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Translation surfaces can be viewed as polygons with parallel and equal sides identified. An affine homeomorphism ϕ from a translation surface to itself is called pseudo-Anosov when its derivative is a constant matrix in $\mathrm{SL}_2(\mathbb{R})$ whose trace is larger than 2 in absolute value. In this setting, the eigendirections of this matrix defines the stable and unstable flow on the translation surface. Taking a transversal to the stable flows, the first return map of the flow induces an interval exchange transformation T. The Sah-Arnoux-Fathi invariant of ϕ is the sum of the wedge product between the lengths of the subintervals of T and their translations. This wedge product does not depend on the choice of transversal. We apply Veech's construction of pseudo-Anosov homeomorphisms to produce infinite families of pseudo-Anosov maps in the stratum $\mathcal{H}(2,2)$ with vanishing Sah-Arnoux-Fathi invariant, as well as sporadic examples in other strata.

New Families of pseudo-Anosov Homeomorphisms with Vanishing Sah-Arnoux-Fathi Invariant

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Chair of the Department of Mathematics

Dean of the Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Hieu Trung Do, Author

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1 Introduction

1.1 A brief history and statement of main results

The Teichmüller modular group Mod(X) of an orientable surface X is the quotient of the group of orientation preserving homeomorphisms by the subgroup of those homeomorphisms isotopic to the identity. By the Nielsen-Thurston classification theorem [FM], every element of Mod(X) is either periodic, reducible, or pseudo-Anosov. Among those, the class of pseudo-Anosov maps remains the most interesting as they are the richest and there are still many unanswered questions.

In 1981, Arnoux-Yoccoz [AY] gave the first example of a pseudo-Anosov homeomorphism whose dilatation was of degree less than twice the genus of the surface on which it is defined. In fact, they gave an infinite family of these, one in each genus $g \geq 3$ with dilatation of degree g. In his Ph.D. dissertation of the same year, Arnoux [A2], see also [A], showed that each of these homeomorphisms has vanishing Sah-Arnoux-Fathi (SAF) invariant.

Pseudo-Anosov homeomorphisms can be realized as affine maps defined on translation surfaces. We are interested in maps on translation surfaces which are closed orientable surfaces endowed with a flat metric having a finite number of conical singularities. Pseudo-Anosov maps with vanishing SAF-invariant are especially interesting for their dynamical properties, see [ABB, LPV, LPV2, Mc2]. However, there are few examples known, see below for a list of these in section §2.5.

Pseudo-Anosov maps with vanishing SAF invariant are characterized by their dilatation λ, sometimes called stretch factor, see Theorem 1.1.1 below. One question about the dilatation is to find the smallest stretch factor in any given genus g. The minimum dilatation λ has only been found for genus 1 and 2 in general, and up to genus 5 for orientable pseudo-Anosov maps, see [LT]. In this paper, we are interested in a particular algebraic property of the dilatation λ.

Various methods of construction of pseudo-Anosov maps were given by Thurston [T], Arnoux-Yoccoz [AY], Casson-Bleiler [CB], Penner [P], Bestvina-Handel [BH]. We focus on Veech's construction [V] using Rauzy-Veech induction on interval exchange transformations (IET).

We are able to provide four new infinite families of speudo-Anosov maps with vanishing SAF invariant in the stratum $\mathcal{H}^{\text{hyp}}(2,2)$, as well as sporadic examples in $\mathcal{H}^{\text{hyp}}(4)$, $\mathcal{H}^{\text{hyp}}(2,2)$,$\mathcal{H}^{\text{hyp}}(6)$, and $\mathcal{H}^{\text{hyp}}(3,3)$. Furthermore, we improve a result in [AS], which was derived from work of Calta-Smillie [CS] as follows.

Theorem 1.1.1. *Suppose that ϕ is an orientable pseudo-Anosov map of a closed compact surface, with dilatation λ. Then ϕ has vanishing Sah-Arnoux-Fathi invariant if and only if the minimal polynomial of λ is not reciprocal.*

The new infinite families of pseudo-Anosov maps with vanishing SAF-invariant are given explicitly by closed loops in the hyperelliptic Rauzy diagram with 7 subintervals.

Theorem 1.1.2. *For each $k \in \mathbb{N}$ with $k \geq 2$, there exists at least four orientable pseudo-Anosov maps in the hyperelliptic component of the stratum $\mathcal{H}(2,2)$ having dilatation of minimal polynomial $x^3 - (2k+4)x^2 + (k+4)x - 1$. In particular, each of these pseudo-Anosov maps has vanishing* SAF*-invariant.*

1.2 Statement of the problems

This dissertation addresses the following questions:

1. How can we better verify that a pseudo-Anosov map has vanishing SAF invariant?

2. Can we use Veech's construction of pseudo-Anosov maps to find (infinitely many) examples with vanishing SAF invariant ? In each genus? In each stratum?

3. Is there a generalized method for constructing vanishing SAF invariant maps using Veech's construction?

1.3 Organization

Section §2 of this dissertation presents background materials. We give definitions and examples of translation surface, pseudo-Anosov homeomorphism, and Sah-Arnoux-Fathi invariant. We compile a list of known examples of SAF-zero maps with brief details.

Section §3 introduces Veech's construction of pseudo-Anosov maps using interval exchange transformations. We follow closely the notation presented in [Fi, L, Vi]. The idea of the construction is simple. However, due to its explicit nature, it is quite involved. We present the notions of interval exchange transformation, Rauzy-Veech induction, Rauzy diagram, closed loop and pseudo-Anosov homeomorphism.

Section §4 discusses the trace field, periodic direction field, and Veech group of a translation surface. In particular, subsection §4.5 characterizes SAF-zero pseudo-Anosov maps. A proof of Theorem 1.1.1 is given.

In subsection §4.6 we apply a construction of pseudo-Anosov homeomorphisms given by Margalit-Spallone [MS] to lend support to a conjecture about the set of all dilatations of pseudo-Anosov homeomorphisms. A real algebraic number α greater than one is called *bi-Perron* if all of its conjugates (other than itself) lie in the annulus $\{||\alpha||^{-1} \le ||z|| < ||\alpha||\}$, where $||z||$ denotes the norm of a complex number. An algebraic integer, thus having monic minimal polynomial with integer coefficients, is a *unit* if its inverse is also an algebraic integer. Fried [Fr] showed that the dilatation of any pseudo-Anosov map is a bi-Perron unit. A conjecture that Farb-Margalit [FM] attribute to C. McMullen (and is a question in [Fr]), states that every bi-Perron unit is the dilatation of some pseudo-Anosov homeomorphism. Exactly when a pseudo-Anosov homeomorphism is orientable, its dilatation is an eigenvalue of the homeomorphism's induced action on first integral homology [FM]. The construction of Margalit-Spallone [MS] shows that any polynomial that passes a certain homological criterion, see subsection §4.3, is the characteristic polynomial of the homology action induced by some pseudo-Anosov map. Using this, we find a partial confirmation of the conjecture, see Theorem 4.6.1.

In subsection §4.7, we answer an implicit question of Birman, Brinkmann and Kawamuro [BBK]. Namely, if ϕ is an orientable pseudo-Anosov map on a genus g compact surface without punctures, then their symplectic polynomial $s(x)$ associated to ϕ is reducible if and only if either ϕ has vanishing SAF-invariant, or ϕ has trace field of degree less than g.

Section §5 gives explicit results of our search for SAF-zero pseudo-Anosov maps. We revisit Lowenstein *et al.*'s study of the Arnoux-Rauzy family of IETs [LPV, LPV2] and the double heptagon example suggested by [AS]. Inspired by the pattern of Arnoux-Rauzy family as well as the similarity between the double heptagon example [CS] and Lanneau's example [Mc2], we discover 4 different families of SAF-zero pseudo-Anosov maps in the stratum $\mathcal{H}^{\text{hyp}}(2,2)$, that is using closed loops in the hyperelliptic Rauzy diagram with 7 subintervals. We also present all other sporadic examples that we found in the hyperelliptic Rauzy diagram with 6, 7, 8 and 9 subintervals.

2 Background

2.1 Pseudo-Anosov map

Suppose that X is an orientable closed real surface of genus $g \geq 2$. The Teichmüller modular group $\mathrm{Mod}(X)$ is the quotient of the group of orientation preserving homeomorphisms by the subgroup of those homeomorphisms isotopic to the identity. By the Nielsen-Thurston classification theorem, every element of $\mathrm{Mod}(X)$ is either periodic, reducible, or pseudo-Anosov. Here, we follow notations from Farb-Margalit [FM].

A mapping class $[\phi] \in \mathrm{Mod}(X)$ is called *pseudo-Anosov*, if there exists a representative $\phi : X \to X$, a pair of invariant transverse measured (singular) foliations $(\mathcal{F}^u, \mu^u), (\mathcal{F}^s, \mu^s)$, and a real number λ, called the dilatation of $[\phi]$, such that ϕ multiplies the transverse measure μ^u (resp. μ^s) by λ (resp. λ^{-1}). The real number $\lambda = \lambda(\phi)$ is called the *dilatation* of the *pseudo-Anosov homeomorphism* ϕ. Some prefer to call λ the *stretch factor* of ϕ.

A pseudo-Anosov homeomorphism ϕ is called *orientable* if either of (and hence both) $\mathcal{F}^u$ or $\mathcal{F}^s$ is orientable (that is, the leaves can be consistently oriented). As recalled in [LT] (see their Theorem 2.4), a pseudo-Anosov homeomorphism ϕ is orientable if and only if its dilatation is an eigenvalue of the standard induced action on first homology $\phi_* : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$.

2.2 Translation surface

Definition 2.2.1. *A **translation surface** is given by a pair* (X,ω) *where* X *is a Riemann surface and* ω *is a holomorphic 1-form (abelian differential) on* X*.*

Equivalently, we can view a translation surface as a finite union of Euclidean polygons $(P_1, P_2, ..., P_n)$ such that

· the boundary of every polygon is oriented so that the polygon lies to the left;

· for every $1 \leq j \leq n$, for every oriented side s_j of P_j there is a $1 \leq k \leq n$ and an oriented side s_k of P_k so that s_j and s_k are parallel and of the same length. They are glued together in the opposite orientation by parallel translation so that as one moves along a glued edge, one polygon appears to the left and the other on the right.

We often use the words *pseudo-Anosov map* to mean an orientable pseudo-Anosov homeomorphism (usually with an emphasis on its translation surface). The pseudo-Anosov ϕ acts affinely with respect to the local Euclidean structure of (X,ω). Furthermore, taking the view of real local coordinates, $\mathrm{SL}_2(\mathbb{R})$ acts on the collection of all translation surfaces by post-composition with the local coordinate maps. The eigenvectors of its derivative $D\phi$ give the direction of the stable and unstable flow, see Definition 2.3.3 below.

2.3 Sah-Arnoux-Fathi invariant

The Sah-Arnoux-Fathi (SAF) invariant was first defined [A2] for any interval exchange transformation (IET). IETs are presented in more detail in section §3.1.

Definition 2.3.1. *An* ***interval exchange transformation*** *(IET) is a bijection, T, from an interval I to itself that permutes, by translation, a finite partition of subintervals I_j of I , $j = 1, ..., d$.*

We can express T as a piecewise function:

$$T(x) = x + \omega_j, \text{for } x \in I_j$$

where ω_j is the translation constant.

Definition 2.3.2. *For an IET T defined as above, the* ***Sah-Arnoux-Fathi invariant*** *of T is the element in $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$, the second exterior product of the $\mathbb{Q}$-vector space $\mathbb{R}$ (commonly called the wedge product over $\mathbb{Q}$ of $\mathbb{R}$ with itself), given by $\sum_{j=1}^{d} \ell_j \wedge \omega_j$, where ℓ_j is the length of I_j.*

Here, $a \wedge_{\mathbb{Q}} b = a \otimes_{\mathbb{Q}} b - b \otimes_{\mathbb{Q}} a$. The wedge product is bilinear and anti-symmetric. We define SAF invariant of a pseudo-Anosov as the SAF invariant of an induced IET resulting from the linear flow in the stable direction.

Definition 2.3.3. *In the plane $\mathbb{R}^2$, the* ***linear flow*** *in the direction of a unit vector v is the map $F_v : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$F_v(t, P) = P + tv$$

Definition 2.3.4. *The* ***linear flow*** *on a translation surface (X, ω) in the direction of a unit vector v is the map $\mathcal{F}_v : \mathbb{R} \times (X, \omega) \to (X, \omega)$ given by the pull-back of the linear flow in $\mathbb{R}^2$ using coordinate charts of (X, ω). The flow $\mathcal{F}_v$ sends (t, P) to a point $P' \in (X, \omega)$ such that the segment $PP' = tv$.*

The function $\mathcal{F}_v$ is not well-defined when the flow reaches a singularity or when P is a singularity. The cone angle $k2\pi$ (for $k > 1$) at a singularity allows multiple choices to continue the flow in the direction of v. In such situation the flow simply stops. A *full transversal* with respect to a linear flow $\mathcal{F}_v$ is a straight segment in (X, ω)

such that for almost every P on (X,ω), the ray $\mathcal{F}_v(t,P)$ for $0 \le t < \infty$ intersects the transversal. The first return map of a linear flow on (X,ω) to a full transversal defines an IET.

In [A2], Arnoux showed that any linear flow on a translation surface defines a family of interval exchange transformations, by taking any appropriately chosen full transversal of the flow, all having the same SAF-invariant.

Definition 2.3.5. *The* ***Sah-Arnoux-Fathi invariant*** *of a pseudo-Anosov homeomorphism ϕ is the SAF invariant of an IET resulting from the flow in the stable direction of ϕ.*

2.4 Known examples

Arnoux-Yoccoz [AY] construct a family of SAF-zero pseudo-Anosov maps (one per genus at least three). For $g \ge 3$, the polynomial $AY(x) = x^g + x^{g-1} + \cdots + x - 1$ has a unique real root α. Indeed, we can see that $AY(0) = -1$ and $AY(1) = g - 1 > 0$ and its derivative $AY'(x) > 0$ for all $x > 0$. For each g, they create an IET f_g using α. The important fact is that f_g is periodic under Rauzy-Veech induction, see definition 3.2.1. This property is also known as *self-similar* in the literature. They are able to build a translation surface (X,ω) (in fact, a so-called zippered rectangle) whose first return map of the vertical flow on a horizontal transversal is exactly f_g. Since f_g is periodic, applying a full period of Rauzy-Veech induction of f_g on (X,ω) will result in an isometric translation surface $(Y,\zeta) = \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{bmatrix} \cdot (X,\omega)$. It follows that (X,ω) is equipped with a pseudo-Anosov map ϕ whose dilatation is α^{-1}, whose minimal polynomial is non-reciprocal. Arnoux [A] proves that ϕ has vanishing SAF invariant. This is easily confirmed by our Theorem 1.1.1.

