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The main result of this dissertation provides rather general conditions under which extensions of zero order propositional calculi inherit the property of having a finite characteristic model. This result is applied to show that if a calculus T is a normal extension of the Heyting calculus H, and if T has a finite characteristic model, then every normal extension of T has a finite characteristic model. Equivalent statements are shown to hold for the normal extensions of the implicational fragment of H, as well as for various extensions of the modal logics S2 and E2.

Discussion of these results is preceded by a detailed study of the properties of finite order Lindenbaum models. It is followed by a summary of known results and methods on the existence or nonexistence of finite characteristic

models. Finally, there is a study of completeness, which provides sufficient conditions on the first order Lindenbaum model of a calculus for the calculus to be complete. (A calculus is said to be complete if its only normal extensions are itself and the calculus in which all words are theorems.)

FINITE MODELS OF ZERO ORDER

PROPOSITIONAL CALCULI

by

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CHAPTER I

INTRODUCTION

This dissertation is concerned with propositional calculi (zero order calculi) and their models. Propositional calculi are described by giving a procedure for generating the elements of the calculus, called its theorems. An immediate problem for any calculus is the "decision problem": Is there an algorithm for deciding whether or not an arbitrary proposition is a theorem? In 1921, Post [21] solved this problem for the Classical calculus by showing that a proposition is a theorem of this calculus if and only if it is valid in the "Classical truth tables." Whenever the decision problem for a calculus has a solution of this kind, the calculus is said to have a finite characteristic model. Thus, the "Classical truth tables" provide a finite characteristic model of the Classical calculus. After 1930, many other philosophically interesting calculi were presented, including the Intuitionist calculus H of Heyting [11] and Lewis' modal logics [15]. Gödel [9] first showed that H does not have a finite characteristic model. Dugundji [5] obtained the same result for the modal logics.

A solution of the decision problem for the Intuitionist calculus H was obtained by Jaskowski [23], who showed that there is a sequence of special finite models such that the theorems of H are precisely those propositions which are valid in all of the models. Similar results were obtained by McKinsey []9] and others for the modal logics. Harrop [10] showed that the techniques used by Jaskowski and McKinsey are essentially the same, the special models used being what he calls "strong models." Harrop also showed that although the existence of such a sequence of strong models for any suitably presented calculus ensures that the calculus is decidable, the converse is not true. Calculi for which such sequences exist are said to have the "finite model property."

The work of Harrop [10], Scroggs [24], Lemmon [16], and others suggest the following conjecture: If a calculus has the finite model property then all its "extensions" also have this property. This conjecture has been shown to hold in various special cases, and Ulrich [28] has announced a general result in which "finite model property" is replaced by the more specialized "finite

Lindenbaum property" (see Remark 4.4). Naturally associated with each calculus is a family of "Lindenbaum models" which, if finite, serve as strong models for the "finite model property" above. Unfortunately, the Intuitionist calculus - a calculus which has the finite model property - escapes the range of application of the above results. An incomplete proof of the above conjecture for the Intuitionist calculus was given in [7].

The conjecture above provided much of the motivation for our work. In addition to an independent proof of Ulrich's result, the discussion chiefly concerns the existence of finite characteristic models - a stronger condition than either the "finite model property" or the "finite Lindenbaum property." The main result of Chapter IV (Proposition 4.13), provides rather general conditions under which extensions of a calculus inherit the property of having a finite characteristic model. In Chapter V, it is shown how this result applies to extensions of several familiar calculi: the Intuitionist calculus (Proposition 5.6), the implicational fragment of the Intuitionist calculus (Proposition 5.14) and the modal systems (Proposition 5.16). Results from the literature on existence (usually nonexistence) of finite characteristic models, together with some applications of our results to nonexistence, are collected at the end of

Chapter V.

A second problem for propositional calculi is that of "completeness": Does a given calculus have an "extension", other than itself, which is consistent in the sense that not every proposition is a theorem? If not, then the calculus is "complete". Apart from the Classical calculus, most familiar calculi are not complete. In Chapter VI our result on Lindenbaum models are applied to the completeness problem. Conditions are given which, for practical purposes, are necessary and sufficient for completeness (Propositions 6.7 and 6.8). As a final application, we show that the implicational fragment of the Classical calculus is complete.

Our results are obtained for general structures which we call "standard" propositional calculi. (The word "standard" is suggested by Ulrich's use of this term; however our meaning of "standard" is somewhat more inclusive than his.) The author is unaware of any philosophically interesting calculi discussed in the literature which are not standard. Some of our results require that a calculus have a distributive lattices of congruences (DLC calculus); a property which is shared by most familiar calculi (see Proposition 5.5).

To avoid many of the problems associated with the fact that various calculi have different sets of logical connectives, we borrow heavily from the terminology of universal algebra (see [3]). Chapter II deals with the necessary algebraic preliminaries, including Ω-algebras, their lattices of congruences, and word algebras. In Chapter III we discuss general properties of calculi and their models. These results are based on a new characterization of the Lindenbaum congruence, as the greatest congruence which preserves the set of theorems (Proposition 3.4). This allows us explicitly to construct isomorphs of the finite order Lindenbaum models for those calculi having finite characteristic models (Proposition 3.16). In applying these results, the strategy is to establish, for the Lindenbaum congruences, analogs of various properties of the propositional calculi. For example, if a calculus T' is a "standard extension" of T, then the Lindenbaum congruence of T' contains that of T (Lemma 4.1). This provides natural homomorphisms between the corresponding finite order Lindenbaum models for T and T', which serve to transfer properties of T to any such extension T'.

To assist the reader, an index of definitions is included (p. 88).

CHAPTER II

ALGEBRAIC PRELIMINARIES

An algebra is to be thought of as a set with certain operations defined on it. Since we will need to compare different algebras, the most convenient way of doing this is to index the operations in each algebra by a given indexing set, which is kept constant in any problem under discussion. An operator-domain is a nonempty set Ω with a mapping n of Ω into the set of natural numbers. The elements of Ω are called operators. If $n(\omega) = 1$ or 2 then $\omega \in \Omega$ is said to be unary or binary respectively; if $n(\omega) = k$ then ω is k-ary.

Notational convention. To simplify the notation we will consistently use the symbol n in place of $n(\omega)$, for an arbitrary $\omega \in \Omega$.

Let A be a nonempty set and Ω an operator domain; then a Ω-algebra structure on A is a family of mappings such that with each (n-ary) operator $\omega \in \Omega$ is associated a function (operation on A) from A^n into A, where A^n is the set of all n-tuples of elements of A. The set A, together with an Ω-algebra structure on A, is called an _Ω-algebra_. The symbol $\mathbf{A}$ will be used to represent an Ω-algebra whose underlying set is A. Except in the discussion of word algebras (see below) we shall not distinguish between an operator ω and its corresponding operation.

Given an Ω-algebra A, let B be a subset of A such that for each n and each $\omega \in \Omega$ the operation ω maps B^n into B. Then the set B together with the operations of A restricted to B is an Ω-algebra; such Ω-algebras are called _subalgebras_ of A.

Let A and B be two Ω-algebras with the same Ω, and σ be a mapping from the set A into the set B. The mapping σ is called a _homomorphism_ from A into B if and only if for all $\omega \in \Omega$ and all $a_1, \ldots, a_n \in A$,

$$\sigma(\omega(a_1, \ldots, a_n)) = \omega(\sigma(a_1), \ldots, \sigma(a_n)).$$

2.1 Remark. Let σ be a homomorphism from the Ω-algebra A into the Ω-algebra B. Then the image of A under the homomorphism is a subalgebra of B. This follows easily from the preceding definitions. It also follows easily that the composition of two homomorphisms is a homomorphism.

By giving fixed values in A to all but one of the arguments of an operation ω, we obtain a mapping $t: A \to A$,

$$t(x) = \omega(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$$

which we call an elementary translation on A. Let Q be an equivalence relation on the underlying set A of an Ω-algebra. A mapping $t: A \to A$ and an equivalence Q are said to be compatible if, whenever $(a, b) \in Q$ we have $(t(a), t(b)) \in Q$. The equivalence relation Q is a congruence on the Ω-algebra A if and only if all elementary translations on A are compatible with Q; i.e., if and only if $(a, b) \in Q$ implies

$$(\omega(c_1,\ldots,c_{i-1},a,c_{i+1},\ldots,c_n),\omega(c_1,\ldots,c_{i-1},b,c_{i+1},\ldots,c_n)) \in Q.$$

2.2 Remark. By the transitive property of a congruence Q, one can show that for each $\omega \in \Omega$ and all sequences $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that $(a_i, b_i) \in Q$, $i = 1, \ldots n$, we have

$$(\omega(a_1, \ldots, a_n), \omega(b_1, \ldots, b_n)) \in Q.$$

If $a \in A$ and Q is a congruence, we denote the equivalence class of a by

$$|a| \bmod Q = \{x \in A \mid (x, a) \in Q\}.$$

If B is a subset of an Ω-algebra A and Q is a congruence we write

$$|B| \bmod Q = \{x \in A \mid (x, b) \in Q \text{ for some } b \in B\}$$

If Q is a congruence then A/Q denotes the usual quotient Ω-algebra, whose underlying set is the set of equivalence classes, and whose operations are defined in the usual way (see remark 2.2). The mapping $h:A \to A/Q$ given by

$$h(a) = |a| \bmod Q$$

is a homomorphism, called the <u>natural homomorphism</u> onto the quotient algebra. If K is any subset of A, we shall also write K/Q for $h(K)$, the set of equivalence classes of members of K.

Let A and B be Ω-algebras, and σ be a homomorphism from A into B. Then the equivalence relation

$$\ker \sigma = \{(x, y) \in A^2 | \sigma(x) = \sigma(y)\}$$

is easily shown to be a congruence on A, called the <u>kernel of σ</u>.

A congruence Q on the Ω-algebra A is said to <u>preserve</u> a subset K of A if $|K| \bmod Q = K$. This means that K is the union of equivalence classes of its members, and is therefore equivalent to the following condition:

if $(x, y) \in Q$ and $y \in K$ then $x \in K$.

It turns out that for each subset K of A, there is a greatest congruence (in the sense of inclusion) which preserves K. We describe a technique for constructing such congruences. Let Γ be any set of mappings from A into A such that:

<u>2.3a</u> Γ contains the identity map and all elementary

translations on A.

<u>2.3b</u> Γ is closed with respect to functional composition.

<u>2.3c</u> Every member of Γ is compatible with each congruence of A. If K is any subset of A, we write

$$Q(K, \Gamma) = \{(x, y) \in A^2 \mid \text{ for all } t \in \Gamma, \\ t(x) \in K \text{ if and only if } t(y) \in K\}$$

<u>2.4 Lemma</u>. Every relation $Q(K, \Gamma)$, as above, is a congruence on the Ω-algebra A. Indeed, $Q(K, \Gamma)$ is the greatest congruence Q which preserves K.

<u>Proof:</u> Clearly $Q(K, \Gamma)$ is an equivalence. The fact that each elementary translation is compatible with this equivalence follows from the fact that Γ contains each elementary translation and is closed under functional composition. Thus $Q(K, \Gamma)$ is a congruence.

Now write $Q = Q(K, \Gamma)$. To show that Q preserves K, suppose that $(x, y) \in Q$ for some $y \in K$. Since Γ contains the identity map, we have $x \in K$.

Finally, let Q' be any congruence which preserves K. To show that $Q' \subseteq Q$, let $(x, y) \in Q'$ and t be any member of Γ. Since t is compatible with Q', we have $(t(x), t(y)) \in Q'$. If $t(x) \in K$ then $t(y) \in |K| \bmod Q' = K$, and vice versa. Thus $t(x) \in K$ if and only if $t(y) \in K$, and $(x, y) \in Q$.

2.5 Remark. Given any Ω-algebra A, there clearly exists a smallest family of functions Γ satisifying conditions 2.3; namely, the closure under functional composition of the elementary translations plus the identity. Thus Lemma 2.4 ensures, for each subset K of A, the existence of the greatest congruence Q which preserves K. This congruence can also be constructed as the join of all congruences preserving K in the lattice of congruences (see below). Lemma 2.4 also shows that the construction $Q(K, \Gamma)$ is independent of Γ.

We now construct a special family of mappings, satisifying conditions 2.3, which is needed for later application. Let A be an Ω-algebra and X be any nonempty set. Let $Fn(X, A)$ denote the set of all mappings from X into A. An Ω-algebra structure on $Fn(X, A)$ can be defined as follows: For every operator $\omega \in \Omega$, $\omega(f_1, \ldots, f_n)$ is the function $h \in Fn(X, A)$ given by $h(x) = \omega(f_1(x), \ldots, f_n(x))$ for all $x \in X$. Now take $X = A$, and let $G_0 \subseteq Fn(A, A)$ denote the set of constant maps plus the identity map on A. Then define $G_m \subseteq Fn(A, A)$ recursively, for $m > 0$, by

$$G_{m+1} = G_m \cup \{g \in Fn(A, A) \mid g = \omega(h_1, \ldots, h_n)$$
$$\text{for some } \omega \in \Omega,\ h_1, \ldots, h_n \in G_m\}.$$

Let $$A^* = \bigcup_{m=0}^{\infty} G_m$$

<u>2.6 Lemma</u>. The set of mappings A^*, defined above, meets conditions 2.3.

<u>Proof</u>: Clearly, A^* is an Ω-algebra - a subalgebra of $Fn(A, A)$. It is also immediate that every elementary translation on A is a member of G_1, and therefore a member of A^*.

We now show by induction that A^* is closed under functional composition. If $f \in A^*$ and $g \in G_0$ then the composition $g \circ f$ is either a constant map or the mapping f. Thus $g \circ f \in A^*$. Let $f \in A^*$ and $g \in G_{m+1} - G_m$. Then $g = \omega(h_1, \ldots, h_n)$ where $h_i \in G_m$, $i = 1, \ldots, n$. By the induction hypothesis $h_i \circ f \in A^*$. Clearly $g \circ f = \omega(h_1 \circ f, \ldots, h_n \circ f)$, and since A^* is an Ω-algebra it follows that $g \circ f \in A^*$.

Finally, we show by induction that every member of A^* is compatible with every congruence Q; i.e., $(a, b) \in Q$ implies $(g(a), g(b)) \in Q$ for all $g \in A^*$. This is clearly true if $g \in G_0$. Let $g \in G_{m+1} - G_m$, say $g = \omega(h_1, \ldots, h_n)$ where $h_i \in G_m$. Now by the induction hypothesis $(a, b) \in Q$ implies $(h_i(a), h_i(b)) \in Q$, $i = 1, \ldots, n$, whence in view of Remark 2.2, $(g(a), g(b)) \in Q$.

We now present three isomorphism theorems; the first two are convenient versions of the first and third isomorphism theorems of Cohn [3]. A homomorphism f from A onto B is an <u>isomorphism</u> between the Ω-algebras A and B if f has an inverse which is also a homomorphism.

