AN ABSTRACT OF THE THESIS OF

Nat Vorayos for the degree of Doctor of Philosophy in Mechanical Engineering presented on June 27, 2000. Title: Laminar Natural Convection within Long Vertical Uniformly Heated Parallel-Plate Channels and Circular Tubes.

Abstract approved: Redacted for Privacy

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The problem of simple mathematical models of laminar *natural* convective flow within a long vertical parallel-plate channels and circular tubes kept at uniformly heated walls is revisited to seek a clear physical understanding of heat transfer mechanisms. A series solution method to analyze the fully developed flow and an integral solution method to analyze the developing flow are used. Chapters 3, 4, and 5 of this dissertation constitute a series of three-paper manuscripts for submission to archival journals.

The channels and circular tubes considered here are assumed to be sufficiently long to yield a fully developed flow thermally as well as hydrodynamically before the exit is encountered. In such fully developed flow situation, the fluid mass flow rate naturally induced into the channel due to buoyancy is found to be a function of the wall heating condition. The predicted average Nusselt number as a function of GrPrD/L not only agrees with the existing

literature but also is found to be in a functional form comparable to that proposed by Elenbaas (1942 a and b). Our results show that, in spite of being driven by buoyancy (rather than by a pump or a blower), the flow and heat transfer characteristics in the fully developed regime are essentially the same as those of fully developed laminar *forced* convection in which the flow is externally driven.

This observation is confirmed to be valid also in the study (Chapter 5) of laminar natural convection in the developing (entrance) region within a *long* vertical parallel-plate channel and circular tube. The mass flow rate, which has to remain invariant with axial location even in the entry region, is determined by the flow in the fully developed region. This is the same mechanism involved in forced convection in which the fluid outside the developing boundary layers (i.e. the core flow) is forced to accelerate in the entrance region. The entrance length of channel natural convection is also discovered to be about the same as that in forced convection.

Laminar Natural Convection within Long Vertical Uniformly Heated Parallel-Plate Channels and Circular Tubes

By

Nat Vorayos

A THESIS

Submitted to

Oregon State University

In partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy
in Mechanical Engineering

Presented June 27, 2000
Commencement June 2001

Doctor of Philosophy thesis of Nat Vorayos presented on June 27, 2000

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ACKNOWLEDGEMENTS

I would like to express my deep and sincere appreciation to my advisor, Dr. A. Murty Kanury who has been guiding my graduate studies and helping me to accomplish my goal since the first day we met. Without his guidance and generosity this work of mine would have not reached this far.

I would like to thank my committee members, Dr. Satish Reddy, Dr. Richard Peterson, Dr. Jim Liburdy, and Dr. Jeffrey Woldstad for their guidance and time. My eternal appreciation goes to my mother and sister whose contribution to my life can not be expressed in words.

Finally, I would like to dedicate this work to my late father, Dr. Tepprasong Vorayos and my late advisers, Dr. Ed Zukoski and Dr. Dwight Bushnell.

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LAMINAR NATURAL CONVECTION WITHIN LONG VERTICAL UNIFORMLY HEATED PARALLEL-PLATE CHANNELS AND CIRCULAR TUBES

CHAPTER 1: INTRODUCTION

1.1 The Problem

This dissertation deals with certain problems of laminar natural convection heat transfer within long vertical uniformly heated parallel-plate channels and circular tubes. The topic of natural convective heat transfer in vertical parallel plate channels and in circular tubes has been studied for over six decades. Wall(s) of the channel or tube may be at a specified constant and uniform wall temperature or wall heat flux. More recent applications of this topic have recently emerged in modern equipment and devices such as in nuclear reactors, solar panels, cooling in buildings, and electronic circuit boards. Even though forced convection is usually a main method of removing excessive heat in such applications, natural convection is always present to some extent. In most situations, natural convection alone is perhaps preferable for carrying out the cooling since the process is spontaneous, simpler, and requires no compressors, fans, blowers, or pumps.

In forced convection, the mass and momentum equations can be solved for the flow distribution before embarking on the solution of the energy equation.

Laminar and turbulent forced convection in channels have been studied for over a century and the associated physics of flow and heat transfer are well understood. In contrast, and, in spite of many studies on the subject, the physics of the pure natural convection process within vertical channels or tubes is not fully understood even in the laminar regime. Since the flow in natural convection is a consequence of the non-uniformity in the density field as caused by the temperature distribution, the flow and energy problems are coupled, whence the complexity of natural convection. Closed-form solutions of natural convection, if any, are difficult to be acquired. Analytical modeling for such a flow, therefore, calls for reasonable simplifications and approximations and some accuracy is prone to be sacrificed.

Our goal in this thesis is to seek solutions to the problem of natural convection heat transfer in vertical ducts.

1.2 Background

The pioneering work on the topic of natural convection in a vertical channel (formed by two parallel plates separated by a distance D) was carried out by Elenbaas (1942a). A semi-empirical correlation of the average Nusselt number as a function of one parameter — the product of Grashof and Prandtl numbers over the aspect ratio of the parallel plates (Gr Pr D/L) — has been proposed and confirmed by experimental data. A similar relation for natural convective flow in a vertical circular tube has also been suggested in a subsequent paper (Elenbaas, 1942b). These two papers of Elenbaas have been referred to, debated, criticized, discussed, and extended in numerous investigations which followed over the decades.

The geometric configuration studied by Elenbaas, as shown in Fig. 1.1, is a vertical channel formed of two very wide parallel plates separated by a distance D (or tube of circular cross-section with diameter D). The words "channel", "pipe", "duct", and "tube" are henceforth used in the present work synonymously unless a specific distinction is called upon. The channel length extends from $x = 0$ (entrance) to $x = L$ (exit). The (radial) coordinate, normal to the axial, is denoted by r. $r = 0$ represents the axis, or plane of symmetry, while $r = R \equiv D/2$ represents the inside surface of the channel. The local acceleration due to gravity is g. This channel is situated in an infinite chamber of quiescent air at a temperature T_∞ and a pressure P_∞. The ambient pressure varies hydrostatically with height x. The channel walls are symmetrically kept at a uniform temperature T_W (larger than T_∞). This prescribed inner surface temperature boundary condition is henceforth denoted as the UWT condition. (In contrast, if the channel walls are kept at a constant/uniform heat flux boundary condition, we denote it as the UHF condition.) The finite positive temperature difference (T_W-T_∞) produces density differences in the air; an upward flow is induced gravitationally within the channel. At the inlet, boundary layers develop over the two walls. Theories of free convection over an isolated single wall may fail to satisfactorily describe the interaction between these boundary layers developing within the channel. If the channel is sufficiently long,

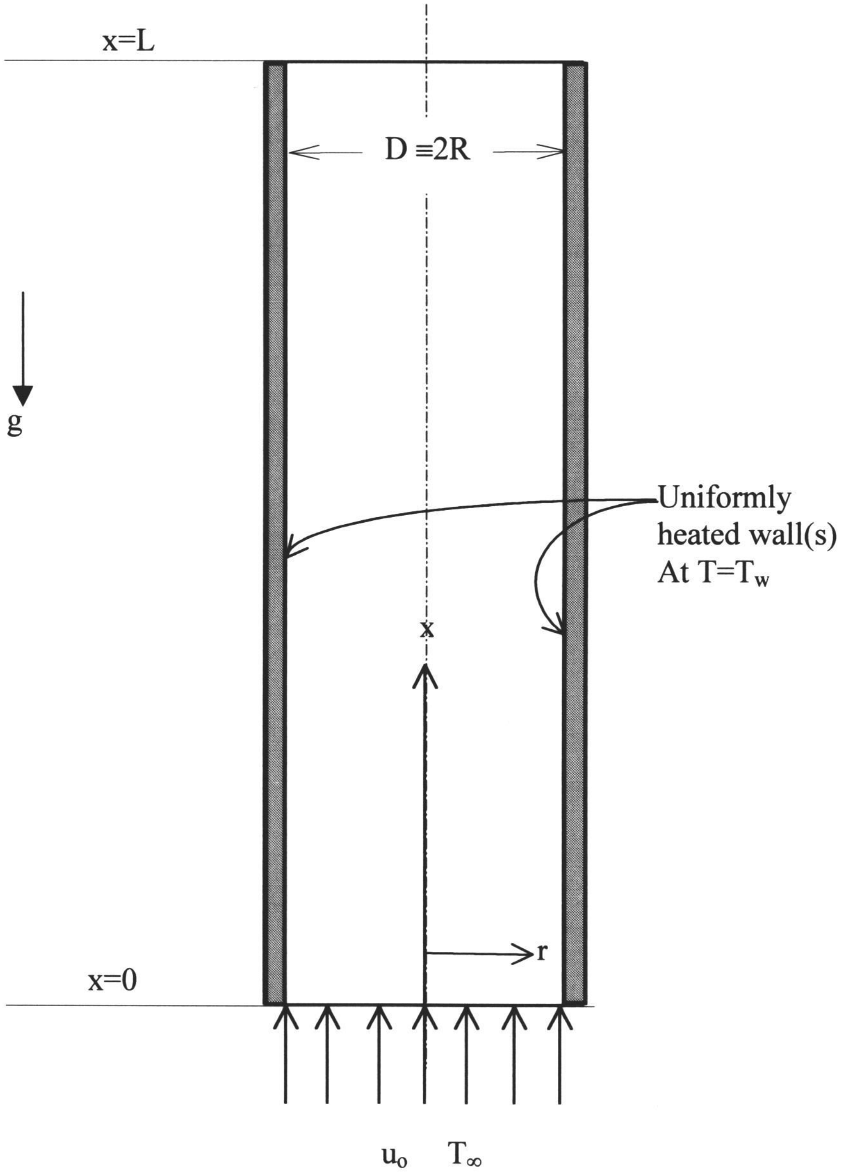


Figure 1.1 Channel configuration and coordinate system

the boundary layers merge at the plane (or axis) of symmetry at some value of x. Characteristics of the natural convection flow within the channel depend on the non-dimensional parameter Gr Pr D/L where Grashof, Gr, and Prandtl, Pr, numbers are defined as usual as

$$Gr = \frac{g\beta(T_w - T_\infty)D^3}{\nu^2} \quad (1.1) \qquad\qquad Pr = \frac{\nu}{\alpha} \quad (1.2)$$

Where β is coefficient of fluid thermal expansion, ν is fluid kinematic viscosity, and α is fluid thermal diffusivity.

- Short Channel:

A specified high Gr Pr D/L, represents flow in a short channel (i.e. small L/D) due to strong heating (i.e. high Gr). In this situation, sketched in Fig. 1.2a, the boundary layer on one side of the channel has relatively small effect on the one developing over the opposing surface. The boundary layers may not merge at or before x = L in this short channel; the mass flow rate is an unknown. Unlike forced convection in a channel, the flow characteristics in this short-channel case result from the laminar natural convection flows over each of the two vertical uniformly heated plates facing each other. As a corollary, as shown in Fig. 1.2a, the velocity distribution is doubled-peaked over a channel cross section. These profile distributions on each side of the flow axis symmetry are similar to those that occur in the laminar natural convection flows over a vertical uniformly heated plate. If the channel is short enough such that the interaction between the boundary layers is

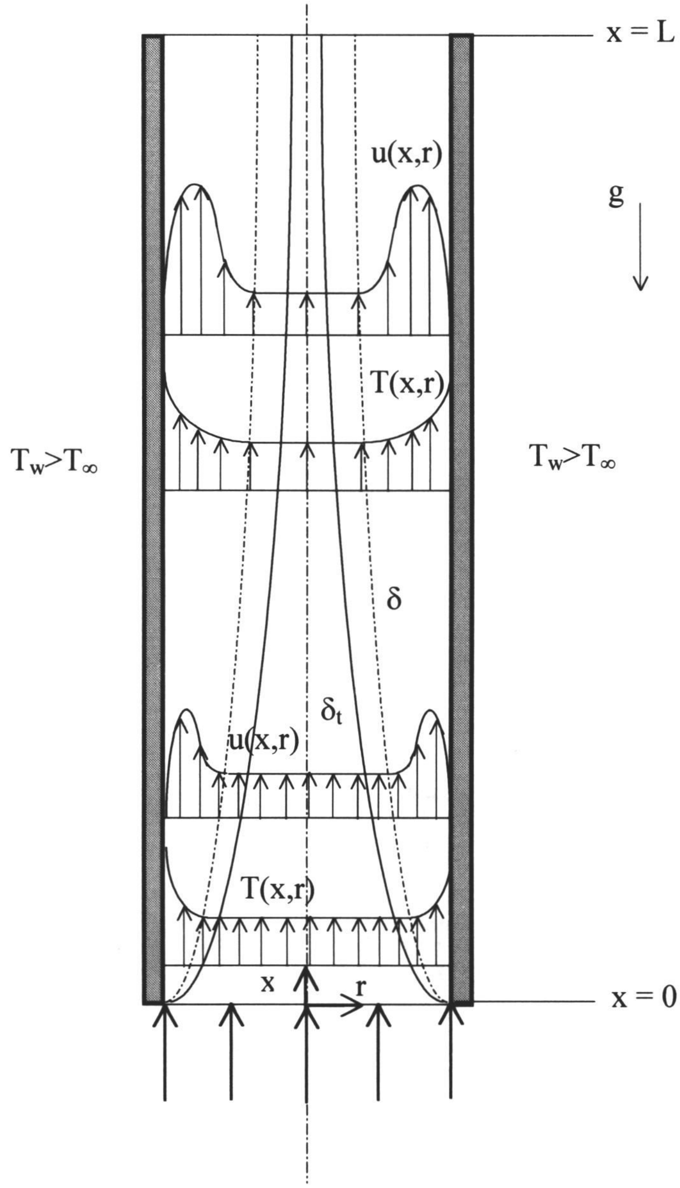


Figure 1.2a Qualitative radial (r) temperature and velocity distributions at various values of axial (x) distance from the inlet for limiting case: wide and short channel, (i.e., high GrPrD/L).

not well established, the fluid temperature and velocity distributions will approach the asymptotic limit of flow over a vertical uniformly heated plate

- Long Channel:

The characteristics of flow are different when the specified Gr Pr D/L is low. In this second limiting case, the channel is either very long (i.e. high L/D) or is weakly heated (i.e. low Gr). In this case, as sketched in Fig. 1.2b, the mass flow rate is maximum. Again and similar to forced convection, there are two flow regimes within the channel; (a) near the entrance, there exists a developing region where the flow and thermal boundary layers gradually develop to merge at the plane (or axis) of symmetry; and (b) a fully developed region where velocity and temperature profiles across the channel are unchanging in relative shape. It will be later shown in Chapter 5 that flow in this fully developed region stipulates the maximum mass flow rate.

Recalling from Prandtl's boundary layer theory, for air ($Pr \approx 0.72$), the heating effect would penetrate into the fluid flow faster than would the friction effect; therefore, thermal boundary layers reach the channel's axis or plane of symmetric slightly sooner than the hydrodynamic (viscous) boundary layers do. The region between $x = 0$ and the value of x at which the thermal boundary layers merge (i.e. thermally developing region) shall henceforth be termed the *first* entrance region.

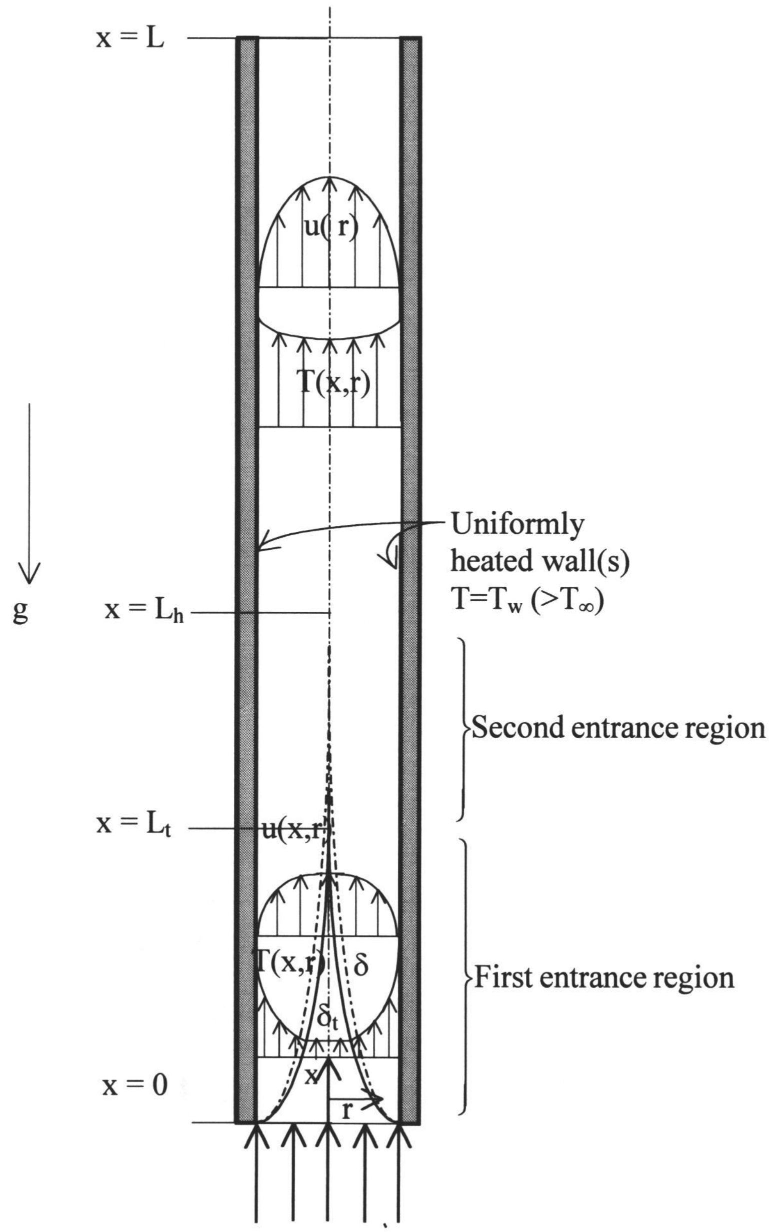


Figure 1.2b Qualitative radial (r) temperature and velocity distributions at various values of axial (x) distance from the inlet for limiting case of narrow and long channel, (i.e., low GrPrD/L). L_h is hydrodynamic developing length, L_t is thermal developing region length.

At the end of this region, the temperature distribution T(r) across the channel cross section will be developed in shape whereas the velocity distribution u(r) may not be yet fully developed. In the subsequent flow, transverse diffusion will transform the developing velocity distribution into the well-known Poiseuille parabolic distribution in laminar flow in a channel. This region (in which the flow confines to develop into the Poiseuille pattern) shall henceforth be termed as the *second* entrance region. Its length is expected to be much smaller than that of the first region since the fluid Prandtl number is close to unity. Following the two entrance regions, the flow is fully developed both thermally and hydrodynamically.

For the UWT boundary condition, simplified mathematical models with adjustments to deal with limiting cases form the basis of the following heat transfer correlations as proposed by Elenbaas (1942a and b). The simplifications are so severe that many later authors called these correlations as "semi-empirical". $\overline{Nu_\infty}$ is the Nusselt number (to be defined and discussed in Chapter 2) and generally correlated as

$$\overline{Nu_\infty} = \frac{1}{K_{Eb}}\left(\frac{2A_c}{DP_w}\right)^3 \frac{Gr\,Pr\,D}{L}\left[1-\exp\left(-K_{Eb}\left(\frac{1}{2}\left(\frac{DP_w}{2A_c}\right)^4 \frac{L}{Gr\,Pr\,D}\right)\right)^{C_A}\right]^{C_B} \tag{1.3}$$

which is an average value over the duct length. A_c is the duct cross-sectional area and P_w is the duct perimeter. K_{Eb} is a constant obtained from an analysis of the flow at extreme limits and found to be 24 for the parallel-plate channel and 16 for circular tube. To match Eq. (1.3) with the experimental results, constants C_A and C_B are arbitrarily chosen to be 3/4 and 1, respectively. For parallel-plate channels

Elenbaas (1942b) also reported that using C_A as 1 and C_B as 3/4 also yield good agreement with experimental data within 10 such that

$$\overline{\mathrm{Nu}}_{\infty} = \frac{1}{24}\frac{\mathrm{Gr\,Pr\,D}}{\mathrm{L}}\left[1-\exp\left(-\frac{35\mathrm{L}}{\mathrm{Gr\,Pr\,D}}\right)\right]^{3/4} \tag{1.4a}$$

For a circular tube, Eq. (1.3) gives

$$\overline{\mathrm{Nu}}_{\infty} = \frac{1}{128}\frac{\mathrm{Gr\,Pr\,D}}{\mathrm{L}}\left[1-\exp\left(-16\left(\frac{8\mathrm{L}}{\mathrm{Gr\,Pr\,D}}\right)^{3/4}\right)\right] \tag{1.4b}$$

For the case of a very long channel, L/D is sufficiently large such that $\overline{\mathrm{Nu}}_b$ then becomes GrPrD/(24L) for UWT parallel plates and GrPrD/(32L) for UWT circular tube. However, these numbers serve as the asymptotic values for the case of a long channel only. They represent the convective heat transfer coefficients of the fluid when the fluid temperature is very close to the channel wall temperature, implying that the heating is almost complete; therefore, they do not really describe any physics of the flow.

The coupled equations of momentum and energy governing pure natural convection are generally solved by a variety of approaches ranging from simplified analyses to numerical analyses. With the advent of the digital computer, the problem of natural convection in vertical channels and tubes has attracted renewed attention in the recent literature, as authors have attempted to validate the work of Elenbaas. (see, for instance, Bodia and Osterle 1962, Aung et al. 1972, and Ramanathan and Kumar1991.)

Numerical methods lead to solutions of complex engineering problems. The mathematical problem is correctly formulated first in terms of the conservation equations, constitutive relations, and boundary conditions, even if in greater detail usually than what is necessary. These equations are then discretized by using finite difference or element techniques. Then, an algorithmic flow chart is developed for a computer program. The numerical solution is processed in the back-end into instructive and impressive tabular and graphical output which is interpreted by the investigator. However, it is fair to recognize that resorting to a numerical method too early in a study generally leads to a sacrifice of the deeper physical understanding of the processes involved.

Churchill and Usagi (1972) proposed a way of combining the extreme limiting cases: the long narrow channel and the short wide channel, to obtain the characteristics in between. A series of publications exploiting this notion are available; see, for example, Raithby and Hollands, 1975, Bar-Cohen and Rohsenow, 1984. Even though much literature of this sort exists, a thorough understanding of the physics of the flow and heat transfer, and of the interactions among relevant transport mechanisms is lacking. *In this dissertation, we seek a clear physical understanding of the mechanisms of naturally induced flow and heat transfer in vertical channels and tubes. To accomplish this goal, we apply boundary layer theory, a series solution method, and integral methods of solutions to the problems.* The governing equations of the considered system remain complex even after making several reasonable simplifications and approximations.

Numerical schemes appear to be unavoidable. However, the use of numerical analysis are limited in this work and used only when absolutely necessary (i.e., when no other paths of analysis seem feasible). The numerical procedures employed are kept as simple as possible while yet delivering acceptable accuracy for comparison with previous work. The results, especially of the heat transfer, is then compared with existing work and discussed.

1.3 Outline of this Dissertation

Following the present introductory chapter, several preliminary concepts and definitions are introduced and reviewed in Chapter 2. The Series Solution Method for fully developed flows is validated in Chapter 3 by applying it to forced convection channel/pipe flow with UWT and UHF boundary conditions. (The method offers an alternative to the iterative method to solve UWT laminar pipe-flow forced convection problems available in most of the convective heat transfer textbooks.) Once thus validated, the Series Solution approach similar to what is done in Chapter 3 is used in Chapter 4 to study the characteristics of fully developed pure natural convection in vertical plate channels as well as pipes with UWT and UHF boundary conditions. Using the flow characteristics in the simpler fully developed flow regime, the developing boundary layer regime of the flow near the inlet of the long vertical channel/tube is analyzed in Chapter 5 for the parallel-plate channel and tube flow with UWT boundary condition by employing the boundary layer theory and integral method of solution. This developing region problem for UHF boundary condition is yet to be solved. Chapter presents a summary of work done here and directions for future work.

CHAPTER 2: PRELIMINARIES

To facilitate a clear and unambiguous discussion of the problem of laminar natural convection within a channel or tube, certain basic definitions and concepts are presented.

2.1 Mass Conservation and Bulk Velocity of Flow

Consider a flow in a uniformly heated parallel-plate channel/circular tube. For steady state flow, conservation of mass requires that the mass flow rate in the channel/tube be constant with respect to x, the axial coordinate. This constant mass flow rate is related to the local velocity distribution u(x,r) at any given value x by

$$\dot{m} = \int_{A_c} \rho u(x,r) dA_c \tag{2.1}$$

where A_c is a cross-sectional area of channel. To represent a constant mean value for velocity distribution u(x,r) at any given value x, one defines the bulk (average) velocity, u_b as

$$u_b \equiv \frac{\dot{m}}{\rho A_c} = \frac{\int_{A_c} \rho u(x,r) dA_c}{\rho A_c} \tag{2.2}$$

For incompressible flow, u_b is a constant with respect to x and leads to the following alternative expression for mass conservation:

$$\int_0^1 \left(\frac{u}{u_b}\right)\left(\frac{r}{R}\right)^j d\left(\frac{r}{R}\right) = \frac{1}{2^j} \tag{2.3}$$

where the index values $j = 0$ and 1 stand for parallel-plate channel and circular tube respectively. This relation holds in both developing and developed regimes, regardless of whether the flow is laminar or turbulent and is forced or natural.

2.2 Overall Energy Conservation and Bulk Temperature of Flow

The arguments of section 2.1 also apply to the energy equation. Since the fluid temperature is possibly dependent on both x and r coordinates, its average value is of a practical interest. The bulk or mean fluid temperature T_b at a given cross section is defined in terms of the thermal energy convected by the fluid at any given cross section. Consider a differential length dx of the heated channel as shown in Fig. 2.1 and write energy equation balance for this fluid element:

$$\dot{q}''_w P_w dx = \dot{m} C_p dT_b, \tag{2.4}$$

where T_b is bulk (or average) temperature of fluid (to be formulated below) and P_w is channel perimeter. $\dot{q}''_w$ is the heat flux at the channel wall. In essence, this

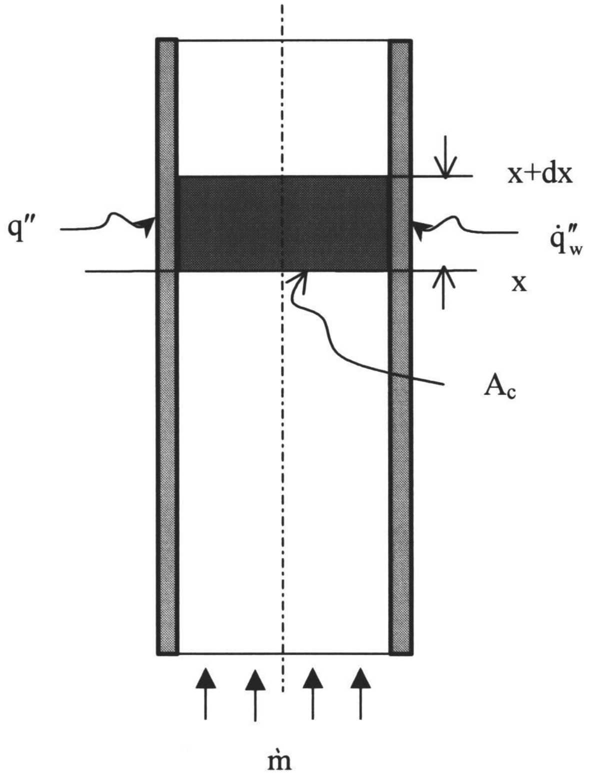


Figure 2.1 Energy balance in a segment of length dx in a channel or tube

equation says that the heat transferred through the periphery area, $P_w dx$, is equal to the differential increase in the enthalpy rate in the flow. The local bulk temperature $T_b(x)$ is related to the local temperature distribution $T(x,r)$ as to convey the local rate of enthalpy flow in the local velocity distribution $u(r)$. This can be express by the equation

$$\dot{m}C_p\left(T_b(x) - T_{ref}\right) = \int_{A_c} \rho u(r)\left(T(x,r) - T_{ref}\right)C_p dA_c \qquad (2.5)$$

where T_{ref} is any chosen reference temperature. Thus, the fluid bulk temperature is defined as

$$T_b(x) - T_{ref} = \frac{\int_{A_c} \rho u(r)\left(T(x,r) - T_{ref}\right)C_p dA_c}{\rho u_b A_c C_p} \qquad (2.6)$$

The numerator on the right hand side is the enthalpy flow rate in the entire channel cross-section at any x as it is distributed over r. The denominator is the heat capacity flow rate. For incompressible flow, this leads to another form of the energy equation, analogous to Eq. (2.3),

$$\int_0^1 \left(\frac{u}{u_b}\right)\left(\frac{T(x,r) - T_{ref}}{T_b(x) - T_{ref}}\right)\left(\frac{r}{R}\right)^j d\left(\frac{r}{R}\right) = \frac{1}{2^j} \qquad (2.7)$$

Bulk temperature variation with x can be obtained from an integration of Eq. (2.4). If the channel wall(s) is at an UHF boundary condition, (i.e., a constant uniform heat flux into the fluid from the channel wall is specified) integration leads to

$$T_b - T_{bi} = \frac{\dot{q}'' P_w}{\dot{m} C_p} x \tag{2.8}$$

where T_{bi} is the uniform inlet temperature at $x = 0$.

For an UWT boundary condition, (i.e., constant, uniform channel wall temperature is specified), from Newton's law of cooling, the wall heat flux is written in terms of the fluid bulk temperature as

$$\dot{q}'' = h_b (T_W - T_b) \tag{2.9}$$

to define the heat transfer coefficient h_b. The corresponding Nusselt number is defined as $Nu_b \equiv h_b D/k$. The bulk temperature variation with x is then obtained by inserting Eq. (2.9) into Eq. (2.4). Separating variables and integrating from the channel inlet to the outlet gives

$$T_b = T_w - (T_w - T_\infty) \exp\left(- \frac{\bar{h}_b P_w}{\dot{m} C_p} \right) \tag{2.10}$$

where h_b is an average over the length $x = 0$ to L, defined as

$$\bar{h}_b = \frac{1}{L} \int_0^L h_b dx \tag{2.11}$$

These relations hold for any steady flow in both developing and developed regimes, for laminar as well as turbulent flow and for forced as well as natural convection.

We note from Eq. (2.2) that while the fluid bulk velocity is invariant with x respect to x to satisfy the conservation of mass because the channel wall(s) is (are) impervious, in contrast, the fluid bulk temperature T_b is not a constant with respect to x. It increases with x due to the heating of the fluid from the channel wall. The temperature of the fluid within a UWT channel, therefore, increases as a function of x and r coordinates. Eventually, when heating in the channel is fully done, the fluid temperature T(x,r) becomes uniform over the entire cross section in equilibrium with the wall temperature. However, for the flow within a UHF channel, heating is never complete as long as the fluid does not reach the end of the channel and as long as the fluid does not change phase.

2.3 Fully Developed Flow Criterion

Near the entrance, as the effects of viscosity and the thermal conductivity diffuse from the channel wall(s), the boundary layers within a uniformly heated channel gradually develop as a function of x as the flow advances forward into the channel. If the channel/tube is long enough for the effects to penetrate to a plane (or axis) of symmetry, the flow then reaches its fully developed regime in which the relative shapes of velocity u(r) and temperature T(r) remain unchanging with x. The flow velocity profile in comparison with some reference value (herein using fluid velocity at the channel wall u_w) then remains invariant in the fully developed region:

$$\frac{\partial}{\partial x}\left(\frac{u(x,r)-u_w}{u_b-u_w}\right)_{FD}=0 \tag{2.12}$$

This implies that the $(u-u_w)/(u_b-u_w)$ is a function of r only in the fully developed flow regime. Since the fluid bulk velocity is a constant to keep a mass flow rate $\dot{m}$ constant and the reference value at the channel wall u_w is zero due to the no-slip condition, Eq. (2.12) is generally stated in heat transfer textbooks and papers simply as $\partial u/\partial x = 0$. The velocity profile u(r) is invariant with x in the fully developed flow. Together with conservation of mass, this leads to the conclusion that the fluid lateral motion is absent in a fully developed flow regime, if the walls are impervious. In the laminar case this is denoted at "Poiseuille flow".

