NEUTRON TOTAL CROSS SECTIONS
USING THE OPTICAL MODEL OF THE NUCLEUS

by

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A THESIS
submitted to
OREGON STATE COLLEGE

in partial fulfillment of
the requirements for the
degree of
MASTER OF ARTS

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Date thesis is presented December 1, 1960

Typed by Diane Burkett
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The optical model of the nucleus employs a complex single-particle potential, $V = V_0 + iV_1$, which acts upon an incoming nucleon. It is assumed that the incident nucleon can enter the nucleus and move within the nuclear boundaries without forming a compound state. The formation of a compound nucleus is then considered to be an absorption described by the complex part of the potential. This model may be used to calculate total and elastic scattering cross sections which are considered to be averaged over many resonance levels of the compound nucleus.

Neutron total cross sections have been calculated using this model of the nucleus and the simplest form of nuclear potential, the square well. A fairly general program has been formulated for the computation of the average total cross section for any useful value of atomic mass number and neutron energies between 1 and 10 Mev. The computations were performed on the Oregon State College Department of Mathematics ALWAC digital computer and resulted in values of total cross sections of the correct order of magnitude.
THE OPTICAL MODEL OF THE NUCLEUS

Before the success of the shell model, it was generally assumed that all nucleons forming the nucleus or incident upon it interact strongly. The strong interaction between nucleons seemed to indicate a quick exchange of energy between an incident particle and the target nucleus. This resulted in the formation of a compound system or compound nucleus in which all or very many nucleons participate collectively. (9, p. 136) Because of the high density of particles in the nucleus it seemed extremely likely that a compound nucleus would be formed before the incident nucleon penetrated beyond the surface layer of the nucleus. But the shell model shows that a nucleon can move comparatively freely within the nuclear volume. (9, p. 144) Neutron cross sections calculated using the assumption that the compound nucleus is formed "immediately" when the incident neutron reaches the nuclear surface are monotonically decreasing functions of energy.

However measurements have indicated that this is not so. These cross sections show a more complicated behavior; the curves of total cross section versus the energy E of the incident neutron exhibit maxima and minima. These maxima and minima "seem to indicate an interference of the incident wave with the outgoing one,
suggesting that the neutron is not completely absorbed into collective motion in one passage through the nucleus." (6, p. 166) It is noted that nuclei with small differences of atomic mass number \( A \) show almost the same behavior; thus, one may conclude that these characteristic shapes do not depend on detailed features of nuclear structure but on some general properties which vary slowly with \( A \), such as the nuclear radius. (5, p. 449) The nuclear radius \( R \) may be assumed to vary as the cube root of \( A \).

If it is assumed that the incident neutron can penetrate into the nucleus and move within the nuclear boundaries without forming a compound state, then the actual formation of a compound nucleus occurs with a probability smaller than unity once the particle has entered the nucleus. There is a finite chance that the particle will leave the nucleus, without having formed a state in which it has exchanged energy or momentum with the rest of the nucleons. The formation of a compound state then would have the aspect of an absorption. Thus if the target nucleus may be described as acting upon the incoming neutron as a potential well, this case may be described by a potential well with absorption. (5, p. 449) The absorption, which can be described by an imaginary part of the potential, should reproduce the compound nucleus formation.
For a square potential well the potential \( V(r) \) is of the form

\[
V(r) = \begin{cases} 
0, & r > R; \\
-V_0, & r \leq R. 
\end{cases}
\]  

(1)

This means that the surface of the nucleus is represented by a sharp discontinuity in the potential experienced by the neutron. But the nucleus does not have a sharp boundary. The potential should not plunge to zero abruptly at \( r=R \), but should change more slowly. (4, p. 83) That is, a rounded or tapered well representing a diffuse nuclear surface would be a better approximation than the square well. Here the square well will be used with \( V(r) = -V_0(1+i\epsilon) \) when \( r \leq R \). The square well is chosen because the computations using this simple form of potential will consume much less machine time than computations using any other form.

This approach is certainly an oversimplification of the actual conditions. This model will not reproduce any resonance phenomena which are connected with the many possible quantum states of the compound system. It will describe only the features of nuclear reactions after averaging over these resonances. Also all spin-orbit terms other than those contributing to \( V(r) \) have been omitted. Use of the square potential well is another simplification but the resulting cross sections should be of the correct order of magnitude.
FORMULATION

In order to develop an expression which is suitable for the calculation of neutron total cross sections, we will first derive a general expression from the wave functions \( \psi(r) \), corresponding to the relative motion of the incident neutron and the target nucleus. Conditions at the nuclear surface, i.e., at the boundary of the potential well, will determine the final form.