<u>2.7 Proposition</u>. Let the mapping f from A onto B be a homomorphism between the Ω-algebras A and B, and let Q be the kernel of f. Then the mapping $g: A/Q \to B$, given by $g(|a| \bmod Q) = f(a)$, is an isomorphism between the quotient algebra A/Q and B.

<u>2.8 Proposition</u>. Let A be an Ω-algebra, and Q, Q' be congruences on A such that $Q \subseteq Q'$. Then the mapping $k: A/Q \to A/Q'$, given by $k(|a| \bmod Q) = |a| \bmod Q'$, is a homomorphism of the quotient algebra A/Q onto A/Q'.

<u>2.9 Lemma</u>. Let A, B be Ω-algebras, K be a subset of A, and h be a homomorphism of A onto B whose kernel preserves K. Let Q and Q' be the greatest congruences which preserve K and $h(K)$ in the algebras A and B respectively. Then there is an isomorphism g between A/Q and B/Q' such that for every subset M of A, $g(M/Q) = h(M)/Q'$.

<u>Proof</u>: Let k be the natural homomorphism of B onto B/Q'. It is shown below that the kernel of $k \circ h$ coincides with Q. Since $k \circ h$ maps A homomorphically onto B/Q', it follows by Proposition 2.7 that the function g given by

$$g(|a| \bmod Q) = k(h(a)) = |h(a)| \bmod Q'$$

is an isomorphism between A/Q and B/Q'. Clearly $g(M/Q) = h(M)/Q'$ for all $M \subseteq A$.

By Proposition 2.7, the mapping $h^{-1}: B \to A/\ker h$ is an isomorphism. Composing this with the homomorphism from $A/\ker h$ onto A/Q guaranteed by Proposition 2.8 (since $\ker h \subseteq Q$) we obtain a homomorphism f from B onto A/Q, given by $f(h(a)) = |a| \bmod Q$. Thus $f \circ h = j$, where j is the natural homomorphism of A onto A/Q. Therefore $Q = \ker j = \ker f \circ h$, and it remains to show that $\ker f \circ h = \ker k \circ h$.

To show that $\ker f \circ h \subseteq \ker k \circ h$, it is clearly sufficient to show that $\ker f \subseteq \ker k$. Since $Q' = \ker k$ is the maximal congruence which preserves $h(K)$, it suffices to show that $\ker f$ preserves $h(K)$. Now if $(h(a), h(b)) \in \ker f$ for some $b \in K$, then $f(h(a)) = f(h(b))$, or $j(a) = j(b)$, so that $(a, b) \in \ker j = Q$. Since Q preserves K, it follows that $a \in K$, whence $h(a) \in h(K)$.

Similarly, to show that $\ker k \circ h \subseteq \ker f \circ h \ Q$, it suffices to show that $\ker k \circ h$ preserves K. Suppose $(a, b) \in \ker (k \circ h)$ for some $b \in K$. Then $(h(a), h(b)) \in \ker k$, and since $\ker k = Q'$ preserves $h(K)$, it follows that $h(a) \in h(K)$. Since, by hypothesis, the kernel of h preserves K, we have $a \in K$.

The reader will recall [1] that a lattice is a partially ordered set in which each pair of elements a, b has a least upper bound (or join) $a \wedge b$, and a greatest lower bound (or meet) $a \vee b$. A lattice is complete if every indexed family $a_\sigma : \sigma \in \Sigma$ has a least upper bound (join) $\bigvee\{a_\sigma : \sigma \in \Sigma\}$ and a greatest lower bound (meet) $\bigwedge\{a_\sigma : \sigma \in \Sigma\}$; it is distributive if either of the identities

$$a\wedge(b\vee d) = (a\wedge b)\vee(a\wedge d), \quad a\vee(b\wedge d) = (a\vee b)\wedge(a\vee d)$$

holds (in which case they both hold).

It is known [1] that the set of congruences of an Ω-algebra, ordered by inclusion, is a complete lattice. In this lattice the meet of a family of congruences is simply the intersection, while the union is contained in the join.

We now give two results about such lattices.

2.10 Proposition. Let A be an Ω-algebra and Q be a congruence on A. Then the lattice of congruences on the quotient algebra A/Q is isomorphic to the lattice of all congruences on A which contain Q.

<u>Proof</u>: For this proof we will write $|x|$ instead of $|x|$ mod Q. Let Δ be the lattice of congruences on A/Q, and D be the set of congruences Q' on A such that $Q \subseteq Q'$. We define a map $f:\Delta \to D$, given by

$$f(\theta) = \{(x, y) \in A^2 | (|x|, |y|) \in \theta\}.$$

To verify that $f(\theta)$ is a congruence on A, observe that $f(\theta)$ is the kernel of the homomorphism from A to $(A/Q)/\theta$ formed by composing the natural homomorphism from A into A/Q and from A/Q into $(A/Q)/\theta$. Clearly the function g defined on D by

$$g(Q') = \{(|x|, |y|) \in (A/Q)^2 | (x, y) \in Q'\}$$

is a left inverse; $g(f(\theta)) = \theta$. Hence f is one-to-one. To show that f is onto D, let $Q' \in D$, and let k be the homomorphism of A/Q onto A/Q' as in Proposition 2.8; $k(|x|) = |x|$ mod Q'. The kernel of k is a congruence on A/Q, whose image under f is clearly equal to Q'. Thus f is one-to-one onto D, with inverse g. In view of the above formulas, f and g are both order-preserving; hence D is a lattice, and f is a lattice isomorphism.

An Ω-algebra A is called a <u>lattice with operators</u> if A contains a pair of operators whose corresponding operations on A are the meet and join of a lattice structure.

2.11 Proposition. If the Ω-algebra A is a lattice with operators, then the lattice of congruences on A is a distributive lattice.

A slightly stronger version of this proposition holds; the lattice of congruences is "Brouwerian" in the sense that the infinite distributive law $Q \wedge \bigvee Q_\sigma = \bigvee (Q \wedge Q_\sigma)$ holds (though its dual may not hold). The proof is the same as that given in Birkhoff [1] (Theorem 9, page 138).

We conclude this chapter with a basic discussion of word algebras. Let Ω be an operator domain and X be a nonempty set which is disjoint from Ω. The word algebra $W(X, \Omega)$ is a particular Ω-algebra, defined as follows: Let $S = \bigcup_{k=1}^{\infty} (X \cup \Omega)^k$. The elements of S are called strings, and are written without commas or parentheses. Let $J_0 = X$ and let J_m, $m > 0$, be defined recursively by

$$J_{m+1} = J_m \cup \{\alpha \in S \mid \alpha = \omega\alpha_1 \ldots \alpha_n \text{ for some } \omega \in \Omega, \text{ with } \alpha_1, \ldots, \alpha_n \in J_m\} .$$

The underlying set of $W(X, \Omega)$ is $\bigcup_{m=0}^{\infty} J_m$. The elements of $W(X, \Omega)$ are called words, and will be denoted by Greek letters $\alpha, \beta, \gamma, \ldots$.

The operations of the word algebra are defined by taking $\omega(\alpha_1, \ldots, \alpha_n)$ to be the string $\omega\alpha_1 \ldots \alpha_n$. A word α of $W(X, \Omega)$ is sometimes written $\alpha(v_1, \ldots, v_n)$ to indicate that $v_1, \ldots, v_n$ are precisely those distinct members of X occuring in α. The <u>rank</u> of a word α in $W(X, \Omega)$ is m if $\alpha \in J_m$ but $\alpha \notin J_i$ for $i < m$.

We now give for easy reference some results concerning word algebras. The following proposition, although easily proved, may be found in [3].

<u>2.12 Proposition.</u> Let A be an Ω-algebra and X be an arbitrary nonempty set. Then each mapping $f:X \to A$ extends uniquely to a homomorphism $f^*:W(X, \Omega) \to A$.

<u>2.13 Proposition.</u> Let g and h be two homomorphisms of $W(X, \Omega)$ into an Ω-algebra A such that the range of h contains the range of g. Then there is a homomorphism s of $W(X, \Omega)$ into itself such that g is the composition of h with s (i.e. $g(\alpha) = h(s(\alpha))$ for all $\alpha \in W(X, \Omega)$).

<u>Proof:</u> Choose any function f on X into $W(X, \Omega)$ such that $f(x) \in h^{-1}(g(x))$ for all x. By Proposition 2.12, f has an extension to a homomorphism s of $W(X, \Omega)$ into itself. The composition $h \circ s$ is a homomorphism into A which coincides with g on X. Hence

by the uniqueness in Proposition 2.12 we have $g = h \circ s$.

<u>2.14 Remark.</u> The question of whether a word algebra $W(X, \Omega)$ can be mapped homomorphically onto a given Ω-algebra A can be answered affirmatively if A has a <u>generating set</u> (i.e., a subset which is not contained in any proper subalgebra) of cardinality less than or equal to the cardinality of X. In view of Remark 2.1, one need only map X arbitrarily onto the generating set, and apply Proposition 2.12.

<u>2.15 Proposition.</u> If X is a subset of Y then $W(X, \Omega)$ is a subalgebra of the word algebra $W(Y, \Omega)$.

<u>Proof:</u> Let J'_m and J_m be the sets of strings used to define $W(X, \Omega)$ and $W(Y, \Omega)$ respectively. Clearly $J'_0 \subseteq J_0$, and one can show by a simple inductive argument that $J'_m \subseteq J_m$ for all m. Thus $W(X, \Omega) \subseteq W(Y, \Omega)$.

If $\omega \in \Omega$ then $\omega(\alpha_1, \ldots, \alpha_n) = \omega\alpha_1 \ldots \alpha_n$ both for $\alpha_1, \ldots, \alpha_n \in W(X, \Omega)$ and for $\alpha_1, \ldots, \alpha_n \in W(Y, \Omega)$. Therefore, the operations of $W(Y, \Omega)$ restricted to $W(X, \Omega)$ agree exactly with the operations on $W(X, \Omega)$. Hence the Ω-algebra $W(X, \Omega)$ is a subalgebra of $W(Y, \Omega)$.

<u>2.16 Proposition.</u> Let X be a subset of Y and A be an Ω-algebra. Then a mapping σ from the word

algebra $W(X, \Omega)$ into A is a homomorphism if and only if σ is the restriction of some homomorphism from $W(Y, \Omega)$ into A.

<u>Proof</u>: Clearly, if σ is the restriction of a homomorphism $\sigma':W(Y, \Omega) \to A$, then σ is a homomorphism from $W(X, \Omega)$ into A.

Let $\sigma:W(X, \Omega) \to A$ be a homomorphism, ρ be the restriction of σ to X, and ρ' be any extension of ρ to Y. By Proposition 2.9 ρ' has a unique extension to a homomorphism $\rho^*:W(Y, \Omega) \to A$. By the last paragraph, the restriction of ρ^* to $W(X, \Omega)$ is a homomorphism $\rho^*:W(X, \Omega) \to A$. This homomorphism is equal to σ, by the uniqueness in Proposition 1.9, since it coincides with σ on X.

CHAPTER III

PROPOSITIONAL CALCULI AND THE LINDENBAUM MODEL

A _substitution_ is a homomorphism from some word algebra $W(X, \Omega)$ into itself. A substitution is determined by its values on X, as follows from Proposition 2.12.

Let $Y = \{p, q, r, \ldots\}$ be a countable infinite set which is to be fixed for the remainder of this dissertation. The members of Y will be called _propositional variables_. Let Ω also be regarded as fixed, although Ω may change from one application to another. A _propositional calculus_ is a nonempty subset T of the set of words $W(Y, \Omega)$, such that for every substitution s, $s(T) \subseteq T$. The members of the subset T are called _theorems_ of the calculus.

We illustrate these definitions with two examples of well-known propositional calculi, the intuitionist calculus of Heyting, and the classical calculus. In each case $\Omega = \{C, D, K, N\}$, where C, D, and K are binary operators representing implication $(\supset)$, disjunction $(\vee)$ and conjunction $(\wedge)$ respectively, and N is a unary operator representing negation ($\sim$). The following propositions:

$$p \supset (q \supset p)$$

$$(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$$

$$p \wedge q \supset p$$

$$p \wedge q \supset q$$

$$(p \supset q) \supset ((p \supset r) \supset (p \supset q \wedge r))$$

$$p \supset p \wedge q$$

$$q \supset p \wedge q$$

$$(p \supset r) \supset ((q \supset r) \supset (p \vee q \supset r))$$

$$(p \supset (\sim q)) \supset (q \supset (\sim p))$$

$$(\sim p) \supset (p \supset q)$$

which are "obvious" theorems of the classical calculus, take the following form in the notation of Chapter II for word algebras (Polish notation):

(3.1) CpCqp

CCpCqrCCpqCpr

CKpqp

CKpqq

CCpqCCprCpKqr

CpDpq

CqDpq

CCprCCqrCDpqr

CCpNqCqNp

CNpCpq

Horn [12] has shown that the <u>intuitionist calculus</u> H of Heyting [11] is the smallest subset T of $W(Y, \Omega)$ such that:

(i) For all α, β, if $\alpha \in T$ and $C\alpha\beta \in T$ then $\beta \in T$.

(ii) For each substitution s, $s(T) \subseteq T$.

(iii) T contains each proposition in the list (3.1).

Horn [12] also states that the classical calculus K is the smallest subset T containing CNNpp i.e., $(\sim\sim p) \supset p$, and which meets conditions (i), (ii), and (iii). Any calculus having a distinguished "implication" operation C, and whose set T of theorems meets condition (i), is said to be closed under the <u>rule of detachment</u> (or modus ponens).

Let A be an Ω-algebra. An <u>assignment</u> from A to a word algebra $W(X, \Omega)$ is a homomorphism from $W(X, \Omega)$ into A. We shall use lower case Greek letters ρ, σ, τ, ... to denote assignments.

<u>3.2 Proposition</u>. Let A be an Ω-algebra and $\alpha = \alpha(p_1, \ldots, p_n)$ be a word in $W(Y, \Omega)$. For every pair of assignments σ and σ' from A to $W(Y, \Omega)$, such that $\sigma(p_i) = \sigma'(p_i)$ for $i = 1, \ldots, m$, we have $\sigma(\alpha) = \sigma'(\alpha)$.

<u>Proof</u>: Let $X = \{p_1, \ldots, p_n\} \subseteq Y$. By induction

on the rank of α one can show that $\alpha \in W(X, \Omega)$. Let ρ and ρ' be the restrictions of σ and σ' respectively to $W(X, \Omega)$. By Propositions 2.12 and 2.16, we have $\rho = \rho'$. Since $\sigma(\alpha) = \rho(\alpha)$ and $\sigma'(\alpha) = \rho'(\alpha)$, we obtain $\sigma(\alpha) = \sigma'(\alpha)$.

