A Generalized Scheme for Creating Regge Calculus Models of General Relativity

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Abstract

Solutions to Einstein’s equations are usually found by considering ideal, simplified models. However, if the real world always matched ideal physics models, then farmers would milk black and white spheres. To combat this, Regge calculus was developed as a numerical approximation scheme for general relativity. In Regge calculus curved manifolds are approximated by a lattice of triangles where spacetime is flat on each individual triangle. Regge intended this scheme to be used for all metrics. As such, a consistent framework must exist for discretizing a manifold without relying on any prior knowledge of the metric or symmetries. The papers which discuss this are often very hard to follow or highly specialized to one specific metric without much general applicability. In this paper, the outlines of such a method of Regge calculus were found. This was done by issuing a few simple constraints on the edge lengths of the lattice. Using these constraints, equations were then derived for the angle deficits, Regge action, and metric coefficients in terms of only the edge lengths.
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1 Introduction

Einstein’s equations are a series of differential equations which show how the mass and energy of a certain region of spacetime are related to the curvature of that region. Solutions to these equations are functions of the geometry of spacetime in terms of nothing but the mass and energy associated with it. In other words, regions of spacetime are curved because the stuff in that region is curving it. This is similar to the way a trampoline’s surface changes when a person is on top of it (adding mass) and begins jumping (adding energy). Solutions to Einstein’s Equations show explicitly how the mass-energy of the stuff is related to the curvature of the region of space that stuff is in.

Conventionally, problems in general relativity are solved using the methods and techniques described in Section 2. Some problems, however, are just too hard to solve by conventional means. Gaussian distributions are well-studied, however any attempt to find a function \( F(x) \) which satisfies \( F(x) = \int e^{-x^2} \, dx \) will result in failure. In fact, the only known way of writing \( F(x) \) is as the integral of \( e^{-x^2} \). In situations like this, mathematicians and scientists will often turn to various inexact solutions. Instead of integrating over infinitesimally small chunks, one can take a Riemann sum of boxes that best approximate the area under the curve \( F(x) \). This is the main idea behind Regge calculus. General relativity is hard enough when dealing with perfectly symmetric spacetimes. However, most of the universe isn’t so neat and tidy, and even seemingly simple scenarios can result in a series of equations that are unsolvable given current techniques. For example, the Schwarzschild metric provides a nice approximation for the spacetime geometry given by a single black hole or star. However, simply giving the black hole a charge and angular momentum turns one of the easiest problems in GR into a nightmare. And it gets even worse when considering binary or systems. So what is one to do if they are accosted in the wild by a complicated spacetime manifold?

The answer is to approximate it, just like one would approximate a difficult integral by taking its Riemann sum. The field of numerical relativity contains numerous approximation schemes, and the simplest one for the beginner to wrap their head around is that created by Tullio Regge in his 1961 paper, *General Relativity without Coordinates*. The idea is fairly simple: consider the sphere shown in Figure 1. Let’s say that we don’t know how to solve Einstein’s Equations for that sphere. What Regge found was that, if one divided that sphere up into a bunch of individual, triangular pieces, then spacetime on each individual piece would behave as ordinary flat spacetime. In other words, all one needs to know to do physics on one of the triangular pieces is special relativity.

This sounds simple enough, however many obstacles can arise in practice. First, a
spherical surface is inherently a two-dimensional being, while general relativity deals with four dimensions. This can make it quite challenging to figure out what’s going on, since most humans tend to live in a universe with only 3 spatial dimensions. The second obstacle is in the formulation of the Regge field equations themselves (being the discretized versions of the Einstein field equations). Tullio Regge showed that it was possible to discretize spacetime manifolds like this, but failed to mention how to go about actually doing so. In the years since his paper was published, there have been multiple physicists approaching Regge calculus approximations with their own different methods of attack. There have been significant successes, such as discretizing the Freidman-Lemaître-Robertson-Walker (FLRW) universe and Schwarzschild blackhole (which will be discussed later on in this paper), however everyone appears to have their own favorite method, and not all methods work for every manifold. The goal of this paper is to provide the reader with a clear and concise method of discretizing any sort of crazy manifold. It may not be an optimal strategy (that’s a topic for a different paper), but it will hopefully be one that works regardless of any special considerations.
2 Theory

2.1 General Relativity

This section is intended to provide a quick description of the basics of general relativity (GR). GR is a massive and difficult subject. It took Einstein eight years to develop the field and OSU two semesters to teach it. What follows here is an attempt to fit all of that into a few pages. For a more thorough description of the subject, see Hartles’ book *Gravity: An Introduction to Einstein’s General Relativity* and/or *Gravitation* by Meisner, Thorne, and Wheeler.

2.1.1 Notation

The biggest obstacle to overcome for the beginning GR student is simply understanding the notation. For those unfamiliar with it, it can be like looking at a foreign language, however a knowledge of Einstein’s index notation is essential.

To start, consider the vectors $u$ and $v$, given as

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{1}$$

The dot product between these two vectors is given as

$$u \cdot v = \sum_{\alpha=0}^{3} u_\alpha v_\alpha. \tag{2}$$

The matrix equivalent of this would be

$$A \cdot B = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} A_{\alpha\beta} B_{\alpha\beta}. \tag{3}$$

Sums like these are everywhere in general relativity, and Einstein did not want to constantly have to write summation symbols for every single equation. He also knew that a slicker version of this could keep one from getting bogged down by a wall of text. So he devised a new notation system. To drop the summation symbol without losing any information will require that all information about the sum be expressed in the indices alone. This means knowing both which indices are being summed over, as well as the range in which they’re
being summed.

Let’s begin with the second requirement. In GR, one is always concerned with the three spatial coordinates and time. If we label time as \( t = x_0 \), and the others as \( x_1, x_2, \) and \( x_3 \), then we can see that there are really only two options one can have when doing summations in GR. The first is a sum over all four coordinates \( (n = 0, 1, 2, 3) \), while the second is a sum over just the spatial coordinates \( (n = 1, 2, 3) \). In GR, Greek indices are used to represent four dimensions while Latin indices are used for three dimensions. The following two equations are examples of this, with \( x \) and \( y \) being arbitrary vectors

\[
\sum_{\alpha=0}^{3} x_{\alpha} y_{\alpha} = x_{\alpha} y^{\alpha} = x_0 y^0 + x_1 y^1 + x_2 y^2 + x_3 y^3 \tag{4}
\]

\[
\sum_{i=1}^{3} x_i y_i = x_i y^i = x_1 y^1 + x_2 y^2 + x_3 y^3. \tag{5}
\]

By now you’ve also probably noticed the superscripts. Those are there to show which indices are being summed over and which are not. If there’s something with an \( \alpha \) subscript being multiplied to something with an \( \alpha \) superscript, then there is a sum over \( \alpha \). If there’s just an \( \alpha \) subscript, then there is no sum going on. This also means that the indices on both sides of an equation must balance out, as demonstrated here

\[
A^\gamma_{\alpha \beta} B_\gamma = A^0_{\alpha \beta} B_0 + A^1_{\alpha \beta} B_1 + A^2_{\alpha \beta} B_2 + A^3_{\alpha \beta} B_3 = C_{\alpha \beta}. \tag{6}
\]

Checking indices can be used in the same way as a unit or dimensions check for GR. The left side of (6) has subscripts \( \alpha, \beta, \) and \( \gamma \) and superscript \( \gamma \). Since same subscripts "cancel out" same superscripts, we should expect the right side of (6) to only contain the subscripts \( \alpha \) and \( \beta \). Indices that are not summed like this are called "free indices." To interpret the answer \( C_{\alpha \beta} \), think back to the original example of matrix dot products. In (3), matrices \( A \) and \( B \) were relabeled as \( A_{\alpha \beta} \) and \( B_{\alpha \beta} \), respectively. Therefore, in (6), one can think of each \( C_{\alpha \beta} \) as representing the \( \alpha^{th} \) component of the \( \beta^{th} \) row. Written out as a matrix, \( C_{\alpha \beta} \) would appear as

\[
C = \begin{pmatrix}
C_{00} & C_{01} & \ldots & C_{0\beta} & \ldots \\
C_{10} & C_{11} & \ldots & C_{1\beta} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
C_{\alpha0} & C_{\alpha1} & \ldots & C_{\alpha\beta} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix}. \tag{7}
\]
Those are the basics, however there are a few important subtleties to this notation. First, up until now, superscripts and subscripts have been used somewhat interchangeably. However, it is incorrect to assume that $A^\alpha = A^\alpha$, and the true relationship between these two quantities will be explained in section 2.1.2. Second, superscripts on the denominator of a fraction acts like a subscript. For example, consider the sum of partial derivatives

$$\frac{dx^\alpha}{dx'^\beta} = \frac{dx^\alpha}{dx'^\beta}. \tag{8}$$

Multiplying both sides of (8) by $dx'^\beta$ gives

$$dx^\alpha = \frac{dx^\alpha}{dx'^\beta} dx'^\beta. \tag{9}$$

In (9), it can be seen that the $\beta$ superscript in the denominator must be cancelling out the other $\beta$ superscript in order to balance the indices.

