PLANE WEDGE FLOWS OF A HOMOGENEOUS INCOMPRESSIBLE ISOTROPIC REINER-RIVLIN FLUID

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A THESIS

submitted to

OREGON STATE COLLEGE

in partial fulfillment of
the requirements for the
degree of

DOCTOR OF PHILOSOPHY

June 1957

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Introduction

In this study we have considered the boundary value problem of a viscous fluid flowing in a wedge. A Reiner-Rivlin Fluid is a fluid which has a nonlinear stress relationship taking the form

$$(1) \qquad t^i{}_j = -p\,\delta^i{}_j + Md^i{}_j + Nd^i{}_k d^k{}_j$$

This is in contrast to the linear relationship of the classical Newton-Cauchy-Poisson Law:

$$(2) \qquad t^i{}_j = (-p + \lambda\, d^k{}_k)\,\delta^i{}_j + 2\mu d^i{}_j$$

A fluid having the generalized stress relation of equation (1) serves to help describe the flows of a high polymer solution more adequately than is possible with the linear relation. In the analysis of the wedge flow it is proved that M is a constant. Furthermore, the stress components are investigated and tables are calculated for evaluating these components under various boundary conditions, yielding a more thorough treatment of the problem than is found in the literature.

2. Preliminary Mathematical Formulation

The studies of fluid dynamics can be partially based upon the equations of motion which were early established and have come down through the years substantially unaltered. The principle of conservation of mass takes the form of Eulers's continuity equation

$$(3) \qquad \frac{\partial \rho}{\partial t} + (\rho \dot{x}^i)_{,i} = 0$$

where ρ is the density of the media and $\dot{x}^i$ are the velocity components of the flow. The principle of the conservation of momentum leads to the existence of a stress matrix T with components $t^i{}_j$, and Cauchy's laws of motion

$$(4) \qquad \rho \ddot{x}_i = \rho f_i + t_i{}^j{}_{,j}$$

where $\ddot{x}_i$ represents the i^{th} component of acceleration, f_i the i^{th} component of extraneous force per unit mass. These equations in themselves are inadequate for they apply to both hydrodynamic and elastic phenomena, whereas the two theories are quite distinct.

To complete the characterization of the fluid in question we must postulate the stress relations. To this end we employ G.G. Stokes's statement of fluidity when he defined the mechanical properties of a fluid in saying, (11, p. 80)

> "That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in this neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P; and that the relative motion due to any motion of rotation may be eliminated without affecting the difference of the pressure above mentioned."

The Newton-Cauchy-Poisson Law is one possible interpretation of Stokes's principle. However, the fluid dynamics then describe the flow without regard to springiness of form, so that when released from all deforming forces except a hydrostatic pressure, they retain their present shapes. This theory would imply that a double rate of deformation, if dynamically possible, would lead to double viscous forces (14, p.126). Such a linear response to the rate of deformation is in general not experienced except as an approximation for small rates of deformation. If in addition to a natural viscosity, a fluid also has a natural elasticity, the linear theory becomes less adequate. Substances such as gelatin or high polymer solutions have characteristics of viscosity and elasticity; the Reiner-Rivlin fluid is a mathematical model for these types of substances.

Theories of fluid dynamics employing a generalized form of Stokes's principle are found in M. Reiner's theory of compressible fluids and R. S. Rivlin's theory of incompressible fluids. C. Truesdell unified these theories into one of a model which he called a Reiner-Rivlin fluid (13, pp. 231-35, and 14, pp. 235-238). Properties of this fluid have been partially investigated by constructing special solutions (14, pp. 239-245). The known solutions are for a rectilinear shearing flow, the Poiseuille flow in a pipe or flow in a tube viscometer, the Couette flow or flow between two concentric rotating cylinders, and parallel plate viscometer or flow between two parallel rotating plates. We propose to discuss yet another special flow, the wedge flow.

Stokes's principle as quoted above can be translated into a matrix equation

$$(5) \qquad T - pI = -V = F(D)$$

where T is the stress matrix which in the classical theory has been assumed to be linear, p a pressure, I the identity matrix, and V is an extra stress, a function of the symmetric rate of deformation matrix D. The components of this symmetric matrix D are represented in tensor notation by $d^i{}_j$ where

$$(6) \qquad d^i{}_j = \tfrac{1}{2}g^{ik}(\dot{x}_{k},{}_j - \dot{x}_{j},{}_k) = \tfrac{1}{2}(\dot{x}^i,{}_j - \dot{x}_j,{}^i)$$

where g^{ik} represents the contravariant components of the metric tensor and $\dot{x}_{i,j}$ represent the gradients of the velocity components.

3. The Reiner-Rivlin Fluid (14, pp. 235-245)

The Reiner-Rivlin fluid is defined to be a continuous medium obeying Stokes's principle in the specific form

(7) $$T = T(D, \mu_n, \Theta, \Theta_o, p, p_o, t_n)$$

where μ_n = natural viscosity of dimension M/(LT)

Θ_o = a reference temperature of dimension Θ

p_o = natural elasticity (as Young's Modulus) of dimension $M/(LT^2)$

t_n = natural time = μ_n/p_o.