In a similar way, the fluid temperature profile T(x,r) at any axial distance x can be related to the channel wall temperature T_w and its mean (bulk) value T_b in the form of

$$\Theta(x,r) = \frac{T(x,r) - T_w(x)}{T_b(x) - T_w(x)}$$

The non-dimensional and normalized temperature Θ becomes independent of x when the flow is fully developed (Kays, 1980). In spite of the fact that the temperature profile continues to change with x due to continuous heating, the relative distribution shape remains unchanging with x. Hence, the condition for thermally fully developed flow can be stated as

$$\frac{\partial}{\partial x}\left(\frac{T(x,r) - T_w(x)}{T_b(x) - T_w(x)}\right)_{FD} = 0 \tag{2.13}$$

Following the analogy between flow and heat transfer this condition for thermally fully developed flow has been incorrectly stated in several existing work as $\partial T/\partial x = 0$. This wrongly implies that a fully developed temperature profile is no longer a function of x coordinate. As mentioned above, in a UWT channel, the only situation in which the flow temperature is invariant with x is when the heating is complete, i.e. the fluid is in thermal equilibrium with the channel wall. The convection heat transfer coefficient calculated from the complete heating region has no meaning and reveals no essential physics since heat convection from the channel wall to the fluid is no longer present. Some of the previous investigations of the flow within the channel used a complete heating condition, $\partial T/\partial x = 0$, instead of Eq. (2.13), e.g. Kageyama and Izumi, (1970), Davis and Perona (1971) and Aung (1972). It is even more confusing (as discussed by Yao, 1987) when the complete heating condition is used for the fully developed flow within a UHF channel in which fluid temperature profile is not possibly independent of x coordinate but, in fact, keeps increasing as the flow advances forward. Eq. (2.13), therefore, introduces and provides more reasonable condition for such a fully developed flow within uniformly heated channel. Its powerful meaning is that the normalized "shape" of the temperature distribution is unchanging with x.

Several characteristics of a fully developed flow are recognized from the thermal condition for fully developed flow, Eq. (2.13). From Newton's law of cooling, Eq. (2.9) and Fourier's law of conduction, $\dot{q}''_w$ = -k$\partial T \partial r$ at r =R, the local Nusselt number can be written as

$$h_b = \frac{\dot{q}''}{(T_w - T_b)}$$

$$\frac{h_b D}{k} \equiv Nu_b = \frac{D}{(T_w - T_b)} \left.\frac{\partial T}{\partial r}\right|_{r=R} = 2\left.\frac{\partial \Theta}{\partial (r/R)}\right|_{r=R} \qquad (2.14)$$

As Θ is invariant with the x coordinate in the fully developed flow regime, the derivatives of Θ with respect to r are also independent of x. It follows from Eq.

(2.14) that h_b and Nu_b are constants independent of both x and r for a fully developed flow. This conclusion holds for forced as well as free convection and for laminar as well as turbulent flow within a channel for all wall-heating conditions.

Carrying out the differentiation in Eq. (2.13) leads to a relation between the derivatives (with respect to x) of fluid temperature T(x,r), bulk temperature, and channel wall temperature as the following,

$$\frac{\partial T}{\partial x} = \Theta \frac{dT_b}{dx} + (1-\Theta)\frac{dT_w}{dx} \tag{2.15}$$

For UWT channels (T_w is a constant), this further simplifies to

$$\left.\frac{dT_b}{dx}\right|_{FD} = \left.\frac{dT_w}{dx}\right|_{FD} = \left.\frac{\partial T}{\partial x}\right|_{FD} \tag{2.16}$$

For flow within UHF channels, Eq. (2.9) shows that $dT_b/dx = dT_w/dx$. Then, from Eq. (2.15), $\partial T/\partial x = dT_b/dx$ so that

$$\left.\frac{\partial T}{\partial x}\right|_{FD} = \Theta(r)\left.\frac{dT_b}{dx}\right|_{FD} \tag{2.16a}$$

2.4 Results for Fully Developed Forced Convection within a Uniformly Heated Channel

In the developing region of a uniformly heated parallel-plate channel or circular tube for forced convection, the velocity distribution of the flow can be determined before attacking the energy equation to obtain the temperature distribution. This does not make finding the solution for the developing flow any easier since both the velocity and temperature fields depend on x and r. The UWT circular tube problem was solved by Graetz in the late nineteenth century. Neglecting axial heat diffusion he solved the energy equation: (a) in 1883 by assuming velocity profile to be uniform across the cross-section (i.e. slug or plug flow) appropriate for low-Pr fluid; and (b) in 1885 by assuming the velocity distribution is given by a Poiseulle flow. The details of the analyses will not be repeated here but can be found in several heat transfer textbooks (e.g. Shah and London, 1978 and Kays and Crawford, 1980).

In the fully developed flow region, the functions u(x) and T(x,r) are relatively much simpler than in the developing flow. Eq. (2.12) can then be applied to the equation of momentum. Setting the lateral velocity component equal to zero in this region, the fully developed velocity profile (Poiseuille flow) is

$$\frac{u(r)}{u_0} = K_1\left(1-\left(\frac{r}{R}\right)^2\right) \tag{2.17}$$

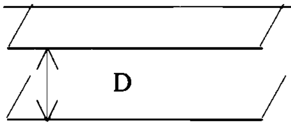
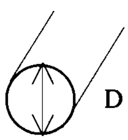
K_1 is 1.5 and 2 for parallel-plate channel and circular tube, respectively, and u_o is the fluid bulk velocity at the inlet.

Using the condition for thermally fully developed flow stated in Eq. (2.13) or Eq. (2.15), and the Poiseuille velocity profile from Eq. (2.17), the energy equation can be solved analytically for the case of UHF channel problem. The UWT problem, however, can be solved only by invoking successive approximations for the temperature profile. Details of this iterative calculation are well known and can be found in most of heat transfer textbooks. Introduce in the following section, we show in Chapter 3 an alternative to this method of successive approximations. The method will be validated in Chapter3 and used in Chapter 4.

Heat transfer to a fully developed flow is then calculated from a temperature distribution obtained from the energy equation. The result is usually given as the Nusselt number defined in Eq. (2.14). Table 2.1 gives a summary of the local Nusselt number results.

It is worth noting that the local Nusselt number, Nu_b, is defined here based on the physical channel spacing or tube diameter, D. This definition differs from the prevailing literature which uses the "hydraulic diameter" as the characteristic dimension for ducts of noncircular cross-section. The use of "hydraulic diameter" in heat transfer has recently attracted much criticism. (See Oosthuizen and Naylor, 1999, pp. 178). Our choice to use the separation distance D of the parallel plates forming the channel is based on its close analogy to the diameter of a tube of circular cross-section and on the closeness of the so expressed Nusselt number Nu_b displayed in Table 2.1. The Nu_b for a parallel plate channel is a mere 3 percent higher than that for a tube of circular cross-section with UWT boundary condition and only 5.5 percent smaller with a UHF boundary condition. The only factor that makes heat transfer from uniformly heated parallel-plate channel to fluid flow much larger than that from uniformly heated circular tube is the difference in the channel perimeter area.

Table 2.1 Local Nusselt number, $Nu_b \equiv h_b D/k = \dot{q}''_w D/[(k(T_w - T_b)]$ for fully developed laminar forced convective flow within uniformly heated channel and tube

Geometry	UWT	UHF
	3.77	4.12
	3.66	4.36

There are a several other ways to define the heat transfer coefficient depending on the particular temperature difference used in Newton's law of cooling. For example, one can use the difference between the wall temperature and the temperature at the axis. Another temperature difference used in previous investigations (including in Elenbaas's work, 1942a and b) is the wall-to-inlet temperature difference (T_W-T_∞) so that the corresponding Nusselt number Nu_∞ is defined as

$$\frac{h_\infty D}{k} \equiv Nu_\infty = \frac{\dot{q}''_w D}{(T_w - T_\infty)k} \tag{2.18}$$

A corresponding average Nusselt number is also defined as

$$\overline{Nu_\infty} \equiv \frac{\bar{h}D}{k} = \frac{D}{kL}\int_0^L h\,dx \tag{2.19}$$

2.5 A Series Solution Method

The series Solution Method (Power Series Method) is well known as a powerful technique having a great utility in solving transport problems in which ordinary differential equations arise in the form of

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)\,y = 0 \tag{2.20}$$

The general solution of this equation is

$$y(x) = \mathbf{A}y_1(x)+\mathbf{B}y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are linearly independent unknown functions and **A** and **B** are constants to be found.

The method is an effective way to solve linear ordinary differential equations with variable coefficients by using a convergent power series expressing the value of the dependent variable y in terms of the independent variable x to any required degree of accuracy. It can be used to determined Υ linearly independent solutions where Υ is the order of the ordinary differential equation. Hence, the fundamental assumption made in solving the second-order ordinary differential equation Eq. (2.20) by the series solution method is such that the solution of the differential equation can be expressed in the form of a power series,

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \ldots \tag{2.21}$$

where c_0, c_1, … are constant coefficients of the series and x_0 is a constant called the center of the series.

If the assumption is valid, the coefficients c_0, c_1, ... are determined in such a manner that Eq. (2.21) does indeed satisfy Eq. (2.20). However, the conditions under which this assumption is valid remain to be established and discussed below. The basic definition and theory related to the power series solution method are introduced in the following.

2.5.1 Basic concepts of power series

Recall from calculus that a power series in the form of Eq. (2.21)

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \tag{2.22}$$

always converges at $x = x_0$ to the sum c_0. It may converge only at this point ,or for all values of x or there may be a positive number $\Re$ such that the series converges absolutely for $|x-x_0| < \Re$ and diverges when $|x-x_0| > \Re$. The number $\Re$ is called the radius of convergence of the power series. The interval of convergence, which may include one or both endpoints $x- x_0 = \pm\Re$. The radius of convergence is mostly obtained from either a ratio test

$$\Re = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| \tag{2.23}$$

or the root test (Kreyszig, 1993)

$$\Re = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|c_n|}} \tag{2.24}$$

Hence, a power series in Eq. (2.22) defines a function f(x) whose domain is the interval of convergence. The value f(x) at each point x is the sum of the series at the point x. It can be shown that if a series f(x) has the radius of convergence $\Re$, then termwise integration and differentiation are valid for every points $x \in (x_0 - \Re, x_0 + \Re)$.

2.5.2 The existence of power series solutions

The remaining important question is whether a differential equation, Eq. (2.20) has a power series solution at all. The existence of power series solution depends on whether the center of the series x_0 is an ordinary point or a singular point.

The point x_0 is classified as an ordinary (regular) point of the differential equation, Eq. (2.20), if all functions $P_n(x)$, n = 1, 2, are analytical at x_0 meaning that they can be represented by a power series in powers of $(x-x_0)$ with some non zero radius of convergence $\Re$. (Thus, $P_n(x)$ is called an analytic function.) If at least one of these functions is not analytic at x_0, then x_0 is called a singular point of the differential equation. However, if functions defined by $(x-x_0)\cdot P_1$ and $(x-x_0)^2\cdot P_1$ are analytical at x_0, then x_0 is called regular singular point. A singular point that is not regular is called otherwise irregular.

- **Solutions about an Ordinary Point**

If all variable coefficients, P_n's, in Eq. (2.20) are analytic at $x = x_0$ then every solution of Eq. (2.20) is also analytic at x_0 and can thus be represented by a power series in powers of $(x - x_0)$ in a form of Eq. (2.21) with some radius of convergence $\Re>0$:

$$y = \mathbf{A}y_1(x) + \mathbf{B}y_2(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \qquad (2.21)$$

where **A** and **B** are arbitrary, and $y_1(x)$ and $y_2(x)$ are now linearly independent series solutions which are analytic at x_0. Further the radius of convergence for each of the series solutions $y_1(x)$ and $y_2(x)$ is at least as large as the minimum of the radii of convergence for the series for $P_1(x)$ and $P_2(x)$.

Power series expansions for dy/dx and d^2y/dx^2 are then obtained by differentiating Eq. (2.21) term by term:

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} nc_n (x - x_0)^{n-1} = c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + \ldots \qquad (2.25)$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1)c_n (x - x_0)^{n-2} = 2c_2 + 3\cdot 2c_3(x - x_0) + 4\cdot 3c_4(x - x_0)^2 + \ldots \qquad (2.26)$$

and these series are substituted back into the given differential equation. Collecting the term involving similar powers of x, ones then obtain Eq. (2.20) in a form

$$k_0 + k_1(x - x_0) + k_2(x - x_0)^2 + \ldots = \sum_{n=0}^{\infty} k_n (x - x_0)^n = 0 \qquad (2.27)$$

where the coefficients k_0, k_1, k_2, … are expressions involving the unknown coefficients c_0, c_1, c_2, …. Since Eq. (2.27) must hold for all values of x in the specified domain, all coefficients k_0, k_1, k_2, … must be zero. It is then possible to determine the coefficients c_0, c_1, c_2, ..., successively. This leads to a set of conditions which must be satisfied by the various coefficients c_n in the series in order that Eq. (2.21) be a solution of the differential equation, Eq. (2.20). If the c_n are chosen to satisfy the set of conditions which thus occurs; mostly, in terms of a recurrence formula, then the resulting series, Eq. (2.21) is the desired solution of the differential equation, Eq. (2.20).

- **Solutions about Singular Points; Method of Frobenius**

However, if x_0 is a either a regular or irregular singular point, the power series solution may not have a solution in a form of Eq. (2.21). It happens that under the condition that x_0 is a regular singular point of Eq. (2.20), then the differential equation has at least one non-trivial solution of the form

$$y=(x-x_0)^m\sum_{n=0}^{\infty}c_n(x-x_0)^n, \tag{2.28}$$

where m is a constant which is to be determined and this solution is valid in some deleted interval $|x-x_0| < \Re$ about x_0. The procedure to solve for c_n's is mostly similar to that previously introduced above and it is known as the Frobenius or extended power series method. The resulting Eq. (2.27) takes the form

$$k_0(x-x_0)^{m+q}+k_1(x-x_0)^{m+q+1}+k_2(x-x_0)^{m+q+2}+...=0 \tag{2.29}$$

where q is a certain integer obtained in the process and the coefficients k_0, k_1, k_2, ... are functions of m and certain of the coefficients c_n of the solution. For Eq. (2.29) to be valid at all x on the interval of convergence, k_0, k_1, k_2, ... must be zero. Upon equating to zero the coefficient k_0 of the lowest power m+q of (x- x_0), an equation for m is obtained, called the indicial equation of the differential equation. Thus the unknown constant m is determined. Roots of the indicial equation, m_1 and m_2 where $m_1 \geq m_2$, are called the exponents of the differential equation and are the only possible values for the constant m in the assumed solution, Eq. (2.28). One then equates to zero the remaining coefficients k_1, k_2, ... and are led to a set of conditions involving constant m, which must be satisfied by the various coefficients c_n in the series. There are three cases to consider for the case of 2nd order linear ordinary differential equation. Recall that the solution is in the form of $y(x) = \mathbf{A}y_1(x)+\mathbf{B}y_2(x)$. It the follows that

Case 1. Distinct roots, $m = m_1$ and m_2, not differing by an integer. Solutions are

$$y_1(x) = (x - x_0)^{m_1}\left(c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + ...\right) \qquad (2.30)$$

and

$$y_2(x) = (x - x_0)^{m_2}\left(C_0 + C_1(x - x_0) + C_2(x - x_0)^2 + ...\right) \qquad (2.31)$$

Case 2. Double root $m_1 = m_2 = m$. Solutions are

$$y_1(x) = (x - x_0)^{m}\left(c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + ...\right) \qquad (2.32)$$

and

$$y_2(x) = y_1(x)\ln(x - x_0) + (x - x_0)^{m}\left(C_0 + C_1(x - x_0) + C_2(x - x_0)^2 + ...\right) \qquad (2.33)$$

<u>Case 3</u>. Roots differing by an integer. Solutions are

$$y_1(x) = (x - x_0)^{m_1}\left(c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \ldots\right) \tag{2.34}$$

and

$$y_2(x) = Dy_1(x)\ln(x - x_0) + (x - x_0)^{m_2}\left(C_0 + C_1(x - x_0) + C_2(x - x_0)^2 + \ldots\right) \tag{2.35}$$

A constant D may turn out to be zero.

Additionally, we note that if $x_0 = 0$ is a regular singular point of the governing equation in a form of

$$\mathbf{P}(x)\frac{d^2y}{dx^2} + \mathbf{Q}(x)\frac{dy}{dx} + \mathbf{R}(x)y = 0 \tag{2.36}$$

where **P**(x), **Q**(x), and **R**(x) are polynomials, then the radius of convergence for the series $y_1(x)$ and $y_2(x)$ is at least equal to the distance from the origin to the nearest zero of **P**.

CHAPTER 3

VALIDATION OF A SERIES SOLUTION METHOD ON FULLY DEVELOPED LAMINAR FORCED CONVECTION WITHIN A UNIFORMLY HEATED VERTICAL PARALLEL-PLATE CHANNEL AND CIRCULAR TUBE

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In Preparation

3.1 Abstract

The familiar problem of forced convection heat transfer in fully developed laminar flow in a channel and tube is solved by a series solution method. This serves to offer an alternative method which alleviates the need for successively approximating of temperature distribution in the flow. It also serves to validate the series solution method for the use in natural convection in channels and tubes in our subsequent work.

3.2 Introduction

Our goal is to validate the use of a series solution method as an alternative approach for solving the well-known problem of fully developed forced convection in a uniformly heated parallel-plate channel and circular tube as shown in Fig. 3.1. This validation will give us confidence in using the method to solve the more complex problem of fully-developed natural convection within similar channel geometries in our later investigation.

The topic of laminar fully developed forced convection in uniformly heated vertical channels and circular tubes has been investigated for over century. It is a topic which these days is included in fundamental heat transfer textbooks due to its importance in numerous practical engineering applications. As a uniform constant

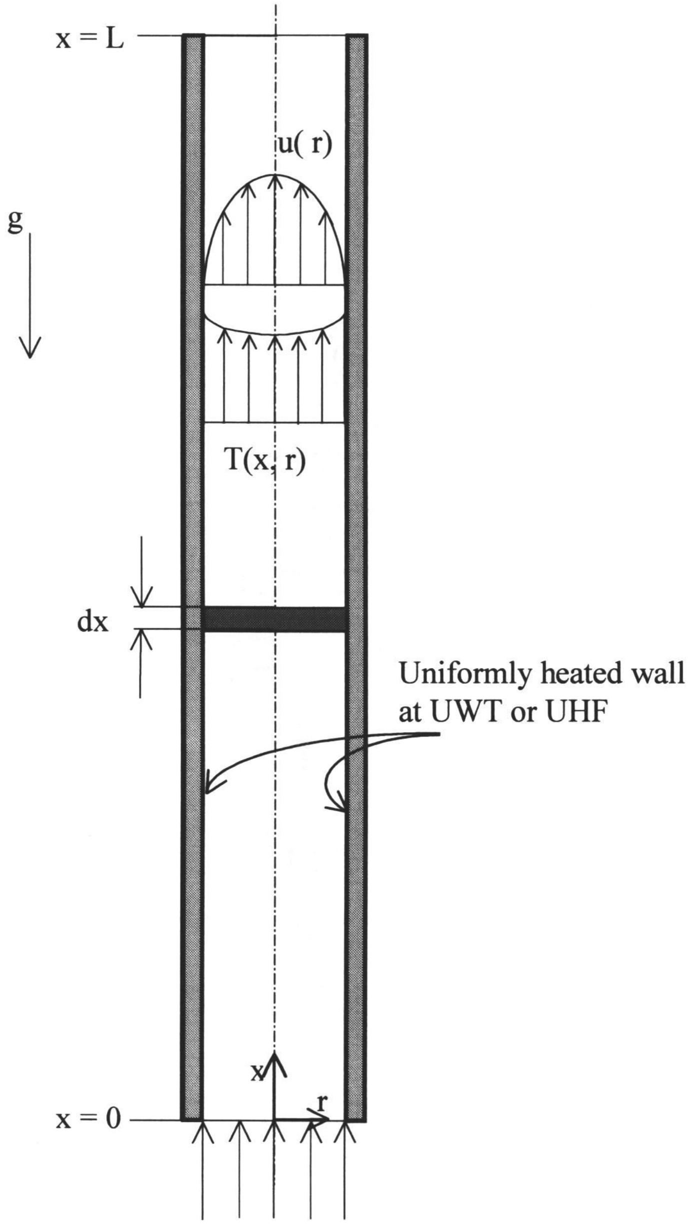


Figure 3.1 Coordinates and configuration used for the problem

inlet velocity is imposed, a forced flow is compelled into and through the duct by inertia. Because our subsequent work deals, in the larger perspective, with natural convection, we consider here the present forced flow in a channel that is vertical but with ignorable gravity.

The channel wall(s) are taken to be at a uniform temperature, T_w, higher than the temperature of the entering fluid. As the flow proceeds upward, the effects of friction and heating diffuse from the uniformly heated surfaces toward the axis of symmetry of the duct. As a result, hydrodynamic and thermal boundary layers develop. At some axial distance from the entrance, these boundary layers grow enough to occupy the entire duct cross-section. Such a portion of the flow is referred to as the developing region. The flow then advances into the fully developed region where the velocity distribution at any cross-section no longer changes with the axial distance. In addition, the fluid temperature distribution normalized by its local bulk (mean) temperature is also independent of the axial distance in the fully developed region. These characteristics define the conditions for a fully developed regime of the flow.

The topic of this paper is thus forced convective heat transfer in the fully developed region in a channel. The channel may be formed by two impervious and wide parallel plates separated by a distance D or by a tube of circular cross-section of diameter D. The channel wall(s) may be kept either at a constant wall temperature (UWT) as mentioned above or at a uniform heat flux (UHF) boundary condition. The flow is fully developed both hydrodynamically and thermally if

$$\frac{\partial}{\partial x}\left(\frac{u-u_W}{u_b-u_W}\right)=0 \tag{3.1}$$

$$\frac{\partial}{\partial x}\left(\frac{T-T_W}{T_b-T_W}\right)=0 \tag{3.2}$$

The subscript "b" stands for bulk mean velocity and temperature at any cross-section while the subscript "w" stands for those at channel wall. ($u_W = 0$ due to the no-slip condition. u_b is a constant at any x for the impervious channels considered here. Thus Eq. (3.1) reduces to the familiar $\partial u/\partial x = 0$.) The similarity of Eq. (3.1) and (3.2) evidently indicates that in the fully developed region, the flow and temperature distributions, upon being normalized and nondimensionalized as shown, retain a functional form, invariant in x, in radial coordinate. Using these conditions, the flow governing equations are solved. The solutions then reveal the characteristics and involved physics of the heat transfer process.

The problem with the constant-uniform-wall-heat-flux boundary condition is simple. A fully developed Poiseuille velocity distribution can be obtained directly from the momentum equation. It is then used in the energy equation to acquire a corresponding closed-form temperature distribution from which the heat transfer coefficient $h_b \equiv \dot{q}''_w/(T_w - T_b)$ can be found. Similar steps are followed to solve the problem with a constant-uniform-wall-temperature boundary condition as well; however, in this case, the energy equation is more complex such that a closed-form solution for the temperature distribution does not exist. In this case, the solutions existing in the literature are obtained by an iterative method carried out either by success approximations or by numerical analysis.

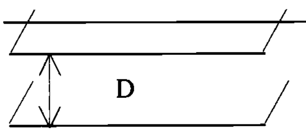
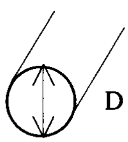
The corresponding heat transfer characteristic of the fully developed flow in both UHF and UWT problems are typically presented in the form of a Nusselt number, Nu_b, based on the wall-to-bulk temperature difference. This Nu_b definition formally is

$$Nu_b \equiv \frac{h_b D}{k} = \frac{\dot{q}''_w D}{(T_W - T_b)k} = \frac{D}{(T_W - T_b)} \left.\frac{dT}{dr}\right|_{r=D/2} \tag{3.3}$$

where D is the plate-to-plate spacing or tube diameter.

From an analysis of fully developed flow (e.g. Incopora and Dewitt, 1990), h_b is known to be a constant with respect to x, the axial position along the duct for both the UHF and UWT boundary conditions. The local wall-to-bulk Nusselt numbers based on D available from previous work for parallel-plate channels and circular tubes, with UHF and UWT boundary conditions, are summarized in Table 3.1.

Table 3.1 Local Nusselt number, Nu_b, for laminar fully developed forced convection flow within uniformly heated channel and tube

Geometry	UWT	UWF
	3.77	4.12
	3.66	4.36

3.3 Problem Analysis and Discussion

Consider steady laminar forced flow within a long vertical channel which is made up of either two parallel plates of very large width W and separated by a spacing distance D; or a tube of circular cross-section of diameter D. The channel (or tube) length is L. As illustrated in Fig. 3.1, the cartesian and cylindrical coordinates are chosen in the descriptions of the parallel channel and circular tube, respectively.

For the sake of convenience, the words "channel", "tube", "pipe", and "duct" are henceforth used as though they are synonyms unless an explicit delineation is called for. The inner surface of the impervious channel wall is kept either at a uniform and constant temperature T_w greater than the inlet ambient air temperature T_∞ or at a uniform and constant heat flux $\dot{q}''_w$. The channel length L is assumed to be large in comparison with the separation distance D such that the entrance region where the flow (viscous) and thermal boundary layers develop is ignorably short compared with the remaining of the channel in which the flow and heat transfer are fully developed. The transverse velocity v is zero in the fully developed region. Assuming constant fluid properties and neglecting axial diffusion, the momentum and energy equations for the steady laminar flow are

$$\frac{1}{r^j}\frac{d}{dr}\left(r^j\frac{du}{dr}\right)=\frac{1}{\mu}\frac{dP}{dx} \tag{3.4}$$

$$\frac{1}{r^j}\frac{\partial}{\partial r}\left(r^j\frac{\partial T}{\partial r}\right)=\frac{u}{\alpha}\frac{\partial T}{\partial x} \tag{3.5}$$

where $j = 0$ and 1 for the parallel-plate channel and circular tube, respectively. The boundary conditions needed to solve Eqs. (3.4) and (3.5) for u(r) and T(x,r) are as follows.

$$u\big|_{r=D/2}=0 \tag{3.6}$$

$$\left.\frac{\partial u}{\partial r}\right|_{x,r=0}=0 \tag{3.7}$$

$$\left.\frac{\partial T}{\partial r}\right|_{x,r=0}=0 \tag{3.8}$$

Additionally, for the uniform wall heat flux (UHF) boundary condition,

$$\dot{q}''_w=k\left.\frac{\partial T}{\partial r}\right|_{r=D/2}=\text{constant} \tag{3.9}$$

whereas for the uniform wall temperature (UWT) boundary condition,

$$T\big|_{r=D/2}=T_w=\text{constant} \tag{3.10}$$

The following nondimensional variables are now introduced.

$$\left.\begin{aligned}
&\phi = \frac{uD}{\nu} \qquad \Pi = \frac{P'D^2}{\rho\nu^2} \qquad \xi = \frac{x}{D} \qquad \eta = \frac{2r}{D} \\
&\theta = \frac{k(T - T_\infty)}{\dot{q}''D} \text{ for the UHF boundary condition} \\
&\theta = \frac{T - T_\infty}{T_w - T_\infty} \text{ for the UWT boundary condition}
\end{aligned}\right\} \tag{3.11}$$

The momentum and energy equations and the boundary conditions then take the following nondimensional forms

$$\frac{1}{\eta^j}\frac{d}{d\eta}\left(\eta^j \frac{d\phi}{d\eta}\right) = \frac{1}{4}\left(\frac{d\Pi}{d\xi}\right) \tag{3.12}$$

$$\frac{1}{\eta^j}\frac{\partial}{\partial\eta}\left(\eta^j \frac{\partial\theta}{\partial\eta}\right) = \frac{Pr}{4}\phi\frac{\partial\theta}{\partial\xi} \tag{3.13}$$

$$\phi\big|_{\eta=1} = 0 \tag{3.14}$$

$$\left.\frac{d\phi}{d\eta}\right|_{\eta=0} = 0 \tag{3.15}$$

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\xi,\eta=0} = 0 \tag{3.16}$$

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\xi,\eta=1} = \frac{1}{2} \text{ for the UHF boundary condition} \tag{3.17}$$

$$\theta\big|_{\xi,\eta=1} = 1 \text{ for the UWT boundary condition} \tag{3.18}$$

Integration of Eq. (3.12) gives the velocity distribution regardless of whether the wall is at the boundary condition of UHF or UWT. Using the boundary conditions in Eqs. (3.14) and (3.15), the velocity distribution in terms of the pressure gradient is

$$\phi = -\frac{1}{2^{3+j}}\frac{d\Pi}{d\xi}\left(1-\eta^2\right) \tag{3.19}$$

A mass flow rate independent of axial distance ξ is obtained by integrating this velocity distribution

$$\dot{m} = \rho u_o A_c = \rho \int_{A_c} u dA_c \tag{3.20}$$

so that an inlet Reynolds number, Re_D, can be obtained in terms of the mass flow rate as the following.

$$Re_D \equiv \frac{u_o D}{\nu} = \frac{\dot{m}D}{\mu A_c} \tag{3.21}$$

where u_o is a uniform inlet velocity and A_c is cross sectional area of the channel. (The bulk velocity u_b is also equal to u_o.) The following relation between the inlet Reynolds number and the pressure gradient is implicit in Eqs. (3.20) and (3.21).

$$Re_D = -\frac{1}{K_0}\frac{d\Pi}{d\xi} \tag{3.22}$$

where K_0 is a constant equal to 12 for the parallel-plate channel and 32 for the circular tube. Equation (3.19), thus, becomes

$$\phi = K_1 Re_D\left(1-\eta^2\right) \tag{3.23}$$

K_1 is 3/2 for the parallel-plate channel and 2 for circular tube.

Further simplification of the energy equation, Eq. (3.13), can be made for the flow in the fully developed region for each boundary condition.

3.3.1 Temperature distribution for problems with uniform-wall-heat flux boundary condition

For the UHF boundary condition, differentiation in Eq. (3.2) leads to, in nondimensional form,

$$\left.\frac{d\theta_b}{d\xi}\right|_{FD} = \left.\frac{d\theta_w}{d\xi}\right|_{FD} = \left.\frac{\partial\theta}{\partial\xi}\right|_{FD} \tag{3.24}$$

Then, an energy balance can be written for the flow over an infinitesimal channel length dx. It follows that

$$\dot{q}''_w P_w dx = \dot{m} C_p dT_b \tag{3.25}$$

$\dot{q}''_w$ is the specified uniform constant wall heat flux. P_w is a channel perimeter. The nondimensional form of this equation is

$$\frac{d\theta_b}{d\xi} = \frac{P_w k}{\dot{m} C_p} = \text{constant} \equiv \lambda \tag{3.26}$$

where k is the fluid conductivity. With the mass flow rate determined from a known Eq. (3.20) into which the velocity profile given by Eq. (3.23) is substituted, λ is

$$\lambda = \frac{K_2}{Re_D \, Pr} \tag{3.27}$$

K_2 is equal to 2 for parallel-plate channel and 4 for circular tube. Neglecting the entrance effects, $\theta_b = 0$ at $\xi = 0$, so that Eq. (3.26) gives

$$\theta_b = \lambda\xi \tag{3.28}$$

and Eq. (3.24) gives

$$\theta = \lambda\xi + \gamma(\eta) \tag{3.29}$$

where $\gamma(\eta)$ is an unknown function of η.

Introducing Eq. (3.29) and the velocity profile given in Eq. (3.23) into the energy equation Eq. (3.13), the unknown function $\gamma(\eta)$ is governed by

$$\frac{1}{\eta^j}\frac{d}{d\eta}\left(\eta^j\frac{d\gamma}{d\eta}\right)=\frac{3}{4}\left(\frac{8}{3}\right)^j\left(1-\eta^2\right) \tag{3.30}$$

This equation can now be integrated by noticing that $\gamma(\eta)$ has to satisfy Eqs. (3.16) and (3.17) as

$$\left.\frac{d\gamma}{d\eta}\right|_{\eta=0}=0 \quad \text{and} \quad \left.\frac{d\gamma}{d\eta}\right|_{\eta=1}=\frac{1}{2}$$

In addition, the definition of bulk temperature θ_b

$$\theta_b=\frac{\int_0^1\phi\theta\eta^j d\eta}{\int_0^1\phi\eta^j d\eta} \tag{3.31}$$

requires γ to fulfill (through Eqs. (3.28) and (3.29)),

$$\frac{\int_0^1\phi\gamma\eta^j d\eta}{\int_0^1\phi\eta^j d\eta}=0 \tag{3.32}$$

Solving Eq. (3.30), with (3.16), (3.17), and (3.32), yields $\gamma(\eta)$. Thus the corresponding temperature distribution thus found is

$$\theta=\frac{K_2}{Re_D\,Pr}\xi+K_3+K_4\eta^2+K_5\eta^4 \tag{3.33}$$

where $K_3 = -13/120$, $K_4 = 3/8$, $K_5 = -1/16$, for parallel-plate channel and $K_3 = -7/48$, $K_4 = 1/2$, and $K_5 = -1/8$ for circular tube. Hence, the temperature of the fluid adjacent to the channel wall θ_w is determined from Eq. (3.33) by setting $\eta=1$. Accordingly, the local Nusselt number Nu_b is calculated, from Eq. (3.3).

$$Nu_b = \frac{1}{\theta_w - \theta_b} = \begin{cases} \dfrac{70}{17} = 4.118 & \text{for UHF parallel - plate channel} \quad (3.34) \\ \\ \dfrac{48}{11} = 4.363 & \text{for UHF circular tube} \quad (3.35) \end{cases}$$

Similar results for the UWT boundary condition are sought to be obtained in the following subsection.