Several different cross sections will be considered, each fitting the general definition

\[
\sigma_i = \frac{\text{number of } x\text{-type events per unit time per nucleus}}{\text{number of incident particles per unit area per unit time}}
\]

All types of events are included in the total cross section.

Cross Sections

The incident neutrons may be considered to be a plane wave that interacts with the potential field of the nucleus producing a secondary scattered wave. If the nuclear potential can be considered as spherically symmetrical, the amplitude of the scattered wave can be expressed as the sum of several partial waves. The physical significance of the decomposition into partial waves is that each corresponds to a certain angular momentum \( \ell \hbar \), where \( \ell \) is the angular momentum quantum number. \( (10, \text{p. } 40) \) The plane wave is \( e^{i \mathbf{k} \cdot \mathbf{r}} = e^{i k z} \)
where \( k = \frac{1}{\hbar} (2mE)^{1/2} \), \( z \) is parallel to \( \vec{k} \), and \( \vec{r} \) is the "channel coordinate"—the vector between the center of the target nucleus and the incident neutron. As indicated above, this plane wave may be expanded into spherical harmonics. For large values of \( kr \),

\[
e^{ikz} = \left( \frac{2}{kr} \right) \sum_{l=0}^{\infty} (2l+1)i^{l+1} \left[ e^{-i(kr-\frac{1}{2}l\pi)} - e^{i(kr-\frac{1}{2}l\pi)} \right] P_l(\cos \theta).
\]

This expansion describes the incident plane wave. The actual wave function in the entrance channel introduces the reflection factor \( \eta \), a complex number describing the amplitude of the outgoing wave with angular momentum \( \ell \hbar \); hence,

\[
\Psi(r) = \left( \frac{2}{kr} \right) \sum_{l=0}^{\infty} (2l+1)i^{l+1} \left[ e^{-i(kr-\frac{1}{2}l\pi)} \eta e^{i(kr-\frac{1}{2}l\pi)} \right] P_l(\cos \theta).
\]

The scattered wave is the difference between the incident wave (2) and the actual wave (3):

\[
\Psi_{sc} = \Psi(r) - e^{ikz} = \left( \frac{2}{kr} \right) \sum_{l=0}^{\infty} (2l+1)i^{l+1}(1- \eta) e^{i(kr-\frac{1}{2}l\pi)} P_l(\cos \theta).
\]

The elastic scattering cross section \( \sigma_{sc} \) is obtained by dividing the number \( N_{sc} \) of scattered particles per unit time, by the number \( N \) of incident particles per unit area per unit time. If \( N_{sc} \) is the flux of \( \Psi_{sc} \) through a large sphere of radius \( r_0 \) about the center of the nucleus, then

\[
N_{sc} = \frac{\hbar}{2im} \int \left( \frac{\partial \Psi_{sc}^*}{\partial r} \Psi_{sc} - \frac{\partial \Psi_{sc}^*}{\partial r} \Psi_{sc} \right) \mid_{r=r_0} r_0^2 \sin \theta d\theta d\phi.
\]
This reduces to \( N_{sc} = \left( v \frac{\sigma}{k^2} \right) \sum_{\ell_0}^{\infty} (2\ell+1) \left| 1 - \eta_{l} \right|^2 \) where \( v \) is the velocity of the particles. The value of the flux \( N \) of a plane wave \( e^{i k x} \) is equal to the velocity \( v \) so that

\[
\sigma_{sc} = N_{sc} / N \quad \text{and} \quad \sigma_{sc}(\ell) = \pi \lambda^2 (2\ell+1) \left| 1 - \eta_{\ell} \right|^2
\]

where \( \sigma_{sc}(\ell) \) is the partial scattering cross section, i.e., \( \sum_{\ell_0}^{\infty} \sigma_{sc}(\ell) = \sigma_{sc} \). The reaction cross section is determined by the number \( N_a \) of particles taken out of the incident beam per unit time. \( N_a \) is thus the net flux into the sphere of radius \( r_0 \). From (3), we have

\[
N_a = - \frac{n}{21M} \int \left( \frac{\partial \psi^*}{\partial r} - \frac{\partial \psi}{\partial r} \right) r^2 \sin \theta \, d\theta \, d\phi
\]

and if \( \sigma_r = N_a / N \), then \( \sigma_r(\ell) = \pi \lambda^2 (2\ell+1) \left( 1 - \left| \eta_{\ell} \right|^2 \right) \). (8)