Proposition 3.2 allows us to write $\alpha(a_1, \ldots, a_n)$ for $\sigma(\alpha)$, when σ is any assignment from the Ω-algebra A to $W(Y, \Omega)$ such that $\sigma(p_i) = a_i$, $i = 1, \ldots, m$ and $\alpha = \alpha(p_1, \ldots, p_n)$. When the Ω-algebra A is $W(Y, \Omega)$ itself, we call $\alpha(a_1, \ldots, a_n)$ a <u>substitution instance</u> of the word α.

The following question arises regarding this notation: Given two sequences of distinct propositional variables, $p_1, \ldots, p_m$ and $q_1, \ldots, q_m$, there is always an assignment (substitution) σ such that $\sigma(p_i) = q_i$, $i = 1, \ldots, m$. Is it true, in keeping with our previous convention, that the word $\alpha(q_1, \ldots, q_m) = \sigma(\alpha(p_1, \ldots, p_m))$ has $q_1, \ldots, q_m$ as its distinct propositional variables? A straightforward inductive proof on the rank of α yields an affirmative answer. Thus there is no conflict of notation; moreover it is now proper to write $\alpha(q_1, \ldots, q_m)$ in cases where the q_i are not distinct.

<u>3.3 Remark</u>. The reader will easily verify that if A, B are Ω-algebras and $h: A \to B$ is a homomorphism,

then for all words $\alpha(p_1, \ldots, p_m)$ and all $a_1, \ldots, a_m \in A, h(\alpha(a_1, \ldots, a_m)) = \alpha(h(a_1), \ldots, h(a_m))$.

Each calculus T induces a natural congruence on the word algebra $W(Y, \Omega)$, called the Lindenbaum Congruence of T. Two words $\alpha, \beta \in W(Y, \Omega)$ are Lindenbaum congruent if and only if for every word $\gamma(p_1, \ldots, p_k)$ and all $\alpha_1, \ldots, \alpha_k \in W(Y, \Omega)$ and all $i = 1, \ldots, k$ the substitution instances $\gamma(\alpha_1,\ldots,\alpha_{i-1},\alpha,\alpha_{i+1},\ldots,\alpha_k)$ and $\gamma(\alpha_1, \ldots, \alpha_{i+1}, \beta, \alpha_{i+1}, \ldots, \alpha_k)$ are either both theorems of the calculus T or neither are theorems.

3.4 Proposition. The Lindenbaum congruence of a calculus T is the greatest congruence which preserves T.

Before proving this proposition (see paragraph preceding Lemma 3.5) we shall rephrase the above traditional definition of the Lindenbaum congruence. Consider an arbitrary word algebra $W(X, \Omega)$. Choose any $p \in X$, and partition the set of substitutions on $W(X, \Omega)$ as follows: σ and σ' are members of the same equivalence class (mod p) if and only if $\sigma(x) = \sigma'(x)$ for all $x \in X - \{p\}$. Now for each equivalence class S (mod p), and each $\gamma \in W(X, \Omega)$, the pair (γ, S) determines a function $S_\gamma : W(X, \Omega) \to W(X, \Omega)$ given by $S_\gamma(\alpha) = \sigma(\gamma)$, where $\sigma \in S$ and $\sigma(p) = \alpha$. Proposition 2.12 guarantees that $S_\gamma(\alpha)$ is defined uniquely for all α. Notice that if p is not one of the propositional variables occurring in γ then S_γ is constant, but if p is one of these variables, say p_i, then $S_\gamma(\alpha)$ has the form $\gamma(\alpha_1, \ldots, \alpha_{i-1}, \alpha, \alpha_{i+1}, \ldots, \alpha_k)$ where each α_i is fixed.

Let $\Gamma(X)$ denote the set of all such functions for variable p, γ, and S.

Now, taking X to be the fixed set Y we have: Two words α, β are Lindenbaum congruent if and only if for every function S_γ in $\Gamma(Y)$, $S_\gamma(\alpha)$ and $S_\gamma(\beta)$ are either both theorems of the calculus T or neither are theorems.

The next lemma shows that $\Gamma(Y)$ coincides with the family of mappings $[W(Y, \Omega)]^*$ as constructed for Lemma 2.6. Therefore $\Gamma(Y)$ satisifies the conditions 2.3 needed to apply Lemma 2.4, and Proposition 3.4 follows as a direct corollary.

3.5 Lemma. For each word algebra $W(X, \Omega)$, $\Gamma(X)$ is a subset of $[W(X, \Omega)]^*$. If X is infinite, then $\Gamma(X)$ coincides with $[W(X, \Omega)]^*$.

Proof: Let J_i and G_i be the sequences occurring in the construction of $W(X, \Omega)$ and $[W(X, \Omega)]^*$ respectively. We now show that for each i,

$$\{S_\gamma \in \Gamma(X) \mid \gamma \in J_i\} \subseteq G_i \; .$$

The proof depends on the following easily verified identity: If $\gamma = \omega(\alpha_1, \ldots, \alpha_n)$, then $S_\gamma = \omega(S_{\alpha_1}, \ldots, S_{\alpha_n})$. This is just what is needed for the induction step in the proof of the above inclusion. For $i = 0$, each $\gamma \in J_0$ is a single variable q, so S_γ is either constant or equal to the identity according as $q \neq p$ or $q = p$ (where S is an equivalence class mod p); hence $S_\gamma \in G_0$.

For the opposite inclusion

$$G_i \subseteq \{S_\gamma \in \Gamma(X) \mid \gamma \in J_i\},$$

consider first the case in which X is countably infinite and Ω is finite or countably infinite. Then $W(X, \Omega)$ is clearly countable. We will show that there is a single equivalence class S such that

$$G_i \subseteq \{S_\gamma \mid \gamma \in J_i\}.$$

Choose any $p \in X$, and any map from $X - \{p\}$ onto $W(X, \Omega)$. Let S be the equivalence class (mod p) of the substitution σ which extends this map onto $W(X, \Omega)$. Then $\{S_\gamma \mid \gamma \in J_0\}$ contains all constant maps and the identity map S_p, and hence coincides with G_0. The inductive step now follows using the identity of the last paragraph.

For the (less interesting) cases in which X or Ω are uncountable, one shows similarly that for each finite sequence $g_1, \ldots g_m \in G_i$ there is a single equivalence class S and $\gamma_1, \ldots, \gamma_m \in J_i$ such that $g_j = S_{\gamma_j}$, $j = 1, \ldots, m$.

<u>3.6 Remark.</u> A useful consequence of Lemmas 3.5 and 2.6 is that every member of $\Gamma(X)$ is compatible with every congruence on $W(X, \Omega)$ (see property 2.3c). As a typical application consider a word of the form $\delta(p, q)$, a congruence Q on $W(X, \Omega)$, and any three words α, β, γ in $W(X, \Omega)$, and let S be an equivalence class (mod p) whose members take q into γ. Then Q is compatible with S_δ, and we obtain the following: If $(\alpha, \beta) \in Q$ then $(\delta(\alpha, \gamma), \delta(\beta, \gamma)) \in Q$.

There is a much simpler characterization of the Lindenbaum congruence which holds for all the familiar calculi, which have the following additional structure: There is a word of the form $\gamma(p, q)$ such that the binary operation C defined on $W(Y, \Omega)$ by $C\alpha\beta = \gamma(\alpha, \beta)$ is distinguished. (This operator is interpreted as "implication"; it may or may not correspond directly to a member of Ω).

For such a calculus T one may define a relation Q^*, called the <u>bi-implicational relation</u>, as follows: For all $\alpha, \beta \in W(Y, \Omega)$, $(\alpha, \beta) \in Q^*$ if and only if $C\alpha\beta \in T$ and $C\beta\alpha \in T$. The next two lemmas give sufficient conditions for Q^* to coincide with the Lindenbaum congruence.

<u>3.7 Lemma</u>. If (i) For all $\alpha \in W(Y, \Omega)$ we have $C\alpha\alpha \in T$.

(ii) For all $\alpha, \beta, \gamma \in W(Y, \Omega)$, if $C\alpha\beta \in T$ and $C\beta\gamma \in T$ then $C\alpha\gamma \in T$.

(iii) For each elementary translation t on $W(Y, \Omega)$ either: (iiia) for all words α, β, if $C\alpha\beta \in T$ then $Ct(\alpha)t(\beta) \in T$, or (iiib) for all words α, β if $C\alpha\beta \in T$ then $Ct(\beta)t(\alpha) \in T$.

Then the bi-implicational relation is a congruence.

It will be demonstrated in Chapter V that the conditions of this lemma are satisified for many well-known calculi. To appreciate condition (iii), the reader may check intuitively that it holds for the Classical calculus, with $\Omega = \{C, K, D, N\}$. For example, taking $t(\alpha) = C\alpha\gamma$ for arbitrary γ, if $C\alpha\beta$ is a theorem, then $Ct(\beta)t(\alpha) = CC\beta\gamma C\alpha\gamma$ is a theorem; taking $t(\alpha) = N\alpha$, if $C\alpha\beta$ is a theorem then $CN\beta N\alpha$ is a theorem. Thus (iiib) holds in these two cases. For all other elementary translations (iiia) hold.

<u>Proof of Lemma 3.7</u>: The relation Q^* is clearly

symmetric; reflexivity, transitivity, and compatibility with each elementary translation follow directly from conditions (i), (ii), and (iii) respectively.

3.8 Lemma. If the bi-implicational relation Q^* is a congruence and the calculus is closed under the rule of detachment, then Q^* coincides with the Lindenbaum congruence Q.

Proof: In view of Proposition 3.4, the inclusion $Q^* \subseteq Q$ will follow once it is shown that Q^* preserves T. This is equivalent to showing that if $C\alpha\beta$ and $C\beta\alpha$ are theorems, then either α and β are both theorems or neither are theorems -- a fact which follows by the rule of detachment.

To show that $Q \subseteq Q^*$, let $(\alpha, \beta) \in Q$. Taking $\gamma(p, q) = Cpq$ for any distinct $p, q \in X$, and substituting α for p, it follows from the definition of Q that either $C\alpha\alpha$ and $C\alpha\beta$ are theorems, or neither are theorems. But $C\alpha\alpha$ is a theorem, since Q^* is reflexive. Hence $C\alpha\beta \in T$. Similarily $C\beta\alpha \in T$, and $(\alpha, \beta) \in Q^*$.

The ideas inherent in Lemmas 3.7 and 3.8 have been exploited frequently (e.g. McKinsey [19]). The following abstraction is suggested by the work of Ulrich [28]. Let Φ be a set of words of the form $\phi(p, q)$. A calculus T is standard with respect to Φ, or Φ-standard provided the following holds for all $\alpha, \beta \in W(Y, \Omega)$: (α, β) belongs to the Lindenbaum congruence of T if and only if $\phi(\alpha, \beta) \in T$ for all $\phi \in \Phi$. The calculus T is <u>standard</u> if it is Φ-standard for some such set Φ. Notice that a calculus satisifying the hypothesis of Lemma 3.8 is standard with respect to $\Phi = \{Cpq, Cqp\}$.

The following lemma, needed for later applications, shows that the above definition of "standard calculus" may be considerably weakened.

<u>3.9 Lemma</u>. Let Q' be a congruence which preserves the calculus T, and let Φ be a set of words such that for all $\alpha, \beta \in W(Y, \Omega)$:

(i) If $\phi(\alpha, \beta) \in T$ for all $\phi \in \Phi$ then $(\alpha, \beta) \in Q'$.

(ii) $\phi(\alpha, \alpha) \in T$ for all $\phi \in \Phi$.

Then T is a standard calculus with respect to Φ.

<u>Proof</u>: By Proposition 3.4 Q' is contained in the Lindenbaum congruence Q of T. Hence $\phi(\alpha, \beta) \in T$, for all $\phi \in \Phi$ implies $(\alpha, \beta) \in Q$. Now let $(\alpha, \beta) \in Q$. Then by Remark 3.6 $(\phi(\alpha, \beta), \phi(\beta, \beta)) \in Q$. Since $\phi(\beta, \beta) \in T$ and Q preserves T we have $\phi(\alpha, \beta) \in T$.

A <u>matrix</u> (or model) is an Ω-algebra A together with a distinguished subset $D \subseteq A$, whose members are called <u>designated elements</u>. As an example of a matrix, consider the "classical truth tables."

C	t	f
t	t	f
f	t	t

D	t	f
t	t	t
f	t	f

K	t	f
t	t	t
f	f	f

N	
t	f
f	t

These tables define an Ω-algebra on the set $\{t, f\}$, with $\Omega = \{C, D, K, N\}$, which becomes a matrix on taking t as the only designated element. (In this example, the symbol D for disjunction unfortunately conflicts the notation $D = \{t\}$ for the set of designated elements.)

Two matrices (A, D) and (B, E) are said to be _isomorphic_ if there is an isomorphism $f:A \to B$ between the Ω-algebras A and B such that $f(D) = E$.

A word α in $W(Y, \Omega)$ is said to be _valid_ in a matrix (A, D) if and only if every assignment from A to $W(Y, \Omega)$ carries α into D. For example, every word in $W(Y, \Omega)$ of the form $C\alpha C\beta\alpha$ is valid in the matrix of classical truth tables, since their possible assignment images, CtCtt, CtCft, CfCtf, CfCff, are all equal to t. The following proposition shows that X may be substituted for Y in the definition of "valid".

3.10 Proposition. Let (A, D) be a matrix and X be a subset of Y. Then a word $\alpha \in W(X, \Omega)$ is valid in the model (A, D) if and only if every assignment from A to $W(X, \Omega)$ carries α into D.

Proof: This is an immediate consequence of Proposition 2.16.

Two matrices are _equivalent_ if and only if they have the same set of valid words. Note that isomorphic matrices are always equivalent. A matrix is said to be a <u>model of the calculus T</u> if and only if every theorem of T is valid in the matrix. A model of a calculus T is called a <u>characteristic model</u> of T if and only if every word which is valid in the model is a theorem of T. For example, the matrix of classical truth tables is a characteristic model of the Classical calculus.

<u>3.11 Remark</u>. Every calculus T has a characteristic model, namely the Ω-algebra $W(Y, \Omega)$ itself, together with the set $D = T$. This matrix is a model of T because the assignments from this algebra are substitutions; it is characteristic since any non-theorem $\alpha \in W(Y, \Omega)$ is carried outside T by the identity assignment. Similarly, every matrix is a characteristic model of the calculus consisting of its set of valid words.

<u>3.12 Lemma</u>. If (A, D) is a model of the calculus T, and B is a subalgebra of A, then $(B, B \cap D)$ is also a model of T.

<u>Proof</u>: Since every assignment from B is an assignment from A, and since every assignment from A carries each theorem into the set D, then each assignment from B must carry every theorem of T into $D \cap B$.

3.13 Lemma. Let (A, D) and (A', D') be matrices, and let A' be the image of A under a homomorphism h. Then:

(i) If $h(D) \subseteq D'$ then every word valid in (A, D) is valid in (A', D').

(ii) If $h^{-1}(D') \subseteq D$ then every word valid in (A', D') is valid in (A, D).

