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Title ON THE MELTING OF CYLINDERS AND SPHERES

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In this thesis we consider boundary and initial value problems associated with the melting of homogeneous cylinders and spheres. The study includes the temperature distribution in cylinders and spheres which are perfectly insulated at the surface and subjected to a heat input respectively along the axis and at the center. Analytical solutions are obtained. The uniqueness question is not discussed.

ON THE MELTING OF CYLINDERS AND SPHERES

by

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NOMENCLATURE

r	=	Space variable
t	=	Time variable
K	=	Thermal conductivity
c	=	Specific heat
ρ	=	Density
k	=	Thermal diffusivity (= $K/\rho c$)
L	=	Latent heat of fusion
a	=	Radius
$Q(t)$	=	Strength of the source of heat per unit time per unit length of the cylinder
$Q(t)$	=	Strength of the source of heat per unit time in the case of the sphere
r'	=	Radius of the source (very, very small)
$T(r,t)$	=	Temperature distribution before melting
$T_L(r,t)$	=	Temperature distribution after melting in the liquid region
$T_S(r,t)$	=	Temperature distribution in the solid
T^*	=	Melting temperature of the material
t^*	=	Time at which the material begins to melt
$R(t)$	=	Position of the boundary between the phases
t_a	=	The time of the complete melting of the cylinder or the sphere

NOMENCLATURE (Continued)

z = $r/R(t)$

ρ_L, c_L, k_L and ρ_S, c_S, k_S are the corresponding quantities in the liquid and solid regions respectively.

$\rho = \rho_L = \rho_S$ (no change in density is assumed)

Units are cgs, calorie and °C.

ON THE MELTING OF CYLINDERS AND SPHERES

INTRODUCTION

Problems involving moving boundaries (known as Stefan Problems) are of great current interest in heat conduction theory. There is a large number of situations involving heat conduction in which moving boundaries result from the change of phase, for example, melting or freezing phenomena and the progress of temperature dependent chemical reactions through a solid. Many practical examples can be also cited, e.g. (1) Decay by evaporation (or growth by condensation) of a liquid drop, (2) Motion of a plane liquid-vapor interface, (3) Freezing of a lake, (4) Structural damage to hypersonic missiles caused by aerodynamic heating etc. Due to the difficulties caused by the non-linearity of these problems, it is generally very difficult to obtain analytical solutions.

There are many classical references to this type of problem, usually involving the melting or solidification of slabs. Problems of this type first seem to have been considered by Franz Neumann in his lectures in the 1860's; Stefan considered the same type of problem in 1891 imposing more restrictions. In 1921, Saitō [22] discussed the temperature distribution, during solidification, in steel ingots of various shape and size subject to different initial and boundary conditions neglecting latent heat of fusion. Lightfoot [16] gave extensive study to Saitō's problem when latent heat of fusion could not be neglected. A simple solution was obtained by the integral equation method. In recent years, high speed computers have been used in order to obtain numerical solutions of the partial differential equation involved. Work done by Landau [15], Citron [3], Lotkin [17], Miranker and Keller [18] among others can be mentioned. Boley's [1] contributions towards this end are remarkable. He developed a new method in which the problem is reformulated so as to involve the solution of two ordinary integro-differential equations. By expanding the equations in powers of the time after melting starts, an exact analytical solution is obtained which is particularly useful for small times. A review of the developments of the past few years in the field of heat conduction was presented by Boley [2] including the application to problems of change of phase and moving boundary. He includes several methods to approximate solutions in his paper. Other methods are given by Evans et al. [7], where the solutions are represented as power series.

Although, this important group of problems has attracted the interest of many authors in the 19th and 20th century, problems involving radial flow in cylindrical and spherical coordinates have received comparatively little attention. The problems of melting are more difficult in the case of cylindrical and spherical flows. A solution of the non-linear problem of the rate of ice formation on a long cylinder whose surface temperature is variable (but always below zero) is given by Pekeris and Slichter [20]. The solution is applied to the specific problem of stabilization of wet soil by freezing. Stewart Paterson [21] has considered the case of fusion when heat propogated from a line source (cylindrical coordinates) of heat and from a point source (spherical coordinates) of heat. Analytical solutions are obtained in both cases. Ingersoll et al.[11] have discussed the problem of freezing around pipes in soil at various conditions, the difficulties faced by piping engineers have been dealt with explicitly. Friedman's method [8] of generalization of one dimensional problems to two and three dimensional cases should be mentioned in this context. Gibson [10] has solved the same problem involving spherical symmetry by using the method of separation of variables with the help of the special transformation

$$T = f(t)G(\xi) + \xi h(t),$$

where t is the time and ξ the space variable and f, G, h are functions of single variables. Later he concluded that the variables are separable only if $R(t) = Ct^{\frac{1}{2}}$, where C is a constant.

An interesting practical problem, solidification of a cylinder and a sphere from the surface, has been presented by Kreith and Romie [14]. The diffusion equation is expressed in dimensionless form, and the solution of the partial differential equation is obtained by means of a method of iterative approximation in which the temperature is expressed in a series

$$T = T_1 + T_2 + T_3 + \cdots + T_n + \cdots \quad .$$

The authors assume that each term of this series may be related to the preceding term by the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_n}{\partial r}\right) = G\frac{\partial T_{n-1}}{\partial \phi}, \quad \phi = 1 + \theta,$$

where θ is a dimensionless time variable, r is dimensionless space variable and G is constant. The last equation is integrated between limits ϕ to r and each term of the series can be determined in this way.

This paper attempts to solve the problem of melting propagating from the axes of cylinders and the centers of spheres due to heat sources of a variable strength. The problem of the sphere is artificial, but the problem of the cylinder has some practical applications, as for example, melting or solidification of soil or ice around a cylindrical heated pipe. Both the cylinder and sphere are assumed to be insulated at the surface, but cases involving infinite radius will also be treated. Unlike the work of some authors, the liquid formed by the melting is not immediately removed, but no convective heat transfer is assumed, i. e. transportation of heat is purely by conduction. The growth of a frozen zone, starting from negligible initial dimensions with either a plane, cylindrical or spherical boundary, into a surrounding super cooled fluid is discussed by Chambrē [6], where convection in the liquid region is taken into account. If the density does not change with the temperature in the liquid region, the assumption of no-convection will be correct. In both cases, analytical solutions are obtained.

In this work, existence and uniqueness will not be discussed, reference can be made to Friedman [8], Kolodner [13] and Miranker [19].

ASSUMPTIONS

In this thesis we shall assume that

1. The material is homogeneous.

2. All thermal properties of the material are uniform and constant; however, the properties of the liquid phase are not necessarily equal to the corresponding properties of the solid phase.

3. There is no convection in the liquid, the transportation of heat is effected only by conduction.

4. Expansion or contraction of volume on melting is neglected.

FORMULATION OF THE PROBLEM

Cylinder with Finite Radius and Infinite Length

Consider a cylinder of radius a, perfectly insulated at the surface, and subjected to a heat input $Q(t)$ per unit time per unit length along the axis. The initial temperature is zero. The radius of the source is r' where $r' << a$. The temperature $T(r, t)$ in the cylinder prior to melting is a solution of the following boundary value problem:

$$\frac{\partial}{\partial r}\left(r \frac{\partial T(r,t)}{\partial r}\right) = \frac{1}{k} \frac{\partial[rT(r,t)]}{\partial t} \qquad r' < r < a, \quad 0 < t \tag{1}$$

$$\begin{aligned} &(a) \quad T(r,0) = 0 && r' < r < a \\ &(b) \quad Q(t) = -2\pi K r \frac{\partial T}{\partial r} && \text{at} \quad r = r' \\ &(c) \quad \frac{\partial T(a,t)}{\partial r} = 0 \end{aligned} \tag{2}$$

Equation 2(a) states that initially the temperature distribution in the cylinder is zero. Conditions 2(b) and 2(c) are boundary conditions, the first one gives the heat input to the cylinder at $r = r'$, the second results from the insulation of the surface. With constant heat input $Q(t) = Q_0$, the foregoing problem could be solved by applying the Laplace transformation method.