### 2.1.2 The Metric

The metric is the fundamental building block needed to construct all other equations of curvature for a specific spacetime geometry. To get a grasp of what precisely this thing is, let’s consider the world of special relativity. Here, spacetime is completely flat, providing the simplest geometry one can work with. From special relativity, one may remember the line element for one spatial and one timelike dimension as

$$ds^2 = -dt^2 + dx^2. \tag{10}$$

The line element for three spatial and one timelike dimension is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{11}$$

Crucially, the line element shows how space and time are connected into one 4-dimensional beast: spacetime. This is in direct opposition to the idea of three spatial dimensions existing independent of time. Furthermore, all timelike dimensions are expressed with the opposite sign as spacelike dimensions. Here, the convention is to make time negative and space positive, although it doesn’t really matter so long as their signs are opposite.

In special relativity, $ds^2$ is the special length preserved for a given coordinate transformation. In other words, even though it’s been defined in (11) in cartesian coordinates, it would be exactly the same length in cylindrical, spherical, or any other coordinate system one can think of. All that matters for $ds^2$ is that there be one timelike coordinate and three
spacelike coordinates. Given this, it may be helpful to think of the line element in terms of completely arbitrary coordinates. Let \( t = x^0, x = x^1, y = x^2, \) and \( z = x^3. \) Then (11) becomes

\[
ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.
\]

In Einstein’s index notation, this would be written as

\[
ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta,
\]

where \( g_{\alpha\beta} \) represents the matrix

\[
g = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The metric is this coefficient matrix \( g, \) or \( g_{\alpha\beta} \) in Einstein notation, that comes from the line element in (13). Multiple metrics can represent the same spacetime. For example, (14) would take the form

\[
g = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2(\theta)
\end{pmatrix}.
\]

These two metrics are identical, the only difference being in the coordinates used. In a curved spacetime, it’s the components of the metric which give the information about how the spacetime is curved.

Before getting into curvature, there are a few properties of the metric which must be discussed. The first is that, for any spacetime geometry, the metric is always symmetric. That means that

\[
g = \begin{pmatrix}
-1 & 0 & 1 & 0 \\
0 & 1 & 0 & r \\
1 & 0 & r^2 & 0 \\
0 & r & 0 & r^2 \sin^2(\theta)
\end{pmatrix}.
\]
could theoretically be a working metric, but

\[
\mathbf{g} = \begin{pmatrix}
-1 & 0 & 1 & 0 \\
0 & 1 & 0 & r \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2(\theta)
\end{pmatrix}
\] (17)

would not. Because all metrics are symmetric, it follows that \( \mathbf{g} = \mathbf{g}^T \). Expressed in Einstein notation, this would be

\[ g_{\alpha \beta} = g_{\beta \alpha}. \] (18)

Due to this symmetry, one will at most be dealing with 10 independent equations at any given time.

Second is the subtle lie told way back when first explaining Einstein’s index notation. In that section, I said that the dot product between two vectors was simply \( u_i v^i \), which is true when considering pure mathematical vectors on a perfectly flat, spacelike manifold. This is certainly true in an orthonormal basis, however if one just wants the scalar product in the coordinate basis, then

\[ u \cdot v = g_{\alpha \beta} u^\alpha v^\beta \] (19)

must be used. This is because the properties of these vectors are defined with respect to the manifold that they’re on. Imagine that a string is tied to the top of a cylinder. If one were to pull on this string at an angle \( \phi \), the effect of that force would be very different than the effect of the force of the same string being pulled at the same angle with the same total force at a point on the opposite side of the cylinder (or for that matter, any point that’s not exactly the same as the first one). This didn’t matter before since flat spacelike manifolds were the only ones being considered, but this is a crucial distinction when considering spacetimes with curvature.

Another property, which is one of the most important concepts to know in GR, is that

\[ g_{\alpha \beta} g^{\beta \gamma} = \delta^\gamma_\alpha, \] (20)

where \( \delta^\gamma_\alpha \) is the Kroenecker delta. This has two implications. The first one is fairly obvious: \( g^{\alpha \beta} \) is the inverse of \( g_{\alpha \beta} \). The second implication is that there now exists a method for finding the relationship between subscripts and superscripts. Consider the matrix \( R_{\alpha \beta} \). To raise the index \( \beta \), perform the operation

\[ R_{\alpha \beta} g^{\beta \gamma} = R^\gamma_\alpha. \] (21)

This would have worked for raising \( \alpha \) as well. To lower the index back down, simply multiply
Finally, if one is only given the coordinates of a system, the metric can be constructed by finding \( \frac{d}{dx^\alpha} \) since
\[
g_{\alpha\beta} = \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta}.
\] (22)

### 2.1.3 Curvature

So now that the metric has been defined, how exactly can it be used to find equations of curvature? Here I will give the pure definitions of equations, since anything more would be enough to fill a textbook. For a more in depth formulation of these equations, please see [8] and [9].

(A quick note before getting started: thinking of scalars as 0-dimensional beasts, vectors as 1-D beasts, and matrices as 2-D beasts, the term "tensor" is used to describe N-D beasts where N is the number of dimensions. A square matrix is a 2-D tensor, so a cube of numbers would be a 3-D tensor)

Okay, so here’s the lightning quick derivation of Einstein’s Field Equations:

The Christoffel Symbols, defined as
\[
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g_{\alpha\beta} \left( \frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta} \right),
\] (23)

are a convenient piece of shorthand that will be used often in this derivation. A geodesic is a straight line on a curved surface. On a globe, lines of longitude are geodesics, while latitude are not. In flat space, two straight lines will never meet, however in curved space two straight lines will meet somewhere. Consider the point where they meet, and then moving along one of the geodesics. The other geodesic will slowly get farther away. Turning around and heading the opposite direction will cause the two geodesics to approach each other. Going back to the globe example, this would be equivalent to starting at the South Pole (where all the lines of longitude intersect), walking away from the South Pole heading due north on the Prime Meridian (0°). Now think about what you’d expect to see if your friend did the same thing at the same time, but travelled along the 1° longitudinal line. You’d both start at the South Pole, but then would quickly find yourselves drifting farther apart. The equation of geodesic deviation describes the manner in which one geodesic "goes away" or "comes towards" the other. The equation for a single geodesic is given as
\[
\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau},
\] (24)
while the equation of geodesic deviation is

\[(\nabla_u \nabla_u \chi)\alpha = -R^\alpha_{\beta\gamma\delta} u^\beta \chi^\gamma u^\delta, \tag{25}\]

where the vector \(\chi\) describes an infinitesimal displacement between an observer on a geodesic and a test particle travelling on another, the vector \(u\) is the observer’s four-velocity (displacement of \(t, x, y,\) and \(z\) with respect to the proper time) and \(\nabla_u f\) is the covariant derivative of \(f\) along the vector \(u\). Let \(v = \nabla_u \chi\) and \(w = \nabla_u v\). Then the components of these vectors are given as

\[v^\alpha = (\nabla_u \chi)^\alpha = \frac{d\chi^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta \chi^\gamma, \tag{26}\]

\[w^\alpha = (\nabla_u v)^\alpha = \frac{dv^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta \chi^\gamma. \tag{27}\]

From (22), (23), and (24), the relationship between the Riemann curvature, \(R^\alpha_{\beta\gamma\delta}\), and the Christoffel symbols can be derived as

\[R^\alpha_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\gamma\epsilon} \Gamma^\epsilon_{\beta\delta} - \Gamma^\alpha_{\delta\epsilon} \Gamma^\epsilon_{\beta\gamma}. \tag{28}\]