(iii) The matrices (A, D) and (A', D') are equivalent if $h(D) = D'$ and the kernel of h preserves D.

Proof: To prove (i), let $\sigma: W(Y, \Omega) \to A'$ be any assignment from A'. Choose any map $\rho: Y \to A$ such that $\rho(x) \in h^{-1}(\sigma(x))$. By Proposition 2.12 we can extend ρ to an assignment ρ^* from A. Since the equation $h(\rho^*(x)) = \sigma(x)$ holds for all $x \in Y$, it also holds for all $\alpha \in W(Y, \Omega)$ by Proposition 2.12. If α is valid in (A, D) then $\rho^*(\alpha) \in D_i$, whence $\sigma(\alpha) = h(\rho^*(\alpha)) \in D'$.

To prove (ii), let $\sigma: W(Y, \Omega) \to A$ be any assignment from A, and let α be valid in (A', D'). Then $h(\sigma(\alpha)) \in D'$, since the composition of h and σ is an assignment from A'. Thus from the fact that $h^{-1}(D') \subseteq D$, we have $\sigma(\alpha) \in D$.

To prove (iii) we show that the hypothesis of (iii) implies those of (i) and (ii). Now (i) follows immediately, since $h(D) = D'$. The fact that the kernel of h

preserves D is equivalent to $h^{-1}(h(D)) \subseteq D$, and we have $h^{-1}(D') \subseteq D$.

Turning now to the definition of the Lindenbaum model, let Q be the Lindenbaum congruence of a calculus T. Consider the matrix $(W(Y, \Omega)/Q, T/Q)$, where Q is the Lindenbaum congruence for T and T/Q is the image of T under the natural homomorphism $h:W(Y, \Omega) \to W(Y, \Omega)/Q$. In view of Proposition 3.4 and Lemma 3.13, part (iii), we see that this matrix is equivalent to $(W(Y, \Omega), T)$. Thus by Remark 3.11 $(W(Y, \Omega)/Q, T/Q)$ is a characteristic model of T, called the <u>Lindenbaum model of T</u>.

In a standard propositional calculus, unless all pairs of words are Linbenbaum congruent, the Lindenbaum model is infinite. For if the Lindenbaum model is finite, there must be two distinct elements p', q' of the infinite set Y which are congruent, and if the words $\phi(p', q')$ in a standard calculus are theorems, so also are all substitution instances $\phi(\alpha, \beta)$; whence α, β are congruent.

Since the primary use of models is to decide whether or not a given word is a theorem of a calculus, the infinite Lindenbaum model is less interesting than certain related models which can be built from subalgebras of $W(Y, \Omega)$. For this purpose we introduce the following notational convention, to be used throughout the remainder of this dissertation.

<u>3.14 Notational Convention</u>. Let T be any calculus, and X be any subset of Y. We will write $T(X)$ to denote $W(X, \Omega) \cap T$. Moreover, $Q(X)$ will denote the greatest congruence on $W(X, \Omega)$ which preserves $T(X)$. (In applications, the symbols T, X, Q may be replaced in a consistent manner by T', X', Q', etc.) We now observe that the matrix $(W(X, \Omega)/Q(X), T(X)/Q(X)$ is a model of T, called a <u>finite order Lindenbaum model of T</u> whenever X is finite. For the matrix $(W(X, \Omega), T(X))$ is, by Lemma 3.12, a model of T. Taking h in Lemma 3.13(iii) to be the natural homomorphism we see that $(W(X, \Omega)/Q(X), T(X)/Q(X))$ is equivalent to $(W(X, \Omega), T(X))$. The <u>order</u> of a finite order Lindenbaum model is the cardinality of the finite set X.

<u>3.15 Proposition</u>. Given an increasing sequence X_m of finite subsets of Y whose union is Y, a word is a theorem of the calculus T if and only if it is valid in each of the finite order Lindenbaum models, $(W(X_m, \Omega)/Q(X_m), T(X_m)/Q(X_m))$.

<u>Proof</u>: It suffices to show that every non-theorem of T is invalid in one of these models of T. Suppose $\alpha \notin T$. Let m be sufficiently large so that $\alpha \in W(X_m, \Omega)$. Since $Q(X_m)$ preserves $T(X_m)$, the

natural homomorphism from $W(X_m, \Omega)$ onto $W(X_m, \Omega)/Q(X_m)$ is an assignment which takes α outside $T(X_m)/Q(X_m)$. Hence α is not valid in the model $(W(X_m, \Omega)/Q(X_m), T(X_m)/Q(X_m))$. It is shown below (see also Løs [17]) that if a calculus T has a finite characteristic model (A, D), then every finite order Lindenbaum model L can be constructed as follows, knowing only (A, D) and the order m of L. Consider the function algebra $Fn(A^m, A)$, as defined in Chapter II. Let B_m be the smallest subalgebra containing all the "coordinate projections" $\pi_i = 1, \ldots, n$, given by $\pi_i(a_1, \ldots, a_m) = a_i$. Since $Fn(A^m, A)$ is finite, this subalgebra can be constructed by any of several algorithms, such as closing the set of projections under the operations of Ω. Let E_m be the set of all functions $f \in B_m$ whose ranges lie in D, and let Q'_m be the greatest congruence on B_m which preserves E_m. An algorithm for constructing Q'_m is suggested by Lemma 2.4 and Remark 2.5. Then $(B_m/Q'_m, E_m/Q'_m)$ is the desired isomorph.

<u>3.16 Proposition</u>. If a calculus T has a finite characteristic model (A, D), then each of its finite order Lindenbaum models of order m is isomorphic to the finite matrix $(B_m/Q'_m, E_m/Q'_m)$ constructed above.

<u>Proof</u>: Let $(W(X, \Omega)/Q(X), T(X)/Q(X))$ be a finite

order Lindenbaum model of order m. We will define a homomorphism $h: W(X, \Omega) \to Fn(A^m, A)$ whose range is B_m, whose kernel preserves $T(X)$, and such that $h(T(X)) = E_m$. The proposition then follows by Proposition 2.9.

To define h, let $X = \{p_1, \ldots, p_m\}$. Then $h(\alpha)$ is the function f given by $f(a_1, \ldots, a_m) = \sigma(\alpha)$, where σ is the assignment from A to $W(X, \Omega)$ such that $\sigma(p_i) = a_i$, $i = 1, \ldots, m$ (see Proposition 2.12). Notice that the range of $h(\alpha)$ is the set of images of α under all such assignments.

To show that h is a homomorphism, let $f = h(\omega(\alpha_1, \ldots, \alpha_n))$ and $g_i = h(\alpha_i)$. Since $\sigma(\omega(\alpha_1, \ldots, \alpha_n)) = \omega(\sigma(\alpha_1), \ldots, \sigma(\alpha_n))$ we have $f(a_1, \ldots, a_m) = \omega(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$ for all $(a_1, \ldots, a_m) \in A^m$. Thus $h(\omega(\alpha_1, \ldots, \alpha_n)) = \omega(h(\alpha_1), \ldots, h(\alpha_n))$.

Let J_i be the sequence used to define $W(X, \Omega)$. Clearly, $h(J_0)$ is the set of coordinate projections. A simple inductive argument shows that $h(J_i)$ must be contained in every subalgebra of $Fn(A^m, A)$ which contains $h(J_0)$. Thus the image of $W(X, \Omega)$ under h is the smallest subalgebra containing $h(J_0)$, that is, B_m.

We now show that the kernel of h preserves $T(X)$, and that $h(T(X)) = E_m$. For all $\alpha, \beta \in W(X, \Omega)$ we have $(\alpha, \beta) \in \ker h$ if and only if $\sigma(\alpha) = \sigma(\beta)$ for all assignments from A to $W(X, \Omega)$. Therefore, since (A, D) is a model for T, if $(\alpha, \beta) \in \ker h$ and $\alpha \in T(X)$ we have $\sigma(\beta) = \sigma(\alpha) \in D$ for all such assignments σ, and since it is a characteristic model we have $\beta \in T(X)$ by Proposition 3.10. Clearly, $h(\alpha) \in E_m$ if and only if α is valid in (A, D). Thus, by Proposition 3.10, $h(T(X)) = E_m$.

The next two results give sufficient conditions for finite order Lindenbaum models to be characteristic.

<u>3.17 Lemma</u>. Let T be a calculus with a characteristic model (A, D) and let X be a subset of Y. Then $(W(X, \Omega), T(X))$ is a characteristic model for T if there is a homomorphism from $W(X, \Omega)$ onto A.

<u>Proof</u>: Let $h:W(X, \Omega) \to A$ be any such homomorphism. Since $(W(X, \Omega), T(X))$ is a model for T, we need only show that each non-theorem of T is carried by a substitution into a member of $W(X, \Omega)$ which is not in $T(X)$. Let $\alpha \notin T$. Since (A, D) is a characteristic model of T, there is an assignment $\sigma:W(Y, \Omega) \to A$ such that $\sigma(\alpha) \notin D$. Choose any map $s:Y \to W(X, \Omega)$ such that $s(x) \in h^{-1}(\sigma(x))$ for all $x \in Y$, and let s^* be the unique extension of s to a homomorphism on $W(Y, \Omega)$ into $W(X, \Omega)$. The composition of h with s^* yields a homomorphism $h \circ s^*:W(Y, \Omega) \to A$ which coincides with

σ on Y. Thus, by Proposition 2.12 $h(s^*(\alpha)) = \sigma(\alpha)$. Since h is an assignment and (A, D) is a model it follows, using Proposition 3.10 that $s^*(\alpha) \notin T(X)$.

<u>3.18 Proposition</u>. Suppose the calculus T has a finitely generated (or finite) characteristic model (A, D). Let m be the cardinality of a generating set for A. Then each finite order Lindenbaum model of order greater than or equal to m is characteristic for T.

<u>Proof</u>: In view of Remark 2.14, if the cardinality of X is greater than or equal to m there is a homomorphism from $W(X, \Omega)$ onto A. Then by Proposition 3.17 $(W(X, \Omega), T(X))$ is a characteristic model which, as noted above, is equivalent to $(W(X, \Omega)/Q(X), T(X)/Q(X))$.

Proposition 3.16 and 3.18 provide a decision algorithm for the equivalence of finite matrices. (Other such algorithms are discussed by Kalicki [13] and Łos [17]). Let (A, D) and (A', D') be two finite matrices, with sets of valid words V, V', respectively. Choose any integer m such that both A and A' have generating sets of cardinality less than or equal to m. Recalling that (A, D) is a characteristic model of the calculus V (see Remark 3.11), construct an isomorph L_m of the <u>mth</u> order Lindenbaum model for V. In accordance with

Proposition 3.15, this can be done without knowing V. Construct the corresponding matrix L'_m for V', and test whether or not L_m and L'_m are isomorphic. Then we have:

<u>3.19 Proposition</u>. Two finite matrices (A, D) and (A', D') are equivalent if and only if the matrices L_m and L'_m of the above construction are isomorphic.

<u>Proof</u>: We are to show that $V = V'$ if and only if L_m is isomorphic to L'_m. If $V = V'$ then L_m, L'_m are both isomorphic to the <u>mth</u> order Lindenbaum model of V, and are therefore isomorphic. If L_m and L'_m are isomorphic, they are certainly equivalent. By Proposition 3.18 they are characteristic of V and V', respectively; hence $V = V'$.

We now give two properties of standard calculi.

<u>3.20 Lemma</u>. If a calculus T is a standard with respect to Φ, and X is any subset of Y, then $Q(X)$ is the restriction of the Lindenbaum congruence to $W(X, \Omega)$. Moreover for all $\alpha, \beta \in W(X, \Omega)$,

$$(\alpha, \beta) \in Q(X) \text{ if and only if } \phi(\alpha, \beta) \in T(X) \text{ for all } \phi \in \Phi.$$

<u>Proof</u>: Note first that if $\alpha, \beta \in W(X, \Omega)$, then $\phi(\alpha, \beta) \in W(X, \Omega)$. This is seen by observing that

$\phi(\alpha, \beta) = s(\phi(p, q))$, where s is any substitution which takes Y into $W(X, \Omega)$ in such a way that $s(p) = \alpha$ and $s(q) = \beta$.

Let Q' be the restriction to $W(X, \Omega)$ of the Lindenbaum congruence. Then $(\alpha, \beta) \in Q'$ if and only if $\alpha, \beta \in W(X, \Omega)$ and $\phi(\alpha, \beta) \in T(X)$, for all $\phi \in \Phi$. Thus we need only show that $Q' = Q(X)$, the greatest congruence on $W(X, \Omega)$ which preserves T, Q' clearly preserves $T(X) = T \cap W(X, \Omega)$. Let Q'' be any congruence which preserves $T(X)$, and let $(\alpha, \beta) \in Q''$. Since Q' is reflexive we have $\phi(\beta, \beta) \in T(X)$, for all $\phi \in \Phi$. Moreover it follows on taking $\delta = \phi$ and $\gamma = \beta$ in Remark 3.6 that $(\phi(\alpha, \beta), \phi(\beta, \beta)) \in Q''$, for all $\phi \in \Phi$. Since Q'' preserves $T(X)$, it follows that $\phi(\alpha, \beta) \in T(X)$, for all $\phi \in \Phi$; whence $(\alpha, \beta) \in Q'$.

<u>3.21 Proposition</u>. Let T be a standard calculus and let X and X' be subsets of Y. If $(\alpha, \beta) \in Q(X)$ then $(s(\alpha), s(\beta)) \in Q(X')$ for all homomorphisms $s: W(X, \Omega) \to W(X', \Omega)$.

<u>Proof</u>: Let $(\alpha, \beta) \in Q(X)$. Since T is standard we have, by Lemma 3.20, $\phi(\alpha, \beta) \in T(X)$, for all $\phi \in \Phi$. Let s be any homomorphism from $W(X, \Omega)$ to $W(X', \Omega)$. By Remark 3.3 we have $s(\phi(\alpha, \beta)) = \phi(s(\alpha), s(\beta))$. Thus, in view of Lemma 3.20, it suffices to show that s takes

$T(X)$ into $T(X')$. By Proposition 2.16, s is the restriction of a homomorphism (substitution) $s^*: W(Y, \Omega) \to W(X', \Omega)$, and the result follows since T is closed under substitution.

<u>3.22 Remark</u>. (i) In the case $X' = X$, Proposition 3.21 asserts that $Q(X)$ is "fully invariant" (or "fully characteristic") in the sense that every homomorphism of $W(X, \Omega)$ into itself is compatible with $Q(X)$ (see [3] p.163). A result due to Neumann (see [1] Theorem 21 p.152) asserts if A is a free algebra generated by the subset B, then its quotient algebra with respect to any fully invariant congruence is freely generated by the equivalence classes of members of B. Thus, in a standard calculus, $W(X, \Omega)/Q(X)$ is freely generated by $X/Q(X)$.