The material constant t_n has been introduced since, "Any body endowed both with viscosity and with elasticity unavoidably possesses also a material constant of the dimension of time." (13, p.231). This new constant we define as natural time. In order that the Reiner-Rivlin fluid obeys the ordinary laws of hydrostatics, it is necessary that

(8) $$T(0, \mu_n, \Theta, \Theta_o, p, p_o, t_n) = -pI.$$

Dimensional analysis requires that

(9) $$T = pf(t_n D, pt_n/\mu_n, p/p_o, \Theta/\Theta_o)$$

The condition of isotropy is exhibited by expressing the stress tensor as

$$(10) \qquad t^i{}_j = -p\,\delta^i{}_j + Md^i{}_j + Nd^i{}_k d^k{}_j$$

where again the $d^i{}_j$ 'es are the components of the rate of displacement matrix. Originally the stress matrix T was assumed to have a power series expansion in terms of the matrix t_nD, however, the Hamilton-Cayley equation makes it possible for us to eliminate all powers of D higher than the second. It then follows that the coefficients M and N in equation (10) depend upon the dimensionless parameters Θ/Θ_o and p/p_o and are analytic functions of the scalar invariants of the matrix t_nD. In other words M and N are power series expansions of the scalar invariants of t_nD and the coefficients of this expansion depend on the dimensionless parameters Θ/Θ_o and p/p_o.

In our development we will consider a second order approximation by which is meant that the coefficients M and N are assumed to be independent of temperature and pressure. It follows that the power series expansions of M and N in terms of the scalar invariants of t_nD have constant coefficients. Under this assumption, we prove that $M = 2\mu$ for the wedge flow where μ is a constant identified as the coefficient of viscosity.

4. The Stress Matrix for Wedge Flows

The wedge flows are defined by

(11) $\dot{r} = \dot{x}^1 = F(r,\theta), \quad \dot{\theta} = \dot{x}^2 = 0, \quad \dot{z} = \dot{x}^3 = 0$

Since the fluid is incompressible, the equation of continuity implies that

(12) $F(r,\theta) = f(\theta)/r$

where $f(\theta)$ is the flow per unit density per radian of wedge angle in a unit thickness.

In 1915, G. B. Jeffery (5, p. 455-465) published a solution to this problem using the Navier-Stokes equation and boundary conditions as the defining relationships; in 1916, Georg Hamel (3,p. 34-60) published an exact solution as a stream function problem giving a good discussion of the problem along with several other related flows. Later writings[1] on this problem have yielded few new contributions; however, S. Goldstein (2, p.105-110) gives a good summary of the results based on the classical theory.

A formal simplification is obtained by expressing

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1. The problem has also been considered by W. J. Harrison, (4, p. 307-312), K. Pohlhausen (10, P.266), T. von Kármán, (6, p. 146), F. Noether (9, pp. 733-736), W. Tollmein (12, pp. 257-260), and W. R. Dean (1, pp. 759-777).

the problem in terms of the symmetrical matrix of physical components[2], then the matrix corresponding to T is

$$(13)\quad (\widehat{ij}) = -p\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \frac{M}{r^2}\begin{pmatrix}-f&\frac{1}{2}f'&0\\\frac{1}{2}f'&f&0\\0&0&0\end{pmatrix} + (f^2+\tfrac{1}{4}f'^2)\frac{N}{r^4}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

Considered equivalently as a plane problem, the matrix of equation (13) can be written as a sum of two matrices, that is

$$(14)\qquad (\widehat{ij}) = -\bar{p}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{M}{r^2}\begin{pmatrix}-f&\frac{1}{2}f'\\\frac{1}{2}f'&f\end{pmatrix}$$

where

$$\bar{p} = p - IIN, \qquad II = -(f^2 + \tfrac{1}{4}f'^2)/r^4$$

and M and N are power series expansions of II. When a solution f has been found, $\widehat{\theta\theta}$ and $\widehat{rr}$ are then immediately related to p, the ambient pressure, recalling that $\widehat{zz} = -p$. We obtain a physical interpretation for $\bar{p}$ by noting that at the walls $(f = 0)$

$$(15)\qquad \widehat{\theta\theta} = \widehat{rr} = -\bar{p}.$$

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2. Physical components are sometimes referred to as components of a vector, or coefficients which are neither covariant nor contravariant. i.e. "Physical components of vectors and tensors referred to general curvilinear co-ordinates are defined and shown to represent quantities possessed of the natural physical dimensions of the field and capable of immediate physical interpretation." C. Truesdell (15, p.345). For a brief discussion see McConnell (8, p.304).

5. The Flow Equations and Stress Relations

The flow equations in tensor notation are

(16) $$t^{ij}{}_{,j} = \ddot{x}^i; \qquad \ddot{x}^1 = f^2/r^3, \quad \ddot{x}^2 = 0.$$

For an incompressible fluid ρ is a constant and the pressure p is a basic unknown. In terms of the physical components these become (7, p.90)

(17)
$$\frac{\partial}{\partial r}\left(-\bar{p} - \frac{fM}{r^2}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{f'M}{2r^2}\right) - \frac{2fM}{r^3} = -\rho\frac{f^2}{r^3}$$

$$\frac{\partial}{\partial r}\left(\frac{f'M}{2r^2}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(-\bar{p} + \frac{fM}{r^2}\right) + \frac{fM}{r^3} = 0$$

Assuming that $M = 2\mu$ = constant,
these equations can be further simplified to

(18) $$-\frac{\partial\bar{p}}{\partial r} + \frac{\mu f'' + \rho f^2}{r^3} = 0, \quad -\frac{\partial\bar{p}}{\partial\theta} + \frac{2\mu f'}{r^2} = 0.$$

It will then be shown that

(19) $$\mu f'' + \rho f^2 = -\mu(4f + k), \qquad k = \text{constant}$$

and hence

(20) $$-\bar{p} + \frac{\mu(4f + k)}{2r^2} = \text{constant}$$

It follows that

$$\widehat{rr} = -\bar{p} - \frac{2\mu f}{r^2} = -\mu\frac{8f + k}{2r^2} + \text{constant}$$

(21) $$\widehat{\theta\theta} = -\bar{p} + \frac{2\mu f}{r^2} = -\frac{\mu k}{2r^2} + \text{constant}$$

$$\widehat{zz} = -p \qquad \widehat{\theta r} = \frac{\mu f'}{r^2}$$

We now need to find the solution $f = f(\theta)$, prove the assertion $M = 2\mu$ is the only admissable solution within the framework of our hypothesis, and investigate the nature of the constant k for various boundary conditions.

