle| \leq \tilde{C}\, ||\phi||_{2|\beta|+4}, \text{ for some constant } \tilde{C}\,.$$

In particular,

$$|\tilde{T}_k| = |\langle T,u_k\rangle| \leq \tilde{C}\, ||u_k||_{2|\beta|+4} = \tilde{C}\,\lambda_k^{|\beta|+2} \quad \blacksquare$$

Now, we can give the extension of H to distributions with compact support. This extension should agree with the original representation of H given in (1.15). The scheme is quite clear. Given the expansion coefficients of a function ϕ (1.15) shows how to define $H\phi$. Similarly, given $T\varepsilon\, D'(\Sigma)$ with compact support, Theorem 2.8 shows that the coefficients $\tilde{T}_k$ play the role of expansion coefficients. So, we will take the definition of H(T) dictated by (1.15), to be

$$H(T) = \lim_{z\downarrow 0} \sum_k \lambda_k^{1/2}\, e^{-\lambda_k^{1/2} z}\, \tilde{T}_k\, u_k \tag{2.3}$$

Since the added features $e^{-\lambda_k^{1/2} z}$ and limit as $z\downarrow o$ played no role in the definition of H within an L^2-setting and as illustrated by the example with δ_R in Lemma 2.6; we can expect that the representation in (2.3) can be simplified by bringing the limit as $z\downarrow o$ inside the series here, also.

Lemma 2.9

If $T\varepsilon\ D'(\Sigma)$ with compact support, then

(i) $H(T) = \lim\limits_{z\downarrow 0} \sum\limits_k \lambda_k^{1/2}\ e^{-\lambda_k^{1/2} z}\ \tilde{T}_k\ u_k$ defines a distribution in $D'(\Sigma)$.

moreover,

(ii) $H(T) = \sum\limits_k \lambda_k^{1/2} \tilde{T}_k\ u_k$

and we have the symmetric property that, for every $\phi\varepsilon D(\Sigma)$,

(iii) $\langle H(T), \phi\rangle = \langle T, H\phi\rangle$.

Proof We will first show that the series in (ii) converges in $D'(\Sigma)$. Here, for every $\phi\varepsilon D(\Sigma)$ with r specified by Theorem 2.8,

$$\left|\sum_{k=n}^{m} \lambda_k^{1/2} \tilde{T}_k \langle u_k, \phi\rangle\right| < \left(\sum_{k=n}^{m} \frac{|\tilde{T}_k|^2}{\lambda_k^{2r+2}}\right)^{1/2} \left(\sum_{k=n}^{m} \lambda_k^{2r+3} |\tilde{\phi}_k|^2\right)^{1/2}$$

$$< C \left(\sum_{k=n}^{m} \frac{1}{\lambda_k^2} \right)^{1/2} ||\phi||_{2r+3} \to 0 \text{ as } n,m \to \infty \text{ (by Thm 2.8 and Lemma 2.4).}$$

Thus, the partial sums of the series in (ii) evaluated at any $\phi \varepsilon D(\Sigma)$, form a Cauchy sequence and converges weak*.

Now, we will show that,

$$\sum_k \lambda_k^{1/2} e^{-\lambda_k^{1/2} z} \tilde{T}_k u_k \to \sum_k \lambda_k^{1/2} \tilde{T}_k u_k, \text{ in } D'(\Sigma), \text{ as } z \downarrow 0;$$

showing that $H(T)$ defined by (i) is in $D'(\Sigma)$ and satisfies (ii). For this, let $\phi \varepsilon D(\Sigma)$, then

$$|<\sum_k \lambda_k^{1/2} \tilde{T}_k u_k, \phi> - <\sum_k \lambda_k^{1/2} e^{-\lambda_k^{1/2} z} \tilde{T}_k u_k, \phi>|$$

$$= | \sum_k \lambda_k^{1/2} (1 - e^{-\lambda_k^{1/2} z}) \tilde{T}_k \tilde{\phi}_k | .$$

From here, the argument that this last series $\to 0$ as $z \downarrow 0$ is exactly analogous to the argument used in the proof of Lemma 1.3. The tail end of the above series can be made small, since, by the 1st part of this proof it is dominated by the series

$$\sum_{k=1}^{\infty} \lambda_k^{1/2} \tilde{T}_k \tilde{\phi}_k = < \sum_k \lambda_k^{1/2} \tilde{T}_k u_k, \phi> \text{ which converges.}$$

Then, the front end of the series, of finite length, can be made small by letting z get close to zero, so that, $1 - e^{-\lambda_k^{1/2} z}$ becomes small. For the last part, recall that by Lemma 2.4, if $\phi \varepsilon D(\Sigma)$, then $H\phi \varepsilon E(\Sigma)$ (i.e. of class $C^{\infty}(\Sigma)$). So, the R.H.S. of (iii) makes sense. By part (ii),

$$<H(T),\phi> = \sum_k \lambda_k^{1/2} \tilde{T}_k \; \tilde{\phi}_k = \sum_k \tilde{T}_k \; (\widetilde{H\phi})_k$$

$$= \sum_k \tilde{T}_k \; <u_k, H\phi> = < \sum_k \tilde{T}_k \; u_k, \; H\phi>$$

$$= <T,H\phi> \text{ (by Theorem 2.8) } \blacksquare$$

III. AN APPLICATION TO SMALL AMPLITUDE WATER WAVES

3.1 Derivation of the Basic Equations

The cross-surface differential operator H developed in chapter I is motivated through several models of physical field situations that contain a setting analogous to that used in the definition of H. In fact, a model for gravity waves on deep water will be presented where the setting is exactly the same as in Defn. 1.1 of H. This is a simple case of the more general model involving the linearized equations for internal water waves which will be presented in this section. The goal of this chapter is to demonstrate that, through the use of H, a single scalar wave equation for these models can be obtained by means of a classical linearized theory, i.e., by considering waves of an infinitesimal amplitude on a surface interface contained in an ideal fluid.

Let Σ, whose boundary γ is a sufficiently smooth line, be a region in the plane satisfying the conditions outlined in section 1.2. Then take B to be the 3-dimensional domain, formed in a rectangular coordinate system, as the cylindrical product of Σ, embedded in the plane z=o, and the interval (-d,d) on the z-axis. The boundary of B will be denoted by Γ which consists of the vertical side walls and the two end faces in the planes $z = \pm d$. Also, the two halves to this cylindrical region will be denoted by,

$$B_1 = \Sigma \times (-d,0) \quad \text{and} \quad B_2 = \Sigma \times (0,d)$$

with boundaries

$$\partial B_i = \Sigma \cup \Gamma_i \ , \text{ for } i = 1,2,$$

so, $\Gamma = \Gamma_1 \cup \Gamma_2$.

Σ is to be the static horizontal surface interface of a homogeneous, incompressible, inviscid non-rotating liquid contained in a basin given by the domain B defined above. The fluid is of density ρ_i in the regions B_i. We want to model the infinitesimal amplitude wave motion of Σ under the influence of gravity and surface tension. We will consider motion of the fluid with velocity vectors, $\overline{V}_i(P,t) = (U_i, V_i, W_i)\,(P,t)$, and pressure, $\overline{P}_i(P,t)$ in B_i, for $i = 1,2$, where a general field point in B is given by $P = (x,y,z)$ with $S = (x,y)$ representing a field point on Σ. Under motion the free fluid surface is denoted by Ω. Then the Eulerian equations of motion for this fluid take the form,

$$\rho \frac{D\bar{v}}{Dt} = -\nabla\bar{p} + \rho g, \text{ on } B \tag{3.1}$$

where $D/_{Dt}$ is the material derivative and g is the acceleration of gravity. Here, we have condensed into equation (3.1) what should be two equations of the same form with indices $i = 1,2$, for the respective halves of B. For brevity, this format will be maintained until the final form for the equations is derived. Assuming no fluid sources these equations have to be adjoined with the condition of incompressibility

$$\nabla \cdot \bar{v} = 0, \text{ on } B \tag{3.2}$$

and, disregarding surface tension for the moment, the boundary conditions are

$$\left.\frac{D\bar{p}}{Dt}\right|_{\bar{p}=0} = 0 \text{ , on } \Omega \tag{3.3}$$

with

$$\tilde{\eta} \cdot \bar{v} = 0, \text{ on } \Gamma \tag{3.4}$$

where $\tilde{\eta}$ is the outward normal to Γ.

The linearization of these equations is by a standard process where it is assumed that only "slow" wave motion of infinitesimal amplitude is to be considered. Also, the effects of the undulating surface Ω on the flow field are ignored but a perturbational pressure effect on Σ is incorporated. That is, if we assume that Ω deviates from Σ by a small vertical amplitude $h(s,t)$ which is positive up, then the pressure is of the form $\overline{p} = p_h + p$ where p_h is the hydrostatic pressure and $p(P,t)$ is a small perturbation pressure assumed to satisfy $|p| << |p_h|$ and can be approximated on Σ by ρgh. Under these assumptions, (3.1) is linearized as

$$\rho\ \partial_t \overline{V} = -\ \nabla p \tag{3.5}$$

which, in conjunction with (3.2) and (3.4), yields

$$-\nabla^2 p = 0, \text{ in } B \quad \text{and} \quad \frac{\partial p}{\partial \tilde{n}} = 0 \text{ on } \Gamma. \tag{3.6}$$

Also, a linearization of the boundary condition (3.3) which is further reduced and simplified by assuming the value of the vertical velocity component W is the same on Σ as on Ω, yields

$$\partial_t h + w = 0, \text{ on } \Sigma. \tag{3.7}$$

Then, inserting W from (3.7) into the vertical component of (3.8) gives the final form of the relation between the two dependent variables h and p as

$$\rho\ \partial_{tt} h = \partial_z p\ , \text{ on } \Sigma.$$

The pressure balance on Σ takes the form

$$p_2\Big|_{z\downarrow 0} = p_1\Big|_{z\uparrow 0} + (\rho_2 - \rho_1)gh.$$

However, this can be generalized to include a surface tension term and allow for an impressed surface pressure on Σ. The force density due to a coefficient of surface tension τ is given by, $\tau\,\nabla^2 h$ [5].

So, in summary, the linearized equations for the infinitesimal amplitude wave motion of an internal surface interface are given by

$$\rho_i\ \partial_{tt}h = \partial_z p_i\Big|_{\Sigma} \tag{3.8}$$

with

$$\nabla^2 p_i = 0, \text{ in } B_i \text{ and } \frac{\partial p_i}{\partial \tilde{n}}\Bigg|_{\Gamma_i} = 0, \text{ for } i=1,2 \tag{3.9}$$

and the pressure balance on Σ is now of the form

$$p_2\Big|_{z\downarrow 0} = p_1\Big|_{z\uparrow 0} + (\rho_2 - \rho_1)gh + \tau\,\nabla^2 h + f \tag{3.10}$$

where $f(S,t)$ represents an impressed surface pressure.

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[5] This is a linearization of the surface tension forces which arise from the curvature of Ω, where we have kept only 2nd order derivatives.

A more simple case of this model, known as gravity waves on deep water, is obtained by considering the top region B_1 (it is typical to orient the basins with the z-axis as positive down) as the atmosphere and letting d go to infinity. That is, we set $\rho_1 = o$ and consider Σ as the static horizontal surface to a liquid contained in a basin of infinite extent where the aim is to model the wave motion of Σ under the influence of gravity alone. Then, dropping the index i=2, the equations governing this wave motion, in the linearized infinitesimal approximation, are easily obtained from (3.8) - (3.10) as

$$\nabla^2 p = 0, \text{ in } B \text{ with } \left.\frac{\partial p}{\partial \tilde{n}}\right|_{\Gamma} = 0 \tag{3.11}$$

$$\rho\, \partial_{tt} h = \left.\partial_z p\right|_{z \downarrow 0} \tag{3.12}$$

with the pressure on Σ given by

$$\left. p \right|_{z \downarrow 0} = \rho g h + \rho g h_0 \tag{3.13}$$

where we have represented the impressed surface pressure in terms of an impressed surface amplitude, $h_0(s,t)$.

3.2 Reformulation in terms of H

The form of the equations governing internal water waves in (3.8) - (3.10) and for gravity waves in (3.11) - (3.13) is unsatisfactory since they include two dependent variables h and p. However, one reason for explicitly writing out the equations for the case of gravity waves on deep water is because it is easily recognized that this is precisely the setting presented in the definition of H in section 1.2. From (3.11), we see that the perturbation pressure p is harmonic on B with a homogeneous Neumann boundary condition on Γ. Hence, from Defn. 1.1, its cross-surface derivative on Σ can be expressed in terms of its values on Σ with the help of the surface operator H. That is,

$$H\left(p \Big|_{z \downarrow 0}\right) = -\partial_z p \Big|_{z \downarrow 0} . \tag{3.14}$$

But, then, using (3.12) and (3.13), the dependency on p can be eliminated with the resulting equation

$$\partial_{tt} h + gHh = f \tag{3.15}$$

where the source term is

$$f = -gHh_0 . \tag{3.16}$$

Equation (3.15) represents a reformulation of the equations for gravity waves on deep water which was our goal. This gives a single scalar equation which we can refer to as the basic gravity wave equation governing this example. The process is the same for the internal water waves, if we again consider the case where d goes to infinity. It is only necessary to recognize that a change of sign is needed in the definition of H if the domain is in the negative z half-space. That is, from (3.9) we have

$$H(p_2 \mid_{z\downarrow 0}) = -\partial_z p_2 \mid_{z\downarrow 0} \text{ and } H(p_1 \mid_{z\uparrow 0}) = \partial_{\bar{z}} p_1 \mid_{z\uparrow 0} . \tag{3.17}$$

Then, using (3.8) and (3.17) above we obtain

$$\rho_2 \, \partial_{tt} h = - H \, (p_2 \mid_{z\downarrow 0}) \text{ and } \rho_1 \, \partial_{tt} h = H(p_1 \mid_{z\uparrow 0}). \tag{3.18}$$

Hence, from the pressure balance equation in (3.10) and using (3.18) above, we can eliminate the dependency on the pressure with the resulting equation

$$(\rho_1+\rho_2) \, \partial_{tt} h + (\rho_2 - \rho_1) gHh + \tau \, H \, \nabla^2 h = -Hf. \tag{3.19}$$

Equations (3.15) and (3.19) illustrate how this reformulation has led to a new type of wave operator. From Chapter I, we have seen that, in the appropriate setting, H behaves as a square-root of the Laplacian. So the equations (3.15) and (3.19) give rise to wave operators where the spatial part of the operator involves fractional powers of the Laplacian, given by

$$\text{(\underline{Gravity Waves})} \qquad \Box_G = \partial_{tt} + gH \tag{3.20}$$

$$\text{(\underline{Internal Water Waves})} \qquad \Box_I = \partial_{tt} + \left(\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\right) gH + \frac{\tau}{\rho_1 + \rho_2} H^3 \tag{3.21}$$

where $H = \Pi^{1/2}$.

3.3 A Representation of Solutions in terms of the Impulse Response and Examples for Various Basins

The remainder of this chapter will be spent considering some specific examples of the model for gravity waves where a formal eigenfunction expansion technique will be employed to explicitly find solutions. Also, a representation of solutions in terms of the Green's function for the gravity wave operator in (3.20) will be obtained illustrating a direct analogy with the classical wave operator.

To solve a mixed boundary value problem involving the wave equation in (3.15) some care is needed to interpret this equation correctly. In the first two chapters several settings were presented where the operator H could be viewed as a surface operator independent of the region B. However, within the context of the water wave model, it is important to recognize that H maintains a dependency on B. For the most part, no attempt will be made to justify the solution technique and say in what sense a solution is given, since the subject of gravity waves on deep water is well developed and the solutions that are presented are usually well known. A more in depth analysis of this problem will be taken up in the last two chapters where the formal treatment in this section will be justified within two classes of solutions.