(ii) Taking $X' = X = Y$ we see that the Lindenbaum congruence of a standard calculus is "fully invariant." It is known (see [3] pages 162-163) that there is a natural correspondence between the fully invariant congruences on $W(X, \Omega)$ and the "equationally definable classes" or "varieties". In this way the standard calculi are in one-to-one correspondence with some set of varieties.

CHAPTER IV

STANDARD EXTENSIONS AND GENERIC MODELS FOR PROPOSITIONAL CALCULI

We now give a definition of "standard extension" which is a generalization of a notion introduced by Ulrich [28].

Let T be a calculus, and T' be a standard calculus with respect to the set of words Φ. Then T' is a <u>standard extension</u> of T if $T \subseteq T'$, and $\phi(p, p) \in T$ for all $p \in Y$ and all $\phi \in \Phi$.

The following lemma, basic to the rest of our discussion, will allow us to transfer properties of a calculus to its standard extensions.

<u>4.1 Lemma</u>. Let T' be a standard extension of the calculus T, and X be any subset of Y. Then $Q(X) \subseteq Q'(X)$ (see Convention 3.14).

<u>Proof</u>: By Remark 3.6 $(\alpha, \beta) \in Q(X)$ implies $(\phi(\alpha, \beta), \phi(\beta, \beta)) \in Q(X)$ for all $\phi \in \Phi$. Since $\phi(\beta, \beta) \in T$ it follows that $\phi(\beta, \beta) \in T(X) \subseteq T'(X)$ for all $\phi \in \Phi$. Since $Q(X)$ preserves $T(X)$ we have

$\phi(\alpha, \beta) \in T(X) \subseteq T'(X)$ for all $\phi \in \Phi$. By Lemma 3.20 we have, $(\alpha, \beta) \in Q(X)$ implies $(\alpha, \beta) \in Q'(X)$.

We say that a calculus has the finite Lindenbaum property if every finite order Lindenbaum model is finite.

4.2 Proposition. If T has the finite Lindenbaum property, and T' is a standard extension of T, then T' has the finite Lindenbaum property.

Proof: The calculus T has the finite Lindenbaum property if and only if for each finite subset X of Y, $Q(X)$ consists of only a finite number of equivalence classes. By Lemma 4.1 $Q'(X)$ has only a finite number of equivalence classes.

Combining the contrapositives of Propositions 4.2 and 3.16 we obtain:

4.3 Remark. Let T' be a standard extension of T. If the calculus T' does not have the finite Lindenbaum property then T does not have a finite characteristic model.

In this paragraph we interrupt the main discussion in order to apply Lemma 4.2 to the decision problem for propositional calculi.

4.4 Remark. A calculus is said to be decidable if there is an algorithm for determining, in a finite number of steps, whether or not an arbitrary word is a theorem of the calculus. Sufficient conditions for a calculus to be decidable can be given provided the calculus is "finitely axiomatizable." Intuitively speaking, a calculus is "finitely axiomatizable" means that it can be described in a manner similar to our description of the Intuitionist and Classical calculi; that is, as the smallest calculus containing a given finite list of words called "axioms" and closed under certain "rules of inference." Łos [17] states that: If a calculus is "finitely axiomatizable" and has the finite Lindenbaum property then the calculus is decidable. In view of Proposition 4.2 and the above result of Łos we have: If T is a calculus which has the finite Lindenbaum property, and T' is a standard extension of T, then T' is decidable if T' is "finitely axiomatizable." This result slightly generalizes a result announced by Ulrich [28] and appears to have been exploited by McKay [18] in a special case, although without explicit formulation.

Let Q be the Lindenbaum congruence of T. An Ω-algebra A is said to be <u>generic</u> (functionally free) for the calculus T if the following conditions are equivalent: i) $(\alpha, \beta) \in Q$, ii) $\sigma(\alpha) = \sigma(\beta)$ for every assignment σ from A. A model (A, D) of T is a <u>generic model</u> of T if A

is generic for T. (The term "generic" is borrowed from Cohn's discussion of varieties [3]; this corresponds to Tariski's notion of "functionally free" in his discussion of equationally definable classes of algebras [25]. There is no conflict between our usage and those discussed above, if the pairs of the Lindenbaum congruence are regarded as "laws".)

<u>4.5 Lemma</u>. If the Ω-algebra A is generic for the calculus T and every two theorems in T are Lindenbaum congruent, then a characteristic model (A, D) for T can be formed from A, in which D consists of a single element of A.

<u>Proof</u>: We will first show that theorems take only one value for all assignments from A. This is trivial in case A has only one element. Suppose A has more than one element. Let $\alpha(p_1, \ldots, p_\ell)$ be a theorem of T and assume there are assignments σ_1 and σ_2 from A such that $\sigma_1(\alpha) \neq \sigma_2(\alpha)$. Since α has only a finite number of variables there are infinitely many assignments which agree with σ_1 on α. Let $p_1', \ldots, p_\ell'$ be members of Y which do not occur in α. Let $s(p_i) = p_i'$ $i = 1, \ldots, \ell$ and $s(p) = p$ for all other $p \in Y$. The mapping s extends uniquely to a substitution s^*. Now $s^*(\alpha)$ is a theorem of T,

whence by hypothesis $(\alpha, s^*(\alpha)) \in Q$. Let σ_3 be an assignment from A such that $\sigma_3(p_i) = \sigma_1(p_i)$ and $\sigma_3(p_i') = \sigma_2(p_i')$ $i = 1, \ldots, \ell$. Now $\sigma_3(\alpha) = \sigma_1(\alpha)$ and $\sigma_3(s^*(\alpha)) = \sigma_2(\alpha)$. Therefore $\sigma_3(\alpha) \neq \sigma_3(s^*(\alpha))$, which contradicts the fact that A is generic for T. Having shown that the theorems of T take a single common value under all assignments from A, we obtain a model of T by taking this common value as the only element of D. If α is a theorem of T and β is not a theorem of T, then $(\alpha, \beta) \notin Q$, since Q preserves T. Since A is generic there is an assignment such that $\sigma(\alpha) \neq \sigma(\beta)$. Hence $\sigma(\beta) \notin D$, and our model is characteristic for T.

A matrix (A, D) is said to be <u>standard</u> with respect to a set Φ of words if for all $a, b \in A$ we have $a = b$ if and only if $\phi(a, b) \in D$ for all $\phi \in \Phi$.

<u>4.6 Lemma</u>. If (A, D) is both a standard matrix with respect to Φ and a characteristic model for a calculus T, then T is standard with respect to Φ.

<u>Proof</u>: Let $Q(A) = \{(\alpha, \beta) | \sigma(\alpha) = \sigma(\beta)\}$ for all assignments $\sigma: W(Y, \Omega) \to A$. Since $Q(A)$ is the meet of congruences consisting of kernels of assignments, it follows that $Q(A)$ is a congruence. Since (A, D) is characteristic for T, clearly $Q(A)$ preserves T.

Thus, by Proposition 3.4, $Q(A) \subseteq Q$, where Q is the Lindenbaum congruence of T. To complete the proof, it suffices to show that the hypotheses of Lemma 3.9 are satisfied, with $Q' = Q(A)$. We show first, if $\phi(\alpha, \beta) \in T$, for all $\phi \in \Phi$ then $(\alpha, \beta) \in Q(A)$. Assume $\phi(\alpha, \beta) \in T$ for all $\phi \in \Phi$. Then $\sigma(\phi(\alpha, \beta)) \in D$. By Remark 3.3 $\sigma(\phi(\alpha,\beta)) = \phi(\sigma(\alpha), \sigma(\beta)) \in D$. Since (A, D) is a standard matrix, $\sigma(\alpha) = \sigma(\beta)$ for all assignments and we have $(\alpha, \beta) \in Q(A)$.

To show that $\phi(\alpha, \alpha) \in T$ for all $\phi \in \Phi$, note that $\sigma(\phi(\alpha, \alpha)) = \phi(\sigma(\alpha), \sigma(\alpha))$ belongs to D by hypothesis, and the result follows since (A, D) is characteristic.

<u>4.7 Lemma</u>. Let (A, D) be a characteristic model of a calculus T, and suppose that T is standard with respect to Φ. Then (A, D) is generic for T provided the following holds for all $a, b \in A$: if $\phi(a, b) \in D$ for all $\phi \in \Phi$ then $a = b$.

<u>Proof</u>: Since T is standard with respect to Φ, we have $\phi(\alpha, \alpha) \in T$ and so, since (A, D) is a model of T, $\phi(a, a) \in D$ for all $a \in A$. Therefore, if $\sigma(\alpha) = \sigma(\beta)$ for all assignments σ from A we have $\phi(\sigma(\alpha), \sigma(\beta)) \in D$ for all assignments. By Remark 3.3, $\sigma(\phi(\alpha, \beta)) = \phi(\sigma(\alpha), \sigma(\beta))$; hence $\phi(\alpha, \beta)$ is valid. Since (A, D) is characteristic for T we have

$\phi(\alpha, \beta) \in T$. Therefore $\sigma(\alpha) = \sigma(\beta)$ for all assignments σ from A implies $(\alpha, \beta) \in Q$. Conversely, let $(\alpha, \beta) \in Q$, so that $\phi(\alpha, \beta) \in T$ for all $\phi \in \Phi$. Now, since (A, D) is a model, we have $\sigma(\phi(\alpha, \beta)) \in D$, and thus $\phi(\sigma(\alpha), \sigma(\beta)) \in D$ for all assignments σ. Thus, by hypothesis, $\sigma(\alpha) = \sigma(\beta)$ for all assignments σ from A.

<u>4.8 Lemma</u>. Let T be a standard calculus with respect to Φ. Then every finite order Lindenbaum model of T is a standard matrix with respect to Φ. Moreover, if T has a finite characteristic model, then one of its finite order Lindenbaum models is a finite, characteristic, and generic, model of T.

<u>Proof</u>: Taking h in Remark 3.3 to be the natural homomorphism from $W(X, \Omega)$ onto $W(X, \Omega)/Q(X)$, we see that $\phi(|\alpha| \bmod Q(X), |\beta| \bmod Q(X)) = |\phi(\alpha, \beta)| \bmod Q(X)$. To show that $(W(X, \Omega)/Q(X), T(X)/Q(X))$ is standard, assume that $\phi(|\alpha| \bmod Q(X), |\beta| \bmod Q(X)) \in T(X)/Q(X)$ for all $\phi \in \Phi$. Then $|\phi(\alpha, \beta)| \bmod Q(X) \in T(X)/Q(X)$. Since $Q(X)$ preserves $T(X)$ we have $|\phi(\alpha, \beta)| \bmod Q(X) \in T(X)/Q(X)$ for all $\phi \in \Phi$. Hence, by Lemma 3.20, $(\alpha, \beta) \in Q(X)$, that is $|\alpha| \bmod Q(X) = |\beta| \bmod Q(X)$. Conversely, assume that $|\alpha| \bmod Q(X) = |\beta| \bmod Q(X)$. Then the identity above yields $|\phi(\alpha, \beta)| \bmod Q(X) = |\phi(\alpha, \alpha)| \bmod Q(X)$. Since

$\phi(\alpha, \alpha) \in T(X)$, and $Q(X)$ preserves $T(X)$, we have $\phi(|\alpha| \bmod Q(X), |\beta| \bmod Q(X)) = |\phi(\alpha, \alpha)| \bmod Q(X) \in T(X)/Q(X)$ for all $\phi \in \Phi$.

By Proposition 3.18 some finite order Lindenbaum model, $(W(X, \Omega)/Q(X), T(X)/Q(X))$ is characteristic for T, and by Proposition 3.16 it is also finite. Since $(W(X, \Omega)/Q(X), T(X)/Q(X))$ is standard with respect to Φ, it follows by Lemma 4.7 that it is also generic.

The next two lemmas are preliminary to our main result, Proposition 4.13.

A homomorphism h from the word algebra $W(X', \Omega)$ into $W(X, \Omega)$ is called a <u>simple homomorphism</u> if $h(X') \subseteq X$.

<u>4.9 Lemma</u>. Let T be a standard calculus, and X, X' be subsets of Y, where X' is finite with cardinality m. Suppose that T has a finite generic algebra with m elements. Then $(\alpha, \beta) \in Q(X)$ if and only if, for all simple homomorphism $s: W(X, \Omega) \to W(X', \Omega)$ we have $(s(\alpha), s(\beta)) \in Q(X')$.

<u>Proof</u>: By Proposition 3.21 $(\alpha, \beta) \in Q(X)$ implies $(s(\alpha), s(\beta)) \in Q(X')$ for all homomorphisms s. Let $\alpha, \beta \in W(X, \Omega)$, and suppose that $(\alpha, \beta) \notin Q(X)$. By Proposition 3.20, $Q(X)$ is the restriction of Q to $W(X, \Omega)$. Hence there is an assignment σ from the

generic algebra to $W(Y, \Omega)$ such that $\sigma(\alpha) \neq \sigma(\beta)$. Let ψ be a bijection from X' onto the generic algebra. By Proposition 2.12, there is a simple homomorphism $s:W(X, \Omega) \to W(X', \Omega)$ such that $s(x) = \psi^{-1}(\sigma(x))$ for all $x \in X$. Also by Proposition 2.12, ψ extends to an assignment ψ^* from the generic algebra to $W(Y, \Omega)$ such that $\psi^* \circ s$ is the restriction of σ to $W(X, \Omega)$. It follows that $\psi^*(s(\alpha)) = \sigma(\alpha)$ and $\psi^*(s(\beta)) = \sigma(\beta)$. Hence $(s(\alpha), s(\beta)) \notin Q$, and $(s(\alpha), s(\beta)) \notin Q(X')$ by Proposition 3.20.

It is easily shown that any mapping $h:U \to V$ can be written as the composition of a mapping $s:U \to U$ and a bijection $g:s(U) \to h(U)$. The following remark applies this result to a special case needed for the next lemma.

4.10 Remark. Let $h:U \to V$ be a mapping on a finite set $U = \{p_1, \ldots, p_k\}$. Then there exists mappings $s:U \to U$ and $f:h(U) \to s(U)$ such that

(i) $h(s(p_i)) = h(p_i)$,

(ii) $f(h(p_i)) = s(p_i)$, for all $i = 1, \ldots, k$.

4.11 Lemma. Let T' be a standard extension of the calculus T, and let X and Z be subsets of Y. Let τ, ρ be the natural homomorphisms from $W(X, \Omega)$ onto $W(X, \Omega)/Q(X)$ and $W(X, \Omega)/Q'(X)$, respectively. If $h:W(Z, \Omega) \to W(X, \Omega)$ is any simple homomorphism, then

$(\ker \tau\circ h) \vee Q'(Z) = \ker(\rho\circ h)$.