Lowenstein, Poggiaspalla, and Vivaldi, in [LPV] and [LPV2], discuss two SAF-zero pseudo-Anosov maps on genus 3 surfaces. Loweinstein *et al.* used the first two IETs from the infinite Arnoux-Rauzy family, which were constructed using compositions of order two IETS in a similar fashion as the Arnoux-Yoccoz IETs. However, instead of increasing the algebraic degree of the dilatation, Arnoux and Rauzy chose a sequence of cubic algebraic numbers. The IETs are not self-similar. Lowenstein *et al.* renormalized the first return map of each IET onto an appropriate subinterval, obtaining a self-similar IET. We believe that the authors intend to extend their calculation to the rest of the family. When we do this, we find that the resulting dilatations are the largest roots of $p_k(x) = x^3 - (3k+4)x^2 + (k+4)x - 1$. Hence the SAF invariant is zero.

Infinitely many examples of pseudo-Anosov maps with vanishing SAF invariant were found by Calta and Schmidt [CS] using Fuchsian group techniques. Sporadic examples were given by Arnoux-Schmidt [AS] and in [CS]; McMullen [Mc2] presents an example in genus three found by Lanneau. Strenner, in [S], shows that whenever we have a pseudo-Anosov map on a nonorientable surface, its so-called oriented double cover has vanishing SAF invariant.

3 Veech's construction of pseudo-Anosov homeomorphisms

Throughout this section we follow Fickenscher's [Fi], Lanneau's [L] (following [MMY]), and Viana's [V] overview of Veech's construction of pseudo-Anosov homeomorphisms using the Rauzy-Veech induction, [V]. In this section we follow standard convention and let λ denote the length vector for an interval exchange transformation. When we refer to a (sub)interval, we mean open on the right and closed on the left (that is $[a, b)$ for some $b > a$).

3.1 Interval Exchange Transformation

Definition 3.1.1. *An* ***interval exchange transformation*** *(IET) is a bijection, T, from an interval I to itself that permutes, by translation, a finite partition of subintervals $I_j, j = 1, ..., d$ of I.*

We denote λ_j to be the length of the subinterval I_j. Without loss of generality, we anchor the interval I at 0, $I = [0, \Sigma\lambda_j)$. We can express T as a piecewise function:

$$T(x) = x + \omega_j, \text{ for } x \in I_j$$

where ω_j is the translation constant.

However, we would like to introduce a more useful notation for T, which is determined precisely by the following data: a permutation π that encodes how the intervals are exchanged, and positive vector λ that encodes the lengths of the subintervals.

Let $\mathbb{R}_+^d$ be the cone of positive vectors in $\mathbb{R}^d$. It's easy to see that $\lambda \in \mathbb{R}_+^d$ and its components are the lengths λ_j. We define the length of λ as $|\lambda| = \sum_{j=1}^d \lambda_j$.

Let $\mathcal{A}$ be a finite alphabet and $d = |\mathcal{A}|$. Let $\pi_i : \mathcal{A} \to \{1, ..., d\}$, $i \in \{0, 1\}$, be bijections. In the partition of I into intervals, we denote the k^{th} interval, when counted from the left to the right, by $I_{\pi_0^{-1}(k)}$. Once the intervals are exchanged, the k^{th} interval is labeled $I_{\pi_1^{-1}(k)}$. We will usually represent the combinatorial datum $\pi = (\pi_0, \pi_1)$ by a table:

$$\pi = \begin{pmatrix} \pi_0^{-1}(1) & \pi_0^{-1}(2) & ... & \pi_0^{-1}(d) \\ \pi_1^{-1}(1) & \pi_1^{-1}(2) & ... & \pi_1^{-1}(d) \end{pmatrix}.$$

In particular, we can calculate the translation components as follow:

$$\omega_\alpha = \sum_{\pi_1(\beta) < \pi_1(\alpha)} \lambda_\beta - \sum_{\pi_0(\beta) < \pi_0(\alpha)} \lambda_\beta$$

Given $\pi = (\pi_0, \pi_1)$, define $\Omega_\pi : \mathbb{R}^\mathcal{A} \to \mathbb{R}^\mathcal{A}$ by

$$\Omega_{\alpha,\beta} = \begin{cases} +1 & \text{if } \pi_1(\alpha) > \pi_1(\beta), \text{and } \pi_0(\alpha) < \pi_0(\beta) \\ -1 & \text{if } \pi_1(\alpha) < \pi_1(\beta), \text{and } \pi_0(\alpha) > \pi_0(\beta) \\ 0 & \text{else.} \end{cases} \tag{3.1}$$

We can see that Ω_π is skew-symmetric, and $\Omega_\pi \cdot \lambda = \omega$.

Definition 3.1.2. *A permutation π is called **irreducible** if $(\pi_1 \circ \pi_0^{-1})\{1, ..., k\} = \{1, ..., k\}$ only when $k = d$.*

Whenever T is reducible, it decomposes into two or more irreducible IETs. Naturally, we are only interested in irreducible IETs. We now denote a IET T as (π, λ).

Remark 3.1.1. *The notation for $T = (\pi, \lambda)$ seems redundant as we can simplify the two-line expression of π (referred to as* labeled IET*) to the one-line expression using the composition: $\pi_1 \circ \pi_0^{-1}$ (referred to as* unlabeled IET*). However, the labeled format is particularly useful for the next section.*

3.2 Rauzy-Veech induction

In this section, we define a family $\mathcal{R}$ of maps on irreducible IETs, known as Rauzy-Veech induction. First introduced in [R], it is the first return map of an IET to an appropriate subinterval J of I.

Let $\alpha_\epsilon = \pi_\epsilon^{-1}(d)$ for $\epsilon \in \{0,1\}$. Define the interval $J = [0, |\lambda| - \min\{\alpha_0, \alpha_1\})$. The image of T by the Rauzy-Veech induction $\mathcal{R}$ is defined as the first return map of T to the subinterval J. This is again an interval exchange transformation, defined on d letters. Thus this defines two maps $\mathcal{R}_0$ and $\mathcal{R}_1$ by $\mathcal{R}(T) = (\mathcal{R}_\epsilon(\pi), \lambda')$, where ϵ is the type of T, as follows.

Definition 3.2.1.

We define **Rauzy-Veech induction** on $T = (\pi, \lambda)$ as $T' = (\pi', \lambda')$ by the following:

1. Assume $\lambda_{\alpha_0} > \lambda_{\alpha_1}$.

   We call this *Rauzy-Veech induction* $\mathcal{R}_0$ *of type 0.*

   Let k be $\pi_1^{-1}(k) = \pi_0^{-1}(d)$ with $k \le d-1$. Then $\mathcal{R}_0(\pi_0, \pi_1) = (\pi_0', \pi_1')$ where $\pi_0 = \pi_0'$ and

$$\pi_1'^{-1}(j) = \begin{cases} \pi_1^{-1}(j) & \text{if } j \le k \\ \pi_1^{-1}(d) & \text{if } j = k+1 \\ \pi_1^{-1}(j-1) & \text{otherwise} \end{cases} ;$$

   that is the following diagram

$$\pi = \begin{pmatrix} \dots & & & \dots & \alpha_0 \\ \dots & \alpha_0 & \beta & \dots & \alpha_1 \end{pmatrix} \to \begin{pmatrix} \dots & & & \dots & \alpha_0 \\ \dots & \alpha_0 & \alpha_1 & \beta & \dots \end{pmatrix} = \pi'$$

and λ is related to λ' by

$$\lambda'_\alpha = \begin{cases} \lambda_{\alpha_0} - \lambda_{\alpha_1} & \text{if } \alpha = \alpha_0, \\ \lambda_\alpha & \text{otherwise.} \end{cases}$$

2. Assume $\lambda_{\alpha_0} < \lambda_{\alpha_1}$.

   We call this *Rauzy-Veech induction* $\mathcal{R}_1$ *of type 1.*

   Let k be $\pi_0^{-1}(k) = \pi_1^{-1}(d)$ with $k \le d-1$. Then $\mathcal{R}_1(\pi_0, \pi_1) = (\pi'_0, \pi'_1)$ where $\pi_1 = \pi'_1$ and

$$\pi_0'^{-1}(j) = \begin{cases} \pi_0^{-1}(j) & \text{if } j \le k \\ \pi_0^{-1}(d) & \text{if } j = k+1 \\ \pi_0^{-1}(j-1) & \text{otherwise} \end{cases} ;$$

   that is the following diagram

$$\pi = \begin{pmatrix} \dots & \alpha_1 & \beta & \dots & \alpha_0 \\ \dots & & & \dots & \alpha_1 \end{pmatrix} \to \begin{pmatrix} \dots & \alpha_1 & \alpha_0 & \beta & \dots \\ \dots & & & & \alpha_1 \end{pmatrix} = \pi'$$

   and λ is related to λ' by

$$\lambda'_\alpha = \begin{cases} \lambda_{\alpha_1} - \lambda_{\alpha_0} & \text{if } \alpha = \alpha_1, \\ \lambda_\alpha & \text{otherwise.} \end{cases}$$

We refer to the larger subinterval among $\{\alpha_0, \alpha_1\}$ as the *winner*, and the smaller one as the *loser*.

Remark 3.2.1. *The Rauzy-Veech induction excluded the case* $\lambda_{\alpha_0} = \lambda_{\alpha_1}$*, as the resulting induced IET is over* $(d-1)$ *symbols. However, such vectors* λ *form a set of codimension one in* $\mathbb{R}_+^{\mathcal{A}}$*. The Lebesgue measure of such a set is zero.*

Let $V_{\alpha,\beta}$ be the matrix $I + E_{(\alpha,\beta)}$ where $E_{\alpha,\beta}$ is the matrix whose only nonzero entry is at (α, β) where the value is 1.

Remark 3.2.2. *The new lengths* λ' *and* λ *are related by a non-negative transition matrix:*

$$V_{\alpha,\beta}\lambda' = \lambda.$$

where $V_{\alpha,\beta} = V_{\alpha_0,\alpha_1}$ *for type 0 induction and* $V_{\alpha\beta} = V_{\alpha_1,\alpha_0}$ *for type 1 induction.*

Definition 3.2.2. *Let* $T = (\pi, \lambda)$ *be an IET and* ∂I_i *denote the left endpoint of the subinterval* I_i *for* $i \in \{1, , ..., d\}$. *Then* T *satisfies the* ***Keane conditionn*** *if for all* $m \geq 1$

$$T^m(\partial I_i) \neq \partial I_j$$

for all $i, j \in \{1, ..., d\}$, *and* $j \neq 1$. *Here,* T^m *means composition of the function* T *with itself* m *times.*

Proposition 3.2.1. *Let* $T^{(n)}$ *be the* n^{th} *iteration of the induction on an IET* T. *Then the following are equivalent:*

- T *satisfies the Keane condition.*
- $T^{(n)}$ *is defined for all* $n \geq 0$.

Iterating the Rauzy-Veech induction $\mathcal{R}$ n times, we obtain a sequence of transition matrices $\{V_k\}$. We define the nth iteration as $\mathcal{R}^{(n)}(\pi, \lambda) = (\pi^{(n)}, \lambda^{(n)})$ with $(\Pi_{k=1}^{n} V_k)\lambda^{(n)} = \lambda$.

3.3 Rauzy diagram

Definition 3.3.1. *Given a permutation* π, *the* ***Rauzy Class*** *of* π *is the orbit of* $\mathcal{R}_0$ *and* $\mathcal{R}_1$ *moves on* π. *The* ***Rauzy Graph*** *of* π *is the graph with vertices in the Rauzy Class of* π *and directed edges corresponding to the inductive moves.*

Example:

Let's start with the permutation $\pi = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$. Its unlabeled form is $(3,2,1)$. We have the following two other elements in its class:

$$\mathcal{R}_0(\pi) = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}, \qquad \mathcal{R}_1(\pi) = \begin{pmatrix} A & C & B \\ C & B & A \end{pmatrix}$$

whose unlabeled forms are $(2,3,1)$ and $(3,1,2)$ respectively. The (unlabeled) Rauzy diagram for π is given in Figure 3.1 below.

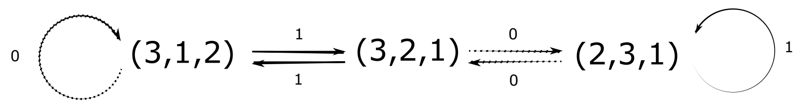

FIGURE 3.1: Rauzy diagram with 3 subintervals

From each vertex, there are two directed outgoing edges labeled 0 and 1 (the type) corresponding to the two combinatorial Rauzy moves.

The definition of a Rauzy Class is dependent on the choice of π, but the next proposition shows that being in the same Rauzy Class is not dependent on the choice of representative.

Proposition 3.3.1. *For any π^1, and π^2 in the Rauzy Class of π, there exists a directed path from π^1 to π^2 in the Rauzy Graph.*

Proof. It suffices to show that for any permutation $\tilde{\pi}$ in the Rauzy Class of π, there exists a directed path from either of its successors $\mathcal{R}_\epsilon(\tilde{\pi})$ back to itself. Then, given any π^1 and π^2, π is a successor of both π^1, and π^2 by definition. In particular, we can inductively find a path from π^1 back to π. Concatenate that with a path from π to π^2, we'll have the desired path from π^1 to π^2.

From the definition of Rauzy-Veech induction, there exists $n > 0$ such that $\mathcal{R}_\epsilon^n(\tilde{\pi}) = \tilde{\pi}$. So $n-1$ moves of type ϵ form a path from $\mathcal{R}_\epsilon(\tilde{\pi})$ to $\tilde{\pi}$. $\square$

Thus, Rauzy Classes are equivalence classes under the relation of eventual successorship. Identifying the permutations that share the same unlabeled format on a Rauzy Graph results in a *Reduced Rauzy Graph* or *Unlabeled Rauzy Diagram.* A summary of Rauzy Classes for $d \leq 5$ are listed in Table 1.

d	representative	# vertices (full class)	# vertices (reduced)
2	(2,1)	1	1
3	(3,2,1)	3	3
4	(4,3,2,1)	7	7
4	(4,2,3,1)	12	6
5	(5,4,3,2,1)	15	15
5	(5,3,2,4,1)		11
5	(5,4,2,3,1)		35
5	(5,2,3,4,1)		10

TABLE 3.1: Rauzy Classes with $d \leq 5$

3.4 Suspension data

We will follow Viana in [Vi] and present a construction of suspended surfaces over an IET.

Fix an IET, $T = (\pi, \lambda)$. Let

$$\mathcal{T}_\pi := \left\{ \tau \in \mathbb{R}^{\mathcal{A}} : \sum_{\alpha:\pi_0(\alpha)\leq k} \tau_\alpha > 0, \sum_{\alpha:\pi_1(\alpha)\leq k} \tau_\alpha < 0, \text{for all } 1 \leq k < d \right\} \tag{3.2}$$

Define vectors $\zeta_\alpha = (\lambda_\alpha, \tau_\alpha) \in \mathbb{R}^2$ for each $\alpha \in \mathcal{A}$. Consider the closed curve

$\Gamma = \Gamma(\pi, \lambda, \tau)$ in $\mathbb{R}^2$ formed by concatenation of

$$\zeta_{\pi_0^{-1}(1)}, \zeta_{\pi_0^{-1}(2)}, ..., \zeta_{\pi_0^{-1}(d)}, -\zeta_{\pi_1^{-1}(d)}, -\zeta_{\pi_1^{-1}(d-1)}, ..., -\zeta_{\pi_1^{-1}(1)}$$

starting at the origin.