Since the cylinder is insulated at the surface, melting will be initiated around the source after the elapse of a certain time $t = t^*$. For $t > t^*$ the cylinder will be divided into two regions, liquid and solid. The position of the phase boundary will be an unknown function of time which must be determined and hence gives rise to a floating boundary value problem. The position of the phase boundary will be denoted by $r = R(t)$ and the temperature at this boundary will be equal to the melting temperature of the material, T^*. The problem for the liquid region in the post melting period, $t > t^*$, $R(t) > r'$, may be formulated as follows:

$$\frac{\partial}{\partial r}\left(r \frac{\partial T_L(r,t)}{\partial r}\right) = \frac{r}{k_L} \frac{\partial T(r,t)}{\partial t} \qquad r' < r < R(t), \quad t^* < t. \tag{3}$$

$$\begin{aligned} &(a) \quad T_L(R(t),t) = T^* \\ &(b) \quad Q(t) = -2\pi r K_L \frac{\partial T_L(r,t)}{\partial r} \quad \text{at} \quad r = r' \end{aligned} \tag{4}$$

The condition 4(a) expresses the fact that at the phase boundary, the temperature is constant and equal to the melting temperature of the material.

In the solid region we have:

$$\frac{\partial}{\partial r}\left(r\,\frac{\partial T_S(r,t)}{\partial r}\right) = \frac{r}{k_S}\,\frac{\partial T_S(r,t)}{\partial t}, \quad R(t) < r < a, \quad t^* \leq t. \tag{5}$$

$$\text{(a)} \quad T_S(R(t),t) = T^*$$

$$\text{(b)} \quad \frac{\partial T_S(a,t)}{\partial r} = 0$$

$$\text{(c)} \quad T_S(r,t^*) = T(r,t^*) \tag{6}$$

$$\text{(d)} \quad -K_L\,\frac{\partial T_L(r,t)}{\partial r} + K_S\,\frac{\partial T_S(r,t)}{\partial r} = L\rho\,\dot{R}(t)$$

$$\text{at} \quad r = R(t).$$

$$\text{(e)} \quad R(t^*) = r'.$$

Conditions 6(a), 6(b) are not new, while 6(c) implies that at $t = t^*$, i.e. when melting starts, the temperature is continuous. The last two conditions provide (d) that part of the heat passing through the liquid region enters the solid region, while part goes towards overcoming the latent heat of fusion L and (e) that there is no liquid region at $t = t^*$.

The condition 6(d) above may be put into an alternative form by considering the curves of constant temperature

$$T_L(r,t) = T^* = T_S(r,t) \quad \text{at} \quad r = R(t).$$

Hence

$$\frac{\partial T_L}{\partial r}\dot{R}dt + \frac{\partial T_L}{\partial t}dt = 0 = \frac{\partial T_S}{\partial r}\dot{R}dt + \frac{\partial T_S}{\partial t}dt \quad \text{at} \quad r = R(t).$$

So equation 6(d) becomes

$$-K_L\frac{\partial T_L(r,t)}{\partial r} + K_S\frac{\partial T_S(r,t)}{\partial r} = -L\rho\frac{\partial T_L/\partial t}{\partial T_L/\partial r} = -L\rho\frac{\partial T_S/\partial t}{\partial T_S/\partial r}$$

$$\text{at} \quad r = R(t) \qquad (7)$$

That the problem is non-linear can be seen from the fact that two different heat input functions $Q_1(t)$ and $Q_2(t)$ will give different phase boundaries $R_1(t)$ and $R_2(t)$, because the movement of the phase boundary depends upon the strength of the source and on other physical constants; and that the solution for the sum $Q_1(t) + Q_2(t)$ cannot be obtained by addition of $R_1(t)$ and $R_2(t)$. The non-linearity of the problem is also apparent from equation (7) above in that form.

In this work we will attempt to determine the quantities t^*, t_a, T_L, T_S and $R(t)$. The last three quantities will be obtained in the next chapter under further restrictions to be explained later. The first two quantities are obtained by Landau [15] for the problem of melting of a slab, by Gauss's Theorem applied to the heat conduction equation, the domain of integration is xt plane. The same quantities for the same problem are determined by Citron [3] by using simple integration method. Here, use is made of Citron's procedure to determine t^* and t_a.

Integrating (3) between the limit $r = r'$ to $r = R(t)$ and using conditions 4(a) and 4(b), we obtain:

$$\int_{r'}^{R(t)} \frac{\partial}{\partial r}\left(r \frac{\partial T_L(r,t)}{\partial r}\right) dr = \frac{1}{k_L} \int_{r'}^{R(t)} \frac{\partial}{\partial t}[r T_L(r,t)]\, dr.$$

or

$$r \frac{\partial T_L(r,t)}{\partial r}\Bigg|_{r=r'}^{R(t)} = \frac{1}{k_L}\left\{ \frac{\partial}{\partial t} \int_{r'}^{R(t)} r T_L(r,t) dr - R(t) T_L(R(t),t) \frac{dR(t)}{dt} \right\}.$$

$T_L(r,t)$ is the temperature distribution in the homogeneous liquid where no convection takes place. Hence $T_L(r,t)$ is a solution of the heat conduction equation and is therefore a smooth function in $r' < r < R(t)$ and hence the order of differentiation and integration can be interchanged.

$$r \frac{\partial T_L}{\partial r}\Bigg|_{r=R(t)} - r \frac{\partial T_L}{\partial r}\Bigg|_{r=r'} = \frac{1}{k_L}\left\{ \frac{\partial}{\partial t} \int_{r'}^{R(t)} r T_L(r,t) dr - R(t) T^* \frac{dR(t)}{dt} \right\}$$

or

$$r \frac{\partial T_L}{\partial r}\Bigg|_{r=R(t)} - r \frac{\partial T_L}{\partial r}\Bigg|_{r=r'} = \frac{1}{k_L} \frac{\partial}{\partial t} \left\{ \int_{r'}^{R(t)} r T_L(r,t) dr - T^* R^2(t)/2 \right\}$$

or

$$r\frac{\partial T_L}{\partial r}\Bigg|_{r=R(t)} = -\frac{Q(t)}{2\pi K_L} + \frac{1}{k_L}\frac{\partial}{\partial t}\left\{\int_{r'}^{R(t)} rT_L(r,t)dr - T^*R^2(t)/2\right\}$$

using 4(b) (8)

Proceeding exactly the same way, integrating (5) between the limit $r = R(t)$ to $r = a$ and using conditions 6(a) and 6(b), one will obtain:

$$-r\frac{\partial T_S}{\partial r}\Bigg|_{r=R(t)} = \frac{1}{k_S}\frac{\partial}{\partial t}\left\{\int_{R(t)}^{a} rT_S(r,t)dr + T^*R^2(t)/2\right\}. \quad (9)$$