The Riemann curvature \(R^\alpha_{\beta\gamma\delta}\) can be manipulated using the metric to give

\[R^\alpha_{\beta\gamma\delta} g^\gamma_\alpha = R^\gamma_{\beta\gamma\delta} = R_{\beta\delta}. \tag{29}\]

Since all the indices are just dummy indices, (26) can be written as \(R_{\alpha\beta}\). This is the Ricci Curvature. The Ricci Curvature can then be manipulated in the exact same way as before to give the scalar curvature

\[R_{\alpha\beta} g^{\alpha\beta} = R_{\alpha\beta} g^{\alpha\beta} = R. \tag{30}\]

The source of space time curvature is the energy-momentum density of matter within the given region of spacetime. This is expressed using the stress-energy tensor, \(T_{\alpha\beta}\). The components of \(T_{\alpha\beta}\) describe the change in momentum (in the \(x^\alpha\) direction) with respect to time experienced across a surface with a normal vector in the \(x^\beta\) direction. The relationship between spacetime curvature and the stress-energy-momentum tensor is known as the Einstein Field Equations, which are given as

\[R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}, \tag{31}\]

where \(c\) is the speed of light, \(G\) is the gravitational constant, and \(\Lambda\) is the cosmological
constant. There’s much debate over whether or not $\Lambda = 0$, however this paper will assume that it is zero. Normally, and for the remainder of this paper, this equation will be expressed in "natural units," which essentially means "units such that $c = G = \hbar = 1,"$ however it’s important to realize that those constants are still there in the equation.

2.1.4 ADM Formulation

The previous section is a very important introduction to Einstein’s theory which one needs in order to understand the contents of this paper, however it is not the formulation of Einstein’s equation used in Regge calculus. Regge calculus uses the ADM Formulation (Arnowitt-Deser-Misner), which uses variation of the action to derive Einstein’s field equations. The following is not a complete discussion of of ADM formalism but rather a broad overview of the concepts needed to understand this thesis. For an in depth look at the formulation, see Ch. 21 of *Gravitation* [9], which will be the primary source throughout this section.

In classical mechanics, both the Lagrangian and Hamiltonian are derived from the action, $S$, defined as

$$S = \int_{t_1}^{t_2} L dt,$$

(32)

where $L$ is the normal Lagrangian. The modification of this equation for GR is given as

$$I = \frac{1}{16\pi} \int R \sqrt{-g} d^4x = \text{extremum},$$

(33)

where $R$ is the Ricci scalar, $g$ is the determinant of the metric, and $d^4x$ is common shorthand used to represent the quantity $dt dx dy dz$. (Side note: $g$ itself is a negative number, meaning that $\sqrt{-g} \in \mathbb{R}$). Variation of this action with respect to the inverse metric ($\frac{\delta I}{\delta g^{\alpha\beta}}$) leads to the Einstein field equations (30) as derived in 2.1.3. In reaching this conclusion, numerous techniques and concepts had to be developed along the way. One of which was an effective way to separate space from time in spacetime.

Recall from special relativity that a timelike vector $\hat{t}$ is one which the magnitude is negative. For example, if $\hat{t}$ is a unit vector pointing in the time direction, then $\hat{t} \cdot \hat{t} = -1$.

As was discussed above, the most generic metric one can have is $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. Equation (12) showed an example of a Minkowskian flat metric. However, while it is true that all metrics are symmetric, it is not true that all metrics are diagonal. An example of this is given on page 168 of Hartle’s book [8]:

$$ds^2 = -dt^2 + [dx - V_s(t)f(r_s) dt]^2 + dy^2 + dz^2.$$

(34)
The functions $V_s(t)$ and $f(r_s)$ can be ignored. As can be seen by the $(dx - V f dt)^2$ term, this is not a diagonal metric. However, this is a perfectly viable one (and of much interest to any wannabe space travelers, as it describes a warp drive). But this poses a problem. In deriving Einstein’s equations from the action principal, it is desirable to have a metric that separates the three spatial coordinates from the time coordinate. But this metric has weird $dx dt$ terms in it. To solve this, consider the geometry of spacetime if time is held constant. This is known as the 3-geometry (or more generally as a hypersurface), and the metric describing the warp-drive’s 3-geometry would be

$$ds^2 = dx^2 + dy^2 + dz^2,$$

(35)

since $dt = 0$ for $t = constant$. This 3-geometry (or 3-surface) is then evolved through time to produce the 4-geometry described in (34). Figure 2 shows how points on one surface of time $t = constant$ will evolve to the next surface at time $t' = t + dt = constant$. As shown, a point on the first surface, located at $x^i$, will go travel a certain spatial and temporal distance from $x^i$ to get to $x^{i+1}$. To describe the lapse in time and shift in space, the lapse function

$$\frac{\partial}{\partial t} = N + \beta.$$  

(36)

Figure 2: Two 3-geometries at times $t$ and $t + dt$, connected by the time-evolved paths from points on the past surface to the future surface. Adapted from pg 506 [9].
Recall that $g_{\alpha \beta} = \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta}$. Therefore, $g_{00} = -N^2 + \beta_k \beta^k$, $g_{0i} = \beta_i$, $g_{i0} = \beta_i$, and $g_{ij} = h_{ij}$ where $i, j = 1, 2, 3$ and $h_{ij}$ is the metric describing the 3-geometry. Thus, the generic line element in terms of the lapse and shift functions is

$$ds^2 = (-N^2 + \beta_k \beta^k) dt^2 + \beta_i dt dx^i + \beta_j dt dx^j + h_{ij} dx^i dx^j.$$  \hspace{1cm} (37)

Notice that there can exist multiple lapse functions and shift vectors which produce the same metric, and thus one is free to choose those which best suit their purposes.

### 2.2 Regge Calculus in Brief

#### 2.2.1 Deficits and Simplices

How could someone with no concept of $\pi$ figure out the perimeter of a circle? A natural first step might be to choose a few points on that circle, connect those points with straight lines, and then measure the lengths of those lines, assuming that the perimeter is approximately equal to the sum of the lengths.

![Figure 3: A circle subdivided into 5 line segments.](image)

To appropriately approximate the circle, the shape’s curvature must be contained somewhere in the straight line approximation. Since the lines themselves are perfectly flat, the only place the curvature could be is in the vertices. Regge calculus is nothing more than this idea applied to higher dimensions and generic curves.
With this in mind, let’s increase the dimensions and think about the sphere. The surface area of this sphere can be approximated using the same idea as before, except now we have an extra dimension to worry about. In Figure 1, it was shown that a sphere could be approximated by using a careful placement of equilateral triangles. Just like before, the curvature of the sphere has been concentrated entirely inside these triangles. But how exactly is curvature described in this triangular lattice?

Consider a single vertex from some triangulated manifold, such as the one shown in figure 4. This structure cannot be flattened without tearing it somewhere. All the triangles around the vertex are equilateral triangles, and thus the angle between any two lines around the vertex is $60^\circ$. Starting at the yellow line, if one were to circumnavigate the vertex and end up back at the yellow line in flat space, then they would expect to have travelled a total of $2\pi$ radians. But this is not the case on the triangular surface in figure 4. Instead, one only needs to travel $\frac{5\pi}{3}$ radians to circumnavigate the vertex. This difference in angular distance is given as

$$\epsilon_p = 2\pi - \sum_p \delta_p$$

where $\delta_p$ is the angle between two adjacent line segments meeting at the vertex. The $\delta_p$’s are referred to as dihedral angles, and $\epsilon_p$ is called the deficit.(Note: Unfortunately, Regge calculus is the frustrating area of GR where Einstein notation and regular notation is abused and mixed).

![Figure 4: Left: Triangles concentrated around a vertex. Right: The same structure after being forced flat. The two yellow lines on the right represent the same line; that is, the yellow line seen on the left. (Note: here $\theta$ is used instead of $\delta$ to describe the dihedral angles).](image)

In his 1961 paper, Tullio Regge found that the curvature of the underlying spacetime
manifold was related to the deficits of the vertices [1]. This relationship will be discussed more in section 3.1.2. He also found that this basic concept can generalize into higher dimensions as well, so long as a few tweaks are made.

To begin the discussion of higher dimensions, consider how one draws a 2-dimensional triangle. First choose a point, then choose a second point and draw a line connecting it to the first point, choose a third point and draw lines connecting that point to the previous two points. What about a 3-dimensional triangle (aka, a tetrahedron)? Well just choose a fourth point and connect that to the other three. In general, to draw an n-dimensional triangle one must choose $n + 1$ points and connect every point to every other point. The common term for such a shape is an n-simplex (for example, a normal triangle would be a 2-simplex and a line would be a 1-simplex).