We will begin by obtaining an eigenfunction expansion for the impulse responce $h_p(S,t)$ of an unperturbed static fluid found by assuming causal conditions and, that, at time $t = o^+$, Σ is hit by a delta-like pressure pulse centered at $R \varepsilon\ \Sigma$, viz., we take

$$h_0(S,t) = \frac{1}{\rho g}\ \delta_+\ (t)\ \ \delta(S-R). \tag{3.22}$$

To obtain a solution to (3.15) with h_o given in (3.22), we assume an expansion for h_p of the form

$$h_p(S,t) = \sum_k a_k(t)\ u_k(S). \tag{3.23}$$

Recall that (see (2.1)) $\delta(S-R)$ has the bilinear expansion

$$\delta(S-R) = \sum_k u_k(R)\, u_k(S).$$

So, h_0 can be written as

$$h_0(S,t) = \frac{1}{\rho g}\, \delta_+(t) \sum_k u_k(R)\, u_k(S). \tag{3.24}$$

Formally applying H by (1.15), equation (3.15) with h_p from (3.23) and h_0 in (3.24) above becomes

$$\sum_k a_k''(t) u_k(S) + \lim_{z \downarrow 0} \sum_k \lambda_k^{1/2} e^{-\lambda_k^{1/2} z} a_k(t) u_k(S) = -\frac{1}{\rho}\, \delta_+(t) \lim_{z \downarrow 0} \sum_k \lambda_k^{1/2} e^{-\lambda_k^{1/2} z} u_k(R) u_k(S).$$

At this point, the solution method is to obtain an equation for the modes separately, by assuming the limit as $z \downarrow o$ can be brought inside the summation, yielding

$$a_k''(t) + g\lambda_k^{1/2} a_k(t) = -\frac{1}{\rho}\, \lambda_k^{1/2} u_k(R)\, \delta_+(t), \text{ for } t>0 \tag{3.25}$$

with $a_k \equiv o$ for $t \leq o$ (by causality).

However, we retain the dependency of H on B (i.e. the method of extending a function harmonically off Σ and then taking a limit back to Σ as $z \downarrow o$) by inserting the solution to (3.25) back into h_p written in the form

$$h_p(S,t) = \lim_{z \downarrow 0} \sum_k e^{-\lambda_k^{1/2} z} a_k(t)\, u_k(S). \tag{3.26}$$

The solution for $a_k(t)$ in (3.25) is straightforward and yields

$$a_k(t) = -\frac{1}{\rho} (\lambda_k^{1/2}/g)^{1/2} u_k(R) \sin(g^{1/2} \lambda_k^{1/4} t), \text{ for } t>0.$$

Thus, plugging this into (3.26) we have our solution. Recapitulating, we have demonstrated the following result.

Lemma 3.1

The impulse response to

$$(\partial_{tt} + gH)\, h_p(S,t) = -g\, H\, h_0(S,t), \text{ for } S \varepsilon \Sigma,\ t>0$$

with $h_p \equiv 0$ for $t<0$, where

$$h_0(S,t) = \frac{1}{\rho g} \delta_+(t)\ \delta(S-R)$$

is an impressed surface amplitude, is given by

$$h_p(S,t) = -\frac{1}{\rho g^{1/2}} \lim_{z \downarrow 0} \sum_k \lambda_k^{1/4} e^{-\lambda_k^{1/2} z} \sin(g^{1/2} \lambda_k^{1/4} t)\ u_k(R)\ u_k(S),\ t>0 \tag{3.27}$$

where $u_k(S)$ is to be interpreted as a regular distribution with (3.27) converging weak*.

Let $\Box = \partial_{tt} + c^2 H$. Then, the general mixed boundary value problem with this wave operator is

$$\Box\, u(S,t) = f(S,t)\ ,\ \text{for } S\varepsilon\Sigma,\ t>0$$

(M.B.V.P.1)

$$\text{with} \quad \begin{matrix} u(S,0) = \phi(S) \\ \partial_t u(S,0) = \psi(S) \end{matrix} \quad \text{and} \quad \frac{\partial u}{\partial \tilde{n}}\bigg|_{\gamma} = 0.$$

A representation for the solution $u(S,t)$ is given in terms of a Green's function $G(S,R;t)$, for the operator $\Box$. $G(S,R;t)$ is a solution to the problem with homogeneous boundary conditions and a δ-like non-homogeneous term. That is $G(s,R;t)$ is the solution to

$$\Box_S\, G(S,R;t) = \delta_+(t)\ \delta(S-R),\ \text{for } S\varepsilon\Sigma,\ t>0$$

(M.B.V.P.2)

$$G\equiv 0 \text{ for } t<0 \text{ and } \frac{\partial G}{\partial \tilde{n}_S}\bigg|_{\gamma} = 0.$$

A solution to M.B.V.P.2 can be obtained in exactly the same manner as the impulse responce h_p to (3.15). However, notice that the only difference in the two problems is that for G, $f(S,t)$ is precisely the δ-like term $\delta_+(t)\ \delta(S-R)$ while for h_p, we first apply $-1/\rho\, H$. This suggests that G can be obtained from the impulse response already found in (3.27) by

$$G(S,R;t) = -\rho\, H^{-1}\, h_p(S,t) \ , \qquad (3.28)$$

where, it is clear that an eigenfunction representation for H^{-1}, obeying the formal calculus rule of bringing the exponent -1 inside to the eigenvalues plus retaining a dependency on B, is given by

$$H^{-1}\phi(S) = \lim_{z\downarrow 0} \sum_k \lambda_k^{-1/2} e^{-\lambda_k^{1/2} z}\, \tilde{\phi}_k\, u_k(S) \ . \qquad (3.29)$$

Applying (3.28) by (3.29) to $h_p(S,t)$ given in (3.27), where c^2 replaces g, and the expansion coefficients for h_p are

$$(\tilde{h}_p)_k = -\frac{1}{c\rho}\, \lambda_k^{1/4} \sin\,(c\ \lambda_k^{1/4} t)\ u_k(R) \ ,$$

we obtain

$$G(S,R;t) = \frac{1}{c} \lim_{z\downarrow 0} \sum_k \lambda_k^{-1/4} e^{-\lambda_k^{1/2} z} \sin(c\ \lambda_k^{1/4} t) u_k(R) u_k(S), \text{ for } t>0. \qquad (3.30)$$

Now, we want to demonstrate the following representation for the solution to M.B.V.P.1.

Lemma 3.2

The solution to M.B.V.P.1 has the formal representation in terms of the Green's function, $G(S,R;_t)$ the solution to M.B.V.P.2 given in (3.30), by

$$u(S,t) = \int_0^t \int_\Sigma G(S,R;\ t-\tau)\ f(R,\tau)\ d\Sigma_R d\tau + \int_\Sigma G(S,R;\ t)\ \psi(R)\ d\Sigma_R$$

$$+ \int_\Sigma \partial_t\ G(S,R;\ t)\ \phi(R)\ d\Sigma_R\ .$$

Proof We will actually obtain the above representation by first solving M.B.V.P.1 through an eigenfunction expansion method and then rewrite the solution by making use of the expansion for $G(S,R;t)$ in (3.30).

Assume that $u(S,t)$ and all data functions have eigenfunction expansions, i.e., take

$$u(S,t) = \sum_k a_k(t)\ u_k(S)$$

with

$$f(S,t) = \sum_k f_k(t)\ u_k(S)\ ,\ \phi(S) = \sum_k \tilde{\phi}_k u_k(S) \text{ and } \psi(S) = \sum_k \tilde{\psi}_k\ u_k(S).$$

Formally plugging these into M.B.V.P.1 we obtain, in the same manner as for the impulse response, the equation for the modes separately as

$$a_k''(t) + c^2\ \lambda_k^{1/2}\, a_k(t) = f_k(t),\ \text{for } t>0$$

with

$$a_k(0) = \tilde{\phi}_k \text{ and } a_k'(0) = \tilde{\psi}_k.$$

Then, (see Vladimorov (1971) pg. 147) $a_k(t)$ is given by

$$a_k(t) = (E_k * f_k)\ (t) + g_k(t),\ \text{for } t>0 \tag{3.31}$$

where $g_k(t)$ is the solution to

$$g_k''(t) + c^2\lambda_k^{1/2}\ g_k(t) = 0,\ \text{for } t<0 \text{ and } g_k(0) = \tilde{\phi}_k,\ g_k'(0) = \tilde{\psi}_k \tag{3.32}$$

and $E_k(t)$ is the fundamental solution to

$$E_k''(t) + c^2\ \lambda_k^{1/2}\, E_k(t) = \delta_+(t), \text{with } E_k \equiv 0 \text{ for } t<0. \tag{3.33}$$

The solutions to (3.32) and (3.33) are straightforward yielding

$$g_k(t) = \tilde{\phi}_k \cos(c\lambda_k^{1/4}\, t) + \tilde{\psi}_k/c\lambda_k^{1/4} \sin(c\ \lambda_k^{1/4}\, t)$$

and

$$E_k(t) = \frac{1}{c\lambda_k^{1/4}} \sin(c\lambda_k^{1/4} t), \text{ for } t>0.$$

Inserting these into (3.31) we obtain

$$a_k(t) = \tilde{\phi}_k \cos(c\lambda_k^{1/4} t) + \frac{\tilde{\psi}_k}{c\lambda_k^{1/4}} \sin(c\lambda_k^{1/4} t) + \frac{1}{c\lambda_k^{1/4}} \int_0^t \sin[c\lambda_k^{1/4}(t-\tau)] f_k(\tau) d\tau \tag{3.34}$$

Thus, the solution $u(S,t)$ is found by plugging (3.34) above into

$$u(S,t) = \lim_{z \downarrow 0} \sum_k e^{-\lambda_k^{1/2} z} a_k(t)\, u_k(S) \text{ , for } t>0 . \tag{3.35}$$

Finally, if we write the expansion coefficients in integral form as

$$\tilde{\phi}_k = \int_\Sigma \phi(R)\, u_k(R)\, d\Sigma_R, \; \tilde{\psi}_k = \int_\Sigma \psi(R)\, u_k(R)\, d\Sigma_R \text{ and}$$

$$f_k(t) = \int_\Sigma f(R,t)\, u_k(R)\, d\Sigma_R$$

and formally bring the summation inside the integral, then (3.35) becomes

$$u(S,t) = \lim_{z\downarrow 0} \Big\{ \int_0^t \int_\Sigma \frac{1}{c} \sum_k \lambda_k^{-1/4} e^{-\lambda_k^{1/2} z} \sin[c\lambda_k^{1/4}(t-\tau)] f(R,\tau) u_k(R) u_k(S)\, d\Sigma_R\, d\tau$$

$$+ \int_\Sigma \sum_k e^{-\lambda_k^{1/2} z} \cos(c\lambda_k^{1/4} t)\; \phi(R)\; u_k(R)\; u_k(S)\; d\Sigma_R$$

$$+ \int_\Sigma \frac{1}{c} \sum_k \lambda_k^{-1/4} e^{-\lambda_k^{1/2} z} \sin(c\lambda_k^{1/4} t)\; \psi(R)\; u_k(R) u_k(S)\, d\Sigma_R \Big\} .$$

Hence, bringing the limit as $z\downarrow 0$ inside the integrals and making use of the expansion for $G(S,R;t)$ in (3.30), we obtain the representation given in the statement of the Lemma ∎

The solution given in Lemma 3.2 resembles the classical Green's function representation to the initial and boundary value problem for the wave equation with the operator $\partial_{tt} - c^2\nabla^2$ (see Duff and Naylor, 1966). We are missing a boundary integral over γ since a homogeneous Neumann condition on γ is specified. It appears that a treatment of boundary conditions on γ must be omitted since there is no formula for H analogous to Green's second formula which is used in the classical problem to handle non-homogeneous boundary conditions. This goes back to the original discussion of a square-root of Π and the appropriate domain for a square-root. In the construction of H, in Chapter I, the treatment of boundary conditions to be included in its domain of definition is avoided. The presentation, there, was more analytic in nature with an emphasis on obtaining an eigenfunction expansion for H giving a square-root of Π, valid in $L^2(\Sigma)$. Hence, it was only necessary to include a growth condition on the expansion coefficients.

So, it seems, an important issue that will have to be avoided concerns the appropriate construction of H within a boundary value problem context. That is, is H suitable for the treatment of mixed boundary value problems, such as M.B.V.P.1, but with non-homogeneous boundary conditions on γ?

Certainly, one way to avoid this difficulty, concerning the model of gravity waves on deep water, is to consider basins where Σ is unbounded. In fact, of special interest is the basin of infinite extent where Σ is the whole plane.

We will finish this section by presenting, explicitly, the impulse response for the basin where $\Sigma = R^2$ and several other basins of interest.

Example 1: <u>**The Unbounded Half-Space Basin**</u>

In this case, $\Sigma = R^2$ and $B = \Sigma \times (o,\infty)$. The eigenfunctions and eigenvalues for Π depend on the continuous index $\vec{k} = (k_1, k_2)$ and are given by

$$u_{\vec{k}}(x,y) = \frac{1}{2\pi} e^{-i(k_1x+k_2y)}, \text{ with } \lambda_{\vec{k}} = k_1^2 + k_2^2 = k^2.$$

Hence, the impulse response is found by inserting these into (3.27) and observing that the summation becomes an integration over $\vec{k}$. This yields

$$h_p(x,y;t) = \frac{1}{4\pi\rho g^{1/2}} \lim_{z\downarrow 0} \int_{-\infty}^{\infty}\int k^{1/2} e^{-kz} \sin(g^{1/2} k^{1/2} t) e^{-i[k_1(x-x')+k_2(y-y')]} d\vec{k}$$

(the complex conjugate of $u_k(R)$ appears since the eigenfunctions were not taken as real-valued). Here, $S = (x,y)$ and the source point is $R = (x',y')$. Placing the source at the origin and using polar coordinates gives

$$h_p(x,y;t) = -\frac{1}{2\pi\rho g^{1/2}} \lim_{z\downarrow 0} \int_0^{\infty} k^{3/2} e^{-kz} \sin(g^{1/2} k^{1/2} t) dk \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \bar{e}^{ikr\cos(\theta-\theta')} d\theta\right),$$

where r is the distance between S and the origin and $\Theta-\Theta'$ is the angle between $\vec{k}$ and $S = (x,y)$. Now, the inner integral can be written as the Bessel function $J_0(kr)$ and by symmetry, since this solution only depends on the distance between the source and field points, the source can be restored to its original position $R = (x',y')$ yielding

$$h_p(r,t) = -\frac{1}{2\pi\rho g^{1/2}} \lim_{z\downarrow 0} \int_0^{\infty} k^{3/2} \bar{e}^{kz} \sin(g^{1/2} kt) J_0(kr) dk, \text{ for } t>0 \qquad (3.36)$$

where r is now the distance between S and the source point R.

Example 2: The Spherical Basin

Here, we take B to be the open ball of radius a centered at the origin and $\Sigma = \partial B$. As with the Half-space problem, this avoids the treatment of boundary conditions. Now, the operator H is to be a square-root of the sperical surface Laplacian

$$-\Delta_s = -\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\psi^2}\right).$$

The eigenfunctions and eigenvalues for $-\Delta_s$ on Σ are of the form

$$P_n^m(\cos\theta)\, e^{im\psi}, \text{ with } \lambda_n = n(n+1) ,$$

involving the Legendre polynominals $P_n^m(\cos\Theta)$ and where we have multiplicity 2n + 1 as m runs from -n to n.