Multiplying 6(d) by r we get

$$-K_L r\frac{\partial T_L}{\partial r}\Bigg|_{r=R(t)} + K_S r\frac{\partial T_S}{\partial r}\Bigg|_{r=R(t)} = L\rho R(t)\cdot\dot{R}(t)$$

and substituting (8) and (9)

$$\frac{Q(t)}{2\pi} - \frac{K_L}{k_L}\frac{\partial}{\partial t}\left\{\int_{r'}^{R(t)} rT_L(r,t)dr - T^*R^2(t)/2\right\}$$

$$-\frac{K_S}{k_S}\frac{\partial}{\partial t}\left\{\int_{R(t)}^{a} rT_S(r,t)dr + T^*R^2(t)/2\right\} = L\rho R(t)\dot{R}(t)$$

or

$$Q(t) = 2\pi\rho c_L \frac{\partial}{\partial t}\{ \int_{r'}^{R(t)} rT_L(r,t)dr - T^*R^2(t)/2\}$$

$$+ 2\pi\rho c_S \frac{\partial}{\partial t}\{ \int_{R(t)}^{a} rT_S(r,t)dr + T^*R^2(t)/2\} + 2\pi\rho LR(t)\dot{R}(t). \qquad (10)$$

$$(\because \rho_L = \rho_S = \rho)$$

Integrating (10) between $t = t^*$ and $t = t_1$

$$\int_{t^*}^{t_1} Q(t)dt = 2\pi\rho c_L \int_{t^*}^{t_1} \frac{\partial}{\partial t} \{ \int_{r'}^{R(t)} rT_L(r,t)dr - T^*R^2(t)/2\}dt$$

$$+ 2\pi\rho c_S \int_{t^*}^{t_1} \frac{\partial}{\partial t} \{ \int_{r'}^{R(t)} rT_L(r,t)dr + T^*R^2(t)/2\} dt$$

$$+ 2\pi\rho L \int_{t^*}^{t_1} R(t)\dot{R}(t)dt.$$

Replacing t_1 by t we get

$$\int_{t^*}^{t} Q(t)dt = 2\pi\rho c_L \{ \int_{r'}^{R(t)} rT_L(r,t)dr - T^*R^2(t)/2 \}$$

$$-2\pi\rho c_L \{ \int_{r'}^{R(t^*)} rT_L(r,t^*)dr - T^*R^2(t^*)/2 \}$$

$$+ 2\pi\rho c_S \{ \int_{R(t)}^{a} rT_S(r,t)dr + T^*R^2(t)/2 \}$$

$$- 2\pi\rho c_S \{ \int_{R(t^*)}^{a} rT_S(r,t^*)dr + T^*R^2(t^*)/2 \}$$

$$+ 2\pi\rho L [R^2(t)/2 - R^2(t^*)/2]$$

$$= 2\pi\rho c_L \{ \int_{r'}^{R(t)} rT_L(r,t)dr - T^*R^2(t)/2 \}$$

$$+ 2\pi\rho c_S \{ \int_{R(t)}^{a} rT_S(r,t)dr + T^*R^2(t)/2 \}$$

$$+ 2\pi L\rho [R^2(t)/2 - r'^2/2]$$

$$-2\pi\rho c_S [\int_{r'}^{a} rT_S(r,t^*)dr + T^*r'^2/2] + \pi\rho c_L T^* r'^2$$

$$\because R(t^*) = r'. \qquad (11)$$

We are interested in obtaining the time for complete melting of the cylinder, t_a. If heating is continued long enough, the time t_a will be obtained observing that $R(t_a) = a$, hence

$$\int_{t^*}^{t_a} Q(t)dt = 2\pi\rho c_L \{ \int_{r'}^{a} rT_L(r,t_a)dr - T^*a^2/2\} + 2\pi\rho c_S\{T^*a^2/2\}$$

$$+ \pi\rho c_L T^* r'^2 + \pi\rho L(a^2 - r'^2) - 2\pi\rho c_S [\int_{r'}^{a} rT_S(r,t^*)dr + T^* r'^2/2]$$

or

$$\int_{t^*}^{t_a} Q(t)dt = \pi\rho [a^2(c_S T^* - c_L T^*) + L(a^2 - r'^2)] + \pi\rho c_L T^* r'^2$$

$$- 2\pi\rho c_S \int_{r'}^{a} rT_S(r,t^*)dr + 2\pi\rho c_L \int_{r'}^{a} rT_L(r,t_a)dr. \tag{12}$$

Proceeding in exactly the same way, equation (1) and the conditions 2(a), 2(b) and 2(c) give

$$- r\frac{\partial T}{\partial r}\bigg|_{r = r'} = \frac{1}{k}\frac{\partial}{\partial t}\int_{r'}^{a} rT(r,t)dr$$

or

$$Q(t) = \frac{2\pi K}{k}\frac{\partial}{\partial t}\int_{r'}^{a} rT(r,t)dr. \tag{13}$$

An expression for t^* may be obtained by integrating (13) between the limit $t = 0$ to $t = t^*$:

$$\int_0^{t*} Q(t)dt = 2\pi\rho c \int_0^{t*} \frac{\partial}{\partial t} \int_{r'}^{a} rT(r,t)dr\,dt$$

$$= 2\pi\rho c \left[\int_{r'}^{a} rT(r,t*)dr - \int_{r'}^{a} rT(r,0)dr \right]$$

$$= 2\pi\rho c \int_{r'}^{a} rT(r,t*)dr \qquad (14)$$

So adding (12) and (14) we get,

$$\int_{t*}^{t_a} Q(t)dt + \int_0^{t*} Q(t)dt = \pi\rho \left[a^2(c_S T* - C_L T*) + L(a^2 - r'^2) \right] + \pi\rho c_L T* r'^2$$

$$-2\pi\rho c_S \int_{r'}^{a} rT_S(r,t*)dr + 2\pi\rho c_L \int_{r'}^{a} rT_L(r,t_a)dr$$

$$+ 2\pi\rho c \int_{r'}^{a} rT(r,t*)dr$$

But $T(r,t*) = T_S(r,t*)$, $c_S = c$.

$$\therefore \quad \int_0^{t_a} Q(t)dt = \pi\rho \left[a^2(c_S T* - C_L T*) + L(a^2 - r'^2) \right] + \pi\rho c_L T* r'^2$$

$$+ 2\pi\rho c_L \int_{r'}^{a} rT_L(r,t_a)dr . \qquad (15)$$

It can be seen that (15) equates the total heat inflow in time $t = 0$ to $t = t_a$ to the heat content in the cylinder at that moment.

In order to be able to estimate t^* on the basis of equation (14) we have to determine $T(r, t^*)$. This may be obtained from equation (1) with conditions (2) by applying the Laplace transform method. A sufficient condition for the existence of the Laplace transform is that $T(r,t)$ is of exponential order, i.e.

$$\left| e^{-pt} T(r,t) \right| < c\, e^{-(p-a)t}$$

where c is any constant. We will assume that this condition holds, and applying Laplace transform to (1) and (2) we get:

$$\frac{d^2\overline{T}}{dr^2} + \frac{1}{r}\frac{d\overline{T}}{dr} = q^2\overline{T} \qquad (1)'$$

where $\overline{T} = \int_0^\infty T(r,t)e^{-pt}dt. \qquad q^2 = \frac{p}{k}$

$$\text{(a)} \quad \frac{Q_0}{p} = -2\pi K r \left.\frac{d\overline{T}}{dr}\right|_{r=r'}$$

$$\text{(b)} \quad \frac{d\overline{T}(a,p)}{dr} = 0 \qquad (2)'$$

From (1') $\overline{T}(r,p) = AI_0(qr) + BK_0(qr)$ where $I_0(qr)$ and $K_0(qr)$ are the modified Bessel's functions of the first and second kind of zero order. Using 2'(a) and 2'(b) we have

$$\overline{T}(r,p) = \frac{Q_0}{2\pi K r' p q}\left[\frac{K_1(qa)I_0(qr) + K_0(qr)I_1(qa)}{K_1(qr')I_1(qa) - K_1(qa)I_1(qr')}\right].$$

Then by the inversion theorem,

$$T(r,t) = \frac{Q_0}{4\pi^2 i K r'} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{K_1(\mu a)I_0(\mu r)+K_0(\mu r)I_1(\mu a)}{K_1(\mu r')I_1(\mu a)-K_1(\mu a)I_1(\mu r')} \frac{e^{\lambda t}}{\mu\lambda}\, d\lambda$$

where $\mu = \sqrt{\frac{\lambda}{k}}$.