![Figure 5: Simplices of 0, 1, 2, 3, and 4 dimensions.](image)

Now let’s return to the 2-dimensional triangulated manifold. Recall that the curvature of this manifold was concentrated on the vertices (0-simplices), the deficits were given in terms of the dihedral angles between lines (1-simplices) meeting at a vertex, and the manifold itself was constructed using a lattice network of triangles (2-simplices). If the manifold were n-dimensional, the manifold would be constructed using a network of n-simplices, the curvature concentrated on (n-2)-simplices, and the dihedral angles being between (n-1)-simplices. For example, in 3-dimensional manifolds curvature would be concentrated on lines, manifolds would be made of tetrahedra and dihedral angles would be between triangles. This concept is shown in figure 6.
Finally, it is quite common to see people referring to line segments in Regge manifolds as "legs" and to see triangles referred to as "bones" or "hinges."

2.2.2 Regge Equations

It is now time to address the question of how exactly the curvature of the smooth manifold is manifested in the discretized lattice. In his paper, Regge found that the angle deficit at a
point p could be related to the Ricci scalar curvature by
\[ R = 2 \rho \epsilon_p, \]  
(39)

where \( \rho \) is the density of triangles around the point. This equivalence leads to the derivation of the Regge-Hilbert-Einstein action,
\[ 8\pi I_R = \sum A_i \epsilon_i, \]  
(40)

the analogous equation to the Einstein-Hilbert action (33) (see [1] for a complete description of the derivation). Here, \( A_i \) is the area of a 2-simplex, \( \epsilon_i \) is the corresponding deficit of that 2-simplex, and the sum is over all simplices in the lattice. In curved spacetimes, the Einstein equations were derived by taking
\[ \frac{\delta I}{\delta g^{\alpha \beta}} = 0. \]  
(41)

Here in discretized spacetime, the analogous Regge-Einstein equations are derived by varying the Regge action with respect to a leg length \( l_j \) and using the principal of minimal action to get
\[ 8\pi \frac{\delta I_R}{\delta l_j} = \sum \frac{\delta A_i}{\delta l_j} \epsilon_i = 0. \]  
(42)

This equation holds for all triangulations of a curved surface.

3 Results

One is in theory free to choose the locations of their vertices. This freedom sounds great to the scientist working with a metric who’s properties they can exploit (for example, knowing that \( r = \text{constant} \) on a sphere). But for the student and the scientist who are encountering a metric about which they know nothing, the question of optimal choice becomes less obvious. In the following sections, a method of doing Regge calculus is derived for a generic ADM metric. Unless stated otherwise, everything from this point on is original work.

3.1 Length Constraints

In Regge calculus, the easiest way to obtain solutions to the Regge equations is to begin by triangulating a hypersurface \( \Sigma_i \) at the time \( t_i \), and then evolving this hypersurface through time. One has the freedom to choose where they wish to place the vertices of the tetrahedra that split up this hypersurface, however more edges of different lengths leads to more
complicated equations later on. Thus, a natural choice of triangulation is to let every edge length on $\Sigma_i$ be equal. This is the first constraint imposed in this method of Regge calculus, and all edges on $\Sigma_i$ are given the length $\ell_i$.

The second constraint is to evolve the vertices of $\Sigma_i$ to $\Sigma_{i+1}$ (ie, from time $t_i$ to time $t_{i+1}$) in a way such that all edges on surface $\Sigma_{i+1}$ have length $\ell_{i+1}$ (and therefore, by induction, all edges on surface $\Sigma_n$ have length $\ell_n$ for all $n \in \mathbb{Z}$). However, the edges connecting a vertex on $\Sigma_i$ to a vertex on $\Sigma_{i+1}$ do not themselves need to be lengths $\ell_i$ or $\ell_{i+1}$.

Figure 7: A 3-D version of the evolution scheme. Here, $\Sigma_i$ is the plane intersecting all the unprimed vertices. Adapted from [13].

The final constraint is to connect vertices on $\Sigma_i$ to vertices on $\Sigma_{i+1}$ using the Sorkin triangulation method, described in [13]: Start by labeling a vertex of a specific tetrahedron in $\Sigma_i$ as [A] and labeling the other vertices attached to [A] as [B], [C], etc. Edges connecting one vertex on a hypersurface to another will be referred to as spacelike edges. Evolve [A] through a strictly timelike path to vertex [A'] on $\Sigma_{i+1}$ and connect [A'] to all the vertices that [A] is connected to. Do this for [B] and [B'], making sure that [B'] is evolved in such a way that the distance between [A'] and [B'] is length $\ell_{i+1}$. Edges connecting a vertex to its primed counterpart will be referred to as timelike edges. Next, connect [A'] to every vertex on $\Sigma_i$ that [A] is connected to, such as [A'] to [B]. Edges like this will be referred to as diagonal edges. While timelike and diagonal edges may or may not be of length $\ell_i$, the freedom of choice in lapse and shift means that all timelike edges can be of length $m_i$ and all diagonal edges can be of length $d_i$. Additionally, the evolution scheme is chosen such that
Figure 8: The line $AB$ on the surface $\Sigma_i$ (at time $t_i$) is evolved through time to the surface $\Sigma_{i+1}$ (at time $t_{i+1}$). The points $A'$ and $B'$ are then connected to $A$ and $B$ respectively, and a diagonal line is drawn to split the trapezoid into two triangles.

all edges either extend or contract, which again amounts to a choice in lapse and shift [13].

To recap, by taking advantage of the allowable freedoms in this problem, a triangulation scheme was constructed using the following constraints:

(i) all edges on the hypersurface $\Sigma_i$ (which a snapshot of the spacetime at the time $t_i$), have length $\ell_i$,
(ii) all edges on the surface $\Sigma_{i+1}$ (the spacetime manifold at time $t_{i+1}$), have length $\ell_{i+1}$,
(iii) all edges connecting a vertex on $\Sigma_i$ to its time-evolved counterpart on $\Sigma_{i+1}$ have length $m_i$, and
(iv) all edges from $\Sigma_i$ to $\Sigma_{i+1}$ which don’t connect a vertex to its primed counterpart have length $d_i$.

3.2 The Basic Process

With these constraints in mind, let’s now turn to the effects they have on Regge’s equations. In Regge calculus, the Regge equations provide the instructions for how to build the lattice. Specifically, if one starts by placing a single tetrahedron in the manifold with some known vertex locations, then the Regge equations provide an exact set of instructions as to where
all other vertices must go in order to properly approximate the region of spacetime. Furthermore, since every quantity can be written in terms of the edge lengths, all one needs to do to get a complete picture of a triangulated spacetime is to specify how long each edge is. The constraints discussed in the previous section will now be put to use in the next few sections to derive a general version of Regge’s equations.

3.2.1 Edge Lengths

Since the process involves building a lattice structure from a few initial vertex locations, a natural assumption may be to start by choosing these vertices and then going on from there. This will be discussed later on when talking about dihedral angles and deficits, however the constraints mentioned above have implications to the Regge equations which are better discussed beforehand.

Recall that all edges on the surface $\Sigma_i$ have length $\ell_i$, and that all edges are evolved such that they either simply expand or simply contract. Thus, the evolutionary path of the line $AB$ to $A'B'$ shown in Figure 8 is representative of all such paths in the lattice. Since $\Sigma_i$ is given as the spacetime manifold at time $t_i$, it follows that the difference in time from one edge to another is zero and thus all edges are spacelike. Because of this, any edge of length $\ell_i$ will be referred to as a spacelike edge. The next edge type connected vertices on $\Sigma_i$ to their primed counterparts on $\Sigma_{i+1}$ and are given the length $m_i$. These edges are called timelike edges. Finally, there were the edges of length $d_i$ which connected vertices on $\Sigma_i$ to those on $\Sigma_{i+1}$, and were called diagonal (notice: these names are not official and are only used here to quickly differentiate between edge types). In all figures from now on, all spacelike edges will be colored red, all timelike edges will be colored green, and all diagonal edges will be colored blue.