Since π_ϵ, $\epsilon \in \{0, 1\}$, are bijections from $\mathcal{A} \to \{1, ..., d\}$, for each $i \in \{1, ..., d\}$, there exists a $j \in \{1, ..., d\}$ such that $\pi_0^{-1}(i) = \pi_1^{-1}(j)$. That means, for each $i \in \{1, ..., d\}$, there exists a $j \in \{1, ..., d\}$ such that $\zeta_{\pi_0^{-1}(i)} = \zeta_{\pi_1^{-1}(j)}$. The concatenation consists of pairs of parallel and equal length segments. It is indeed a closed curse in $\mathbb{R}^2$.

Restricting $\tau \in \mathcal{T}_\pi$ and $\lambda \in \mathbb{R}_+^{\mathcal{A}}$, we get that the endpoints of all $\zeta_{\pi_0^{-1}(1)} + \zeta_{\pi_0^{-1}(2)} + ... + \zeta_{\pi_0^{-1}(k)}$ are in the upper half plane; and the endpoints of all $\zeta_{\pi_1^{-1}(1)} + \zeta_{\pi_1^{-1}(2)} + ... + \zeta_{\pi_1^{-1}(k)}$ are in the lower half plane, for every $1 \le k \le d - 1$.

Assuming that the closed curve is simple, it defines a polygon with $2d$ sides which consists of pairs of parallel and equal sides. Let $S = S(\pi, \lambda, \tau)$ be the translation surface obtained by identifying the sides in each of the pairs.

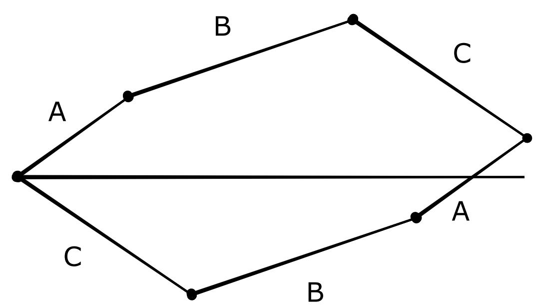

FIGURE 3.2: Suspension surface over $\pi = (3, 2, 1)$.

Figure 3.2 is an example of a suspension surface over $\pi = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$, whose unlabeled form is $(3, 2, 1)$. For simplicity, we have labeled ζ_α by α.

By definition, the leftmost point is $(0, 0)$. Define $I = [0, |\lambda|) \times \{0\}$. The interval exchange transformation $T = (\pi, \lambda)$ is realized as the first return map to I of the vertical flow in S. See Figure 3.3 below.

Each of the identifications on these surfaces is a translation. Therefore, the

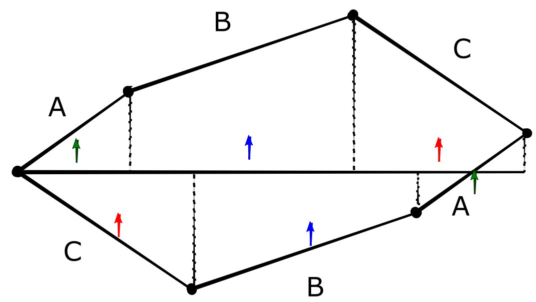

FIGURE 3.3: Vertical flow on suspension surface.

standard form dz descends to a holomorphic 1-form on the surface with zeroes, if any, at the vertex equivalent classes. Each vertex class is called a singularity of degree k, where k is the degree of the corresponding zero of the differential and the total angle around the singularity is $2\pi(k+1)$.

The Rauzy-Veech induction can be extended to the translation surfaces. Let $\epsilon \in \{0,1\}$ be such that $\lambda_{\alpha_\epsilon} > \lambda_{\alpha_{1-\epsilon}}$. We define a new surface $S' = (\pi', \lambda', \tau')$ where (π', λ') is defined as in Section §3.2 and τ' is defined as

$$\tau'_\alpha = \begin{cases} \tau_\alpha, & \text{if } \alpha \neq \alpha_\epsilon \\ \tau_\alpha - \tau_{\alpha_{1-\epsilon}} & \text{if } \alpha = \alpha_\epsilon \end{cases}$$

That is

$$V^{-1}(\tau) = \tau' \tag{3.3}$$

We see that V^{-1} acts on $\mathcal{T}_\pi$. In particular

$$V^{-1}(\mathcal{T}_\pi) \subset \mathcal{T}_{\pi'} \tag{3.4}$$

This procedure is "cutting and pasting" by translation from $S = (\pi, \lambda, \tau)$ to $S' = (\pi', \lambda', \tau')$. The two translation surfaces S and S' are isometric, i.e. they define the same surface in the moduli space.

Remark 3.4.1. *$S = (\pi, \lambda, -\tau)$ defines a suspension surface in which the IET (π, λ) is realized as the first return map to I of the vertical flow in S in the negative direction.*

Proposition 3.4.1. *The number and degrees of singularities, consequently the genus, are constant over a Rauzy Class.*

This follows from counting before and after each inductive move to verify that the number and degrees of singularities do not change. A summary of translation surfaces for $d \leq 5$ are listed in Table 2.

d	representative	# vertices	angles	orders	genus	χ
2	(2,1)	1	2π	0	1	0
3	(3,2,1)	3	2π, 2π	0,0	1	0
4	(4,3,2,1)	7	6π	2	2	-2
4	(4,2,3,1)	6	2π, 2π, 2π	0,0,0	1	0
5	(5,4,3,2,1)	15	4π, 4π	1,1	2	-2
5	(5,3,2,4,1)	11	6π, 2π	2, 0	2	-2
5	(5,4,2,3,1)	35	6π, 2π	2, 0	2	-2
5	(5,2,3,4,1)	10	2π, 2π, 2π, 2π	0,0,0,0	1	0

TABLE 3.2: Suspension surfaces for $d \leq 5$.

Here χ is the Euler characteristic of the surface S, $\chi(S) = 2 - 2g(S)$.

Starting from $d = 5$, distinct Rauzy classes may give rise to translation surfaces of the same number and orders of singularities.

It is possible (however, rare) that $S(\pi, \lambda, \tau)$ has a polygon representation that is non-simple, as in Figure 3.4. In this case, we want to extend the definition of the suspension surface. Consider the polygon obtained by removing the self-intersections as described in Figure 3.4. We obtain a translation surface by identifying sides of the new polygon in the same way as before. The horizontal segment I is still a cross-section to the vertical flow on the translation surface. The first return map of the vertical flow still coincides with $T = (\pi, \tau)$. The examples presented in Section §5 are not in this situation.

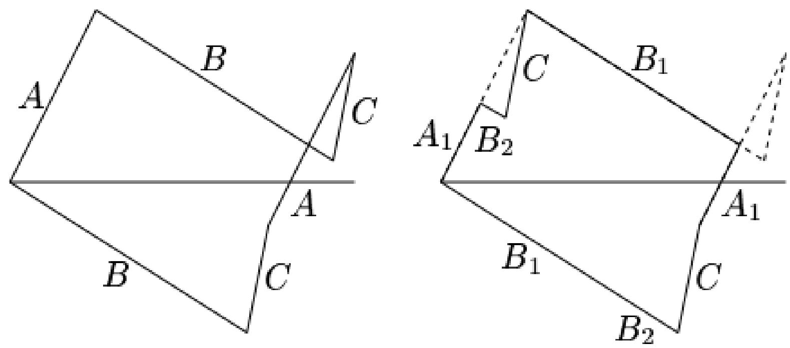

FIGURE 3.4: Self-intersecting suspension surface.

Remark 3.4.2. *Under Rauzy-Veech induction on a translation surface* $S(\pi, \lambda, \tau)$*, whose polygon representation is non-simple, there exists* $n \in \mathbb{N}$ *such that* $S(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$ *is not self-intersecting (see [Vi], remark 18.3).*

3.5 Review of Perron-Frobenius theorem

Before giving the Veech's construction of pseudo-Anosov maps, we review some properties of matrices in $\mathbb{R}^{n\times n}$, leading to a well-known theorem of Perron-Frobenius. The construction relies on key properties of primitive matrices, which are stated and proved in Theorem 3.5.1 below. Statements and proofs in this subsection mainly follow [HJ].

Definition 3.5.1. *An matrix* $A \in \mathbb{R}^{n\times n}$ *is called* ***primitive*** *if there exists a positive integer* k *such that* $(A^k)_{ij} > 0$ *for all* $1 \leq i, j \leq n$*. That is, all entries of the* k^{th} *power of* A *are positive.*

Lemma 3.5.1. *If* v *is a non-negative eigenvector of a positive matrix* A *(that is, all entries of* A *are positive), then* v *is necessarily strictly positive.*

Proof. Since v is an eigenvector (corresponding to an eigenvalue λ), it is non-zero. Suppose $v_i > 0$. We have

$$(\lambda v)_i = (Av)_i \geq A_{ii}v_i > 0, \qquad \text{since } A \text{ is positive}$$

Thus, $\lambda > 0$. Furthermore,

$$(\lambda v)_j = (Av)_j \geq A_{ji}v_i > 0, \qquad \text{for } j \neq i$$

We conclude that $v_j > 0$. Since j is arbitrary, v is necessarily strictly positive. We can extend this lemma to a primitive matrix as well. □

For two vectors v and w, we adopt the notation $v > w$ if $v_i > w_i$ for all $1 \leq i \leq n$. We'll need the following lemma in the proof of Theorem 3.5.1.

Lemma 3.5.2. *If A is a positive matrix and v and w are non-equal vectors such that $v \geq w$ then $Av > Aw$.*

Proof. Consider the vector $v - w$. It is a non-zero vector whose entries are non-negative. We have

$$(A(v-w))_i = \sum_j A_{ij}(v-w)_j > 0$$

since $A_{ij} > 0$ and at least one of $(v-w)_j$ is positive. Thus, $Av > Aw$. □

Definition 3.5.2. *Let A be an $n \times n$ matrix. The **spectral radius** of A is defined as follows.*

$$\rho(A) = \max_j \{|\lambda_j|, \lambda_j \text{ is an eigenvalue of } A\}$$

Next, we'll state a well-known result known as Gelfand's formula, see [HJ] Corollary 5.6.14.

Proposition 3.5.1. *Let A be an $n \times n$ matrix and $\rho(A)$ be its spectral radius. For any matrix norm $\|\cdot\|$, we have*

$$\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$$

Corollary 3.5.1. *If A and B are non-negative $n \times n$ matrices such that $A_{ij} \le B_{ij}$ for all $1 \le i, j \le n$ then $\rho(A) < \rho(B)$.*

Corollary 3.5.1 follows from proposition 3.5.1 using the matrix norm where $||A||_\infty = \max\{|a_{ij}|, \text{ for } 1 \le i, j \le n\}$.

Corollary 3.5.2. *If A is a non-negative $n \times n$ matrix then for any positive integer m, we have $\rho(A^m) \le (\rho(A))^m$.*

Corollary 3.5.2 follows from Proposition 3.5.1 using any submultiplicative matrix norm, e.g. $|||A||| = n||A||_\infty$. See below from [HJ], see the section entitled *Matrix Norm.*

$$
\begin{aligned}
|||AB||| &= n \max_{1 \le i,j \le n} |\sum_{k=1}^{n} a_{ik} b_{kj}| \\
&\le n \max_{1 \le i,j \le n} \sum_{k=1}^{n} |a_{ik} b_{kj}| \\
&\le n \max_{1 \le i,j \le n} \sum_{k=1}^{n} ||A||_\infty ||B||_\infty \\
&= n||A||_\infty n||B||_\infty \\
&= |||A||| \cdot |||B|||
\end{aligned}
\tag{3.5}
$$

Thus,

$$
\rho(A^m) = \lim_{k \to \infty} |||(A^m)^k|||^{1/k} = \lim_{k \to \infty} |||(A^k)^m|||^{1/k} \le \lim_{k \to \infty} (|||A^k|||^m)^{1/k} = (\rho(A))^m \tag{3.6}
$$

Theorem 3.5.1 (Perron-Frobenius, [HJ] simplified). *Suppose that $A \in \mathbb{R}^{n \times n}$ is primitive, then:*

- *There exists an eigenvalue $\lambda_{PF} > 0$ which has a positive corresponding eigenvector.*
- *For any other eigenvalue λ, we have $|\lambda| < \lambda_{PF}$.*
- *λ_{PF} is a simple eigenvalue.*

Proof. Suppose that A is a primitive matrix $n \times n$ matrix, and that A^k is a positive matrix.

We first show the second statement.

Case $k = 1$:

Without loss of generality, we may assume that the spectral radius $\rho(A) = 1$ (if not, consider $A/\rho(A)$). Thus, all eigenvalues lie in the closed unit disk. Also, there exists an eigenvalue λ_{PF} with $|\lambda_{PF}| = \rho(A) = 1$. We'll show that $\lambda_{PF} = 1$ by contradiction. Assume $\lambda_{PF} \neq 1$. There exists a positive integer m such that the real part of λ_{PF}^m is negative. Note that A^m is also a positive matrix. Let ϵ be a small positive number such that $\epsilon < \min\{A_{ii}, 1 \leq i \leq n\}$. Consider the positive matrix $B = A^m - \epsilon I$. If $Av = \lambda_{PF} v$, we have

$$\begin{aligned} B(x) &= A^m v - \epsilon v \\ &= \lambda_{PF}^m v - \epsilon v \\ &= (\lambda_{PF}^m - \epsilon) v \end{aligned} \tag{3.7}$$

By the choice of m, the eigenvalue $\lambda_{PF}^m - \epsilon$ of B lies outside of the unit circle. We get $\rho(B) > 1$. However, all entries of B are positive and not larger than entries of A. By Corollary 3.5.1 and Corollary 3.5.2, we have $\rho(B) \leq \rho(A^m) \leq (\rho(A))^m = 1$. Contradiction. We must have $\lambda_{PF} = 1$.

Case $k > 1$:

We can repeat a similar argument for A^k. We have established the second statement.

To show the first statement, let v be an eigenvector corresponding to λ_{PF}.

Entries of v may not necessarily be real. Consider the vector w where $w_i = |v_i|$.

$$\begin{aligned}(Aw)_i &= \sum_{j=1}^{n} A_{ij}w_j \\ &= \sum_{j=1}^{n} A_{ij}|v_j| \\ &\geq |\sum_{j=1}^{n} A_{ij}v_j| \\ &= |\lambda_{PF}v_i| \\ &= \lambda_{PF}w_i\end{aligned} \tag{3.8}$$

We have $Aw \geq \lambda_{PF}w$. If $Aw = \lambda_{PF}w$, then w is a non-negative eigenvector. By Lemma 3.5.1, v is strictly positive. We'll show that there is a contradiction otherwise. Suppose otherwise, using Lemma 3.5.2, we have $A(Aw) > \lambda_{PF}Aw$. In particular, there exists a positive ϵ such that for all $1 \leq i \leq n$:

$$A(Aw) \geq \lambda_{PF}(1+\epsilon)Aw$$

So,

$$\begin{aligned}A^{n+1}w &= A^{n-1}A^2w \\ &\geq \lambda_{PF}(1+\epsilon)A^{n-1}Aw \\ &\geq \dots \\ &\geq \lambda_{PF}^{n}(1+\epsilon)^nAw\end{aligned} \tag{3.9}$$

Using Gelfand's formula in Proposition 3.5.1 and the fact that $\lambda_{PF} = \rho(A)$ from above, we get

$$\rho(A) \geq \rho(A)(1+\epsilon), \qquad \text{contradiction.}$$

Lastly, given a positive eigenvector v corresponding to an eigenvalue λ, suppose λ has a second linearly independent eigenvector w. Assuming w has a positive entry (otherwise multiply by -1), we consider $u = v - \alpha w$. Take the maximum α such that u is non-negative. Then at least one component of u is 0, otherwise α is not maximal.