The integrand, when expressed in series form, will contain only even powers of μ, therefore it is a single valued function. So we need to take the contour of integration as shown in Figure 1. We take the radius of the semi-circle as $k(n+\frac{1}{2})^2\pi^2/a^2$ so that it does not pass through a pole. When $n \to \infty$ the integral over BB'CA'A tends to zero [5]. So the line integral in question will be equal to the sum of the residues of the poles at $\lambda = 0$ and $K_1(\mu r')I_1(\mu a) - K_1(\mu a)I_1(\mu r') = 0$. There are double poles at $\lambda = 0$. The residues at $\lambda = 0$ will be given by

$$\frac{d}{d\lambda}\left[\lambda^2 \frac{K_1(\mu a)I_0(\mu r)+K_0(\mu r)I_1(\mu a)}{K_1(\mu r')I_1(\mu a)-K_1(\mu a)I_1(\mu r')} \frac{e^{\lambda t}}{\mu\lambda}\right]_{\substack{\lambda=0 \\ \mu=\sqrt{\frac{\lambda}{k}}}}$$

$$= k \frac{d}{d\lambda}\left[\frac{\mu K_1(\mu a)I_0(\mu r)+\mu K_0(\mu r)I_1(\mu a)}{K_1(\mu r')I_1(\mu a)-K_1(\mu a)I_1(\mu r')} \cdot e^{\lambda t}\right]_{\lambda=0}$$

$$= \frac{2ar'}{a^2-r'^2}\left[\frac{r^2-a^2}{4a}+\frac{a}{2}\log\frac{a}{r}+\frac{kt}{a}\right] - \frac{4a^2r'^3}{4(a^2-r'^2)^2}\log\frac{r'}{a} - \frac{r'(a^2+r'^2)}{4(a^2-r'^2)}.$$

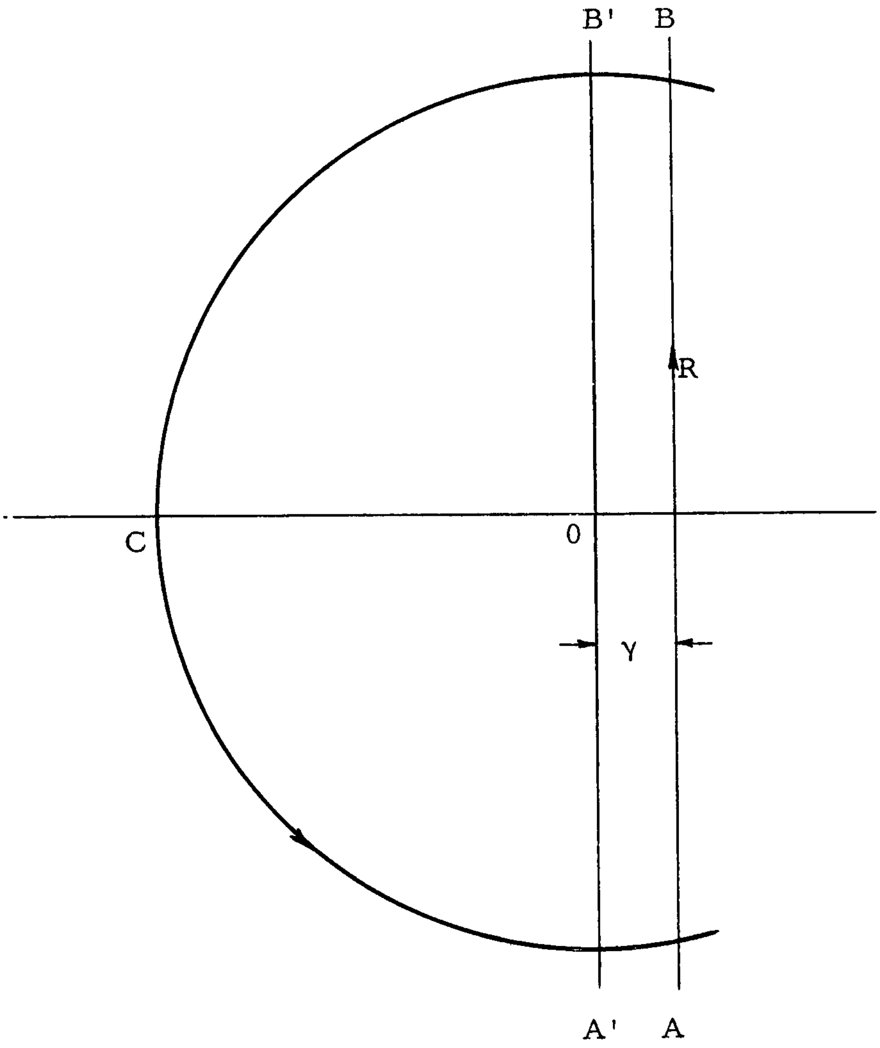

Figure 1.

Neglecting powers of r' greater than one, since r' is small in comparison with a, we find that the residue at $\lambda = 0$ is

$$= \frac{2r'}{a}\left[\frac{r'^2 - a^2}{4a} + \frac{a}{2}\log\frac{a}{r} + \frac{kt}{a}\right] - \frac{r'}{4}.$$

The residues at

$$K_1(\mu r')I_1(\mu a) - K_1(\mu a)I_1(\mu r') = 0:$$

The poles are $\lambda = -k\alpha_1^2, \; -k\alpha_2^2, \cdots, -k\alpha_n^2, \cdots$ where $\alpha_1, \alpha_2, \alpha_3, \cdots$ are the roots of

$$Y_1(\alpha r')J_1(\alpha a) - Y_1(\alpha a)J_1(\alpha r') = 0 \quad \text{1/}$$

Hence the residues at $\lambda = -k\alpha_n^2$

$$= \frac{k}{\lambda^2} \left. \frac{\mu[K_1(\mu a)I_0(\mu r) + K_0(\mu r)I_1(\mu a)]\, e^{\lambda t}}{\frac{d}{d\lambda}[K_1(\mu r')I_1(\mu a) - K_1(\mu a)I_1(\mu r')]} \right|_{\lambda = -k\alpha_n^2}$$

$$= \pi \frac{[J_1(\alpha_n a)Y_0(\alpha_n r) - Y_1(\alpha_n a)J_0(\alpha_n r)]J_1(a\alpha_n)J_1(r'\alpha_n)e^{-k\alpha_n^2 t}}{\alpha_n [J_1^2(a\alpha_n) - J_1^2(r'\alpha_n)]} \quad \text{2/}$$

1/ $\alpha_1, \alpha_2, \alpha_3, \cdots$ are all real and simple. See Gray and Mathews, Treatise on Bessel Functions. 2nd Edition, 1922. p. 82.

2/ See [5] page 173.

$$\therefore \quad T(r,t) = \frac{Q_0}{2\pi K}\left[\left(\frac{r^2 - a^2}{2a^2} + \log\frac{a}{r} + \frac{2kt}{a^2} - \frac{1}{4} \right) \right.$$

$$\left. + \frac{\pi}{r'} \sum_{n=1}^{\infty} \frac{Y_0(\alpha_n r)J_1(a\alpha_n) - Y_1(a\alpha_n)J_0(\alpha_n r)}{\alpha_n[J_1^2(a\alpha_n) - J_1^2(r'\alpha_n)]} \cdot J_1(a\alpha_n)J_1(r'\alpha_n)e^{-k\alpha_n^2 t} \right].$$

This is a rapidly converging series for higher values of time. So the sum of finite number of terms will give the result with sufficient accuracy. From (16), the time when melting starts at $r = r'$ can be obtained:

$$T(r',t^*) = \frac{Q_0}{2\pi K}\left[\left(\frac{r'^2 - a^2}{2a^2} + \log\frac{a}{r'} + \frac{2kt^*}{a^2} - \frac{1}{4} \right) \right.$$

$$\left. + \frac{\pi}{r'} \sum_{n=1}^{\infty} \frac{Y_0(\alpha_n r')J_1(a\alpha_n) - Y_1(a\alpha_n)J_0(\alpha_n r')}{\alpha_n[J_1^2(a\alpha_n) - J_1^2(r'\alpha_n)]} \cdot J_1(a\alpha_n)J_1(r'\alpha_n)e^{-k\alpha_n^2 t^*} \right]$$

or

$$T^* = T(r', t^*) = \frac{Q_0}{2\pi K}\Big[\Big(\log\frac{a}{r'} + \frac{2kt^*}{a^2} - \frac{3}{4}\Big)$$

$$+ \frac{\pi}{r'}\sum_{n=1}^{\infty}\frac{Y_0(\alpha_n r')J_1(a\alpha_n) - Y_1(a\alpha_n)J_0(\alpha_n r')}{\alpha_n[J_1^2(a\alpha_n) - J_1^2(r'\alpha_n)\mu}\cdot J_1(a\alpha_n)J_1(r'\alpha_n)e^{-k\alpha_n^2 t^*}\Big]. \tag{17}$$

In this series only unknown quantity is t^*; so taking finite number of terms, an approximate value of t^* could be obtained with considerable accuracy.