The first step will be to find equations for $m_i$ and $d_i$ in terms of only our spacelike edges, $\ell_i$ and $\ell_{i+1}$, and the times $t_i$ and $t_{i+1}$. Consider the time evolution of a single spacelike edge, such as the one shown in figure 9. The black lines show the paths that points [A] and [B] would take should they remain at rest. Since the time at $\Sigma_i$ is $t_i$ and the time at $\Sigma_{i+1}$ is $t_{i+1}$, the length of these lines is $\delta t_i = t_{i+1} - t_i$. Since the line $AB'$ is purely spacelike, it must be perpendicular to the black $\delta t_i$ lines. Thus, the total length of line $AB'$ must be

$$\ell_{i+1} = \ell_i + \gamma_{i+1} + \beta_{i+1}.$$  

(43)

Using the hyperbolic version of the pythagorean theorem on the two triangles in figure 9 gives

$$m_i^2 = \gamma_{i+1}^2 - \delta t_i^2 = \beta_{i+1}^2 - \delta t_i^2,$$

(44)
implying that $\gamma_{i+1} = \beta_{i+1}$. Plugging this into equation (43), it is found that

$$\beta_{i+1} = \gamma_{i+1} = \frac{1}{2}(\ell_{i+1} - \ell_i).$$  \hspace{1cm} (45)

Combining this with equation (44) gives

$$m_i^2 = \left(\frac{1}{4}\ell_i^2 - 1\right)\delta t_i^2,$$  \hspace{1cm} (46)

where $\ell = \frac{\ell_{i+1} - \ell_i}{t_{i+1} - t_i}$.

![Figure 9: Time evolved path of $AB$ to $A'B'$.](image)

To find the length $d_i$, draw a connection from one corner to the other. Since $|AA'| = |BB'| = m_i$ and $\gamma_i = \beta_i$, it does not matter which of the two diagonals are chosen since they’d both be equal lengths. Chosen here is the diagonal from $[A]$ to $[B']$, which is shown in figure 10. More hyperbolic trig can be used to show that

$$d_i^2 = (\ell_i + \beta_{i+1})^2 - \delta t_i^2.$$  \hspace{1cm} (47)

Plugging in equation (45) for $\beta_{i+1}$ and doing some algebraic manipulation results in

$$d_i^2 = \frac{1}{4}(\ell_{i+1} + \ell_i) - \delta t_i^2,$$  \hspace{1cm} (48)

or equivalently

$$d_i^2 = \ell_{i+1}\ell_i + m_i^2.$$  \hspace{1cm} (49)
These relationships will be important to remember when it comes time to relate the edge lengths to the metric coefficients.

3.2.2 Triangle Types

Let’s take a moment now to think about what these "same edge type" constraints mean for the lattice. The Regge-Einstein equations, (42), only ask for three parameters: the edge lengths, the areas of the triangles, and the deficits. There exist only finitely many ways to construct a triangle using three different edge lengths, and therefore only finitely many triangle areas. In this section, it will be shown that only three kinds of triangles exist on the manifold.

Consider the edge $\overline{AB}$. This is a spacelike edge since vertices $[A]$ and $[B]$ are both on $\Sigma_i$. Since $\Sigma_i$ is 3-dimensional, $[A]$ and $[B]$ must share at least two other vertices on $\Sigma_i$. Figure 11 shows one of these vertices as vertex $[C]$. The edges $\overline{AC}$ and $\overline{BC}$ must be spacelike since $[C]$ is on $\Sigma_i$. Now recall that the vertex $[A']$ on $\Sigma_{i+1}$ is attached $[A]$ by a timelike edge and to every vertex that $[A]$ is attached to by a diagonal edge. Therefore, $[A']$ must be attached to $[B]$ by a diagonal edge. A similar argument can be made for $[B']$ and $[C']$. Finally, vertices on the surface $\Sigma_{i-1}$ must be considered. These are denoted with the prime on the left side, and given the construction the exact same arguments can be made for which vertices are attached to $\overline{AB}$ and by which edge types.

![Figure 10: The triangle made by connecting points $[A]$, $[B]$, and $[B']$.](image-url)
Figure 11: The spacelike edge $\overline{AB}$ and all possible triangle types that can be connected to it. Tick marks denote lines which are of the same type but not necessarily the same length. Note: not all connections are shown.

Notice that, in figure 11, there are only three different kinds of triangles. The first has all spacelike edges, the second has two diagonal edges and one spacelike edge, and the third has one timelike edge, one spacelike edge, and one diagonal edge. Figures 12 and 13 show the results of the same analysis when applied to diagonal and timelike edges (keeping in mind that vertices from $\Sigma_{i-1}$ won’t be connected to those on $\Sigma_{i+1}$). It can be seen from these figures that no other triangle types exist on the manifold.

Figure 12: The diagonal edge $\overline{AB'}$ and all possible triangle types that can be connected to it. Tick marks denote lines which are of the same type but not necessarily the same length. Note: not all connections are shown.

Ultimately what this means is that every single triangle in the manifold can be broken down into having one of three different edge configurations and one of five different
Figure 13: The timelike edge $\overline{AA'}$ and all possible triangle types that can be connected to it. Tick marks denote lines which are of the same type but not necessarily the same length. Note: not all connections are shown.

Area equations. If a triangle has three spacelike edges, it will be referred to as an $S\triangle$. Triangles of this type have area

$$A_i^S = \frac{\sqrt{3}}{4} \ell_i^2,$$  \hfill (50)

which can be found using Heron’s formula [13].

If a triangle has two diagonal edges and one spacelike edge, it will be referred to as a $D\triangle$. Heron’s formula can again be used, and thus its area will be either

$$A_i^{Df} = \frac{1}{4} \sqrt{2d_i^2 \ell_i^2 + \ell_i^4}$$  \hfill (51)

or

$$A_i^{Dp} = \frac{1}{4} \sqrt{2d_i^2 \ell_{i+1}^2 + \ell_{i+1}^4}. \hfill (52)$$

Finally, if a triangle has a timelike edge it will be referred to as a $T\triangle$. The areas of these triangles can are

$$A_i^{Tf} = \frac{1}{4} \sqrt{(m_i^2 + \ell_i^2 + d_i^2)^2 - 4\ell_i^2 d_i^2}$$  \hfill (53)

and

$$A_i^{Tp} = \frac{1}{4} \sqrt{(m_i^2 + \ell_{i+1}^2 + d_i^2)^2 - 4\ell_{i+1}^2 d_i^2} \hfill (54)$$

It’s important to stress that, due to the way in which it’s constructed, these are the only kinds of triangles which can exist in the entire manifold. Even though $\ell_{i+23}$ may be much greater or smaller than $\ell_i$, the triangles which have edge length $\ell_{i+23}$ are proportionally identical to those with edge length $\ell_i$. They are just scaled up or down.
3.2.3 The Refined Regge Equations

Let’s now turn back to the Regge equations for a moment. The Regge-Hilbert-Einstein action is given as

\[ I_R = \frac{1}{8\pi} \sum_i A_i \epsilon_i \]  

where \( A_i \) is the \( i \)th triangle in the manifold and \( \epsilon_i \) is the deficit corresponding to that triangle. However, it is now known that there exists only five unique kinds of triangle in the manifold. Therefore (55) can be rewritten using (50) to (54) as

\[ 8\pi I_R = \sum_i A_i^S \epsilon_i^S + \sum_i A_i^{Df} \epsilon_i^{Df} + \sum_i A_i^{Dp} \epsilon_i^{Dp} + \sum_i A_i^{Tf} \epsilon_i^{Tf} + \sum_i A_i^{Tp} \epsilon_i^{Tp}. \]  

Now recall that the Regge-Einstein equations are defined as the variation of \( I_R \) with respect to some edge \( l_j \). Furthermore, since spacetime is flat on each individual triangle, this variation must be equal to zero. This is stated in (42). Solutions to the Regge-Einstein equations need one to know what \( \frac{\delta A_i}{\delta l_j} \) is and what \( \epsilon_i \) is. So now that all \( A_i \)'s are known, the first parameter can be taken care of.

To begin, let’s consider what the value of \( \frac{\delta A_i}{\delta l_j} \) is if \( A_i \) does not contain \( l_j \). Since the edge \( l_j \) is not part of the triangle, the triangle can not vary with respect to that edge and
therefore $\frac{\delta A}{\delta m^j} = 0$. This implies that, even if a triangle has an edge of length $\ell_{j+1}$, it still wouldn’t vary with respect to $\ell_j$ since $\ell_{j+1} \neq \ell_j$. Thus, the variational derivative $\frac{\delta A}{\delta m^j}$ behaves like a weird Kronecker delta, knocking out every term in the sum that doesn’t have an $\ell_j$ in it. (Note: the two are completely unrelated, the $\delta$ notation is coincidental). With these factors in mind, let’s begin taking derivatives.