Any boundary value $\phi(\theta,\psi)$ defined on Σ has the expansion (see Duff and Naylor, 1966 pg. 345) for a treatment of eigenfunctions of a spherical surface and expansions in terms of them)

$$\phi(\theta,\psi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\tilde{\phi}_{nm}P_n^m(\cos\theta)\, e^{im\psi}$$

with expansion coefficients

$$\tilde{\phi}_{nm} = \frac{(-1)^m (n+1/2)}{2\pi} \int_\Sigma \phi(\theta,\psi)\, \bar{P}_n^m(\cos\theta)\, e^{-im\psi}\, d\Sigma_{(\theta,\psi)}$$

(here, $d\Sigma_{(\theta,\psi)} = \sin\theta d\theta d\psi$). The harmonic extension to B is given by

$$u(r,\theta,\psi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \tilde{\phi}_{nm} \left(\frac{r}{a}\right)^n P_n^m(\cos\theta)\, e^{im\psi}.$$

Hence, a representation for H, analogous to (1.15), can now be given by

$$H\phi(\theta,\psi) = \lim_{r\uparrow a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [n(n+1)]^{1/2} \left(\frac{r}{a}\right)^n \tilde{\phi}_{nm} P_n^m(\cos\theta)\, e^{im\psi}. \tag{3.37}$$

A source function for the operator $\square = \partial_{tt} + c^2 H$ is obtained by solving

$$\square_{(\theta,\psi)} G(\theta,\psi;\theta',\psi';t) = \delta_+(t) \frac{\delta(\theta-\theta')\ \delta(\psi-\psi')}{\sin\theta}, \text{ for } t>0 \tag{3.38}$$

with $G \equiv o$ for $t<o$.

we assume an expansion for G by

$$G(\theta,\psi;\ \theta',\psi';t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} g_{nm}(t)\ P_n^m(\cos\theta)\ e^{im\psi}$$

and expand the R.H.S. of (3.38) as

$$\delta_+(t) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(-1)^m(n+\frac{1}{2})}{2\pi}\ P_n^m\ (\cos\theta)\ \bar{P}_n^m(\cos\theta')\ e^{im(\psi-\psi')}.$$

Formally plugging these into equation (3.38), a differential equation for the modes separately is obtained as

$$g''_{nm}(t) + c^2[n(n+1)]^{1/2}\ g_{nm}(t) = \frac{(-1)^m(n+\frac{1}{2})}{2\pi}\ \bar{P}_n^m\ (\cos\theta')\ e^{-im\psi'}\ \delta_+(t).$$

The solution to this is given by

$$g_{nm}(t) = \frac{(-1)^m(n+\frac{1}{2})}{2\pi\ c[n(n+1)]^{1/4}}\quad \sin\ [c\ [n(n+1)]^{1/4}\ t],\ \text{for } t>0.$$

Thus, the source function is given by

$$G(\theta,\psi;\theta',\psi',t)=\frac{1}{2\pi c}\lim_{r\to a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^m(n+\tfrac{1}{2})[n(n+1)]^{1/4}\,(\tfrac{r}{a})^n \sin\ [c[n(n+1)]^{1/4}\,t] \tag{3.39}$$

$$P_n^m\ (\cos\theta)\ \bar{P}_n^m(\cos\theta')\ e^{im(\psi-\psi')}.$$

Example 3: The Layered space or Basin of Finite Depth

In this case, we take $\Sigma = R^2$ and $B = \Sigma x(o,d)$. We now have an extra boundary face and write, $\partial B = \Sigma \cup \Gamma \cup \Sigma_d$, where $\Sigma_d = \Sigma x\{d\}$. The development of H goes exactly as in Chapter I, except now, we include the added boundary condition

$$\left.\frac{\partial u}{\partial \tilde{n}}\right|_{\Sigma_d} = 0$$

in B.V.P.* of section 1.2. Hence, the harmonic extension of a function $\phi(S)$ defined on Σ takes on a slightly different form than given in (1.11). A reflection factor will be included. That is, as before, assuming an eigenfunction expansion for u(P) and plugging into the boundary value problem now yields for the modes $a_k(z)$ a differential equation in the form

$$a_k''(z) = \lambda_k a_k(z), \text{ for } 0<z<d, \text{ with } a_k(0) = \tilde{\phi}_k \text{ and } a_k'(d) = 0.$$

The solution to this can be written in terms of hyperbolic trigonometric functions as

$$a_k(z) = \tilde{\phi}_k \frac{\cosh\left(\lambda_k^{1/2}(z-d)\right)}{\cosh\left(\lambda_k^{1/2} d\right)} .$$

Hence, an expansion for H is given by

$$H\phi(S) = \lim_{z\downarrow 0} \sum_k \lambda_k^{1/2} \frac{\sinh(\lambda_k^{1/2}(d-z))}{\cosh(\lambda_k^{1/2} d)} \tilde{\phi}_k \, u_k(S) \ . \tag{3.40}$$

The model for gravity waves goes as before with (3.15) the governing equation. The impulse response is found by assuming expansions for $h_p(S,t)$ and $h_o(S,t)$ as given before in (3.22) and (3.23) and obtaining the equation for the modes separately, by plugging these into (3.15) with the aid of (3.40), as

$$a_k''(t)+g\frac{\sinh(\lambda_k^{1/2} d)}{\cosh(\lambda_k^{1/2}\ d)}\lambda_k^{1/2}\, a_k(t) = -\frac{1}{\rho}\, \frac{\sinh(\lambda_k^{1/2} d)}{\cosh(\lambda_k^{1/2} d)}\, \lambda_k^{1/2}\, u_k(R)\ \ \delta_+(t) . \tag{3.41}$$

As before, we insert the solution to (3.41) back into $h_p(S,t)$ written in the form

$$h_p(S,t) = \lim_{z\downarrow 0} \sum_k \frac{\cosh(\lambda_k^{1/2}(z-d))}{\cosh(\lambda_k^{1/2}\ d)}\ a_k(t)\ u_k(S) \ . \tag{3.42}$$

The solution to (3.41) is given by

$$a_k(t) = -\frac{1}{\rho g^{1/2}} \tanh^{1/4}(\lambda_k^{1/2} d)\, \lambda_k^{1/4}\, u_k(R)\ \sin[g^{1/2} \tanh^{1/2}(\lambda_k^{1/2} d)\ \lambda_k^{1/4}\, t], \text{ for } t>0.$$

Thus, the Impulse response has the expansion

$$h_p(S,t) = - \frac{1}{\rho g^{1/2}} \lim_{z \downarrow 0} \sum_k \lambda_k^{1/4} \frac{\tanh^{1/2}(\lambda_k^{1/2} d)}{\cosh(\lambda_k^{1/2} d)} [\cosh \lambda_k^{1/2} (z-d)] \quad (3.43)$$

$$\sin [g^{1/2} \tanh^{1/2} (\lambda_k^{1/2} d) \lambda_k^{1/4} t] \; u_k(R) \; u_k(S) \; ,$$

which is valid for t>0. Now, exactly as before, the eigenfunctions and eigenvalues are

$$u_{\vec{k}}(x,y) = \frac{1}{2\pi} e^{-i(k_1 x + k_2 y)}, \text{ with } \lambda_{\vec{k}} = k^2 = k_1^2 + k_2^2$$

and these are inserted into (3.43) where the summation is replaced by an integration over $\vec{k}$. Again, this is first simplified by placing the source at the origin and then making use of radial symmetry in $\vec{k}$-space to obtain the Impulse response for a basin of infinite extend and finite depth as

$$h_p(r,t) = - \frac{1}{2\pi\rho g^{1/2}} \lim_{z \downarrow 0} \int_0^\infty k^{3/2} \frac{\tanh^{1/4}(kd)}{\cosh (kd)} \cosh [k(z-d)]$$

$$\sin[g^{1/2} \tanh^{1/2}(kd) \; k^{1/2} t] \; J_0(kr) \; dk,$$

$$(3.44)$$

valid for t>o, where $r = |S-R|$.

Example 4: **The Swimming Pool Basin**

For this problem we take Σ to be the rectangular region given by, $o\leq x\leq a$, $o\leq y\leq b$, embedded in the plane $z = o$, with $B = \Sigma x(o,d)$. We can make use of all the work in the previous example, since there, we first obtained the expansion for H in (3.40) and the Impulse response $h_p(S,t)$, in (3.43) for a general region Σ with eigenfunctions $u_k(S)$ and eigenvalues λ_k. By a separation of variables, these are easily shown to be

$$u_{nm}(x,y) = \sqrt{\frac{4\ \gamma_{nm}}{ab}}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \tag{3.45}$$

with

$$\lambda_{nm} = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

where, $n,m = 0,1,2,\ldots$ and $\gamma_{00} = 1/4,\ \gamma_{n0} = \gamma_{0m} = 1/2$ with $\gamma_{nm} = 1$ for both $n>0$ and $m>0$.

Now, the Impulse response for the swimming pool basin can easily be obtained by inserting (3.45) into (3.43) yielding

$$h_p(x,y;\ x',y';t) = -\frac{4\ \pi^{1/2}}{ab\rho g^{1/2}}\lim_{z\to 0}\sum_{n,m=0}^{\infty}\gamma'_{nm}\ \Gamma_{nm}^{1/4}\ \frac{\tanh^{1/2}(\pi\ \Gamma_{nm}^{1/2}d)}{\cosh\ (\pi\ \Gamma_{nm}^{1/2}d)}\ \times$$

$$\times \cosh(\pi\, \Gamma_{nm}^{1/2}(z-d)\ \sin[(\pi g)^{1/2} \tanh^{1/2}(\pi\, \Gamma_{nm}^{1/2} d)\, \Gamma_{nm}^{1/2} t] \times$$

$$\times \cos(\frac{m\pi x}{a})\cos(\frac{m\pi x'}{a})\cos(\frac{n\pi y}{b})\cos(\frac{n\pi y'}{b})$$

(3.46)

where $\gamma'_{nm} = \gamma_{nm}$ as above except, $\gamma'_{00}=0$ and $\Gamma_{nm} = \frac{m^2}{a^2} + \frac{n^2}{b^2} = \frac{\lambda_{nm}}{\pi^2}$.

IV. A MIXED PROBLEM INVOLVING THE WAVE OPERATOR $\partial_{tt} + c^2H$

4.1 An Adaptation of the Classical Method of Energy Integrals

In this chapter we will take up an analysis of the operator $\square = \partial_{tt} + c^2H$ within a general initial and boundary value problem given as

$$\square h(S,t) = f(S,t), \text{ for } (S,t) \in \Sigma \times (0,\infty)$$

<u>W.E.1</u> with

$$h\Big|_{t=0^+} = h_0 \text{ and } \partial_t h\Big|_{t=0^+} = h_1 \text{ and } \alpha\, h + \beta \frac{\partial h}{\partial \tilde{n}}\Big|_{\gamma} = 0 \; ^{(6)}$$

We have seen numerous examples of an eigenfunction expansion technique applied to wave equations, such as W.E.1, to obtain solutions. Here, the goal is to justify in what sense these expansions converge and define solutions. This will be done from two points of view. In one approach, it will be shown that the series expansions converge in $L^2(\Sigma)$ uniformly w.r.t. time on bounded intervals. Then, the series is a solution defining a function which is continuous in $L^2(\Sigma)$ w.r.t. $t \in [0.\infty)$. Within this class of functions, an exis-

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(6) We will now be considering eigenfunctions and eigenvalues for Π with this boundary condition where α and β satisfy the conditions originally spelled out in Section 1.2

tence, uniqueness and continuous dependence on the data theorem will be presented. A second approach, taken up in Chapter V, will be to consider a generalized wave equation where it will be proven, under mild restrictions imposed on the data, that a formal eigenfunction expansion yields a distributional solution.

In both of the above settings, H is treated as a surface operator independent of the region B. That is, within an L^2 setting, the definition of H is obtained from Lemma 1.3 (where $D_H = \{\phi:$ condition (A) holds $\}$) as

$$H\phi(S) = \sum_k \lambda_k^{1/2} \tilde{\phi}_k \; u_k(S) \;, \tag{4.1}$$

and, as an operator on distributions with compact support, the definition of H is given by (ii) of Lemma 2.9 as

$$H(T) = \sum_k \lambda_k^{1/2} \; \tilde{T}_k \; u_k, \text{ where } \tilde{T}_k = \langle T,u_k \rangle. \tag{4.2}$$

In either case, the factor $e^{-\lambda_k^{1/2} z}$ and limit as $z \downarrow o$ are omitted since, within these settings, it has been shown that it is permissable to pass the limit through the summation. This justifies the earlier solution technique used to obtain the equation governing the modes separately. Moreover, in the final form of the solutions, the series expansions remain independent of B, with the exponential factor and limit omitted there, also,

The first approach is analogous to the classical method of solving a mixed problem for an equation of hyperbolic type, with the operator $\partial_{tt} - c^2\Delta$. It will be helpful to begin by first illustrating the techniques to be used on this classical wave equation. This will also suggest how to adjust these techniques to apply to the new problem in W.E.1.

The setting is generalized to include the operator L defined earlier in section 1.2 as

$$L = -\mathrm{div}(p\ \mathrm{grad}) + q.$$

It is also possible to incorporate the constant c^2 into a general density function $\rho(S)$, defined on Σ using the operator $\rho\partial_{tt} + L$, where the work is carried out in the space $L^2(\Sigma;\rho)$. However, the definition of H forces us to restrict to the case where ρ is constant. To avoid changing the setting when L is replaced by H and since there are previous examples involving $\partial_{tt} + c^2H$, the classical problem will be illustrated using the operator $\partial_{tt} + c^2L$.

The problem under consideration is

$$\partial_{tt} u + c^2 L u = F, \text{ on } U_\infty$$

W.E.2

$$\text{with} \quad u\Big|_{t=0^+} = v_0 \qquad\qquad \text{and} \quad \alpha u + \beta \frac{\partial u}{\partial \tilde{n}}\Big|_\gamma = 0.$$

$$\partial_t u\Big|_{t=0+} = v_1$$

(where U_∞ is the time cylinder $\Sigma \times (o,\infty)$).

A classical solution to W.E.2 is defined as a solution of class $C^2(U_\infty) \cap C^1(\bar{U}_\infty)$. Notice, if u is a classical solution then, necessarily, we have

$$F \varepsilon C(U_\infty) \ , \ v_0 \varepsilon C^1(\bar{\Sigma}), \ v_1 \varepsilon C(\bar{\Sigma}) \text{ and } \alpha\, v_0 + \beta \frac{\partial v_0}{\partial \tilde{n}}\Big|_\gamma = 0.$$

These conditions will always be adjoined to the problem when dealing with classical solutions.

The approach to this problem makes use of the method of energy integrals which represents the sum of the kinetic and potential energy of an oscillating system at time t. Given a classical solution u to W.E.2, the magnitude

$$J^2(t) = \tfrac{1}{2} \int_\Sigma [(\partial_t u)^2 + c^2 p\, |\nabla u|^2 + c^2 q u^2] + \tfrac{1}{2} \int_{\gamma_0} c^2 p \frac{\alpha}{\beta} u^2, \qquad (4.3)$$

where γ_0 is that portion of γ such that $\alpha, \beta > o$, is known as the Energy Integral for u. A key result for this approach is to obtain a relationship between $J^2(t)$ and the data. This easily allows for L^2- estimates on u, $\partial_t u$ and ∇u in terms of the data which immediately yields uniqueness and continuous dependence on the data. These results won't be presented now since analogous results, for the adaptation to the surface operator H, will be given later (for further details see Vladimirov, 1971).

Existence within the class of classical solutions presents some difficulties. Since existence is sought via an eigenfunction expansion fairly harsh conditions must be imposed on the data to obtain a classical solution. Instead, the notion of a generalized solution, which is well suited for these expansions and involves functions continuous in $L^2(\Sigma)$ w.r.t. time, is introduced.

Pertinent definitions and preliminary results that are also needed for the adaptation of this approach to W.E.1 will be given now along with a brief sketch of the results leading up to the main theorem on existence for the classical problem in W.E.2.

Definition 4.1

For every $t \varepsilon [a,b]$ let the function $u(S,t)$ belong to $L^2(\Sigma)$. Then, u is said to be continuous in $L^2(\Sigma)$ w.r.t. $t \varepsilon [a,b]$ iff for every $t \varepsilon [a,b]$

$$u(S,t') \to u(S,t), \text{ as } t' \to t, \text{ in } L^2(\Sigma).$$

The following preliminary results are immediately obtained from this definition.

Lemma 4.2

If u is continuous in $L^2(\Sigma)$ w.r.t. $t \varepsilon [a,b]$ then

(i) $||u||_2$ is a continuous function of time for $t \varepsilon [a,b]$

(ii) for every $f \varepsilon L^2(\Sigma)$, the scalar product (u,f) is continuous for $t \varepsilon [a,b]$

(iii) $u \varepsilon L^2(\Sigma \times [a,b])$

Proof (i) follows from the inequality

$$\left| \, ||u(S,t')||_2 - ||u(S,t)||_2 \right| \leq || \, u(S,t') - u(S,t)||_2 \; ,$$

which is a consequence of the Minkowski inequality and where the R.H.S. above goes to zero as $t' \to t$ by assumption.

(ii) This follows from Schwarz's inequality

$$|(u(S,t'),f) - (u(S,t),f)| = |(u(S,t') - u(S,t),f)| \leq$$

$$||u(S,t') - u(S,t)||_2 \; ||f||_2$$

(iii) here,

$$\int_a^b \int_\Sigma |u(S,t)|^2 \, d\Sigma_S dt = \int_a^b ||u(S,t)||_2^2 \, dt,$$

and this last integral is finite, since by (i), $||u||_2^2$ is continuous on $[a,b]$ ∎

Definition 4.3

The sequence of functions $u_k(S,t)$ is said to converge to $u(S,t)$ in $L^2(\Sigma)$ uniformly w.r.t. $t \varepsilon [a,b]$ iff

$$||u_k(S,t) - u(S,t)||_2 \to 0, \text{ as } k \to \infty, \text{ uniformly for } t \varepsilon [a,b].$$

Uniform convergence on a set Ω will be denoted by

$$u_k \overset{s\varepsilon\Omega}{\Longrightarrow} u \text{ as } k \to \infty .$$

Hence, the convergence in Defn. 4.3 will be denoted by

$$u_k \overset{t\varepsilon[a,b]}{\Longrightarrow} u \text{ as } k \to \infty, \text{ in } L^2(\Sigma) .$$

Definition 4.4

The sequence of functions $u_k(S,t)$ is said to converge in itself in $L^2(\Sigma)$ uniformly w.r.t. $t\varepsilon[a,b]$ iff

$$(u_n - u_m) \overset{t\varepsilon[a,b]}{\Longrightarrow} 0 \text{ as } n,m \to \infty, \text{ in } L^2(\Sigma) .$$

Lemma 4.5

If $u_k \overset{t\varepsilon[a,b]}{\Longrightarrow} u$ as $k \to \infty$, in $L^2(\Sigma)$, then

(i) $u_k \to u$ as $k \to \infty$, in $L^2\ (\Sigma \times [a,b])$,

and

(ii) $||u_k(S,t)||_2 \overset{t\varepsilon[a,b]}{\Longrightarrow} ||u(S,t)||_2$ as $k \to \infty$

Proof (ii) By Minkowski's inequality

$$\big|\, ||u_k||_2 - ||u||_2 \big| \leq ||u_k(S,t) - u(S,t)||_2 \to 0 \text{ as } k \to \infty$$

uniformly on [a,b] by assumption.