However, since u is also an eigenvector, by Lemma 3.5.1, u must be strictly positive. Contradiction. We conclude that the eigenspace of the Perron-Frobenius eigenvalue has dimension 1.

□

3.6 Veech's construction of pseudo-Anosov maps

First, we follow Viana [Vi] to discuss the symplecticity of the transition matrix V, a key ingredient in the construction. From the definition of Ω_π in equation (3.1), we can see that it is skew-symmetric:

$$\Omega_\pi^* = -\Omega_\pi$$

where Ω_π^* is the adjoint operator, relative to the usual inner product on $\mathcal{R}^{\mathcal{A}}$. Let $\mathbb{H}_\pi = \Omega_\pi(\mathbb{R}^{\mathcal{A}})$, define:

$$\omega_\pi : \mathbb{H}_\pi \times \mathbb{H}_\pi \to \mathbb{R}, \qquad \omega_\pi(\Omega_\pi(u), \Omega_\pi(v)) = u \cdot \Omega_\pi(v)$$

Proposition 3.6.1. *The above defines a symplectic form. In other words, ω_π is a non-degenerate, alternating bilinear form.*

Proof. Since $\Omega_\pi^* = -\Omega_\pi$, the orthogonal complement $\mathbb{H}_\pi^\perp$ coincides with $\ker(\Omega_\pi)$. Suppose $\Omega_\pi(u) = \Omega_\pi(u')$, then $u - u' \in \ker(\Omega_\pi)$. Thus,

$$u \cdot \Omega_\pi(v) = u' \cdot \Omega_\pi(v)$$

for every $v \in \mathbb{R}^{\mathcal{A}}$. We have that ω_π is well-defined. It is clear that ω_π is bilinear. And,

$$\omega_\pi(\Omega_\pi(u), \Omega_\pi(v)) = u \cdot \Omega_\pi(v) = v \cdot \Omega_\pi^*(u) = -\omega_\pi(\Omega_\pi(v), \Omega_\pi(u))$$

so it is alternate. Finally,

$$\omega_\pi(\Omega_\pi(u), \Omega_\pi(v)) = 0 \quad \forall v \quad \leftrightarrow \quad u \cdot \Omega_\pi(v) = 0 \quad \forall v \quad \leftrightarrow \quad u \in \mathbb{H}_\pi^\perp$$

□

Proposition 3.6.2. *If* $(\pi', \lambda') = \mathcal{R}(\pi, \lambda)$ *and* V *is the transition matrix under Rauzy-Veech induction then* $V^*\Omega_\pi V = \Omega_{\pi'}$*. In particular, the operator* V^* *induces a symplectic isomorphism from* $\mathbb{H}_\pi$ *onto* $\mathbb{H}_{\pi'}$*.*

Proof. We have already seen that $V(\lambda') = \lambda$. Let $\omega = \Omega_\pi(\lambda)$ and $\omega' = \Omega_\pi(\lambda')$. It's not hard to see that $V^*(\omega) = \omega'$. Then

$$\Omega_{\pi'}(\lambda') = \omega' = V^*(\omega) = V^*\Omega_\pi(\lambda) = V^*\Omega_\pi V(\lambda')$$

Moreover, $V^* : \mathbb{H}_\pi \to \mathbb{H}_{\pi'}$ is symplectic:

$$\begin{aligned} \omega_{\pi'}(V^*\Omega_\pi(u), V^*\Omega_\pi(v)) &= \omega_{\pi'}(\Omega_{\pi'}V^{-1}(u), V^*\Omega_\pi(v)) \\ &= V^{-1}(u) \cdot V^*\Omega_\pi(v) \\ &= u \cdot \Omega_\pi(v) \\ &= \omega_\pi(\Omega_\pi(u), \Omega_\pi(v)) \end{aligned} \tag{3.10}$$

□

Let π be an irreducible permutation and let γ be a closed loop in the Rauzy diagram associated to π. We obtain the matrix V as above; let us assume that V is primitive and let $\theta > 1$ be its Perron-Frobenius eigenvalue. We choose a positive eigenvector λ for θ. Now, V is appropriately symplectic, allowing one to choose τ an eigenvector for the eigenvalue θ^{-1} with $\tau_{\pi_0^{-1}(1)} > 0$.

We show that (λ, τ) is a suspension data for π below, on page 28. Thus, with a minor abuse of notation,

$$\begin{aligned} S(\pi', \lambda', \tau') &= S(\pi', V^{-1}(\lambda), V(\tau)) \\ &= S(\pi', \theta^{-1}\lambda, \theta\tau) \\ &= \begin{bmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{bmatrix} \cdot S(\pi, \lambda, \tau) \\ &= g_t \cdot S(\pi, \lambda, \tau) \end{aligned} \tag{3.11}$$

where $t = \log(\theta) > 0$.

The two surfaces $S(\pi, \lambda, \tau)$ and $g_t \cdot S(\pi, \lambda, \tau)$ differ by some element of the mapping class group. In other words there exists a homeomorphism ϕ, with respect to the translation surface $S(\pi, \lambda, \tau)$, such that $D\phi = g_t$. Indeed, we can obtain an affine homeomorphism from the construction ("cutting and pasting"). The fact that this affine homeomorphism is a pseudo-Anosov follows from section §4.2 (The trace of $D\phi$ is greater than 2). In particular the dilatation of ϕ is θ. Note that by construction ϕ fixes the zero on the left of the interval I and also the separatrix adjacent to this zero (namely the interval I). Veech proved Theorem 3.6.1 in section 8 of [V] using zippered rectangles. Following [L], an alternative version of Veech's theorem is stated below.

Theorem 3.6.1 (Veech, [V] reduced)**.** *Let γ be a closed loop, beginning at the vertex corresponding to π, in a labeled Rauzy diagram and V be the associated transition matrix. If V is primitive, then let λ be a positive eigenvector for the Perron eigenvalue θ of V and τ be an eigenvector (with $\tau_{\pi_0^{-1}(1)} > 0$) for the eigenvalue θ^{-1} of V. We have*
(1) ζ with $\zeta_\alpha = (\lambda_\alpha, \tau_\alpha)$ is a suspension datum for $T = (\pi, \lambda)$;
(2) The matrix $A = \begin{pmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{pmatrix}$ is the derivative map of a pseudo-Anosov homeomorphism ϕ on $X(\pi, \zeta)$;
(3) The dilatation of ϕ is θ;
(4) Up to conjugation, all orientable pseudo-Anosov homeomorphisms fixing a separatrix are obtained by this construction.

Veech's theorem was originally stated for loops in a unlabeled Rauzy diagram, whose lift to the labeled Rauzy diagram may not necessarily be a closed loop. In such cases, we have to multiply the transition matrix with a an appropriate permutation matrix. In this paper, we consider only hyperelliptic Rauzy diagram, see definition 3.7.1 below. The labeled and unlabeled Rauzy diagram of a hyperelliptic IET are exactly the same. We note that switching types 0 and 1 throughout γ results in the inverse pseudo-Anosov homeomorphism.

Sketch of proof of the first statement of Theorem 3.6.1. We only need to show that τ is an element of $\mathcal{T}_\pi$ (defined in equation (3.2)). From equation (3.3) and (3.4), the matrix V^{-1} sends $\mathcal{T}_\pi$ to itself. Also, λ is the largest eigenvalue of V^{-1}. It is simple due to the fact that V is symplectic and primitive. Indeed, the largest eigenvalue of V is simple (from Perron-Frobenius), so is its smallest (due to symplecticity). We then apply Von Mises' iteration. Take any vector $\tau_0 \in \mathcal{T}_\pi$. The sequence

$$\tau_n = \frac{A\tau_{n-1}}{||A\tau_{n-1}||}, \qquad \text{for } n \geq 1$$

converges to a multiple of τ, say $\tau' = c\tau$ (see lemma 3.6.1 below). Since $\mathcal{T}_\pi$ is invariant under the action of V^{-1}, we have $\tau' \in \mathcal{T}_\pi$. Thus $\tau'_{\pi_0^{-1}(1)} > 0$ from the definition of $\mathcal{T}_\pi$. We chose τ with $\tau_{\pi_0^{-1}(1)} > 0$. Therefore $c > 0$, and $\tau \in \mathcal{T}_\pi$.

Lemma 3.6.1 (Von Mises [MP])**.** Known as Von Mises' iteration

Let A be an $n\times n$ matrix such that its largest eigenvalue $\lambda > 0$ is simple, and λ is strictly larger than all other eigenvalues in absolute value. Let v_0 be the $n\times 1$ column vector. Then the sequence

$$v_k = \frac{Av_{k-1}}{||Av_{k-1}||}, \qquad \textit{for } n \geq 1$$

converges to an eigenvector corresponding to λ. The choice of v_0 is arbitrary except on a set of measure 0.

Proof. Consider the Jordan canonical form PJP^{-1} of A, where $J_{11} = \lambda$. Choose v_0 such that $P^{-1}v_0 = c_1e_1 + c_2e_2 + \cdots + c_ne_n$ where $c_1 \neq 0$ and e_i, $i \in \{1, ..., n\}$, are the standard basis vectors. Since P^{-1} is invertible, only a measure 0 set of v_0 gives $(P^{-1}v_0)_1 = 0$.

$$
\begin{aligned}
v_k &= \frac{PJ^kP^{-1}v_0}{\|PJ^kP^{-1}v_0\|} \\
&= \frac{PJ^k(c_1e_1 + c_2e_2 + \cdots + c_ne_n)}{\|PJ^k(c_1e_1 + c_2e_2 + \cdots + c_ne_n)\|} \\
&= \frac{PJ^k(c_1e_1) + PJ^k(c_2e_2 + \cdots + c_ne_n)}{\|PJ^k(c_1e_1) + PJ^k(c_2e_2 + \cdots + c_ne_n)\|} \\
&= \frac{P\lambda^k(c_1e_1) + PJ^k(c_2e_2 + \cdots + c_ne_n)}{\|P\lambda^k(c_1e_1) + PJ^k(c_2e_2 + \cdots + c_ne_n)\|} \\
&= \frac{\lambda^k c_1}{\|\lambda^k c_1\|} \cdot \frac{Pe_1 + \frac{1}{\lambda^k c_1}PJ^k(c_2e_2 + \cdots + c_ne_n)}{\|Pe_1 + \frac{1}{\lambda^k c_1}PJ^k(c_2e_2 + \cdots + c_ne_n)\|}
\end{aligned}
\tag{3.12}
$$

Since λ is the dominant eigenvalue, we have $(\frac{1}{\lambda}J)^k(c_2e_2 + \cdots + c_ne_n) \to 0$, the zero vector. Thus $v_k \to v = \frac{Pe_1}{\|Pe_1\|}$ (independent of choice of v_0) and

$$
Av = PJP^{-1}v = PJP^{-1}\frac{Pe_1}{\|Pe_1\|} = \lambda\frac{Pe_1}{\|Pe_1\|} = \lambda v
$$

□

3.7 Hyperelliptic Rauzy diagram

Our new families of examples of pseudo-Anosov maps with vanishing SAF-invariant are constructed using hyperelliptic diagrams.

Definition 3.7.1. *An interval exchange transformation T is called* ***hyperelliptic*** *if the corresponding permutation is such that $\pi_1 \circ \pi_0^{-1}(i) = d + 1 - i, \forall i = 1, ..., d.$*

A particular example of such a combinatorial datum is $\varepsilon_d := \begin{pmatrix} 1 & 2 & \cdots & d \\ d & d-1 & \cdots & 1 \end{pmatrix}$, with corresponding unlabeled form $(d, d-1, \ldots, 1)$.

Definition 3.7.2. *A* ***hyperelliptic Rauzy diagram*** *is one that contains a combinatorial datum π of a hyperelliptic IET.*

Exactly when a Rauzy diagram is hyperelliptic, the labeled and unlabeled diagrams are isomorphic directed graphs. See Figure 3.5 for the unlabeled Rauzy diagram with four subintervals.

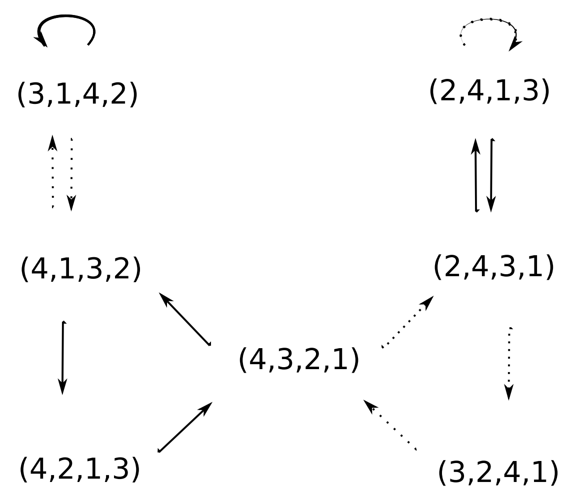

FIGURE 3.5: Unlabeled hyperelliptic Rauzy diagram with 4 subintervals.

Here and throughout, type 1 moves are shown by dotted lines, type 0 by solid.

In our examples, we always choose the "central" vertex of the hyperelliptic diagram at hand to be the initial vertex of our path. (The resulting pseudo-Anosov map is a conjugate homeomorphism of that given from taking any other initial vertex along the path.) The following justifies that normalization, confer Figures 3.5, 5.1, 5.3.

Lemma 3.7.1. *Suppose that γ is a closed path in a hyperelliptic Rauzy diagram such that the corresponding transition matrix $V = V(\gamma)$ is primitive. Then γ must pass through the vertex corresponding to ε_d.*

Proof. First, we show that when V is primitive, every letter must win at least once. By contradiction, suppose that letter α is never a winner on the path γ. That means, in each of the transition matrices V_k (defined in §3.2, where $V = \prod V_k$), row α has only one non-zero entry at (α, α) of value 1. Thus, row α of V also has only one non-zero entry at (α, α). So does V^k for any positive k, contradicting V being primitive. Thus each letter of $\mathcal{A}$ must be a winner at least once.

In the hyperelliptic diagram of ϵ_d, there is exactly one cycle in which d is a winner, and exactly one cycle in which 1 is a winner. The cycles are of type 1 and type 0 respectively. These two cycle share one common vertex, that is ϵ_d. The omission of which disconnects the Rauzy Graph. Thus any path that has both d and 1 as a winner will have to go through ϵ_d. □

3.8 Component of strata and Rauzy classes

Let $g \geq 2$ be the genus of the Riemann surface X, the non-zero abelian differentials on X have zeros whose multiplicities sum to $2g-2$, as guaranteed by the Riemann-Roch theorem. Let κ be a partition of $2g-2$, the *stratum* $\mathcal{H}(\kappa)$ is the set (modulo the action of the mapping class group) of abelian differentials whose zeros have the multiplicities of κ. For each partition κ, the stratum $\mathcal{H}(\kappa)$ is realizable [CS].