In order to apply equation (15) to the problem of determining the time of total melting t_a, we have to evaluate the integral

$$\int_{r'}^{a} rT_L(r, t_a)dr$$

where $T_L(r, t_a)$ is the temperature distribution in the cylinder at the time t_a. An exact evaluation of this integral therefore requires a solution of the melting problem for the cylinder.

However, the following approximate method can be used in order to arrive at a fairly good estimation of the fixed t_a. The right hand side of (15) is nothing but the total heat content of the just molten cylinder. As this quantity involves an integration in space, we can use an approximate temperature distribution. One way of arriving at such an approximation is to use the fact that the latent heat of fusion L can be regarded as a part of the internal energy of the material and define an average specific heat of the material for temperatures above the melting point. Hence for $T_0 > T^*$ we define the average specific heat (See Figure 2 and Figure 3)

$$c_{av} = \frac{c_S T^* + L + C_L(T_0 - T^*)}{T_0} .$$

For constant conductivity K, the heat conduction equation for an incompressible material can be written as:

$$K\nabla^2 T = \rho \frac{\partial E}{\partial t} = \rho \frac{dE}{dT} \cdot \frac{\partial T}{\partial t} = \rho c \frac{\partial T}{\partial t}$$

where E is the total internal energy. In general, the specific heat c is a function of temperature T and has to include the heat of fusion. As pointed out previously, the equation is non-linear and can not be solved explicitly in general. But as a first approximation we may use c_{av} for c:

$$K\nabla^2 T = \rho c_{av} \frac{\partial T}{\partial t}$$

and solve this equation in the case of a constant c_{av}. For temperatures close to the melting temperature, we can use the value

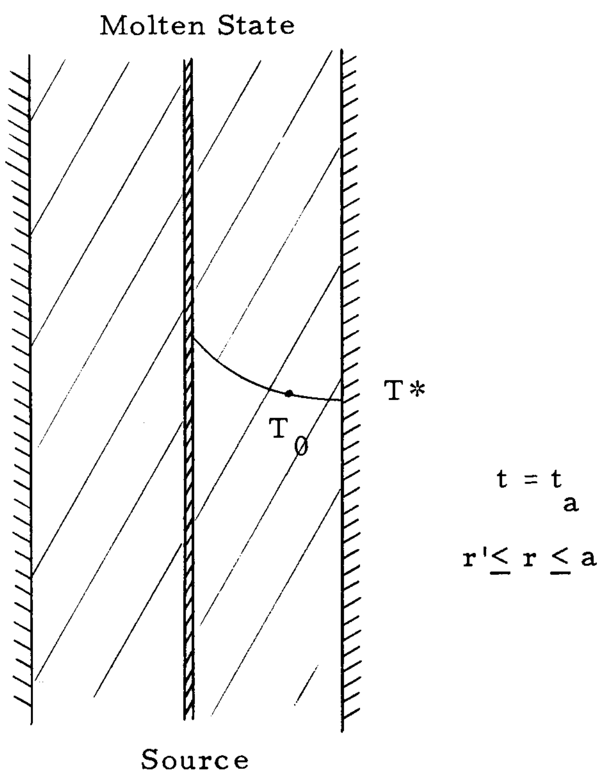

Figure 2.

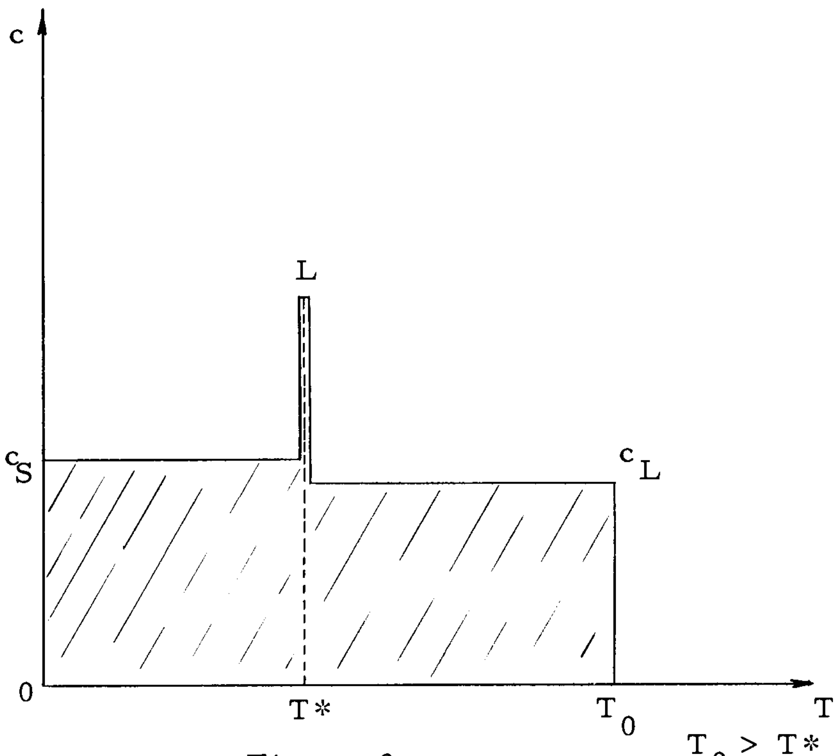

Figure 3.

$$c_{av} \approx c_S + \frac{L}{T^*} \cdot$$

So a solution identical to (16) can be obtained, and since it is a uniformly convergent series, term by term integration can be performed.

If the liquid thus formed is removed at once, then from (15)

$$\int_0^{t_a} Q(t)dt = \pi\rho[a^2 cT^* + L(a^2 - r'^2)] = \pi\rho a^2(cT^* + L) \quad (r'^2 \approx 0)$$

If $Q(t) = Q_0$, then $t_a = \pi\rho a^2(cT^* + L)/Q_0$ which is similar to the expression obtained by Landau in the case of melting of a slab. Also this corresponds to Citron's expression for the complete melting time for the slab.