The simplest Regge equation to compute at this stage is

$$8\pi \frac{\delta I_R}{\delta m^j} = \sum_i \frac{\delta A_i^S}{\delta m^j} \epsilon_i^S + \sum_i \frac{\delta A_i^{Df}}{\delta m^j} \epsilon_i^{Df} + \sum_i \frac{\delta A_i^{Tp}}{\delta m^j} \epsilon_i^{Tp} = 0. \quad (57)$$

No $S\triangle$’s or $D\triangle$’s contain any edges of length $m_j$ and thus the first three sums are all zero. The variation of $A_i^{Tf}$, given as the partial derivative with respect to $m_j$, is

$$\frac{\delta A_i^{Tf}}{\delta m^j} = \frac{m_j(d_j^2 + \ell_j^2 + m_j^2)}{2\sqrt{-4d_j^2\ell_j^2 + (d_j^2 + \ell_j^2 + m_j^2)^2}}. \quad (58)$$

The only difference between (58) the partial derivative $\frac{\delta A_i^{Tf}}{\delta m^j}$ is the use of $\ell_{j+1}$ instead of $\ell_j$. Notice also that the denominator in (58) is equal to $8A_j^{Tf}$, and similarly the denominator in $\frac{\delta A_i^{Tf}}{\delta m^j}$ is $8A_j^{Tp}$. Thus, the Regge equation for timelike edges $m_j$ is

$$8\pi \frac{\delta I_R}{\delta m^j} = \frac{m_j(d_j^2 + \ell_j^2 + m_j^2)}{8A_j^{Tf}} \sum_k \epsilon_k^{Tf} + \frac{m_j(d_{j+1}^2 + \ell_{j+1}^2 + m_j^2)}{8A_j^{Tp}} \sum_k \epsilon_k^{Tp} = 0. \quad (59)$$

Notice that the sum is now over $k$, and that everything apart from the deficits $\epsilon_k$ can be pulled out of the sum. This is because the values of edge the edge lengths $\ell_j$, $m_j$, and $d_j$, and therefore the areas of the triangles with those edge lengths, remain fixed throughout all of $\Sigma_j$. However, the deficit associated with a triangle in one region of $\Sigma_j$ is not necessarily identical to the value of a triangle in a different region of $\Sigma_j$. Essentially, edge lengths don’t contain curvature information, but deficits do. The $k$ subscript is used to differentiate between the $i$ and $j$ subscripts, since $i$ and $j$ refer to edges at different points in time while the variation between the deficits can occur over different points in space and time. After some reordering, factoring, and cancellation, (59) becomes

$$(d_j^2 + m_j^2)\left(\frac{1}{A_j^{Tf}} \sum_k \epsilon_k^{Tf} + \frac{1}{A_j^{Tp}} \sum_k \epsilon_k^{Tp}\right) + \left(\ell_j^2 + \ell_{j+1}^2\right)\left(\frac{1}{A_j^{Tf}} \sum_k \epsilon_k^{Tf} + \frac{1}{A_j^{Tp}} \sum_k \epsilon_k^{Tp}\right) = 0. \quad (60)$$

This is about as far as one can go algebraically without knowing what the deficits are, so (60) will be kept as is for now. As an aside, notice that, while all the sums over $\epsilon_k$ may look
intimidating, those are really just fancy ways of writing a number. All $\epsilon_k$’s are constants since they’re evaluated at a specific point in spacetime.

The same procedure of variation and algebraic manipulation can be done for $8\pi \frac{dI_R}{dp}$ to produce a second Regge equation. Skipping past all the partial derivatives and algebra, the result is

$$(d_j^2 + m_j^2)(\frac{1}{A^{df}_j} \sum_k \epsilon^j_k + \frac{1}{A^{tp}_j} \sum_k \epsilon^j_k^{Tp}) + \epsilon^j_1(\frac{1}{A^{df}_j} \sum_k \epsilon^j_k - \frac{1}{A^{tp}_j} \sum_k \epsilon^j_k) + \epsilon^j_{j+1}(\frac{1}{A^{dp}_j} \sum_k \epsilon^j_k^{Dp} - \frac{1}{A^{tp}_j} \sum_k \epsilon^j_k^{Tp}) = 0. \quad (61)$$

Again, this is about as far as one can go to simplify the equation without knowing the deficits.

Finally, let’s move on to $8\pi \frac{dI_R}{dp}$. The procedure here is technically identical, however one must take care to not overlook certain subtleties. While $A^{dp}_j$ and $A^{tp}_j$ do not have any $\ell_j$ terms in them, the same can not be said for $A^{dp}_{j-1}$ and $A^{tp}_{j-1}$, since

$$A^{dp}_{j-1} = \frac{1}{4} \sqrt{2d_{j-1}^2 \ell^4_j + \ell^4_j} \quad (62)$$

and

$$A^{tp}_{j-1} = \frac{1}{4} \sqrt{(d_{j-1}^2 + \ell^2_j + m_{j-1}^2)^2 - 4\ell^2_j d_{j-1}^2}. \quad (63)$$

Accounting for this, the last Regge equation, after algebraic simplification, is

$$4\sqrt{3} \sum_k \epsilon^S_k + \ell^2_j(\frac{1}{A^{df}_j} \sum_k \epsilon^j_k^{Df} + \frac{1}{A^{dp}_j} \sum_k \epsilon^j_k^{Dp} + \frac{1}{A^{tp}_j} \sum_k \epsilon^j_k^{Tp} + \frac{1}{A^{tp}_j} \epsilon^j_k^{Tp})$$

$$+ d_j^2(\frac{1}{A^{df}_j} \sum_k \epsilon^j_k^{Df} - \frac{1}{A^{df}_j} \sum_k \epsilon^j_k^{Tf}) + d_{j-1}^2(\frac{1}{A^{dp}_j} \sum_k \epsilon^j_k^{Dp} - \frac{1}{A^{tp}_j} \sum_k \epsilon^j_k^{Tp})$$

$$+ m_j^2 \frac{1}{A^{df}_j} \sum_k \epsilon^j_k^{Tf} + m_{j-1}^2 \frac{1}{A^{tp}_j} \sum_k \epsilon^j_k^{Tp} = 0. \quad (64)$$

These are not the final forms of the Regge equations, however the equations as they are are true for any spacetime manifold given the lattice construction scheme. Once the angle deficits are known, all that has to be done is solve these equations for $\ell_{j+1}$. Since $m_i$ and $d_i$ are defined in terms of $\ell_i$ and $\ell_{i+1}$ above, there exists three equations for essentially one unknown.
3.3 Connections, Deficits, and Dihedral Angles

3.3.1 Preamble

Recall from section 2.2.1 that the deficit of a triangle in a 4-D simplicial manifold is the quantity which relates the edge lengths and vertex locations to the curvature of the manifold being approximated. This idea can be generalized to $n$ dimensions, where the curvature of an $n$-dimensional manifold of $n$-simplices is contained entirely within the $n-2$-simplex "faces" (Note: "faces" could refer to vertices, lines, triangles, tetrahedra, etc., however since this paper deals with 4-simplices, the $n-2$-simplices containing the curvature information are all triangles). As shown in Figure 4, the two-dimensional version of Regge calculus deals with curvature contained entirely in the vertices. Recall that the deficit is given as

$$\epsilon_i = 2\pi - \sum_i \delta_i$$  \hspace{1cm} (65)

where $\delta_i$ is the angle between two lines meeting at the vertex in question and the sum is over all such angles meeting at this vertex. It can be immediately seen that flat spacetime regions are those such that $\sum_i \delta_i = 2\pi$, while regions with positive curvature have $\sum_i \delta_i < 2\pi$ and regions with negative curvature have $\sum_i \delta_i > 2\pi$.