Computations by Veech and then Arnoux using Rauzy classes showed that general strata have more than one connected component. Kontsevich and Zorich [KZ] determined all possible components. They showed that any stratum has at most three components: there may be a hyperelliptic component where both X is hyperelliptic and the hyperelliptic involution preserves ω; and possibly two more components, differentiated by the parity of an appropriate notion of spin, these components are thus called "even" and "odd", correspondingly. One denotes the various components by $\mathcal{H}^{\text{hyp}}(\kappa)$, $\mathcal{H}^{\text{even}}(\kappa)$ and $\mathcal{H}^{\text{odd}}(\kappa)$.

Our examples are in low genus, thus we recall only (part of) the second theorem of [KZ]: Each of $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$ is connected (and coincides with its hyperelliptic component), while each of $\mathcal{H}(4)$ and $\mathcal{H}(2,2)$ has two connected components: a hyperelliptic component, and an odd spin component.

Each Rauzy class corresponds to a single component (see [B] for details on this correspondence), and indeed one finds that the number of intervals d is equal to $2g + \sigma - 1$, where σ equals the total number of zeros of the corresponding abelian differentials. This accords with the fact that local coordinates on $\mathcal{H}(\kappa)$ are given by period coordinates, which one can view as the integration of ω over a basis of relative homology $H_1(X, \Sigma, \mathbb{C})$, where Σ is the set of zeros of ω. One can take the basis to be the union of an integral symplectic basis of absolute homology with a set of paths from a chosen zero to each of the other zeros.

The transition matrix $V = V(\gamma)$ for a closed path gives the action of the element of the mapping class group on relative homology. In the pseudo-Anosov case, there is some power of the map that fixes all of Σ and hence this power changes any path connecting zeros by an element of absolute homology. On absolute homology, the pseudo-Anosov (and perforce any of its powers) acts integrally symplectically, thus the action on relative homology of the power decomposes naturally into a block form with the block corresponding to pure relative homology being an identity. Thus, the characteristic polynomial of this action is the product of a degree $2g$ reciprocal polynomial times a power of $(x - 1)$. Therefore, the action of the original pseudo-Anosov has a similar decomposition, as seen in our examples below.

4 Characterization of vanishing SAF invariant, and its implications

We aim to prove that a pseudo-Anosov map has vanishing SAF-invariant exactly when a certain algebraic condition holds; we thus naturally first gather some algebraic results.

4.1 Galois theory

We begin with a result using elementary Galois theory.

Definition 4.1.1. *A polynomial* $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ *is called* ***reciprocal*** *when* $a_i = a_{n-i}$ *for all* $1 \le i \le n$.

Lemma 4.1.1. *Suppose that* $p(x) \in \mathbb{Z}[x]$ *is reciprocal and of odd degree (greater than one). Then* $p(x)$ *is reducible.*

Proof. Suppose $p(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + ... + a^{2n}x + a_{2n+1}$. Then:

$$\begin{aligned} p(x) &= a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + ... + a^{2n}x + a_{2n+1} \\ &= a_{2n+1}(x^{2n+1} + 1) + a_{2n}(x^{2n} + x) + ... + a_{n+1}(x^{n+1} + x^n) \end{aligned} \tag{4.1}$$

Obviously $p(-1) = 0$. Since the degree of $p(x)$ is at least 3, we conclude that $p(x)$ is reducible.

□

Proposition 4.1.1. *Suppose that* α *is a non-zero (irrational) algebraic number. The minimal polynomial of* α *over* $\mathbb{Q}$ *is reciprocal if and only if* $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\alpha + \alpha^{-1})$.

Proof. Let $p(x) \in \mathbb{Q}[x]$ be the minimal polynomial of α. Let $q(x) \in \mathbb{Q}[x]$ be the minimal polynomial of $\alpha + \alpha^{-1}$. Denote the degree of $q(x)$ by n. Since α satisfies $x^2 - (\alpha + \alpha^{-1})x + 1$ and of course $\mathbb{Q}(\alpha) \supseteq \mathbb{Q}(\alpha + \alpha^{-1})$, the degree of $p(x)$ is either n or $2n$.

($\Leftarrow$) Set $\tilde{q}(x) = x^n q(x + x^{-1})$. Then $\tilde{q}(x) \in \mathbb{Q}[x]$ is monic of degree $2n$. Of course, $\tilde{q}(x)$ has α as a root. Therefore, $p(x)$ divides $\tilde{q}(x)$, and by the restrictions on the degree of $p(x)$, either $p(x) = \tilde{q}(x)$ or else $p(x)$ has degree n. If $p(x)$ is not reciprocal, then it cannot equal $\tilde{q}(x)$, as this latter is clearly reciprocal; it then follows that $n = [\mathbb{Q}(\alpha) : \mathbb{Q}] = [\mathbb{Q}(\alpha + \alpha^{-1}) : \mathbb{Q}]$, and thus $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha + \alpha^{-1})$.

($\Rightarrow$) Suppose now that $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha + \alpha^{-1})$. Any root of $q(x)$ is the image of $\alpha + \alpha^{-1}$ under some field embedding (fixing $\mathbb{Q}$), $\mathbb{Q}(\alpha+\alpha^{-1}) \hookrightarrow \mathbb{C}$. Since $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha+\alpha^{-1})$, each such field embedding sends α to some root of $p(x)$. This field equality also implies that $\deg p(x) = n$, and thus we conclude that the roots of $q(x)$ are of the form $\beta + \beta^{-1}$ with β a root of $p(x)$. However, under the further supposition that $p(x)$ is reciprocal (which implies that n is even, see Lemma 4.1.1), there are only $n/2$ *distinct* values in the set of the $\beta + \beta^{-1}$. Hence, the degree of $q(x)$ must in fact be at most $n/2$, and we have reached a contradiction.

□

We draw some immediate conclusions from Proposition 4.1.1. Let us introduce a non-standard definition: Call $\mathbb{Q}(\alpha + \alpha^{-1})$ the trace field of the algebraic number α. The (algebraic) norm of an algebraic number is the product of all of its conjugates over $\mathbb{Q}$.

Corollary 4.1.1. *If α is of norm one with quadratic trace field, then*

$$\mathbb{Q}(\alpha) \neq \mathbb{Q}(\alpha + \alpha^{-1})$$

.

Proof. Field equality would imply that α is quadratic, and hence with minimal polynomial of the form $p(x) = x^2 + nx + 1$ for some $n \in \mathbb{Z}$. But $p(x)$ is reciprocal of even degree and by Proposition 4.1.1 field equality cannot hold.

□

Definition 4.1.2. *A* ***Pisot*** *number (or a Pisot-Vijayaraghvan number) is a real irrational algebraic number larger than 1 such that all of its other Galois conjugates lie inside the unit disk.*

Corollary 4.1.2. *If α is a non-quadratic Pisot number, then the minimal polynomial of α is non-reciprocal. Moreover, $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha + \alpha^{-1})$.*

Proof. Since the minimal polynomial $p(x)$ of α has degree greater than two, it has another root $\beta \neq \alpha^{-1}$ with $||\beta|| < 1$. Therefore $||\beta^{-1}|| > 1$. Since $p(x)$ has only α as a root that has norm greater than one, we conclude that $p(x)$ is not a reciprocal polynomial. Thus, we can invoke Proposition 4.1.1 to find that also the second statement holds.

□

Definition 4.1.3. *A* ***Perron*** *number α is a real algebraic number larger than 1 such that all of its other Galois conjugates lie inside the disk $\{z : ||z|| < \alpha\}$.*

Definition 4.1.4. *A* ***bi-Perron*** *number α is a real algebraic number larger than 1 such that all of its other Galois conjugates lie inside the annulus $\{z : \alpha^{-1} < ||z|| < \alpha\}$.*

Motivated by the last result, we now show that every non-quadratic Pisot unit is bi-Perron (which presumably is well-known).

Lemma 4.1.2. *If α is a non-quadratic Pisot unit, then α is bi-Perron.*

Proof. Let $\alpha = \alpha_1, \dots, \alpha_n$ be the roots of the minimal polynomial of α. Then we have that $||\alpha_1 \cdots \alpha_n|| = 1$, $||\alpha_1|| > 1$ and for each $j > 1$, $||\alpha_j|| < 1$. Therefore for each $i > 1$ we have

$$||\alpha_i|| = \frac{1}{||\alpha_1||} \frac{1}{\prod_{j>1, j\neq i} ||\alpha_j||}$$

and thus $||\alpha_i|| > 1/||\alpha_1||$ and the result holds. □

Lemma 4.1.3. *The set of cubic bi-Perron units is exactly the set of cubic Pisot units.*

Proof. Suppose α_1 is a bi-Perron unit with minimal polynomial $x^3+ax^2+bx+c \in \mathbb{Z}[x]$ with $c = \pm 1$, which has non-zero roots $\alpha_1, \alpha_2, \alpha_3$. Then $||\alpha_2\alpha_3|| = 1/||\alpha_1||$, and hence α_2 satisfies $||\alpha_2|| > 1/||\alpha_1||$ if and only if $||\alpha_3|| < 1$ and similarly with the roles of α_2, α_3 exchanged. Thus, its two conjugates have norm greater than α^{-1} if and only if they both lie in the unit disk. Since $-c = \alpha_1\alpha_2\alpha_3$. Thus, we find that every cubic bi-Perron unit is indeed a Pisot unit. Of course, the previous result gives the other inclusion. □

Remark 4.1.1. *On the other hand, not every non-reciprocal Perron unit is a Pisot number. For example: take* $f(x) = x^4 - 4x^3 + 3x + 1$*. Its roots are approximately* $3.7703, 1.1668, -.4685 \pm 0.0883i$*. One can verify that* $\alpha \approx 3.7703$ *is a bi-Perron unit (as* $1/\alpha \approx 0.2652$*) but not a Pisot unit (as it has a conjugate lying outside of the unit disk).*

We verify a property required for a certain construction of pseudo-Anosov elements, see subsection §4.3.

Lemma 4.1.4. *If* α *is a non-reciprocal bi-Perron number, let* $\tilde{q}(x)$ *be the product of the minimal polynomial of* α *with the minimal polynomial of* α^{-1}*. Then* $\tilde{q}(x) = f(x^k)$ *with* $f(x) \in \mathbb{Z}[x]$ *and* $k \in \mathbb{N}$ *implies* $k = 1$*.*

Proof. Suppose $\tilde{q}(x) = f(x^k)$. Since $\tilde{q}(\alpha) = 0$, $f(\alpha^k) = 0$. Also $f((\alpha\zeta_k^n)^k) = 0$ where ζ_k is the principal kth root of unity and $n = 0, ..., k-1$. Thus $\tilde{q}(\alpha\zeta_k^n) = 0$ for $n = 0, ..., k-1$. But α is bi-Perron. When $k > 1$, we have a contradiction to the fact that all of its other Galois conjugate must lie inside the annulus $\{z : \alpha^{-1} < ||z|| < \alpha\}$. So k must be 1. □

Similarly, we have the following.

Lemma 4.1.5. *If* α *is a reciprocal bi-Perron number and* $p(x)$ *its minimal polynomial, then* $p(x) = f(x^k)$ *with* $f(x) \in \mathbb{Z}[x]$ *and* $k \in \mathbb{N}$ *implies* $k = 1$*.*

Proof. Here also, the polynomial in question has α as its only root that is of complex norm $\|\alpha\|$. Thus, the argument used to prove the previous Lemma applies. □

4.2 Trace field, periodic direction field, Veech group

The *Veech group* $\mathrm{SL}(X,\omega) \subset \mathrm{SL}_2(\mathbb{R})$ is the group of matrix parts of (orientation-preserving) affine diffeomorphisms of (X,ω). It is the image of a homomorphism from $\mathrm{Aff}(X,\omega)$ into $\mathrm{SL}_2(\mathbb{R})$, given by the function that takes an affine diffeomorphism f to its derivative Df. The Veech group $\mathrm{SL}(X,\omega)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ [V2]. When (X,ω) has genus $g > 1$, the kernel of the homomorphism is finite. It is not hard to check the the Veech group $\mathrm{SL}(X,\omega)$ is the $\mathrm{SL}_2(\mathbb{R})$-stabilizer of (X,ω). For any matrix $A \in \mathrm{SL}_2(\mathbb{R})$, the Veech group of (X,ω) and $A \cdot (X,\omega)$ are conjugate:

$$\mathrm{SL}(Y,\xi) = A \cdot \mathrm{SL}(X,\omega) \cdot A^{-1}$$

Elements of the Veech group are closely connected with elements of the modular group $\mathrm{Mod}(X)$. An affine diffeomorphism f of (X,ω) is periodic if and only if its matrix part is a parabolic element of $\mathrm{SL}_2(\mathbb{R})$ (i.e. $Tr(Df) = 2$); reducible if and only if its matrix part is elliptic (i.e. $Tr(Df) < 2$); pseudo-Anosov if and only if its matrix part is hyperbolic (i.e. $Tr(Df) > 2$), see [T, V].

The *trace field* of a group $\Gamma \in \mathrm{SL}_2(\mathbb{R})$ is a subfield of $\mathbb{R}$ geneerated by $tr(A)$, with $A \in \Gamma$. The *trace field* of the translation surface (X,ω) is defined to be the trace field of its Veech group.

Theorem 4.2.1 (Kenyon, Smillie [KS])**.** *The trace field of a surface (X,ω) has degree at most its genus, g, over $\mathbb{Q}$. Assume that the affine diffeomorphism group of (X,ω) contains a pseudo-Anosov element f with expansion factor λ. Then the trace field of (X,ω) is $\mathbb{Q}(\lambda + \lambda^{-1})$.*

We define the *holonomy vectors* to be the integral of ω along saddle connections (geodesics joining two singularities). Denote $\Lambda = \Lambda(\omega)$ the subgroup of $\mathbb{R}^2$ generated by the holonomy vectors

$$\Lambda = \int_{H_1(X,\mathbb{Z})} \omega$$

Let $e_1, e_2 \in \Lambda$ be non-parallel vectors in $\mathbb{R}^2$. We define the *holonomy field* k to be the smallest subfield of $\mathbb{R}$ such that every element of Λ can be written as $ae_1 + be_2$ with $a, b \in k$.

Theorem 4.2.2 (Kenyon, Smillie [KS]). *The trace field of* (X,ω) *coincides with* k. *The space* $\Lambda \otimes \mathbb{Q} \subset \mathbb{C}$ *is a 2-dimensional vector space over* k.

Note that Gutkin and Judge, in [GJ], have a different approach to this theorem. These results are also reproved by McMullen in [Mc, Mc3].

Theorem 4.2.3 (Calta, Smillie [CS]). *If a translation surface has at least three directions of vanishing SAF-invariant, then it has infinitely many. If we choose coordinates so that the SAF-zero directions have slope 0,1, and* ∞ *then there exists a field* K *such that all SAF-zero directions are exactly those with slopes in* $K \cup \{\infty\}$. *Moreover,* K *is a number field with deg(*K*)* $< g$.

The field K as described above is called the *periodic direction field.*

Theorem 4.2.4 (Calta, Smillie [CS]). *When a translation surface* (X,ω) *has an affine pseudo-Anosov map then its periodic direction field coincides with its trace field.*

Thus when a translation surface (X,ω) has an affine pseudo-Anosov map with dilatation λ, all three fields: holonomy field, trace field, and periodic direction field coincide, $k = K = \mathbb{Q}(\lambda + \lambda^{-1})$.