3.3.2 Temperature distribution for problems with uniform-wall-temperature boundary condition

Now, consider the UWT boundary condition. The thermal condition of fully developed flow in nondimensional form is

$$\frac{\partial}{\partial \xi}\left(\frac{1-\theta}{1-\theta_b}\right)\Bigg|_{FD} = 0$$

implying that

$$1-\theta = \gamma_o(\eta)\ s(\xi) \tag{3.36}$$

and that

$$1\text{-}\theta_b = s(\xi) \tag{3.37}$$

where $\gamma_o(\eta)$ and $s(\xi)$ are unknown functions of η and ξ respectively. The energy equation can now be written in term of $\gamma_o(\eta)$ and $s(\xi)$.After separation of variables, it follows that

$$\frac{4}{\Pr\phi\gamma_o}\frac{1}{\eta^j}\frac{d}{d\eta}\left(\eta^j\frac{d\gamma_o}{d\eta}\right)=\frac{1}{s}\frac{ds}{d\xi}=-\Lambda \tag{3.38}$$

Λ is an unknown constant. Since $\theta_b = 0$ at $\xi = 0$, $s(\xi)$ is readily obtained as

$$s=\exp(-\Lambda\xi) \tag{3.39}$$

From an energy balance over an infinitesimal length dx of the duct,

$$k\left.\frac{\partial T}{\partial r}\right|_{r=D/2} P_w dx = \dot{m}C_p dT_b \tag{3.40}$$

In nondimensional form, this equation can be written in terms of $\gamma_o(\eta)$ from Eq. (3.36) as

$$\left.\frac{d\gamma_o}{d\eta}\right|_{\eta=1}=-\frac{1}{2}\frac{\dot{m}C_p\Lambda}{kP_w} \tag{3.41}$$

Equations. (3.36) and (3.37) introduced into Eq. (3.3) reveal that

$$\left.\frac{d\gamma_o}{d\eta}\right|_{\eta=1}=-\frac{Nu_b}{2} \tag{3.42}$$

Λ is then written in terms of nondimensional numbers from Eq. (3.41) and (3.42) as

$$\Lambda = K_6\frac{Nu_b}{Re_D\Pr} \tag{3.43}$$

where K_6 is 2 for the case of a parallel-plate channel and 4 for a circular tube.

Substituting Eqs. (3.23) and (3.43) into Eq. (3.38),

$$\frac{1}{\eta^j}\frac{d}{d\eta}\left(\eta^j\frac{d\gamma_o}{d\eta}\right)=\frac{3}{4}\left(\frac{8}{3}\right)^j Nu_b\gamma_o\left(\eta^2-1\right) \tag{3.44}$$

When j = 1, Eq. (3.44) is similar to the equation given in Bejan (1995) for the case of UWT circular tube. The boundary conditions needed to solve this equation are Eqs. (3.16) and (3.18). Equation (3.42) is also needed for determining Nu_b. In terms of $\gamma_o(\eta)$, these conditions become

$$\left.\frac{d\gamma_o}{d\eta}\right|_{\eta=0}=0, \qquad \left.\gamma_o\right|_{\eta=1}=0, \qquad \left.\frac{d\gamma_o}{d\eta}\right|_{\eta=1}=-\frac{Nu_b}{2} \tag{3.45}$$

Eqs. (3.44) and (3.45) now complete the problem statement for the flow in the UWT channel. These equations have been solved in the existing literature by successive approximations or by numerical finite differences.

We will present below the use of a classical series solution method as an alternative to the successive approximation method to solve this problem. Providing that the temperature distribution is symmetry across the channel/tube, it is assumed that $\gamma_o(\eta)$ can be represented by a polynomial of degree n, which satisfies Eq. (3.44) and the conditions in Eq. (3.45).

$$\gamma_o=A_0+A_1\eta+\ldots+A_n\eta^n=\sum_{n=0}^{n}A_n\eta^n \tag{3.46}$$

A_n denote coefficients yet to be determined. Introducing Eq. (3.46) into Eq. (3.44) with a few steps of manipulation, it is clear that A_1 and A_3 have to be zero to satisfy the energy equation and A_2 is a function of A_0 and of the Nusselt number as

$$A_2=-K_8Nu_bA_o \tag{3.47}$$

The constant K_8 is 3/8 for the a parallel-plate channel and 1/2 for a circular tube, respectively. All other coefficients can be found from the recurrence relation,

$$A_n = \frac{3}{4}\left(\frac{8}{3}\right)^j \frac{Nu_b(A_{n-4} - A_{n-2})}{n(n-1+j)} \qquad n \geq 4 \qquad (3.48)$$

Half of the coefficients vanish due to the recurrence and those remaining can be written in terms only of A_0 and Nu_b. As A_1 is zero, the condition $d\gamma_o/d\eta=0$ at $\eta=0$ is satisfied. Hence, the calculation is left with only two boundary condition equations and two unknowns; A_0 and Nu_b. Instead of using infinite series to approximate the solution, a polynomial γ_o with a particular small number of terms is chosen such that its corresponding solution, Nusselt number, shows convergence.

For the sake of an illustration, if a sixth degree polynomial is used to approximate a function γ_o, for the case of UWT circular tube, from γ_o $(\eta=1) = 0$, the recurrence relation shows that

$$1 - \frac{3}{8}Nu_b + \frac{1}{36}Nu_b^2 - \frac{1}{288}Nu_b^3 = 0$$

from which the solution, Nu_b, is 3.1033. Then A_0=0.875 is retrieved by using the condition $d\gamma_o/d\eta = -Nu_b/2$ at $\eta=1$. From Eq. (3.36), the fluid temperature is

$$\theta = 1 - (0.875 - 1.358\eta^2 + 0.866\eta^4 - 0.383\eta^6)\exp\left(-\frac{14.66\xi}{Re_D\, Pr}\right)$$

The calculation can be repeated with more than six terms used in the polynomial γ_o (η). As more terms are used in the polynomial, the Nusselt number becomes closer to 3.657 for the UWT circular tube. This is graphically illustrated in Fig. 3.2. This result is very close to what has been known in the literature from iterative and numerical solutions.

A similar process is carried out also for the case of UWT parallel-plate channel. The corresponding plot is shown in Fig. 3.3 indicating that the Nusselt number Nu_b for this case converges to an anticipated value of 3.7704.

The number of terms needed in the series to obtain the nearly converged results is not really very large. Figures 3.2 and 3.3 show that about 17 terms are needed for the case of parallel-plate channel and 23 terms for the circular tube, respectively. Moreover, half of the terms are zeros and coefficients of higher-degree terms are much smaller than those of the lower-degree terms.

Thus , the series solution method gives,

$$Nu_b = 3.770 \quad \text{for the UWT parallel-plate channel} \tag{3.49}$$

$$Nu_b = 3.657 \quad \text{for the UWT circular tube} \tag{3.50}$$

These results stand next to the UHF results embodied in Eqs. (3.34) and (3.35). Evidently, the Nusselt numbers acquired from this analysis based on series solution method agree very well with the existing literature. This agreement validates our use of the series solution method for a fully developed convection within uniformly heated parallel-plate channel and circular tube.

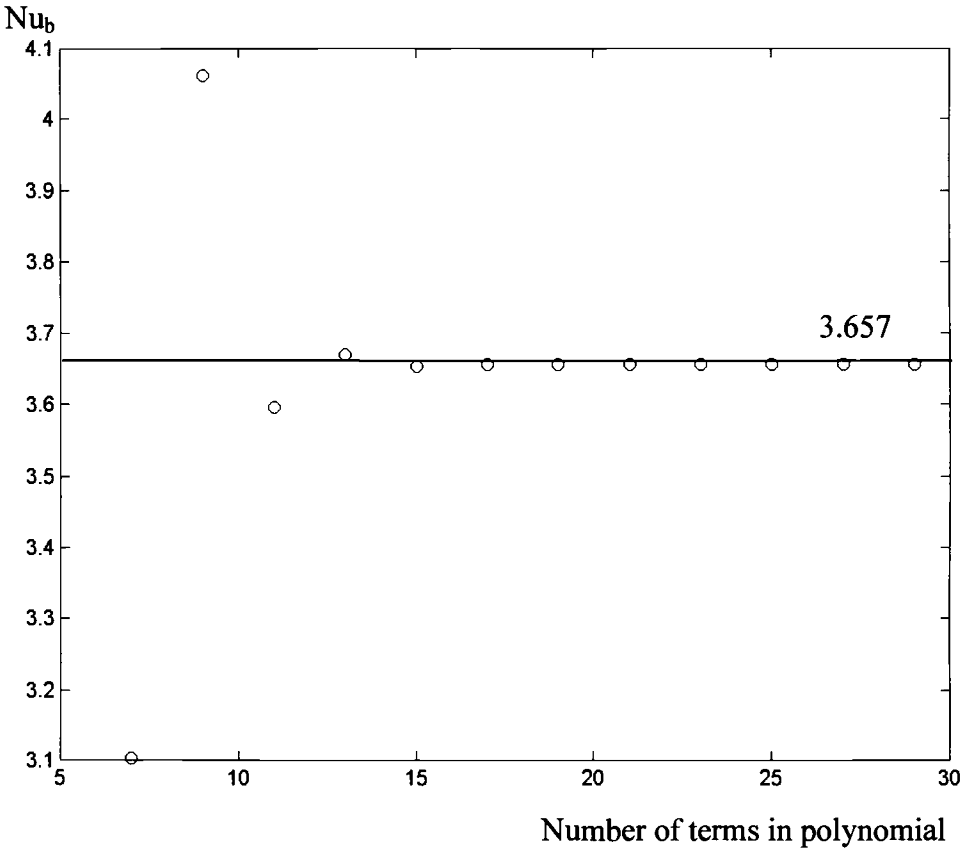


Figure 3.2 Convergence of local Nusselt number solved by series solution method as a function of number of terms used in the polynomial approximation. (UWT circular tube)

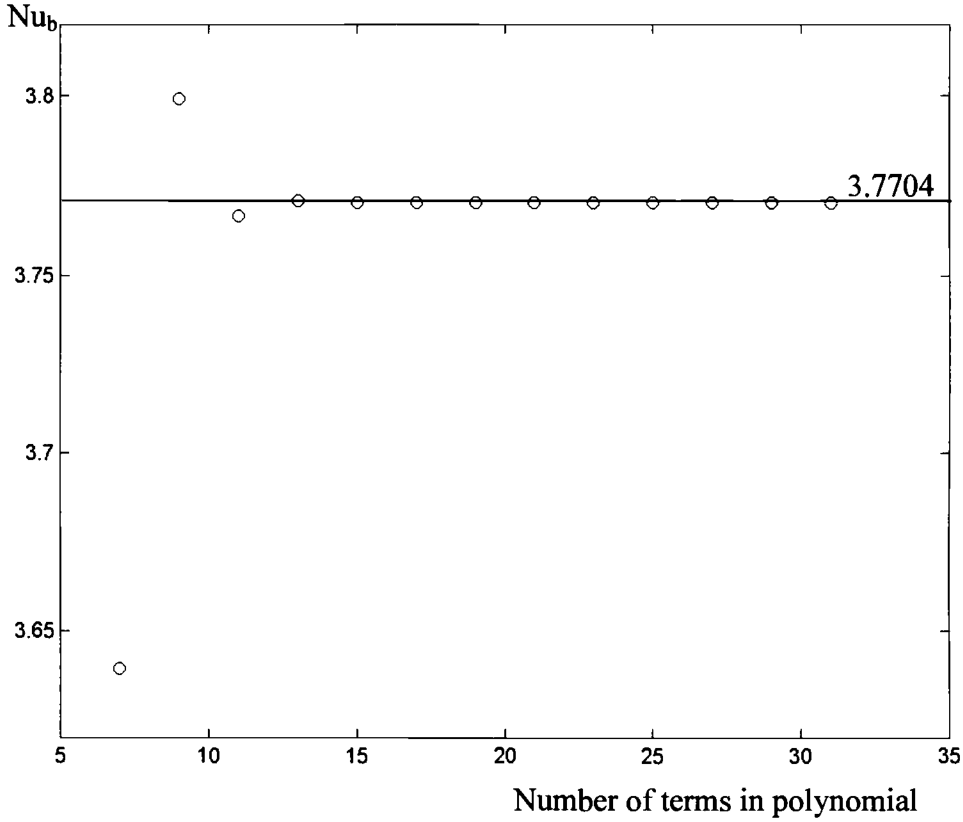


Figure 3.3 Convergence of local Nusselt number solved by series solution method as a function of number of terms used in the polynomial approximation. (UWT parallel-plate channel)

3.4 Conclusion

The series solution method of solving the energy equation in fully-developed forced convection heat transfer in laminar flow in channels/tube seems to be versatile, efficient, and accurate. It is clear that this method has not only alleviated the need for an iteration or numerical method for the UWT forced convection problem but also gives us the confidence in applying the technique to problem of fully developed *natural convection* flow in vertical channels. Such an application is described in an account of our later investigations.

3.5 References

1. Bejan, A., 1995, "Convection Heat Transfer," John Willey & Sons, New York.
2. Duffy, D. G., 1998, "Advanced Engineering Mathematics," CRC Press, Boca Raton.
3. Incopora, F.P. and DeWitt, D.P., 1990, "Introduction to Heat Transfer," John Wiley & Sons, New York.
4. Ross, S. L., 1966, "Introduction to Ordinary Differential Equations," Xerox, Lexington.
5. Kreyszig, E., 1993, "Advanced Engineering Mathematics," 7th edition, John Wiley & Sons, New York.
6. Derrick, W.R. and Grossman, S.I., 1987, "A First Course in Differential Equations," 3rd edition, West Publishing, St. Paul.
7. Kells, L.M., 1960, "Elementary Differential Equations," McGraw-Hill, New York.

8. Rabenstein, A.L., 1975, "Elementary Differential Equations with Linear Algebra," 2nd edition, Academic Press, New York.

9. Rainville, E.D. and Bedient, P.E., 1969, "Elementary Differential Equations," 4th edition, Collier-Macmillan, New York.

CHAPTER 4

A SERIES SOLUTION METHOD FOR FULLY DEVELOPED LAMINAR NATURAL CONVECTION WITHIN UNIFORMLY HEATED VERTICAL PARALLEL-PLATE CHANNELS AND CIRCULAR TUBES

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In Preparation

4.1 Abstract

The equations which mathematically describe the problem of fully developed laminar natural convective flow in a uniformly heated vertical parallel-plate channel and tube of circular cross-section are formulated and simplified. A series solution method is used to solve them to obtain the velocity and temperature profiles, pressure distributions and heat transfer characteristics. Average Nusselt numbers based on wall-to-ambient temperature difference $\overline{Nu_\infty}$ are found to be functions of a single parameter — a combination of the Grashof and Prandtl numbers and the aspect ratio of the channel (Gr Pr D/L). The results are in good agreement with the existing literature. Local Nusselt numbers based on wall-to-bulk temperature difference Nu_b are found to be constants; this is precisely as in the very familiar fully developed laminar *forced* convection flow in channels. Additionally and perhaps surprisingly, the numerical values of Nu_b for fully developed natural convection are exactly the same as for fully developed forced convection. The natural and forced convection in uniformly heated vertical parallel-plate channels and circular tube approach to the same fully-developed states, if the length of the channels/tubes is long enough, as the inertia will eventually prevail. This indicates that same mechanism driving both fully developed forced and natural convective flow within long channel/tube.

4.2 Nomenclature

C_p	specific heat at constant pressure
D	plate spacing and diameter of tubes
D_h	hydraulic diameter
Gr	Grashof number;
	for UWT channel, $g\beta(T_w\text{-}T_\infty)D^3/\nu^2$,
	for UHF channel, $g\beta\dot{q}''_w D^4/(k\nu^2)$
h	coefficient of convection heat transfer;
	$h_b \equiv \dot{q}''_w/(T_w - T_b)$ and $h_\infty \equiv \dot{q}''_w/(T_w - T_\infty)$
k	thermal conductivity of the fluid
L	channel/tube length
$\dot{m}$	mass flow rate
Nu	Nusselt number, (hD/k)
Nu_b	local Nusselt number of h_b
Nu_∞	local Nusselt number of h_∞
$\overline{Nu}_\infty$	average Nusselt number of average h_∞
P	pressure
P′	pressure difference between fluid pressure and hydrostatic pressure outside the channel and tube, $P' = P\text{-}P_\infty$
P_w	channel/tube wetted perimeter
Pr	Prandtl number, ν/α

$\dot{q}''_w$ heat flux at the wall

T absolute fluid temperature

u, v axial and lateral components of fluid velocity,

W width of parallel-plate channels

x, r axial and radial coordinates used in the analysis

α thermal diffusivity

β coefficient of thermal expansion

ϕ nondimensional axial velocity, (uD/ν)

ν kinematic viscosity

Π nondimensional pressure difference, $P'D^2/(\rho\nu^2)$

θ nondimensional temperature,

$\theta = (T - T_\infty)/(\dot{q}''D/k)$ for the UHF problems,

$\theta = (T-T_\infty)/(T_W - T_\infty)$ for the UWT problems

ρ fluid density

ξ,η nondimensional x and r coordinates

ξ_L channel aspect ratio, L/D

Subscripts

b bulk

w wall

o inlet

∞ ambient fluid

4.3 Introduction

Laminar natural convection heat transfer within uniformly heated vertical parallel-plate channels and circular tubes has been a subject of many studies. The pioneering papers of Elenbaas (1942a, b) describe this problem which has continued to be in focus by many researchers who followed. Applications of this problem have been recognized in several practical areas ranging from heat exchangers, nuclear reactors, solar panels and buildings, to recently evolved practical equipments such as those involving electronic circuit boards.

Even though forced convection is usually the primary method to remove excess heat in these applications, natural convection is always present. In most situations, natural convection alone is perhaps preferable to carry out the cooling since it is spontaneous, simpler, and requires no compressor, pump, or a blower. Whereas the subject of pure *forced* convection has been long studied for over a century and the associated physics of flow and heat transfer are well understood, that of pure natural convection within vertical channels or tubes has not been as extensively studied and is less clearly understood.

The work of Elenbaas has been referred to, quoted, debated, criticized, discussed, and extended in the studies of numerous investigators. With the advent of modern computer, the problem of natural convection in vertical channels and tubes has attracted a renewed and redoubled attention to validate the work of

Elenbaas (see Bodia and Osterle 1962, Aung et al. 1972, and Ramanathan and Kumar1991).

The configuration used in the study by Elenbaas (1942a) is shown in Fig. 4.1. A vertical channel is formed by two very wide impervious parallel plates separated by a distance D. The channel extends from x = 0 (entrance) to x = L (exit). Alternately, the channel is a vertical tube of length x = 0 to x = L and diameter D. The transverse coordinate r is normal to the flow-wise or axial coordinate x so that r = 0 represents the axis or plane of symmetry while r = R ≡ D/2 represents the inside surface of the channel. The local acceleration due to gravity is g. This channel is situated in an infinite chamber of quiescent air at pressure P_∞ (which varies with height location x hydrostatically) and temperature T_∞. The channel walls are symmetrically kept (either) at a uniform temperature T_w ($>T_\infty$) (or at a prescribed uniform heat flux $\dot{q}''_w$). Henceforth, the problems with prescribed inner surface temperature boundary condition are abbreviated as UWT problems while those with the channel walls kept at a constant/uniform heat flux boundary condition are abbreviated as UHF problems. Due to the finite positive temperature difference (T_w-T_∞), density differences arise in the air and thereby an upward flow is induced gravitationally within the channel/tube. In Elenbaas paper, a simplified mathematical model of a flow was formulated and solved. The solution is compared with and then adjusted with the experimental data. This "adjustment" made the subsequent researchers declare that Elenbaas correlations are "semi-empirical". In these semi-empirical correlations the average Nusselt

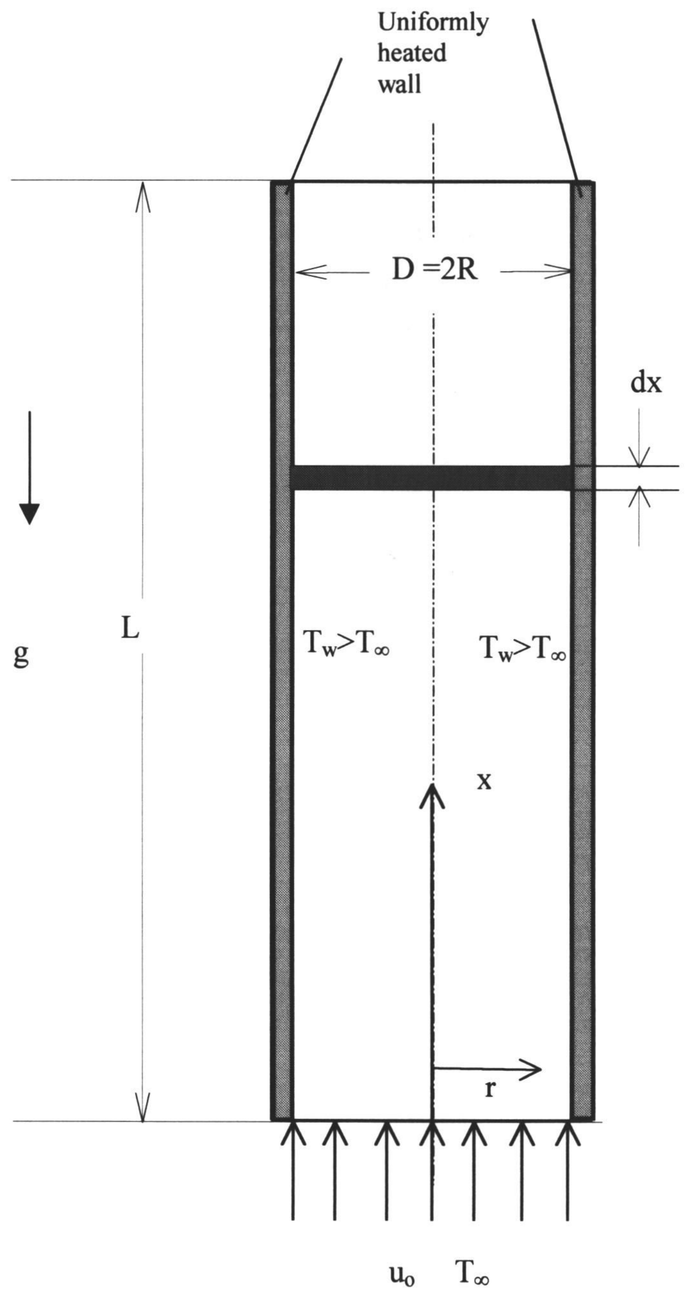

Figure 4.1 Problem configuration and coordinate system

number is found to be a function of only one parameter; the product of Grashof and Prandtl numbers over the aspect ratio of the channel/ tube [Gr Pr /(L/D)]:

$$\overline{\mathrm{Nu}_\infty} = \frac{1}{24}\frac{\mathrm{Gr\,Pr\,D}}{\mathrm{L}}\left[1-\exp\left(-\frac{35\mathrm{L}}{\mathrm{Gr\,Pr\,D}}\right)\right]^{3/4} \quad : \text{parallel - plate channel} \tag{4.1}$$

$$\overline{\mathrm{Nu}_\infty} = \frac{1}{128}\frac{\mathrm{Gr\,Pr\,D}}{\mathrm{L}}\left[1-\exp\left(-16\left(\frac{8\mathrm{L}}{\mathrm{Gr\,Pr\,D}}\right)^{3/4}\right)\right] \quad : \text{circular tube} \tag{4.2}$$

The Grashof[1] and Prandtl numbers are defined as usual as follows:

$$\mathrm{Gr} = \frac{g\beta(T_w - T_\infty)D^3}{\nu^2} \tag{4.3}$$

$$\mathrm{Pr} = \frac{\nu}{\alpha} \tag{4.4}$$

Two asymptotic limiting conditions of extremely high and low [Gr Pr D/L] can be pointed out for the heat transfer characteristics of the channel/tube flow. A specified high [Gr Pr/(L/D)] represents flow in a short and wide channel with strong heating. In this limit, the flow and heat transfer characteristics of the fluid on (either of) the channel wall(s) will correspond to those of laminar free convection along an isolated UWT vertical flat plate. Interaction between boundary layers along the "opposing" walls is ignorable if at all existent.

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[1] Eq. (4.3) holds for UWT cases. For the UHF cases, the following definition of Grashof number holds; $\mathrm{Gr} \equiv g\beta D^4 \dot{q}''_w/(\nu^2 k)$.

Characteristics of the flow are different when the specified [Gr Pr D/L] value is small. In this second limiting case, the channel is long and narrow with weak heating. Near the entrance there exists a developing or entrance region where the flow and thermal boundary layers gradually develop to eventually merge at the plane or axis of symmetry of the channel to enter into the fully developed region. In the entrance region, the effect of viscosity and conductivity exists in the flow field confined only to these boundary layers. Later, once both effects diffuse across the entire cross section and the channel being longer than the entry length then the fully developed region is realized. In this region the velocity and temperature distributions across the channel cross-section remain unchanging in relative shape such that

$$\frac{\partial}{\partial x}\left(\frac{u-u_w}{u_b-u_w}\right)_{FD}=0 \tag{4.5}$$

$$\frac{\partial}{\partial x}\left(\frac{T-T_w}{T_b-T_w}\right)_{FD}=0 \tag{4.6}$$

Here T and u are the temperature and the x- component velocity of the flow, respectively. Both T and u may be functions of the axial distance x from the inlet and transverse distance r from channel's axis or plane of symmetry. Subscript "w" refers to the T or u at the wall r = R, whereas subscript "b" refers to the bulk (mean) temperature or velocity. The velocity of the fluid adjacent to the solid wall, u_w, is zero due to the no-slip condition. The bulk velocity, u_b, is constant and equal to the uniform inlet velocity (denoted by u_o) for an incompressible flow. Fluid bulk (mean) temperature T_b is defined as

$$T_b(x) = \frac{\int_A \rho C_p u T dA}{\int_A \rho C_p u dA} \tag{4.7}$$

The definition becomes simpler if the fluid's volumetric heat capacity ρC_p is constant.

Generally, if the product [Gr Pr D/L] is high enough, it may well be that the entrance region length is ignorably small so that flow in the entire length of the channel can be considered fully developed.

Churchill and Usagi (1972) proposed a way of combining the large and small [GrPrD/L] limiting cases to obtain the characteristics for the intermediate range of the parameter. A series of publications exploiting this notion are available; see, for example, Raithby and Hollands, 1975, and Bar-Cohen and Rohsenow, 1984. In spite of rather extensive existing literature of this sort, an understanding of the physics of the flow and heat transfer and of the interactions among relevant transport mechanisms, even in the fully developed flow regime, is not complete.

In this paper, we study the flow's second limiting case, i.e. the *fully developed* laminar natural heat convection within vertical channels and tubes with UWT and UHF boundary conditions. This is the second of a series three papers written on our research which seeks an enhanced physical understanding of the mechanisms of naturally induced steady laminar flow and heat transfer in vertical channels and tubes.

Several publications in the existing literature deal with the fully developed laminar natural convective flow in vertical channels. Aung (1972) studied the flow in a vertical parallel-plate channel heated symmetrically (and also asymmetrically) with both UWT and UHF boundary conditions. Instead of what is given in Eq. (4.6), Aung's solutions for UWT case are obtained by different thermal condition of fully developed flow as $\partial T/\partial x=0$. The condition implies that the fluid temperature no longer adjusts its profile and the heating is so completely done that the bulk temperature is nearly a constant $\approx T_w$. The heat transfer coefficient calculated from the analysis based on this condition appears to have little or no physical significance since there is essentially no heat transfer.

We employ here, as our primary tool, a series solution method. It is a basic and alternative approach to solve an ordinary differential equation. As shown later, the system of partial differential equations governing a laminar natural convective flow in the fully developed region are simplified and condensed into a single ordinary differential equation to which the series solution method is applied to arrive at reasonably accurate results. The series solution method has been used by Willie (1996) to study a parallel plate channel problem with UWT boundary condition. A closed form solution has been obtained by Aktan (1996) (in a manner very similar to the work of Aung (1972)) on the same geometry but with UHF boundary condition. Both these problems are here redone using the series method. The method has then been extended here to UWT and UHF problems of natural convective flow in a tube of circular cross-section.

4.4 Mathematical Model

Consider, following Elenbaas, laminar fully developed natural convective flow within a long vertical channel made up of two parallel plates of a very large width W separated by a spacing D; or tube of circular cross-section with diameter D. The channel (or tube) length is L. The configuration used for both the parallel-plate channel and circular tube is similar to that illustrated in Fig. 4.1. The channel is described in cartesian coordinates whereas the circular tube is described in cylindrical coordinates. The inner surface of the channel (or tube) is kept either at a uniform and constant temperature T_w greater than the inlet ambient air temperature T_∞ or at a uniform and constant heat flux $\dot{q}''_w$. The involved temperature difference between the hot tube wall and the cooler air results in a density distribution in the air. An upward flow is thus driven by the resultant buoyancy force.

Assume that the channel is very long within which the flow is fully developed both hydrodynamically and thermally. Also assume that the developing (entry) length is small in comparison with the channel length. At the inlet x = 0, the inlet air velocity u_0 is taken to be uniform across the channel cross section. This velocity comes out as a part of the our solution. For an incompressible fluid, mass conservation reveals that u_0 equals to u_b, the bulk velocity of the flow. The inlet temperature is also assumed to be uniform at T_∞. Boussinesq approximation is applied such that all fluid properties are constant except for the fluid density appearing in the gravitational term of the x-component momentum equation. Other details of Boussinesq approximation need not to be repeated here but can be found in the natural heat convection literature (Bejan ,1995, for example). For the steady, two-dimensional (axisymmetric), and constant property flow, noting the fully-developed flow criteria given by Eqs. (4.5) and (4.6), we arrive at the following x-component momentum and energy conservation equations.

$$\frac{1}{r^j}\frac{d}{dr}\left(r^j\frac{du}{dr}\right)=\frac{1}{\mu}\frac{dP'}{dx}-\frac{g\beta}{\nu}\left(T-T_\infty\right) \tag{4.8}$$

$$\frac{1}{r^j}\frac{\partial}{\partial r}\left(r^j\frac{\partial T}{\partial r}\right)=\frac{u}{\alpha}\frac{\partial T}{\partial x} \tag{4.9}$$

The index $j = 0$ is for the problem of the parallel-plate channel and $j = 1$ is for the circular tube problem. The pressure defect, P', is defined as

$$P' = P - P_\infty \tag{4.10}$$

where $P_\infty = P_{\infty o} - \rho g x$, the hydrostatic pressure of air outside the channel. Pressure distribution within the fully developed flow channel is a function only of x, distance from the inlet. In the fully developed regime, both heat and momentum diffusions in the axial (a flow-wise) direction are negligible. The boundary conditions for Eqs. (4.8) and (4.9) are:

$$u\big|_{r=D/2}=0 \tag{4.11}$$

$$\left.\frac{du}{dr}\right|_{r=0}=0 \tag{4.12}$$

$$\left.\frac{\partial T}{\partial r}\right|_{x,r=0}=0 \tag{4.13}$$

$$P'\big|_{x=0 \text{ and } x=L} = 0 \tag{4.14}$$

$$\dot{q}'' = \dot{q}''_w = k\left.\frac{\partial T}{\partial r}\right|_{r=D/2} = \text{constant} \quad \text{for UHF problems} \tag{4.15}$$

$$T\big|_{r=D/2} = T_w = \text{constant} \quad \text{for UWT problems} \tag{4.16}$$

Equation (4.11) is for the no-slip condition at the channel wall. Equations. (4.12) and (4.13) are for symmetry. Equation (4.14) gives the pressure condition at the inlet and exit. Equations (4.15) or (4.16) correspond to the channel wall heating conditions.