(i) $$\int_a^b \int_\Sigma |u_k(S,t) - u(S,t)|^2 \, d\Sigma_S dt = \int_a^b ||u_k(S,t) - u(S,t)||_2^2 \, dt$$

which converges to zero as $k \to \infty$ by assumption, since the integrand converges to zero uniformly on [a,b] ∎

Lemma 4.6

Let $u_k(S,t)$, $k = 1,2,\ldots$ be a sequence of functions continuous in $L^2(\Sigma)$ w.r.t. $t \varepsilon [a,b]$ with

$$u_k \xrightarrow[\quad]{t \varepsilon [a,b]} u \text{ as } k \to \infty, \text{ in } L^2(\Sigma);$$

Then, u is continuous in $L^2(\Sigma)$ w.r.t. $t \varepsilon [a,b]$.

Proof Let $\varepsilon > o$ be given. Then, there exists an integer m such that

$$||u_m(S,t) - u(S,t)||_2 < \frac{\varepsilon}{3}, \text{ for every } t \varepsilon [a,b].$$

Since, u_m is continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon[a,b]$, given $t_0\varepsilon[a,b]$ there exists $\delta>o$ such that

$$||u_m(S,t') - u_m(S,t_0)||_2 < \frac{\varepsilon}{3} \text{ whenever } |t' - t_0| < \delta .$$

Then, by Minkowski's inequality

$$||u(S,t')-u(S,t_0)||_2 < ||u(S,t')-u_m(S,t')||_2 + ||u_m(S,t')-u_m(S,t_0)||_2$$

$$+ ||u_m(S,t_0) - u(S,t_0)||_2 < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon ,$$

whenever $|t'-t_0|<\delta$. This shows that $u(S,t')\to u(S,t_0)$ as $t'\to t_0$ in $L^2(\Sigma)$. Since $t_0\ \varepsilon[a,b]$ is arbitary, we have shown that u is continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon[a,b]$ ∎

Lemma 4.7 (A completeness result concerning uniform cauchy sequences in $L^2(\Sigma)$ w.r.t. time)

If the sequence of functions $u_k(S,t)$, $k = 1,2,\ldots$, converges, in itself, in $L^2(\Sigma)$ uniformly w.r.t. $t\varepsilon[a,b]$; then, there exists a function $u(S,t)$ continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon[a,b]$ such that

$$u_k \overset{t\varepsilon[a,b]}{\Longrightarrow} u \text{ as } k \to \infty, \text{ in } L^2(\Sigma).$$

Proof By the Riesz-Fischer Theorem, for every $t\varepsilon[a,b]$ there exists a function $u(S,t)$ in $L^2(\Sigma)$ with

$$u_k \to u \text{ as } k \to \infty, \text{ in } L^2(\Sigma) .$$

Moreover, it is possible to choose a subsequence $u_{k_i}(S,t)$, $i = 1,2,\ldots$ such that

$$||u_{k_{(i+1)}}(S,t) - u_{k_i}(S,t)||_2 < \frac{1}{2^i}, \text{ for every } t\varepsilon[a,b].$$

Here,

$$u(S,t) = \lim_{p\to\infty} u_{k_p}(S,t) = u_{k_i}(S,t) + (u_{k_{i+1}}(S,t) - u_{k_i}(S,t)) + \ldots,$$

which implies that

$$||u - u_{k_i}||_2 \leq ||u_{k_{i+1}} - u_{k_i}||_2 + ||u_{k_{i+2}} - u_{k_{i+1}}||_2 + \ldots$$

$$\leq \frac{1}{2^i} + \frac{1}{2^{i+1}} + \ldots = \frac{1}{2^{i-1}} .$$

Hence, the subsequence converges to u in $L^2(\Sigma)$ uniformly w.r.t. $t\varepsilon[a,b]$ and since

$$||u - u_k||_2 \leq ||u-u_{k_i}||_2 + ||u_{k_i} - u_k||_2 ,$$

by assumption, we in fact have u_k converging to u in $L^2(\Sigma)$ uniformly w.r.t. $t\epsilon[a,b]$ ∎

Now, we can give the definition of a generalized solution to the classical problem in W.E.2.

Definition 4.8

$u(S,t)$ is a generalized solution to W.E.2 iff it is the limit in $L^2(\Sigma)$, uniform w.r.t. $t\epsilon[0,T]$, for every $T>0$, of a sequence of classical solutions $u_k(S,t)$ to the problems

$$\partial_{tt}u_k + c^2 L\, u_k = F_k, \text{ on } U_\infty$$

with

$$u_k \Big|_{t = 0^+} = v_0^k \; , \; \partial_t u_k \Big|_{t = 0^+} = v_1^k \text{ and } \alpha u_k + \beta \frac{\partial u_k}{\partial \tilde{n}} \Big|_{\gamma} = 0 \; ,$$

where we have

$$F_k \overset{t\epsilon[0,T]}{\Longrightarrow} F \text{ as } k\to\infty, \text{ in } L^2(\Sigma); \; v_0^k \to v_0 \text{ as } k \to \infty, \text{ in } L^2(\Sigma)$$

$$\text{grad } v_0^k \to \text{grad } v_0 \text{ and } v_1^k \to v_1 \; , \text{ both as } k \to \infty, \text{ in } L^2(\Sigma)$$

With this definition in hand, the next step is to show that the L^2-estimates for u, $\partial_t u$ and grad u where u is a classical solution also hold for generalized solutions. This imediately allows a uniqueness and continuous dependence on the data theorem for generalized solutions. Moreover, this notion of a solution is well suited for a formal eigenfunction expansion solution technique where it is verified, under the appropriate restrictions on the data, that this expansion yields a generalized solution. This theorem will be presented here so it can be compared with the analogous result for W.E.1.

Theorem 4.9 (Existence)

If $v_0 \varepsilon D_L$ (7), $v_1 \varepsilon L^2(\Sigma)$ and F is continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon[0,\infty)$, then, the solution to W.E.2. obtained, formally, by an eigenfunction expansion is a generalized solution.

Moreover, the following properties of a generalized solution can be established.

---

(7) D_L consists of functions u of class $C^2(\Sigma) \cap C^1(\bar{\Sigma})$ with $\Delta u \varepsilon L^2(\Sigma)$ and satisfying the boundary condition $\alpha u+\beta \frac{\partial u}{\partial \tilde{n}}\Big|_\gamma = 0$.

Lemma 4.10

If u(S,t) is a generalized solution to W.E.2, then the following properties hold

(i) u(S,t) is a distributional solution, i.e., given $\phi \in D(U_\infty)$

$$\langle u, \partial_{tt} \phi + c^2 L\phi\rangle = \langle F,\phi\rangle$$

(ii) The first (generalized or distributional) derivatives of u, $\partial u/\partial t$ and gradu, are continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon[0,\infty)$, and for every T>0

$$\partial_t u_k \overset{t\varepsilon[0,T]}{\Longrightarrow} \partial_t u, \text{ grad } u_k \overset{t\varepsilon[0,T]}{\Longrightarrow} \text{grad } u, \text{ as } k \to \infty, \text{ in } L^2(\Sigma).$$

Again, the proofs and details for some of these results have been omitted since they are analogous to results that will be proven when we consider the surface operator H in W.E.1.

Now, we want to adapt the previous theory to the problem in W.E.1, as originally given at the beginning of this section. The main step in this approach is to obtain a representation of the energy integral $J^2(t)$ in terms of the data given by

$$J^2(t) = J^2(0) + \int_0^t \int_\Sigma F(S,\tau)\ \partial_\tau\ u(S,\tau)d\Sigma_S d\tau \tag{4.4}$$

where,

$$J^2(0) = \tfrac{1}{2}\int_{\Sigma} [v_1^2 + c^2 p\ |\nabla v_0|^2 + c^2 q {v_0}^2] + \tfrac{1}{2}\int_{\gamma_0} c^2 p\, \frac{\alpha}{\beta}\, {v_0}^2.$$

An immediate problem here is that this step makes use of Green's second formula for the operator L and it is not clear how an anlogous formula for H should go. However, the energy associated to a symmetric operator L is given by the expression

$$\tfrac{1}{2}(Lu,u),$$

which is, in fact, used in the definition of the energy integral since, by Green's first formula

$$\tfrac{1}{2}(Lu,u) = \tfrac{1}{2}\int_{\Sigma} uLu = \tfrac{1}{2}\int_{\Sigma} [p\ |\nabla u|^2 + qu^2] + \tfrac{1}{2}\int_{\gamma_0} p\, \frac{\alpha}{\beta}\, u^2.$$

This suggests that we define an energy integral associated with the operator H in terms of $\tfrac{1}{2}(Hu,u)$. This will entail some changes on conditions specified in the definitions of classical and generalized solutions. For example, we will now be interested in the behavior of Hu as opposed to ∇u. This leads to slightly different results but we will show that the previous theory, essentially, goes through the same yielding a very nice setting for the treatment of eigenfunction expansion solutions to W.E.1.

Analogous to classical solutions to the wave equation, W.E.2, "nice" solutions to W.E.1 are defined which are to be sufficiently smooth so that all operations are obtained continuously. This will also reflect the necessary changes as imposed on Hu.

Definition 4.11

A "nice" solution to W.E.1, to be referred to as an N-solution, is a solution $h(S,t)$ satisfying the following conditions;

(i) h is twice continuously differentiable w.r.t. t on $\Sigma \times (0,\infty)$,

(ii) Hh is continuous on $\Sigma \times (0,\infty)$,

(iii) $h \varepsilon C^1(\bar{\Sigma} \times [0,\infty))$, and

(iv) for every $t>o$, $Hh(S,t) \varepsilon L^2(\Sigma)$ where $\lim_{t \downarrow o} Hh(S,t)$ exists in $L^2(\Sigma)$, to be denoted as $(Hh)_0(S)$.

Note: If h is an N-solution to W.E.1 then, necessarily, we have $F \varepsilon C(U_\infty)$, $h_0 \varepsilon C^1(\bar{\Sigma})$, $h_1 \varepsilon C(\bar{\Sigma})$ with $\alpha h_0 + \beta\, \partial h_0/\partial \tilde{n} = 0$ on γ. These assumptions will always be adjoined to the problem W.E.1 when considering N-solutions.

The following result is immediate from this definition.

Lemma 4.12

If $h(S,t)$ is an N-solution to W.E.1 then h and $\partial_t h$ are continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon[0,\infty)$ and Hh is continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon(0,\infty)$.

Now, we will define the energy integral for N-solutions to W.E.1 analogous to the definition in (4.3).

Definition 4.13

Given an N-solution $h(S,t)$ to W.E.1, the magnitude

$$J^2(t) = \tfrac{1}{2} \int_\Sigma (\partial_t h)^2 + \frac{c^2}{2} (Hh,h), \text{ for } t>0$$

with

$$J^2(0) = \tfrac{1}{2} \int_\Sigma h_1^2 + \frac{c^2}{2} ((Hh)_0, h_0)$$

defines the Energy Integral, $J(t)$, for h.

Definitions 4.11 and 4.13 illustrate the adaptation of the classical method of energy integrals that will be applied to the problem in W.E.1. The remainder of this chapter will be concerned with developing results, analogous to the previous theory applied to the problem in W.E.2, culminating in the main theorems of existence (via a formal eigenfunction expansion), uniqueness and continuous dependence on the data for this class of generalized solutions.

4.2 L^2-Estimates on Solutions and Uniqueness and Continuous Dependence on the Data

A key result within the method of energy integrals is to obtain a representation for $J^2(t)$ analogous to (4.4). Once this has been proven then the rest of the material leading up to and including uniquenss and continuous dependence on the data for N-solutions is fairly straightforward. Due to the inner product that appears in the definition of J(t), in Defn. 4.13, the following result is needed before giving this key result.

Lemma 4.14

Let u and v be continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon[a,b]$. Then the following hold;

(i) The scalar product (u,v) is a continuous function of t on [a,b] where the bilinear expansion

$$(u,v) = \sum_k \tilde{u}_k(t)\, \overline{\tilde{v}_k(t)}$$

converges uniformly for $t\varepsilon[a,b]$.

(ii) The eigenfunction expansion

$$u(S,t) = \sum_k \tilde{u}_k(t)\, u_k(S)$$

converges, uniformly for $t\varepsilon[a,b]$, in $L^2(\Sigma)$.

<u>**Proof**</u> The first part of (i) is just like Part (ii) of Lemma 4.2. Denote $u(S,t)$ by $u_t(S)$, then

$$|(u_{t'}, v_{t'}) - (u_t,v_t)| < |(u_{t'} - u_t, v_{t'})| + |(u_t, v_{t'} - v_t)|$$

$$< ||u_{t'} - u_t||_2 \; ||v_{t'}||_2 + ||v_{t'} - v_t||_2 \; ||u_t||_2$$

and, by assumption, this converges to zero as $t' \to t$.

Before finishing part (i) we will verify part (ii). Certainly, for each fixed $t\varepsilon[a,b]$, since $u(S,t)\varepsilon L^2(\Sigma)$, it has the eigenfunction expansion

$$u(S,t) = \sum_k \tilde{u}_k(t)\; u_k(S)$$

converging in $L^2(\Sigma)$. We must demonstrate the convergence in $L^2(\Sigma)$, uniformly w.r.t. $t\varepsilon[a,b]$, or, equivalently (by Lemma 4.7), that

$$|| \sum_{k=n}^{m} \tilde{u}_k(t)\; u_k(S)||_2^2 = \sum_{k=n}^{m} |\tilde{u}_k(t)|^2 \; {}^{t\varepsilon[a,b]}{}_0, \text{ as } m,n \to \infty.$$

It is known that, pointwise, on $[a,b]$

$$\sum_{k=1}^{\infty} |\tilde{u}_k(t)|^2 \text{ converges to } ||u||_2^2 \,.$$

By (i) of Lemma 4.2, $||u||_2^2$ is continuous on [a,b] and by (ii) of Lemma 4.2

$$|\tilde{u}_k(t)|^2 = |(u,u_k)|^2$$

defines a positive continuous function on [a,b]. Hence, by Dini's Lemma (see Vladimivov [1971] pg. 5), we in fact have

$$\sum_{k=1}^{\infty} |\tilde{u}_k(t)|^2 \text{ converges uniformly to } ||u||_2^2 \text{ , on } [a,b].$$

So, the sequence of partial sums forms a uniform cauchy sequence on [a,b] which verifies (ii).