The complex reflection factor is a complicated function of the energy of the incoming particle, exhibiting rapid fluctuations coming from numerous closely spaced resonances of the compound nucleus. We must assume that an average can be obtained in the form of

\[
\eta_{\ell} = \left( 1 / I \right) \int_{\epsilon_{\ell}}^{\epsilon_{\ell+1}} \eta_{\ell}(\epsilon) \, d\epsilon,
\]

where \( I \) is an interval including many resonances. Then

\[
\sigma_{re}(\ell) = \pi \lambda^2 (2\ell+1) \left| 1 - \eta_{\ell} \right|^2
\]

and

\[
\sigma_r(\ell) = \pi \lambda^2 (2\ell+1) \left( 1 - \left| \eta_{\ell} \right|^2 \right);
\]
and as
\[ \sigma_{e_1}(l) + \sigma_{e_2}(l) = \sigma_{e_3}(l), \]  
we have
\[ \sigma_{e_3}(l) = n \chi^2(2l+1) \left[ 1 - \xi \right]^2 + 1 - |\eta|^2. \]  

When the neutron energy is in the continuum region the cross sections and the phases are no longer rapidly varying functions of energy and the averaged values are equal to the actual ones.

Some elastic scattering may occur after the formation of a compound system. This is known as compound elastic scattering and may be described by a cross section \( \sigma_{ce} \). Then the cross section for the formation of a compound system \( \sigma_c \) may be given as
\[ \sigma_c = \sigma_{r}^{+} \sigma_{ce}. \]  
In the continuum region the compound elastic scattering is negligible so that \( \sigma_c = \sigma_r \), and \( \sigma_e = \sigma_c^+ \sigma_{e_3} \) from (11). Now \( \sigma_{e_3} \) can be said to describe scattering which does not change the quantum state of the nucleus, and \( \sigma_c \) describes all the reactions which do change the quantum state.

Application of Boundary Conditions

The reflection factor \( \eta \) must be connected with conditions at the nuclear surface. Consider an incident beam of neutrons represented outside the nucleus by a
wave number $k$. Near the nuclear surface this wave is joined with equal value and derivative to the wave function inside the nucleus. Therefore the logarithmic derivative $f_k$ of the partial wave function $u_k(r)$ at the nuclear boundary is introduced (8, p. 146):

$$f_k = \left[ \frac{du_k(r)/dr}{u_k(r)} \right]_{r=R}.$$

(14)

The magnitude of $f$ is completely defined by conditions within the nucleus, but its value can be found in terms of quantities which are dependent on the conditions outside the nucleus. (3, p. 332)

Start with the Schroedinger equation for the wave outside the nucleus,

$$\nabla^2 \psi + (2m/\hbar^2) [(E+V(r)) \psi = 0,$$

and let

$$\psi(r,\theta) = \sum_{l=0}^{\infty} \left[ \frac{(2l+1)}{4\pi} \right] u_k(r) P_l(\cos \theta),$$

where $P_l$ are the Legendre polynomials; then the radial equation in terms of $u_k(r)$ is

$$\frac{d^2 u_k(r)}{dr^2} + \left[ k^2 - \frac{2m}{\hbar^2} V(r) - \frac{\ell(\ell+1)}{r^2} \right] u_k(r) = 0,$$

(17)

where $k^2 = (2m/\hbar^2)E$.

If a square potential well is used $V(r)=0$, for $r>R$, so that

$$\frac{d^2 u_k(r)}{dr^2} + \left[ k^2 - \frac{\ell(\ell+1)}{r^2} \right] u_k(r) = 0.$$
A solution of this equation is

\[ u(\lambda) = krj(\lambda kr) + ik\text{rn}(\lambda kr) = krh(\lambda kr), \]  

where \( j_\lambda \), \( n_\lambda \), and \( h_\lambda \) are the spherical Bessel, Neumann, and Hankel functions. (See Appendix for definitions)

Let

\[ u_\lambda(+) = krj(\lambda kr) + ik\text{rn}(\lambda kr), \]
\[ u_\lambda(-) = krj(\lambda kr) - ik\text{rn}(\lambda kr). \]  