Now the following dimensionless variables can be introduced.

$$\left.\begin{array}{ll}
\xi \equiv \dfrac{x}{D} & \eta \equiv \dfrac{2r}{D} \\[2ex]
\phi \equiv \dfrac{uD}{\nu} & \Pi \equiv \dfrac{P'D^2}{\rho\nu^2} \\[2ex]
\theta \equiv \dfrac{(T - T_\infty)k}{\dot{q}''D} & \text{for UHF problems} \\[2ex]
\theta \equiv \dfrac{T - T_\infty}{T_w - T_\infty} & \text{for UWT problems}
\end{array}\right\} \tag{4.17}$$

ϕ is nondimensional velocity, Π is nondimensional pressure, and θ is nondimensional temperature. The governing equations and boundary conditions then take the following nondimensional forms.

$$\frac{1}{\eta^j}\frac{d}{d\eta}\left(\eta^j\frac{d\phi}{d\eta}\right)=\frac{1}{4}\left(\frac{d\Pi}{d\xi}-Gr\theta\right) \tag{4.18}$$

$$\frac{1}{\eta^j}\frac{\partial}{\partial\eta}\left(\eta^j\frac{\partial\theta}{\partial\eta}\right)=\frac{Pr}{4}\phi\frac{\partial\theta}{\partial\xi} \tag{4.19}$$

$$\phi\big|_{\eta=1}\equiv\phi_w=0 \tag{4.20}$$

$$\left.\frac{d\phi}{d\eta}\right|_{\eta=0}=0 \tag{4.21}$$

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\xi,\eta=0}=0 \tag{4.22}$$

$$\Pi\big|_{\xi=0\ \text{and}\ \xi_L}=0 \tag{4.23}$$

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\xi,\eta=1}=\frac{1}{2}\quad\text{for UHF problems} \tag{4.24}$$

$$\theta_{\xi,\eta=1}=1\qquad\text{for UWT problems} \tag{4.25}$$

In Eq. (4.23), $\xi_L \equiv L/D$ is the channel's aspect ratio. Also the fully developed flow criteria given by Eqs. (4.5) and (4.6) take the nondimensional form:

$$\left.\frac{\partial}{\partial\xi}\left(\frac{\phi-\phi_w}{\phi_b-\phi_w}\right)\right|_{FD}=0 \tag{4.26}$$

$$\left.\frac{\partial}{\partial\xi}\left(\frac{\theta-\theta_w}{\theta_b-\theta_w}\right)\right|_{FD}=0 \tag{4.27}$$

For the constant property fluid, conservation of mass requires that the flow

rate

$$\dot{m} = \rho A u_b = \rho \int_A u(x,r) dA \tag{4.28}$$

be independent of x. Now, an energy balance may be applied over the control volume of differential length dx as shown in Fig. 4.1 to determine how the fluid's bulk temperature varies with the axial distance along the channel due to the convective heat transfer from the warm solid wall of the channel to the fluid.

$$\dot{q}''_w P_w dx = \dot{m} C_p dT_b = \rho C_p \int_{A_c} uT dA_c \tag{4.29}$$

The local heat transfer coefficient is defined on the basis of the temperature difference between the wall temperature and the local bulk temperature.

$$\dot{q}''_w = h_b (T_w - T_b) \tag{4.30}$$

In the fully developed regime, this coefficient is a constant with respect to x (Incopera and DeWitt, 1990). P_w is the wetted perimeter (= 2W for the parallel-plate channel and = πD for the circular tube), and A_c is a flow cross sectional area (= WD for the plate channel and = $\pi D^2/4$ for the tube). The local Nusselt number is defined as

$$Nu_b \equiv \frac{h_b D}{k} \tag{4.31}$$

4.4.1 The uniform-wall-heat-flux (UHF) problem

For the UHF problem, $\dot{q}''_w$ is a prescribed constant. Equations (4.27), (4.29) and the derivative of Eq. (4.30) show, in nondimensional forms,

$$\left.\frac{d\theta_b}{d\xi}\right|_{FD} = \left.\frac{d\theta_w}{d\xi}\right|_{FD} = \left.\frac{\partial\theta}{\partial\xi}\right|_{FD} = \frac{kP_w}{C_p \dot{m}} \equiv \text{constant}, \ \lambda \tag{4.32}$$

Since $\theta_b = 0$ at $\xi = 0$, this equation leads to

$$\theta_b = \lambda\xi \tag{4.33}$$

$$\theta = \lambda\xi + \gamma(\eta) \tag{4.34}$$

where $\gamma(\eta)$ is an unknown function of η. $\theta_w(\xi)$ is obtained by evaluating $\theta(\xi,\eta)$ at $\eta = 1$.

Introduce Eq. (4.34) into the momentum equation given by Eq. (4.18) and then differentiate both sides with respect to ξ to arrive at

$$\frac{d^2\Pi}{d\xi^2} = Gr\lambda \tag{4.35}$$

This result describes the pressure distribution in the fully developed flow regime. It can be solved with the boundary conditions appearing in Eq. (4.23).

$$\Pi = \frac{Gr\lambda}{2}\left(\xi^2 - \xi_L\xi\right) \tag{4.36}$$

Recall that ξ_L is the channel's aspect ration, L/D.

Substituting Eqs. (4.32) and (4.34), equation of energy given by Eq. (4.19) can be rewritten as

$$\phi = \frac{4}{Pr\lambda}\left(\frac{1}{\eta^j}\frac{d}{d\eta}\left(\eta^j\frac{d\gamma}{d\eta}\right)\right) \tag{4.37}$$

Substitute Eqs. (4.36) and (4.37) back into momentum equation, Eq. (4.18) to obtain the following equation for the unknown part, $\gamma = \gamma(\eta)$, of the temperature θ in Eq. (4.34).

$$\frac{1}{\eta^j}\frac{d}{d\eta}\left(\eta^j\frac{d}{d\eta}\left(\frac{1}{\eta^j}\frac{d}{d\eta}\left(\eta^j\frac{d\gamma}{d\eta}\right)\right)\right)=-\frac{1}{32}\frac{\mathrm{Gr\,Pr\,D}}{\mathrm{L}}\lambda^2\left(1+\frac{2\gamma}{\lambda\xi_L}\right) \quad (4.38)$$

Since the channel is very long to ensure the fully developed flow in most of the length, it is reasonable to assume that $\lambda\xi_L$ to be much greater than unity. Consequently, $1+2\gamma/(\lambda\xi_L) \approx 1$. Hence, Eq. (4.38) can then be integrated analytically to obtain γ as a fourth degree polynomial function such that

$$\gamma(\eta) = A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3 + A_4\eta^4$$

Knowing that the fluid temperature and velocity distributions can be written in terms of γ from Eqs. (4.34) and (4.37), the four unknown constant coefficients of integration A_1 to A_4 are then obtained from the boundary conditions given by Eqs. (4.20), (4.21), (4.22) and (4.24). It follows that

$$\theta = \lambda\xi + A_0 + A_2\eta^2 + A_4\eta^4 \quad (4.39)$$

$$\phi = \frac{4}{\mathrm{Pr}\lambda}\left(2(1+j)A_2 + 4(3+j)A_4\eta^2\right) \quad (4.40)$$

A_1 and A_3 are zero whereas $A_2 = (3+j)/8$ and $A_4 = -(1+j)/16$. As the fluid bulk temperature distribution is given by Eq.(4.33), the remaining constant A_o is acquired from introducing Eq. (4.39) into Eq. (4.7) and is found to be $-39/560$ for the parallel-plate channel and $-7/48$ for the circular tube. As a corollary, for use in Eqs. (4.33) and (4.34), we also obtained that

$$\lambda = 4(1+j)\sqrt{(3+j)\frac{\mathrm{D}}{\mathrm{Gr\,Pr\,L}}} \quad (4.41)$$

The heat transfer characteristics of the laminar free convective flow in the fully developed region can now be reckoned in terms of the Nusselt number defined in Eq. (4.31),

$$Nu_b = \frac{h_b D}{k} = \frac{1}{\theta_w - \theta_b} \tag{4.42}$$

From Eqs. (4.33) and (4.39) it follows that this Nusselt number is

$$Nu_b = \frac{70}{17} = 4.118 \qquad \text{for UHF parallel-plate channel} \tag{4.43}$$

$$Nu_b = \frac{48}{11} = 4.363 \qquad \text{for UHF circular tube} \tag{4.44}$$

4.4.2 The uniform-wall-temperature (UWT) problem

The criteria for the fully developed flow (Eqs. (4.26) and (4.27)) give a simpler relation between θ and θ_b for the UWT problem as well. Since θ_w is a constant equal to unity in the UWT problem, Eq. (4.27) implies that

$$1 - \theta = \gamma_o(\eta)s(\xi) \tag{4.45}$$

$$1 - \theta_b = s(\xi) \tag{4.46}$$

$\gamma_o(\eta)$ and $s(\xi)$ are unknown functions of η and ξ, respectively. Introduce Eq. (4.45) into Eq. (4.19) and separate variables.

$$\frac{4}{Pr\,\phi\gamma_o}\frac{1}{\eta^j}\frac{d}{d\eta}\left(\eta^j\frac{d\gamma_o}{d\eta}\right) = \frac{1}{s}\frac{ds}{d\xi} = -\Lambda, \text{ a constant} \tag{4.47}$$

At $\xi = 0$, $\theta_b = 0$ so that $s(\xi=0) = 1$ immediately leads to

$$s = \exp\left(-\Lambda\xi\right) \tag{4.48}$$

The pressure distribution can be approximated from the momentum equation. Differentiate Eq. (4.18) with respect to ξ and integrate it over the channel length. It follows that

$$\Pi = \frac{\gamma_o \mathrm{Gr}}{\Lambda}\left(\exp(-\Lambda\xi) - 1 + \frac{\xi}{\xi_L}\left(1 - \exp\left(-\Lambda\xi_L\right)\right)\right) \tag{4.49}$$

Substitute this expression in the momentum equation and rewrite it as

$$\frac{1}{\eta^j}\frac{d}{d\eta}\left(\eta^j \frac{d\phi}{d\eta}\right) = -\frac{\mathrm{Gr}}{4}\left(1 - \frac{\gamma_o}{\Lambda\xi_L}\left(1 - \exp(-\Lambda\xi_L)\right)\right) \tag{4.50}$$

Further simplification can be made as done earlier in the UHF problem; since the channel aspect ratio is large, $1-\gamma_o(1-\exp(-\Lambda\xi_L))/\Lambda\xi_L \approx 1$. With this simplification, Eq. (4.50) can be easier integrated to obtain the fully developed velocity profile as

$$\phi = \frac{1}{1+j}\frac{\mathrm{Gr}}{8}\left(1 - \eta^2\right) \tag{4.51}$$

The corresponding mass flow rate, Eq. (4.28), in the channel is found by integration of Eq. (4.51).

$$\frac{\dot{m}}{\mu D} = \frac{\pi^j}{2^{2+j}(1+j)^2(3+j)}\left(\frac{W}{D}\right)^{1-j}\mathrm{Gr}$$

Implicitly, since $\dot{m}$ is $\rho u_b A_c$, this result gives a relation between the nondimensional inlet velocity ϕ_0 and Grashof numbers as

$$\phi_0 \equiv \frac{u_b D}{\nu} = \frac{\mathrm{Gr}}{2^{2-j}(1+j)^2(3+j)} \tag{4.52}$$

from which ϕ_0 = Gr/12 for UWT parallel-plate channel and Gr/32 for UWT circular tube. Recall that $\dot{q}''_w = k\partial T/\partial r$ at r = D/2. Then, nondimensional forms of Eqs. (4.29) and (4.30) are respectively

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=1} = \frac{1}{2}\frac{\dot{m}C_p}{P_w k}\frac{d\theta_b}{d\xi}$$

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=1} = \frac{Nu_b}{2}(1-\theta_b)$$

Written in terms of the unknown functions $\gamma_o(\eta)$ and $s(\xi)$, these relations give

$$\left.\frac{d\gamma_o}{d\eta}\right|_{\eta=1} = -\frac{Nu_b}{2} = -\frac{1}{2}\frac{\dot{m}C_p\Lambda}{P_w k} \tag{4.53}$$

Nu_b can then be rewritten in terms of Gr and Λ from Eqs. (4.52) and (4.53) as

$$Nu_b = \frac{Gr\,Pr\,\Lambda}{8(1+j)^2(3+j)} \tag{4.54}$$

Since the velocity profile is already known from Eq. (4.51), the energy equation can be solved for the distribution $\gamma_o(\eta)$. From Eqs. (4.45), (4.46),(4.48),(4.51), and (4.54), this equation is

$$\frac{1}{\eta^j}\frac{d}{d\eta}\left(\eta^j\frac{d\gamma_o}{d\eta}\right) = \frac{3}{4}\left(\frac{8}{3}\right)^j Nu_b\gamma_o\left(\eta^2-1\right) \tag{4.55}$$

with the boundary conditions

$$\gamma_o\big|_{\eta=1} = 0 \tag{4.56}$$

$$\left.\frac{d\gamma_o}{d\eta}\right|_{\eta=0} = 0 \tag{4.57}$$

Furthermore, Eq. (4.53) gives one addition relation between $\gamma_o(\eta)$ and Nu_b as

$$\left.\frac{d\gamma_o}{d\eta}\right|_{\eta=1} = -\frac{Nu_b}{2} \tag{4.58}$$

Eq. (4.55) can be solved by several approaches. Successive iteration has been used for the case of the fully developed *forced* convection in a UWT circular tube (Bejan, 1995 and Incopora and DeWitt, 1990).

In the present investigation, a series solution method is chosen due to its simplicity. Assuming that there exists a polynomial $\gamma_o(\eta)$ of degree n which satisfies Eq. (4.55) and its boundary conditions,

$$\gamma_o = C_0 + C_1\eta + \ldots + C_n\eta^n = \sum_{n=0}^{n} C_n\eta^n \tag{4.59}$$

Introducing this polynomial into Eq. (4.55) and collecting terms of equal powers of η, it follows that

$$\left.\begin{aligned} &\left(\frac{3}{4}\left(\frac{8}{3}\right)^j Nu_b C_o + 2(1+j)C_2\right) + \left(\frac{3}{4}\left(\frac{8}{3}\right)^j Nu_b C_1 + 3(2+j)C_3\right)\eta + \ldots \\ &\sum_{n=4}^{n}\left[\left(n(n-1+j)C_n + \frac{3}{4}\left(\frac{8}{3}\right)^j Nu_b\left(C_{n-2} - C_{n-4}\right)\right)\eta^{n-2}\right] = 0 \end{aligned}\right\} \tag{4.60}$$

This equation is satisfied only if the coefficients of all terms are zero. It is obvious that C_1 has to be zero to satisfy the symmetry condition, Eq. (4.57). Then, for equation above to be valid on any η within the channel, C_3 is zero, C_2 is a function of C_o,

$$C_2 = -\frac{1}{1+j}\left(\frac{8}{3}\right)^{j-1} Nu_b C_o \tag{4.61}$$

and, additionally, a recurrence relation is obtained such that

$$C_n = \frac{3}{4}\left(\frac{8}{3}\right)^j \frac{Nu_b(C_{n-4} - C_{n-2})}{n(n-1+j)} \quad n \geq 4 \qquad (4.62)$$

from which half of the coefficients vanish and all the remaining coefficients can be written in terms of C_o and Nu_b. Thus, since $\gamma_o(\eta)$ has to satisfy Eqs. (4.56) and (4.58), leaving two equations for two unknowns that are solvable analytically. C_o and Nu_b can be obtained manually when the number of terms used in $\gamma_o(\eta)$ are only a few. When n is large, however, calculations are accomplished with the assistance of a computer program since the task at hand is too large to be carried out manually even if it is a routine process. Local Nusselt numbers based on bulk-to-wall temperature, Nu_b, for different values of n, the degree of the polynomial used and their corresponding C_o are shown in Figs. 4.2 (a) to (d). The solutions converge to a constant Nu_b as expected:

$$Nu_b = 3.770 \quad \text{for parallel-plate channel} \qquad (4.63)$$

$$Nu_b = 3.657 \quad \text{for circular tube} \qquad (4.64)$$

This completes the solution.

The solutions – Eqs. (4.43) and (4.44) for UHF boundary condition and Eqs. (4.63) and (4.64) for UWT boundary are summarized in Table 4.1 from which it is evident that the Nu_b of the flow within the fully developed regime of laminar natural convection is exactly the same as that of laminar forced convection despite the fact that the flow is driven by different mechanisms. If the flow is allowed to be fully developed, the local heat transfer characteristics are then significantly influenced

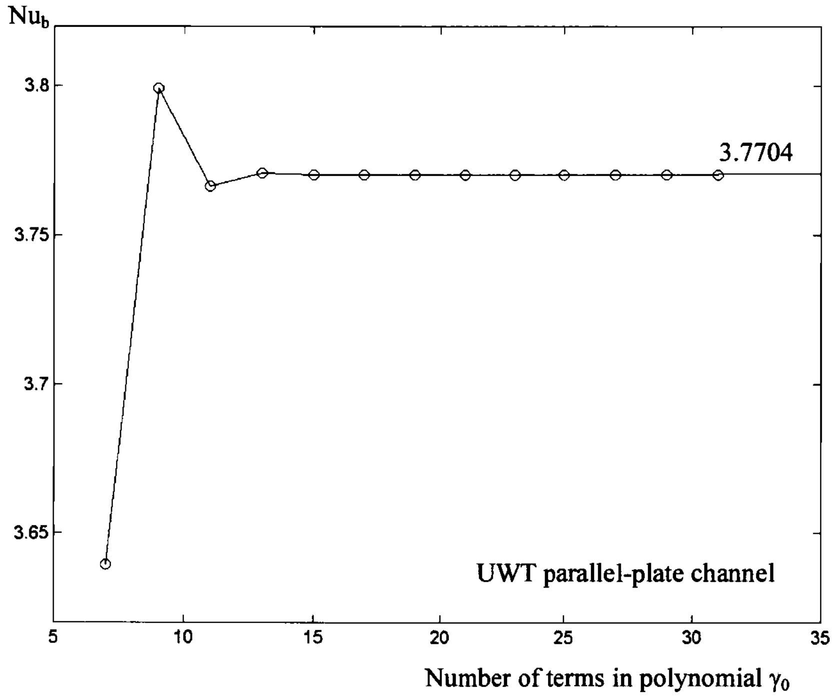


Figure 4.2 (a) Convergence of local Nusselt number solved by series solution method as a function of number of terms of polynomial used in approximation, UWT parallel-plate channel

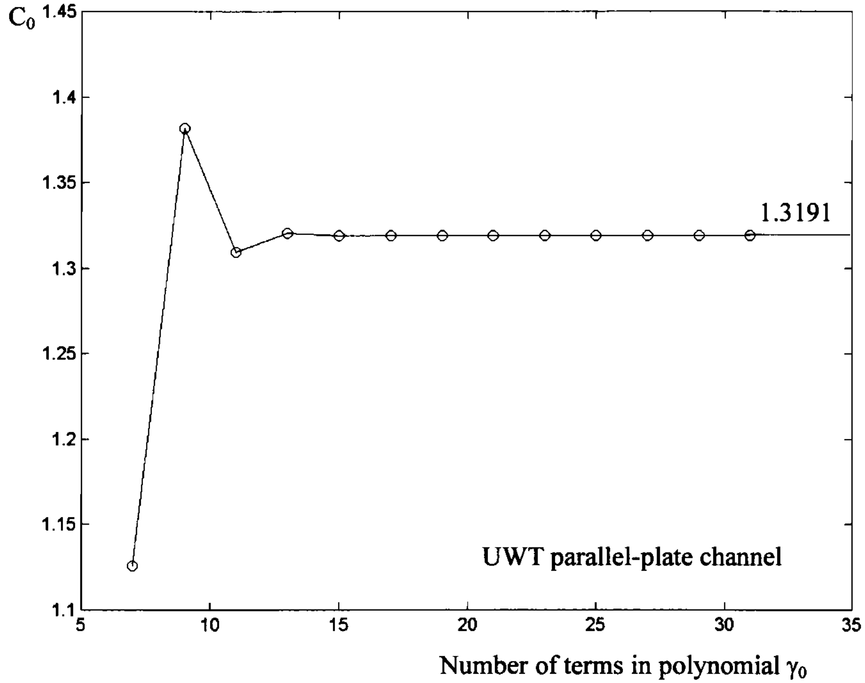


Figure 4.2 (b) Convergence of constant C_0 solved by series solution method as a function of number of terms of polynomial used in approximation, UWT parallel-plate channel

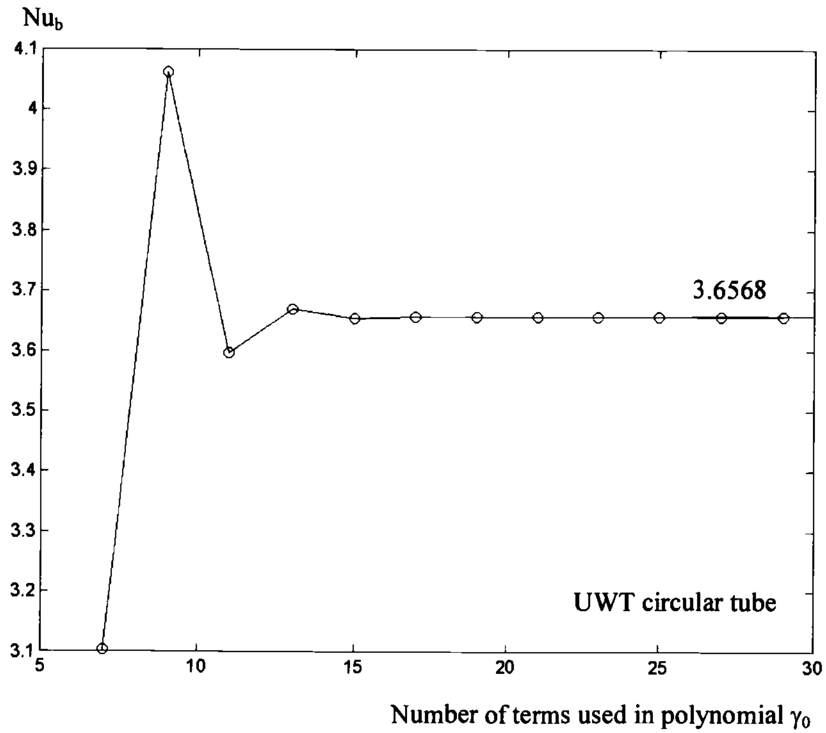


Figure 4.2 (c) Convergence of local Nusselt number solved by series solution method as a function of number of terms of polynomial used in approximation, UWT circular tube

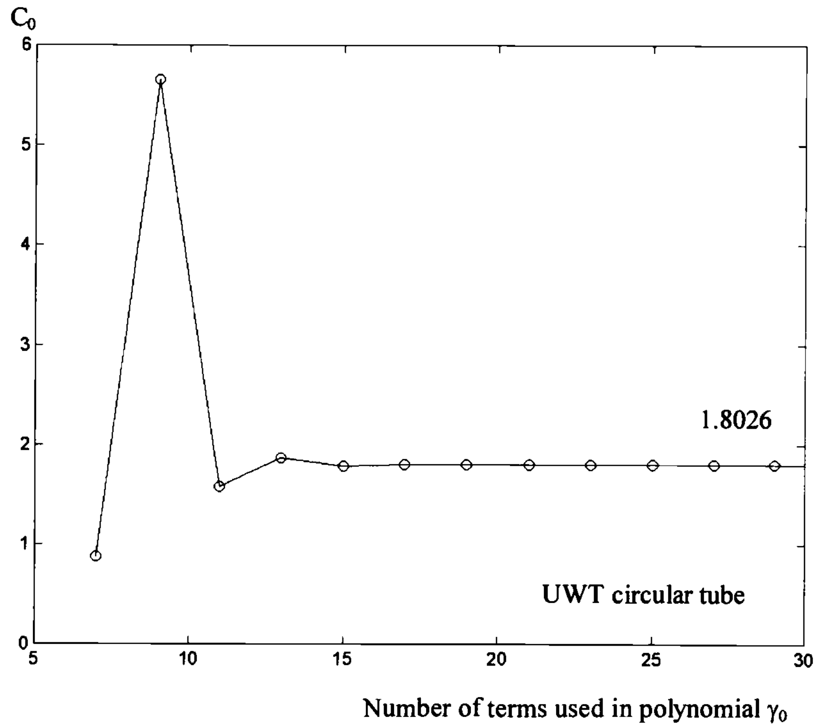
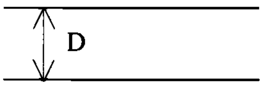
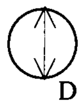


Figure 4.2 (d) Convergence of a constant C_0 solved by series solution method as a function of number of terms of polynomial used in approximation, UWT circular tube

by the same mechanisms as those effective in the fully developed laminar forced convection. Further discussion follows.

Table 4.1 Local Nusselt number, $Nu_b \equiv h_b D/k = \dot{q}''_w D/[(k(T_w - T_b)]$ for fully developed laminar natural convective flow within uniformly heated channel and tube

Geometry	UWT	UHF
	3.770	4.118
	3.657	4.363

4.5 Results and Discussions

In forced convection, the momentum equation can be solved independently of the energy equation. The same is not possible in natural convection since the momentum and energy equations are coupled. The problem is, therefore, more complicated even in the relatively simpler regime of fully developed flow. The criteria for fully developed flow in a long channel leads to simplifications which permit two partial differential equations to be combined into a single ordinary differential equation, i.e., Eq. (4.38) for the UHF problem and Eq. (4.55) for the UWT problem.

Vorayos and Kanury (2000) validated a series solution method by applying it to the well-known problem of fully developed laminar forced convection within long UHF and UWT parallel-plate channel and circular tube. The governing equation given by Eq. (4.55) is the same as that governing the fully developed *forced* convection flow within a UWT channel/tube. Hence, it solidifies the fact that the forced convection mechanisms dominate over the fully developed laminar natural convection in long channels/tubes. A similar observation can be drawn for the case of the UHF problem. For each boundary condition imposed on the channel wall, the flow characteristic in the fully developed region of a very long channel/tube are unique and independent of how the flow is driven; by the density differences in natural convection or by the specified inlet inertia in forced convection.

4.5.1 Velocity distribution

Fully developed x-dependent velocity profile of the laminar natural convective flow within uniformly heated parallel-plate channel and circular tube, for both UHF and UWT boundary conditions, is found to be parabolic as expected.

For UHF case, a relation between a constant λ and inlet velocity is acquired from Eqs. (4.28), (4.29), and (4.33). This relation is

$$\lambda = \frac{P_w D}{\phi_0 \Pr A_c} \tag{4.65}$$

It allows the velocity profile expression, Eq. (4.40), to be rewritten in terms of inlet velocity,

$$\phi = \frac{(1+j)(3+j)}{2^{1+j}}\phi_0\left(1-\eta^2\right) \tag{4.66}$$

Equate Eqs. (4.41) and (4.65) to arrive at the relation between prescribed Gr and induced fluid inlet velocity ϕ_0.

$$\phi_0 = \frac{1}{2^{1-j}(1+j)}\sqrt{\frac{1}{3+j}\cdot\frac{L\,Gr}{D\,Pr}} \tag{4.67}$$

For the case of the UWT problem too, substituting Eq. (4.52) into the velocity distribution given by Eq. (4.51), an approximate velocity profile can be obtained in terms of the inlet velocity to find it be exactly the same as Eq. (4.66). These velocity distributions for both the heating conditions are graphically illustrated in Fig. 4.3.

Furthermore, fully-developed velocity profiles obtained from the work of Vorayos and Kanury (2000) on forced convection within a long UWT and UHF parallel-plate channel and a long circular tube are also similar to Eq. (4.66). Hence, the fully developed velocity profile written in terms of the inlet velocity ϕ_0 is the same for both natural and forced duct flow. A matter of difference is that the nondimensional inlet velocity ϕ_0 for forced convection is a parameter specified as the inlet Reynolds number. For natural convection, on the other hand, it is

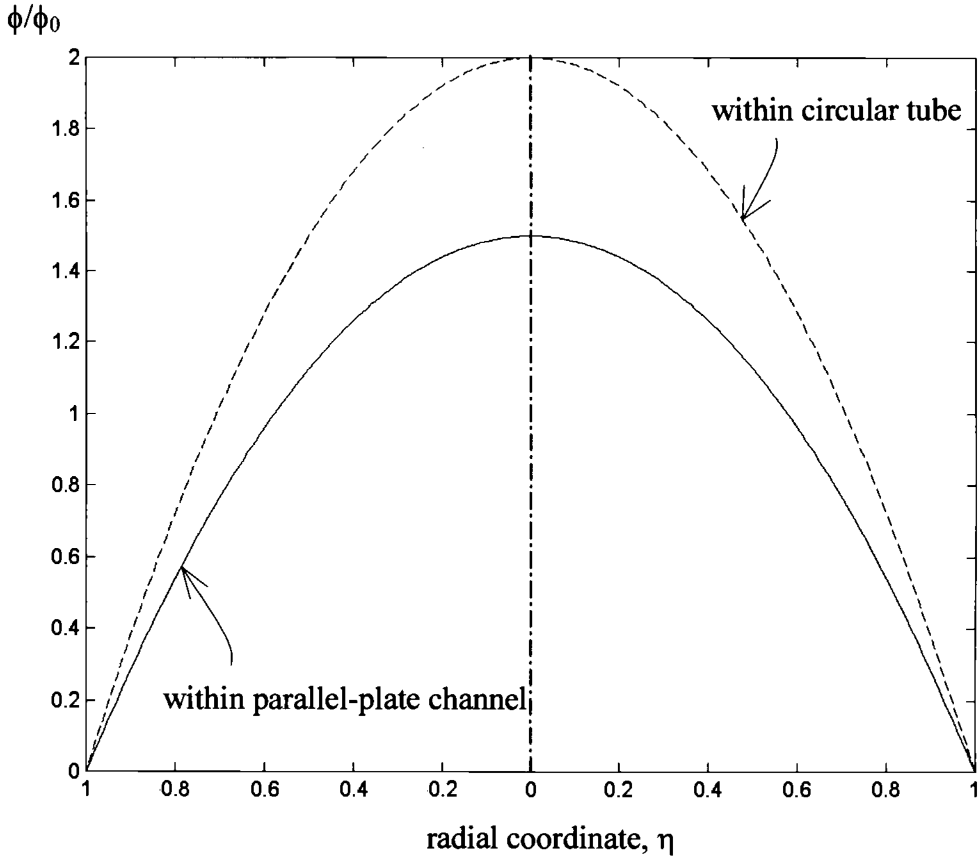


Figure 4.3 Velocity profiles of fully developed flow within uniformly heated channel/tube (same for both UHF and UWT boundary conditions).

determined by the specified wall heating condition, and can be written in terms of the imposed Grashof number as evident in Eqs. (4.52) and (4.67).

4.5.2 Temperature distribution

The fully developed temperature profile of UHF problem is shown in Eq. (4.39). Substituting λ from Eq. (4.41), this distribution becomes

$$\theta = 4(1+j)\sqrt{(3+j)\frac{L}{\mathrm{Gr\,Pr\,D}}\frac{\xi}{\xi_L}} + A_0 + \frac{(3+j)}{8}\eta^2 - \frac{(1+j)}{16}\eta^4$$

For the UWT problem, Eq. (4.60) shows that

$$\theta = 1 - \left(1.319 + 1.8650\eta^2 + ...\right)\exp\left(-90.48\frac{L}{\mathrm{GrPrD}}\frac{\xi}{\xi_L}\right) \tag{4.68}$$

for the parallel-plate channel and

$$\theta = 1 - \left(1.803 + 3.296\eta^2 + ...\right)\exp\left(-468.48\frac{L}{\mathrm{GrPrD}}\frac{\xi}{\xi_L}\right) \tag{4.69}$$

for the circular tube.

These distributions are plotted in comparison in Fig. 4.4. It can be seen that the nondimensional fluid temperature increases the flow progresses upward due to the wall heating condition. The temperature of the flow with larger GrPrD/L is higher than that with lower GrPrD/L in a UHF boundary condition while larger GrPrD/L brings the fluid temperature closer to the specified wall temperature.