Proof: By Lemma 4.1, $Q(X) \subseteq Q'(X)$. Since τ, ρ are the natural homomorphisms, we have $\ker \tau = Q(X) \subseteq Q'(X) = \ker \rho$, and it follows that $\ker \tau\circ h \subseteq \ker \rho\circ h$. By Proposition 3.21 $(\alpha, \beta) \in Q'(Z)$ implies $(h(\alpha), h(\beta)) \in Q'(X)$. Hence $\rho(h(\alpha)) = \rho(h(\beta))$, and it follows that $Q'(Z) \subseteq \ker \rho\circ h$. Thus $(\ker \tau\circ h) \vee Q'(Z) \subseteq \ker(\rho\circ h)$.

For the opposite inclusion, we will show that there is a substitution $s^*:W(Y, \Omega) \to W(Z, \Omega)$ such that for all $(\alpha, \beta) \in \ker \rho\circ h$, the pairs $(\alpha, s^*(\alpha))$ and $(\beta, s^*(\beta))$ are members of $\ker \tau\circ h$, while $(s^*(\alpha), s^*(\beta)) \in Q'(Z)$. The inclusion then follows since $(\ker \tau\circ h) \vee Q'(Z)$ is a (transitive) equivalence relation containing both $\ker \tau\circ h$ and $Q'(Z)$.

To determine s^*, let j be the restriction of h to Z. Choose a representative point from each of the (mutually disjoint) sets $j^{-1}(j(q))$. For each $p \in Z$, let $s(p)$ denote the chosen representative of $j^{-1}(j(p))$. This determines a function $s:Z \to Z$ which, in view of Proposition 2.12, has an extension to a homomorphism $s^*:W(Y, \Omega) \to W(Z, \Omega)$. Notice that

$$h(s^*(p)) = h(p) \quad \text{for all} \quad p \in Z.$$

In view of the construction of s, the equation

$$f(h(p)) = s(p)$$

defines a function $f:h(Z) \to Z$. Since h is simple, the domain $h(Z)$ of f is a subset of X. Therefore f can be extended to a homomorphism $f^*: W(Y, \Omega) \to W(Z, \Omega)$. Moreover h can be extended to a homomorphism $h^*: W(Y, \Omega) \to W(X, \Omega)$.

Now the equations $h^*(s^*(p)) = h^*(p)$ and $f^*(h^*(p) = s^*(p)$ hold for all $p \in Z$. It follows by Proposition 3.2 that for all $\alpha \in W(Z, \Omega)$ we have $h^*(s^*(\alpha)) = h^*(\alpha)$ and $f^*(h^*(\alpha)) = s^*(\alpha)$. Since h^* is an extension of h and s^* is into $W(Z, \Omega)$ it follows that

(a) $h(s^*(\alpha)) = h(\alpha)$ for all $\alpha \in W(Z, \Omega)$.

Similarly, if g is the restriction of f^* to $W(X, \Omega)$, we have

(b) $g(h(\alpha)) = s^*(\alpha)$ for all $\alpha \in W(Z, \Omega)$.

Now let $(\alpha, \beta) \in \ker \rho \circ h$. Then $\alpha, \beta \in W(Z, \Omega)$, and it follows from equation (a) that $(\alpha, s^*(\alpha))$ and $(\beta, s^*(\beta))$ are elements of $\ker h$, and hence belong also to $\ker \tau \circ h$. To show that $(s^*(\alpha), s^*(\beta)) \in Q'(Z)$, notice that $(h(\alpha), h(\beta)) \in \ker \rho = Q'(X)$. Combining this with equation (b), for β as well as α, it follows by Proposition 3.21 that $(s^*(\alpha), s^*(\beta)) \in Q'(Z)$.

A calculus is a <u>DLC calculus</u> if for each finite subset $X \subseteq Y$ the lattice of congruences of the corresponding quotient algebra $W(X, \Omega)/Q(X)$ is distributive. (DLC stands for "distributive lattice of congruences. It is shown in Chapter V that nearly all the well-known calculi are DLC.)

4.12 Proposition. Let T be a standard DLC calculus which has a finite generic algebra of cardinality m. If T' is a standard extension of T, then for each finite subset $X \subseteq Y$ of cardinality m, $W(X, \Omega)/Q'(X)$ is a finite generic algebra for T'.

Proof: Let Q and Q' be the Lindenbaum congruences of T and T' respectively. Let $\tau: W(X, \Omega) \to W(X, \Omega)/Q(X)$ and $\rho: W(X, \Omega) \to W(X, \Omega)/Q'(X)$, be the corresponding natural homomorphisms. In the notation of lattice theory, the fact that $W(X, \Omega)/Q'(X)$ is generic may be stated as the equation

$Q' = \bigwedge\{\ker \sigma : \sigma \text{ is a homomorphism from } W(Y,\Omega) \text{ to } W(X,\Omega)/Q'(X)\}$.

Now every such homomorphism σ can be written as a composition $\sigma = \rho \circ h$, where ρ is the above natural homomorphism, and h is a homomorphism from $W(Y, \Omega)$ into $W(X, \Omega)$. To see this, choose any function $g: Y \to W(X, \Omega)$, such that for all $p \in Y$, $g(p) \in \sigma(p)$. Let h be the extension of g to a homomorphism. Since $\sigma(p)$ is an equivalence class mod $Q'(X)$, we have $\rho(h(p)) = \sigma(p)$ for all $p \in Y$; whence $\rho \circ h = \sigma$. Moreover, for each homomorphism $h: W(Y, \Omega) \to W(X, \Omega)$, the composition $\rho \circ h$ is a homomorphism. Therefore the required equation is equivalent to

$Q' = \bigwedge\{\ker \rho\circ h:$ h is a homomorphism from $W(Y, \Omega)$ to $W(X, \Omega)$.

Now let Z be any subset of Y. Notice, in view of Proposition 3.21 and the equation $\ker \rho = Q'(X)$, that for all homomorphisms $h:W(Z, \Omega) \to W(X, \Omega)$ we have $Q'(Z) \subseteq \ker \rho\circ h$. Similarly, by Lemma 4.9, $(\alpha, \beta) \in Q(Z)$ if and only if $(\alpha, \beta) \in \ker \tau\circ h$ for all simple homomorphisms $h:W(Z, \Omega) \to W(X, \Omega)$. Restating these facts in lattice notation we have, for all $Z \subseteq Y$,

(a) $Q'(Z) \subseteq \bigwedge\{\ker \rho\circ h \mid h$ is a homomorphism: $W(Z, \Omega) \to W(X, \Omega)\}$,

(b) $Q(Z) = \bigwedge\{\ker \tau\circ h \mid h$ is a simple homomorphism: $W(Z, \Omega) \to W(X, \Omega)\}$.

Taking $Z = Y$ in (a), we obtain

$Q' \subseteq \bigwedge\{\ker \rho\circ h:$ h is a homomorphism: $W(Y, \Omega) \to W(X, \Omega)\}$

and to complete the proof we must establish the opposite inclusion.

Suppose that $(\alpha, \beta) \notin Q'$. We need only show that $(\alpha, \beta) \notin \ker \rho\circ h$ for some homomorphism $h:W(Y,\Omega) \to W(X,\Omega)$. This condition may be replaced by $(\alpha, \beta) \notin \ker \rho\circ h$ for some homomorphism $h:W(Z, \Omega) \to W(X, \Omega)$, where Z is any subset of Y such that $\alpha, \beta \in W(Z, \Omega)$; for by Proposition 2.16 any such h can be extended to a homomorphism from $W(Y, \Omega)$ to $W(X, \Omega)$. Choose for Z any finite subset of Y containing all the propositional variables of both

α and β. We will use the distributive law to show that for all finite $Z \subseteq Y$,

(c) $Q'(Z) = \bigwedge\{\ker \rho \circ h \mid h$ is a simple homomorphism: $W(Z, \Omega) \to W(X, \Omega)\}$.

Since T' is standard, Lemma 3.20 shows that $Q'(Z)$ is the restriction of Q' to $W(Z, \Omega)$. Hence $(\alpha, \beta) \notin Q'(Z)$, and the required condition holds by (c).

To prove (c) when Z is finite, notice that there are only finitely many maps from Z into the finite set X. Hence by Proposition 2.12 there are only finitely many simple homomorphisms $h : W(Z, \Omega) \to W(X, \Omega)$. Therefore the right member of equation (b) is the meet of only finitely many congruences. Therefore the distributive law yields

$Q'(Z) \vee Q(Z) = \bigwedge\{Q'(Z) \vee (\ker \tau \circ h) \mid h$ is a simple homomorphism from $W(Z, \Omega)$ into $W(X, \Omega)\}$.

By Lemma 4.1, $Q(Z) \subseteq Q'(Z)$; hence $Q'(Z) \vee Q(Z) = Q'(Z)$. Equation (c) now follows by Lemma 4.11.

To show that $W(X, \Omega)/Q'(X)$ is finite, let Q^* be the restriction of the Lindenbaum congruence Q to $W(X, \Omega)$. Clearly, Q^* preserves $T(X)$. Hence, $Q^* \subseteq Q(X)$. By Lemma 4.1, $Q(X) \subseteq Q'(X)$. We now show that Q^* has only a finite number of equivalence classes, from which it follows, since $Q^* \subseteq Q'(X)$, that $W(X, \Omega)/Q'(X)$ is finite. Let A be a finite generic algebra for T. It follows, by Proposition 2.16, that $(\alpha, \beta) \in Q^*$ if and only if $\sigma(\alpha) = \sigma(\beta)$ for all assignments $\sigma: W(X, \Omega) \to A$. Thus Q^* is the meet (intersection) of the kernels of all assignments from A to $W(X, \Omega)$. Since, by Proposition 2.12, each assignment σ is uniquely determined by its values on X and since both X and A are finite there are only a finite number of distinct assignments σ. Since A is finite each congruence $\ker \sigma$ has only a finite number of equivalence classes. Now each equivalence class mod Q^* is the intersection over all σ of the equivalence classes mod $\ker \sigma$ which contain it. But only finitely many such intersections can be formed. Therefore Q^* has only finitely many equivalence classes.

The following is our main result, which we shall apply in Chapter V to particular calculi.

4.13 Proposition. Let T be a standard DLC calculus with a finite characteristic model. If T' is a standard extension of T in which every two theorems are Lindenbaum congruent, then T' has a finite characteristic model.

Proof: By Lemma 4.8, T has a finite generic algebra. By Proposition 4.12, T' has generic algebra, $W(X, \Omega)/Q'(X)$, where X is some finite subset of Y.

Since T has a finite characteristic model, by Proposition 3.16, every finite order Lindenbaum model is finite. Hence, by Proposition 4.2, $W(X, \Omega)/Q'(X)$ is finite. The result now follows on taking $A = W(X, \Omega)/Q'(X)$ in Lemma 4.5.

We conclude this chapter with a result on DLC calculi needed in Chapter V.

4.14 Lemma. Let T be a DLC calculus, and T' be any calculus such that $Q(X) \subseteq Q'(X)$ for each finite subset X of Y. Then T' is a DLC calculus.

Proof: For any finite subset X of Y, let L and L' denote the lattices of congruences on $W(X,\Omega)/Q(X)$ and $W(X, \Omega)/Q'(X)$, respectively. By Proposition 2.10 L is isomorphic to the lattice of congruences Q on $W(X, \Omega)$ such that $Q(X) \subseteq Q$, and similarly for L'. Since $Q(X) \subseteq Q'(X)$ by Lemma 4.1, it follows that L' is a sublattice of L, and hence is also distributive.

Combining this result with Lemma 4.1, we obtain:

4.15 Proposition. Every standard extension T' of a DLC calculus is a standard DLC calculus.

CHAPTER V

APPLICATIONS

In this section we will show that most familiar calculi are standard DLC calculi, and that this property holds also for their "normal" extensions. Then Proposition 4.13 shows that whenever a normal extension T of one of these calculi has a finite characteristic model, so also does each normal extension of T. For this purpose we will introduce two special DLC calculi; namely, LL which underlies the Intuitionist calculus of Heyting, and ML which underlies most modal logics. It is also shown that the implicational fragment of the Intuitionist calculus is a standard DLC calculus.

Let the operator domain Ω contain operators C, D, K, and N, where C, D, K are interpreted as implication, disjunction, conjunction, respectively, and N is interpreted as negation. Consider the following list of words, which are theorems of most familiar calculi:

5.1 (a) Cpp

(b) CCpqCCqrCpr

(c) CCpqCCrpCrq

(d) CCpqCNqNp

(e) CKpqp

(f) CKpqq

(g) CCpqCCprCpKqr

(h) CpDpq

(i) CqDpq

(j) CCprCCqrCDpqr

5.2 Proposition. Let T be a calculus containing the list of words 5.1. For each $X \subseteq Y$, let the relation "$\leq$" be defined on $W(X, \Omega)$ by $\alpha \leq \beta$ if and only if $C\alpha\beta \in T(X)$. This relation has, for all $\alpha, \beta \in W(X, \Omega)$, the following properties:

(a) $\alpha \leq \alpha$

(b) If $\alpha \leq \beta$ and $\beta \leq \gamma$ then $\alpha \leq \gamma$.

(c) If $\alpha \leq \beta$ then $C\beta\gamma \leq C\alpha\gamma$ and $C\gamma\alpha \leq C\gamma\beta$.

(d) If $\alpha \leq \beta$ then $N\beta \leq N\alpha$.

(e) $K\alpha\beta \leq \alpha$ and $K\alpha\beta \leq \beta$.

(f) If $\alpha \leq \beta$ and $\alpha \leq \gamma$ then $\alpha \leq K\beta\gamma$.

(g) $\alpha \leq D\alpha\beta$ and $\beta \leq D\alpha\beta$.

(h) If $\alpha \leq \gamma$ and $\beta \leq \gamma$ then $D\alpha\beta \leq \gamma$.

(i) If $\alpha \leq \beta$ then $D\alpha\gamma \leq D\beta\gamma$ and $D\gamma\alpha \leq D\gamma\beta$.

(j) If $\alpha \leq \beta$ then $K\alpha\gamma \leq K\beta\gamma$ and $K\gamma\alpha \leq K\gamma\beta$.

Proof: Property (a) follows from 5.1(a) by substitution. Property (b) follows from 5.1(b) and two applications of the rule of detachment, and similar arguments are used for (c) through (h). To prove the first part of (i), note that by (g), $\alpha \leq \beta \leq D\beta\gamma$. Hence by (b), $\alpha \leq D\beta\gamma$, and by (g) $\gamma \leq D\beta\gamma$. Finally $D\alpha\gamma \leq D\beta\gamma$ follows by (h). Similar arguments are used to prove the second part of (i) and (j).

5.3 Proposition. Let T be a calculus which contains the list of words 5.1 and is closed under the rule of detachment. If the bi-implicational relation ($\alpha \leq \beta$ and $\beta \leq \alpha$) is a congruence, then T is a DLC calculus, which is standard with respect to $\Phi = \{Cpq, Cqp\}$.