Using (15), equation (3) may be rewritten as

\[ u_\lambda(r) = 2^{\lambda+1}(2\lambda + 1)\frac{\sin(\pi/\lambda)}{\pi} \left[ u_\lambda(-)(r) - \eta_\lambda u_\lambda(+)\right]. \]  

In order to relate \( \eta_\lambda \) and \( f_\lambda \), the real numbers \( \Delta_\lambda \) and \( s_\lambda \) are defined by

\[ R \frac{du_\lambda(+) / dr}{u_\lambda(+) r=R} = \Delta_\lambda + is_\lambda, \]  

as well as the phase constant \( \delta_\lambda \) by

\[ e^{-2i\delta_\lambda} = \frac{u_\lambda(+)R}{u_\lambda(-)R}. \]  

By substituting (20) into (14) and using the two preceding relations to simplify the result, an expression for \( \eta_\lambda \) is obtained: (3, p. 333)

\[ \eta_\lambda = \frac{f_\lambda - \Delta_\lambda + is_\lambda}{f_\lambda - \Delta_\lambda - is_\lambda} e^{-2i\delta_\lambda}. \]
This relation may be used to evaluate $n_\ell$ using quantities dependent on the potential well and conditions at the nuclear surface.

Formulas used in Computation

The formula used to compute $\bar{\sigma}_\ell^{(\nu)}$ may be obtained by substituting the equation derived above into equation (12) obtained earlier for $\bar{\sigma}_\ell^{(\nu)}$. For simplification of notation two real numbers are introduced: (5, p. 454)

$$M_\ell = s_\ell - iMf_\ell \quad \text{and} \quad N_\ell = -A_\ell + Rf_\ell.$$  \hspace{1cm} (24)

Then

$$\bar{\sigma}_\ell^{(\nu)} = \pi l^2 (2l+1) \left[ (1 - n_\ell) + l - |n_\ell|^2 \right]$$  \hspace{1cm} (12)

$$= \frac{4\pi i}{k^2} (2l+1) \left[ \sin^2 \delta_\ell + s_\ell \frac{M_\ell \cos 2\delta_\ell - N_\ell \sin 2\delta_\ell}{M_\ell^2 + N_\ell^2} \right].$$  \hspace{1cm} (25)

But $\delta_\ell$ and $f_\ell$ must be evaluated before these cross sections may be computed.

To evaluate the phase constant $\delta_\ell$, the definition (22) is used:

$$e^{-2i\delta_\ell} = \frac{j_\ell(kR) - j_{\nu}_{\ell}(kR)}{j_\ell(kR) + j_{\nu}_{\ell}(kR)}. \hspace{1cm} (22^*)$$

Substituting (22*) into

$$\tan \delta_\ell = -i \left[ \frac{e^{2i\delta_\ell} - 1}{e^{2i\delta_\ell} + 1} \right]$$

gives

$$\tan \delta_\ell = -j_\ell(x)/n_\ell(x) \hspace{1cm} (26)$$
where $x = kR$.

To evaluate $f_\ell$, consider conditions just inside the surface of the nucleus. Inside the nucleus, the radial Schrödinger equation in terms of $r^{\ell} \psi \propto U_\ell(r, \theta)$ is

$$\frac{\partial^2 U_\ell(r)}{\partial r^2} + \left[ k^2 - \frac{2m}{\hbar^2} V(r) - \ell(\ell+1) \right] U_\ell(r) = 0. \quad (27)$$

Since $V(r) = V_0(l+1)$ for $r \ll R$, let $k^2 = k^2 + (2m/\hbar^2)V_0(l+1)$ so that

$$\frac{\partial^2 U_\ell(r)}{\partial r^2} + \left[ k^2 - \frac{\ell(\ell+1)}{r^2} \right] U_\ell(r) = 0. \quad (28)$$

A solution to this equation is $U_\ell(r) = rK\ell(rR)$.