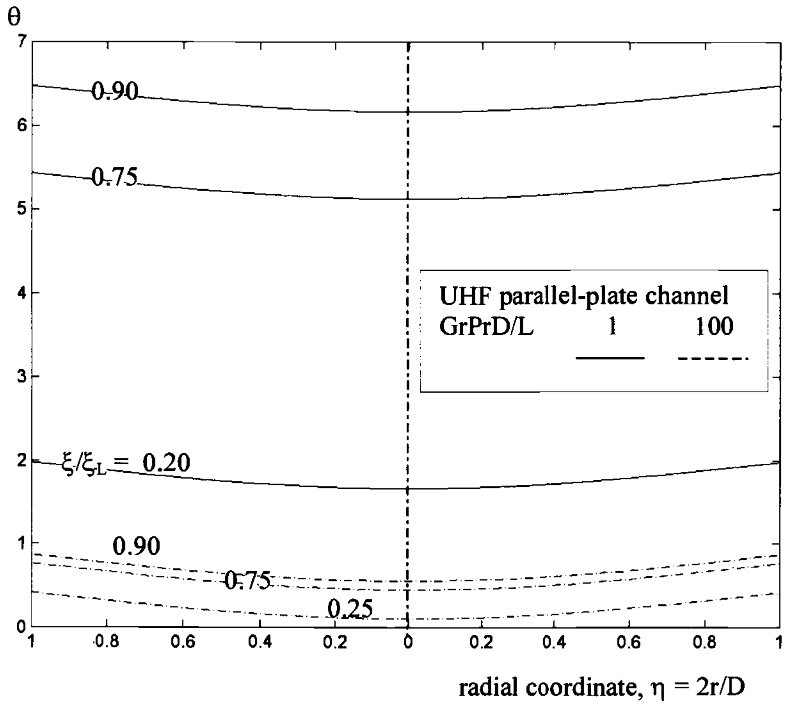


Figure 4.4 (a) Non-dimensional temperature distribution when GrPrD/L = 1 and 100 at different axial positions, UHF parallel-plate channel

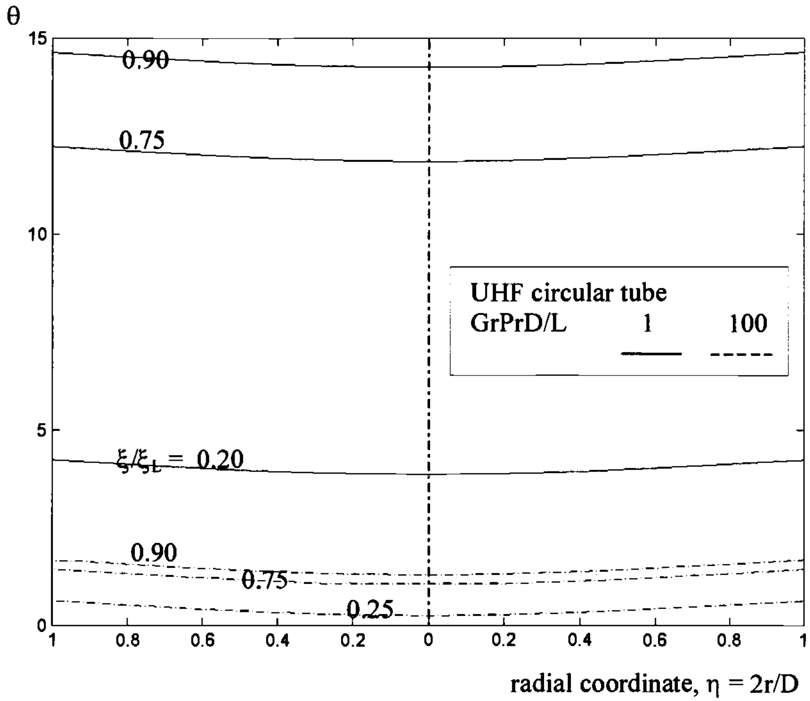


Figure 4.4 (b) Non-dimensional temperature distribution when GrPrD/L = 1 and 100 at different axial positions, UHF circular tube

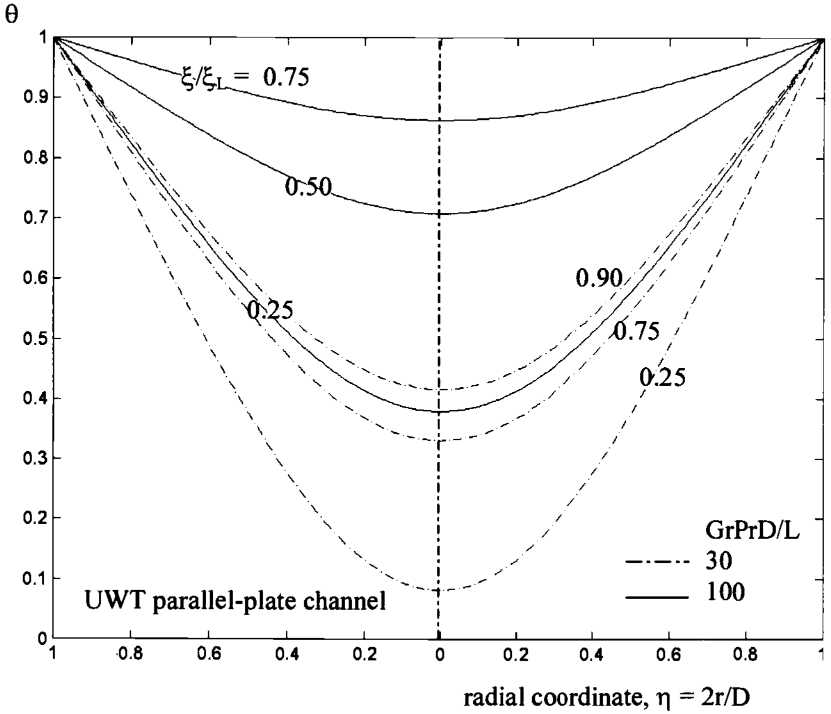


Figure 4.4 (c) Non-dimensional temperature distribution when GrPrD/L = 30 and 100 at different axial positions, UWT parallel-plate channel

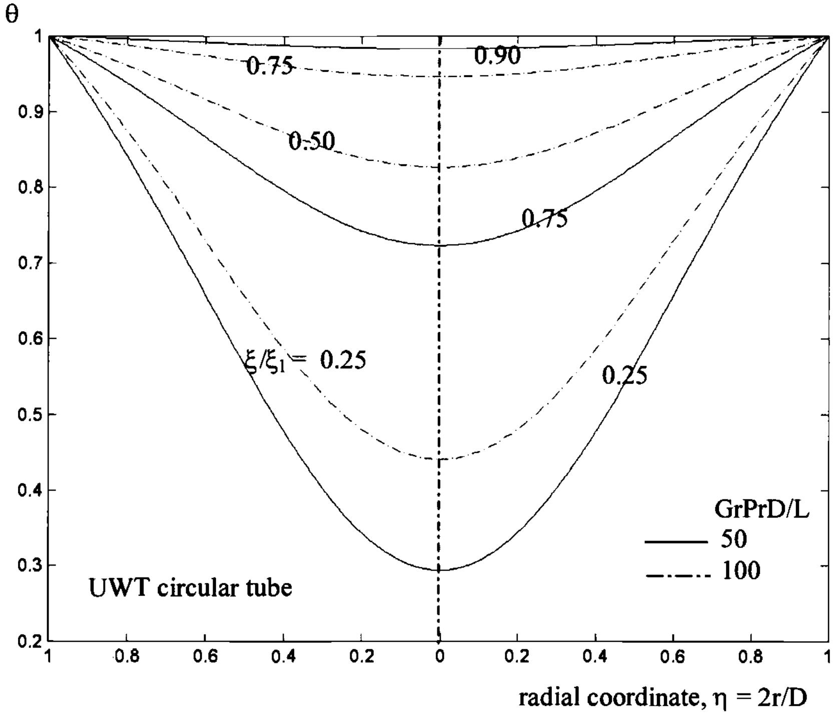


Figure 4.4 (d) Non-dimensional temperature distribution when GrPrD/L = 50 and 100 at different axial positions, UWT circular tube

4.5.3 Heat transfer characteristics

The local Nusselt number, Nu_b, based on wall-to-bulk temperature difference (T_w-T_b) and channel spacing (diameter) D, of fully developed laminar natural convection flow is shown above in Table 4.1 for UHF and UWT plate channels/tubes. The numerical values of Nu_b determined here for natural convection are found to be the same as those for forced convection.

Furthermore, the values of Nu_b for the parallel-plate channel are quite close to those of a circular tube provided that the characteristic dimensions used in the definitions of Nusselt number is the separation distance of the channel plates D and the tube diameter D. This closeness holds for both the UHF and UWT boundary conditions. (Quantitatively the channel and tube results are within 6% of each other for UHF problems and within 3% for UWT problems.) This simple finding is apparently not discussed in the existing literature in which the Nusselt number for the parallel-plate channel is usually defined on the basis of the "hydraulic diameter" D_h, for forced as well as natural convection. Since $D_h = 2D$ for the parallel plates, the resultant Nusselt number magnitudes are about twice the values for a circular-section pipe thus obscuring the physical significance of the separation distance D as the characteristic dimension. Notwithstanding the strong debate in the existing literature (e.g. In Oosthuizen and Naylor, 1999, pp. 177-178) about the meaninglessness of the hydraulic diameter in heat transfer, the same literature absurdly continues to present Nusselt number results in terms of the hydraulic diameter. Our study strongly advocates the use of physical channel spacing D in defining the Nusselt number. The hydraulic diameter should be abandoned.

The Nusselt number may be defined in other ways as well to facilitate a comparison with the literature. For *UHF* problems, the local Nusselt number based on the inlet temperature difference is defined as

$$Nu_{\infty} = \frac{h_{\infty} D}{k} = \frac{\dot{q}''_{w} D}{(T_{w} - T_{\infty})k} = \frac{1}{\theta_{w}} \tag{4.70}$$

Knowing the temperature distribution Eqs. (4.39), it follows that,

$$Nu_{\infty}(\xi) = \left(4(1+j)\sqrt{(3+j)\frac{L}{Gr\,Pr\,D}\frac{\xi}{\xi_{L}}} + K_{h} \right)^{-1} \tag{4.71}$$

K_h is a constant of 17/70 for parallel-plate channel and 11/48 for circular tube. This correlation is valid only for a fully developed flow and is not valid for flow within a short channel, i.e. small aspect ratio L/D or GrPrD/L. Our results, plotted in Fig. 4.5 for GrPrD/L between 0.01 and 1000, are in a fairly good agreement with the existing solutions. Wirtz and Stutzman (1982) and Bar-Cohen and Rohsenow (1984) reported their results in terms of Nusselt number, Nu_{∞}, at x = L (at channel exit) as a function of GrPrD/L for laminar convection in a UHF channel including for the effect of the entry region. Our result, Eq. (4.72), on the other hand, deals only with the fully developed flow. The downward departure of our result from the literature at high GrPrD/L is thus understandable. It agrees well also with the work of Aktan (1996) in which a closed form solution was obtained for a laminar fully developed natural convection flow within a UHF parallel-plate channel. Furthermore, in fully developed flow within an UHF circular tube, our work

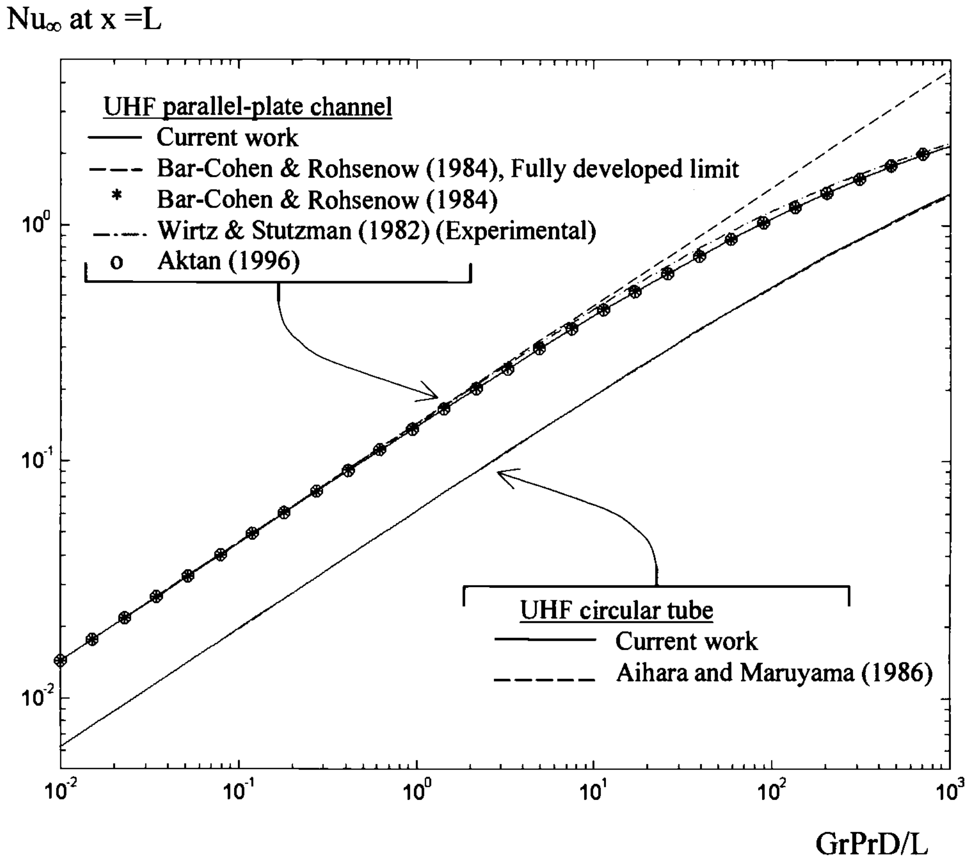


Figure 4.5 Nusselt number based on wall-to-inlet temperature difference at the channel/tube exit for UHF parallel-plate channel and circular tube

compares well with that of Aihara and Maruyama (1986) up to GrPrD/L=1000 with a difference less than 2%.

The nusselt number can be alternatively written in terms of average value as well, i.e.

$$\overline{Nu_\infty} = \frac{1}{L}\int_0^L Nu_\infty dx \tag{4.72}$$

where Nu_∞ is defined in Eq. (4.71). The average Nusselt number $\overline{Nu_\infty}$ from the present solution is compared with the existing works of Aktan (1996) and Dyer (1975) in Fig. 4.6. In spite of the fact that the solution of Dyer (1975) includes heat transfer in the vicinity of the entrance, it agrees quite well with the current solution up to GrPrD/L=1000 within 10% of error.

A local Nusselt number based on inlet temperature for *UWT* problems is found to be

$$Nu_\infty = \frac{h_\infty D}{k} = \frac{\dot{q}''_w D}{(T_w - T_\infty)k} = \frac{D}{(T_w - T_\infty)}\left.\frac{\partial T}{\partial r}\right|_{r=R} = 2\left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=1}$$

Since $\theta = 1 - \gamma_0(\eta)s(\xi)$, the local Nusselt number based on inlet temperature for UWT channel is

$$Nu_\infty = K_{w1}\exp\left(-K_{w2}\frac{L}{Gr\,Pr\,D}\left(\frac{\xi}{\xi_L}\right)\right) \tag{4.73}$$

where K_{w1} and K_{w2} are respectively 3.770 and 90.48 for a parallel-plate channel and 3.657 and 468.48 for a circular tube. Substitute Eq. (4.74) into Eq. (4.73) to obtain, for UWT pallelel-plate channel,

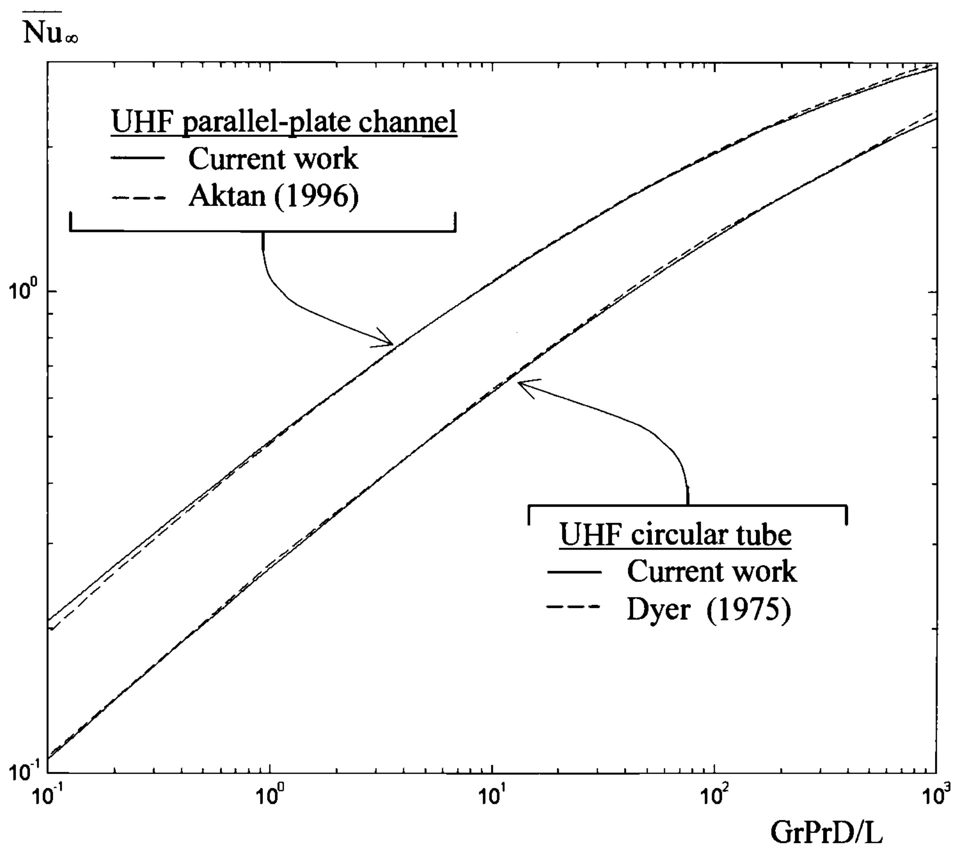


Figure 4.6 Average Nusselt number of UHF parallel-plate channel and circular tube

$$\overline{\mathrm{Nu}}_{\infty} = \frac{1}{24}\frac{\mathrm{Gr\,Pr\,D}}{\mathrm{L}}\left(1-\exp\left(-90.48\frac{\mathrm{L}}{\mathrm{Gr\,Pr\,D}}\right)\right) : \text{for UWT parallel-plate channel} \quad (4.74)$$

and

$$\overline{\mathrm{Nu}}_{\infty} = \frac{1}{128}\frac{\mathrm{Gr\,Pr\,D}}{\mathrm{L}}\left(1-\exp\left(-468.48\frac{\mathrm{L}}{\mathrm{Gr\,Pr\,D}}\right)\right) : \text{for UWT circular tube} \quad (4.75)$$

Equations (4.75) and (4.76) are in a form directly comparable to those proposed by Elenbaas (1942a and b), so much so that the only difference is in the constants within the argument of the exponential. None of the existing analyses of fully-developed natural convection flow have shown this important resemblance. A graphic comparison is shown in Fig. 4.7. Although the average Nusselt number obtained in the current work altogether ignores the developing region, our solutions agree well with Elenbaas's solutions of UWT circular tube (1942b) with an error less than 10 percent if GrPrL/D is less than 450. The agreement is good with Elenbaas (1942a) for a UWT parallel-plate channel to a lower range of GrPrL/D; an error of 10% when GrPrL/D is less than 20. These agreements appear to suggest that a great tribute is to be paid to the vision embodied in Elenbaas's correlations which were to harshly characterized as "semi-empirical" by the investigators who followed. Agreement is reasonable also with Aung (1972) and Martin et al. (1991) whose analyses account for the development of the flow in the vicinity of the entrance.

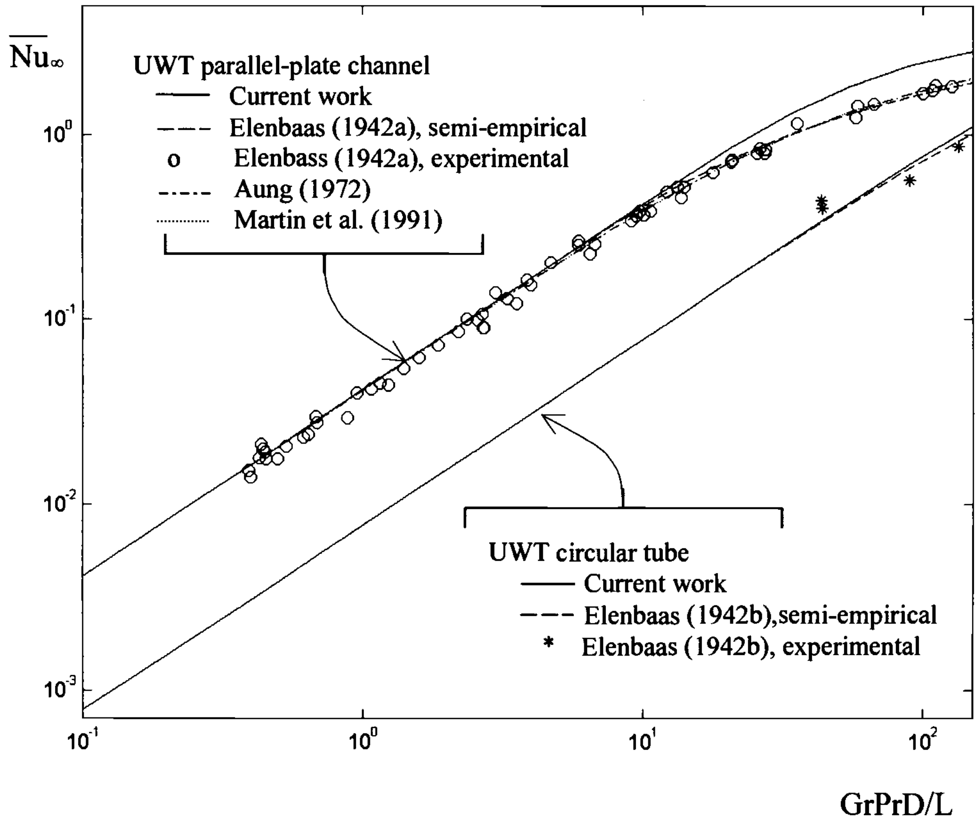


Figure 4.7 Average Nusselt number of UWT parallel-plate channel and circular tube

4.6 Conclusion

Heat transfer characteristics of steady fully developed laminar natural convection flow within long vertical UHF and UWT parallel-plate channels and circular tubes are studied. The corresponding governing equations are simplified with reasonable assumptions and temperature and velocity profiles of the flow are approximated as polynomial functions for use in a series solution method. Laminar natural convection flow within uniformly heated channel/tube is a spontaneous process in which fluid motion is gravitationally induced by a density variation of a fluid within a flow field; nevertheless, as has been shown in the paper, if the channel is sufficiently long to allow the flow to become fully developed, the flow characteristics in the regime are not different from those of a fully developed laminar *forced* convection in which the process is mandated by a specified inlet velocity. Fully developed velocity profiles of a laminar flow within the channel/tube are parabolic function similar for both forced and natural convection. While an inlet Reynolds number of forced convective flow is prescribed because of the uniform inlet velocity, that of natural convective flow is part of solution of the problem, dependent upon the boundary condition at the channel/tube's wall(s). The heating condition is prescribed by the Grashof number. Similar fully-developed temperature profiles are also presented and found to depend on the product of Grashof number, Prandtl number, and the channel's geometry: GrPrD/L. The resemblance between fully developed laminar natural and forced convective flow is highlighted with the finding that their local heat transfer coefficients, Nu_b, are remarkably close to one another. In spite of being initiated by different mechanisms, the flow physics in the fully developed region are independent of how the flow is driven. Also shown in this paper is the need to avoid using the hydraulic diameter in the description and definition of convective heat transfer in ducts of noncircular cross-section, especially in a parallel-plate channel.

4.7 References

1. Aihara, T., and Maruyama S., 1986, "Laminar Free Convective Heat Transfer in Vertical Uniform-Heat-Flux Ducts (Numerical Solutions with Constant/Temperature-Dependent Fluid Properties," Heat Transfer-Japanese Research, Vol. 15, pp. 69-86.

2. Aktan, G., 1996 "Fully Developed Laminar Natural Convection in a Vertical Parallel-Plate Duct with Constant Wall Heat Transfer," M.S. Thesis, Oregon State University.

3. Aung, W., 1972, "Fully Developed Laminar Free Convection Between Vertical Plates Heated Asymmetrically," International Journal of Heat and Mass Transfer, Vol. 15, pp. 1577-1580.

4. Aung, W., Fletcher, L.S., And Sernas, V., 1972, "Developing Laminar Free Convection Between Vertical Plates With Asymmetric Heating," International Journal of heat and Mass Transfer, Vol. 16, pp. 2293-2308.

5. Bar-Cohen, A., and Rohsenow, W.M., 1984, "Thermally Optimum Spacing of Vertical Natural Convection Cooled Parallel Plates," ASME Journal of Heat Transfer, Vol. 106, pp. 116-123.

6. Bejan, A., 1995, "Convection Heat Transfer" 2nd edition, John Wiley & Sons, New York.

7. Bodia, J. R., and Osterle, J.F., 1962, "The Development of Free Convection Between Heated Vertical Plates," ASME Journal of Heat Transfer, Vol. 84, pp. 40-44.

8. Çengel, Y. A., 1998, "Heat Transfer: A Practical Approach," McGraw Hill, New York.

9. Churchill, S. W., and Usagi, R., 1972, "A General Expression for the Correlation of Rates of Heat Transfer and Other Phenomena," Journal of American institute of Chemical Engineering, Vol 18, pp. 1121-1138

10. Duffy, D. G., 1998, "Advanced Engineering Mathematics," CRC Press, Boca Raton.

11. Elenbaas, W., 1942 a, "Heat Dissipation of Parallel Plates by Free Convection," Physica, Vol. 9, pp. 1-28.

12. Elenbaas, W., 1942 b, "The Dissipation of Heat by Free Convection: The Inner Surface of Vertical Tubes of Different Shapes of Cross-Section," Physica, Vol. 9, pp. 865-874.

13. Hallman, T. M., 1956, "Combined Forced and Free-Laminar Heat Transfer in Vertical Tubes with Uniform Internal Heat Generation," Transaction of ASME, Vol. 78, pp. 1831-1841.

14. Johnson, C. E., 1986, "Evaluation of Correlations for Natural Convection Cooling of Electronic Equipment," Heat Transfer in Electronic Equipment, A Bar-Cohen, ed., AIAA/ASME 4th Thermophysics and Heat Transfer Conference, Boston, June 2-4.

15. Martin, L., Raithby, G. D., and Yovanovich M. M., 1991, "On the Low Rayleigh Number Asymptote for Natural Convection Through an Isothermal, Parallel-Plate Channel, " Transactions of ASME, Vol. 113, pp. 899-905.

16. Miyatake, O., Fujii, T., Fujii, M., and Tanaka, H., 1973, "Natural Conevctive Heat Transfer Between Vertical Parallel Plates-One Plate with a Uniform Heat Flux and the Other Thermally Insulated," Heat Transfer-Japanese Research, Vol. 2, pp. 25-33.

17. Oosthuizen, P.H., and Naylor, D., 1999, "Introduction to Convective Heat Transfer," McGraw-Hill, New York.

18. Raithby, G.D., Hollands, K. G. T., 1975, "A General Method of Obtaining Approximate Solutions to Laminar and Turbulent Free Convection Problems," in: Advances in Heat Transfer, T.F. Irvine, Jr. and J.P. Hartnett, eds., Vol. 11 pp. 265-315.

19. Ramanathan S., and Kumar R. , 1991, "Correlations for Natural Convection between Heated Vertical Plates, " ASME Journal of Heat Transfer, Vol. 113, pp. 97-107.

20. Ross, S. L., 1966, "Introduction to Ordinary Differential Equations," Xerox, Lexington.

21. Seban, R.A. and Simazaki, T.T., 1951, "Heat Transfer to a Fluid Flowing Turbulently in a Smooth Pipe With Walls at Constant Temperature," Transactions of the ASME, Vol. 73, pp. 803-809

22. Sobel, N., Landis, F., and Mueller, W., 1966, "Natural Convection Heat Transfer in Short Vertical Channels, Including the Effects of Stagger," Proceedings, Third International Heat Transfer Conference, Vol. 2 pp. 121-125

23. Sparrow, E.M., and Gregg, J.L., 1956, "Laminar Free Convection from a Vertical Plate with Uniform Surface Heat Flux," Transactions of ASME, Vol. 78, pp. 435-440.

24. Vorayos, N. and Kanury, A.M. 2000, "Validation of a Series Solution Method on Fully Developed Laminar Forced Convection within a Uniformly heated Vertical Parallel-Plate Channel and Circular Tube," in preparation.

25. Webb, B. W., and Hill, D. P.,1989, "High Rayleigh Number Laminar Natural Convection in an Asymmetically Heated Vertical Channel," Transaction of ASME, Vol. 111, pp 649-656.

26. Willie, R. H., 1996, "Fully Developed Laminar Natural Convection in a Vertical Parallel Plate Channel with Symmetric Uniform Wall Temperature," M.S. Thesis, Oregon State University.

27. Wirtz, R. A., and Stutzman, R. J., 1982, "Experiments on Free Convection Between Vertical Plates with Symmetric Heating," ASME Journal of Heat Transfer, Vol. 104, pp. 501-507.

CHAPTER 5

DEVELOPING LAMINAR NATURAL CONVECTION OF AIR WITHIN LONG VERTICAL PARALLEL-PLATE CHANNELS AND CIRCULAR TUBES

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In Preparation

5.1 Abstract

This is the third of a series of three papers in which we revisit the famous investigation of Elenbaas (1942a and 1942b).

The developing laminar natural convection of air (Pr = 0.7) within a vertical parallel plate channel and a circular tube is investigated. Our approach involves obtaining the development of the viscous and thermal boundary layers near the channel inlet using the well-known integral method. Use of classical boundary layer concepts, the Boussinesq approximation, and the integral method of solution gives us a better perspective of the interactions among the relevant mechanisms. It is presumed that the channel or tube, kept at a uniform wall temperature, is sufficiently long to ensure a fully developed flow. Under such long channel/tube conditions, the physics of flow and heat transfer is found to be the same as that involved in forced convection. Comparison of our results with previous investigations indicates a close agreement. The entrance length of such a flow can be approximated as $L_e/L \approx 0.05GrPrD/(KL)$ where K is a constant equal to 12 and 32 for the parallel plate channel and circular tube, respectively.

5.2 Nomenclature

D	plate spacing or tube diameter
Gr	Grashof number; based on constant temperature difference, $Gr = g\beta(T_w - T_\infty)D^3/\nu^2$,
h	coefficient of convection heat transfer
k	thermal conductivity
L	parallel-plate-channel or circular-tube length
L_e	entrance length
Nu_b	local Nusselt number based on temperature difference between wall temperature and bulk temperature
$\overline{Nu_b}$	average Nusselt number
P	pressure
P'	pressure difference between fluid pressure and hydrostatic pressure outside, $P' = P - P_\infty$
Pr	Prandtl number, ν/α
R	half of channel spacing or tube radius
Re_D	inlet Reynolds number based on D, u_oD/ν
T	absolute temperature
u, v	axial and transverse component of velocity vector
u_o	inlet velocity
u_c	core velocity; velocity of the fluid outside the boundary layer

W	width of parallel plates
x, r	the coordinates used in the analysis (see Fig. 1)
Δ	nondimensionalhydrodynamic boundary layer thickness, δ/R
Δ_T	nondimensionalthermal boundary layer thickness, δ_T/R
α	thermal diffusivity
β	coefficient of thermal expansion
δ	hydrodynamic (viscous) boundary layer thickness
δ_T	thermal boundary layer thickness
ϕ	nondimensionalvelocity, axial component, $\phi = u/u_o$
ϕ_c	nondimensionalcore velocity, u_c/u_o
ν	kinematic viscosity
Π	nondimensionalpressure difference, $\Pi = P'/(\rho u_o^2)$
θ	nondimensionaltemperature, $\theta = (T - T_\infty)/(T_w - T_\infty)$
ρ	fluid density
ζ	ratio between thermal and hydrodynamic boundary layer thickness, (Δ_T/Δ)
ξ,η	nondimensionalquantities of the coordinates used in the analysis,
ψ	nondimensionalvelocity, transverse component, $\psi = vD/\nu$

Subscripts

$(\ )_b$	bulk
$(\ )_w$	wall
$(\ )_o$	inlet

5.3 Introduction

The topic of natural convective heat transfer in vertical parallel plate channels and in circular tubes whose walls are kept at a uniform constant temperature has been studied for over sixty years. Newer applications of this topic recently emerged in numerous modern equipments and devices, as for example in nuclear reactors, solar panels, and electronic circuit boards. Even though forced convection is usually the primary method to remove excessive heat in these and other applications, natural convection is always present. In most situations, natural convection alone is perhaps preferable to carry out the cooling since the process is spontaneous, simpler, requires no compressor or pump.

Laminar (and turbulent) *forced* convection in channels and in circular cross-section tubes has been studied for over a century so that the associated physics of flow and heat transfer are reasonably well understood. In contrast, and in spite of considerable numbers of studies on the subject, the physics of pure natural convection process within vertical channels or tubes is not as clearly understood. The pioneering work in this area has been that which was carried out by Elenbaas (1942a) on a vertical parallel plate channel. The average Nusselt number has been correlated in a "semi-empirical" manner as a function of one parameter — the product of Grashof and Prandtl numbers divided by the aspect ratio of the parallel-plates channels — has been developed and confirmed with experiments. Elenbaas also suggested similar relations for the channel geometry of circular tube in his subsequent paper (Elenbaas, 1942b). These works of Elenbaas have been referred to, debated, criticized, discussed, and extended in the studies of many investigators who followed.

Let us briefly describe the problem at hand by considering a vertical channel formed of two very wide parallel plates separated by a distance $D \equiv 2R$. The channel extends from $x = 0$ (entrance) to $x = L$ (exit). The local acceleration due to gravity is g. This channel is situated in an infinite chamber of quiescent air at pressure P_∞ (which varies hydrostatically with height location x) and temperature T_∞. The channel walls are symmetrically kept at a uniform temperature T_W ($>T_\infty$). Due to the finite positive temperature difference (T_W-T_∞), density differences arise in the air and thereby an upward flow is induced gravitationally within the channel/tube. It is this flow (after it attains a steady state) upon which we focus our attention in this investigation.

At the inlet, boundary layers develop over the two walls. Little or no interaction or influence exists between these boundary layers very near the inlet. Here, the boundary layer growth is quite satisfactorily described by the theories of free convection over a single wall. At larger distances x from the entrance, however, the boundary layers grow thicker, thus the growth is influenced by the confinement of the flow. Eventually the boundary layers merge at the plane or axis of symmetry.

For air, whose Prandtl number, Pr, is approximately 0.7, the region between $x = 0$ and the value of x at which the thermal boundary layers merge shall henceforth be termed the "first entrance region". At the end of this region, the temperature distribution T(r) across the channel exhibits a minimum at the axis or plane of symmetry whereas the velocity distribution u(r) is still developing or double-peaked depending on the channel aspect ratio and wall heating condition.

In subsequent flow, transverse diffusion will transform the developing velocity distribution into the well-known Poiseuille parabolic distribution in laminar flow in a channel. This region is henceforth termed the "second entrance region"; its length is assumed to be ignorably small compared to the length of the first region. To quantitatively describe the flow and heat transfer in the entrance region is the objective of this present paper.

Following the two entrance regions, the flow is fully developed both thermally and hydrodynamically. We assume in this work that our channel is sufficiently long to result in fully developed flow at the exit of the channel/tube.

Because the governing equations for pure natural convection are complex and coupled, an explicit closed-form mathematical solution is difficult, if not impossible. Characteristics of the flow are therefore generally sought to be deduced from variety of approaches ranging from simplified mathematical analyses to experimental studies.

With the advent of numerical analyses, the problem of natural convection in vertical channels and tubes has attracted a great deal of attention to validate the work of Elenbaas (for instance, Bodia and Osterle 1962, Aung et al. 1972, and Ramanathan and Kumar 1991). Numerical methods lead to solutions of complex engineering problems which are otherwise insoluble. First, the mathematical problem is correctly formulated in terms of the conservation equations, constitutive relations, and boundary conditions, even if in a greater detail than generally necessary. Second, these equations and boundary conditions are then discretized by using finite difference or finite element techniques. Third, an algorithmic flow chart is developed and a computer program is written in one language or another. (It is more likely that a contemporary researcher would use a commercially available solver/program.) The numerical solution is processed in the back-end into impressive tabular and graphical output which is interpreted by the investigator. However, choosing to work with a numerical method too soon tends to sacrifice the level of understanding gained of the physics of the involved processes.