Sphere with Finite Radius

Consider a sphere of radius a, insulated over its surface. A constant heat input (point source) Q_0 units per unit time is at the centre of the sphere; the initial temperature distribution is zero. The temperature distribution in the sphere before melting occurs will be given by the following boundary value problem:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{1}{k} \frac{\partial}{\partial t}(r^2 T) \qquad 0 < r < a, \quad 0 \leq t \tag{18}$$

(a) $T(r,0) = 0$

(b) $Q_0 = -4\pi K r^2 \left.\frac{\partial T}{\partial r}\right|_{r=0}$ (19)

(c) $\frac{\partial T}{\partial r}(a,t) = 0$

This problem can be solved by applying Laplace's transform. The subsidiary equations are:

$$\frac{d^2\bar{v}}{dr^2} = q^2\bar{v}, \qquad v = Tr \tag{18'}$$

(b) $\frac{Q_0}{p} = -4\pi K r^2 \left[\frac{1}{r}\frac{d\bar{v}}{dr} - \frac{\bar{v}}{r^2}\right]\Big|_{r=0}$

(19)'

(c) $\left.\frac{d\bar{v}}{dr}\right|_{r=a} = \frac{\bar{v}(a,p)}{a}$

Solving (18)' and (19)' we get,

$$\bar{v} = \frac{Q_0}{4\pi K p}\left[\frac{\sinh q(a-r) - aq\cosh q(a-r)}{\sinh qa - aq\cosh qa}\right]$$

By inversion formula,

$$T(r,t) = \frac{Q_0}{8\pi^2 kir} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\sinh\mu(a-r) - a\mu\cosh\mu(a-r)}{\sinh\mu a - a\mu\cosh\mu a} \frac{e^{\lambda t}}{\lambda} d\lambda .$$

$$\mu = \sqrt{\frac{\lambda}{k}}$$

The integrand when expanded will contain only terms with even powers of μ, so it is a single-valued function. So we take the same contour as shown in Figure 1. The radius of the circle is large enough to include all the poles and we will assume that the circle will not pass through any one of the poles. As the radius of the circle tends to infinity the integral over the circumference tends to zero. So the line integral in question will be equal to the sum of the residues at the poles. We have residues at $\lambda = 0$:

$$= \frac{(a-r)^2(2a+r)}{2a^3} + \frac{3krt}{a^3} .$$

The poles at $\sinh\mu a - a\mu\cosh\mu a = 0$ are

$$\lambda = -k\alpha_1^2/a^2, \quad -k\alpha_2^2/a^2, \quad \cdots$$

where $\mu a = \pm i\alpha_n$ are the roots of $\sinh\mu a - a\mu\cosh\mu a = 0$. The residues at $\lambda = -k\alpha_n^2/a^2$ are:

$$= \frac{\sinh\mu(a-r) - a\mu\cosh\mu(a-r)}{\lambda \frac{d}{d\lambda}[\sinh\mu a - a\mu\cosh\mu a]} e^{\lambda t} \quad \text{at} \quad \lambda = -k\alpha_n^2/a^2$$

$$= 2\,\frac{\sin[\alpha_n(a-r)/a] - \alpha_n\cos[\alpha_n(a-r)/a]}{\alpha_n^2 \sin\alpha_n} e^{-k(\alpha_n^2/a^2)t} .$$

$$\therefore \quad T(r,t) = \frac{Q_0}{4\pi K r}\Big[\frac{3krt}{a^3} + \frac{(a-r)^2(2a+r)}{2a^3}$$

$$+ 2\sum_{n=1}^{\infty} \frac{\sin[\alpha_n(a-r)/a] - \alpha_n\cos[\alpha_n(a-r)/a]}{\alpha_n^2 \sin\alpha_n} e^{-k(\alpha_n^2/a^2)t}\Big] \tag{20}$$

After melting occurs, our problem will satisfy the following set of equations :

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial T_L}{\partial r}\right) = \frac{1}{k_L}\frac{\partial}{\partial t}\left(r^2 T_L\right) \qquad 0 < r < R(t), \quad t^* < t. \tag{21}$$

(a) $T_L(R(t),t) = T^*$

(22)

(b) $Q_0 = -4\pi K_L r^2 \frac{\partial T_L}{\partial r}\Big|_{r=0}$

and

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial T_S}{\partial r}\right) = \frac{1}{k_S} \frac{\partial}{\partial t}(r^2 T_S), \qquad R(t) < r < a, \quad t^* \leq t . \tag{23}$$

(a) $T_S(R(t), t) = T^*$

(b) $T_S(r, t^*) = T(r, t^*)$

(c) $\frac{\partial}{\partial r}[T_S(a, t)] = 0$ (24)

(d) $-K_L \frac{\partial T_L}{\partial r} + K_S \frac{\partial T_S}{\partial r} = L\rho \dot{R}(t) \quad \text{at} \quad r = R(t)$

(e) $R(t^*) = 0$

Proceeding exactly the same way as in the case of the cylinder, we are able to get the time for complete melting:

$$Q_0 t_a = \frac{4}{3}\pi a^3[\rho c T^* - \rho_L c_L T^* + L\rho] + 4\pi\rho_L c_L \int_0^a r^2 T_L(r, t_a) dr. \tag{25}$$

When the liquid is removed as soon as it forms,

$$t_a = \frac{4}{3}\pi a^3 \rho [cT^* + L]/Q_0 . \tag{26}$$

For an approximation of $T_L(r, t_a)$, let T_0 be the average temperature, and c_{av}, the average specific heat, then as before,

$$c_{av} = \frac{c_S T* + L + c_L(T_0 - T*)}{T_0} .$$

With this specific heat, assuming no melting occurs, we have from (18) and (19),

$$T_L(r,t) = \frac{Q_0}{4\pi K r}\left[\frac{3k' r t_a}{a^3} + \frac{(a-r)^2(2a+r)}{2a^3}\right.$$

$$\left. + 2\sum_{n=1}^{\infty} \frac{\sin[\alpha_n(a-r)/a] - \alpha_n \cos[\alpha_n(a-r)/a]}{\alpha_n^2 \sin \alpha_n} e^{-k' \alpha_n^2 t_a / a^2}\right]$$

where $k' = \dfrac{K}{\rho c_{av}}$. The time when the sphere starts to melt will be given by (20):

$$T* = \frac{Q_0}{4\pi K}\left[\frac{3kt*}{a^3} - \frac{3}{2a} - \frac{2}{a}\sum_{n=1}^{\infty} \frac{\cos \alpha_n + \alpha_n \sin \alpha_n}{\alpha_n \sin \alpha_n} e^{-k\alpha_n^2 t*/a^2}\right]. \quad (27)$$

CASE OF AN INFINITELY LONG CYLINDER AND A SPHERE

Cylinder with a Very Large Radius

Our main problem is to determine the location of the moving boundary (phase boundary) at any time $t^* < t < t_a$. This is a difficult task due to the non-linear character of the problem. However, the melting proceeds only a relatively small distance into the solid in many cases occurring in practice; so it will be sufficiently accurate to assume $a = \infty$. Landau seems to make this assumption in the case of the melting of slabs. The assumption is applicable, for example, in the case of melting or solidification of wet soils around a cylindrical pipe. In the case of melting, the input Q_0 will represent a source, while in the case of solidification Q_0 is a sink (negative source).

When $a = \infty$, the boundary condition at infinity is physically equivalent to

$$T_S(\infty, t) = T_0 .$$

So our problem now is the following:

$$\frac{\partial}{\partial r}\left(r\frac{\partial T_L(r,t)}{\partial r}\right) = \frac{1}{k_L}\frac{\partial}{\partial t}\left(r\, T_L(r,t)\right) \qquad 0 < r < R(t), \quad t^* < t \tag{28}$$

$$\text{(a)} \quad Q_0 = -2\pi K_L r \left.\frac{\partial T_L}{\partial r}\right|_{r=0} \;\; \text{3/}$$

$$\text{(b)} \quad T_L(r,t) = T^* \quad \text{for} \quad r = R(t) \;. \tag{29}$$

And

$$\frac{\partial}{\partial r}\left(r\frac{\partial T_S(r,t)}{\partial r}\right) = \frac{1}{k_S}\frac{\partial(r\,T_S(r,t)}{\partial t} \qquad R(t) < r < \infty, \quad t^* \leq t \tag{30}$$

$$\text{(a)} \quad T_S(\infty,t) = T_0$$

$$\text{(b)} \quad T_S(R(t),t) = T^* \tag{31}$$

$$\text{(c)} \quad -K_L\frac{\partial T_L}{\partial r} + K_S\frac{\partial T_S}{\partial r} = L\rho\dot{R}(t) \quad \text{at} \quad r = R(t)$$