Proof: By Proposition 3.8 the bi-implicational relation Q^* coincides with the Lindenbaum congruence. Thus, T is standard with respect to $\Phi = \{Cpq, Cqp\}$. By Lemma 3.20, for each $X \subseteq Y$, $Q(X)$ is the restriction of Q^* to $W(X, \Omega)$; therefore by Proposition 5.2a, b, the relation "$\leq$" induces a partial order on the quotient algebra $W(X, \Omega)/Q(X)$. Proposition 5.2c, f, g, h show that this partially ordered set is a lattice with operators, the meet and join being given by the quotient operations $|K\alpha\beta| \bmod Q^*$ and $|D\alpha\beta| \bmod Q^*$. The lattice of congruences on the quotient algebra is distributive by Proposition 2.11. Therefore T is a DLC calculus.

5.4 Remark. To apply Proposition 5.3, it is necessary to verify that the bi-implicational relation is a congruence. Our tool for doing this is Lemma 3.7; and in fact Proposition 5.2(a), (b) are equivalent to parts (i), (ii) of the hypothesis of Lemma 3.7, while (c), (d), (i), (j) imply part (iii) for those elementary translations involving C, K, D, N. The next proposition applies this observation to a class of calculi which include the Classical and Intuitionist cases, while Proposition 5.17 applies it to Modal calculi.

Taking $\Omega = \{C, K, D, N\}$, let LL be the smallest calculus which contains the list of words 5.1 and is closed under the rule of detachment. Then those calculi to which Proposition 5.3 applies are precisely the "normal extensions" of LL in the following sense: A calculus T' is a normal extension of T if and only if $T \subseteq T'$ and T' is closed under the rule of detachment. (This is the weakest sense of "normal extension" used in the literature.) Thus, in view of Remark 5.4, we have:

5.5 Proposition. Every normal extension of LL, including LL itself, is a DLC calculus, and is a standard extension of LL with respect to $\Phi = \{Cpq, Cqp\}$.

Kleene [14] shows that 5.1a, b, c, d are theorems of the Intuitionist calculus; the remaining members of List 5.1 are explicitly given in 3.1. Thus, the Intuitionist calculus is a normal (and hence standard) extension of LL. Since the Classical calculus is clearly a normal extension of the Intuitionist calculus, both are DLC calculi.

Now consider two calculi T, T', where T is a standard extension of LL, and T' is a normal extension of T which, like the Intuitionist calculus, has CpCqp as a theorem. We will apply Proposition 4.13 to show that if T has a finite characteristic model, then so also does T'. First, T is a standard DLC calculus by Lemma 4.15. By Proposition 5.5 and the fact that Cpp $\in$ LL $\subseteq$ T, we see that T' is a standard extension of T. The fact that all pairs $\alpha, \beta \in$ T' are Lindenbaum congruent follows from Proposition 5.5 and the fact that $C\beta C\alpha\beta$, $C\alpha C\beta\alpha \in$ T', so that $C\alpha\beta$, $C\beta\alpha \in$ T'. In particular we have:

<u>5.6 Proposition.</u> If T is a standard extension of the Intuitionist calculus which has a finite characteristic model, then every normal extension of T has a finite characteristic model.

We turn our attention next to the implicational fragment H_I of the Intuitionist calculus. Axioms for H_I are given by Horn [12]. Let $\Omega = \{C\}$. Then H_I is the smallest calculus T such that:

(a) CpCqp $\in$ T,

(b) CCpCqrCCpqCpr $\in$ T,

and T is closed under the rule of detachment. We will show (Proposition 5.12) that H_I is a standard DLC calculus. Our proof is based on the work of Diego [4], but in order to apply his results we must first establish a correspondence between the Lindenbaum congruences and Diego's "deductive systems."

Let $\Omega = \{C\}$, A be an Ω-algebra, and D be a nonempty subset of A. The pair (A, D) is called a <u>pre-Hilbert algebra</u> if for all $x, y, z \in A$:

<u>5.7</u> (a) $CxCyx \in D$

(b) $CCxCyzCCxyCxz \in D$

(c) $x \in D$ and $Cxy \in D$ implies $y \in D$.

<u>5.8 Lemma</u>. Let (A, D) be a pre-Hilbert algebra. The relation

$$R = \{(x, y) \in A^2 \mid Cxy \in D \text{ and } Cyx \in D\}$$

is a congruence on A. Moreover, $a, b \in D$ implies $(a, b) \in R$.

<u>Proof</u>: We show: (i) $Cxx \in D$.

Substituting Cxx for y and x for z in 5.7(a), (b), we obtain $CxCCxxx$ and $CCxCCxxxCCxCxxCxx \in D$. Applying (c) gives $CCxCxxCxx \in D$. From (a) we have $CxCxx \in D$. Thus by (c), $Cxx \in D$.

(ii) If $y \in D$ then $Cxy \in D$.

This follows by 5.7(a) and (c).

(iii) If $Cxy \in D$ then $CCzxCzy \in D$.

By (ii) we have $CzCxy \in D$. Now (b) gives $CCzCxyCCzxCzy \in D$. Hence, by (c) we obtain $CCzxCzy \in D$.

(iv) $Cxy \in D$ and $Cyz \in D$ implies $Cxz \in D$.

By (iii) $Cyz \in D$ implies $CCxyCxz \in D$. Applying (c) gives $Cxz \in D$.

(v) $Cxy \in D$ implies $CCyzCxz \in D$.

By (a) we have $CCyzCxCyz \in D$. Combining this with (b) and applying (iv) yields $CCyzCCxyCxz \in D$. Substituting Cyz for x, Cxy for y, and Cxz for z in (b) and applying (c) we obtain $CCCyzCxyCCyzCxz \in D$. By (ii), we have $CCyzCxy \in D$ and another application of (c) yields (v).

The relation R is reflexive, symmetric, and transitive by (i), the definition of R, and (iv), respectively. The fact that each elementary translation is compatible with R follows by (iii) and (v). From (ii) we have: If $x, y \in D$ then $Cxy, Cyx \in D$; whence $(x, y) \in R$.

Let $\Omega = \{C\}$ and A be an Ω-algebra such that for all $x, y, z \in A$,

5.9 (i) $CCxxx = x$

(ii) $Cxx = Cyy$

(iii) $CxCyz = CCxyCxz$

(iv) $CCxyCCyxx = CCyxCCxyy$.

Any such Ω-algebra is called a Hilbert algebra. A nonempty subset D of a Hilbert algebra is called a deductive

<u>system</u> if $x \in D$ and $Cxy \in D$ imply $y \in D$, and for some x, $Cxx \in D$. If A is a Hilbert algebra, and Q is a congruence on A, we define:

$$N(Q) = \{x \in A \mid (x, Cxx) \in Q\}.$$

<u>5.10 Lemma</u>. Let A be a Hilbert algebra and Q be a congruence on A. Then $N(Q)$ is a deductive system. Moreover,

$$Q = \{(x, y) \in A^2 \mid Cxy \text{ and } Cyx \in N(Q)\}.$$

<u>Proof</u>: In view of 5.9(ii) we may denote Cxx by 1 for all $x \in A$. Assume x and $Cxy \in N(Q)$. Then $(x, 1)$ and $(Cxy, 1) \in Q$. Since Q is a congruence, $(x, 1) \in Q$ implies $(Cxy, C1y) \in Q$. By the transitive property of Q, $(C1y, 1) \in Q$. By 5.9(i), it follows that $C1y = y$; whence $y \in N(Q)$. Since $C11 = 1$, we have $C11 \in N(Q)$, so $N(Q)$ is a deductive system.

Let $(x, y) \in Q$. Since Q is a congruence, we have $(Cxy,Cxx) = (Cxy, 1) \in Q$; whence $Cxy \in N(Q)$. Similarly, $Cyx \in N(Q)$.

For the opposite inclusion, let $(Cxy, 1)$ and $(Cyx, 1) \in Q$. Since $(Cyx, 1)$, $(x, x) \in Q$ we have (see Remark 2.2), $(CCyxx, C1x) \in Q$. Repeating this operation with $(Cxy, 1)$ and $(CCyxx, C1x)$, we obtain $(CCxyCCyxx, C1C1x) \in Q$. Similarly $(CCyxCCxyy, C1C1y) \in Q$. By 5.9(iv) it follows that $(C1C1x, C1C1y) \in Q$, and since

$Clx = x$ for all x, it follows that $(x, y) \in Q$.

5.11 Lemma. The lattice of congruences on a Hilbert algebra is distributive (Brouwerian).

Proof: Diego [4] (Theorem 6, p.19) proves that the deductive systems of a Hilbert algebra A, ordered by inclusion, is a Brouwerian lattice. Let N be the function, from the lattice of congruences on A to the lattice of deductive systems, whose value at Q is $N(Q)$. We will show that N is a lattice isomorphism. To show that N is onto, let D be any deductive system on A. We will show that $R = \{(x, y) \in A^2 | Cxy, Cyx \in D\}$ is a congruence on A; whence $D = N(R)$. This follows from Lemma 5.8 once we have shown that (A, D) is a pre-Hilbert algebra. By 5.9(ii) and (iii), we have $Cxx = C(CxCyz)(CCxyCxz)$. Thus, $CCxCyzCCxyCxz \in D$. Setting $y = z$ in (iii) and using (ii) we obtain $CxCzz = CCxzCxz = Cxx \in D$. Setting $z = x$ in (iii) gives $CxCyx = CCxyCxx$, and it follows, on substituting Cxy for x and x for z above, that $CxCyx \in D$.

Lemma 5.10 shows that N is one-to-one. Clearly, both N and its inverse are order preserving. Thus N is a lattice isomorphism.

5.12 Proposition. The implicational fragment H_I of the Intuitionist calculus is a standard DLC calculus.

Proof: Recall that $H_I(X) = W(X, \Omega) \cap H_I$. Since the calculus H_I is closed under substitutions and the rule of detachment, the model $(W(X, \Omega), H_I(X))$ is a pre-Hilbert algebra. By Lemma 5.8, the bi-implicational relation is a congruence on $W(X, \Omega)$. Taking $X = Y$, and applying Lemma 3.8 it follows that H_I is standard with respect to $\Phi = \{Cpq, Cqp\}$. Diego ([4] Theorems 1 and 3) shows that for every pre-Hilbert algebra (A, D), the quotient algebra A/Q^*, where $Q^*=\{(x, y) | Cxy, Cyx \in D\}$, is a Hilbert algebra. Taking $A = W(X, \Omega)$ and $D = H_I(X)$, Lemma 3.20 shows that $Q^* = Q(X)$; hence $W(X, \Omega)/Q(X)$ is a Hilbert algebra. It now follows by Lemma 5.11 that H_I is a DLC calculus.

5.13 Lemma. If T is a normal extension of H_I then T is a standard extension of H_I with respect to $\Phi = \{Cpq, Cqp\}$.

Proof: Since $H_I \subseteq T$, $(W(Y, \Omega), T)$ is a pre-Hilbert algebra. By Lemma 5.8, the bi-implicational relation of T is a congruence. It now follows by Lemma 3.8 that T is standard with respect to $\Phi = \{Cpq, Cqp\}$. Since $Cpp \in H_I$, the result follows.

5.14 Proposition. Let T be a standard extension of the implicational fragment H_I of the Intuitionist calculus. If T has a finite characteristic model then every normal extension of T has a finite characteristic model.

<u>Proof</u>: Let T' be a normal extension of T. Then $H_I \subseteq T \subseteq T'$; whence, T' is a normal extension of H_I. By Lemma 5.12, T' is a standard extension of T. By Lemma 4.15, T is a standard DLC calculus, and the result now follows by Proposition 4.13.

We now consider the modal logics. In addition to the usual logical operators, implication, conjunction, etc., modal logics have an additional unary operator L. Among the various interpretations for L, the simplest is to regard "Lp" as meaning "p is necessary." If T is a calculus defined on $W(Y, \Omega)$, where $L \in \Omega$, it is possible to interpret both the words Cpq and LCpq as implication. In these calculi, Cpq is referred to as "material implication" and LCpq as "strict implication." The reader is referred to Lemmon [15] for simplified definitions and discussion of the well-known modal calculi S2-S5, E2-E5 and D2-D5: Although his definitions use only the operators C, N, L, regarding K, D as derived operators, we may take $\Omega = \{C, K, D, N, L\}$. Instead of listing rules and axioms for these systems, we simply note the following properties: (1) These calculi are ordered so that $Xm \subseteq Yn$ if $m \leq n$ and

$X \leq Y$ in the alphabetic ordering. (2) They all contain the classical calculus, and are closed under detachment. (3) S2-S5 are closed under strict detachment (i.e., if α and $LC\alpha\beta$ are theorems, then β is a theorem), and in S2-S5 strict bi-implication is a congruence (a property referred to by Lemmon and others as "substitutability of strict equivalents"). (4) The rule

<u>5.15</u> For all α, β, if $C\alpha\beta$ is a theorem then so also is $CL\alpha L\beta$, holds for E2-E5 and D2-D5.

By analogy with the calculus LL used above, taking $\Omega = \{C, K, D, N, L\}$ let ML be the smallest calculus containing the list of words 5.1, and which is closed under the rule 5.15 as well as detachment. In view of Remark 5.4, the rule 5.15 is precisely what is needed to obtain:

<u>5.16 Proposition</u>. Every normal extension of ML (including ML itself) which is closed under rule 5.15, is a DLC calculus, and is standard with respect to $\Phi = \{Cpq, Cqp\}$.

In view of properties (2) and (4) above, this result applies to all the modal calculi E2-E5 and D2-D5. By direct analogy with the discussion preceeding Proposition 5.6, we obtain:

<u>5.17 Proposition</u>. Let T be a standard extension of ML, and T' be a normal extension of T in which CpCqp is a theorem and which is closed under rule 5.15. If T has a finite characteristic model, then so also does T'.

Note that, since all the modal logics contain the Classical calculus, the condition "CpCqp is a theorem" is satisified whenever T' is an extension of a modal calculus.

<u>5.18 Lemma</u>. Every normal extension of ML, including each of the systems S2-S5, is a DLC calculus. Moreover, S2-S5 are standard with respect to $\Phi = \{LCpq, LCqp\}$.

<u>Proof</u>: Let T' be a normal extension of ML, and let Q, Q' be the Lindenbaum congruences for ML and T', respectively. We will show that for each subset X of Y, $Q(X)$ preserves $T'(X)$; whence $Q(X) \subseteq Q'(X)$, and T' is a DLC calculus by Lemma 4.14. Let $\alpha \in T'(X)$ and $(\alpha, \beta) \in Q(X)$. Since $Q(X)$ is a congruence, $(C\alpha\alpha, C\alpha\beta) \in Q(X)$, and since $C\alpha\alpha \in ML(X)$ while $Q(X)$ preserves $ML(X)$, we have $C\alpha\beta \in ML(X) \subseteq T'(X)$. It follows by detachment that $\beta \in T'(X)$.

By property (2) above, S2-S5 are normal extensions of ML. By property (3) and Lemma 3.8, they are standard with respect to the stated set Φ.