From equations (17) we obtain an integrability condition on M by eliminating p to get

(22) $$\frac{f'}{2r^3}\frac{\partial^2 M}{\partial \theta^2} - \frac{2f}{r^2}\frac{\partial^2 M}{\partial r \partial \theta} - \frac{f'}{2r}\frac{\partial^2 M}{\partial r^2} - \frac{3f'}{2r^2}\frac{\partial M}{\partial r} + \frac{2f + f''}{r^3}\frac{\partial M}{\partial \theta} + \frac{\frac{1}{2}f''' + 2f'}{r^3} M + \frac{2\mu f f'}{r^3} = 0.$$

Define

(23) $$y = -(f^2 + \tfrac{1}{4}f'^2) = y(\theta), \qquad x = II = y/r^4$$

Then since $f = f(\theta)$ and $M = M(x)$, we can use primes to denote differentiation of a function with respect to its argument without ambiguity of notation; hence

(24) $$f' = \frac{df}{d\theta}, \qquad M' = \frac{dM}{dx}.$$

The integrability condition (22) now takes the form

$$(25) \qquad 2\rho f f' + AM + \frac{x}{y} BM' + \frac{x^2}{y^2} CM'' = 0$$

where

$$A = \tfrac{1}{2} f''' + 2f'$$

$$(26) \qquad B = (64f^2 + 32ff'' + 3f''^2 + f'f''')\, f'/4$$

$$C = (64f^4 + 32f^3 f'' + 16f^2 f'^2 + 16ff'^2 f'' + f'^2 f''^2 - 4f'^4)\, f'/8$$

In order that M be an analytic function of x with non-zero viscosity μ

$$(27) \qquad M = 2\mu + \sum_{n=1}^{\infty} a_n x^n$$

We get as a necessary and sufficient condition

$$(H) \qquad 2\rho f f' + 2\mu A = 0$$

$$(*) \qquad [A + n\, B/y + n(n-1)\, C/y^2]\, a_n = 0, \qquad n \geq 1.$$

Suppose now that there exists three $n \geq 1$, say n_1, n_2, and n_3 for which $a_n \neq 0$. Then (*) is compatible for A, B, and C not identically zero if and only if

$$\begin{vmatrix} 1 & n_1 & n_1(n_1-1) \\ 1 & n_2 & n_2(n_2-1) \\ 1 & n_3 & n_3(n_3-1) \end{vmatrix} = (n_1-n_2)(n_2-n_3)(n_3-n_2) = 0$$

but this is impossible, hence there are four cases.

I $\quad a_n = 0$ for $n \geq 1$ then f must satisfy H and that is all.

II $\quad a_n = 0$ for $n \geq 1$ except $n = n_1$, then f must satisfy both H and

(H^1) $\quad n_1 B/y + n_1(n_1-1)C/y^2 = \rho f f'/\mu$

III $\quad a_n = 0$ for $n \geq 1$ except for $n = n_1$, and $n = n_2$, then f must satisfy H, H^1, and

(H^2) $\quad yB + (n_1 + n_2 - 1)\,C = 0$

IV $\quad$ There are three a_n, $n \geq 1$ which do not vanish then f must satisfy H and

(H^3) $\quad A = 0, \quad B = 0, \quad C = 0.$

We exclude case IV since it implies $ff' = 0$, only the trivial case.

In equation H let $\nu = \mu/\rho$ where ν is the kinematical viscosity, μ the coefficient of viscosity and ρ the density, then first and second integrals are

(28) $\quad f'' + 4f + \frac{1}{\nu} f^2 + k = 0$

(29) $\quad f'^2 = \frac{2}{3\nu}(h - 3k\nu f - 6\nu f^2 - f^3)$

6. Coefficient M

Let us digress to prove the contention that $M = 2\mu$ = constant. This we do by showing that case II and hence case III cannot be satisfied. Case II can be equivalently expressed as

$$Q = 8\left[n(n-1)C + nyB - \frac{1}{\nu} ff'y^2\right]/f' = 0$$

where A, B, and C are defined by equations (26). Substitute equation H and its first and second integrals (28) and (29) to eliminate all derivatives and obtain a polynomial of degree 7 in f. This polynomial must vanish identically in a closed interval, which is possible only if all coefficients vanish identically. The coefficient of the seventh degree term is $-n\nu(2n+1)/3$ which vanishes only for $n = 0, -\frac{1}{2}$ or $\nu = 0$. It follows that $M = 2\mu$ as asserted.

7. Flow Classification from the Equation

In considering equation (29), Hamel (3, pp. 34-60) made the following observations. If f_1, f_2, and f_3 are roots of the equation, it follows that

$$
(30)\quad \begin{aligned} & f_1 + f_2 + f_3 = -6\nu \\ & f_1 f_2 + f_2 f_3 + f_3 f_1 = 3k\nu \\ & f_1 f_2 f_3 = h. \end{aligned}
$$

The first of these tell us that at least one root has a negative real part. Order the roots such that

$$R(f_1) \geq R(f_2) \geq R(f_3)$$

Since at least one root of the cubic is real, we have the following cases.