We can apply the so-called Boltzmann transformation $z = r/R(t)$; so at $r = R(t)$ i.e. at the phase boundary $z = 1$. Or in other words, we fix the moving boundary at $z = 1$. Then equations (28), (29), (30), and (31) become:

3/ The source is assumed to be line source here.

$$\frac{d^2T_L}{dz^2} + \left(\frac{1}{z} + \frac{R(t)\dot{R}(t)}{k_L} \cdot z\right)\frac{dT_L}{dz} = 0, \quad 0 < z < 1 \tag{28)'}$$

(a) $$Q_0 = -2\pi K_L z \frac{dT_L}{dz}\bigg|_{z=0}$$

(29)'

(b) $$T_L(1) = T^*$$

and

$$\frac{d^2T_S}{dz^2} + \left(\frac{1}{z} + \frac{R(t)\dot{R}(t)}{k_S}\; z\right)\frac{dT_S}{dz} = 0, \quad 1 < z < \infty \tag{30)'}$$

(a) $$T_S(\infty) = T_0$$

(b) $$T_S(1) = T^* \tag{31)'}$$

(c) $$-K_L\frac{dT_L}{dz} + K_S\frac{dT_S}{dz} = L\rho\, R(t)\dot{R}(t) \quad \text{at} \quad z = 1.$$

The above equations could be solved explicitly if $R(t)\dot{R}(t)$ = constant. Or $R = ct^{\frac{1}{2}}$ where c is an arbitrary constant to be determined. If the value of c is known, the location of the phase boundary is known at any time t. This result has been applied by other authors [4, 9, 12]. Solving (28)':

$$\frac{dT_L}{dz} = A \frac{e^{-z^2c^2/4k_L}}{z} \tag{32}$$

$$\therefore \quad T_L = A \int_0^z \frac{e^{-z^2c^2/4k_L}}{z} dz + B.$$

Using conditions (29)' (a), (b) we have

$$T_L(z) - T^* = \frac{Q_0}{2\pi K_L} \int_z^1 \frac{e^{-z^2c^2/4k_L}}{z} dz. \tag{33}$$

From (33) we get, putting $\beta = zc/2\sqrt{k_L}$ $\left(\beta = \frac{r}{2\sqrt{k_L t}}\right)$,

$$T_L(r,t) - T^* = \frac{Q_0}{2\pi K_L} \int_{\frac{r}{2\sqrt{k_L t}}}^{\frac{c}{2\sqrt{k_L}}} \frac{e^{-\beta^2}}{\beta} d\beta \tag{34}$$

Similarly from (30)' and (31)'

$$\frac{dT_S}{dz} = \frac{A_1 e^{-z^2c^2/4k_S}}{z} \tag{35}$$

$$\therefore \quad T_S(z) = A_1 \int_1^z \frac{e^{-z^2c^2/4k_S}}{z} dz + B_1$$

And using conditions (31)' (a), (b) and putting $\beta = zc/2\sqrt{k_S}$ we get,

$$T_S(r,t) = T^* - (T^* - T_0) \int_{\frac{c}{2\sqrt{k_S}}}^{\frac{r}{2\sqrt{k_S t}}} \frac{e^{-\beta^2}}{\beta} d\beta \Big/ \int_{\frac{c}{2\sqrt{k_S}}}^{\infty} \frac{e^{-\beta^2}}{\beta} d\beta \tag{36}$$

Now we will be able to satisfy the boundary condition (31)' (c):

$$-K_L \frac{dT_L}{dz} + K_S \frac{dT_S}{dz} = L\rho c^2/2 . \tag{37}$$

Utilizing (32) and (35) in (37) we obtain:

$$\frac{Q_0}{2\pi} e^{-c^2/4k_L} - K_S(T^* - T_0)e^{-c^2/4k_S} \Big/ \int_{\frac{c}{2\sqrt{k_S}}}^{\infty} \frac{e^{-\beta^2}}{\beta} d\beta = \rho L c^2/2 \tag{38}$$

Let $\beta^2 = u$, and $c = 2\lambda\sqrt{k_L}$.

$$\therefore \int_{\lambda\sqrt{\frac{k_L}{k_S}}}^{\infty} \frac{e^{-\beta^2}}{\beta} d\beta = \frac{1}{2} \int_{\lambda^2(k_L/k_S)}^{\infty} \frac{e^{-u}}{u} du$$

$$= -\frac{1}{2} \int_{\infty}^{\lambda^2(k_L/k_S)} \frac{e^{-u}}{u} du$$

$$= -\frac{1}{2} Ei\left(-\frac{\lambda^2 k_L}{k_S}\right).$$

So from (38):

$$\frac{Q_0}{2\pi} e^{-\lambda^2} + \frac{2K_S(T^* - T_0)e^{-\lambda^2(k_L/k_S)}}{Ei(-\lambda^2 k_L/k_S)} = 2\rho L\lambda^2 k_L$$

or

$$\frac{Q_0}{4\pi} e^{-\lambda^2} + \frac{K_S(T^* - T_0)e^{-\lambda^2(k_L/k_S)}}{Ei(-\lambda^2 k_L/k_S)} = L\rho\lambda^2 k_L , \qquad (39)$$

which is the result given by Carslaw and Jaeger [4, p. 296]. The left hand side decreases monotonically from $Q_0/4\pi$ to $-\infty$ as λ increases from 0 to ∞. So it has one and only one real solution.

Sphere with Infinite Radius

Now to determine the location of the solid liquid interface and the temperature T_L and T_S, we make use of the same assumption that the sphere is of infinite radius and that the temperature of the surface is held constant. So our new set of equations and conditions are as follows:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial T_L}{\partial r}\right) = \frac{1}{k_L}\frac{\partial}{\partial t}(r^2 T_L), \quad 0 < r < R(t), \quad t^* < t \qquad (40)$$

$$\text{(a)} \quad Q_0 = -4\pi K_L r^2 \frac{\partial T_L}{\partial r}\bigg|_{r=0} \qquad (41)$$

$$\text{(b)} \quad T_L(R(t), t) = T^*$$

and

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial T_S}{\partial r}\right) = \frac{1}{k_S}\frac{\partial}{\partial t}(r^2 T_S), \qquad R(t) < r < \infty, \quad t^* \leq t \tag{42}$$

$$\begin{aligned} &(a) \quad T_S(\infty, t) = T_0 \\ &(b) \quad T_S(R(t), t) = T^* \\ &(c) \quad -K_L \frac{\partial T_L}{\partial r} + K_S \frac{\partial T_S}{\partial r} = L\rho \dot{R}(t) \ \text{ at } \ r = R(t) . \end{aligned} \tag{43}$$

Letting $z = r/R(t)$ as before, we get from (40), (41), (42) and (43):

$$\frac{d^2 T_L}{dz^2} + \left(\frac{2}{z} + \frac{z}{k_L} R(t)\dot{R}(t)\right)\frac{dT_L}{dz} = 0, \quad 0 < z < 1 \tag{40$'$}$$

$$\begin{aligned} &(a) \quad Q_0 = -4\pi K_L z^2 R(t) \frac{dT_L}{dz}\bigg|_{z=0} \\ &(b) \quad T_L(1) = T^* \end{aligned} \tag{41$'$}$$

$$\frac{d^2 T_S}{dz^2} + \left[\frac{2}{z} + \frac{z}{k_S} R(t)\dot{R}(t)\right]\frac{dT_S}{dz} = 0, \quad 1 < z < \infty \tag{42$'$}$$

$$\begin{aligned} &(a) \quad T_S(\infty) = T_0 \\ &(b) \quad T_S(1) = T^* \\ &(c) \quad -K_L \frac{dT_L}{dz} + K_S \frac{dT_S}{dz} = L\rho R(t)\dot{R}(t) \ \text{ at } \ z = 1 \end{aligned} \tag{43$'$}$$