The parameter [Gr Pr D/L] plays an important role in natural convection in vertical channel with Gr and Pr respectively denoting the Grashof and Prandtl numbers of the flow and L/D is channel/tube aspect ratio. (a) Large values of this parameter correspond to a short and wide channel with strong wall heating where the solution approaches the limiting case of laminar free convection along a vertical flat plate. (b) Small values of the parameter [Gr Pr D/L] correspond to a long and narrow channel with low wall heating where the solution approaches the limiting case of laminar fully developed channel flow. Churchill and Usagi (1972) proposed a way of combining the limiting case to obtain the characteristics in between. A series of publications exploiting this notion are available such as, for example, that of Bar-Cohen and Rohsenow (1984). In spite of rather extensive literature of this sort, only the heat transfer characteristics between extreme limits are predicted but the physics of the flow and heat transfer and interactions among relevant transport mechanisms have not been well understood.

In this paper, we seek to obtain an improved physical understanding of the mechanisms of naturally induced laminar flow and heat transfer in vertical channels and tubes. We use boundary layer theory and integral methods of solutions. The governing equations of such systems are complex even after making several reasonable simplifications and approximations. Numerical schemes appear to be unavoidable. The use of numerical analysis will be limited in our work and used only when absolutely necessary; even the numerical procedures employed will be kept relatively simple while yet delivering acceptable accuracy for comparison with previous work. The results, especially of the heat transfer characteristics, are then compared with existing work and discussed.

This is the third in a series of three stand-alone papers. The use of a series solution method on the problem of laminar fully developed *forced* convection within uniformly heated parallel-plate channels and circular tubes is validated in the first paper. In the second paper, the series solution method is applied to laminar fully developed natural convection within a uniformly heated vertical parallel-plate channel and a tube of circular cross-section. The channel/tube is so long that the flow is fully developed and that the length of the developing region is ignorably small.

In the present paper we apply boundary layer theory to the flow near the entrance of a long vertical channel or circular tube whose wall(s) is kept at a constant and uniform temperature. By employing an integral method, we analyze the development of the hydrodynamic and thermal boundary layers. The channel/tube is assumed to be long enough for the flow to be fully developed at its upper end. Mass flow rate in the channel tube is then prescribed by the fully developed flow[1].

5.4 Mathematical Model

Consider steady laminar natural convective flow within a long vertical channel made up of two very wide parallel plates separated by a spacing $D \equiv 2R$; or a long vertical tube of circular cross-section with diameter $D \equiv 2R$. The words "channel" and "tube" are henceforth used in the present work synonymously unless a distinction needs to be made. The channel or tube length is L. As illustrated in Fig. 5.1, cartesian and cylindrical coordinates are chosen in the descriptions of the parallel channel and circular tube, respectively. The inner surface(s) of the channel wall(s) is (are) kept at a uniform and constant temperature T_W greater than the inlet ambient air temperature T_∞. This prescribed inner surface temperature boundary

[1] If the channel/tube is so short as not to yield a fully developed flow, the mass flow rate is a smaller unknown quantity.

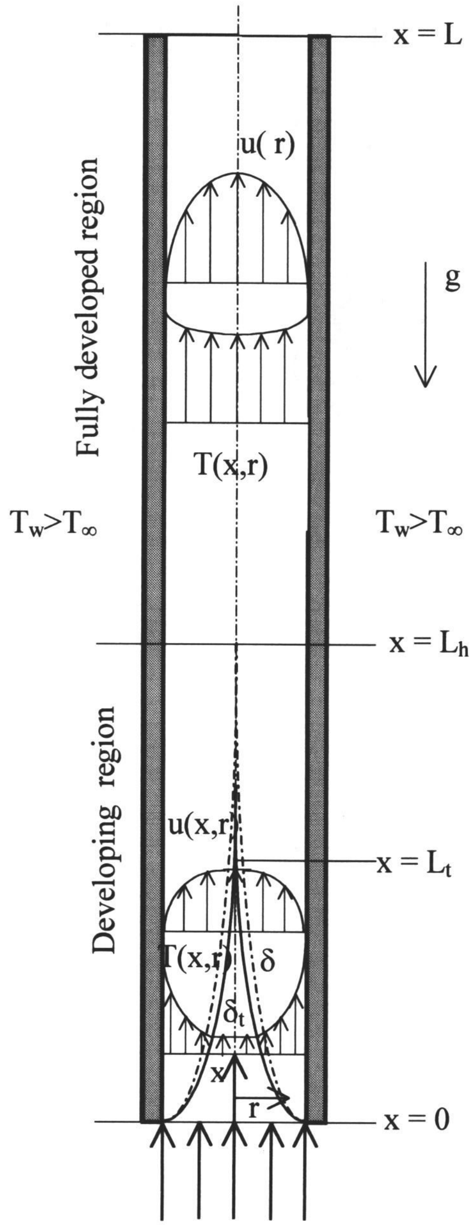

Figure. 5.1 Configuration and coordinate used in an analysis with qualitative radial (r) temperature and velocity distributions along axial (x). L_h and L_t are viscous (hydrodynamic) and thermal developing lengths, respectively

condition is henceforth referred as the UWT condition[2]. At the inlet $x = 0$, the air velocity is to be determined and taken to be a constant and uniform, $u(x = 0) = u_0$, across the cross-section $0 < r < R$. The inlet temperature is also assumed to be uniform at T_∞. The mass flow rate is gravitationally induced upward and is part of the unknowns. However, since the channel is very long, the flow is fully developed beyond some axial location which is sufficiently downstream from the entrance. The dynamics of this fully developed natural convective flow as studied by Vorayos and Kanury (2000b) give the unknown mass flow rate.

The following simplifications are made in the present analysis. All the properties are constant. Even the density is constant everywhere except in the buoyancy term of the momentum equation, thus the well-known Boussinesq approximation (Schlichting, 1968) is adopted. The fluid density in the gravitational term is expressed in terms of the fluid temperature. An order-of-magnitude analysis of the viscous and thermal boundary layers indicates that the axial (x) component of fluid velocity u is much larger than the lateral (r) component v. Additionally, lateral gradients and the resultant diffusions are much larger than the axial ones. These observations allow the governing equations to be simplified as follows:

Conservation of mass

$$\frac{1}{r^j}\frac{\partial}{\partial r}\left(r^j v\right)+\frac{\partial u}{\partial x}=0 \tag{5.1}$$

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[2] The problem of uniform-wall-heat-flux (UHF) boundary condition is not considered here.

Conservation of momentum

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial P'}{\partial x}+g\beta(T-T_{\infty})+\frac{\nu}{r^{j}}\frac{\partial}{\partial r}\left(r^{j}\frac{\partial u}{\partial r}\right) \tag{5.2}$$

Conservation of Energy

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial r}=\frac{\alpha}{r^{j}}\frac{\partial}{\partial r}\left(r^{j}\frac{\partial T}{\partial r}\right) \tag{5.3}$$

When $j = 0$, this system of equations pertains to the flow within the vertical parallel-plate channel. When $j = 1$, flow in a tube of circular cross-section is described. The pressure defect P' is defined as

$$P' = P - P_{\infty} \tag{5.4}$$

where $P_{\infty} = P_{\infty 0} - \rho g x$, is the hydrostatic pressure of an air outside the channel or tube. $P_{\infty 0}$ is a reference ambient pressure. All calculations are carried out based on the presumption that the fluid within channel is air whose Prandtl number is about 0.7. The boundary conditions needed to solve the foregoing governing equations are as below.

$$u(x=0,r)=u_{o} \tag{5.4a}$$

$$v(x=0,r)=0 \tag{5.4b}$$

$$T(x=0,r)=T_{\infty} \tag{5.4c}$$

$$u(x,r=R)=0 \tag{5.5a}$$

$$v(x,r=R)=0 \tag{5.5b}$$

$$T(x,r=R)=T_{w} \tag{5.5c}$$

$$\frac{\partial}{\partial r}\left(r^j\frac{\partial u}{\partial r}\right)_{r=R} = -\frac{R}{4}\left(-\frac{1}{\rho}\frac{dP'}{dx} + g\beta(T_w - T_\infty)\right) \tag{5.6a}$$

$$\frac{\partial}{\partial r}\left(r^j\frac{\partial T}{\partial r}\right)_{r=R} = 0 \tag{5.6b}$$

The boundary conditions given by Eqs. (5.6a) and (5.6b) are the momentum and energy equation evaluated at the wall; $r = R$, giving the conditions for the second derivatives of fluid velocity and temperature. The conditions outside the boundary layers where the flow is not yet affected by viscosity and heating are:

$$u(x, R \le r \le R - \delta) = u_c \tag{5.7a}$$

$$\left.\frac{\partial u}{\partial r}\right|_{0 \le r \le R-\delta} = 0 \tag{5.7b}$$

$$T(x, 0 \le r \le R - \delta) = T_\infty \tag{5.8a}$$

$$\left.\frac{\partial T}{\partial r}\right|_{0 \le r \le R-\delta_T} = 0 \tag{5.8b}$$

Here, u_c is defined as fluid velocity outside the boundary layer, i.e., in the "core" of the flow. It is an unknown quantity; a function of axial distance. The core half-thickness is R-δ.

By making the following substitutions

$$\left.\begin{aligned} &\phi = \frac{u}{u_o} \qquad \psi = \frac{vD}{\nu} \qquad \xi = \frac{x}{DRe_D} \qquad \eta = \frac{r}{R} \\ &\Pi = \frac{P'}{\rho u_o^2} \qquad \theta = \frac{T - T_\infty}{T_w - T_\infty} \end{aligned}\right\} \tag{5.9}$$

The Reynolds number Re_D is nondimensionaluniform inlet velocity u_o defined by

$$\mathrm{Re}_D = \frac{u_0 D}{\nu} \tag{5.10}$$

The governing equations can now be rewritten in nondimensionalform as

Conservation of mass

$$\frac{2}{\eta^j}\frac{\partial}{\partial\eta}\left(\eta^j\psi\right)+\frac{\partial\phi}{\partial\xi}=0 \tag{5.11}$$

Conservation of momentum

$$\phi\frac{\partial\phi}{\partial\xi}+2\psi\frac{\partial\phi}{\partial\eta}=-\frac{\partial\Pi}{\partial\xi}+\left(\frac{\mathrm{Gr}}{\mathrm{Re}_D}\right)\theta+\frac{4}{\eta^j}\frac{\partial}{\partial\eta}\left(\eta^j\frac{\partial\phi}{\partial\eta}\right) \tag{5.12}$$

Conservation of energy

$$\phi\frac{\partial\theta}{\partial\xi}+2\psi\frac{\partial\theta}{\partial\eta}=\frac{4}{\mathrm{Pr}}\frac{1}{\eta^j}\frac{\partial}{\partial\eta}\left(\eta^j\frac{\partial\phi}{\partial\eta}\right) \tag{5.13}$$

The Grashof number Gr and Prandtl number Pr are defined as

$$\mathrm{Gr}=\frac{g\beta(T_w-T_\infty)D^3}{\nu^2} \tag{5.14}$$

$$\mathrm{Pr}=\frac{\nu}{\alpha} \tag{5.15}$$

The boundary conditions become

$$\phi(\xi=0,\eta)=1 \tag{5.16a}$$

$$\psi(\xi=0,\eta)=0 \tag{5.16b}$$

$$\theta(\xi=0,\eta)=0 \tag{5.16c}$$

$$\phi(\xi,\eta=1)=0 \tag{5.17a}$$

$$\psi(\xi,\eta=1)=0 \tag{5.17b}$$

$$\theta(\xi,\eta=1)=1 \tag{5.17c}$$

$$\frac{\partial}{\partial \eta}\left(\eta^{j}\frac{\partial \phi}{\partial \eta}\right)_{\eta=1} = -\frac{G}{4} \tag{5.18a}$$

$$\frac{\partial}{\partial \eta}\left(\eta^{j}\frac{\partial \theta}{\partial \eta}\right)_{\eta=1} = 0 \tag{5.18b}$$

$$\phi(\xi, 1 \le \eta \le 1-\Delta) = \phi_c \tag{5.19a}$$

$$\left.\frac{\partial \phi}{\partial \eta}\right|_{0 \le \eta \le 1-\Delta} = 0 \tag{5.19b}$$

$$\theta(\xi, 0 \le \eta \le 1-\Delta_T) = 0 \tag{5.20a}$$

$$\left.\frac{\partial \theta}{\partial \eta}\right|_{0 \le \eta \le 1-\Delta_T} = 0 \tag{5.20b}$$

Here, Δ and Δ_T are nondimensionalforms of viscous and thermal boundary layer thicknesses δ and δ_T respectively, and defined as $\Delta = \delta/R$, $\Delta_T = \delta_T/R$. G is defined as

$$G = -\frac{d\Pi}{d\xi} + \frac{Gr}{Re_D} \tag{5.21}$$

Outside the boundary layer(s), i.e., in the core, friction and heating effects are absent. The flow here is governed simply by Bernoulli's equation. In a nondimensionalform, this relation is

$$\phi_c \frac{d\phi_c}{d\xi} = -\frac{d\Pi}{d\xi} \tag{5.22}$$

So that Eq. (5.21) can be rewritten as

$$G = \phi_c \frac{d\phi_c}{d\xi} + \frac{Gr}{Re_D} \tag{5.23}$$

Recall that the lateral component of fluid velocity is zero both at the channel wall and at the axis or plane of symmetry. Integrating conservation of mass, Eq. (5.11) from $\eta = 0$ to 1, it follows that

$$\frac{d}{d\xi}\int_0^1 \eta^j \phi d\eta = \frac{d}{d\xi}\left(\int_0^{1-\Delta} \eta^j \phi d\eta + \int_{1-\Delta}^1 \eta^j \phi d\eta\right) = 0 \qquad (5.24)$$

This is nothing but the statement that the sum of mass flow rates in the boundary layer and in the core is a constant.

Since mass is conserved,

$$\dot{m} = \rho u_o A_c = \int_{A_c} \rho u dA_c = \text{a constant with respect to x} \qquad (5.25)$$

for our incompressible flow, this equation is rewritten in nondimensionalform as

$$\int_{1-\Delta}^1 \eta^j \phi d\eta + \frac{\phi_c}{1+j}(1-\Delta)^{1+j} = \frac{1}{1+j} \qquad (5.26)$$

This is a useful alternative form of Eq. (5.24). Integration of the conservation of mass, Eq. (5.11), from $\eta=(1-\Delta)$ to 1, i.e. for the portion of the flow inside the viscous boundary layer δ, gives the lateral velocity component at the edge of the boundary layer ($\eta=1-\Delta$) as

$$\psi(\eta = 1 - \Delta) = -\frac{\phi_c}{2}\frac{d\Delta}{d\xi} - \frac{1}{2(1-\Delta)^j}\frac{d}{d\xi}\int_0^{1-\Delta} \eta^j \phi d\eta \qquad (5.27)$$

Multiply the momentum and energy conservation equations, Eqs. (5.12) and (5.13), by η^j. With a few steps using the continuity equation given by Eq. (5.11), the Bernoulli equation of the core flow given by, Eq. (5.23), and the lateral-component velocity Eq. (5.27), the momentum and energy equations take the following forms.

$$\frac{\partial}{\partial\xi}\left(\eta^j\phi^2\right)+2\frac{\partial}{\partial\eta}\left(\eta^j\psi\phi\right)=\eta^j\frac{d\phi}{d\xi}+\left(\frac{Gr}{Re_D}\right)\eta^j\theta+4\frac{\partial}{\partial\eta}\left(\eta^j\frac{\partial\phi}{\partial\eta}\right) \quad (5.28)$$

$$\frac{\partial}{\partial\xi}\left(\eta^j\phi\theta\right)+2\frac{\partial}{\partial\eta}\left(\eta^j\psi\theta\right)=\frac{4}{Pr}\frac{\partial}{\partial\eta}\left(\eta^j\frac{\partial\theta}{\partial\eta}\right) \quad (5.29)$$

Integrating Eq. (5.28) from $\eta = (1\text{-}\Delta)$ to 1,i.e., across the boundary layer, it follows that

$$\frac{d}{d\xi}\int_{1-\Delta}^{1}\eta^j\phi(\phi-\phi_c)d\eta+\frac{d\phi_c}{d\xi}\int_{1-\Delta}^{1}\eta^j(\phi-\phi_c)d\eta=\frac{Gr}{Re_D}\int_{1-\Delta}^{1}\eta^j\theta d\eta+4\left.\frac{\partial\phi}{\partial\eta}\right|_{\eta=1} \quad (5.30)$$

Similarly integrating Eq. (5.29) from $\eta = (1\text{-}\Delta_T)$ to 1, i.e., across the thermal boundary layer, we obtain

$$\frac{d}{d\xi}\int_{1-\Delta_T}^{1}\eta^j\theta\phi d\eta=\frac{4}{Pr}\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=1} \quad (5.31)$$

The integrated form of the governing equations given by Eqs. (5.26), (5.30) and (5.31) will allow the transformation of a system of three partial differential equations into three simpler ordinary differential equations for the three unknown functions Δ, Δ_T and ϕ_c as dependent on ξ. An approximate solution can be then obtained if the shape of the velocity and temperature distributions is assumed. This approach is pioneered by T. von Karman as shown by Mill (1992) which has been followed by several investigators, for example, the work of Spalding (1954) on laminar flow with mass transfer on a flat plate and Sparrow (1955) on laminar forced convection heat transfer in entrance region of flat rectangular ducts.

Assume that the velocity and temperature distributions of the flow inside the boundary layer, ϕ and θ, can be represented by third-degree polynomial functions of η as

$$\theta = A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3 \tag{5.32}$$

$$\phi = B_0 + B_1\eta + B_2\eta^2 + B_3\eta^3 \tag{5.33}$$

where the coefficients may be functions of axial coordinate ξ and can be determined by boundary conditions of the problem as specified in Eqs. (5.17a), (5.18a), (5.19a), and (5.19b) for ϕ and in Eqs. (5.17c), (5.18b), (5.20a), and (5.20b) for θ. It follows that

$$\theta = 1 - \frac{6}{(4+j\Delta_T)}\left(\frac{1-\eta}{\Delta_T}\right) - \frac{3j\Delta_T}{(4+j\Delta_T)}\left(\frac{1-\eta}{\Delta_T}\right)^2 + 2\left(\frac{1+j\Delta_T}{4+j\Delta_T}\right)\left(\frac{1-\eta}{\Delta_T}\right)^3 \tag{5.34}$$

$$\phi = \frac{G}{4}\frac{\Delta^2}{(4+j\Delta)}\left[\left(\frac{1-\eta}{\Delta}\right) - 2\left(\frac{1-\eta}{\Delta}\right)^2 + \left(\frac{1-\eta}{\Delta}\right)^3\right] +$$
$$\frac{\phi_c}{(4+j\Delta)}\left[6\left(\frac{1-\eta}{\Delta}\right) + 3j\Delta\left(\frac{1-\eta}{\Delta}\right)^2 - 2(1+j\Delta)\left(\frac{1-\eta}{\Delta}\right)^3\right] \tag{5.35}$$

Δ and Δ_T are functions of ξ, yet to be determined. A knowledge of Δ and Δ_T will lead to the heat transfer characteristics of the flow in the channel.

Vorayos and Kanury (2000b) have shown that if the channel is long enough to yield full development of the flow thermally and hydrodynamically, the resultant Poiseuille flow velocity distribution in the fully developed region is given for a UWT heating boundary condition in the channel by

$$\phi = \frac{1}{1+j}\frac{Gr}{Re_D}\frac{\left(1-\eta^2\right)}{8} \tag{5.36}$$

Integration of this distribution give the mass flow rate in the channel which has to be ξ, including within the developing region. Therefore, the induced Reynolds number Re_D at the entrance can be obtained as a function of the wall heating condition embodied in the UWT Grashof number. Introducing Eqs. (5.36) into (5.25), this relation is

$$Re_D = \frac{Gr}{K} \tag{5.37}$$

where K is 12 for a parallel-plate channel and 32 for a circular tube.

The velocity distribution in the fully developed region Eq. (5.36) and the associated mass flow rate Eq. (5.37) reveal that the velocity at the axis (core velocity) increases from unity at the entrance $\xi = 0$ to 1.5 (for a parallel-plate channel) or 2 (for a tube of circular cross-section) in the fully developed region. Hence, flow at the axis is accelerated in the developing region.

Three ordinary equations for the three ξ-dependent variables, ϕ_c, Δ, and Δ_T then emerge from the introduction of the velocity and temperature profiles, Eq. (5.34) and (5.35) into the integral conservation equations; namely, the mass equation given by Eqs. (5.26), momentum equation given by Eq. (5.30), and energy equation given by Eq. (5.31). Omitting details of this substitution and horrendously tedious algebraic manipulation, these three equations are given as follows:

Conservation of mass

$$\frac{G\Delta^2}{4}\left(j\frac{\Delta}{12}-F_1\right)+\phi_c(F_2-F_3)=\frac{4+j\Delta}{j+1}\left(1-\phi_c(1-\Delta)^{1+j}\right) \tag{5.38}$$

where F_1, F_2, and F_3 are functions of Δ given by

$$F_1=(-\Delta)^{j+1}\left(\frac{1}{j+2}-\frac{2}{j+3}-\frac{1}{j+4}\right) \tag{5.38a}$$

$$F_2=j\left(\frac{5}{2}\Delta+\frac{1}{2}\Delta^2\right) \tag{5.38b}$$

$$F_3=(-\Delta)^{j+1}\left(\frac{6}{j+2}+\frac{3j\Delta}{j+3}-\frac{2+2j\Delta}{j+4}\right) \tag{5.38c}$$

Conservation of Momentum

$$\frac{d}{d\xi}\left(\frac{G^2F_4+G\phi_cF_5+\phi_c^2F_6}{(4+j\Delta)^2}-F_7\right)+\left(\frac{1-\phi_c}{j+1}\right)\frac{d\phi_c}{d\xi}=GF_8+\phi_cF_9+\frac{Gr}{Re_D}\left\{j\Delta F_{10}-(-\Delta)^{j+1}F_{11}\right\} \tag{5.39}$$

where F_4 to F_{11} are also functions of Δ given by

$$F_4=j\frac{1}{16\times105}\Delta^5+(-1)^{j+2}\frac{J_1}{16}\Delta^{j+5} \tag{5.39a}$$

$$F_5=j\left(\frac{19}{210}\Delta^3+\frac{13}{840}\Delta^4+\frac{1}{120}\Delta^5\right)+(-1)^{j+2}\frac{J_2}{2}\Delta^{j+4} \tag{5.39b}$$

$$F_6=j\left(\frac{272}{35}\Delta+\frac{117}{35}\Delta^2-\frac{74}{35}\Delta^3-\frac{2}{7}\Delta^4\right)+(-1)^{j+2}J_3\Delta^{j+1} \tag{5.39c}$$

$$F_7=\frac{1}{j+1}\left(\phi_c-\phi_c^2(1-\Delta)^{1+j}\right) \tag{5.39d}$$

$$F_8 = -\frac{\Delta}{(4+j\Delta)} \tag{5.39e}$$

$$F_9 = -\frac{24}{\Delta(4+j\Delta)} \tag{5.39f}$$

$$F_{10} = 1 - \frac{1}{2}\left(\frac{6}{4+\zeta\Delta}\right)\frac{1}{\zeta} - \frac{1}{3}\left(\frac{3\zeta\Delta}{4+\zeta\Delta}\right)\frac{1}{\zeta^2} + \frac{1}{2}\left(\frac{1+\zeta\Delta}{4+\zeta\Delta}\right)\frac{1}{\zeta^3} \tag{5.39g}$$

$$F_{11} = \frac{1}{j+1} - \frac{1}{j+2}\left(\frac{6}{4+j\zeta\Delta}\right)\frac{1}{\zeta} - \frac{1}{j+3}\left(\frac{3j\zeta\Delta}{4+j\zeta\Delta}\right)\frac{1}{\zeta^2} + \frac{2}{j+4}\left(\frac{1+j\zeta\Delta}{4+j\zeta\Delta}\right)\frac{1}{\zeta^3} \tag{5.39h}$$

Conservation of Energy

$$\frac{d}{d\xi}\left\{\frac{G\left(jF_{12} + (-1)^{j+2}\Delta^{j+3}F_{13}\right) + \phi_c\left(jF_{14} + (-1)^{j+2}\Delta^{j+1}F_{15}\right)}{(4+j\Delta)(4+j\zeta\Delta)}\right\} = \frac{24}{\Pr}\frac{1}{(4+j\zeta\Delta)\zeta\Delta} \tag{5.40}$$

where F_{12} to F_{15} are also functions of Δ given by

$$F_{12} = \left(\frac{1}{120}\Delta^3\right)\frac{1}{\zeta^3} + \left(\frac{1}{120}\Delta^4 - \frac{1}{336}\Delta^5\right)\frac{1}{\zeta^2} + \left(-\frac{1}{20}\Delta^3 - \frac{1}{80}\Delta^4 + \frac{1}{140}\Delta^5\right)\frac{1}{\zeta} + \frac{1}{12}\Delta^3 + \left(\frac{1}{48}\Delta^4 - \frac{1}{120}\Delta^5\right)\zeta \tag{5.40a}$$

$$F_{13} = J_4\frac{1}{\zeta^3} + J_5\frac{1}{\zeta} + J_6 \tag{5.40b}$$

$$F_{14} = \left(-\frac{6}{35}\Delta - \frac{1}{14}\Delta^2 + \frac{3}{70}\Delta^3\right)\frac{1}{\zeta^3} + \left(-\frac{6}{35}\Delta^2 + \frac{1}{35}\Delta^3 + \frac{3}{70}\Delta^4\right)\frac{1}{\zeta^2} + \left(\frac{12}{5}\Delta + \frac{7}{5}\Delta^2 - \frac{16}{35}\Delta^3 - \frac{3}{28}\Delta^4\right)\frac{1}{\zeta} + \left(-6\Delta - 2\Delta^2 + \frac{3}{5}\Delta^3\right) + \left(6\Delta - \frac{1}{10}\Delta^3 + \frac{3}{20}\Delta^4\right)\zeta + \left(2\Delta^2 + \frac{1}{10}\Delta^3\right)\zeta^2 + \left(-\frac{3}{5}\Delta^3 - \frac{3}{20}\Delta^4\right)\zeta^3 \tag{5.40c}$$

$$F_{15} = J_7\frac{1}{\zeta^3} + J_8\frac{1}{\zeta} + J_9 + J_{10}\zeta^{j+1} \tag{5.40d}$$

ζ is the ratio of thermal to viscous boundary layer thicknesses, $\zeta \equiv \Delta_T/\Delta$. J_1 to J_{10} are constants and listed in Table 5.1.

The viscous and thermal boundary layers are quite thin, especially in the vicinity of the inlet. Thereby, the high-order terms of Δ and Δ_T in the foregoing equations are ignorably small. One can thus obtain approximate solution for Δ and ϕ_c, as dependent on ξ for small values of ξ, from Eqs. (5.38) to (5.39). With a few steps of calculation under an additional assumption that ζ is close to unity in the vicinity of the inlet, it can be shown that

$$\Delta = \sqrt{C_1\left(1 - e^{-C_2\xi}\right)} \tag{5.41}$$

$$\phi_c = \frac{1}{\left(1 - C_3\xi\right)} \tag{5.42}$$

Values of the constants C_I (i = 1, 2, and 3) are listed in Table 5.2. These small-ξ results serve as the starting solutions in dealing with Eqs. (5.38), (5.39), and (5.40) to obtain ϕ_c, Δ and Δ_T at larger values of ξ.

At this juncture, use of a numerical method is unavoidable to solve the system of Eqs. (5.38)-(5.40). That this system is complex is an obvious understatement. We choose a simple numerical approach namely the Runge-Kutta method. All numerical calculation work involved was done on a solver of an ordinary differential equation (ODE45) of the commercially available software MATLAB. Equations. (5.41) and (5.42) are used to start the calculation in the domain close to the inlet to avoid dealing with the associated with the inlet condition; $\Delta(\xi = 0)$ and Δ_T $(\xi = 0)$ equal to zero.

Table 5.1 Values of constant J's appearing in Eqs. (5.39) and (5.40)

	Parallel-plate channel j = 0	Circular tube j = 1
J_1	1/105	1/280
J_2	19/105	37/420
J_3	272/35	11/2
J_4	1/210	1/336
J_5	-1/20	-1/40
J_6	1/12	1/30
J_7	-6/35	-1/10
J_8	12/5	1
J_9	-6	-8/5
J_{10}	6	8/5

Table 5.2 Values of constant C's appearing in Eqs. (5.41) and (5.42)

	Parallel-plate channel j = 0	Circular tube j = 1
C_1	4	3/2
C_2	35/6	140/9
C_3	3/8	3/4

The heat transfer characteristics of the region can be observed in a variety of ways as seen from existing literature. For the sake of comparison with the available works, we define here local and average Nusselt numbers as follows:

$$Nu_b = \frac{h_b D}{k} = \frac{\left(\frac{\partial T}{\partial r}\right)_{r=D/2}}{(T_w - T_b)} \tag{5.43}$$

$$\overline{Nu}_b = \frac{\bar{h}_b D}{k} = \frac{D}{xk}\int_0^x h_b dx \tag{5.44}$$

5.5 Results and Discussions

ODE45, a Runge-Kutta scheme in MATLAB, is deployed to solve Eqs. (5.38)-(5.40). The start up solution given by Eqs. (5.41) and (5.42) is used up to $\xi=10^{-7}$ after which the Runge-Kutta scheme takes over the calculation.