4.3 Homological criterion, Margalit-Spallone construction

Margalit and Spallone [MS] give a construction of pseudo-Anosov classes in the Teichmüller modular group. A monic reciprocal polynomial with integral coefficients is called *symplectically irreducible* if it is not the product of reciprocal polynomials of strictly lesser degree.

The *homological criterion* for a monic reciprocal polynomial $q(x)$ of even degree is that all of the following hold.

- $q(x)$ is symplectically irreducible,
- $q(x)$ is not cyclotomic, and
- $q(x)$ is not a polynomial in x^k for any integral $k > 1$.

Margalit-Spallone extend a result of Casson-Bleiler [CB]: For any f representing a class of the modular group of a closed surface X of genus at least two, let $q_f(x)$ be the characteristic polynomial for the action on first integral homology induced by f. If $q_f(x)$ passes the homological criterion, then the class of f is pseudo-Anosov. Furthermore, by considering words in explicit elements of the modular group, for any $q(x)$ passing the homological criterion Margalit-Spallone build a homeomorphism f whose homological action has characteristic polynomial $q(x)$. Hence the class of f (and indeed all of its so-called Torelli group coset) is pseudo-Anosov.

4.4 Proof of Theorem 1.1.1: Characterizing SAF-zero pseudo-Anosov maps

Theorem 1.1.1. *Suppose that ϕ is an orientable pseudo-Anosov map of a closed compact surface, with dilatation λ. Then ϕ has vanishing Sah-Arnoux-Fathi invariant if and only if the minimal polynomial of λ is not reciprocal.*

Proof. Suppose that ϕ is a pseudo-Anosov map with dilatation λ. By the results of Calta-Smillie reviewed in Subsection §4.2, we can assume that ϕ is an affine diffeomorphism on (X,ω) with matrix part being hyperbolic in $k = \mathbb{Q}(\lambda + \lambda^{-1})$ and that ϕ has vanishing SAF-invariant if and only if its stable direction has slope in k.

The fixed points under the Möbius action on $\mathbb{R} \cup \{\infty\}$ of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ hyperbolic in $\mathrm{SL}_2(\mathbb{R})$ are $(a-d) \pm \sqrt{(a+d)^2 - 4}/(2c)$. Due to the projective nature of this action, the inverses of these fixed points are the slopes of the corresponding eigenvectors. Thus, these eigenvectors are of slope in k exactly when $(a+d)^2 - 4$ is a square in k. When M is the matrix part of the affine diffeomorphism ϕ on (X,ω), the eigenvectors of M give the direction of the stable and unstable foliations for ϕ on (X,ω), and the trace of M equals $\lambda + \lambda^{-1}$. Thus, these foliations have directions in the trace field exactly when $(\lambda + \lambda^{-1})^2 - 4$ is a square in k. That is, ϕ has vanishing SAF-invariant if and only if $(\lambda + \lambda^{-1})^2 - 4$ is a square in k.

On the other hand, it is obvious that $\mathbb{Q}(\lambda) \supset k$ and that λ is a zero of $(x - \lambda)(x - \lambda^{-1}) = x^2 - (\lambda + \lambda^{-1})x + 1 \in k[x]$. Hence $\mathbb{Q}(\lambda) = k$ if and only if the discriminant of $x^2 - (\lambda + \lambda^{-1})x + 1$ is a square in k, and otherwise $[\mathbb{Q}(\lambda) : k] = 2$. However, the discriminant of $x^2 - (\lambda + \lambda^{-1})x + 1$ is $(\lambda + \lambda^{-1})^2 - 4$. Thus, we find that ϕ has vanishing SAF-invariant if and only if $\mathbb{Q}(\lambda) = \mathbb{Q}(\lambda + \lambda^{-1})$.

Our result now follows from Proposition 4.1.1. □

Remark 4.4.1. *Since the stable and unstable foliations of the pseudo-Anosov map correspond to the fixed points of the linear part, it follows from the above proof that either both are of vanishing SAF-invariant, or else neither is.*

4.5 Some implications

Recall that if λ is the dilatation of a (orientable) pseudo-Ansov map ϕ, then we call $\mathbb{Q}(\lambda + \lambda^{-1})$ the *trace field* of ϕ.

Corollary 4.5.1. *If an orientable pseudo-Anosov map ϕ has quadratic trace field, then ϕ has non-vanishing* SAF*-invariant.*

Proof. The dilatation of ϕ is a unit, and hence has norm $\mathrm{N}(\alpha) \pm 1$. When $\mathrm{N}(\alpha) = 1$, Corollary 4.1.1 applies. When $\mathrm{N}(\alpha) = -1$, the minimal polynomial (being monic) cannot be reciprocal. □

Remark 4.5.1. *Kenyon-Smillie [KS] showed that if (X,ω) supports an affine pseudo-Anosov map, then the trace field of the map is the trace field of (X,ω). We can thus compare Corollary 4.5.1 with McMullen's Theorem A.1 of the appendix in [Mc]. In our language, McMullen shows that under the hypothesis that the Veech group of (X,ω) is a lattice (which certainly implies the existence of affine pseudo-Anosov maps), the trace field of (X,ω) being quadratic implies that the only directions of flow with vanishing SAF-invariant are those for which the flow is periodic. (By Veech's dichotomy [V2], these are the directions in which (X,ω) decomposes into cylinders). McMullen's result thus certainly excludes the stable foliation of pseudo-Anosov maps from having vanishing SAF-invariant. That is, the hypothesis of lattice Veech group allows one also to rule out (from having vanishing SAF-invariant) those non-periodic directions which are not the stable foliations of pseudo-Anosov maps.*

Remark 4.5.2. *We point out that if a pseudo-Anosov map ϕ is of vanishing SAF-invariant and its dilatation is not totally real, then its trace field is also not totally real. This holds, as vanishing SAF-invariant implies equality of the trace field with the field generated over $\mathbb{Q}$ by the dilatation. This can be applied to allow a minor simplification in the existence arguments of [HL].*

4.6 Every bi-Perron unit has its minimal polynomial dividing the characteristic polynomial of some pseudo-Anosov's homological action

Theorem 4.6.1. *Suppose that α is a bi-Perron unit. Then the minimal polynomial of α divides the characteristic polynomial of the action on first integral homology induced by some pseudo-Anosov map.*

Proof. If α is a bi-Perron unit whose minimal polynomial $p(x)$ is reciprocal (and hence of even) degree say $2g$, then $p(x)$ is obviously (symplectically) irreducible. That $p(x)$ is not cyclotomic is clear. That $p(x) = f(x^k)$ is only trivially possible is shown in Lemma 4.1.5. Thus, the hypotheses are all satisfied for the Margalit-Spallone construction of [MS] to give an explicit pseudo-Anosov element, (indeed a full coset of the Torelli group) in the mapping class group of the genus g surface, whose induced action on homology has characteristic polynomial $p(x)$.

If α is a bi-Perron unit of degree g whose minimal polynomial $p(x)$ is not reciprocal, let $\hat{p}(x)$ be the minimal polynomial of α^{-1}. And once again let $q(x)$ be the minimal polynomial of $\alpha + \alpha^{-1}$, which by Theorem 1.1.1 is also of degree g. Let $\tilde{q}(x) = x^g q(x + x^{-1})$. Since both α, α^{-1} are roots of $\tilde{q}(x)$, degree considerations give that $\tilde{q}(x) = p(x)\hat{p}(x)$.

Lemma 4.1.4 shows that $\tilde{q}(x)$ is not equal to any non-trivial $f(x^k)$. That $\tilde{q}(x)$ has no cyclotomic roots is clear, as its only roots are those of $p(x), \hat{p}(x)$ and each of these is an irreducible polynomial with a root that is of absolute value greater than one. Again, the hypotheses are all satisfied for the Margalit-Spallone construction, so that there exist pseudo-Anosov homeomorphisms whose induced action on homology is of characteristic polynomial $\tilde{q}(x)$.

□

Remark 4.6.1. *If any of the pseudo-Anosov maps guaranteed by [MS] is orientable, then its dilatation is an eigenvalue of the action on homology. But, the dilatation must then be* α*, and we have realized* α *as a dilatation.*

As recalled in [LT], the dilatation of a non-orientable pseudo-Anosov homeomorphism cannot be an eigenvalue for the induced action on homology. Thus, if none of the pseudo-Anosov maps guaranteed by [MS] is orientable, even after applying the standard double cover construction (see say the text [FM]) we can say little more than stated in Theorem 4.6.1.

4.7 A problem of Birman *et al.*

Birman, Brinkmann and Kawamuro [BBK] associate to a pseudo-Anosov map ϕ of dilatation λ a symplectic polynomial $s(x)$ that has λ as its largest real root. They write "its relationship to the minimum polynomial of λ is not completely clear at this writing." We give an explanation in the setting that ϕ is orientable (and defined on a surface without punctures).

Theorem 4.7.1. *Suppose that* ϕ *is an orientable pseudo-Anosov map on a surface of genus* g*. Let* $s(x)$ *be the symplectic polynomial associated to* ϕ *in [BBK]. Then* $s(x)$ *is reducible if and only if either* ϕ *has vanishing* SAF*-invariant or has trace field of degree strictly less than* g*.*

Proof. Let λ be the dilatation of ϕ and $p(x)$ be the minimal polynomial of λ. Since $s(x) \in \mathbb{Z}[x]$ is monic and has λ as a root, of course $p(x)$ divides $s(x)$. As well, since $s(x)$ is a reciprocal polynomial, whenever some α is a root of $s(x)$ so also is α^{-1} a root.

If $s(x)$ is irreducible then it equals $p(x)$. Thus, $p(x)$ is in particular reciprocal. Therefore, by Theorem 1.1.1 the SAF-invariant of ϕ does not vanish.

Suppose now that $s(x)$ is reducible but symplectically irreducible. Were $p(x)$ reciprocal, then there would exist some other factor of $s(x)$, but this factor would perforce be reciprocal. This contradiction shows that in this case $p(x)$ is not a reciprocal polynomial. By Theorem 1.1.1 the SAF-invariant of ϕ vanishes. Furthermore, the minimal polynomial $\hat{p}(x)$ of λ^{-1} is distinct from $p(x)$. But since λ is a root of $s(x)$, so is λ^{-1} and hence $\hat{p}(x)$ also divides $s(x)$. That is $\tilde{q}(x) = p(x)\hat{p}(x)$ divides $s(x)$. The existence of any further factor of $s(x)$ would lead to a contradiction of the symplectic irreducibility of $s(x)$. That is, whenever $s(x)$ is reducible but symplectically irreducible it is exactly the product $\tilde{q}(x) = p(x)\hat{p}(x)$ and $p(x)$ is not reciprocal.

Finally, suppose that $s(x)$ is symplectically reducible. We have that either $p(x)$ is reciprocal or that $\tilde{q}(x) = p(x)\hat{p}(x)$ divides $s(x)$. In either case, there is some other reciprocal factor of $s(x)$. Thus the degree of $p(x)$ or $\tilde{q}(x)$ is correspondingly of degree less than $2g$ and as the trace field $\mathbb{Q}(\lambda + \lambda^{-1})$ has dimension over $\mathbb{Q}$ equal to one-half of the degree of $p(x)$ or $\tilde{q}(x)$ in these respective cases, we indeed find that the trace field of ϕ has degree strictly less than g. □

Remark 4.7.1. *In particular, Example 5.2 of [BBK] shows that the "monodromy of the hyperbolic knot* 8_9*" leads to an orientable pseudo-Anosov map with* $s(x) = (x^3 - 2x^2 + x - 1)(x^3 - x^2 + 2x - 1)$*. Here the dilatation* λ *is the real root of* $x^3 - x^2 + 2x - 1$*, the second factor is the minimal polynomial of* $1/\lambda$*. Using its minimal polynomial, one easily shows that* λ *equals* $-(\lambda + \lambda^{-1})^2 + 3(\lambda + \lambda^{-1}) - 1$*, implying that indeed* $\mathbb{Q}(\lambda) = \mathbb{Q}(\lambda + \lambda^{-1})$.

5 Spinning about small loops

5.1 Rediscovering the Arnoux-Rauzy family of $\mathcal{H}^{\mathbf{odd}}(2,2)$

Mimicking the construction of [AY], Arnoux and Rauzy [AR] constructed an infinite family of IETs, the first two of which Lowenstein, Poggiaspalla, and Vivaldi [LPV, LPV2] studied in detail, as these lead to SAF-zero pseudo-Anosov maps. Indeed, by making an appropriate adjustment, Lowenstein *et al.* renormalized these first two IETs in such a way that each was periodic under Rauzy induction. Each corresponds to a cycle passing through the same 29 vertices in the 294-vertex Rauzy class of 7-interval IETs, and under the Veech construction leads to a pseudo-Anosov homeomorphism. The dilatations of these are the largest root of $x^3-7x^2+5x-1=0$ and $x^3-10x^2+6x-1=0$ respectively.

Presumably, Lowenstein *et al.* intend that one follow their recipe for constructing pseudo-Anosov homeomorphisms for the remainder of the Arnoux-Rauzy family. This seemed somewhat daunting to us. However, we found that one can succeed by adjusting the cycle given by the first Arnoux-Rauzy IET by spinning about certain small cycles. Since the Arnoux-Yoccoz pseudo-Anosov homeomorphism in genus 3 corresponds to an abelian differential in $\mathcal{H}^{\text{odd}}(2,2)$ (for this and much more see [HLM]), all of these examples (since they arise from the same Rauzy class) are in this same connected component.

More precisely, the path $\rho_k = 00001010(111111)^{k-1}1101(00)^{k-1}010100111$, for each $k \geq 1$, starting from the permutation (7354621), gives these maps. (Here and throughout, exponents as in the expression for ρ_k indicate repeated concatenation of the correspondingly grouped symbols.) One then finds that the characteristic polynomial of the induced transition matrix for γ_k is

$$p_k(x) = (x^3 - (3k+4)x^2 + (k+4)x - 1)(x^3 - (k+4)x^2 + (3k+4)x - 1)(x-1)$$

For ease of calculation, we take the alphabet set $\mathcal{A}$ to be $\{1, ..., 7\}$. We break up ρ_{k+1} into five paths corresponding to $00001010, (111111)^k, 1101, (00)^k, 010100111$, and compute their respective transition matrices. Using the notation $V_{\alpha,\beta}$ introduced in section §3.2, the transition matrices of the paths are given respectively as follow:

$$V_{7,1}.V_{7,2}.V_{7,6}.V_{7,4}.V_{5,7}.V_{6,5}.V_{3,6}.V_{7,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$(V_{1,7}.V_{1,5}.V_{1,4}.V_{1,6}.V_{1,3}.V_{1,2})^k = \begin{pmatrix} 1 & k & k & k & k & k & k \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$V_{1,7}.V_{1,5}.V_{4,1}.V_{2,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(V_{6,2}.V_{6,5})^k = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & k & 0 & 0 & k & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{And, } V_{6,2}.V_{5,6}.V_{3,5}.V_{2,3}.V_{4,2}.V_{4,6}.V_{1,4}.V_{1,3}.V_{1,2} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

From this, one easily shows that the associated matrix for γ_{k+1} is the matrix

$$V_k = \begin{pmatrix} k & k^2+3k-3 & k^2+2k-2 & 3k-2 & k^2 & k^2+2k-2 & k \\ 0 & 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & k & k+1 & 0 & k & k & 0 \\ 1 & 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & k & k & 0 & k & k+1 & 0 \\ k+1 & k^2+4k+1 & k^2+3k+1 & 3k+1 & k^2+k+1 & k^2+3k+1 & k+1 \end{pmatrix}$$

whose characteristic polynomial is $p_{k+1}(x)$.