(a) Three real roots

(i) $-\infty < f < f_3 \leq -2\mu$

(ii) $f_2 \leq f \leq f_1$

(b) One real root f_1, which may be positive

$-\infty \leq f \leq f_1$

The differential equation admits two possible flows, (1) no fixed walls, thus a source or sink in an infinite fluid. In this case f must be a periodic function of θ, $f = -\infty$ is admitted, $f = 0$ cannot appear. This implies that the three roots are real and $f_2 \leq f \leq f_1$. (2) two fixed walls on which $f = 0$

(a) Three real roots

(i) $f_2 \leq f \leq 0$

(ii) $0 \leq f \leq f_1$

(b) One real root, f must be positive and

$0 \le f \le f_1$.

Case (1) admits both convergent and divergent flows; however, these are not under consideration. In the following we will consider the convergent flows of 2ai and the divergent flows of 2aii and 2b.

Since $f'^2 \geq 0$ and the wall condition is to be included in the flow it follows that $h \geq 0$. By Descartes rule of signs it follows that if $h > 0$, $f'^2 = 0$ has at most one positive root and two negative roots. If $h = 0$, one root would be zero and one positive if $k < 0$, otherwise no positive root. These features are summarized in Table I and Figure 1 on page 16.

Imposing the condition that the flow is to be symmetrical about $\theta = 0$, i.e. $f(\theta) = f(-\theta)$ it follows that

(31) $$f'(\theta)\Big|_{\theta = 0} = 0$$

Furthermore, if the fluid is to be stationary at the walls as already stated, and the angle of the wedge is $2\theta_0$ it follows that

(32) $$f(\theta_0) = 0,$$

a value which must be in the flow region.

TABLE I

	Condition on roots of Equation (29)	Possible flows	Corresponding Coordinate Axes on Figure 1
$h < 0$		no flows	
$h = 0$	$f_1>0, f_2=0, f_3<0$	divergent flow	(1,b)
	$k=0,\ f_1=f_2=0$	no flows	(2,c)
	$k > 0\quad f_1=0$		
	$k<3\quad f_3 \leq f_2<0$	convergent flow	(3,b)
	$k>3\quad f_2 = f_3$	no flows	(4,a)
$h > 0$	$f_1 > 0$		
	$(6+f_1)^2 > 4h$ $f_3 \leq f_2 < 0$	both types	(2,b)
	$(6+f_1) \leq 4h$ $f_2 = \bar{f}_3$	divergent flows	(2,a)

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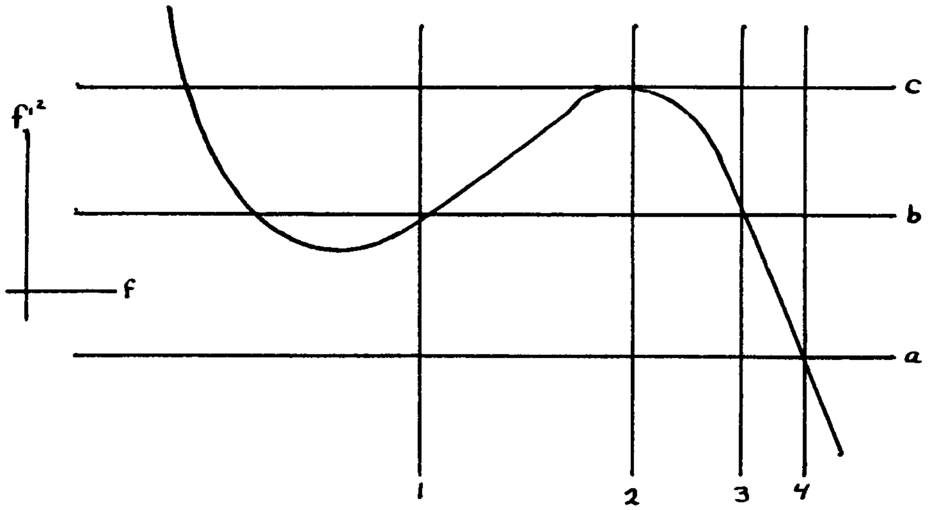

Figure 1

If the flow is divergent let $f^* = f_1$ and if convergent let $f^* = f_2$. Equation (29) can now be written in the factored form

$$(33) \qquad f'^2 = \frac{2}{3\nu}\,(f^* - f)\left[\frac{h}{f^*} + (6\,\nu + f^*)f + f^2\right]$$

where

$$(34) \qquad h = f^*\left[3k\,\nu + f^*\,(6\nu + f^*)\right].$$

The equation is made dimensionless by defining

$$(35) \qquad R = \frac{|f^*|}{\nu}\,,\quad K = \frac{k}{|f^*|}\,,\quad H = \frac{h}{|f^*|^3}\,,\quad w = \frac{f}{|f^*|}\,.$$

8. Convergent Flows

For convergent flows the dimensionless form of equation (33) is

$$(36) \qquad w'^2 = -\,\frac{2R}{3}\,(w + 1)\,\left(w^2 + \frac{6-R}{R}\,w - H\right),$$
$$H = (-3K + 6 - R)/R$$

where the boundary conditions are transformed as follows

$$f = 0,\quad w = 0;\quad f = f^*,\quad w = -1.$$

According to Table I on page 16, there are two cases to consider. If $h = 0$, the equation takes the form

$$(37) \qquad w'^2 = -\,\frac{2R}{3}\,w\,(w + 1)\,\left(w + \frac{6-R}{R}\right)$$

where the roots are ordered

$$-(6-R)/R \leq -1 < 0.$$

This in turn imposes the conditions that $R \leq 3$ and $1 \leq K \leq 2$.