Now, for the second part of (i), we will show that the sequence of partial sums of the bilinear expansion for (u,v) converges in itself uniformly w.r.t. $t\varepsilon[a,b]$. But, here

$$\left|\sum_{k=n}^{m} \tilde{u}_k(t)\, \overline{\tilde{v}_k(t)}\right| \leq \left(\sum_{k=n}^{m} |\tilde{u}_k(t)|^2\right)^{1/2} \left(\sum_{k=n}^{m} |\tilde{v}_k(t)|^2\right)^{1/2} \xrightarrow{t\varepsilon[a,b]} 0,$$

as $n,m \to \infty$, by part (ii) ∎

<u>Lemma 4.15</u> (Key Result)

$J^2(t)$ given in Defn. 4.13 is continuous for $t \geq 0$ and if we assume $F \varepsilon C(\bar{U}_\infty)$ then

$$J^2(t) = J^2(0) + \int_0^t \int_\Sigma F(S,\tau)\ \partial_\tau\ h(S,\tau)\ d\Sigma_S\ d\tau\ ,\ \text{for every } t\geqslant 0. \quad (4.5)$$

<u>**Proof**</u> Since, h and Hh are continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon\ (0,\infty)$, by (i) of Lemma 4.14, $J^2(t)$ is continuous for $t>0$. Also, by (iv) of Defn. 4.11,

$$\lim_{t\downarrow 0}\ (Hh,h) = \big((Hh)_0,\ h_0\big) \text{ which implies that } \lim_{t\downarrow 0} J^2(t) = J^2(0).$$

Now, assume $F\varepsilon C(\bar{U}_\infty)$ and multiply the equation h = f, in W.E.1, by $\partial_t h$ and integrate over Σ x (ε,T) for $\varepsilon>0$, $T>0$ to obtain

$$\int_{\Sigma X(\varepsilon,T)} F(S,t)\ \partial_t h(S,t) d\Sigma_S dt = \int_\Sigma \int_\varepsilon^T \partial_{tt} h(S,t)\ \partial_t h(S,t)\ dt\ d\Sigma_S$$

$$+\ c^2 \int_\varepsilon^T \int_\Sigma \partial_t h(S,t)\ Hh(S,t)\ d\Sigma_S dt.$$

Here, the first integral on the R.H.S. which we denote as I, is simplified

$$I = \tfrac{1}{2} \int_\Sigma (\partial_t h(S,t))^2 \Big|_\varepsilon^T d\Sigma_S = \tfrac{1}{2} \int_\Sigma (\partial_t h)^2 \Big|_{t=T} - \tfrac{1}{2} \int_\Sigma (\partial_t h)^2 \Big|_{t=\varepsilon}$$

while, for the second integral, we have

$$II = c^2 \int_\varepsilon^T (Hh,\ \partial_t h) = c^2 \int_\varepsilon^T \sum_k (Hh,u_k)\ (\partial_t h,u_k) \quad {}^{(8)}$$

---

(8) We stay within a real setting, to omit the complex conjugate, by considering only real-valued solutions and we know the eigenfunctions for this problem can be chosen real-valued.

$$= \quad c^2 \sum_k \int_\varepsilon^T (Hh,u_k)\ (\partial_t h,u_k),$$

Since, by (i) of Lemma 4.14 the bilinear expansion for $(Hh,\ \partial_t h)$ converges uniformly on $[\varepsilon,T]$.

This can be further simplified to

$$II = \quad c^2 \sum_k \lambda_k^{1/2} \int_\varepsilon^T \tilde{h}_k(t)\ \partial_t \tilde{h}_k(t)\ dt = c^2 \sum_k \lambda_k^{1/2} (1/2\, (\tilde{h}_k(t))^2 \Big|_\varepsilon^T)$$

$$= \quad \frac{c^2}{2} \sum_k \lambda_k^{1/2} (\tilde{h}_k(T))^2 - \frac{c^2}{2} \sum_k \lambda_k^{1/2} (\tilde{h}_k(\varepsilon))^2 .$$

Now, letting $\varepsilon \downarrow o$ and replacing T by t, by the continuity of F, $\partial_t h$ and (Hh,h) at $t=o$ $\left(\text{recall, } \lim_{t\downarrow o} (Hh,h) = ((Hh)_o,h_o)\right)$ where

$$(Hh,h) = \sum_k \lambda_k^{1/2} (\tilde{h}_k(t))^2 \quad \text{(by (i) of Lemma 4.14)}$$

we obtain

$$\int_0^t \int_\Sigma F(S,\tau)\partial_\tau\, h(S,\tau) d\Sigma_S d\tau = 1/2 \int_\Sigma (\partial_t h)^2 + \frac{c^2}{2} \sum_k \lambda_k^{1/2} (\tilde{h}_k(t))^2$$

$$- \left(1/2 \int_\Sigma h_1^2 + \frac{c^2}{2} \lim_{\varepsilon\downarrow 0} \sum_k \lambda_k^{1/2} (\tilde{h}_k(\varepsilon))^2 \right.$$

$$= J^2(t) - J^2(0) \blacksquare$$

Note: This shows $J^2(t)$ is continuously differentiable for $t \geq o$.

Using Lemma 4.15 it is now an easy matter to obtain

L^2-estmates on h and $\partial_t h$ in terms of the data to W.E.1.

Lemma 4.16

If h(S,t) is an N-solution to W.E.1, then the following estimates hold

(i) $||\partial_t h||_2 \leq \sqrt{2}\, J(0) + \int_0^t ||F(S,\tau)||_2 d\tau$, for every $t \geq 0$

(ii) $||h||_2 \leq ||h_0||_2 + \sqrt{2}\, J(0)\, t + \int_0^t (t-\tau)\, ||F(S,\tau)||_2 d\tau$, for every $t \geq 0$

where

(iii) $J(0) \leq \frac{1}{\sqrt{2}} (||h_1||_2 + c||(Hh)_0||_2 + c||h_0||_2)$

Proof

$$||\partial_t h||_2^2 = \int_\Sigma (\partial_t h)^2 \leq 2J^2(t)$$

since, $\frac{c^2}{2}(Hh,h) = \frac{c^2}{2} \sum_k \lambda_k^{1/2} (\tilde{h}_k(t))^2 \geq 0.$

Hence,

$$||\partial_t h||_2 \leq \sqrt{2}\, J(t) .$$

Differentiating equation (4.5) of Lemma (4.15) and using the Cauchy-

Schwarz inequality, we obtain

$$2\, J\, J' \leq ||F||_2\ ||\partial_t h||_2 \leq \sqrt{2}\, J\ ||F||_2 ,$$

Which implies that

$$J'(t) \leq \frac{1}{\sqrt{2}}\ ||F||_2 \text{ , for every } t \geq 0.$$

Upon integration this yields

$$J(t) \leq J(0) + \frac{1}{\sqrt{2}} \int_0^t ||F(S,\tau)||_2\, d\tau.$$

substituting this back into the estimate for $||\partial_t h||_2$ gives (i).

For (ii), differentiation of

$||h||_2^2 = \int_\Sigma h^2(S,t)d\Sigma_S$ implies that $2||h||_2\ ||h||_2 \leq 2||h||_2\ ||\partial_t h||_2$

or

$$||h||_2 \leq ||\partial_t h||_2 \leq \sqrt{2}\, J(0) + \int_0^t ||F||_2.$$

Then, integration of this yields

$$||h||_2 \leq ||h||_2\Big|_{t\downarrow 0} + \sqrt{2}\, J(0)\, t + \int_0^t \int_0^{t'} ||F||_2\, d\tau dt' .$$

By (i) of Lemma 4.2, $\lim\limits_{t\downarrow 0} ||h||_2 = ||h_o||_2$. Hence, interchanging the order of integration in the above gives the result (ii).

For (iii), we have

$$2\, J^2(0) = \int_\Sigma h_1^2 + c^2\, ((Hh)_0, h_0)$$

$$\leq ||h_1||_2^2 + c^2\, ||(Hh)_0||_2\, ||h_0||_2$$

$$\leq (||h_1||_2 + c\, ||(Hh)_0||_2^{1/2}\, ||h_0||_2^{1/2})^2$$

$$\leq (||h_1||_2 + c\, ||(Hh)_0||_2 + c\, ||h_0||_2)^2$$

(where we have used the facts that $a^2+b^2 \leq (a+b)^2$ and $ab \leq a^2+b^2$ for $a,b \geq 0$). Taking square-roots gives result (iii) ∎

At this point, we can immediately obtain a theorem on uniqueness and continuous dependence on the data for N-solutions to W.E.1.

Theorem 4.17 (Uniqueness and Continuous Dependence on the Data for N-solutions)

The N-solution to W.E.1. is unique and it and its first time derivative depend continuously on the data h0, h_1, and F in the sense that if for any T>0 F, $\tilde{F} \varepsilon C(\bar{U}_T)$ with

$||F-\tilde{F}||_2 \leq \varepsilon$ for every $0 \leq t \leq T$, $||h_0-\tilde{h}_0||_2 \leq \varepsilon_0$, $||h_1 - h_1||_2 \leq \varepsilon_1$

and

$$||(Hh)_0-(H\tilde{h})_0||_2 \leq \varepsilon_0' \;,$$

Then, for the corresponding solutions, h and $\tilde{h}$

$$||h-\tilde{h}||_2 \leq \tilde{c}\ ((1+T)\ \varepsilon_0 + T\varepsilon_1 + T\varepsilon_0' + T^2\varepsilon),\ \text{for all } 0 \leq t \leq T \tag{4.6}$$

and for some constant $\tilde{c} < \infty$, not depending on h_0, h_1, F, t or T. While, for $\partial_t h$ and $\partial_t \tilde{h}$, the following estimate holds;

$$||\partial_t h - \partial_t \tilde{h}||_2 \leq \tilde{c}\ (\varepsilon_0 + \varepsilon_1 + \varepsilon_0' + T\varepsilon),\ \text{for all } 0 \leq t \leq T. \tag{4.7}$$

Proof For uniqueness, it suffices to show that if h is an N-solution to W.E.1 with all homogeneous data, then h = o (in $L^2(\Sigma)$). But, in this case, $||h_0||_2 = o$, J(o) = o and $||F||_2 = o$, which implies by (ii) of Lemma 4.16 that $||h||_2 = o$, for all $t \geq o$.

For continuous dependence on the data, let $\eta = h-\tilde{h}$ and $\tilde{J}(t)$ be the energy integral associated with η. Then, by (iii) of Lemma 4.16

$$\tilde{J}(0) \leq \frac{1}{\sqrt{2}}\ (||h_1 - \tilde{h}_1||_2 + c||h_0 - \tilde{h}_0||_2 + c||(Hh)_0 - (\tilde{H}h)_0||_2) \tag{4.8}$$

and for every T > 0, by (ii) of Lemma 4.16, for all $0 \leq t \leq T$

$$||\eta||_2 = ||h-\tilde{h}||_2 \leq ||h_0-\tilde{h}_0||_2 + \sqrt{2}\, T\, \mathcal{J}(0) + \int_0^t (t-\tau)\, ||F-\tilde{F}||_2 d\tau$$

using (4.8) above and the estimate

$$\int_0^t (t-\tau)\, ||F-\tilde{F}||_2 d\tau \leq \frac{T^2}{2}\, \varepsilon, \text{ for all } 0 \leq t \leq T,$$

yields,

$$||h-\tilde{h}||_2 \leq \varepsilon_0 + T\, (\varepsilon_1 + c\varepsilon_0 + c\, \varepsilon_0') + \frac{T^2}{2}\, \varepsilon$$

$$\leq \tilde{c}\, ((1+T)\, \varepsilon_0 + T\varepsilon_1 + T\varepsilon_0' + T^2\varepsilon), \text{ for all } 0 \leq t \leq T$$

(where $\tilde{C} = \max\, \{1,c\}$).

Also, by (i) of Lemma 4.16

$$||\partial_t \eta||_2 = ||\partial_t h - \partial_t \tilde{h}||_2 \leq \sqrt{2}\, \mathcal{J}(0) + \int_0^t ||F-\tilde{F}||_2$$

$$\leq (\varepsilon_1 + c\varepsilon_0 + c\varepsilon_0') + T\varepsilon$$

$$\leq \tilde{c}\, (\varepsilon_0 + \varepsilon_1 + \varepsilon_0' + T\varepsilon), \text{ for all } 0 \leq t \leq T \blacksquare$$

4.3 The Main Results of Existence, Uniqueness and Continuous

Dependence on the Data in a Class of Generalized Solutions

As was mentioned in section 4.1, the notion of a generalized solution analogous to Defn. 4.8 will be given that is well suited for obtaining existence through a formal eigenfunction expansion solution technique. Less stringent conditions need be imposed on the data for the series expansion to converge yielding a generalized solution as opposed to an N-solution to W.E.1. Moreover, the properties of a generalized solution, that will be given in the main theorems of this section and in the last section, illustrate its appropriateness for this problem.

Definition 4.18

If we have a sequence of N-solutions $f_k(S,t)$ to the problems

$$\partial_{tt} f_k + c^2 H f_k = F_k, \text{ on } U_\infty$$

with

$$f_k\Big|_{t=0^+} = f_0^k \ , \ \partial_t f_k\Big|_{t=0^+} = f_1^k \text{ and } \alpha f_k + \beta \frac{\partial f_k}{\partial \tilde{n}}\Big|_\gamma = 0$$

where, for all T>0

$$F_k \overset{t\varepsilon[0,T]}{\Longrightarrow} F \text{ as } k\to\infty, \text{ in } L^2(\Sigma),\ f_0^k \to h_0,\ f_1^k \to h\ , \text{ in } L^2(\Sigma)$$

and

$$\{(Hf_k)_0\} \text{ converges in } L^2(\Sigma) \;,$$

satisfying;

$$f_k \overset{t\varepsilon[0,T]}{\Longrightarrow} h \text{ as } k \to \infty, \text{ in } L^2(\Sigma)$$

then, h(S,t) is called a <u>Generalized Solution</u> to W.E.1. We will denote

$$\lim_{k\to\infty} (Hf_k)_0 = (Hh)_0 \;.$$

<u>Lemma 4.19</u>

If h(s,t) is a generalized solution to W.E.1 then, the estimates in (ii) and (iii) of Lemma 4.16 are also valid for h.

<u>Proof</u> Since, for each k, $f_k(S,t)$ is an N-solution to W.E.1, the estimates of Lemma 4.16 hold for it. Hence, making use of the convergence

$$F_k \overset{t\varepsilon[0,T]}{\Longrightarrow} F, \text{ in } L^2(\Sigma) \text{ which implies } ||F_k||_2 \overset{t\varepsilon[0,T]}{\Longrightarrow} ||F||_2$$

and

$$||f_k||_2 \overset{t\varepsilon[0,T]}{\Longrightarrow} ||h||_2, \text{ holding for all } T>0$$

along with

$$f_0^k \to h_0,\ f_1^k \to h_1\ ,\ (Hf_k)_0 \to (Hh)_0\ ,\ \text{all in } L^2(\Sigma)\ ,$$

we merely need to pass to the limit in k, on the estimates for f_k, to obtain these estimates for h. For example, if we first define J(o), for h, by

$$J^2(0) = \tfrac{1}{2} \int_\Sigma h_1^2 + \frac{c^2}{2}\,((Hh)_0, h_0)$$

Then, for each k

$$J_k^2(0) = \tfrac{1}{2} \int_\Sigma (f_1^k)^2 + \frac{c^2}{2}\,((Hf_k)_0, f_0^k) \underset{\text{as } k\to\infty}{\to} J^2(0)\ .$$

Thus, from the estimate for f_k

$$||f_k||_2 \leq ||f_0^k||_2 + \sqrt{2}\, J_k(0)\ t + \int_0^t (t-\tau)\ ||F_k(S,\tau)||_2 d\tau$$

letting $k\to\infty$, we obtain

$$||h||_2 \leq ||h_0||_2 + \sqrt{2}\, J(0)\ t + \int_0^t (t-\tau)\ ||F(S,\tau)||_2\, d\tau\ ,$$

which verifies that (ii) holds for h. Result (iii) is obtained in exactly the same fashion, by letting $k\to\infty$ in

$$J_k(0) \leqslant \frac{1}{\sqrt{2}}\left(||f_1^k||_2 + c\ ||f_0^k||_2 + c\ ||(Hf_k)_0||_2\right) \blacksquare$$

Theorem 4.20 (Uniqueness and Continuous Dependence on the Data for Generalized Solutions)

The generalized solution to W.E.1 is unique and depends continuously on the data, h_o, h_1, and F in exactly the same sense as for an N-solution to W.E.1 as given in theorem 4.17.

Proof Since Lemma 4.19 verifies that parts (ii) and (iii) of Lemma 4.16 (estimates on $||h||_2$ and $J^2(o)$ in terms of the data) hold, in the exact same form, for a generalized solution as for an N-solution; the very same proof in theorem 4.17, for uniqueness and continuous dependence on the data for h, applies here $\blacksquare$

Before presenting the other main result within this theory, that of existence to W.E.1, it will be shown that to obtain a generalized solution it is only necessary to have the sequence of N-solutions satisfying the appropriate conditions given in Defn. 4.18. That is, if there exists a sequence of N-solutions $f_k(S,t)$ to W.E.1 with data f_o^k, f_1^k and F_k (as in the defn.) satisfying

$$F_k \overset{t\varepsilon[0,T]}{\Longrightarrow} F,\ f_0^k \to h_0,\ f_1^k \to h_1\ ,\ \text{all in } L^2(\Sigma)$$

along with, $\{(Hf_k)_0\}$ converging in $L^2(\Sigma)$

where $T>0$ is arbitrary; then the following result holds.