5.2 Two known examples in $\mathcal{H}^{\mathbf{hyp}}(4)$

Veech [V2] constructed an infinite family of translation surfaces with Veech groups that are lattices in $\mathrm{SL}_2(\mathbb{R})$. For each $n \geq 5$, his construction is to identify, by translation, parallel sides of a regular n-gon and its mirror image. In the case of $n = 7$, one finds a genus 3 surface with exactly one singularity of cone angle 8π. Veech shows that the Veech group here is generated by $S = \begin{pmatrix} \cos(\pi/7) & -\sin(\pi/7) \\ \sin(\pi/7) & \cos(\pi/7) \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 2\cot(\pi/7) \\ 0 & 1 \end{pmatrix}$. In [AS], it is pointed out that results on (Rosen) continued fractions of Rosen and Towse [RT] imply that on this surface there is a SAF-zero pseudo-Anosov. Indeed, this is the map, say ψ, of linear part $D\psi = TST^{-1}S^{-1}$.

Explicitly taking a transversal to the flow in the expanding direction for ψ, and following Rauzy induction on the corresponding IET, we found that ψ results from the loop displayed in Figure 5.1. The letters in Figure 5.1 indicate the winner letter of that cycle. We take note that we have exactly one winning cycle for each letter.

The associated transition matrix is given by

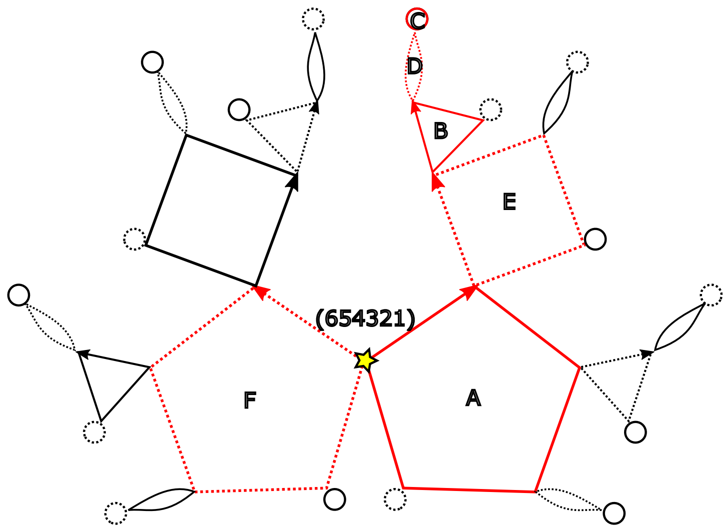

FIGURE 5.1: Red loop representing ψ, a pseudo-Anosov on the double heptagon.

$$V = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 1 \\ 0 & 2 & 2 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 \\ 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

whose characteristic polynomial is $(x-1)(x^3-6x^2+5x-1)(x^3-5x^2+6x-1)$. Its largest eigenvalue is a root of (x^3-6x^2+5x-1), verifying that the SAF invariant vanishes.

Lanneau's example, given in [Mc2], has as its dilatation the largest root of x^3-8x^2+6x-1. We noticed that both ψ and this example correspond to paths passing through the same 15 vertices of the hyperelliptic Rauzy graph of 6-interval IETs. These paths only differ in that Lanneau's has added spins (indeed, the "top right" 1-cycle is repeated four times).

The associated transition matrix is given by

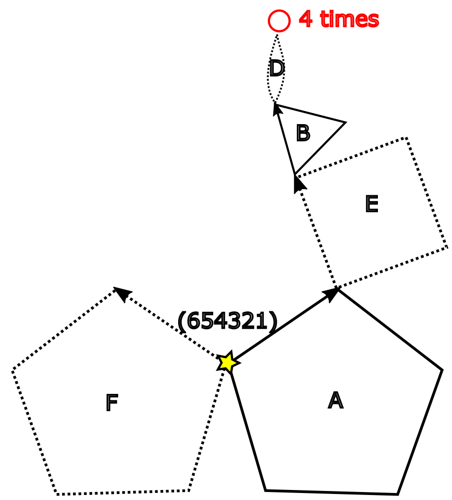

FIGURE 5.2: Lanneau's SAF-zero example

$$V = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 1 \\ 0 & 2 & 2 & 2 & 1 & 0 \\ 0 & 0 & 5 & 4 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 \\ 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

whose characteristic polynomial is $(x-1)(x^3-8x^2+6x-1)(x^3-6x^2+8x-1)$, verifying that the SAF invariant vanishes.

5.3 New families of pseudo-Anosov maps in $\mathcal{H}^{\mathbf{hyp}}(2,2)$

Theorem 1.1.2 *For each $k \in \mathbb{N}$ with $k \geq 2$, there exists at least four orientable pseudo-Anosov maps in the hyperelliptic component of the stratum $\mathcal{H}(2,2)$ having dilatation of minimal polynomial $x^3 - (2k+4)x^2 + (k+4)x - 1$. In particular, each of these pseudo-Anosov maps has vanishing* SAF*-invariant.*

Motivated by the previous examples, we sought an infinite family of pseudo-Anosov with vanishing SAF invariant. Arnoux-Rauzy's family has the dilatation being the largest root of $x^3 - (3k+4)x^2 + (k+4)x - 1$. Since the double heptagon example has dilatation being a root of $x^3 - 6x^2 + 5x - 1$ and Lanneau's example has dilatation being a root of $x^3 - 8x^2 + 6x - 1$, we wished to find an infinite family whose dilatation is the largest root of $x^3 - (2k+4)x^2 + (k+4)x - 1$.

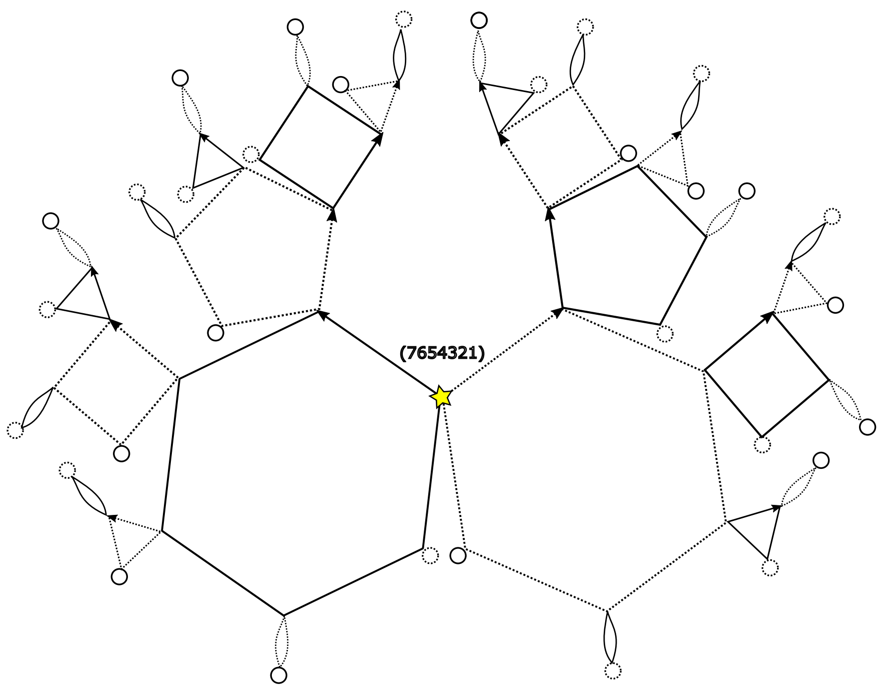

FIGURE 5.3: Hyperelliptic Rauzy diagram with 7 subintervals.

We found such a family, but rather by taking certain paths in the hyperelliptic Rauzy graph of 7-intervals IETs. This graph is shown in Figure 5.3. We found in fact four distinct families, and thus new examples of pseudo-Anosov maps with vanishing SAF-invariant. Naturally enough (by Lemma 1), we describe the paths as starting at the vertex of $\pi = (7654321)$.

5.3.1 Closed loops α_k

For $k \geq 2$, let α_k be given by $10101(0^{k-1})10011100001111100000(1^{k-1})0$, see Figure 5.9. Similarly to the infinite examples from the Arnoux-Rauzy family, to verify this, we simply break the path α_k into 5 shorter paths and calculate their transition matrix.

$$V_{1,7}.V_{6,1}.V_{2,6}.V_{5,2}.V_{3,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(V_{4,3})^k = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & k & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$V_{3,4}.V_{5,3}.V_{5,4}.V_{2,5}.V_{2,4}.V_{2,3}.V_{6,2}.V_{6,3}.V_{6,4}.V_{6,5}.V_{1,6}.V_{1,5}.V_{1,4}.V_{1,3}.V_{1,2}.V_{7,1}.V_{7,2}.V_{7,3}.V_{7,4}.V_{7,5}$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$(V_{6,7})^k = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & k \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{And, } V_{7,6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

We obtain the following transition matrix for α_k

$$V_{\alpha_k} = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & k+1 & k \\ 0 & 2 & 2 & 2 & 2 & k & k-1 \\ 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & k-1 & k & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 2 & 0 & 0 \\ 1 & 2 & 2 & 2 & 2 & 2k & 2k-2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This has characteristic polynomial

$$(x^3 - (2k+4)x^2 + (k+4)x - 1)(x^3 - (k+4)x^2 + (2k+4)x - 1)(x-1).$$

Hence, the corresponding pseudo-Anosov map ϕ_{α_k} has vanishing SAF-invariant.

5.3.2 Closed loops β_k

Let $\beta_k : 11010101(0^{k-1})1000111100000(1^{k-1})0$ for $k \geq 2$, see Figure 5.9. We obtain the transition matrix

$$V - \beta_k = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & k+1 & k \\ 0 & 2 & 2 & 2 & 1 & 0 & 0 \\ 0 & 2k-2 & 2k & 1 & 0 & 0 & 0 \\ 0 & k & k+1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k & k-1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

whose characteristic polynomial is

$$(x^3 - (2k+4)x^2 + (k+4)x - 1)(x^3 - (k+4)x^2 + (2k+4)x - 1)(x-1).$$

. Hence, the corresponding pseudo-Anosov map ϕ_{β_k} has vanishing SAF-invariant.

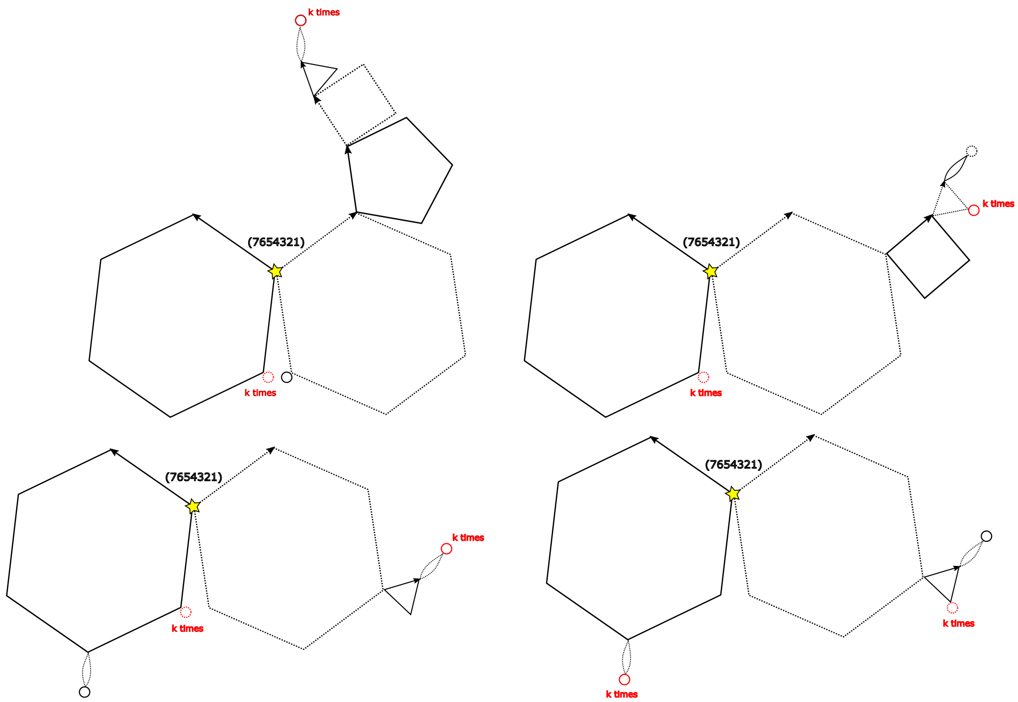

FIGURE 5.4: Clockwise from top left: The paths for $\alpha_{k+1}, \beta_{k+1}, \gamma_{k+1}, \delta_{k+1}$.

5.3.3 Closed loops γ_k

Let γ_k : $11101010(1)^{k-1}011100001(0)^{k-1}100$ for $k \geq 2$, see Figure 5.9. We obtain the transition matrix

$$V_{\gamma_k} = \begin{pmatrix} 2 & 2 & 2 & 2 & 4 & 3k+2 & 3k-1 \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & k-1 & k & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & k+1 & k \\ 0 & 0 & 0 & 0 & 1 & 2k & 2k-2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

whose characteristic polynomial is

$$(x^3 - (2k+4)x^2 + (k+4)x - 1)(x^3 - (k+4)x^2 + (2k+4)x - 1)(x-1).$$

Hence, the corresponding pseudo-Anosov map ϕ_{γ_k} has vanishing SAF-invariant.

5.3.4 Closed loops δ_k

Let $\delta_k : 11101(0^{k-1})10011100001010(1^{k-1})0$ for $k \geq 2$, see Figure 5.9. We obtain the transition matrix

$$V_{\delta_k} = \begin{pmatrix} 2 & 2 & 2 & 2 & k+2 & k+3 & 2 \\ 0 & 2 & k+1 & k & 0 & 0 & 0 \\ 0 & 1 & 2k & 2k-2 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & k-1 & k & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

whose characteristic polynomial is

$$(x^3 - (2k+4)x^2 + (k+4)x - 1)(x^3 - (k+4)x^2 + (2k+4)x - 1)(x-1).$$

Hence, the corresponding pseudo-Anosov map ϕ_{δ_k} has vanishing SAF-invariant.

5.4 Examples in other strata

The similarities in patterns of the paths for the double heptagon example and Lanneau's example in $\mathcal{H}^{\text{hyp}}(4)$ and those of our examples in $\mathcal{H}^{\text{hyp}}(2,2)$, motivated us to investigate similar patterns in the Rauzy hyperelliptic diagrams for 8 and 9 subintervals. We discovered several isolated examples of SAF-zero maps here, i.e. in the components $\mathcal{H}^{\text{hyp}}(6)$ and $\mathcal{H}^{\text{hyp}}(3,3)$. However, an infinite family is yet to be found. Below, we also include other examples in genus 3.