The above equations can be solved explicitly only when $R(t) = ct^{\frac{1}{2}}$, where c is any constant.

$$\therefore \; R(t)\dot{R}(t) = c^2/2 \, .$$

But the condition (41)'(a) still presents difficulties. However if we assume that the heat input is $Q_0 t^{\frac{1}{2}}$, an analytical solution can be obtained, [see Paterson[21]]. With this assumption we have the solution as follows:

From (40)' and (41)',

$$\frac{dT_L}{dz} = A \frac{e^{-z^2c^2/4k_L}}{z^2} \tag{44}$$

$$\therefore \quad T_L(z) = A \int_0^z \frac{e^{-z^2c^2/4k_L}}{z^2} \, dz + B$$

$$\therefore \quad T_L(z) = T* + \frac{Q_0}{4\pi K_L c} \int_z^1 \frac{e^{-z^2c^2/4k_L}}{z^2} \, dz$$

And letting $\beta = \dfrac{zc}{2\sqrt{k_L}}$,

$$T_L(r,t) = T* + \frac{Q_0}{4\pi K_L c} \int_{\frac{r}{2\sqrt{k_L t}}}^{\frac{c}{2\sqrt{k_L}}} \frac{e^{-\beta^2}}{\beta^2} \, d\beta \Big/ \frac{2\sqrt{k_L}}{c} \, .$$

$$\therefore \quad T_L(r,t) = T^* + \frac{Q_0}{8\pi K_L \sqrt{k_L}} \int_{\frac{r}{2\sqrt{k_L t}}}^{\frac{c}{2\sqrt{k_L}}} \frac{e^{-\beta^2}}{\beta^2}\, d\beta. \tag{45}$$

Similarly we have from (42)' and (43)':

$$\frac{dT_S}{dz} = \frac{A_1 e^{-z^2c^2/4k_S}}{z^2} \tag{46}$$

$$\therefore \quad T_S(z) = A_1 \int_1^z \frac{e^{-z^2c^2/4k_S}}{z^2}\, dz + B .$$

Evaluating the constants, we finally get

$$T_S(r,t) = T^* + (T_0 - T^*) \int_{\frac{c}{2\sqrt{k_S}}}^{\frac{r}{2\sqrt{k_S t}}} \frac{e^{-\beta^2}}{\beta^2}\, d\beta \Bigg/ \int_{\frac{c}{2\sqrt{k_S}}}^{\infty} \frac{e^{-\beta^2}}{\beta^2}\, d\beta \tag{47}$$

Then (43)'(c) gives:

$$\frac{Q_0}{4\pi} e^{-c^2/4k_L} - \frac{cK_S(T^*-T_0)e^{-c^2/4k_S}}{\int_1^\infty \frac{e^{-z^2c^2/4k_S}}{z^2}\,dz} = L\rho c^3/2 \qquad (48)$$

Letting $\beta = \frac{zc}{2\sqrt{k_S}}$ where $z = \frac{r}{ct^{\frac{1}{2}}}$ we get

$$\int_1^\infty \frac{e^{-z^2c^2/4k_S}}{z^2}\,dz = e^{-c^2/4k_S} - \frac{c}{\sqrt{k_S}} \int_{\frac{c}{2\sqrt{k_S}}}^\infty e^{-\beta^2}\,d\beta$$

$$= e^{-c^2/4k_S} - \frac{c}{\sqrt{k_S}} \frac{\sqrt{\pi}}{2} \operatorname{erfc}\left(\frac{c}{2\sqrt{k_S}}\right)$$

So (48) will be

$$\frac{Q_0}{4\pi} - \frac{cK_S(T^* - T_0)}{1 - \frac{c}{2}\sqrt{\frac{\pi}{k_S}}\, e^{c^2/4k_S} \operatorname{erfc}\left(\frac{c}{2\sqrt{k_S}}\right)} = L\rho c^3/2 \qquad (49)$$

which is identical with the equation obtained by Paterson [21]. The left hand side of (49) decreases monotonically from $\frac{Q_0}{4\pi}$ to $-\infty$ as c increases from 0 to ∞. Hence it has always one and only one real root. We have derived a solution for propagation of heat from a point source where strength increases as $\sqrt{t}$. Due to this fact, however, this solution is of less practical importance.

We have obtained the same results as Paterson, but the method of approach is different.

BIBLIOGRAPHY

1. Boley, Bruno A. A method of heat conduction analysis of melting and solidification problems. Journal of Mathematics and Physics, 40, 4:300-313. Dec. 1961.

2. ________________. The analysis of problems of heat conduction and melting. High temperature structures and materials. Proceedings of the 3rd Symposium on Naval Structural Mechanics held in New York. Jan. 1963.

3. Citron, Stephen J. Heat conduction in a melting slab. Journal of the Aerospace Science, 27, 3:219-228. March 1960.

4. Carslaw, H.S. and J.C. Jaeger. Conduction of heat in solids. Oxford University Press, 1959. p. 282.

5. ________________. Operational methods in applied mathematics. Dover Publications, Inc. N.Y. 1963. p. 76.

6. Chambrē, P.L. On the dynamics of phase growth. Quarterly Journal of Mechanics and Applied Mathematics, 9, 2:224-233. June 1956.

7. Evans II, G.W., E. Issacson and J.K.L. Macdonald. Stefan like problem. Quarterly of Applied Mathematics, 8, 3:312-319. Oct. 1950.

8. Friedman, Avner. Free boundary problems for parabolic equations. I. Melting of solids. Journal of Mathematics and Mechanics, 8, 4:499-518. July 1959.

9. Field, A.L. Solidification of steel in the ingot mould. Transactions of the American Society of Steel Treating, 11:p. 264. 1927.

10. Gibson, R.E. A heat conduction problem involving a specified moving boundary. Quarterly of Applied Mathematics, 16, 4:426-430. Jan. 1959.

11. Ingersoll, L.R., F.T. Adler, H.J. Plass and A.C. Ingersoll. Theory of earth heat exchangers for the heat pump. Heating, Piping and Air Conditioning, 22, 5:113-122. May 1950.

12. Ingersoll, L.R., Otto J. Zobel, A.C. Ingersoll. Heat conduction with engineering, geological and other applications. University of Wisconsin Press, 1954. p. 190.

13. Kolodner, I.I. Free boundary problem for the heat equation with applications to problems of change of phase. Communication on Pure and Applied Mathematics, 9, 1:1-31. Feb. 1956.

14. Kreith, F., F.E. Romie. A study of the thermal diffusion equation with boundary conditions corresponding to solidification or melting of materials initially at the fusion temperature. Proceedings of the Physical Society, London. Series B, 68:377-391. 1955.

15. Landau, H.G. Heat conduction in a melting solid. Quarterly of Applied Mathematics, 8, 1:81-94. 1950.

16. Lightfoot, N.M.H. The solidification of molten steel. Proceedings of the London Mathematical Society. Series 2, 31:97-116. 1930.

17. Lotkin, Mark. Calculation of heat flow in melting solid. Quarterly of Applied Mathematics, 18, 1:79-85. 1960.

18. Miranker, Willard L. and Joseph B. Keller. The Stefan problem for a non-linear equation. Journal of Mathematics and Mechanics, 9, 1:67-70. Jan. 1960.

19. Miranker, W.L. Free boundary value problem for the heat equation. Quarterly of Applied Mathematics, 16, 2:121-130. July 1958.

20. Pekeris, C.L. and L.B. Slichter. Problem of ice formation. Journal of Applied Physics, 10:135-137. Feb. 1939.

21. Paterson, S. Propagation of a boundary of fusion. Proceedings of the Glasgow Mathematical Association, Vol. 1, :42-47. Jan. 1952.

22. Saitō. On the distribution of temperature in steel ingots during cooling. Tōhoku Imperial University Science Reports, 10, 1921.