This is the two dimensional case. As shown in figure 3, moving up from 2-D to 3-D also means moving everything else up a dimension. Now, curvature is concentrated around the edges, while the angles being plugged in to (65) are not the angles between two lines which share a vertex, but two planes which share an edge. Such angles are known as dihedral angles. The term "dihedral angle" is also used to describe the 4-dimensional case

![Figure 15: The dihedral angle $\delta_i$ between two triangular planes sharing a common edge.](image)

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as well. Unfortunately, we do not live in a universe with four spatial dimensions, and thus
depicting the dihedral angle between two tetrahedra which share a triangle is not an easy
task. A common means of doing this, seen in both references [3] and [12], is shown in Figure
16. Here, if one is looking at the 4-simplex with vertices [A], [B], [C], [D], and [E], and
wants to know the dihedral angle between \(ABCD\) and \(ABCE\), then the two tetrahedra are
represented as two lines and the triangle \(\triangle ABC\) is represented as a circle. The advantage
of diagrams like Figure 16 is that normal vectors to tetrahedra can be represented easier.
However it can be easy to misinterpret this sort of diagram. Since tetrahedra become lines,
but not all lines are tetrahedra (see line \(DE\)). And a similar statement can be made regarding
points and vertices. Information about the kinds of connections between [D] and [A], [B],
and [C] are also lost. There is no way to visually distinguish \(AD\) from \(BD\) or \(CD\), and the
same can be said for \(AE\) from \(BE\) or \(CE\).

Figure 16: The dihedral angle \(\delta_i\) between tetrahedra \(ABCD\) and \(ABCE\), where the circle
represents \(\triangle ABC\)

Figure 17 presents an alternative means of representation. The full triangle which
is shared by the two tetrahedra is represented, as are all the lines of the simplex. However,
the lines from [D] and [E] are not drawn completely out, freeing up the bottom and interior
of the diagram for labels and from excess clutter. Colors are also used to indicate whether
an edge is spacelike (ie, of length \(\ell_i, \ell_{i+1}\), etc.), timelike (ie, of length \(m_i, m_{i+1}\), etc.), or
diagonal (ie, of length \(d_i, d_{i+1}\), etc.). The top and bottom lines are also placed such that they
point towards the vertex on the triangle they’re connected to, while the middle line is placed
roughly half way between the top and bottom (this is due to spatial issues. If it were to
point directly at its corresponding vertex, it could overlap the top or bottom). In Figure 17,
[A’] was used in place of [E] in order to show how the coloring scheme works. While Figure
16 can be very helpful in the vector analysis that’s about to come up, Figure 17 will be much
better than Figure 16 when it comes time to assign dihedral angles to triangle types.

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3.3.2 Dihedral Angles

With the above considerations in mind, let’s now turn attention back to (65). In order to calculate the defect of a given triangular face, one must first know what the dihedral angles are between all adjacent tetrahedra sharing that face. However, finding angles between 3-D faces in four dimensions sounds like a daunting task, so perhaps it would be helpful to begin by stepping down a few dimensions.

Consider the points \([A], [B],\) and \([C],\) and let \(\delta_i\) describe the angle between \(AB\) and \(BC\). If one knows the coordinates of \([A], [B]\) and \([C],\) then \(\delta_i\) can be determined by

\[
\vec{AB} \cdot \vec{AC} = |\vec{AB}| |\vec{AC}| \cos(\delta_i).
\]

(66)

Now consider the tetrahedron \(ABCD\). If one wanted to find the dihedral angle \(\delta_i\) between \(\triangle ABC\) and \(\triangle ABD\) by applying (66), they would be immediately confronted with a problem: what vector describes a triangle? (They may also have noticed the requirement of having coordinates when none have been defined, but let’s stick to one issue at a time.) The not-so-obvious answer to this question is the vector which is normal to the face of that triangle. To find the normal vector, it can be helpful to start thinking of one of the vertices as the origin of some coordinate system. Since the common edge between \(\triangle ABC\) and \(\triangle ABD\) is \(\overline{AB}\), a natural choice for an origin would be either \([A]\) or \([B],\) Since the answer doesn’t depend
on our choice of origin, let’s just pick [A]. The vectors $\vec{AB}$ and $\vec{AC}$ are all that’s needed to describe $\triangle ABC$, since $\vec{BC} = \vec{AC} - \vec{AB}$. A vector which is normal to $\triangle ABC$ must by definition be normal to both $\vec{AB}$ and $\vec{AC}$. Thus, if such a vector is called $\vec{Q}$, then

$$\vec{Q} = \vec{AB} \times \vec{AC}. \quad (67)$$

The same argument can be made for finding a normal vector to $\triangle ABD$. A dot product can then be taken between these two vectors in order to find the dihedral angle.

Now let’s move on to the 4-D case. Consider the 4-simplex $ABCDE$ with the entirely generic vertices [A], [B], [C], [D], and [E] (here, entirely generic means that the vertices could be located anywhere in the manifold and not necessarily exclusive to $\Sigma_i$). Let [A] be the origin. Since all vectors from [A] to some point [X] can be described as $\vec{AX}$, the "A" in front is superfluous and therefore will be dropped (ie $\vec{X} := \vec{AX}$). The only time this is not the case is for $\vec{Q}$, which maintains the definition given above. Let’s consider the dihedral angle $\delta_i$ between the tetrahedra $ABCD$ and $ABCE$. Once again the goal here is to find two vectors, $\vec{N}_D$ and $\vec{N}_E$, which are normal to $ABCD$ and $ABCE$ respectively. Cross products can again be used to obtain this goal, however there is a subtle snag to be considered. In Figure 18, it can clearly be seen that two vectors exist which are normal to the tetrahedron $ABCD$. In order to avoid this problem, $\vec{N}_D$ and $\vec{N}_E$ must be constructed following

$$\vec{N}_X = (\vec{B} \times \vec{C}) \times \vec{X} \quad (68)$$

where $[X] = [D], [E]$. This will ensure that the angle between the two vectors is still $\delta_i$ due to the 4-D equivalent of the right-hand-rule.

Figure 18: The normal vectors between two tetrahedral faces of the 4-simplex $ABCDE$. 

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Since no ambiguity exists given (68), $\delta_i$ can be found using
\[
cos(\delta_i) = \frac{\vec{N}_D \cdot \vec{N}_E}{|\vec{N}_D||\vec{N}_E|}.
\] (69)

If one knows the coordinates of $[A]$, $[B]$, $[C]$, $[D]$, and $[E]$, then (69) may be sufficient. However, taking cross products of cross products and then dotting them with other cross products of cross products is computationally inefficient. Furthermore, the coordinates of these vertices haven’t yet been defined. So let’s see if this expression can be put into terms which align more to the spirit of Regge calculus.

First, let’s consider the numerator,
\[
\vec{N}_D \cdot \vec{N}_E = (\vec{B} \times \vec{C}) \times (\vec{D} \times (\vec{B} \times \vec{C}) \times \vec{E}).
\] (70)

To make this more manageable, let’s use (67) to replace $\vec{B} \times \vec{C}$ with $\vec{Q}$, giving
\[
\vec{N}_D \cdot \vec{N}_E = (\vec{Q} \times \vec{D}) \cdot (\vec{Q} \times \vec{E}).
\] (71)

It is well known that, for any vectors $\vec{U}$, $\vec{V}$, and $\vec{W}$,
\[
\vec{U} \cdot (\vec{V} \times \vec{W}) = \vec{V} \cdot (\vec{W} \times \vec{U}) = \vec{W} \cdot (\vec{U} \times \vec{V})
\] (72)

and
\[
\vec{U} \times (\vec{V} \times \vec{W}) = \vec{V}(\vec{U} \cdot \vec{W}) - \vec{W}(\vec{U} \cdot \vec{V}).
\] (73)

These identities can be used to rewrite (71) as
\[
\vec{N}_D \cdot \vec{N}_E = (\vec{Q} \cdot \vec{Q})(\vec{D} \cdot \vec{E}) - (\vec{Q} \cdot \vec{D})(\vec{Q} \cdot \vec{E}),
\] (74)

which itself is equivalent to the expression
\[
\vec{N}_D \cdot \vec{N}_E = (|\vec{Q}|^2)(|\vec{D}||\vec{E}|cos(\Phi_{ED})) - (|\vec{Q}|^2)|\vec{D}||\vec{E}|cos(\Phi_{QD})cos(\Phi_{QE}),
\] (75)

where $\Phi_{ED}$ is the angle between the lines $\overline{AE}$ and $\overline{AD}$, and $\Phi_{QX}$ is the angle between the line $\overline{AX}$ and $\triangle ABC$. All three of these angles can easily be found using nothing but Heron’s formula and some basic trig (in other words, they can be expressed purely in terms of the edge lengths). Furthermore, recall that the cross product between two vectors describes the area of a parallelogram with edge lengths equal to the magnitude of those vectors. Thus, the area of the triangle created using those two vectors is simply half of the area of that
parallelogram. Therefore,

$$|\vec{Q}| = 2A_{ABC}$$  \hspace{1cm} (76)

where $A_{ABC}$ is the area of $\triangle ABC$. Finally, the magnitude of the vector $\vec{X}$ is equal to the length of edge $AX$. If we call this length $L_{AX}$, then

$$|\vec{D}| = L_{AD}, |\vec{E}| = L_{AE}.$$  \hspace{1cm} (77)