Then

$$f_\ell = R \left[ \frac{U'_\ell(r)}{U_\ell(r)} \right]_{r=R} = 1 + \frac{X\ell_\ell^0(X)}{J_{\ell^0}(X)} \quad (29)$$

where $X = kR$. Using the relation $j_0(x) = \sin x / x$, we obtain $f_0 = X \cot X$ which can be divided into

$$\text{Re} f_0 = \frac{X_1 \sin 2X_1 + X_2 \sinh 2X_2}{\cosh 2X_2 - \cos 2X_1} \quad (30)$$

and

$$\text{Im} f_0 = \frac{X_2 \sin 2X_1 - X_1 \sinh 2X_2}{\cosh 2X_2 - \cos 2X_1}$$

where $X = X_1 + iX_2$. For other values of $f_\ell$, the recurrence relation $f_\ell = x^2/((\ell + 1) - \ell - \ell)$ is used:
\begin{align*}
\text{Re}_2 &= (x_1^2 - x_2^2) \cdot (\ell - \text{Re}_{\ell-1}) - 2x_1x_2\text{Im}_\ell - (\ell - \text{Re}_{\ell-1})^2 + (\text{Im}_\ell)^2 \\
\text{Im}_\ell &= (x_1^2 - x_2^2)\text{Im}_{\ell-1} + 2x_1x_2(\ell - \text{Re}_{\ell-1}) \\
&= \frac{(x_1^2 - x_2^2)\text{Im}_{\ell-1} + 2x_1x_2(\ell - \text{Re}_{\ell-1})}{(\ell - \text{Re}_{\ell-1})^2 + (\text{Im}_{\ell-1})^2} 
\end{align*}

For the usual real potential, \( V(r) = V_0 \) for \( r < R \), the values of \( K, X, \) and \( f_\ell \) will not be complex. From (23) it can be seen that \( |\eta_\ell|^2 \) will equal unity if \( f_\ell \) is not complex. Then according to the relation (10), \( \sigma_r(\nu) \) must be zero. Thus it can be seen that the total cross section using a real potential will equal the elastic scattering section.
COMPUTATIONS

In order to compute the average neutron total cross sections using a square potential well, a somewhat general program was designed. While the program could employ a variety of parameters, one set of values for R, V₀, and r was chosen in order to make tables of total cross sections. To show how the value of r affects the total cross sections, computations were made for three values of A using several values of r.

The Program

The program used to compute the average total cross sections was designed especially for the ALWAC digital computer. Floating-point arithmetic was used and the available floating-point subroutines were used extensively.

Evaluation of functions. The spherical Bessel functions were evaluated using the recurrence relation

\[ j_{p-1} = \frac{(2p+1)/x}{j_p - j_{p+1}}. \]

In order to find \( j_n(x) \), a value k larger than n or x was chosen. It was assumed that \( j_{k-1} = 0 \), and \( j_k = b \neq 0 \) where b was a small arbitrarily chosen constant. The sequence \( j_{k-1}, j_{k-2}, \ldots, j_1, j_0 \) was generated using the recurrence relation given. If k was chosen sufficiently large, \( j_p = c j \) for any value of p from p=0 to p=n. The fact that \( j_0 = \sin x/x \) was used to determine c from \( j_0 = c j_0 \). The degree of accuracy could be improved by choosing b smaller or k larger.
Spherical Neumann functions were computed in ascending order using the recurrence relation

\[ n_{p+1} = -(2p+1)/(x^2) n_p - n_{p-1}. \]

Starting with \( n_0 = \cos x/x \) and \( n_1 = \sin x/x - \cos x/x^2 \), the sequence \( n_3, n_4, \ldots, n_n \) was generated. (16, p. 256-257)

Polynomial expansions were used to compute \( \alpha_x \) and \( \sigma_p \). These polynomials were derived by using the definition

(21) and

\[ h_x(x) = \frac{e^{ix}}{ix} \sum_{m=0}^{\ell} \frac{(\ell-m-1)!}{\ell!(\ell+m)!} \left( \frac{i}{2x} \right)^m \]

since \( u_x^+(r) = kr \cdot h_x(kr) \). Subroutines were available for all other functions used.

**Input.** Any chosen set of parameters could be inserted into the program using an alternate input routine.

The program normally asked only for values of \( E \) (in Mev) and \( A \) as input. The program started with \( E+1 \) and continued through \( n \) cycles until \( E+n \) was greater than ten. Fractional values of \( E \) could have been used.

**Output.** The value of \( E+1 \) would be typed out first. After "debugging," the other output was limited to the values of \( \lambda \) used to find the average total cross section and the total cross section itself. The program was designed to limit the number of terms summed so that the error resulting from the omission of further terms would be in the order of ten percent. The program could be modified so that the last partial cross section to be calculated would be typed out also.
Values of Parameters

The parameters which had to be evaluated were the nuclear radius $R$, the well depth $V_0$, and the absorption constant $\delta$. Tables were made using these values:

$R = 1.45 \times 10^{-13} \text{ cm}$,
$V_0 = 42 \text{ MeV}$,
$\delta = 0.03$.