When $H = 0$ it follows that at the walls where $w = 0$, that $w' = 0$ also which is the condition necessary for an extreme value. It also follows that the second derivative (equation 28) has the sign of $-K$ at this point. Since K is positive it follows that f is a maximum at the walls. Figure 2 is a geometrical representation of this relationship. For small wedge angles and Reynold's numbers the velocity distribution is approximately parabolic, with increasing speeds the distribution tends to become flatter in the middle of the channel with the drop in velocity to zero taking place in a layer near the wall which decreases in thickness as R increases.

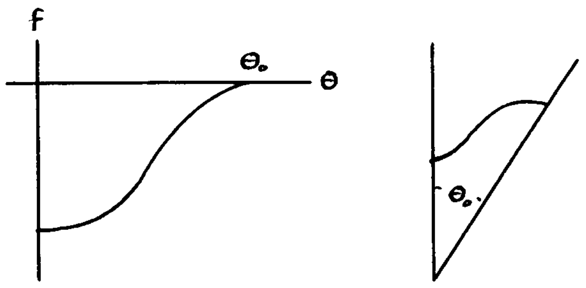

Figure 2

The equation (37) integrates to

$$\frac{6 - 2R}{wR + 6-R} = dn^2 \left(\sqrt{\frac{6-R}{6}}\, \theta \,\middle|\, \frac{R}{6-R} \right) \qquad (38)$$

or

$$\frac{1 - m}{mw + 1} = dn^2 \left(\sqrt{\frac{R}{6m}}\, \theta \,\middle|\, m \right), \quad m = \frac{R}{6-R}.$$

Under the assumption that m dn u = 0, this simplifies to

$$w = - cd^2 \left(\sqrt{\frac{6-R}{6}}\, \theta \,\middle|\, \frac{R}{6-R} \right)$$

The stress components (equation 21) take on the form

$$\widehat{rr} = -\mu |f_2| \frac{w+K}{r^2} + \text{constant}$$

$$= -\mu |f_2| \frac{3w + 6-R}{3r^2} + \text{constant}.$$

$$\widehat{\theta\theta} = -\mu |f_2| \frac{K}{2r^2} + \text{constant} \qquad (39)$$

$$= -\mu |f_2| \frac{6-R}{6r^2} + \text{constant}$$

$$\widehat{zz} = -p, \qquad \widehat{\theta r} = \frac{\mu |f_2| w'}{r^2}$$

where $0 \leq R \leq 3$ and w is a negative quantity. We note that $|f_2| = r\, u_{max}$, where u_{max} is the maximum velocity of the fluid for a given radial distance r. Releasing the condition that H = 0 we have the more general equation

$$w'^2 = -\frac{2R}{3}(w + 1)\left(w^2 + \frac{6-R}{R}w - H\right)$$

(40)

$$H = (-3K + 6-R)/R > 0, \quad \text{i.e.} \quad K < (6-R)/3$$

Since all the roots of the equation must be real, the quadratic part can be factored. The roots of the cubic equation can then be ordered

(41) $$-\tfrac{1}{2}\left[\frac{6-R}{R} + \sqrt{\left(\frac{6-R}{R}\right)^2 + 4H}\right] \le -1 < 0 < -\tfrac{1}{2}\left[\frac{6-R}{R} - \sqrt{\left(\frac{6-R}{R}\right)^2 + 4H}\right]$$

This ordering imposes the further condition that

(42) $$H \ge 2 - \frac{6}{R} = \frac{2(R-3)}{R}$$

If $R > 3$, this is a new lower bound on H, which in turn produces a new upper bound on K, namely

(43) $$K \le 4 - R$$

Geometrically these bounds are portrayed by the shaded region of Figure 3 representing the RK plane.

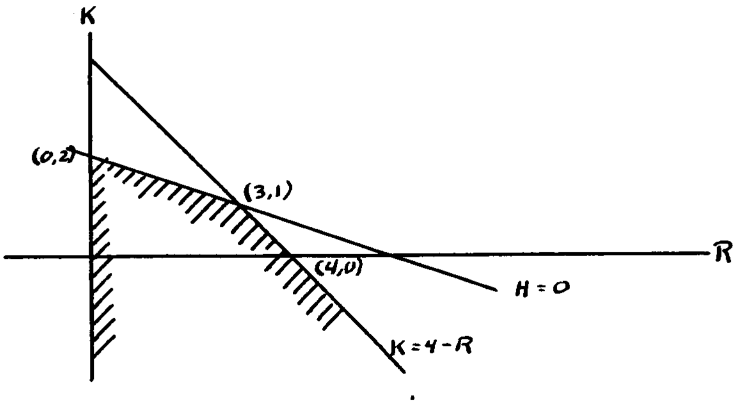

Figure 3

The differential equation integrates to

$$\frac{6-3R+\sqrt{(6-R\ )^2+4R^2H}}{2Rw+6-R+\sqrt{(6-R\ )^2+4R^2H}} = dn^2\left(\left[3(6-R)^2+2R^2H\ /18\right]^{\frac{1}{4}}\theta \mid m\right)$$

(44)

$$m = \frac{\sqrt{(6-R)^2+4R^2H}\ -6-3R}{2\sqrt{(6-R)^2+4R^2H}}$$

It should be observed that the case of $H = 0$ formally follows from the more general case by setting $H = 0$ and reducing the expression.

Table II is a partial table showing half the wedge angles as a function of the Reynold's number R and the constant of integration K. These relationships are also exhibited in graphical form in Figure 4.