Lemma 4.21

Given the above situation, there exists a function $h(S,t)$ continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon[0,\infty)$ such that for every $T>0$

$$f_k \overset{t\varepsilon[0,T]}{\Longrightarrow} h \text{ as } k\to\infty\ ,\ \text{in } L^2(\Sigma)\ .$$

Proof Given $T>0$, by applying (4.6) of theorem 4.17 to the difference $f_n - f_m$, yields for every $0<t\leqslant T$

$$||f_n - f_m||_2 \leqslant \tilde{c}\,(||f_0^n - f_0^m||_2(1+T) + T\ ||f_1^n - f_1^m||_2$$

$$+ T\ ||(Hf_n)_0 - (Hf_m)_0||_2 + T^2\ ||F_n - F_m||_2).$$

By the assumptions on the data, this shows that the sequence of N-solutions converges in itself, in $L^2(\Sigma)$, uniformly w.r.t. $t\varepsilon[0,T]$, for all $T>0$. Hence, by Lemma 4.7 there exists a function $h(S,t)$ satisfying the requirements of this Lemma ∎

Now, we will explicity solve W.E.1 by a formal eigenfunction expansion method. Here, we try a solution of the form

$$h(S,t) = \sum_k a_k(t)\, u_k(S) \,, \tag{4.9}$$

where it is assumed we have expansions for the data as

$$F(S,t) = \sum_k \tilde{F}_k(t)\, u_k(S),\ h_0(S) = \sum_k (\tilde{h}_0)_k u_k(S),\ h_1(S) = \sum_k (\tilde{h}_1)_k u_k(S). \tag{4.10}$$

Then, by a method that is quite familiar by now, formally plugging (4.9) and (4.10) into W.E.1, gives for the modes separately, the equation

$$a_k''(t) + c^2 \lambda_k^{1/2} a_k(t) = \tilde{F}_k(t) \ , \text{ for } t>0, \text{ with}$$

$$a_k(0) = (\tilde{h}_0)_k \text{ and } a_k'(0) = (\tilde{h}_1)_k \ . \tag{4.11}$$

The solution to this (see (3.31) - (3.34) of Lemma 3.2) is given by

$$a_k(t) = (\tilde{h}_0)_k \cos(c\lambda_k^{1/4}\ t) + \frac{(\tilde{h}_1)_k}{c\lambda_k^{1/4}} \sin(c\lambda_k^{1/4} t) + \\ + \frac{1}{c\lambda_k^{1/4}} \int_0^t \sin[c\lambda_k^{1/4}(t-\tau)]\, \tilde{F}_k(\tau)\, d\tau \ , \tag{4.12}$$

holding for $t>0$. Inserting (4.12) back into (4.9) gives the formal solution.

The idea of a generalized solution fits well with the eigenfunction expansion for a solution in (4.9), since, it is clear that the partial sums of this series are to play the role of the sequence of N-solutions. Moreover, the tools are already at hand to guarantee the data for these partial sums converges appropriately under analogous restrictions imposed as in the classical problem in theorem 4.9.

Theorem 4.22 **(Existence of a Generalized Solution)**

If $h_0 \in D_H$ [9], $h_1 \in L^2(\Sigma)$ and F is continuous in $L^2(\Sigma)$ w.r.t. $t \in [0,\infty)$; then, the formal eigenfunction expansion of $h(S,t)$ given in (4.9) (with the modes $a_k(t)$ given in (4.12)) defines a generalized solution to the wave equation, W.E.1.

Proof Take, $f_n(S,t) = \sum_{k=1}^{n} a_k(t)\, u_k(S)$, $F_n(S,t) = \sum_{k=1}^{n} \tilde{F}_k(t)\, u_k(S)$

and

$$f_0^n(S) = \sum_{k=1}^{n} (\tilde{h}_0)_k u_k(S), \quad f_1^n(S) = \sum_{k=1}^{n} (\tilde{h}_1)_k\, u_k(S) .$$

(9) H is defined by equation (1.15) and its domain is given in Lemma 1.13 by condition (A).

Then, clearly $f_n(S,t)$ solves the problem

$$f_n = F_n, \text{ on } U_\infty$$

with

$$f_n \Big|_{t=0^+} = f_0^n , \quad \partial_t f_n \Big|_{t=0^+} = f_1^n \text{ and } \alpha f_n + \beta \frac{\partial f_n}{\partial \tilde{n}} \Big|_{\gamma} = 0 ,$$

where f_n satisfies (i) - (iii) of Defn. 4.11 for an N-solution. Also, for every $t>0$

$$Hf_n(S,t) = \sum_{k=1}^{n} \lambda_k^{1/2} a_k(t)\, u_k(S) \in L^2(\Sigma)$$

where

$$\lim_{t\downarrow 0} Hf_n(S,t) = (Hf_n)_0 = \sum_{k=1}^{n} \lambda_k^{1/2} (\tilde{h}_0)_k\, u_k(S), \text{ exists in } L^2(\Sigma).$$

Hence, each f_n is an N-solution to the above problem.

From the assumption on F and by (ii) of Lemma 4.14, the eigen-function expansion for F converges in $L^2(\Sigma)$, uniformly w.r.t.

$t\in[0,T]$, to F, for every $T>0$. That is,

$$F_n \overset{t\in[0,T]}{\Longrightarrow} F \text{ as } n\to\infty, \text{ in } L^2(\Sigma).$$

Also, since, h_0, $h_1 \varepsilon L^2(\Sigma)$ we have that

$$f_0^n \to h_0 \ , \ f_1^n \to h_1 \ , \text{ as } n\to\infty, \text{ in } L^2(\Sigma) \ .$$

Finally, $h_0 \varepsilon D_H$, by Lemma 1.3, implies that

$$Hh_0(S) = \sum_k \lambda_k^{1/2} \ (\tilde{h}_0)_k \ u_k(S) \ , \text{ with convergence in } L^2(\Sigma) \ .$$

Hence,

$$(Hf_n)_0 = \sum_{k=1}^{n} \lambda_k^{1/2} \ (\tilde{h}_0)_k \ u_k(S) \to Hh_0(S) \ , \text{ as } n\to\infty, \text{ in } L^2(\Sigma).$$

(we see that in this case, $(Hh)_0$ is given by Hh_0).

Thus, the data converges in the proper manner given in the definition of a generalized solution, hence, by Lemma 4.21, the sequence, $f_n(S,t)$ (and consequently the formal series (4.9)) converges in $L^2(\Sigma)$, uniformly w.r.t. $t\varepsilon[0,T]$, for every $T>0$, to the generalized solution $h(S,t)$ ∎

4.4 Further Properties of Generalized Solutions

Lemma 4.23

Let $h(S,t)$ be a generalized solution to W.E.1. Then the following properties can be verified;

(i) The (generalized) derivative $\partial_t h$ is continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon[0,\infty)$ and for every $T>0$

$$\partial_t\, f_k \overset{t\varepsilon[0,T]}{\Longrightarrow} \partial_t h \text{ as } k\to\infty, \text{ in } L^2(\Sigma)$$

(where $\{f_k\}$ is the sequence of N-solutions given in the definition of a generalized solution).

(ii) The derivative $\partial_t h$ depends continuously on the data h_0, h_1 and F in the same sense given in (4.7) of theorem 4.17 for N-solutions.

(iii) h is a distributional solution to W.E.1[10], i.e., for every $\phi\varepsilon D(U_\infty)$,

$$\langle h, \Box\phi\rangle = \langle F,\phi\rangle$$

Proof (i) Applying (4.7) to the difference $f_n - f_m$ for any $T>0$ we have

$$||\partial_t f_n - \partial_t\, f_m||_2 < \tilde{c}\,(||f_0^n - f_0^m||_2 + ||f_1^n - f_1^m||_2$$

$$||(Hf_n)_0 - (Hf_m)_0||_2 + T \max_{t\varepsilon[0,T]} ||F_n - F_m||_2),$$

(10) Later, a different meaning to the notion of a distributional solution to W.E.1 which includes the intial conditions will be given.

which, by the assumptions on the data given in Defn. 4.18, implies that the sequence $\{\partial_t f_n\}$ converges in itself, in $L^2(\Sigma)$, uniformly w.r.t. $t\epsilon[o,T]$, for all $T>0$. Then, by Lemma 4.7, there exists a function $\tilde{h}(S,t)$ continuous in $L^2(\Sigma)$ w.r.t. $t\epsilon[0,\infty)$ such that for all $T>0$

$$\partial_t f_n \overset{t\epsilon[0,T]}{\Longrightarrow} \tilde{h} \text{ as } n\to\infty, \text{ in } L^2(\Sigma) .$$

On the other hand, by definition,

$f_n \overset{t\epsilon[0,T]}{\Longrightarrow} h$ as $n\to\infty$, in $L^2(\Sigma)$ which implies that $f_n \to h$, in $D'(U_\infty)$

$$(\text{ie. } \int_0^\infty \int_\Sigma f_n \cdot \phi \to \int_0^\infty \int_\Sigma h\, \phi \quad \text{for all } \phi\epsilon D(U_\infty)).$$

By the continuity of ∂_t on $D'(U_\infty)$ we also have

$$\partial_t f_n \to \partial_t h \text{ , in } D'(U_\infty) .$$

Thus, the distributional or generalized derivative, $\partial_t h$, agrees with $\tilde{h}$ which is our result.

(ii) Now that we have the convergence of the derivatives of the sequence of N-solutions to $\partial_t h$, given by (i), we can verify that part (i) of Lemma 4.16 (an estimate on $||\partial_t h||_2$ in terms of the data) holds for generalized solutions as well as for N-solutions. This is done in the same manner that it was shown that part (ii) of Lemma (4.16) holds for generalized solutions in the proof of Lemma 4.19. That is, by (i) of Lemma 4.16, for all n

$$||\partial_t f_n||_2 < \sqrt{2}\ \ J_n(0) + \int_0^t ||F_n||_2 \ .$$

Then, by (i) above and the assumptions on the data, passing to the limit as $n \to \infty$ yields

$$||\partial_t h||_2 < \sqrt{2}\ \ J(0) = \int_0^t ||F||_2 \ .$$

Now, the proof of continuous dependence on the data for N-solutions given in (4.7) of theorem 4.17 carries over, exactly, for $\partial_t h$.

(iii) Let $\tilde{h}(S,t)$ be the solution to W.E.1 given by the eigenfunction expansion in (4.9) with the mode $a_k(t)$ in (4.12) satisfying the differential equation in (4.11). Then for every $\phi \in D(U_\infty)$

$$<\tilde{h}, \Box\phi> = \int_{U_\infty} \Big(\sum_k a_k(t)\ u_k(S)\Big)\Box\phi(S,t) = \sum_k \int_{U_\infty} a_k(t)\ u_k(S)\Box\phi(S,t) .$$

Since ϕ is zero in a neighborhood of the boundary of U_∞, in particular, near t=o integration by parts yields

$$\langle \tilde{h}, \Box\phi \rangle = \sum_k \int_{U_\infty} \Box(a_k(t)\ u_k(S))\ \phi(S,t)$$

$$= \sum_k \int_{U_\infty} (a_k''(t) + c^2 \lambda_k^{1/2}\ a_k(t))\ u_k(S)\ \phi(S,t)$$

$$= \sum_k \int_{U_\infty} F_k(t)\ u_k(S)\ \phi(S,t) = \langle \sum_k F_k(t)\ u_k(S), \phi \rangle$$

$$= \langle F, \phi \rangle .$$

But, by uniqueness in theorem 4.20, for any generalized solution $h(S,t)$ to W.E.1,

$$||h-\tilde{h}||_2 = 0, \quad \text{for all } t \geq 0$$

which implies that

$$\int_{U_\infty} h(S,t) \Box\phi(S,t) = \int_{U_\infty} \tilde{h}(S,t) \Box\phi(S,t) \text{ , for every } \phi \varepsilon \underset{\sim}{D}_\infty (U) \text{ ,}$$

which is our result ■

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We will finish up this chapter by considering a direct analysis of the convergence of the eigenfunction expansion for a solution, given in (4.9), along with two results that show when $H\phi(S,t)$ defines a function continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon[0,\infty)$ and when it is of class $C(\Sigma \times [0,\infty))$.

From the explicit form of the modes $a_k(t)$ given in (4.12), a direct analysis of the series expansion for $h(S,t)$ given in (4.9) will show that certain properties of h can be obtained under weaker hypotheses than given in the existence theorem. For example, convergence of this series can be obtained by assuming $h_o \varepsilon L^2(\Sigma)$ (satisfying the growth condition, $\sum_k |\tilde{\phi}_k|^2 < \infty$, which is weaker than condition (A) for $h_o \varepsilon D_H$) along with the same conditions on h_1, and F, as in theorem 4.22. This follows from the estimate on $a_k(t)$, in 4.12), given by

$$|a_k(t)|^2 < M \left(|(\tilde{h}_0)_k|^2 + \frac{|(\tilde{h}_1)_k|^2}{\lambda_k^{1/2}} + \frac{T^2}{\lambda_k^{1/2}} \max_{t\varepsilon[o,T]} |\tilde{F}_k(t)|^2\right),$$

for some constant M and for any T>o, holding for all o<t<T. But, this implies that

$$||f_n - f_m||_2^2 = ||\sum_{k=n}^{m} a_k(t)\ u_k(S)||_2^2$$

$$= \sum_{k=n}^{m} |a_k(t)|^2 \xrightarrow{t\varepsilon[0,T]} 0, \text{ as } n,m\to\infty$$

(by the assumptions on h_o, h_1 and F). This shows that the partial sums of h(S,t) converge, in themselves, in $L^2(\Sigma)$, uniformly w.r.t. $t\varepsilon[0,T]$, T>0. Hence, h(S,t) given in 4.9 defines a function continuous in $h^2(\Sigma)$ w.r.t. $t\varepsilon[0,\infty)$ and, the proof of (iii) of Lemma 4.23 holds, so it also gives a distributional solution to W.E.1.

Similarly, a direct analysis of the derivatives of the modes yields

$$|a_k'(t)|^2 \leq M(\lambda_k^{1/2}|(\tilde{h}_0)_k|^2 + |(\tilde{h}_1)_k|^2 + T^2 \max_{t\varepsilon[0,T]}|\tilde{F}_k(t)|^2),$$

which holds for some constant M uniformly on[o,T] for any T>o. Then, using

$$||\partial_t f_n - \partial_t f_m||_2^2 = ||\sum_{k=n}^{m} a_k'(t)\ u_k(S)||_2^2 = \sum_{k=n}^{m} |a_k'(t)|^2 ,$$

we see that to show $\partial_t h$ is continuous in $L^2(\Sigma)$ w.r.t. $t\varepsilon[0,\infty)$ with

$$\partial_t f_n \overset{t\varepsilon[0,T]}{\Longrightarrow} \partial_t h \text{ as } n\to\infty, \text{ in } L^2(\Sigma), \text{ for all } T>0 \ ;$$

we need to impose the, even stronger, growth condition on the expansion coefficients of h_o (than required for $h_o \varepsilon L^2(\Sigma)$) of

$$\sum_k \lambda_k^{1/2} |(\tilde{h}_0)_k|^2 < \infty.$$

This is also weaker than condition (A) for $h_o \varepsilon D_H$. However, the data for the partial sums will not necessarily converge in the appropriate manner so that h defines a generalized solution and allow for uniqueness and continuous dependence on the data for either h or $\partial_t h$.

Conditions (ii) and (iv) in definition 4.11 for an N-solution to W.E.1 involves the operation of H on functions depending on time. This posed no problem in the existence theorem, since H was only applied to finite series where the time dependency was factored out. In general, we want to know when $H\phi\,(S,t)$ defines a function continuous in $L^2(\Sigma)$ w.r.t. time and when it is continuous on $\Sigma \times [0,\infty)$.

It is easy to see how to adapt conditions (A) and (B) with $H\phi \,\varepsilon\, L^2(\Sigma)$ or $H\phi \,\varepsilon\, C(\bar{\Sigma})$, respectively, to conditions for functions depending on time. Condition (A) required

$$\sum_k \lambda_k |\tilde{\phi}_k|^2 < \infty ,$$

and condition (B) required

$$\sum_k \lambda_k^3 |\tilde{\phi}_k|^2 < \infty.$$

Similarly, the following result holds.