These values of the square-well parameters have been found to give reasonably good fit to total cross sections. (4, p. 83)

**Radius.** The effective nuclear radius can be established by scattering experiments. From the inelastic scattering of neutrons, $R$ is found to be (1.4 to 1.5) $\times 10^{-13} \text{ A}^2 \text{ cm}$. (1, p. 944) The value used here is in the middle of that range. The experimental value of the radius is dependent on the assumptions made about the well depth.

**Well depth.** The depth of the potential well best representing the nucleus can be derived from consideration of bound states or from a consideration of free nucleons. These methods give $V_0 = 40 \text{ MeV}$ at $A = 120$. (1, p. 945-949) The various values which have been used for calculating neutron cross sections using a square well have been in general between 40 and 43 MeV. The value used here, 42 MeV, is within this range and has been used previously. (10, p. 51)
Absorption constant. The value $\zeta = 0.03$ was chosen because it seemed to be accepted as resulting in a reasonable fit to total cross section curves at lower energies. (9, p. 147) This "constant" is energy-dependent, however. "The experiments indicate definitely an increase of $\zeta$ with $E$." (18, p. 956) It has been shown that $\zeta$ varies between 0.03 and approximately 0.12 as $E$ varies between one and ten Mev. (12, p. 69-71)(13, p. 1524) Computations were made using values of $\zeta$ in this range for several values of $A$ so that the effect of increasing $\zeta$ could be shown. An increase of $\zeta$ should flatten the maxima and minima. It has been found that at the higher energies of our range the calculated cross sections are only mildly dependent on the parameters $V_0$ and $\zeta$; and therefore the best fit parameters can not be determined very sharply. (12, p. 75)

Number of Terms Summed

Because of limits on program size and the computer time involved, a maximum of seven terms were summed. That a sum of partial cross sections through the one involving $\lambda=6$ should nearly be sufficient for the energy range covered can be seen from the following argument. While a classical argument is used here, the same result can be obtained by using quantum mechanics.

If we again consider the incident neutrons to be a
plane wave, we may divide up the incident beam into cylindrical zones. Let the width of each zone be $\alpha$. Then if the impact parameter is defined to be the distance between the center of the incident neutron and the center of the target nucleus, the innermost zone will contain all neutrons with impact parameters less than $\alpha$. The $\ell$th zone will contain all neutrons with impact parameters between $\ell \alpha$ and $(\ell+1)\alpha$. (3, p. 318) Then if the nuclear radius is less than $\xi$, the $\ell$th partial wave will hardly be affected by the potential, the phase shift will be very small and the contribution to the scattering will be negligible. It follows then that the scattering cross section consists of a series of terms extending from $\ell=0$ to a maximum that is of the order of $kR$ or $\xi$. (15, p. 106) Therefore, because the maximum value of $\xi$ required for the computations was 0.35, summing the partial cross sections through the one involving $\ell=6$ should have been nearly sufficient.

Because the computer time required to calculate and sum seven partial cross sections was about two minutes, it was necessary to avoid computing those terms which would result in a very small contribution to the total cross section. For given values of $E$ and $A$, the partial cross sections will increase to some maximum value as $\ell$ increases, and then decrease with further increases in $\ell$. 
Therefore the following method for limiting the number of terms summed was used in the program.

When each partial cross section was computed and added to the sum of those found previously, it was also compared to one tenth of the new sum. If the nth partial cross section was greater than one tenth of the sum of partial cross sections through \( \ell = n \), the partial cross section involving \( \ell = n+1 \) was computed. This procedure was repeated until the last partial cross section was less than one tenth of the sum and then the sum was typed out.

The argument above indicated that the contribution to the scattering cross section is very small for those terms involving values of \( \ell > x \), and that the value of these terms would decrease quite rapidly as \( \ell \) becomes much larger than \( x \). Therefore if the total cross section that was typed out was the sum of the partial cross sections from \( \ell = 0 \) to \( \ell = m \), the terms for \( \ell = m+1 \) will be smaller than that for \( \ell = m \). The term involving \( \ell = m \) is less than one tenth of total cross section that was typed out, therefore if the values of the terms for \( \ell > m \) decrease fairly rapidly, the error in neglecting them should be about ten percent.