If the limit is taken for H approaching $H = 2-\frac{6}{R}$, the differential equation reduces to the form

(45) $$w'^2 = -\frac{2R}{3}(w+1)^2(w-2)$$

which integrates to

(46) $$w = 3\tanh^2\left[\sqrt{\frac{R}{2}}(\theta-\theta)-\beta\right]-2, \quad \beta = \tanh^{-1}\sqrt{\frac{2}{3}}$$

Since tanh x is asymptotic to 1 for large values of the argument it follows that $u \approx u_{max}$ except for a narrow layer near each wall, of thickness proportional to

TABLE II

Semi-Wedge Angles for Convergent Flows

K	R=1	R=2	R=3	R=4	R=5	R=6
1	1.13	1.44		-	-	-
0	.89	1.04	1.34		-	-
-1	.73	.84	.97	1.14		-
-2	.66	.72	.80	.86	1.15	
-3	.60	.64	.70	.73	.88	1.08
-4	.55	.58	.63	.65	.74	.86
-5	.51	.54	.57	.58	.66	.72
-10	.40	.41	.42	.43	.45	.47
-15	.34	.34	.35	.36	.37	.38
-20	.30	.30	.31	.31	.32	.33
-30	.25	.25	.25	.25	.26	.26
-40	.22	.22	.22	.22	.22	.23
-50	.20	.20	.20	.20	.20	.20
-100	.14	.14	.14	.14	.14	.14

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$R^{-\frac{1}{2}}$ (2, p. 106). It should be observed that this is a limiting situation, the case where

$$K = 4 - R.$$

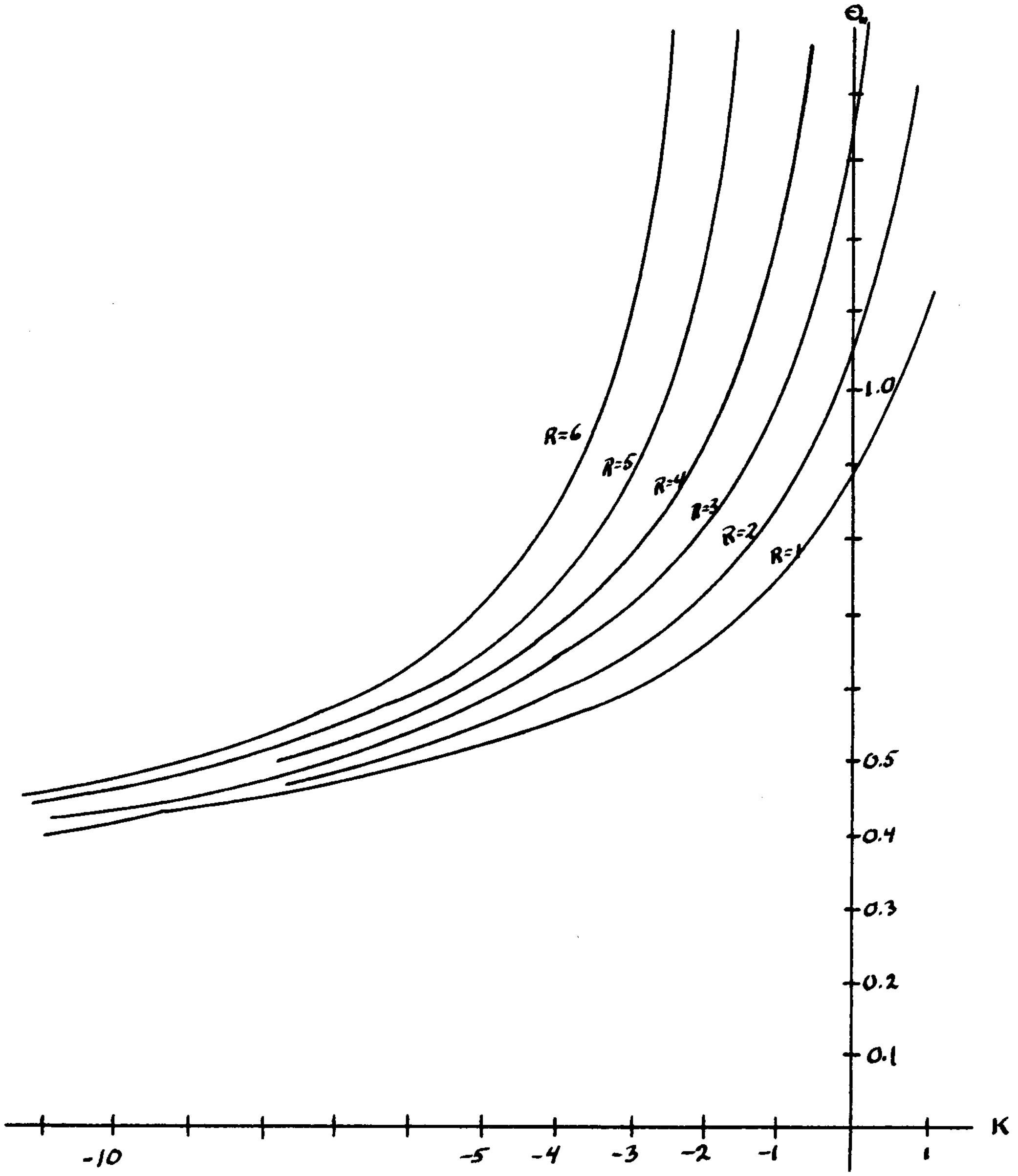

Figure 4

These curves represent the graphical relationship between the constant of integration K and the semi-wedge angle for convergent flows with Reynold's numbers R = 1, 2, 3, 4, 5, and 6 as given by Table II.