<u>**Lemma 4.24**</u>

If $\phi(S,t)$ satisfies

(a) $\sum_k \lambda_k \, |\tilde{\phi}_k(t)|^2$ converges uniformly on $[0,T]$, for all $T>0$

or

(b) $\sum_k \lambda_k^3 \, |\tilde{\phi}_k(t)|^2$ converges uniformly on $[0,T]$, for all $T>0$,

then,

$$H\phi(S,t) = \sum_k \lambda_k^{1/2} \, \tilde{\phi}_k(t) \, u_k(S) \quad , \tag{4.13}$$

with convergence in $L^2(\Sigma)$, uniform w.r.t. $t\varepsilon[0,T]$ for all $T>0$ when condition (a) holds (hence, $H\phi$ is cont. in $L^2(\Sigma)$ w.r.t. $t\varepsilon[0,\infty)$) and uniform convergence on $\bar{\Sigma} \times [0,T]$ for all $T>0$ when condition (b) holds (hence, $H\phi \; \varepsilon \; C(\bar{\Sigma} \times [0,\infty))$).

<u>**Proof**</u> Assume (a) holds. Then,

$$\left|\left| \sum_{k=n}^{m} \lambda_k^{1/2} \, \tilde{\phi}_k(t) \, u_k(S) \right|\right|_2^2 = \sum_{k=n}^{m} \lambda_k |\tilde{\phi}_k(t)|^2 \xrightarrow{t\varepsilon[0,T]} 0, \text{ as } n,m\to\infty$$

for all $T>0$. So, by Lemma 4.7, the R.H.S. of (4.13) converges in $L^2(\Sigma)$, uniformly w.r.t. $t \; \varepsilon \; [o,T]$ for all $T>0$, to a function which is continuous in $L^2(\Sigma)$ w.r.t. $t \in [0,\infty)$. However, condition (a) and Lemma 1.3 implies that this function is $H\phi(S,t)$.

Now, assume (b) holds. Then,

$$\left|\sum_{k=n}^{m} \lambda_k^{1/2} \tilde{\phi}_k(t) u_k(S)\right| \leq \left(\sum_{k=n}^{m} \lambda_k^3 \, |\tilde{\phi}_k(t)|^2\right)^{1/2} \left(\sum_{k=n}^{m} \frac{|u_k(S)|^2}{\lambda_k^2}\right)^{1/2}$$

where the R.H.S.converges uniformly on $[0,T] \times \bar{\Sigma}$ to 0 as $n,m \to \infty$. This shows the sequence of partial sum form a uniform Cauchy sequence on $\bar{\Sigma}$ X $[0,T]$, for all $T>0$ which by Lemma 1.4 establishes the second result ∎

V. A GENERALIZED MIXED PROBLEM FOR THE WAVE EQUATION

5.1 Two Examples of a Formal Eigenfunction Expansion Method

The last chapter built up an L^2-theory for W.E.1 and involved an analysis of the convergence of a formal eigenfunction expansion solution to W.E.1 within this L^2-setting. This chapter will be devoted to an investigation of this eigenfunction expansion as a solution to W.E.1 and to a generalized wave equation from a distributional point of view.

We will begin the treatment of distributional solutions to the wave equation with the operator $\Box = \partial_{tt} + c^2 H$, by obtaining the Green's function solution for $\Box$ through an eigenfunction expansion. By now, the expansion method is quite familiar not yielding anything new. However, here, we will verify that this expansion does give a distributional solution (in a sense to be specified shortly). Also, this will demonstrate that in the several examples given in Chapter III, if the impulse response or Green's function solutions are viewed from a distributional point of view, then the exponential factor and limit as $z \downarrow 0$ can be omitted.

<u>**Lemma 5.1**</u>

The solution to the problem

$$\Box\, h(S,t) = \delta_+(t)\ \delta_R(S), \text{ on } U_\infty$$

is given by

$$h(S,t) = \frac{1}{c} \sum_k \lambda_k^{-1/4} \sin(c\, \lambda_k^{1/4} t)\, u_k(R)\, u_k(S), \text{ for } t>0 \qquad (5.1)$$

where $u_k(S)$ is to represent a regular distribution with (5.1) converging weak*.

Before giving a proof of this, we need to be more explicit on how the equation $h = \delta_+(t)\, \delta_R(S)$ is to be interpreted. The use of $\delta_+(t)$ signifies the assumption of causal conditions which entails the condition that $h \equiv 0$ for $t<0$. That is, if $\psi \in D(\Sigma \times (-\infty, \infty))$ with supp ψ contained in $\Sigma \times (-\infty, 0]$, then $\langle h, \psi \rangle = 0$. We will denote $D(\Sigma \times (-\infty, \infty))$ as D_∞ and distributions in D'_∞ satisfying causal conditions by D'_+. Then, a <u>distributional solution</u> to the problem in Lemma 5.1 will refer to a distribution $h \in D'_+$ such that

$$\langle h, \Box\psi \rangle = \langle \delta_+(t)\, \delta_R, \psi \rangle = \psi(R, 0^+), \text{ for all } \psi \in D_\infty.$$

Proof From the above note, the claim that h(S,t) given in 5.1 is a solution to $\Box h = \delta_+(t)\, \delta_R$ means that if we extend h to be zero for $t<0$, then h is a distributional solution in the sense given above.

(5.1) is obtained, in the usual manner, by assuming

$$h(S,t) = \sum_k a_k(t)\, u_k(S).$$

Formally plugging this into the equation in Lemma 5.1 yields for the modes seperately

$$a_k''(t) + c^2 \lambda_k^{1/2} a_k(t) = u_k(R)\ \delta_+(t). \tag{5.2}$$

The solution to this is given by

$$a_k(t) = u_k(R)\ z_k(t)\ \theta(t)\ \left(\theta(t) \text{ is the heavyside function } = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}\right)$$

where $z_k(t)$ satisfies

$$z_k''(t) + c^2 \lambda_k^{1/2}\ z_k(t) = 0, \text{ for } t>0,$$

and

$$z_k(0) = 0,\ z_k'(0) = 1.$$

Hence,

$$a_k(t) = \frac{1}{c}\ \lambda_k^{-1/4} \sin\left(c\lambda_k^{1/4} t\right) u_k(R), \text{ for } t>0$$

$$a_k \equiv 0, \text{ for } t<0\ ,$$

and inserting this into h(S,t) gives (5.1).

Before showing that h(S,t) is a distributional solution, it must be verified that the series in (5.1) converges weak* hence defining a distribution in D'_+. For this, it is sufficient to show

$$\left| < \sum_{k=n}^{m} a_k(t)\, u_k(S),\ \psi > \right| \to 0 \text{ as } n,m \to \infty, \text{ for all } \psi \in D_\infty.$$

To obtain this we will make use of a result, similar to Lemma 2.4, stating that

$$\sum_k \lambda_k^r \, |\tilde{\psi}_k(t)|^2, \tag{5.3}$$

converges uniformly on [-T,T], for every T>0 and positive integer r. This holds since, for all integers $m \geq 0$ if $\psi \in D_\infty$ then $\Pi^m \psi \in D_\infty$ implying that

$$\Pi^m \psi(S,t) = \sum_k \lambda_k^m \, \tilde{\psi}_k(t)\, u_k(S) \ ,$$

with convergence in $L^2(\Sigma)$, uniformly w.r.t. $t \in [-T,T]$, for all T>0 (by Lemma 4.14). Hence,

$$||\Pi^m \psi||_2^2 = \sum_k \lambda_k^{2m} \, |\tilde{\psi}_k(t)|^2 \ ,$$

with uniform convergence on [-T,T], for all T>0. This verifies the result for all even integers and since, eventually, $\lambda_k \geq 1$, the result holds for any positive integer r.

Now, we can obtain the estimate

$$\left| < \sum_{k=n}^{m} a_k(t)\ u_k(S),\ \psi> \right| = \left| \int_{\Sigma x(0,T)} \sum_{k=n}^{m} a_k(t)\ u_k(S)\ \psi(s,t) \right|$$

(by compact support of ϕ)

$$\leq \int_0^T \left| \sum_{k=n}^{m} a_k(t)\ \tilde{\psi}_k(t) \right| dt \leq \int_0^T \left(\sum_{k=n}^{m} \lambda_k^{3/2}\ |\tilde{\psi}_k(t)|^2 \right)^{1/2} \left(\sum_{k=n}^{m} \frac{|a_k(t)|^2}{\lambda_k^{3/2}} \right) dt.$$

Where we have used the estimate $|a_k(t)| \leq \dfrac{|u_k(R)|}{c\ \lambda_k^{1/4}}$.

Hence the above is $\leq \dfrac{1}{c} \displaystyle\int_0^T \left(\sum_{k=n}^{m} \lambda_k^{3/2}\ |\psi_k(t)|^2 \right)^{1/2} \left(\sum_{k=n}^{m} \frac{|u_k(R)|^2}{\lambda_k^2} \right)^{1/2} dt$

$$\leq \frac{MT}{c} \max_{t\epsilon[0,T]} \left(\sum_{k=n}^{m} \lambda_k^{3/2}\ |\tilde{\psi}_k(t)|^2 \right)^{1/2} \to 0 \text{ as } n,m \to \infty ,$$

by the result in (5.3), where

$$M = \max_{R\epsilon\bar{\Sigma}} \left(\int_\Sigma |N(S,R)|^2\ d\Sigma_S \right)^{1/2} \geq \left(\sum_{k=n}^{m} \frac{|u_k(R)|^2}{\lambda_k^{\ 2}} \right)^{1/2} .$$

Finally, we will show that (5.1), extended to be identically zero for $t<0$, is a distributional solution to the problem in Lemma 5.1.

For this, let $\psi \in D_\infty$. Then,

$$<h,\Box\psi> = \sum_{k=1}^{\infty} \int_{U_\infty} a_k(t)\ u_k(S)\ (\partial_{tt}\psi + c^2\ H\ \psi)$$

$$= \sum_k \lim_{\varepsilon \downarrow 0} \{ \int_\Sigma u_k(S) \int_\varepsilon^\infty u_k(R)\; z_k(t)\; \partial_{tt}\psi + c^2 \int_\varepsilon^\infty u_k(R)\; z_k(t) \int_\Sigma u_k(S)\; H\psi \} \;.$$

Here,

$$\int_\varepsilon^\infty z_k(t)\;\partial_{tt}\psi = z_k\;\partial_t\psi - z_k'\;\psi \Big|^\infty \;+ \int_\varepsilon^\infty z_k''\;(t)\;\psi \;.$$

Hence, letting $\varepsilon \downarrow 0$ and using $z_k(0) = 0$, $z_k'\;(0) = 1$, we obtain

$$\lim_{\Sigma \downarrow 0} \int_\varepsilon^\infty z_k(t)\;\;\partial_{tt}\psi = \psi \Big|_{\,t=0^+} + \int_0^\infty z_k''(t)\;\;\psi \;.$$

Also,

$$\int_\Sigma u_k(S)\;H\psi = (H\psi,\; u_k) = (\psi,\; Hu_k) = \lambda_k^{1/2} \int_\Sigma u_k(s)\;\;\psi.$$

Inserting these back in yields,

$$\langle h, \Box\psi\rangle = \sum_k \{ \int_\Sigma \psi \Big|_{t=0^+} u_k(R)\;\;u_k(S)\;\;d\Sigma_S + u_k(R) \int_0^\infty \int_\Sigma (z_k'' + c^2 \lambda_k^{1/2} z_k)\;\;u_k(S)\psi \}$$

and since, $z_k'' + c^2 \lambda_k^{1/2} z_k = 0$, we have

$$\langle h, \Box\psi\rangle = \sum_k \tilde{\psi}_k(0^+)\;\;u_k(R) = \psi(R,0^+) \;\blacksquare$$

It is an interesting problem to generalize Lemma 5.1 by allowing for an arbitrary distribution $T \varepsilon D'(\Sigma)$, with compact support, in place of δ_R. This will give an application of the eigenfunction expansion for T (see Theorem 2.8) as

$$T = \sum_k \tilde{T}_k u_k \ , \text{ where } \tilde{T}_k = \langle T, u_k \rangle \ , \tag{5.4}$$

and u_k is to be a regular distribution with the series converging weak* to T.

Lemma 5.2

The distributional solution to

$$\Box h(S,t) = \delta_+(t)\ T$$

is given by $h(S,t) = \theta(t)\ \tilde{h}(S,t)$, where $\tilde{h}$ is obtained through a formal eigenfunction expansion as

$$\tilde{h}(S,t) = \frac{1}{c} \sum_k \lambda_k^{-1/4} \tilde{T}_k \sin(c\lambda_k^{1/4} t)\ u_k(S).$$

Proof Assuming an eigenfunction expansion for h(S,t) as

$$h(S,t) = \sum_k a_k(t)\ u_k(S),$$

by the expansion in (5.4) for T, the modes $a_k(t)$ are governed by

$$a_k''(t) + c^2 \lambda_k^{1/2} a_k(t) = \tilde{T}_k \; \delta_+(t).$$

The solution to this is given by

$$\begin{aligned} &a_k(t) = \theta(t)\, \tilde{T}_k \; z_k(t), \text{ where} \\ &z_k'' + c^2 \; \lambda_k^{1/2} \; z_k = 0, \text{ with } z_k(0) = 0 \text{ and } z_k'(0) = 1, \end{aligned} \tag{5.5}$$

which yields

$$a_k(t) = \frac{1}{c} \, \lambda_k^{-1/4} \; \tilde{T}_k \; \sin(c\lambda_k^{1/4} t) \; \theta(t), \tag{5.6}$$

giving the required expansion for $h(S,t)$.

Next, to show $h \in D_+'$, it is sufficient to show that for every $\psi \in D_\infty$,

$$\left|< \sum_{k=n}^{m} a_k(t) \; u_k(S), \; \psi>\right| \to 0, \text{ as } n,m \to \infty.$$

This follows using the same steps as in the proof of weak* convergence of (5.1) in Lemma 5.1. From (5.6),

$$|a_k(t)| \leq \frac{|\tilde{T}_k|}{c\lambda_k^{1/4}} \text{ , for all } t \text{ .}$$

Hence, from the finite order of $|\tilde{T}_k|$ w.r.t. λ_k in theorem 2.8 along with the result in (5.3) of the last proof, we obtain

$$|<\sum_{k=n}^{m} a_k(t)\ u_k(S)\ ,\ \psi>|\ <\ \int_0^T \left(\sum_{k=n}^{m} \lambda_k^r\ |\tilde{\psi}_k(t)|^2\right)^{1/2} \left(\sum_{k=n}^{m} \frac{|a_k(t)|^2}{\lambda_k{}^r}\right)^{1/2} dt$$

$$< \frac{T}{c} \left(\sum_{k=n}^{m} \frac{|\tilde{T}_k|^2}{\lambda_k{}^{r+1/2}}\right) \max_{t\varepsilon[0,T]} \left(\sum_{k=n}^{m} \lambda_k^r\ |\tilde{\psi}_k(t)|^2\right)^{1/2} \to 0,\ \text{as } n,m \to \infty.$$

Finally, for all $\psi \ \varepsilon\ D_\infty$, we have

$$<h,\Box\psi> = \int_{U_\infty} \tilde{h}\ \ \psi = \sum_k \int_{U_\infty} a_k(t)\ u_k(S)\ (\partial_{tt}\psi + c^2\ H\psi)$$

$$\text{(by 5.5)} = \sum_k \lim_{\varepsilon\downarrow 0} \left[\ \int_\Sigma u_k(S) \int_\varepsilon^\infty \tilde{T}_k z_k(t)\,\partial_{tt}\psi + c^2 \int_\varepsilon^\infty \tilde{T}_k\ z_k(t) \int_\Sigma u_k(S)\ H\psi\ \right]$$

$$= \sum_k \left[\ \int_\Sigma \tilde{T}_k\ \psi\Big|_{t=0^+} u_k(S) + \tilde{T}_k \int_{U_\infty} (z_k'' + c^2\lambda_k^{1/2} z_k)\ u_k(S)\ \psi\ \right]$$

$$= \sum_k \tilde{T}_k <u_k,\psi\Big|_{t=0^+}> = <T,\ \psi\Big|_{t=0^+}> = <T,<\delta_+(t),\psi>> = <\delta_+(t)T,\psi>\ \blacksquare$$

5.2 Formulation and Existence to the Generalized Wave Equation

For easy reference the general wave equation problem in W.E.1 will be repeated here.

$$\Box\, h(S,t) = (\partial_{tt} + c^2 H)\, h(S,t) = F(S,t), \text{ on } U_\infty$$

W.E.1 with

$$h \Big|_{t=0^+} = h_0 \ , \ \partial_t h \Big|_{t=0^+} = h_1$$

and

$$\alpha h + \beta \frac{\partial h}{\partial \tilde{n}} \Big|_{\gamma} = 0 \ .$$

The goal is to demonstrate that under very mild restrictions on the data h_0, h_1 and F, a formal eigenfunction expansion method yields a distributional solution to W.E.1. But first, we must specify what is meant by a distributional solution to W.E.1. Unlike part (iii) of Lemma 4.23, the initial conditions are to play a role.