5.5.1 Velocity and temperature distributions

The viscous and thermal boundary layers obtained as functions of x from such calculations are plotted Figs. 5.2 (for a channel) and 5.3 (for a tube). Velocity and temperature distribution at three different cross-sections along the axis are shown in Figs. 5.4 for the channel and 5.5 for the tube. The distance ξ required for the thermal boundary layer(s) to reach the axis of symmetry (i.e., $\delta = R$) is 0.0145 for parallel-plate channel and 0.0159 for circular tube. At this ξ-value, the

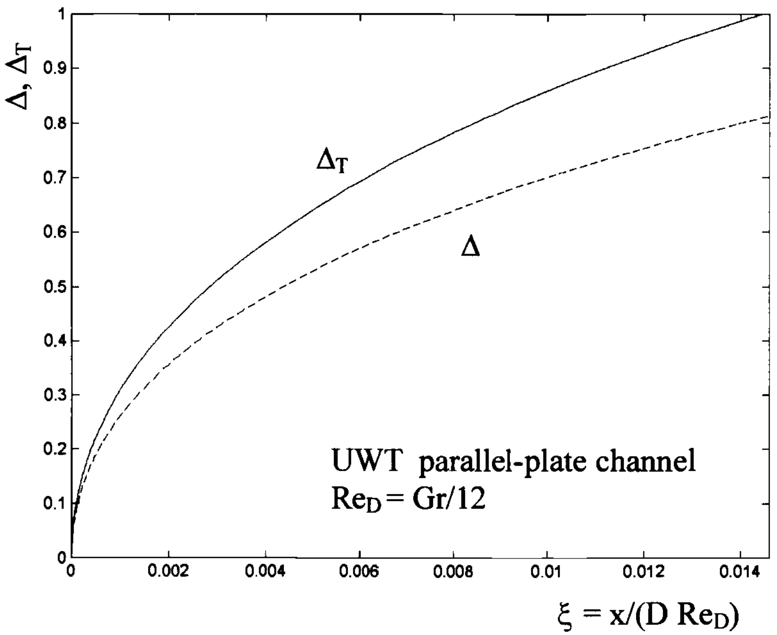

Figure. 5.2 Developing viscous and thermal boundary layers, case of UWT parallel-plate channel

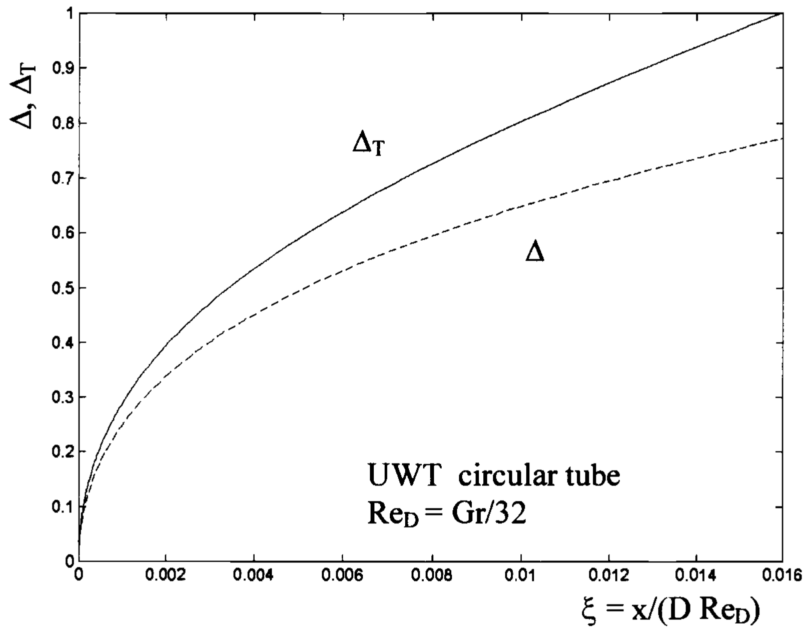

Figure. 5.3 Developing viscous and thermal boundary layers, case of UWT circular tube

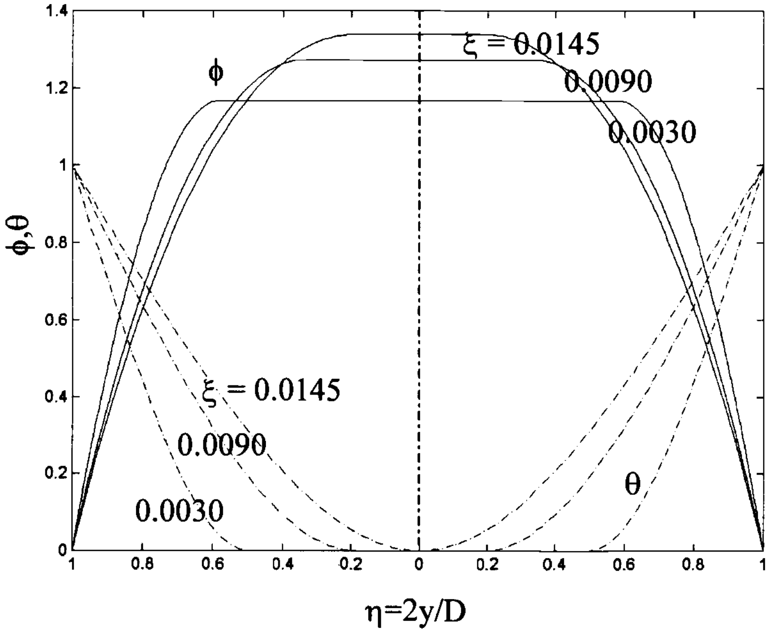

Figure 5.4 Velocity and temperature distribution along the flow axis, case of UWT parallel-plate channel

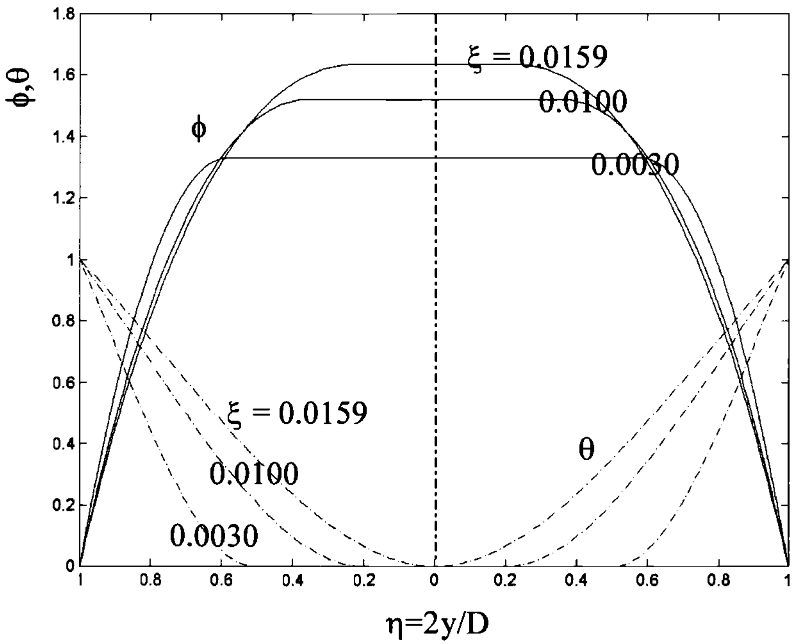

Figure 5.5 Velocity and temperature distribution along the flow axis, case of UWT circular tube

velocity boundary layer has yet to penetrate about a 1/4 (for a channel) to 1/3 (for a tube) of the cross-section. However, for a fluid of $Pr \approx 1$, the flow will develop quite soon after the thermal development. This can be seen in Figs. 5.4 and 5.5 – the velocity at the axis at the location of thermal development is quite close to the fully developed values of 1.5 for a channel and 2 for a tube.

At any cross section, the velocity and the temperature distributions inside the boundary layers are approximated by third degree polynomials. It can be noted from Eqs. (5.34) and (5.35) that the velocity profile comprises of two distinct parts; one with G as a multiplier and the other with ϕ_c as a multiplier. These two parts deal with natural and forced convection respectively. In fact, the well-known polynomial representations of the velocity distributions in free convection and forced convection can be recovered from Eq. (5.34).

If the core flow is small and buoyancy is large, the velocity distribution is little influenced by the core flow. In the limit of $\phi_c \to 0$, the velocity distribution tends to be

$$\phi = \frac{Gr}{4}\frac{\Delta^2}{(4 + j\Delta)}\left[\left(\frac{1-\eta}{\Delta}\right) - 2\left(\frac{1-\eta}{\Delta}\right)^2 + \left(\frac{1-\eta}{\Delta}\right)^3\right]$$

which is the same as the natural convective velocity distribution in the boundary layer over a UWT vertical parallel plate (see Thomas, 1999) in which the velocity profile are shown to be double-peaked.

In contrast, if the core flow dominates over that of buoyancy (see Sparrow, 1955), a forced-convection-like velocity distribution will result.

$$\phi = \frac{\phi_c}{(4+j\Delta)}\left[6\left(\frac{1-\eta}{\Delta}\right)+3j\Delta\left(\frac{1-\eta}{\Delta}\right)^2-2(1+j\Delta)\left(\frac{1-\eta}{\Delta}\right)^3\right]$$

which is the same as the forced convective velocity distribution in the boundary layer over a parallel-plate channel ($j = 0$) in a laminar flow of free stream speed ϕ_c. The details of upward flow inside the boundary layer thus depend not only on the heat diffusion near the wall(s) but also on the forced-convection-like flow outside the boundary layer. While the heat diffusion near the wall(s) results to a *local* buoyancy (to be reckoned in terms of local Grashof number $Gr_x \equiv g\beta(T_w-T_\infty)x^3/\nu^2$), the forced-convection-like flow outside the boundary layer is the result of overall buoyancy due to the heating in the fully-developed flow regime of our very long channel (to be reckoned in terms of inlet Reynolds number Re_D). These driving forces is also reported by Metai and Eckert (1964) in which the two parameters are used to determine whether the laminar flow within vertical tubes was forced, mixed, or free convective. Hence, in this developing region of a long channel where Eqs. 5.37 holds, the ratio Gr_x / Re_D^2 is much greater than one. Mills (1992) shows that in such the limit the effect of overall buoyancy will dominate over the local buoyancy. This leads us to anticipate the prevailing of the *overall* buoyancy over the *local* buoyancy as soon as the fluid enters channel at x = 0 until it leaves the exit at x = L such that the flow and heat transfer characteristics in the long channel will close to those of channel forced convection. This is confirmed from Figs. 5.4 and 5.5, in which the velocity profiles are not in double-peaked shapes as what one would obtain from the natural convection over the vertical flat plate where Gr_x / Re_D^2 is much smaller than one but, in fact, they are similar to those of channel forced convection.

5.5.2 Pressure variation

According to boundary layer theory, variation of pressure across the cross-section at any axial location is negligible. Pressure is thus independent of the transverse coordinate, r, it is only a function of x. The pressure distribution in terms of P′ ($\equiv P_{\infty 0} - P_\infty = P_{\infty 0} - \rho g x$) is simply determined from Bernoulli equation, Eq. (5.22), applied to the flow outside the boundary layer, i.e., at the axis. The result is shown in Fig. 5.6. The pressure difference rapidly decreases near the entrance. It decreases more rapidly in a circular tube than in a channel. When the flow is fully developed, the pressure drop with x also reaches its fully developed form, i.e., parabolic form for UHF boundary condition and exponential form for UWT boundary condition as shown by Vorayos and Kanury (2000b). The corresponding core velocity ϕ_c is calculated by the Bernoulli equation given in Eq. (5.22) and shown in Fig. 5.6. As the fluid advances, the boundary layers grow and the core region gets smaller resulting to the acceleration of nondimensional core velocity from 1 at the inlet to a fully developed value of 1.5 for a parallel-plate channel and 2 for a circular tube at some location x sufficiently far from the channel entrance. The acceleration of the core flow resulting from the pressure difference between the fluid inside and outside the channel is similar to what occurs

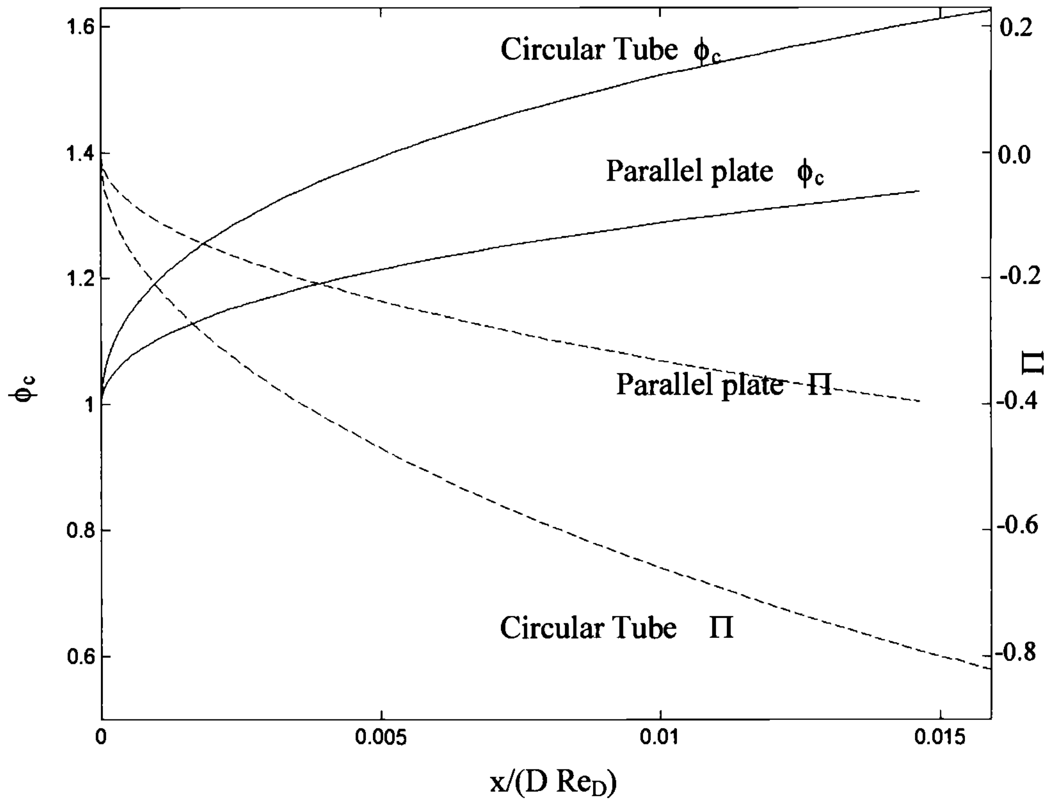

Figure 5.6 Core velocity and corresponding pressure distribution along flow axis; Re_D is Gr/12 for a parallel-plate channel and Gr/32 for a circular tube.

in channel forced convective flow where the fluid is forced into the channel by the specified inertia.

5.5.3 Heat transfer

The heat transfer results determined are as shown in Figs. 5.7 and 5.8. for the case of parallel plates and in Figs. 5.9 and 5.10 for the case of circular tube. Local and average Nusselt numbers defined in Eqs. (5.43) and (5.44) are plotted as functions of $x/(D\ Re_D Pr)$; an inverse of the Graetz number. Vorayos and Kanury (2000b) have shown that the Nusselt number of fully developed flow within the long vertical UWT parallel-plate and circular tube is not dependent on whether the flow is driven by natural or forced convection. Local Nusselt number based on wall-to-bulk temperature difference and physical D, Nu_b, is found in the fully developed flow to be a constant in both cases and equal to 3.770 and 3.657 for the parallel-plate channel and circular tube respectively. As seen in Fig. 5.7 and 5.9, the local Nu_b is large near the entrance and gradually decreases with an increase in x to asymptotically approach the fully developed constant value of 3.770 for the parallel-plate channel and 3.657 for circular tube.

For the case of parallel plates shown in Figs. 5.7 and 5.8, our developing flow solutions of pure natural convection agree fairly well with the forced convection results of Kays, Stephan (1959), and Hwang and Fan (1964) (the first and the third are taken from Rohsenow and Hartnett's Handbook of Heat Transfer, 1971). Kays solved the problem of forced convection with the fully developed

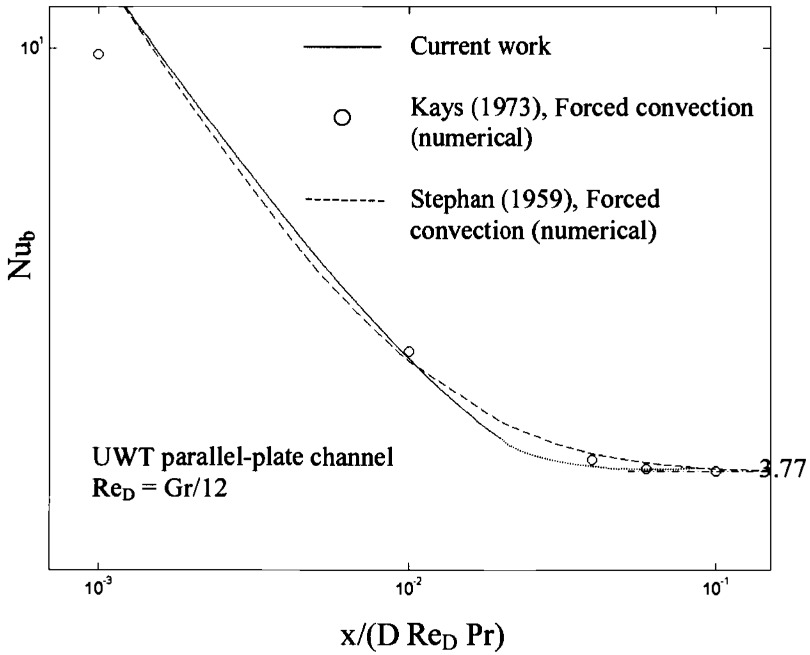

Figure 5.7 Local Nusselt number, Nu_b, as a function of $x/(D\ Re_D\ Pr)$, parallel-plate channel

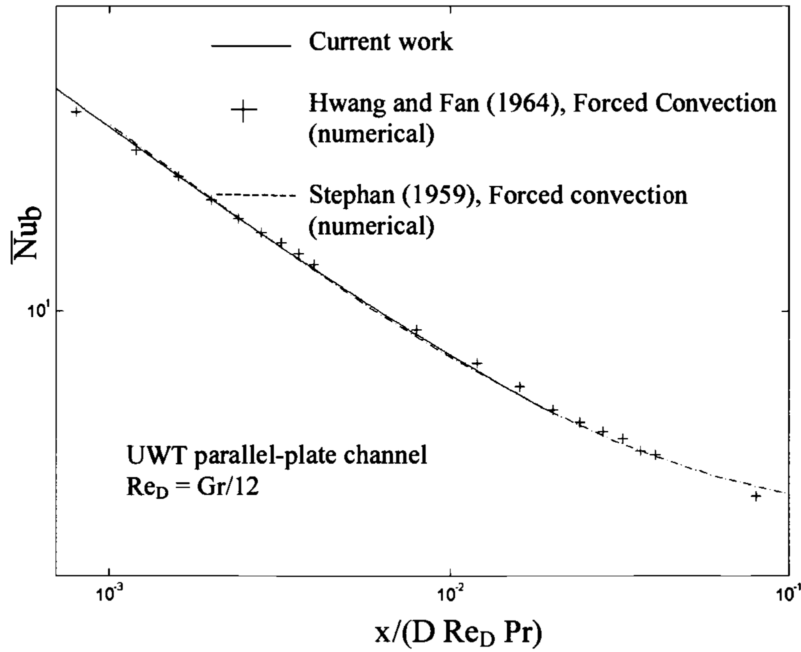

Figure. 5.8 Average Nusselt number, $\overline{Nu}_b$, as a function of $x/(D\ Re_D\ Pr)$, paralle-plate channel

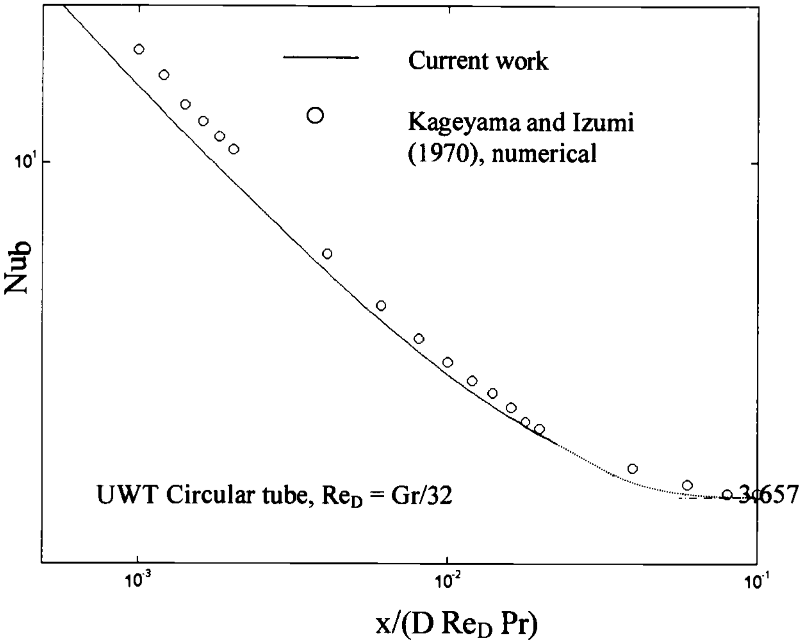

Figure 5.9 Local Nusselt number, Nu_b, as a function of $x/(D\ Re_D\ Pr)$, circular tube

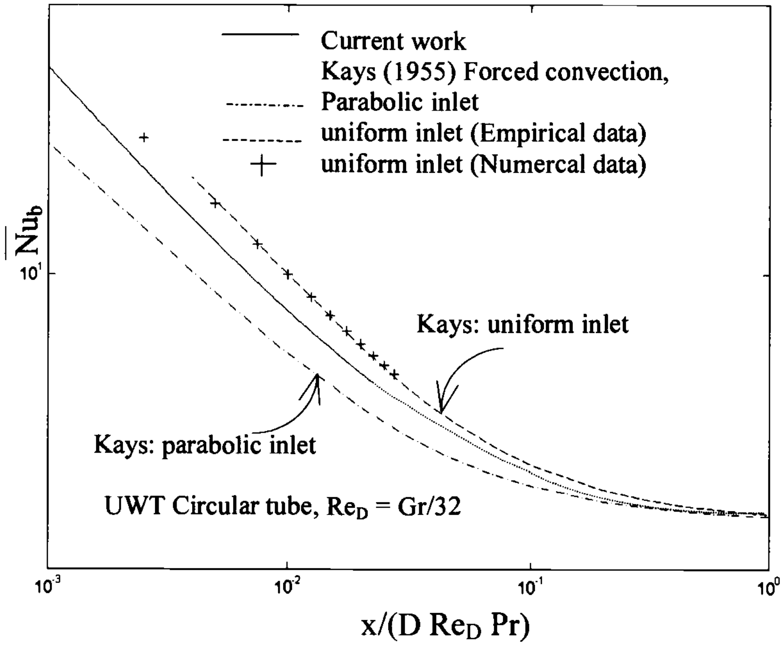

Figure 5.10 Average Nusselt number, $\overline{Nu_b}$, as a function of $x/(D\ Re_D\ Pr)$, circular tube

parabolic inlet velocity distribution. When compared with the results of Kays, our solution is slightly over-predicted. This departure is reasonable since the parabolic velocity inlet yields lesser intensity of heat transfer than does a uniform inlet velocity distribution. The results of Stephan (1959) and Hwang and Fan (1964) based on the assumption of a uniform inlet velocity shows an especially close agreement with our predictions.

Kays (1955) provided the numerical work in terms of average Nusselt number from x = 0 to x as a function of Graetz number [Re_D Pr/ (x/D)] for forced convection in circular tube with a parabolic inlet velocity and give the its best-fitted expression for it as

$$\overline{Nu_b} = 3.66 + \frac{0.104\left(\dfrac{Re_D\,Pr}{x/D}\right)}{1 + 0.016\left(\dfrac{Re_D\,Pr}{x/D}\right)^{0.8}} \tag{5.45}$$

This relation is usually given to in most standard textbooks of convective heat transfer. It is this relation which is plotted in Fig. 5.10 for the sake of comparison over present work. Our result is within 8 per cent when compared with Eq. (5.45) in a range of x/(D Re_D Pr) larger than 0.001, i.e. Graetz number lower than 1000.

A numerical finite-difference solution of heat transfer in pure natural convection within circular tube is also available in the work of Kageyama and Izumi (1970). As seen in Fig. 5.9, our solution is in good agreement with this work. Some difference is noted at high Graetz number, i.e., the vicinity of the inlet, where the boundary layer theory is not accurate due to the neglect of the essential axial conduction. Our start-up solution used to enter the Runge-Kutta scheme without having to deal with the singularity at $x = 0$ may also possibly contribute to this departure. In spite of this, it can be said that the heat transfer characteristics in the developing region of natural convection flow in both long parallel plates and circular tube are not only similar but also quite close to those in the developing region of forced convection.

As discussed in previous sections that the mechanisms of forced convection dominate over those of natural convection, the observation is consolidated again by the fact that the heat transfer characteristics of the natural convection in vertical channels are close to those of channel forced convection. Hence, despite the fact that the flow is naturally induced by wall heating condition, its flow and heat transfer characteristics in terms of the velocity and temperature distributions and Nusselt numbers are similar to those of forced convective flow. The distinction between channel forced and natural convection is that, in channel forced convection, the core flow is forced into the channel by a specified inertia in terms of inlet Reynolds number Re_D but, in channel natural convection, the core flow is naturally induced by the condition of heating wall. As such, the inlet Reynolds number in natural convection is a function of the prescribed wall heating condition.

It should be emphasized again that the channel has to be sufficiently long to assure fully developed flow at the upper end and the dominance of overall buoyancy over local buoyancy. Inlet Re_D of the flow is then governed by the flow in the fully developed region and equal to Gr/K where K is a constant 12 for a parallel-plate channel and 32 for a circular tube. If Re_D exceeds this limit, the flow is then forced into the geometry rather than naturally induced. The flow problem then is categorized either as "mixed convection" or even as pure forced convection if the Re_D is significantly large.

5.5.4 Entry length

From the boundary layer thickness determined as a function of x, we can now approximate the entry length. In spite of the fact that the boundary layer theory does not allow the calculation to proceed beyond where the thermal boundary layers merge at the axis of symmetry, by extrapolating the $Nu_b(x)$ solution curve in Figs. 5.7 and 5.9 to asymptotically approach the fully developed flow Nu_b value, we can determine the entry length (or developing length) L_e to be approximately

$$\frac{L_e}{D} = 0.05\,\mathrm{Re}_D\,\mathrm{Pr} \tag{5.46}$$

for both the parallel-plate channel and circular tube. This is quite the same relation as that is known in channel forced convection flow (Langhaar, 1942) which, once again, support the fact that the flow is dominated by the forced-convection-like mechanisms. Since Re_D is a function of Gr, Eq. (5.37), the above equation for entry length becomes

$$\frac{L_e}{D} = \frac{0.05}{K}\,\mathrm{Gr}\,\mathrm{Pr} \tag{5.47}$$

Again, recall that K is a constant 12 for a parallel-plate channel and 32 for a circular tube. Hence, if the aspect ratio L/D of the vertical UWT channel is larger than this L_e/D, the channel is long enough for the flow to become fully developed at the upper end such that the heat transfer and flow characteristics are then determined by the forced-convection mechanisms described in this paper. In other word, for the channel with specified aspect ratio to be sufficiently long to assure a fully developed flow, the wall heating condition has to satisfy the requirement that

$$\mathrm{Gr} < \frac{1}{\mathrm{Pr}}\frac{K}{0.05}\frac{L}{D} \tag{5.48}$$

for which the present results are applied.

5.6 Conclusion

Laminar natural convection heat transfer of air in the entry (developing) region of a long UWT parallel-plate channel and a circular tube is studied in this paper to gain an understanding of the physics involved in the evolution of thermal and viscous boundary layers. The problem is studied from the view-point of the boundary layer theory with an integral method of solving the governing equations.

The resulting ordinary differential equations are highly nonlinear and could only be solved numerically although with a simple commercially available solver known as MATLAB. The solutions agree reasonably well with the existing literature in spite of the fact that velocity and temperature distributions within the boundary layer(s) are approximated by polynomials merely of third-degree.

It is evident that the flow which is induced by buoyancy within a long vertical UWT heated channel becomes fully developed at an axial distance sufficiently downstream from the entrance. It is this fully developed flow that determines the flow rate which has to remain invariant with x even in the entry region. Both the entry flow and the fully developed flow follow heat transfer physics which is identical to the physics of forced convection heat transfer.

As the boundary layer develops within a long channel, it interacts with the core flow, which is unaffected by channel wall friction and heating, such that the fluid in the core is entrained across the viscous boundary layers inducing the core flow to accelerate with the axial distance. Fluid mass is drawn into the channel at a rate dictated by the flow in the fully developed regime. This mechanism is similar to that of forced convection in which the core flow is forced to accelerate as the boundary layer(s) develop(s). In our long tubes, this forced convection mechanism dominates the natural convection mechanism. This observation is affirmed by the excellent agreement of the Nusselt numbers from the present study with those appearing in the existing duct-flow- forced-convection literature. The solutions compare well also with the existing numerical solutions of the forced convection problem.

Our present model also suggests that if the duct inlet Reynolds number Re_D smaller than Gr/K, where K is a constant 12 for a parallel-plate channel and 32 for a circular tube, then the results of this paper become inapplicable, for then the flow is either (a) not fully developed in the (short) tube or (b) the natural convection in the boundary layer(s) dominates to yield a velocity distribution development significantly different from the manner of development described in this paper pertaining to long ducts in which the flow gets to be fully developed. The entry length has been reasonably predicted and is in excellent agreement with the result of forced-convection-duct-flow available in the literature. It can be then deduced that, for the channel with the specified aspect ratio L/D, the wall heating condition should not exceed some particular limit for the flow to be fully developed.

5.7 References

1. Aung, W., 1972, "Fully Developed Laminar Free Convection Between Vertical Plates Heated Asymmetrically," International Journal of Heat and Mass Transfer, Vol. 15, pp. 1577-1580.

2. Aung, W., Fletcher, L.S., And Sernas, V., 1972, "Developing Laminar Free Convection Between Vertical Plates With Asymmetric Heating," International Journal of heat and Mass Transfer, Vol. 16, pp. 2293-2308.

3. Bar-Cohen, A., and Rohsenow, W.M., 1984, "Thermally Optimum Spacing of Vertical Natural Convection Cooled Parallel Plates," ASME Journal of Heat Transfer, Vol. 106, pp. 116-123.

4. Bejan, A., 1995, "Convection Heat Transfer," Second Edition, John Wiley & Sons, New York.

5. Bodia, J. R., and Osterle, J.F., 1962, "The Development of Free Convection Between Heated Vertical Plates," ASME Journal of Heat Transfer, Vol. 84, pp 40-44.

6. Elenbaas, W., 1942 a, "Heat Dissipation of Parallel Plates by Free Convection," Physica, Vol. 9, pp. 1-28.

7. Elenbaas, W., 1942 b, "The Dissipation of Heat by Free Convection: The Inner Surface of Vertical Tubes of Different Shapes of Cross-Section," Physica, Vol. 9, pp. 865-874.

8. Hewitt, G.F., 1990, "Hemisphere Handbook of Heat Exchanger Design," Hemisphere publishing, New York.

9. Kays, W.M., 1955, "Numerical Solutions for Laminar-Flow Heat Transfer in Circular Tubes," Transaction of the ASME, Vol. 77, pp. 1265-1274.

10. Langhaar, H.L., 1942, "Steady Flow in the Transition Length of a Straight Tube," Journal of Applied Mechanics, Vol. 9, pp. A55-A58.

11. Metais, B. and Eckert, E.R.G., 1964, "Forced, Mixed, and Free Convection Regimes," Journal of Heat Transfer, Vol. 86, pp. 295-296.

12. Mills, A.F., 1992, "Basic Heat and Mass Transfer," Irwin, Chicago.

13. Ramanathan S., and Kumar R. , 1991, "Correlations for Natural Convection between Heated Vertical Plates, " ASME Journal of Heat Transfer, Vol. 113, pp. 97-107.

14. Rohsenow, W.M., and Hartnett, J.P., 1973, "Handbook of Heat Transfer," McGraw-Hill, New York.

15. Shampine, L.F. and Reichelt, M.W., 1997, "The Matlab ODE suite," SIAM Journal of Scientific Computing, Vol. 18, No. 1, pp. 1-22.

16. Spalding, D.B., 1954, "Mass Transfer in Laminar Flow," Proc. Roy. Soc. A, Vol. 221, pp. 78-99

17. Sparrow, E.M., 1955, "Analysis of Laminar Forced-Convection Heat Transfer in Entrance Region of Flat Rectangular Ducts," NACA TN 3331.

18. Thomas, L.C., 1999, "Heat Transfer – Professional Version," Second Edition, Prentice-Hall, New Jersey.

19. Vorayos, N. and Kanury, A.M. 2000a, "Validation of a Series Solution Method on Fully Developed Laminar Forced Convection within a Uniformly Heated Vertical Parallel-Plate Channel and Circular Tube," in preparation.

20. Vorayos, N. and Kanury, A.M. 2000b, "A Series Solution Method for Fully Developed Laminar Natural Convection within Uniformly Heated Vertical Parallel-plate Channels and Circular Tubes ," in preparation.

21. Willie, R. H., 1996, "Fully Developed Laminar Natural Convection in a Vertical Parallel Plate Channel with Symmetric Uniform Wall Temperature," M.S.Thesis, Oregon State University.

CHAPTER 6: SUMMARY AND FUTURE WORK

6.1 Summary of the Dissertation

Heat transfer due to convection within a uniformly heated vertical channel has been a topic of many theoretical and experimental studies due to its importance in practical applications. When the amount of heat to be transferred is large, forced convection is usually favored over natural convection. In applications where the heating/cooling rates and loads are not large, natural convection is preferable since it requires no prime-movers. The equations governing channel natural convection are coupled, and so much more complex. Closed form solutions become rarely possible. However, many experimental and numerical studies exist in the literature attempting to predict heat transfer and flow characteristics of natural convection within a vertical channel.

In this dissertation, we sought to gain a clear physical understanding of the mechanisms of naturally induced flow and heat transfer in vertical channels and tubes. A series of problems have been considered beginning with the flow in fully developed regime and consequently advancing to the developing flow regime. Three stand-alone papers have been prepared and presented in Chapters 3, 4, and 5. A series solution method is proposed in Chapter 3 and validated for fully developed laminar *forced* convective flow within a uniformly heated vertical parallel-plate channel and tube.

Using the series solution method, Chapter 4 is devoted to the study of the flow and heat transfer characteristics of *steady-state fully developed laminar natural convection* within long vertical uniformly heated parallel-plate channels and circular tubes. The velocity and temperature distributions of the fluid are obtained; and, from them, the heat transfer characteristics are deduced. An inlet Reynolds number, representing mass flow rate of the channel, is determined as one of the unknowns. In the fully developed regime, no matter whether the flow is forced or naturally induced, the velocity and temperature distributions are same or similar and local Nusselt number, Nu_b, are very close. Thus, we condlude that the flow in the fully developed region is altogether independent of how the it is driven.

Following the study of fully-developed channel natural convection we investigated, as reported in Chapter 5, flow in the developing regime of laminar natural convection within a very long vertical UWT parallel-plate channel or tube. The flow rate obtained from the fully developed flow analysis of Chapter 4 is employed as a specified input into the problem of flow and heat transfer in the developing region near the inlet. Closed-form solutions for the developing flow are impossible to this problem of natural convection analog of the Graetz problem in forced flow in the tube. An integral method of solution is sought to arrive at the development of thermal and viscous boundary layers in a form of ordinary differential equations. Since even this system of ordinary differential equations are highly nonlinear, a Runge-Kutta numerical integration scheme is engaged to solve them. The results thus obtained show that the heat transfer (Nusselt number) is a function of the axial distance x from the inlet plane, asymptotically approaching the fully developed flow results which confirm those obtained from Chapter 4.

The velocity profile distribution across the channel at locations close to the entrance reveals no resemblance to the velocity profile in the natural convection boundary layer over a single vertical plate, i.e., we found no double-peaked velocity distribution in the channel as suggested by Elenbaas. Although the flow in the channel is definitely driven by buoyancy, it is the fully developed regime that determines the flow rate. The inertia of the fluid thus drawn into the channel is the mechanism that dominates the character of the local natural convection in the entrance region. This is as though the flow is forced.