We note that since S4 and S5 are closed under the rule

$$\alpha \in S \text{ implies } L\alpha \in S$$

(rule (a) of Lemmon [15]), it follows by Lemma 3.9 that S4 and S5 are also standard with respect to $\Phi = \{Cpq, Cqp\}$.

We now collect and prove some results on the existence and nonexistence of finite characteristic models.

Scroggs [24] has shown that every normal extension of Lewis' modal logic S5 has a finite characteristic model. Recently Dunn [8] has proved that all normal extensions of R-mingle have finite characteristic models, and has announced a similar result for Dummett's calculus LC (LC is a normal extension of H).

There are two standard techniques for demonstrating the nonexistence of a finite characteristic model for all calculi between T and some extension T'. The first, due to Gödel [9], consists of exhibiting a sequence of words α_n all of which are non-theorems of T', but each α_n is valid in any finite model of T with fewer than n elements. In this way Dugundji [5] shows that no calculus between S1 and S5 has a finite characteristic model. Thomas [27] exhibits such a sequence for the implicational calculus IIC, and thus shows (without explicit statement) that no calculus between IIC and LIC (the implicational fragment of Dummett's LC) has a finite characteristic model. Prior [22] shows that the calculus OIC of Bull [2] is strictly between IIC and LIC. Thus, the result of Thomas has a nontrivial application.

The second method, due to McKinsey [19], consists of exhibiting a sequence of words α_n such that for $i \neq j$, α_i and α_j are not Lindenbaum congruent in T', and which all belong to $W(X, \Omega)$ for some finite subset X of Y. Thus, if T' is standard, Lemma 3.20 shows that some finite order Lindenbaum model of T' is infinite. It follows by Remark 4.3 that neither T', nor any calculus T of which T' is a standard extension, has a finite characteristic model. McKinsey [19] shows that the calculus S4 has such a sequence, and remarks that hence none of the weaker systems S1-S3 has a finite characteristic model. In view of our definition of "standard extension," and the fact that S4 is standard with respect to {LCpq, LCqp} as well as {Cpq, Cqp}, the same is true of all weaker calculi having either LCpp or Cpp as a theorem.

McKinsey and Tarski ([20], Theorem 4.5), show that the first order Lindenbaum model of the Intuitionist calculus H is infinite. Thus we may conclude, by Remark 4.3, that no calculus T such that $T \subseteq H$ and $Cpp \in T$

has a finite characteristic model.

Proposition 4.13 allows us to prove some similar results on the non-existence of finite characteristic models, based on the fact that Dummett's calculus LC does not have a finite characteristic model (see Dummett [6], Theorem 6). The calculus LC is the smallest normal extension of H containing the additional word DCpqCqp. Thus, by Proposition 5.5, LC is a standard extension of LL which contains the word CpCqp. If T is any calculus between LL and LC which is closed under detachment, it follows by Proposition 5.5 and the paragraph preceeding Proposition 5.6 that T does not have a finite characteristic model.

CHAPTER VI

COMPLETE CALCULI

It is well-known that the classical calculus K has a maximal property, called by Tarski [26] "complete"; that is, the only normal extension of K is the trivial calculus in which all words are theorems. This can easily be shown as follows: Assume that α is a theorem of some normal extension T of K, and $\alpha \notin K$. Since $\alpha \notin K$ there is an assignment σ from the Classical truth tables such that $\sigma(\alpha) = f$. Now define a substitution s such that

$$s(x) = \begin{cases} Cpp & \text{if } \sigma(x) = t \\ NCpp & \text{if } \sigma(x) = f \end{cases}$$

for all $x \in Y$. Let $\alpha' = s(\alpha)$, and σ' be any assignment from the Classical truth tables. Since the composition $\sigma' \circ s(x) = \sigma(x)$ for all $x \in Y$, we have $\sigma'(\alpha') = f$. Thus, $\sigma'(N\alpha') = t$ for all assignments σ', and so $N\alpha' \in K$. Since T is closed under substitutions, $\alpha' \in T$. Now $CpCNpq \in K$. Therefore, since $K \subseteq T$, we have $N\alpha', \alpha', C\alpha'CN\alpha'\beta \in T$ for every word β. It now follows by detachment that $\beta \in T$.

In keeping with the terminology of Chapter III, a calculus T' will be called a <u>Φ-standard extension</u> of a calculus T if T' is Φ-standard and $\phi(p, p) \in T$ for all $\phi \in \Phi$. A calculus T is <u>consistent</u> if $T \neq W(Y, \Omega)$. We will call T <u>Φ-complete</u> if T is consistent and T has no consistent Φ-standard extension T' such that $T \neq T'$. We denote by <u>$V(T, X)$</u> the calculus consisting of the valid words of the Lindenbaum matrix $(W(X, \Omega)/Q(X), T(X)/Q(X))$, where $X \subseteq Y$, and X is finite.

<u>6.1 Lemma</u>. If T is a consistent Φ-standard calculus, then $V(T, X)$ is a consistent Φ-standard extension of T.

<u>Proof</u>: Since T is consistent, $p \notin T(X) \subseteq T$ for any $p \in X$. Since $Q(X)$ preserves $T(X)$, $|p| \bmod Q(X) \notin T(X)/Q(X)$. Hence $p \notin V(T, X)$. Therefore $V(T, X)$ is consistent.

By Lemma 4.8, $(W(X, \Omega)/Q(X), T(X)/Q(X))$ is a standard matrix with respect to Φ. By Lemma 4.6, $V(T, X)$ is a Φ-standard calculus; it is a Φ-standard extension of T since T is Φ-standard and $T \subseteq V(T, X)$.

<u>6.2 Proposition</u>. If T is a Φ-complete, Φ-standard calculus, then the first order Lindenbaum model is characteristic for T.

Proof: This is an immediate consequence of Lemma 6.1 and the definition of Φ-complete.

6.3 Lemma. Let $L(\Phi, T)$ be the set of all Φ-standard extensions of the calculus T. Then $L(\Phi, T)$ is a complete lattice, ordered by inclusion. If Φ is a finite set then the join of an increasing sequence is its union.

Proof: The first half of the proof depends on the following result ([1], Theorem 3, p.112): If P is a partially ordered set and every subset (including the void subset) has a g.l.b in P, then P is a complete lattice. Assume $L(\Phi, T)$ is nonempty; otherwise the theorem is vacuously true. In the partially ordered set $L(\Phi, T)$, $W(Y, \Omega)$ is the g.l.b for the empty set.

Let $T_\sigma : \sigma \in \Sigma$ be any nonempty family from $L(\Phi, T)$, and let $T^* = \cap\{T_\sigma | \sigma \in \Sigma\}$. Clearly, T^* is a calculus which contains T. Moreover, if T^* is Φ-standard, then T^* is the g.l.b of the family T_σ. We now show, using Lemma 3.9, that T^* is Φ-standard. Let Q_σ be the Lindenbaum congruence of T_σ, and let $Q = \cap\{Q_\sigma | \sigma \in \Sigma\}$. Since each calculus T_σ is Φ-standard, we have $(\alpha, \beta) \in Q$ if and only if $\phi(\alpha, \beta) \in T^*$ for all $\phi \in \Phi$. Hence, we need only show that Q preserves T^*. Assume $(\alpha, \beta) \in Q$ and $\alpha \in T^*$ then $(\alpha, \beta) \in Q_\sigma$ and $\alpha \in T_\sigma$ for all $\sigma \in \Sigma$.

Since Q_σ preserves T_σ, we have $\beta \in T^*$. Thus, $L(\Phi, T)$ is a complete lattice.

Now assume that Φ is finite, and let T_i be a sequence in $L(\Phi, T)$ such that $T_i \subseteq T_{i+1}$. Let Q_i be the Lindenbaum congruence of T_i. Clearly, T_{i+1} is a standard extension of T_i; whence, by Lemma 4.1, $Q_i \subseteq Q_{i+1}$. We now apply Lemma 3.9, with Q' equal to the union of the congruences Q_i, to show that $T^* = \bigcup_i T_i$ is Φ-standard; whence T^* is the join of the family T_i. To see that Q' is a congruence, let $(\alpha, \beta) \in Q'$ and $(\beta, \gamma) \in Q'$. Thus, $(\alpha, \beta) \in Q_i$ and $(\beta, \gamma) \in Q_i$ for sufficiently large index i; whence $(\alpha, \gamma) \in Q'$. The remaining properties of a congruence hold for the union of any set of congruences. The congruence Q' preserves T^*. For if $\alpha \in T^*$ and $(\alpha, \beta) \in Q'$ then $\alpha \in T_j$ and $(\alpha, \beta) \in Q_k$; whence $\alpha \in T_i$ and $(\alpha, \beta) \in Q_i$ for $i \geq j, k$. Thus β is a member of $T_i \subseteq T^*$. If $\phi(\alpha, \beta) \in T^*$ for all $\phi \in \Phi$ then each $\phi(\alpha, \beta) \in T_i$ for some i. Taking the index i sufficiently large it follows, since Φ is finite, that $\phi(\alpha, \beta) \in T_i$ for all $\phi \in \Phi$. Since T_i is Φ-standard we have $(\alpha, \beta) \in Q_i \subseteq Q'$, and $\phi(\alpha, \alpha) \in T_i \subseteq T^*$ for all $\phi \in \Phi$ and all words α. Therefore by Lemma 3.9, T^* is Φ-standard.

The following theorem is a variation of a result of

Lindenbaum (see [26]).

<u>6.4 Proposition</u>. Let T be a consistent Φ-standard calculus. If the set Φ is finite, then T has a (consistent) Φ-complete, Φ-standard extension.

<u>Proof</u>: Let the set $W(Y, \Omega) = \{\alpha_1, \alpha_2, \ldots\}$ be indexed by the integers. Consider the sequence of calculi defined by:

$$T_0 = T$$

and for all integers i,

$$T_{i+1} = \begin{cases} T_i & \text{if } \alpha_{i+1} \text{ is not a member of any consistent} \\ & \Phi\text{-standard extension of } T_i, \\ \text{g.l.b. of the consistent } \Phi\text{-standard extensions} & \\ \quad \text{of } T_i \text{ containing } \alpha_{i+1}, \text{ otherwise.} & \end{cases}$$

In view of Lemma 6.3, the calculi T_i are well-defined Φ-standard extensions of T; since they are clearly increasing, the union $T^* = \bigcup_i T_i$ is a Φ-standard extension of T. We will show that T^* is Φ-complete. To see that T^* is consistent, note that, since each T_i is consistent, $p \notin T_i$ for all i; whence $p \notin T^*$. To show that T^* is Φ-complete, let T^{**} be any consistent Φ-standard extension of T^* (hence also of each T_i) such that $T^{**} \neq T^*$. Then there is a word α_j in T^{**} which is not in T^*, contradicting the construction of T_j.

<u>6.5 Lemma</u>. Let T' be a Φ-standard extension of T, and let X be a subset of Y. Then there is a homomorphism h, between the algebras of the first order Lindenbaum models for T and T', such that

$$h(T(X)/Q(X)) \subseteq T'(X)/Q'(X).$$

<u>Proof</u>: By Lemmas 4.1 and 2.8, the function given by $h(|\alpha| \bmod Q(X)) = |\alpha| \bmod Q'(X)$ is such a homomorphism. The last inclusion follows from the fact that $h(T(X)/Q(X)) = T(X)/Q'(X)$, where $T(X) \subseteq T'(X)$.

A matrix (A, D) is called <u>Φ-complete</u> if and only if for every matrix (B, E), such that B is the image of A under a homomorphism h, and $h(D) \subseteq E$, one of the following conditions holds:

<u>6.6</u> (a) The model (B, E) is not standard with respect to Φ.

(b) Every word is valid in (B, E).

(c) The model (B, E) is equivalent to the model (A, D).

<u>6.7 Proposition</u>. Let T be a consistent calculus and Φ be a finite set. If the first order Lindenbaum model of T is both Φ-complete and characteristic for T, then T is Φ-complete.

<u>Proof</u>: Assume that T is not Φ-complete. Then T has a consistent Φ-standard extension T' different from

T. By Proposition 6.4 T' has a consistent, Φ-complete, Φ-standard extension T''. Thus, T'' is also a Φ-complete, Φ-standard extension of T. Taking (B, E) in 6.6 to be the first order Lindenbaum model for T'', and h to be the homomorphism given by Lemma 6.5, we will show that none of conditions 6.6 are met, contrary to the hypothesis that the first order Lindenbaum model for T is Φ-complete. By Lemmas 4.8 and 6.1, respectively, conditions (a) and (b) are not met. By hypothesis and Lemma 6.2 the first order Lindenbaum model for T and T'' are characteristic for T and T''. Since T'' is complete and T is not complete, condition 6.6(c) is not met.

<u>6.8 Proposition</u>. Let T be a Φ-standard calculus. If T is Φ-complete then the first order Lindenbaum model of T is Φ-complete and characteristic for T.

<u>Proof</u>: It follows directly from Proposition 6.2 that if T is Φ-complete then the first order Lindenbaum model (A, D) is characteristic for T. Assume that (A, D) is not complete. Then there exists a model (B, E) and a homomorphism h such that $h(A) = B$ and $h(D) \subseteq E$, and the model (B, E) fails to satisfy each of the conditions 6.6. Let V be the set of valid words of the model (B, E). We will show that V is a consistent Φ-standard extension of T such that $T \neq V$, contradicting the

hypothesis that T is Φ-complete. By Proposition 6.2, (A, D) is characteristic for T. Since (B, E) is characteristic for V, it follows from the negation of 6.6(c) that $T \neq V$. By Lemma 3.13 $T \subseteq V$. By Lemma 4.6 and the negation of condition 6.6(a), V is a Φ-standard extension of T. It follows from the negation of 6.6(b) that V is consistent.

By the implicational fragment K_I of the Classical calculus we mean the set of valid words of the matrix whose algebra is defined by the truth table

<u>6.9</u>

C	t	f
t	t	f
f	t	t

and whose only designated element is t. The following result was first obtained by Wajsberg [29] using a technique similar to that on page 76.

<u>6.10 Proposition</u>. The implicational fragment of the Classical calculus is complete.

<u>Proof</u>: Taking $\Phi = \{Cpq, Cqp\}$ we will, using Proposition 6.7, show that K_I is Φ-complete. The proposition then follows once it is shown that every normal extension T of K_I is Φ-standard. Since the words CpCqp and CCpCqrCCpqCpr are clearly valid in the matrix 6.9, T is a normal extension of H_I; whence T is Φ-standard by Lemma 5.13.

Now the first order Lindenbaum model for K_I is isomorphic to the matrix 6.9. This follows in view of Proposition 3.16 and construction preceding it, with $m = 1$. Since the matrix 6.9 is characteristic for K_I, it remains only to show that this matrix is Φ-complete.

Clearly, the only homomorphic image (B, E) of 6.9 is either the trivial algebra of only one element or an isomorph of 6.9. In the former case condition (b) of 6.6 holds, while in the latter case either (b) or (c) holds, depending on whether or not $E = B$.

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APPENDIX

Index of Terms