There were some cases where the partial cross section for \( \ell = 6 \) was greater than one tenth of the sum of the partial cross sections, then the output was modified so
that the partial cross section for $l=6$ was typed out along with the sum. This would give some basis for estimating the error of neglecting further terms.
RESULTS

Average neutron cross sections for values of $E$ between 1 and 10 Mev and values of $A$ between 1 and 250 will be found in Table I in the appendix. Several graphs have been made to show how these cross sections compare with experimental values and the data for these will be found in Table II. Three of these graphs compare the cross sections resulting when different values of $\zeta$ are used. The starred values in the tables indicate those cases where the optional output was necessary and, therefore the error in these values will be greater than ten percent.

Discussion of Figures

Figures 1 and 2 show average neutron cross sections versus $E$ for values of $A=20$ where poor agreement may be expected. The calculated values are found to be too small. From Figure 1 it can be seen that using a smaller value of $\zeta$, $\zeta=0.01$, increases the calculated values, but only slightly.

Figure 3 shows average neutron cross sections versus $E$ for $A=27$ for each of three different values of $\zeta$: $\zeta=0.01$, 0.03, and 0.05. The curves of the calculated values correspond much more closely to the experimental curve than those in Figures 1 and 2. It can be seen that larger values of $\zeta$ broaden the peak and probably lower it. The curves for
Figure 1. Average total cross sections for $A = 1$

- $\Delta$ (11, P = 74)
- $\triangledown$ $\zeta = 0.03$
- $\square$ $\zeta = 0.01$
Figure 2. Average total cross sections for $A = 6$

$\sigma_t$ - BARNs

$E$ - MEV

$\triangle$ (11, P. 86)

$\circ$ $\zeta = 0.03$
Figure 3. Average total cross sections for $A = 27$.

- $(11, P = 119)$
- $\zeta = 0.01$
- $\zeta = 0.03$
- $\zeta = 0.05$
the three values of $\zeta$ are very close together. At the lower values of $E$, the curve using $\zeta = 0.01$ seems to be a closer fit to the experimental curve, but the curves using the higher values of $\zeta$ fit better at higher values of $E$. Therefore this figure would seem to indicate an increase of $\zeta$ with an increase of $E$.

Figures 4, 5, and 6 are for $A=60$, $A=107$, and $A=197$ respectively. The curves of the calculated values show that more peaks appear as $A$ increases. These peaks are too large and are displaced with respect to those in the experimental curves. Because the curves are drawn through a minimum of points it is hard to define the peaks. Therefore these curves were drawn by referring to Table I and considering the way in which the cross sections vary with $E$ and $A$.

Figure 7 contains curves for $A=235$. Here the optional output had to be used for $E=8\text{MeV}$. The error in these values resulting from summing only seven terms was estimated to be about 20 percent except near the maximum of the third peak where the error was at least 35 percent. Here the curves of the calculated cross sections have three sharp peaks while the curve of the experimental data has only one broad peak. These peaks in the curves of the calculated cross sections are placed approximately where maximum values of the elastic scattering cross sections would be expected.
Figure 4. Average total cross sections for $A = 60$
Figure 5. Average total cross sections for \( A = 107 \)
Figure 6. Average total cross sections for $A = 197$. 

\[ \bar{\sigma}_t - \text{Barns} \]
Included in Figure 5 is a curve using values obtained with $\xi = 0.10$ and in Figure 7 one with $\xi = 0.30$. These curves are very near the curves using $\xi = 0.03$. At energies somewhat higher than two Mev, the total cross sections become less sensitive to the constants and as $E$ increases they have very little dependence on the imaginary part of the potential. (9, p. 155)

This set of figures has shown that the values calculated for the average neutron total cross sections are of the correct order of magnitude when compared with experimental results. It has indicated that $\xi$ is energy dependent since at the lower values of $E$ the curves made using the lower values of $\xi$ best fit the experimental curves. At the higher values of $E$, the curves of the calculated cross sections seem to be nearly the same for all values of $\xi$ used. The curves of the calculated cross sections show more pronounced maxima and minima than the curves of the experimental values of neutron total cross sections.

Square Well Approximation

The curves show that the calculated cross sections are of the correct order of magnitude. Better correspondence between the calculated and the experimental values of neutron total cross sections would be obtained if the diffuse surface of the nucleus were taken into consideration.
Figure 7. Average total cross sections for $A = 235$.