In this situation the stress components become

$$\widehat{rr} = -\mu\, u_{max}\, \frac{w-4-R}{r} + \text{constant}$$

(47) $$\widehat{\theta\theta} = -\mu\, u_{max}\, \frac{4-R}{6r} + \text{constant}$$

$$\widehat{zz} = -p \qquad \widehat{\theta r} = \frac{\mu\, u_{max} w'}{r}$$

where R is unrestricted and w is still a negative quantity.

In the physical situation of a wedge, the wedge angle has an upper bound of $\pi/2$ radians and we note from Table II that the above limiting case is not realizable since this angle of the wedge becomes infinite.

9. Divergent Flows

The divergent flows are those for which $f \geq 0$. It follows that f^* is positive and hence the dimensionless form of equation (33) becomes

(48) $$w'^2 = \frac{2}{3}\left[- R(w - 1)(w^2 + \frac{6+R}{R}\, w + H\right],$$

$$H = (3K + 6+R)/R$$

The boundary conditions $f(\theta_o) = 0$ and $f(o) = f^*$ become $w(\theta_o) = 0$ and $w(0) = +1$.

If all of the roots of the above equation are real, this integrates to

$$\frac{2Rw+(6+R-\sqrt{(6+R)^2-4R^2H})}{2R+(6+R-\sqrt{(6+R)^2-4R^2H})} = cn^2\left(\tfrac{1}{2}\sqrt{2+R+\sqrt{(6+R)^2-4R^2H}/3}\,\theta \,\middle|\, m\right)$$

(49)

$$m = \frac{6+3R-\sqrt{(6+R)^2-4R^2H}}{6+3R+\sqrt{(6+R)^2-4R^2H}}$$

There are two special cases of interest, if $H = 0$, the equation simplifies to

(50) $$w = cn^2\left(\sqrt{\frac{3+R}{3}}\,\theta \,\middle|\, \frac{R}{6+2R}\right)$$

and the relationship between the Reynold's number R, semi-wedge angle θ_0 and constants K is given by Table IIIa. This table also gives the upper bounds on the semi-wedge angle for various Reynold's numbers.

TABLE IIIa

R	0	0.75	1	2	3	4	5	6	12
θ_0	1.57	1.44	1.41	1.28	1.19	1.12	1.05	1.00	0.79
K	-2	$-\frac{27}{12}$	$-\frac{7}{3}$	$-\frac{8}{3}$	-3	$-\frac{10}{3}$	$-\frac{11}{3}$	-4	-6

* * * * * * * *

On the other hand if $H = (6+R)^2/4R^2$, the bounding case of equation (48) having real or complex roots, its solution can be expressed in terms of the hyperbolic secant, namely

(51) $$w = 3\left(\frac{2+R}{R}\right) \operatorname{sech}^2 \left(\frac{\sqrt{2+R}}{2}\,\theta\right) - \frac{6+R}{R}.$$

Table IIIb gives the relationship between the R, θ_o, and K for integral values of R along the curve in the RK plane for which this solution applies.

TABLE IIIb

R	1	2	3	4	5	6	12
θ_o	.43	.48	.49	.49	.48	.46	.40
K	$\frac{7}{4}$	0	$-\frac{3}{4}$	$-\frac{5}{4}$	-1	-2	$-\frac{15}{4}$

* * * * * * * *

On the other hand if equation (48) has complex roots, it integrates to

(52)
$$\frac{\sqrt{3R(4+R+K)} - R(1-w)}{\sqrt{3R(4+R+K)} + R(1-w)} = \operatorname{cn}\left(\left[4R(4+K+R)/3\right]^{\frac{1}{4}}\,\theta \,|\, m\right)$$

$$m = \frac{2\sqrt{3R(4+K+R)} + 6+3R}{4\sqrt{3R(4+K+R)}}$$

Following is Table IIIc, relating the semi-wedge angle to the Reynold's number R and constant of integration K for divergent flows without regard to the special cases above mentioned.

TABLE IIIc

(R) (K)	1	2	3	4	5	6	12
-6	-	-	-	-	-	-	.79
-5	-	-	-	-	-	-	.60
-4	-	-	-	-	-	1.00	.54
-3	-	-	1.19	.90	.79	.71	.50
-2	1.08	.91	.81	.73	.67	.47	.46
-1	.84	.75	.61	.64	.60	.57	.44
0	.67	.48	.62	.58	.55	.52	.42
1	.64	.59	.57	.55	.51	.49	.40
2	1.20	.55	.52	.50	.48	.46	.38
3	.64	.51	.49	.47	.45	.44	.37
4	.50	.48	.46	.44	.43	.42	.36
5	.47	.45	.44	.42	.41	.40	.35
10	.38	.37	.36	.35	.34	.34	.30
15	.32	.32	.31	.31	.30	.30	.27
20	.29	.28	.28	.28	.27	.27	.25
25	.26	.26	.26	.25	.25	.25	.23
50	.19	.19	.19	.19	.19	.19	.18
100	.14	.14	.14	.14	.14	.14	.13

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The lack of uniformity in Table IIIc is better exhibited in the graph of Figure 5. The discontinuity in the