Definition 5.3

By a distributional solution to W.E.1, we will refer to a distributional solution $h \in D_+'$ to the equation

$$\Box\, h = \tilde{F} + h_0 \delta_+'(t) + h_1 \delta_+(t), \text{ on } U_\infty \ , \qquad (5.7)$$

where $\tilde{F} = \theta(t)\, F$. That is, as was defined after the statement of Lemma 5.1, for every $\psi \in D_\infty$

$$\langle h, \Box\psi \rangle = \langle \tilde{F} + h_0\, \delta_+'(t) + h_1 \delta_+(t),\ \psi \rangle$$

with $h \equiv 0$ for $t < 0$.

We will continue to denote problems in the format of W.E.1 even if a distributional solution is sought in which case the equation in (5.7) with the initial data injected into the new source term is to be the interpretation. It is often the case that an explicit representation for $h(S,t)$, valid for $t > 0$, is given to be a distributional solution. This will always refer to the extension of h as being identically zero for $t < 0$.

The motivation for equation (5.7) is seen in the following Lemma.

Lemma 5.4

If $h(S,t)$ is an N-solution to W.E.1 with $F \in C(U_\infty)$, $h_0 \in C^1(\bar{\Sigma})$ and $h_1 \in C(\Sigma)$; then,

$$\Box\, \tilde{h}(S,t) = \tilde{F}(S,t) + h_0(S)\ \delta_+'(t) + h_1(S)\ \delta_+(t), \text{ in } D_+' , \tag{5.8}$$

where $\tilde{F} = \theta(t)\, F$ and $\tilde{h} = \theta(t)\, h$.

Proof

First, by assumption, both sides of (5.8) define distributions in D_+'. Let $\psi \in D_\infty$. Then,

$$<\Box \tilde{h}, \psi> = <\tilde{h}, \Box \psi> = \int_0^\infty \int_\Sigma h(S,t) \ (\partial_{tt}\psi + c^2 H\psi)$$

$$= \lim_{\varepsilon \downarrow 0} \left[\int_\Sigma (h\partial_t\psi - \partial_t h\psi)\Big|_{t=\varepsilon}^{t=\infty} + \int_\varepsilon^\infty \int_\Sigma (\partial_{tt}h + c^2 Hh)\psi \right],$$

by the smoothness properties of an N-solution and the symmetric nature of H. Then, since $h(S,t)$ is an N-solution to W.E.1, letting $\varepsilon \downarrow 0$ yields

$$<\Box \tilde{h}, \psi> = \int_{U_\infty} F(S,t) \ \psi(s,t) \ d\Sigma_S dt - \int_\Sigma h_0(S) \ \partial_t \psi(S,t)\Big|_{t=0^+} d\Sigma_S$$

$$+ \int_\Sigma h_1(S) \ \psi(S,t)\Big|_{t=0^+} d\Sigma_S$$

$$= <\tilde{F} + h_0 \delta_+'(t) + h_1 \delta_+(t), \ \psi> \ \blacksquare$$

At this point it is easy to strengthen the result in part (iii) of Lemma 4.23, where it was shown that the generalized solution (Defn. 4.18) to W.E.1 is a solution to $\Box h = F$ in $D'(U_\infty)$, to obtain that the generalized solution gives a distributional solution in the sense of Defn. 5.3. This result will incorporate the initial conditions unlike part (iii) of Lemma 4.23 which avoids a treatment of initial and boundary conditions due to the compact support of the test functions contained in $\Sigma \times (0,\infty)$.

Lemma 5.5

The generalized solution to W.E.1 with the data satisfying the hypotheses of Theorem 4.22 (Existence), represented by the eigenfunction expansion in (4.9) with the modes given in (4.11) and (4.12), is a distributional solution to W.E.1 in the sense of Definition 5.3.

Proof Set $\tilde{h} = \theta(t)\, h$. We must verify that for every $\psi \in D_\infty$

$$\langle \tilde{h}, \Box\psi \rangle = \langle \tilde{F} + h_0\, \delta_t'(t) + h_1\, \delta_t(t), \psi \rangle .$$

The technique is exactly as given in the earlier two examples in Section 5.1, except here, the limit as $\varepsilon \downarrow 0$ is not needed since $a_k(t)$ is smooth for $t \geq 0$. Giving only the main steps, we have

$$\langle \tilde{h}, \Box\psi \rangle = \sum_k \Big[\int_\Sigma u_k(S)\, (a_k\, \partial_t \psi - a_k' \psi) \Big|_{t=0}^{t=\infty} + \int_0^\infty \int_\Sigma (a_k'' + c^2 \lambda_k^{1/2} a_k)\, u_k(S) \psi \Big]$$

$$= \sum_k \Big[\int_0^\infty \int_\Sigma \tilde{F}_k(t) u_k(S) \psi - \int_\Sigma (\tilde{h}_0)_k \partial_t \psi \Big|_{t=0^+} u_k(S) + \int_\Sigma (\tilde{h}_1)_k \psi \Big|_{t=0^+} u_k(S) \Big]$$

$$= \int_0^\infty \int_\Sigma F(S,t)\ \psi(S,t) - \int_\Sigma h_0(S)\partial_t \psi\Big|_{t=o^+} + \int_\Sigma h_1(S)\ \psi\Big|_{t=o^+}$$

$$= \langle \tilde{F} + h_0\ \delta_+'(t) + h_1 \delta_+(t),\ \psi\rangle\ ,$$

where we have made use of the known expansions for the data along with the compact support of ψ to interchange the order of summation and integration.

Now, of course, it is also required that the series expansion for h(S,t) in (4.9) converges weak*, showing that $\tilde{h}(S,t)$ defines a distribution in D_+'. For this, let $\psi \varepsilon D_\infty$, then

$$\left|\langle \sum_{k=n}^{m} a_k(t)\ u_k(S),\ \psi\rangle\right| \leq \int_0^T \Big(\sum_{k=n}^{m} |\tilde{\psi}_k(t)|^2\Big)^{1/2} \Big(\sum_{k=n}^{m} |a_k(t)|^2\Big)^{1/2}\ dt.$$

Using the estimate

$$|a_k(t)| \leq |(\tilde{h}_o)_k| + |(\tilde{h}_1)_k| + T \max_{t\varepsilon[0,T]} |\tilde{F}_k(t)|, \text{ for all } 0\leq t\leq T$$

for any T>0 (from (4.12), eventually holding for k sufficiently large), where

$\sum_k |(\tilde{h}_0)_k|^2$, $\sum_k |(\tilde{h}_1)_k|^2$ are finite $\left(h_0 \varepsilon D_H,\ h_1 \varepsilon L^2(\Sigma)\right)$

and

$\sum_k |\tilde{F}_k(t)|^2$ converges uniformly on $[0,T]$ for any $T>0$

(F is continuous in $L^2(\Sigma)$ w.r.t. $t \in [0,\infty)$) along with the result in (5.3) in the proof of Lemma 5.1 which states that

$$\sum_k |\tilde{\psi}_k(t)|^2 \text{ converges uniformly on } [0,T], \text{ for any } T>0;$$

we see that this last integral converges to zero as $n,m \to \infty$. Hence, the series

$$h(S,t) = \sum_k a_k(t)\ u_k(S) \text{ converges weak* } \blacksquare$$

Now, we want to extend the setting for W.E.1. Notice, from (5.7) in Definition 5.3, if we wish to consider distributional solutions to W.E.1 then there is no need to require the data to be classical functions. This suggests the following definition.

Definition 5.6

The Generalized Wave Equation (or G.W.E.1) will refer to the problem in W.E.1 where equation (5.7) is to be the interpretation and

$$h_0,\ h_1 \in D'(\Sigma) \text{ with } \tilde{F} \in D'_+ .$$

In the above proof of weak* convergence of the generalized solution given in the series expansion of (4.9), we did not make use of the full strength of $\psi \in D_\infty$. That is, the result in (5.3) states $\sum_k \lambda_k^r |\tilde{\psi}_k(t)|^2$ converges uniformly on [0,T] for any T>0 and for any positive integer r. This allows for far less stringent growth conditions on the expansion coefficients of the data and suggests that the eigenfunction expansion should yield a distributional solution to G.W.E.1 as long as the distributional data allows eigenfunction expansions with a finite growth condition on their expansion coefficients w.r.t. λ_k.

Using Theorem 2.8, such a result will, in fact, be given under the following conditions. Let

$$T,\ h_0,\ h_1 \in D'(\Sigma) \text{ with compact support and } f \in L^1_{LOC}(R) \text{ with } f \equiv 0 \text{ for } t<0. \tag{5.9}$$

Then, by Theorem 2.8, the following expansions hold

$$h_i = \sum_k (\tilde{h}_i)_k\, u_k, \text{ for } i=0,1 \text{ and } \tilde{F} = f(t)\ T = f(t) \sum_k \tilde{f}_k\, u_k\ , \tag{5.10}$$

where $\tilde{F}$ defines a distribution in D'_+ and all series converge weak*. Also, by the second part of Theorem 2.8, there exists a positive integer r and constant $\tilde{c}$ (both independent of k) such that

$$|(\tilde{h}_i)_k|, \ |\tilde{f}_k| \leq \tilde{c} \ \lambda_k{}^r, \text{ for all } k \text{ and for } i = 0,1. \qquad (5.11)$$

Theorem 5.7 **(Existence to the Generalized W.E.1)**

By a formal eigenfunction expansion, h(S,t) given by

$$h(S,t) = \sum_k a_k(t) \ u_k(S) \text{ with}$$

$$a_k(t) = \tilde{f}_k \ f * E_k + (\tilde{h}_0)_k \ E'_k + (\tilde{h}_1)_k \ E_k$$

where (5.12)

$$E''_k + c^2 \lambda_k^{1/2} \ E_k = \delta_+(t) ,$$

yields a distributional solution to G.W.E.1 with the data h_0, h_1 and $\tilde{F}$ given in (5.9).

Proof If we assume h(s,t) is of the form

$$h(S,t) = \sum_k a_k(t) \ u_k(S) ,$$

and formally plug this into G.W.E.1, the equation governing the modes is given by

$$a_k''(t) + c^2 \lambda_k^{1/2} a_k(t) = \tilde{T}_k \; f(t) \quad , \text{ for } t>0, \text{ with}$$

$$a_k(0^+) = (\tilde{h}_0)_k \text{ and } a_k'(0^+) = (\tilde{h}_1)_k .$$

The solution to this can be written in terms of the fundamental solution $E_k(t)$ as given in (5.12) (see Vladimirov (1971) pg. 171). For $t>0$, this can be further simplified to

$$a_k(t)=(\tilde{h}_0)_k\cos(c\lambda_k^{1/4} t)+\frac{(\tilde{h}_1)_k}{c\lambda_k^{1/4}}\sin(c\lambda_k^{1/4} t)+\frac{\tilde{T}_k}{c\lambda_k^{1/4}}\int_0^t \sin[c\lambda_k^{1/4}(t-\tau)]f(\tau)d\tau, \tag{5.13}$$

where

$$E_k(t) = \frac{\theta(t)}{c\lambda_k^{1/4}} \; \sin(c\lambda_k^{1/4} t).$$

As it has been demonstrated many times, if the series expansion for $h(S,t)$ converges weak* then, for any $\psi \in D_\infty$, with $\tilde{h} = \theta(t)\, h$, we have

$$\langle \tilde{h},\Box\psi\rangle = \sum_k \Big[\int_\Sigma u_k(S) \; (a_k \partial_t \psi - a_k' \; \psi)\Big|_{t=0}^{t=\infty} + \int_0^\infty \int_\Sigma (a_k''+c^2 \lambda_k^{1/2} a_k)u_k(S)\psi\Big]$$

$$= \sum_k \Big[\int_0^\infty \int_\Sigma \tilde{f}_k f(t)\; u_k(S)\; \psi(S,t) - \int_\Sigma (\tilde{h}_0)_k\; \partial_t\; \psi\Big|_{t=0^+} u_k(S)$$

$$+ \int_\Sigma (\tilde{h}_1)_k\; \psi\Big|_{t=0^+} u_k(S) \Big]$$

$$= \sum_k \Big[\tilde{f}_k < u_k\;,\; \int_0^\infty f(t)\; \psi(S,t)> - (\tilde{h}_0)_k <u_k,\; \partial_t \psi\Big|_{t=0^+}>$$

$$+ (\tilde{h}_1)_k <u_k,\; \psi\Big|_{t=0^+}> \Big]$$

$$= \sum_k \Big[\tilde{f}_k <u_k,\; <f,\psi>> + (\tilde{h}_0)_k <u_k,\; <\delta_+',\psi>>$$

$$+ (\tilde{h}_1)_k <u_k,\; <\delta_+,\; \psi>> \Big] .$$

Here, since $<f,\psi>$, $<\delta_+',\; \psi> = -\partial_t\; \psi(S,0^+)$ and $<\delta_+,\; \psi> = \psi(S,0^+)$ are all test functions in $D(\Sigma)$, using the expansions of the data in (5.10), yields

$$<\tilde{h},\Box\psi> = <T,\; <f,\psi>> + <h_0,\; <\delta_+',\psi>> + <h_1,\; <\delta_+,\; \psi>>$$

$$= <\tilde{F} + h_0\,\delta_+' + h_1\,\delta_+,\; \psi> .$$

For weak* convergence of the series expansion for $\hbar(S,t)$, following the same steps as in the proofs of weak* convergence in the previous Lemmas (Lemma 5.1, Lemma 5.2 or Lemma 5.5), we arrive at the inequality

$$\left| < \sum_{k=n}^{m} a_k(t)\, u_k(S),\, \psi> \right| \leq \int_0^T \Big(\sum_{k=n}^{m} \lambda_k^p\, |\tilde{\psi}_k(t)|^2 \Big)^{1/2} \Big(\sum_{k=n}^{m} \frac{|a_k(t)|^2}{\lambda_k{}^p} \Big)^{1/2} dt ,$$

valid for any positive integer p. Now from (5.13),

$$|a_k(t)| \leq |(\tilde{\hbar}_0)_k| + \frac{1}{c\lambda_k^{1/4}} |(\tilde{\hbar}_1)_k| + \frac{T}{c\lambda_k^{1/4}} |\tilde{f}_k| \int_0^T |f(\tau)|\, d\tau$$

which implies, using (5.11) that

$$|a_k(t)| \leq c' \lambda_k^r \text{ , for some constant } c' \text{ and for all } t\epsilon[0,T].$$

(here, c' can be any constant $\geq 3 \max \{\tilde{c}, \frac{\tilde{c}}{c}, \frac{T}{c} \int_0^T |f(\tau)|\, d\tau\}$). Then, taking $p \geq 2r + 2$, implies that

$$\Big(\sum_{k=n}^{m} \frac{|a_k(t)|^2}{\lambda_k^p} \Big)^{1/2} \leq c' \Big(\sum_{k=n}^{m} \frac{1}{\lambda_k{}^2} \Big)^{1/2} \to 0, \text{ as } n,m \to \infty.$$

Therefore, by the result in (5.3),

$$\left| \left< \sum_{k=n}^{m} a_k(t)\, u_k(S) \;,\; \psi \right> \right| \to 0, \text{ as } n,m \to \infty \;\blacksquare$$

One final comment is in order concerning the development of the theories for an analysis of W.E.1 within its various settings. In the previous chapter an adaptation of the method of energy integrals for the classical wave equation was presented. This gave a nice theory within an L^2-setting. However, even in the classical problem, no mention was given for the sense in which a generalized solution satisfies the boundary condition. It can easily be shown that the generalized solution satisfies the initial conditions in $L^2(\Sigma)$ (this also holds in the new problem involving the operator H) but the question of boundary conditions is far more complicated requiring a more precise definition. At this time, a treatment of the boundary conditions for the new wave equation, W.E.1, within the class of generalized solutions, must be avoided since it is not clear how to adapt the treatment in the classical case to this setting. Similarly, concerning the development of a distributional theory to W.E.1, in this chapter, a treatment of the boundary conditions is avoided. Here, the theory is based on the test function space, D_∞. By restricting the test functions to have their support contained in $\Sigma \times (-\infty,\infty)$, it is guaranteed that they are zero on a neighborhood of γ.

An appropriate treatment of the boundary conditions for this problem poses some interesting problems for future research. This, perhaps has its roots at the very beginning of this paper with the question of a suitable definition of a square-root of the Laplacian within a boundary value problem context.

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