Implementing this into (75) gives

$$\vec{N}_D \cdot \vec{N}_E = 4A^2_{ABC}L_{AD}L_{AE}(\cos(\Phi_{DE}) - \cos(\Phi_{QD})\cos(\Phi_{QE})).$$  \hspace{1cm} (78)

Now let’s take a look at the denominator of (69). Consider just $|\vec{N}_D|$. This is given as

$$|\vec{N}_D| = |\vec{Q} \times \vec{D}|.$$  \hspace{1cm} (79)

This is equivalent to the expression

$$|\vec{N}_D| = |\vec{Q}||\vec{D}|\sin(\Phi_{QD}).$$  \hspace{1cm} (80)

Now recall that the magnitudes of these vectors are already known as $2A_{ABC}$ and $L_{AD}$. After doing this for $|\vec{N}_E|$, it is found that the denominator of (69) is

$$|\vec{N}_D||\vec{N}_E| = 4A^2_{ABC}L_{AD}L_{AE}\sin(\Phi_{QD})\sin(\Phi_{QE}).$$  \hspace{1cm} (81)

Plugging (78) and (81) into (69), the dihedral angle $\delta_i$ between the two tetrahedra can be given as

$$\cos(\delta_i) = \frac{\cos(\Phi_{DE}) - \cos(\Phi_{QD})\cos(\Phi_{QE})}{\sin(\Phi_{QD})\sin(\Phi_{QE})}.$$  \hspace{1cm} (82)

Notice that, because of the way it was derived, (82) is true regardless of what coordinates are being used and tetrahedra being considered. Furthermore, as stated previously, all angles labeled with a $\Phi$ can be computed using only the edge lengths of the 4-simplex. Obviously, $\cos(\delta_i)$ will be different depending on which simplex is being looked at, what type of triangular face is the tetrahedra share, etc., and computing every single $\delta_i$ even for one triangle is still a long and tedious task. However, (82) is perfect for a computer program, and given the means with which the manifold was constructed, multiple $\Phi$’s will be equal to each other. Unfortunately, the time constraints of this project have forced such computations to be left as either an exercise to the reader or the subject of a different paper.
3.3.3 The Metric and the Edges

Note: Before beginning this section, it should be mentioned that section 6 of Brewin’s paper "Friedmann Cosmologies via the Regge Calculus" [14] provides an alternative method to relate the edge lengths to the metric in a way that can be easily applied to all metrics. However, his method is not easily compatible with the work done in sections 3.2.1-3.2.3. Thus, any implementation will have to be left for a future paper.

So far, everything done has made no use of the line element or the metric coefficients, but they must be involved somehow. How else would one know the difference between a Schwarzschild triangulation and flat space? This section will provide an answer to that question. This method will not be applicable, however, to every single metric that one could be handed, since it relies on the fact that the coordinates being used are already known (ie spherical, cartesian, polar, etc.) Likewise, this method also assumes that the line element is known. However, this method can be easily implemented even if any information about the curvature of the manifold is unknown or difficult to compute.

First, recall that the coordinates of any point \([C]\) in some spacetime can be given as \([C] = (x^0_C, x^1_C, x^2_C, x^3_C)\). In a smooth, continuous spacetime, the proper distance \(ds\) between the points \([X]\) and \([X + dx]\) is

\[
    ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{83}
\]

Let \(N\) denote the lapse function and \(\sigma_\mu\) denote a component of the shift vector. Then (83) in the ADM formulation is

\[
    ds^2 = (N + \sigma_k \sigma^k) dt^2 + \sigma_u dt dx^u + \sigma_v dt dx^v + \gamma_{uv} dx^u dx^v \tag{84}
\]

where \(k = 0, 1, 2, 3\) and \(u, v = 1, 2, 3\). Now let’s focus on two vertices, \([A]\) and \([B]\). Let’s call the proper distance between these vertices \(\Delta s\), and let’s call the Regge edge length between \([A]\) and \([B]\) \(L_{AB}\). In Regge calculus, it is assumed that \([A]\) and \([B]\) are close enough such that \(L_{AB} \approx \Delta s\). From this assumption, the fundamental relationship equation can be derived as

\[
    L^2_{AB} = g_{\mu\nu} \Delta x^\mu_{AB} \Delta x^\nu_{AB}, \tag{85}
\]

where \(\Delta x^\mu_{AB} := x^\mu_B - x^\mu_A\). In ADM language, this becomes

\[
    L^2_{AB} = (N + \sigma_k \sigma^k) \Delta t^2_{AB} + \sigma_u \Delta t_{AB} \Delta x^u_{AB} + \sigma_v \Delta t_{AB} \Delta x^v_{AB} + \gamma_{uv} \Delta x^u_{AB} \Delta x^v_{AB}. \tag{86}
\]

In theory, all one needs to do from here is specify the coordinates of ten adjacent
vertices in the manifold. Doing so produces ten equations of the form

\[ L^2_{\alpha\beta} = \ell_i^2 = \gamma_{uv} \Delta x_{\alpha\beta}^u \Delta x_{\alpha\beta}^v, \quad (87) \]

\[ L^2_{\alpha\alpha'} = m_i^2 = (N + \sigma_k \sigma^k) \Delta t_{\alpha\beta}^2 + \sigma_u \Delta t_{\alpha\beta} \Delta x_{\alpha\beta}^u + \sigma_v \Delta t_{\alpha\beta} \Delta x_{\alpha\beta}^v + \gamma_{uv} \Delta x_{\alpha\beta}^u \Delta x_{\alpha\beta}^v, \quad (88) \]

or

\[ L^2_{\alpha\alpha'} = d_i^2 = (N + \sigma_k \sigma^k) \Delta t_{\alpha\beta}^2 + \sigma_u \Delta t_{\alpha\beta} \Delta x_{\alpha\beta}^u + \sigma_v \Delta t_{\alpha\beta} \Delta x_{\alpha\beta}^v + \gamma_{uv} \Delta x_{\alpha\beta}^u \Delta x_{\alpha\beta}^v. \quad (89) \]

These equations can then be solved to put the \( g_{\mu\nu} \) in terms of \( \ell_i, m_i, \) and \( d_i \). In [3] and [14], a nearly identical method is used to do this, and [14] contains the final form of these relationship equations for a generic metric. Unfortunately, further work is still needed to find the final form of these equations.
4 Conclusions

The goal of this paper was to derive a general method of Regge calculus using the constraints given in section 3.1. These "same-edge-length" constraints provided the nucleus for a derivation of both the Regge equations themselves, as well as the the angle deficits associated with them, without ever specifying a metric. The Regge equations derived were given as (60), (61), and (64), while the equation for the dihedral angles (the only variables of the deficit equations) was found as (82). Finally, a method for relating the metric coefficients to the edge lengths was derived, the final result being (87), (88), and (89). Other papers in this subject will either stick to the general equations, leaving any elaboration as an exercise to the reader, or get so specific as to lose all applicability to any other metric apart from the one being focused on. This paper, however, sacrifices the freedom of differing leg lengths in order to find a nice middle ground between the overly general and the overly specific. Further work is still needed, and an obvious area of future research would be to address the open issues discussed in sections 3.3.2 and 3.3.3, such as implementing the work of Brewin’s paper and writing a program for calculating the dihedral angles and angle deficits. Once all the kinks have been worked out, applications could range from the modelling of galactic systems, binary blackholes, as well as additional uses of the technique in fields outside GR which nonetheless require the use of complicated geometric spaces, such as fluid dynamics. In any case, with the power to bring impossible equations down to manageable chunks, a reliable algorithm for doing Regge calculus can help scientists get approximate answers to problems that may otherwise go unsolved.
5 References