$\Delta$ (11, p. 335)
$\circ$ $\zeta = 0.03$
$\square$ $\zeta = 0.30$
An example of a rounded well uses the form of potential:

\[ V(r) = 0, \quad r > R \]
\[ V(r) = V_0 (1 + i\xi) \left[ 1 + \exp \left( \frac{(r-r_0)}{a} \right) \right]^{-1}, \quad r \leq R \]

where \( r_0 \) is a measure of nuclear size and \( a \) is a measure of the width of rounding. (19, p. 57) Calculations of total cross sections using this well approximate the experimental values more closely than those using a square well. Some of the peaks in the cross section curves are modified by the use of this form of potential. The reflectivity of the square well is too large causing the cross section for the formation of the compound nucleus to be too small. The values of \( \sigma_2 \) are larger when a rounded well is used, indicating that the reaction cross section is sensitive to the nature of the surface of the nucleus. (4, p. 83) Then a rounded well would be expected to decrease the size of the peaks which result from elastic scattering. A number of similar potentials have been used to account for the diffuse nuclear surface. (4, p. 78-100)

It can be seen that the assumption that the calculations were carried out in the continuum region where the compound elastic scattering cross section is negligible is nearly correct. Only in Figure 3 can one see rapid fluctuation of the experimental curve with \( E \). It was also assumed that the omission of the spin-orbit terms would
not affect the overall qualitative features of the total cross sections. Inclusion of these terms would improve the quantitative agreement with experimental values. This is especially noticeable at higher energies. (2, p. 1298) Summing more terms would reduce the error especially for higher values of $E$ and $A>200$. But these are relatively unimportant until better qualitative agreement is reached.

While the curves of the calculated values may over-emphasize the fluctuations of the total cross sections with $E$ and $A$, it must be remembered that these values were obtained by just adding a complex part to the nuclear potential and that the same computations using a purely real potential would result in a monotonically decreasing function of $E$ whose form is rather similar for all values of $A$. (5, p. 449)
SUMMARY

The results may be summarized as follows:

1. The calculated values of the average neutron total cross sections are of the proper order of magnitude and display E and A relationships similar to those found experimentally, even though several simplifying approximations were made.

2. The average neutron total cross sections for higher values of E and A are not very sensitive to the values of E and A increase.

3. While a reasonably general program can be designed for a computer similar to the ALWAC for the calculation of these cross sections, time becomes an important factor in limiting the number of trials which may be made. This is a result of the increasingly large number of terms which must be summed as E and A increase.

4. Taking the diffuse nuclear surface into consideration by using a rounded well should result in values of average neutron total cross sections closer to the experimental values. Because this would involve a more intricate form of potential, the resulting program would be very complicated and therefore very time consuming.


APPENDIX
Table I
Neutron Total Cross Sections
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Neutron Total Cross Sections
\( V(r) = \gamma_2 (1 + 0.03i) \)

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Data for Figures

\[ V(r) = V_0 (1 + i \xi) \]

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Spherical Bessel-type Functions

The spherical Bessel-type functions satisfy the following equation:

\[
\frac{d^2 f}{dz^2} + 2 \frac{df}{dz} + \left[1 - \frac{n(n+1)}{z^2}\right] f = 0
\]

Definitions

**Spherical Bessel functions**

\[ j_n(z) = (\pi/2z)^{1/2} J_{n+\frac{1}{2}}(z) \]

**Spherical Neumann functions**

\[ n_n(z) = (\pi/2z) N_{n+\frac{1}{2}}(z) \]

**Spherical Hankel functions**

\[ h_n(z) = j_n(z) \pm in_n(z) \]

Asymptotic Relations

\[ j_n(z) \rightarrow (1/z) \cos \left[\frac{z}{2} - \frac{1}{2} \pi(n+1)\right] \]

\[ n_n(z) \rightarrow (1/z) \sin \left[\frac{z}{2} - \frac{1}{2} \pi(n+1)\right] \]

\[ h_n(z) \rightarrow (1/z) \Gamma(n-1) e^{iz} \]

Zero Order Values

\[ j_0 = (\sin z)/z \]

\[ n_0 = (\cos z)/z \]

\[ h_0 = (-i/z) e^{iz} \]

In terms of Legendre polynomials

\[ j_n(z) = (1/2^n) \int_0^\pi e^{iz\cos \phi} P_n(\cos \phi) \sin \phi \, d\phi \]