Interactions between the developing boundary layers and the core of the flow near the channel's axis of symmetry are found to be no different from those occuring in the developing channel *forced convective flow*. The entrance length of channel natural convection is also discovered to be the same as that in channel forced convection. However, boundary layer theory is applicable up to the point before the boundary layers reach the axis or plane of symmetric only. For fluids whose Pr is of the order of unity, e.g: air, viscous and thermal development lengths are not significantly far apart.

6.2 Future Work

The studies presented in this dissertation are intended to be used for fluid whose Pr is of the order of unity, e.g: air. Extension of this work for fluids whose Pr is much larger or lesser than one is not now foreseen but possible.

The parallel-plate channel problems described here deal with symmetric heating boundary heating conditions. Extension of the studies to asymmetric boundary conditions is possible.

The work described in Chapter 5 is for a problem in which the channel wall is kept at a prescribed uniform wall temperature, (i.e., the UWT condition). The problem has yet to be solved for the boundary condition of uniform wall heat flux (i.e., UHF condition).

Nonuniform wall heating boundary conditions appear to be important in a number of newly evolving applications. Extension of the present work to such problem seems possible but probably very difficult.

Also related to Chapter 5, if the UWT channel is not sufficiently long to allow the flow to fully develop, the mass flow rate within the channel becomes one of the unknowns of the problem in which the flow is not fully developed at the exit. The work of Chapter 5 then needs a large effort to adapt to such short channels. The physics will most likely involve the local natural convection (showing double peak velocity distribution). This is the problem of short, wide, and intensively heated channel with small Gr Pr D/L mentioned in Chapter 1. The flow characteristics over the channel wall are then expected to asymptotically reach those of laminar natural convection over a single vertical plate. Work is yet to be done to verify and quantify these expected physical features of flow and heat transfer in short wide channels.

6.3 Closure

Thus, in the current study, we have reported the flow and heat transfer characteristics of laminar natural convection of air within long uniformly heated vertical parallel-plate channels and circular tubes as obtained from a combination of mathematical techniques. Our accomplishments are:

- A series solution method is proposed and shown to yield acceptably accurate results offering an alternative way to solve fully developed channel forced and natural convection problems with UWT and UHF boundary conditions.
- Eventhough natural convection in vertical channels and ducts is driven by buoyancy, the results in the fully developed region are very nearly the same as those for forced convection channel flow. These results are summarized once again in Table 6.1.
- Laminar natural convection heat transfer in the developing region of a long UWT channel and tube is found to be quite the same as that in laminar forced convection. Additionally, the entry length for forced and naturalconvection

within UWT parallel-plate channel and circular tube is reasonably approximated as

$$\frac{L_e}{D} = \frac{0.05}{K} \mathrm{Gr}\,\mathrm{Pr}$$

where K is a constant 12 for a parallel-plate channel and 32 for a circular tube.

Table 6.1 Comparison of local Nusselt number $Nu_b \equiv h_b D/k = \dot{q}''_w D/[(k(T_w - T_b)]$ for fully developed laminar forced and natural convection within uniformly heated parallel-plate channels and tubes. D is channel spacing or tube diameter.

Geometry	Parallel-Plate Channel		Circular Tube	
Wall heating condition	UWT	UHF	UWT	UHF
Forced Convection	3.770	4.118	3.657	4.363
Natural Convection	3.770	4.118	3.657	4.363

BIBLIOGRAPHY

1. Aihara, T., and Maruyama S., 1986, "Laminar Free Convective Heat Transfer in Vertical Uniform-Heat-Flux Ducts (Numerical Solutions with Constant/Temperature-Dependent Fluid Properties," Heat Transfer-Japanese Research, Vol. 15, pp. 69-86.

2. Aktan, G., 1996 "Fully Developed Laminar Natural Convection in a Vertical Parallel-Plate Duct with Constant Wall Heat Transfer," Thesis, Oregon State University.

3. Aung, W., 1972, "Fully Developed Laminar Free Convection Between Vertical Plates Heated Asymmetrically," International Journal of Heat and Mass Transfer, Vol. 15, pp. 1577-1580.

4. Aung, W., Fletcher, L.S., And Sernas, V., 1972, "Developing Laminar Free Convection Between Vertical Plates With Asymmetric Heating," International Journal of heat and Mass Transfer, Vol. 16, pp. 2293-2308.

5. Bar-Cohen, A., and Rohsenow, W.M., 1984, "Thermally Optimum Spacing of Vertical Natural Convection Cooled Parallel Plates," ASME Journal of Heat Transfer, Vol. 106, pp. 116-123.

6. Bejan, A., 1995, "Convection Heat Transfer," 2nd edition, John Wiley & Sons, New York.

7. Bodia, J. R., and Osterle, J.F., 1962, "The Development of Free Convection Between Heated Vertical Plates," ASME Journal of Heat Transfer, Vol. 84, pp 40-44.

8. Çengel, Y. A., 1998, "Heat Transfer: a Practical Approach," McGraw Hill, New York.

9. Churchill, S. W., and Usagi, R., 1972, "A General Expression for the Correlation of Rates of Heat Transfer and Other Phenomena," Journal of American institute of Chemical Engineering, Vol 18, pp. 1121-1138.

10. Derrick, W.R. and Grossman, S.I., 1987, "A First Course in Differential Equations," 3rd edition, West Publishing, St. Paul.

11. Duffy, D. G., 1998, "Advanced Engineering Mathematics," CRC Press, Boca Raton.

12. Elenbaas, W., 1942 a, "Heat Dissipation of Parallel Plates by Free Convection," Physica, Vol. 9, pp. 1-28.

13. Elenbaas, W., 1942 b, "The Dissipation of Heat by Free Convection: The Inner Surface of Vertical Tubes of Different Shapes of Cross-Section," Physica, Vol. 9, pp. 865-874.

14. Hallman, T. M., 1956, "Combined Forced and Free-Laminar Heat Transfer in Vertical Tubes with Uniform Internal Heat Generation," Transaction of ASME, Vol. 78, pp. 1831-1841.

15. Hewitt, G.F., 1990, "Hemisphere Handbook of Heat Exchanger Design," Hemisphere publishing, New York.

16. Incopora, F.P. and DeWitt, D.P., 1990, "Introduction to Heat Transfer," John Wiley & Sons, New York.

17. Johnson, C. E., 1986, "Evaluation of Correlations for Natural Convection Cooling of Electronic Equipment," Heat Transfer in Electronic Equipment, A Bar-Cohen, ed., AIAA/ASME 4th Thermophysics and Heat Transfer Conference, Boston, June 2-4.

18. Kageyama, M., and Izumi, R.,1970, "Natural Convection in a Vertical Circular Tube," Bull. Japan Soc., Mech. Eng. Vol.13, pp. 382-394.

19. Kays, W.M. and Crawford, M.E., 1980, "Convective Heat and Mass Transfer," 2nd edition, McGraw-Hill, New York.

20. Kays, W.M., 1955, "Numerical Solutions for Laminar-Flow Heat Transfer in Circular Tubes," Transaction of the ASME, Vol. 77, pp. 1265-1274.

21. Kells, L.M., 1960, "Elementary Differential Equations," McGraw-Hill, New York.

22. Kreyszig, E., 1993, "Advanced Engineering Mathematics," 7th edition, John Wiley & Sons, New York.

23. Langhaar, H.L., 1942, "Steady Flow in the Transition Length of a Straight Tube," Journal of Applied Mechanics, Vol. 9, pp. A55-A58.

24. Martin, L., Raithby, G. D., and Yovanovich M. M., 1991, "On the Low Rayleigh Number Asymptote for Natural Convection Through an Isothermal, Parallel-Plate Channel, " Transactions of ASME, Vol. 113, pp. 899-905.

25. Mills, A.F., 1992, "Basic Heat and Mass Transfer," Irwin, Chicago.

26. Miyatake, O., Fujii, T., Fujii, M., and Tanaka, H., 1973, "Natural Conevctive Heat Transfer Between Vertical Parallel Plates-One Plate with a Uniform Heat Flux and the Other Thermally Insulated," Heat Transfer-Japanese Research, Vol. 2, pp. 25-33.

27. Oosthuizen, P.H., and Naylor, D., 1999, "Introduction to Convective Heat Transfer," McGraw-Hill, New York.

28. Rabenstein, A.L., 1975, "Elementary Differential Equations with Linear Algebra," 2nd edition, Academic Press, New York.

29. Rainville, E.D. and Bedient, P.E., 1969, "Elementary Differential Equations," 4th edition, Collier-Macmillan, New York.

30. Raithby, G.D., Hollands, K. G. T., 1975, "A General Method of Obtaining Approximate Solutions to Laminar and Turbulent Free Convection Problems," in: Advances in Heat Transfer, T.F. Irvine, Jr. and J.P. Hartnett, eds., Vol. 11 pp. 265- 315.

31. Ramanathan S., and Kumar R. , 1991, "Correlations for Natural Convection between Heated Vertical Plates, " ASME Journal of Heat Transfer, Vol. 113, pp. 97-107.

32. Rohsenow, W.M., and Hartnett, J.P., 1973, "Handbook of Heat Transfer," McGraw-Hill, New York.

33. Ross, S. L., 1966, "Introduction to Ordinary Differential Equations," Xerox, Lexington.

34. Seban, R.A. and Simazaki, T.T., 1951, "Heat Transfer to a Fluid Flowing Turbulently in a Smooth Pipe With Walls at Constant Temperature," Transactions of the ASME, Vol. 73, pp. 803-809.

35. Shah, R.K. and London, A.L., 1978, " Laminar Flow Forced Convection in Ducts", Academic Press, New York .

36. Shampine, L.F. and Reichelt, M.W., 1997, "The Matlab ODE Suite," SIAM Journal of Scientific Computing, Vol. 18, No. 1, pp. 1-22.

37. Sobel, N., Landis, F., and Mueller, W., 1966, "Natural Convection Heat Transfer in Short Vertical Channels, Including the Effects of Stagger," Proceedings, Third International Heat Transfer Conference, Vol. 2 pp. 121-125.

38. Spalding, D.B., 1954, "Mass Transfer in Laminar Flow," Proc. Roy. Soc. A, Vol. 221, pp. 78-99

39. Sparrow, E.M., 1995 "Analysis of Laminar Forced-Convection Heat Transfer in Entrance Region of Flat Rectangular Ducts," NACA TN 3331, Jan. 1955.

40. Sparrow, E.M., and Gregg, J.L., 1956, "Laminar Free Convection from a Vertical Plate with Uniform Surface Heat Flux," Transactions of ASME, Vol. 78, pp. 435-440.

41. Thomas, L.C., 1999, "Heat Transfer – Professional Version," 2nd edition Prentice Hall, New Jersey.

42. Vorayos, N. and Kanury, A.M. 2000a, "Validation of a Series Solution Method on Fully Developed Laminar Forced Convection within a Uniformly heated Vertical Parallel-Plate Channel and Circular Tube," in preparation.

43. Vorayos, N. and Kanury, A.M. 2000b, "A Series Solution Method for Fully Developed Laminar natural Convection within Uniformly Heated Vertical Parallel-Plate channels and Circular Tubes," in preparation.

44. Vorayos, N. and Kanury, A.M. 2000b, "Developing Laminar Natural Convection of Air within Long Vertical Parallel-Plate Channels and Circular Tubes, " in preparation

45. Webb, B. W., and Hill, D. P.,1989, "High Rayleigh Number Laminar Natural Convection in an Asymmetically Heated Vertical Channel," Transaction of ASME, Vol. 111, pp 649-656.

46. Willie, R. H., 1996, "Fully Developed Laminar Natural Convection in a Vertical Parallel Plate Channel with Symmetric Uniform Wall Temperature," Thesis, Oregon State University.

47. Wirtz, R. A., and Stutzman, R. J., 1982, "Experiments on Free Convection Between Vertical Plates with Symmetric Heating," ASME Journal of Heat Transfer, Vol. 104, pp. 501-507.

48. Yao, L.S., 1983, "Free and Forced Convection in the Entry Region of a Heated Vertical Channel," Int. J. Heat Mass Transfer. Vol. 26, No.1.

APPENDICES

APPENDIX A: RADIUS OF CONVERGENCE OF THE POLYNOMIAL FUNCTION γ_0 FOR UWT PROBLEMS

In Chapter 4, the polynomial function γ_0 used to approximate the temperature distribution in the problem of laminar fully-developed natural convection flow within UWT problem is

$$\gamma_o(\eta) = C_0 + C_1\eta + C_2\eta^2 + \ldots$$

for which the coefficients C_n are solved using a series solution method. Specifically, C_n with n is an odd number is found to be zero. This leaves

$$\gamma_o(\eta) = C_0 + C_2\eta^2 + C_4\eta^4 + \ldots$$

To verify if the polynomial above is convergent over the cross-section area of the channel, i.e. on $0 \le \eta \le 1$, the radius of convergence has to be calculated by using the ratio test such that,

$$\Re = \lim_{m\to\infty}\left|\frac{C_{2m}}{C_{2m+2}}\right|$$

Define the ratio $R_m \equiv |C_{2m}/C_{2m+2}|$ so that $\Re = \lim_{m\to\infty}|R_m|$. Hence, the radius of convergence $\Re$ is obtained by plotting R_m as seen in Figures A.1 and A.2. For $n\to\infty$, it is suggested from the plot that the convergence interval of γ_0 is infinite; therefore, the polynomial γ_0 is convergence everywhere including the domain of interest; $0 \le \eta \le 1$.

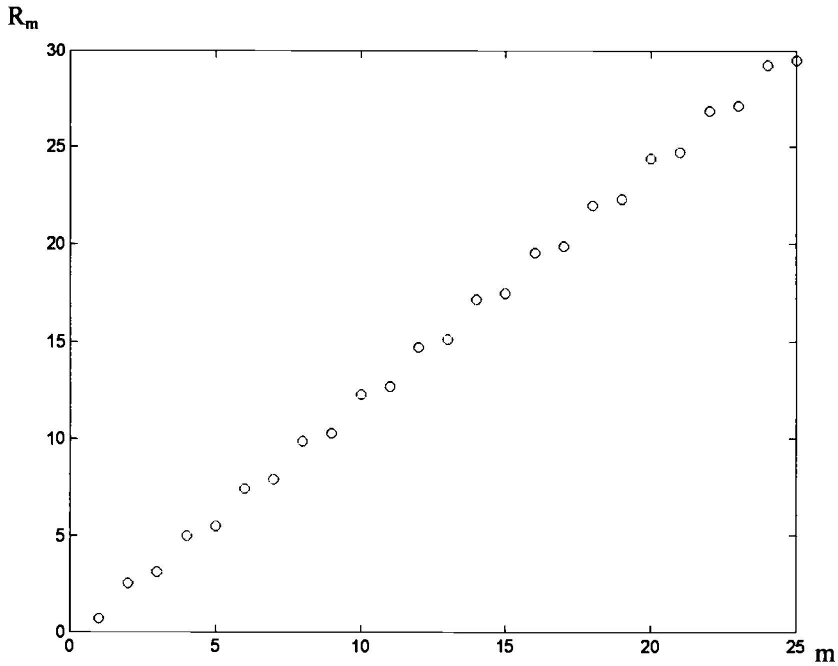

Figure A.1 Distribution of a ratio $R_m \equiv |C_{2m}/C_{2m+2}|$ for the UWT parallel-plate problem

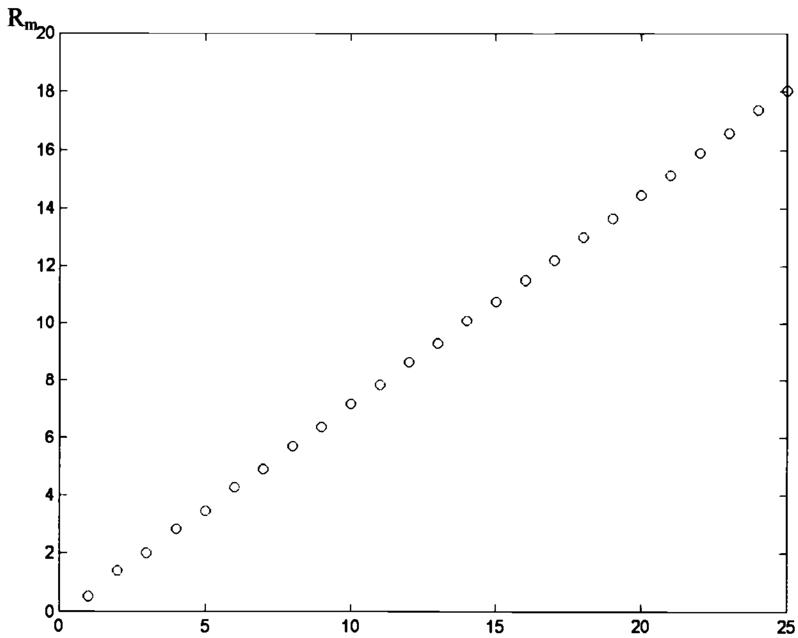

Figure A.2 Distribution of a ratio $R_m \equiv |C_{2m}/C_{2m+2}|$ for the UWT circular tube problem

APPENDIX B: NUMERICAL SUBROUTINES USED FOR A SERIES SOLUTION METHOD IN CHAPTER 4

In Chapter 4, the fluid temperature distribution for the problem of laminar fully developed natural convection within UWT parallel-plate channel and circular tube can be simplified as

$$\theta = 1 - \gamma_o(\eta)s(\xi)$$

While s(ξ) is solved analytically as shown in Chapter 3 of this paper, γ_o is solved by a series solution method from which the funcition is approximated by a polynomial function,

$$\gamma_o = C_0 + C_1\eta + \ldots + C_n\eta^n$$

Substituting this function into the energy equation,

$$\frac{1}{\eta^j}\frac{d}{d\eta}\left(\eta^j\frac{d\gamma_o}{d\eta}\right) = \frac{3}{4}\left(\frac{8}{3}\right)^j Nu_b\gamma_o\left(\eta^2 - 1\right))$$

Then the coefficients of the function can be found and C_1, C_3, C_5, … are zeros whereas the remaining coefficients can be written in terms of the coefficients before them;

$$C_2 = -\frac{1}{1+j}\left(\frac{8}{3}\right)^{j-1} Nu_b C_o$$

and

$$C_n = \frac{3}{4}\left(\frac{8}{3}\right)^j \frac{Nu_b(C_{n-4} - C_{n-2})}{n(n-1+j)} \quad n \geq 4$$

These coefficients has to satisfy the conditions such that

$$\gamma_o\big|_{\eta=1} = C_0 + C_2 + C_4 + ... = 0$$

$$\left.\frac{d\gamma_o}{d\eta}\right|_{\eta=1} = 2C_2 + 4C_4 + 6C_6 + ... = -\frac{Nu_b}{2}$$

If the low degree polynomial function γ_o is used, the calculation can be definitely carried out by hands. However, for the higher degree used, the calculation becomes time consuming and, therefore, a numerical subroutine are used. The detail of the program written in MATLAB is hence shown in Figure B.1. for the UWT parallel-plate problem and in Figure B.2. for the UWT circular tube problem.

```
NuOld=3.6;
NoTmax=31;
AC(1)=1;

for jdex=7:2:NoTmax
   NuDiff=1;
  while (NuDiff>0.00001)
     A(1)=1;
     A(2)=0;
     A(3)=-3/8*NuOld;
      A(4)=0;
      A(5)=3/4*NuOld*(A(1)-A(3))/12;
      A(6)=0;
     NuNew=8/3*(1+A(5));
     for idex=7:jdex
        A(idex)=3/4*NuOld*(A(idex-4)-A(idex-2))/(idex-1)/(idex-2);
        idex=idex+1;
        A(idex)=0;
        NuNew=NuNew+8/3*(A(idex)+A(idex-1));
     end
     NuDiff=abs(NuNew-NuOld);
     NuOld=NuNew;
  end

   NuNew
  Nub((jdex-5)/2)=NuNew;
  NoTerm((jdex-5)/2)=jdex;

   AC(2)=0;
   AC(3)=-3*NuNew/8;
   AC(4)=0;
   AC(5)=1/16*NuNew*(1-A(3));
   AC(6)=0;
   SumM=2*AC(3)+4*AC(5);
   for kdex=7:jdex
      AC(kdex)=0.75*NuNew*(AC(kdex-4)-AC(kdex-2))/(kdex-1)/(kdex-
2);
      SumM=SumM+(kdex-1)*AC(kdex);
      kdex=kdex+1;
      AC(kdex)=0;
   end
   AC(1)=-NuNew/2/SumM;
   A0((jdex-5)/2)=AC(1);

end

figure(1);plot(NoTerm,Nub,'ko');
figure(2);plot(NoTerm,A0,'ko');
```

Figure B.1 MATLAB subroutine used to calculate C_n in UWT parallel-plate channel problem.

```
NuOld=3.2;
AC(1)=1;
NoTMax=29;
for jdex=7:2:NoTMax

   NuDiff=1;
   while (NuDiff>0.00001)
      A(1)=1;
      A(2)=0;
      A(3)=-1/2*NuOld;
      A(4)=0;
      A(5)=2/16*NuOld*(A(1)-A(3));
      A(6)=0;
      NuNew=2+2*A(5);
      for idex=7:jdex
         A(idex)=2*NuOld*(A(idex-4)-A(idex-2))/(idex-1)^2;
         idex=idex+1;
         A(idex)=0;
         NuNew=NuNew+2*(A(idex)+A(idex-1));
      end
      NuDiff=abs(NuNew-NuOld);
      NuOld=NuNew;
   end
   NuNew
   Nub((jdex-5)/2)=NuNew;
   NoTerm((jdex-5)/2)=jdex;

   AC(2)=0;
   AC(3)=-NuNew/2;
   AC(4)=0;
   AC(5)=NuNew/8*(1-AC(3));
   AC(6)=0;
   SumM=2*AC(3)+4*AC(5);
   for kdex=7:jdex
      AC(kdex)=2*NuNew*(AC(kdex-4)-AC(kdex-2))/(kdex-1)^2;
      SumM=SumM+(kdex-1)*AC(kdex);
      kdex=kdex+1;
      AC(kdex)=0;
   end
   AC(1)=-NuNew/2/SumM;
   A0((jdex-5)/2)=AC(1);

end

figure(1);plot(NoTerm,Nub,'ko');
figure(2);plot(NoTerm,A0,'ko');
```

Figure B.2 MATLAB subroutine used to calculate C_n in UWT circular tube problem.

APPENDIX C: NUMERICAL SUBROUTINES USED TO SOLVE THE PROBLEM OF DEVELOPING FLOW IN CHAPTER 5

In Chapter 5, it follows that the conservation equations reduced from a system of partial differential equations into a system of three non-linear ordinary differential equations given by Eqs. (5.38), (5.39), and (5.40). Runge-Kutta numerical schemes written in MATLAB are used to solve these equations for three unknowns; namely, ϕ_c, Δ_T, and Δ, using Eqs. (5.41) and (5.42) as start-up solutions from $\xi = 0$ to 10^{-7} to avoid singularities due to the fact that, at the inlet, $\Delta(\xi=0) = 0$ and $\Delta_T(\xi=0) = 0$. The detail of main-body programs is shown in Figure C.1 for the UWT parallel-plate channel and Figure C.2 for the UWT circular tube. There are a relating functions called out from the main-body programs are also shown in Figure C.3 and C.4.

```
Pr=0.7
psistt=1e-7
psistp=0.024;

GrByPhO=12;
BLSt=sqrt(4*(1-exp(-35/6*psistt)));
VelSt=1/(1-3/8*BLSt);
RtSt=1;

[psi, y]=ode45('G1011c',[psistt psistp],[VelSt BLSt
RtSt],[],(GrByPhO),(Pr));
```

Figure C.1 Program subroutine written in MATLAB and used to solve the problem of developing laminar natural convection flow within a UWT parallel-plate channel.

```
Pr=0.7
psistt=1e-7
psistp=0.024;


GrByPhO=32;
BLSt=sqrt(1.5*(1-exp(-140/9*psistt)));
VelSt=1/(1-3/4*BLSt);
GSt=35/4*(VelSt^3)*exp(-140/9*psistt)/BLSt+GrByPhO;
RtSt=1;

[psi, y]=ode45('G1015a',[psistt psistp],[GSt VelSt BLSt
RtSt],[],(GrByPhO),(Pr));
```

Figure C.2 Program subroutine written in MATLAB and used to solve the problem of developing laminar natural convection flow within a UWT circular tube.

```
function dy=G1011c(psi,y,flag,GrByRd,Pr)
dy=zeros(3,1);
dy(1)=24*(1-y(1)+0.75*y(1)*(y(2)/2)-
((y(2)/2)^3)*GrByRd/24)/(y(1)*((y(2)/2)^3));

Bf=-24/35/((y(2)/2)^2)*(1-
y(1)+0.75*y(1)*(y(2)/2))^2+18/35*y(1)/(y(2)/2)*(1-
y(1)+0.75*y(1)*(y(2)/2))-3/28*(y(1)^2);
Cf=24/35/(y(2)/2)*(1-y(1)+0.75*y(1)*(y(2)/2))*(-
1+0.75*(y(2)/2))+19/35*(1-41/38*y(1)+5/76*y(1)*(y(2)/2));

if Pr>1
   Df=-0.25*GrByRd*(y(2)/2)-
1.5*y(1)/(y(2)/2)+3/8*GrByRd*y(3)*(y(2)/2);
   Ef=3*((y(3)^2)/5-(y(3)^3)/6+3/70*(y(3)^4))*(0.75*(y(2)/2)-
1)+3/4*(y(2)/2)*((y(3)^2)/5-(y(3)^4)/70);
   Hf=9/4*((y(3)^2)/5-
(y(3)^3)/6+3/70*(y(3)^4))*y(1)+0.75*y(1)*((y(3)^2)/5-
(y(3)^4)/70);
   IIf=3*(1+y(1)*(0.75*(y(2)/2)-1))*(2/5*y(3)-
0.5*(y(3)^2)+12/70*(y(3)^3))+...
      0.75*y(1)*(y(2)/2)*(2/5*y(3)-2/35*(y(3)^3));
else
   Df=-0.25*GrByRd*(y(2)/2)-
1.5*y(1)/(y(2)/2)+GrByRd*(y(2)/2)*(1-0.75/y(3)+1/8/(y(3)^3));
   Ef=-8/35+87/280*(y(2)/2);
   Hf=87/280*y(1)+(-1+3/8*y(3)+0.75/y(3)-1/8/(y(3)^3));
    IIf=(y(2)/2)*(3/8+0.75/(y(3)^2)+3/8/(y(3)^4));
end

dy(2)=2/Bf*(Df-Cf*dy(1));
dy(3)=1/IIf*(3/2/Pr/y(3)/(y(2)/2)-Ef*dy(1)-Hf*dy(2)/2);
```

Fig. C.3 Function subroutine called out from the main-body program for the problem of developing natural convection flow within UWT parallel-plate channel.

```
functiondy=G1015a(psi,y,flag,GrByRd,Pr)

D=y(3);D2=y(3)^2;D3=y(3)^3;D4=y(3)^4;D5=y(3)^5;D6=y(3)^6;

Z=y(4);Z2=y(4)^2;Z3=y(4)^3;Z4=y(4)^4;Z5=y(4)^5;Z6=y(4)^6;

FpD=(4+D);

dy=zeros(4,1);

delt=y(3)*y(4);

F1=(D3/24-D4/60)/FpD;
F2=(4-2*D-D2/5+3/10*D3)/(4+D);
F3=y(1)*(D2/2-11/60*D3-1/20*D4)/(FpD^2)+y(2)*(-12-
8/5*D+17/5*D2+3/5*D3)/(FpD^2);
F4=y(1)*(D5/105/8-D6/280/8)/(FpD^2)+y(2)*(19/210*D3-1/35*D4-
1/120*D5)/(FpD^2);
F5=y(1)*(19/210*D3-1/35*D4-1/120*D5)/(FpD^2)+y(2)*((544/35*D-
151/35*D2-148/35*D3-4/7*D4)/(FpD^2)+(1-D)^2-1/2);
F6=(y(1)^2)*(D4/84-1/280*D5-
1/1120*D6)/(FpD^3)+y(1)*y(2)*(38/35*D2-11/30*D3-47/210*D4-
1/40*D5)/(FpD^3)+...
(y(2)^2)*((1088/35-876/35*D-888/35*D2-234/35*D3-
4/7*D4)/(FpD^3)-(1-D));
ifPr<=1
F7=-y(1)*(D/FpD)-y(2)*24/D/FpD+...
GrByRd*D*((1/2-2/5*D)/Z3+(1/2*D-2/5*D2)/Z2+(-
3+D+3/4*D2)/Z+(4-2*D)+(D-D2/2)*Z)/(4+delt);
F8=((D3/210-D4/336)/Z3+(D4/210-D5/336)/Z2+(-
D3/20+D4/80+D5/140)/Z+(D3/12-D4/30)+Z*(D4/48-
D5/120))/FpD/(4+delt);
F9=((-6/35*D+1/35*D2+3/70*D3)/Z3+(-
6/35*D2+1/35*D3+3/70*D4)/Z2+(12/5*D+2/5*D2-16/35*D3-
3/28*D4)/Z+...
(-6*D-2/5*D2+3/5*D3)+Z*(6*D-
1/10*D3+3/20*D4)+Z2*(2/5*D2+1/10*D3)+Z3*(-3/5*D3-
3/20*D4))/FpD/(4+delt);
F10=y(1)*((D2/70-D3/84)/Z3+(2/105*D3-5/336*D4)/Z2+(-
3/20*D2+D3/20+D4/28)/Z+(D2/4-2/15*D3)+Z*(D3/12-D4/24)-...
F8*(4+2*delt+4*Z))/FpD/(4+delt)+y(2)*((-
6/35+2/35*D+9/70*D2)/Z3+(-12/35*D+3/35*D2+6/35*D3)/Z2+...
(12/5+4/5*D-48/35*D2-3/7*D3)/Z+(-6-4/5*D+9/5*D2)+Z*(6-
```

Fig. C.4 Function subroutine called out from the main-body program for the problem of developing natural convection flow within UWT circular tube.

```
3/10*D2+3/5*D3)+Z2*(4/5*D+3/10*D2)+Z3*(-9/5*D2-3/5*D3)-...
      F9*(4+2*delt+4*Z))/FpD/(4+delt);
   F11=y(1)*((-D3/70+D4/112)/Z4+(-D4/105+D5/168)/Z3+(D3/20-
D4/80-D5/140)/Z2+(D4/48-D5/120)-F8*(4*D+D2))/FpD/(4+delt)+...
      y(2)*((18/35*D-3/35*D2-9/70*D3)/Z4+(12/35*D2-2/35*D3-
3/35*D4)/Z3+(-12/5*D-2/5*D2+16/35*D3+3/28*D4)/Z2+...
      (6*D-1/10*D3+3/20*D4)+Z*(4/5*D2+1/5*D3)+Z2*(-9/5*D3-
9/20*D4)-F9*(4*D+D2))/FpD/(4+delt);
else
   F7=-y(1)*(D/FpD)-
y(2)*24/D/FpD+GrByRd*D*Z*(3/2+Z*(1/10*D)+Z2*(-
3/20*D2))/(4+delt);
   F8=(Z2*(D3/10)+Z3*(-1/12*D3-1/240*D4)+Z4*(3/140*D3+D4/105-
D5/60)+Z5*(-1/280*D4+1/56*D5)+...
      Z6*(-3/560*D5))/FpD/(4+delt);
   F9=(Z2*(12/5*D)+Z3*2/5*D2+Z4*(-6/35*D-6/35*D2-
16/35*D3)+Z5*(1/35*D2+1/35*D3-3/28*D4)+...
      Z6*(3/70*D3+3/70*D4))/FpD/(4+delt);
   F10=y(1)*(Z2*(3/10*D2)+Z3*(-1/4*D2-
1/60*D3)+Z4*(9/140*D2+4/105*D3-D4/12)+Z5*(-
1/70*D3+5/56*D4)+...
      Z6*(-3/112*D4)-
F8*(4+4*Z+2*Z*D))/FpD/(4+delt)+y(2)*(Z2*(12/5)+Z3*(4/5*D)+Z4*(
-6/35-12/35*D-48/35*D2)+...
      Z5*(2/35*D+3/35*D2-3/7*D3)+Z6*(9/70*D2+6/35*D3)-
F9*(4+4*Z+2*delt))/FpD/(4+delt);
   F11=y(1)*(Z*(D3/5)+Z2*(-1/4*D3-
1/80*D4)+Z3*(3/35*D3+4/105*D4-D5/15)+Z4*(-
1/56*D4+5/56*D5)+Z5*(-9/280*D5)-...

F8*(4*D+D2))/FpD/(4+delt)+y(2)*(Z*(24/5*D)+Z2*(6/5*D2)+Z3*(-
24/35*D-24/35*D2-64/35*D3)+...
      Z4*(1/7*D2+1/7*D3-15/28*D4)+Z5*(9/35*D3+9/35*D4)-
F9*(4*D+D2))/(4+delt)/FpD;
end

dy(2)=1/y(2)*(y(1)-GrByRd);
dy(3)=(F1*F7-dy(2)*(F5*F1-F2*F4))/(F6*F1-F3*F4);
dy(1)=1/F1*(-dy(2)*F2-dy(3)*F3);
dy(4)=1/F11*(24/Pr/delt/(4+delt)-dy(1)*F8-dy(2)*F9-dy(3)*F10);
```

Fig. C.4 (Continued) Function subroutine called out from the main-body program for the problem of developing natural convection flow within UWT circular tube.