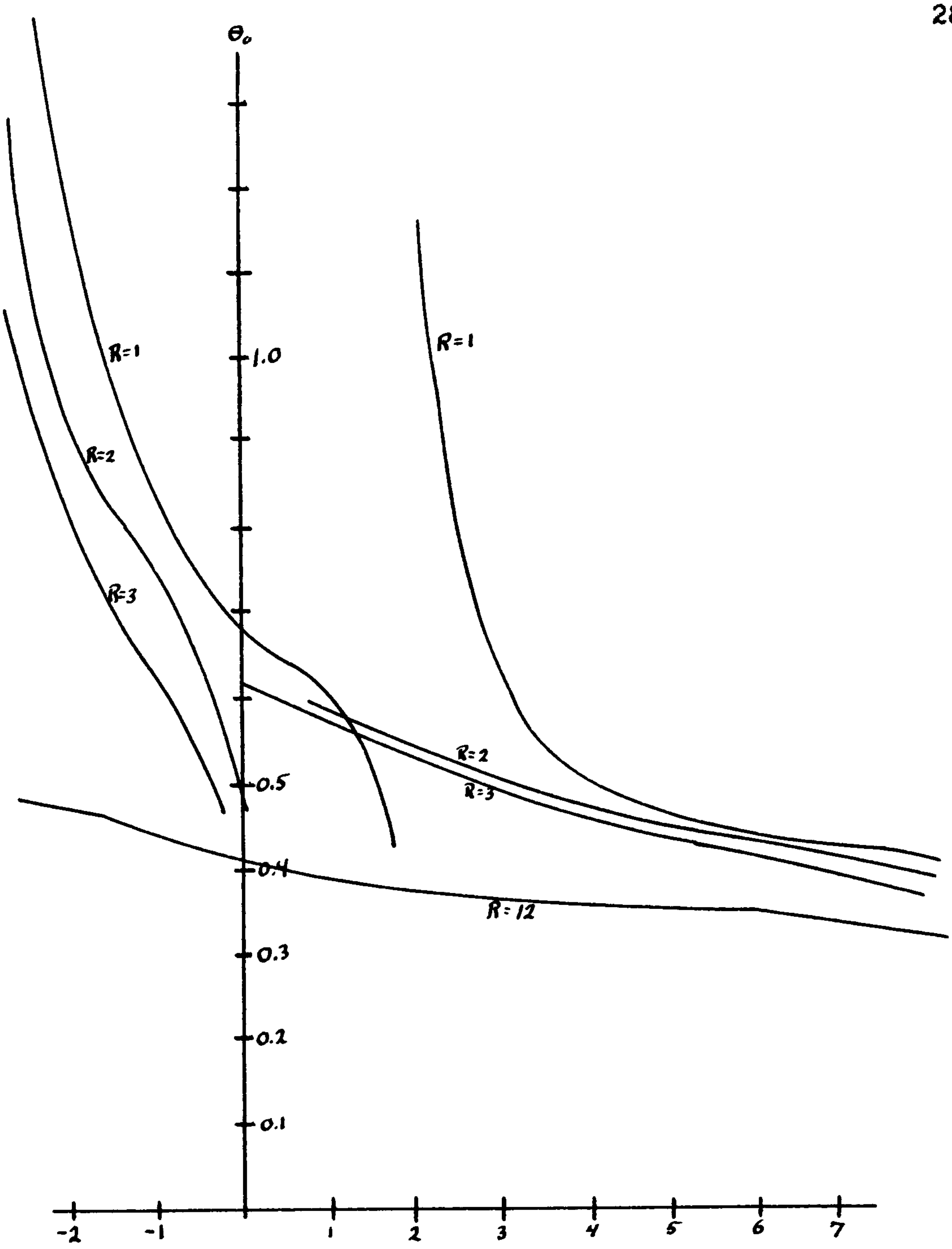

Figure 5

These curves represent the graphical relationship between the constant of integration K and the semi-wedge angle θ_0 for divergent flows with Reynold's numbers R = 1, 2, 3, and 12 as given in Tables III.

curve appears with the transition of the differential equation having all real roots or a pair of complex roots.

10. Summary

For convergent flows the constant of integration which enters into the calculation of the stress components is well defined as a function of the wedge angle and the Reynold's number. The normal components of stress at the walls in the r and θ directions are increased by an additive term which is proportional to the square of the shear stress $\widehat{r\theta}$. The stress components at the walls can be written

$$\widehat{rr} = \widehat{\theta\theta} = -p + |f^*|\, N(12 - 6K - 2R)/12r^4 \qquad \widehat{zz} = -p$$

(53)

$$\widehat{r\theta} = \frac{\mu |f^*|}{r^2}\sqrt{\frac{12 - 6K - 2R}{3}} \qquad \widehat{rz} = \widehat{\theta z} = 0$$

where K is determined from Table II. It should be observed that the flow dictated no restriction on the nature of N; however, there is evidence to believe that $N > 0$.(14, p. 243) This conjecture could be experimentally tested by correctly designed equipment.

In considering the divergent flows we determine the value of K from Table IIIc. In this case the equations for the stresses are

$$\widehat{rr} = \widehat{\theta\theta} = -p + |f^*|\, N(6K + 2R + 12)/12r^4 \qquad \widehat{zz} = -p$$

(54)

$$\widehat{r\theta} = \frac{\mu |f^*|}{r^2} \sqrt{\frac{6K + 2R + 12}{3}} \qquad \widehat{rz} = \widehat{\theta z} = 0$$

Again the flow places no restriction on N. We do note that Tables III do not uniquely define K for all wedge angles and Reynold's numbers. In some regions there seem to be two possible states permitted. The solutions as given by equations (49), (50), and (51) yield one set of values and equation (52) gives another set of values for some of the wedge angles. This is most evident with the tabled values of R = 1. This existence of two possible states is another conjecture which may admit to experimental verification.

The existence of the positive shear and the increase of the normal stresses near the vertex of the wedge for both flows indicates that the convergent flow has a tendency to close the wedge and increase the wedge angle, while the divergent flow tends to do just the opposite, --that is open the wedge and decrease the wedge angle. These observations are not found in the classical theory.

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