5.4.1 Five genus 3 examples in $\mathcal{H}^{\textbf{hyp}}(4)$

We look further into the Rauzy hyperelliptic diagram with 6 subintervals. As usual, we start at the hyperelliptic pair $\pi = (6,5,4,3,2,1)$. Motivated by the double heptagon example, we sought after the closed path that has exactly one winning cycle for each letter. We find that H6V1 and H6V2 as described below share the same characteristic polynomial as the double heptagon loop. Thus, SAF invariant is 0.

$$\begin{aligned}\text{H6V1} &: 110101001110000010;\\ &(x^3-6x^2+5x-1)(x^3-5x^2+6x-1)\end{aligned}$$

$$\begin{aligned}\text{H6V2} &: 1110101100010100;\\ &(x^3-6x^2+5x-1)(x^3-5x^2+6x-1)\end{aligned}$$

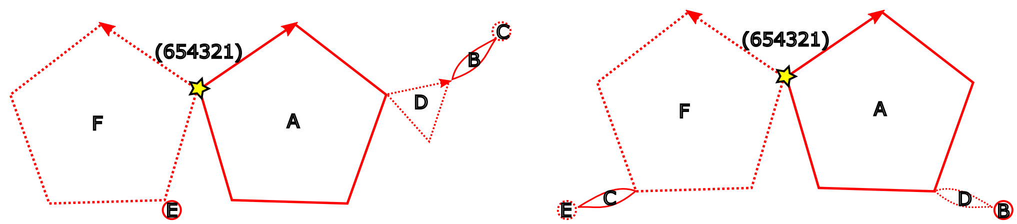

FIGURE 5.5: Examples H6V1 and H6V2 in the stratum $\mathcal{H}^{\text{hyp}}(4)$.

Of course, we take advantage of the similarity between Lanneau's example and the double heptagon example. We spin the 1-cycle in H6V1 and H6V2 to find the following 3 closed path, who shares the same characteristic polynomial as Lanneau's example.

H6L1 :11010000100111000010;

$(x^3 - 8x^2 + 6x - 1)(x^3 - 6x^2 + 8x - 1)$

H6L2 :11010100111000011110;

$(x^3 - 8x^2 + 6x - 1)(x^3 - 6x^2 + 8x - 1)$

H6L3 :1110111101100010100;

$(x^3 - 8x^2 + 6x - 1)(x^3 - 6x^2 + 8x - 1)$.

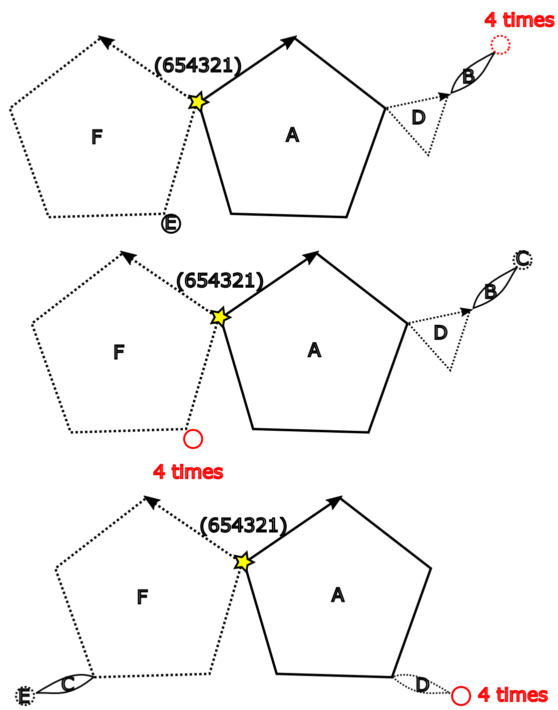

FIGURE 5.6: Examples H6L1, H6L2, and H6L3 in the stratum $\mathcal{H}^{\text{hyp}}(4)$.

5.4.2 Seven genus 4 examples in $\mathcal{H}^{\text{hyp}}(6)$

We look further into the Rauzy hyperelliptic diagram with 8 subintervals. The following examples start at the hyperelliptic pair $\pi = (8, 7, 6, 5, 4, 3, 2, 1)$. We will give the paths and the characteristic polynomial of the associated matrix. Hence, we can see the loops produce SAF-zero pseudo-Anosov from the cubic factors of the characteristic polynomials.

H8_1 : 10101010110001111000001011111110000000;

$$(x^3 - 9x^2 + 6x - 1)(x^3 - 6x^2 + 9x - 1)(x - 1)^2$$

H8_2 : 10101010110001110100000011111110000000;

$$(x^3 - 9x^2 + 6x - 1)(x^3 - 6x^2 + 9x - 1)(x - 1)^2$$

H8_3 : 11010101001110000011110100000010;

$$(x^3 - 9x^2 + 6x - 1)(x^3 - 6x^2 + 9x - 1)(x - 1)^2$$

H8_4 : 11010101001110001011111100000010;

$$(x^3 - 9x^2 + 6x - 1)(x^3 - 6x^2 + 9x - 1)(x - 1)^2$$

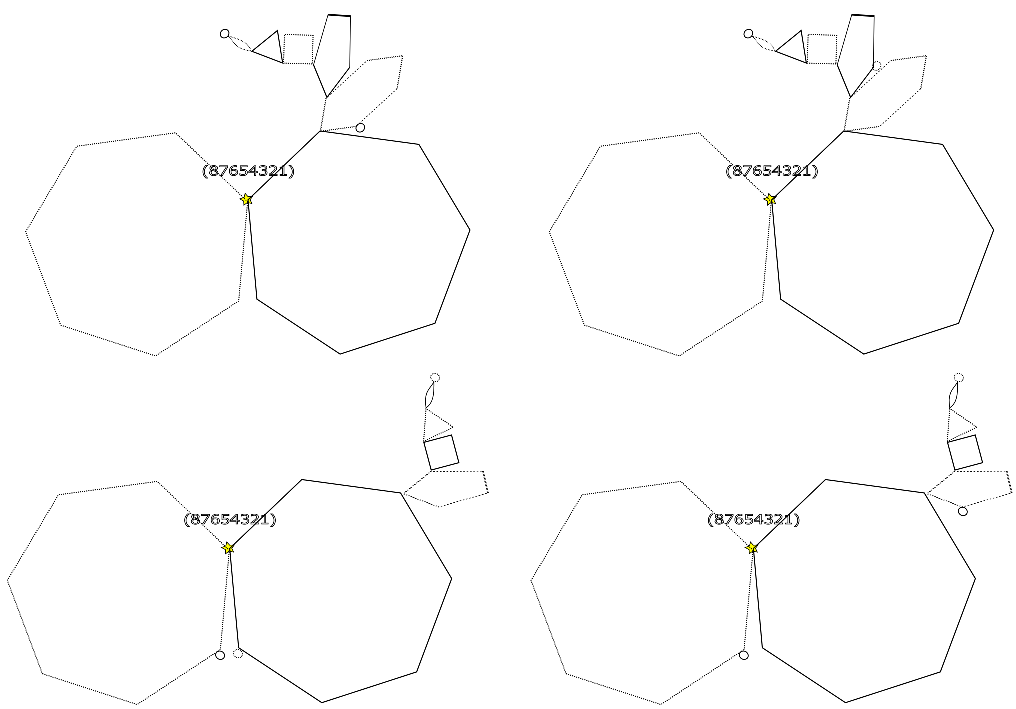

FIGURE 5.7: From top to bottom, left to right: examples H8_1, H8_2, H8_3, and H8_4 in the stratum $\mathcal{H}^{\text{hyp}}(6)$.

H8_5 : 110101001001110000111111000000110;

$$(x^2 - 6x + 1)(x^3 - 6x^2 + 5x - 1)(x^3 - 5x^2 + 6x - 1)$$

H8_6 : 111010110110001111000000100100;

$$(x^2 - 6x + 1)(x^3 - 6x^2 + 5x - 1)(x^3 - 5x^2 + 6x - 1)$$

H8_7 : 11110100100111000010110110000;

$$(x^2 - 6x + 1)(x^3 - 6x^2 + 5x - 1)(x^3 - 5x^2 + 6x - 1).$$

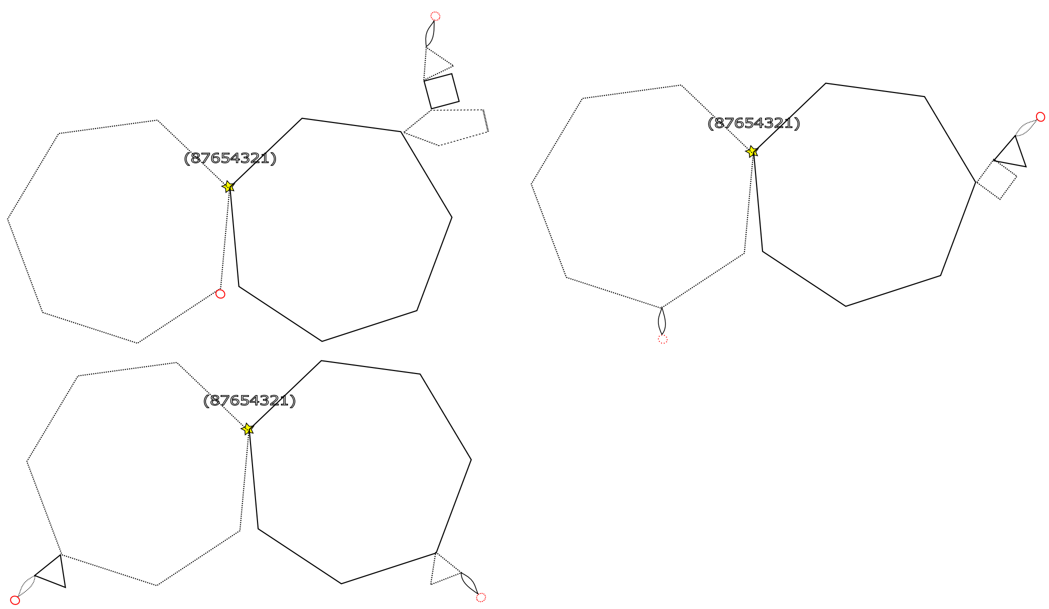

FIGURE 5.8: From top to bottom, left to right: Examples H8_5, H8_6, and H8_7 in the stratum $\mathcal{H}^{\text{hyp}}(6)$, where the red 1-cycle is used twice.

5.4.3 Three genus 4 examples in $\mathcal{H}^{\text{hyp}}(3,3)$

We look into the Rauzy hyperelliptic diagram with 9 subintervals. The following examples start at the hyperelliptic pair $\pi = (9,8,7,6,5,4,3,2,1)$. We can see the loops produce SAF-zero pseudo-Anosov from the non-reciprocal quartic factors of the characteristic polynomials. This is the first time we see a dilatation of non-reciprocal minimal polynomial of degree 4.

H9_1 : 110101010110110001111000001111101000000110;

$$(x-1)(x^4-9x^3+22x^2-11x+1)(x^4-11x^3+22x^2-9x+1)$$

H9_2 : 110101010110110001110100000111111000000110;

$$(x-1)(x^4-9x^3+22x^2-11x+1)(x^4-11x^3+22x^2-9x+1)$$

H9_3 : 111101010110110001110100000101101100 0;

$$(x-1)(x^4-9x^3+22x^2-11x+1)(x^4-11x^3+22x^2-9x+1).$$

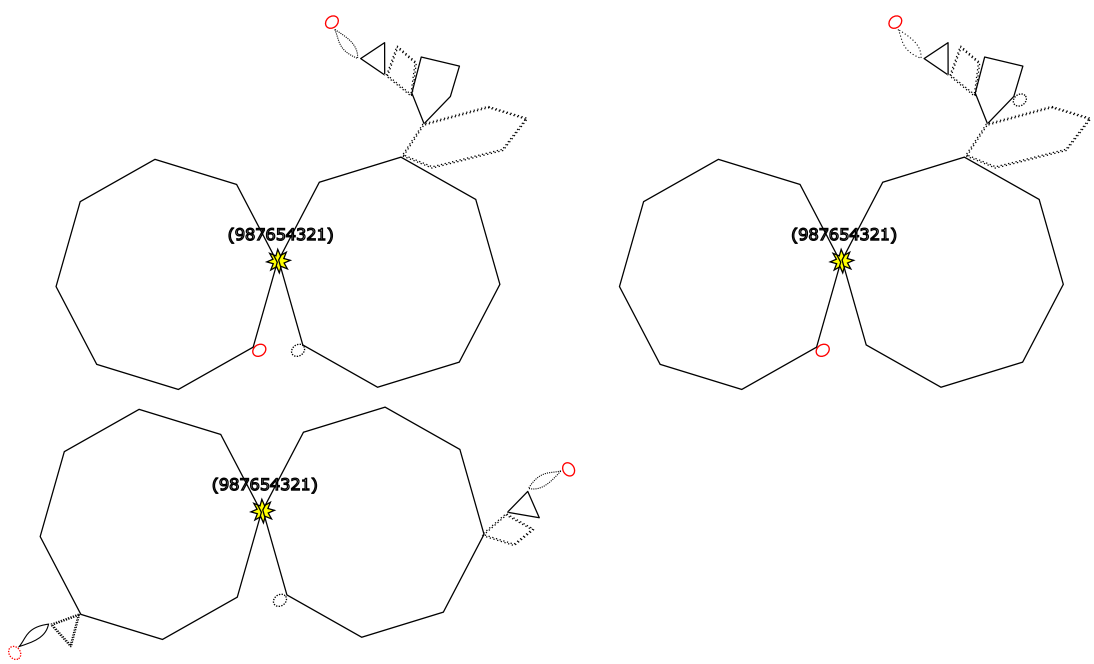

FIGURE 5.9: From top to bottom, left to right: Examples H9_1, H9_2, and H9_3 in the stratum $\mathcal{H}^{\text{hyp}}(3,3)$, where the red 1-cycle is used twice.

6 Conclusion

We have characterized the condition for a pseudo-Anosov map to have vanishing SAF invariant. In particular, SAF invariant vanishes exactly when the minimal polynomial for its dilatation is non-reciprocal.

We have confirmed that there are infinite families of such maps in the strata $\mathcal{H}^{\text{hyp}}(2,2)$, and $\mathcal{H}^{\text{odd}}(2,2)$.

Furthermore, using Veech's construction, we have given genus 3 and genus 4 examples in the strata $\mathcal{H}^{\text{hyp}}(4)$, $\mathcal{H}^{\text{hyp}}(2,2)$,$\mathcal{H}^{\text{hyp}}(6)$, and $\mathcal{H}^{\text{hyp}}(3,3)$.

Question: How is the construction related to the "special" dilatation?

Analyzing the closed paths of these SAF-zero pseudo-Anosov, we can see that they are extremely similar in patterns.

· The two largest cycles corresponding to the first and the last letter of the alphabet $\mathcal{A}$ appear exactly once.

· The same goes for the (only) second largest cycle.

· Most letters have only one winning cycle.

· If any part of the path is to be repeated, it is a cycle of length 1.

We follow these criteria in hope of finding more examples in higher genera.

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