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Various problems in physics and engineering lead to integral equations of the Fredholm type and second kind. Generally speaking, Fredholm's solution of such equations is given in terms of the ratio of two infinite series. This method is not customarily thought of as useful for computation because the direct calculation of the terms of these series is formidably difficult. However, the successive terms of the two series can be calculated from recursion formulas as suggested by G. C. Evans. The principal result contained in this thesis is the bound for the error at the n^{th} step of this process. Techniques of F. Tricomi are adapted to this problem. An illustrative example is worked, and tables are given which will facilitate use of the error bound by others.

AN ERROR BOUND FOR AN ITERATIVE METHOD OF SOLVING FREDHOLM INTEGRAL EQUATIONS

by

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AN ERROR BOUND FOR AN ITERATIVE METHOD OF SOLVING FREDHOLM INTEGRAL EQUATIONS

INTRODUCTION

Physical problems are usually formulated as differential equations or integral equations. Although differential equations are more familiar, many problems can be conveniently expressed as integral equations. The calculus of variations is a rich source of such equations and the use of the Green's function for the differential equations of a boundary value problem results in an integral equation. Considering everything, there is reason to suspect that the extensive use of differential equations is more habitual than expedient. A boundary value problem, when expressed as an integral equation, may appear rather cumbersome as compared with the corresponding differential equation. However, it must be remembered that the integral equation expresses not only the "local" condition of the problem, but also the boundary conditions which must be satisfied. For instance, a linear boundary value problem in one variable can usually be expressed as

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt$$

where $K(x,t)$ and $f(x)$ contain the boundary conditions.

This is known as a Fredholm type linear integral equation of the second kind and it represents the class which will be considered here. The function $f(x)$ and the "kernel" $K(x,t)$ are known. λ is a known parameter and $u(x)$ is the unknown function.

The solution of this equation, if it exists, can be expressed as (6, pp.19-20),

$$u(x) = f(x) + \int_a^b H(x,t;\lambda)\, f(t)\, dt$$

with $H(x,t;\lambda)$ as the "resolvent kernel".

There is a well-known expansion for the resolvent kernel which was derived by Fredholm (3, pp.365-390). This expansion gives the resolvent kernel as the quotient of two integral power series and, generally speaking, both series contain an infinite number of terms. If the series contain only a finite number of terms, it is possible to compute the exact solution. In the general case, however, actual computation of a solution by Fredholm's method must result in an approximation. It is our purpose here to determine a bound for the error contained in this approximate solution.

One of the major objections to the use of Fredholm's expansion for the resolvent kernel has been the amount of labor involved. The coefficient of the n^{th} term of each series involves n successive integrations of a determinant of order n. Beyond the second or third term, the direct computation of these coefficients is far too laborious to be practical. However, relationships exist which lead to two relatively simple recursion formulas. This method of computing the series is not usually found in the literature and, therefore, it seemed that it should appear here.

The necessary relationships for this method appear in the works of Fredholm (3, p.371), as part of his derivation of the solution. Lalesco (5, pp.25-33), used the same relationships to derive Fredholm's resolvent kernel from the iterated or Neuman resolvent kernel. They also appear in several books which present the Fredholm theory but the suggestion that they be used for computation appears very seldom in the literature.

Evans, in a veview of a book written by Lovitt, apparently was the first to suggest the use of the relationships for computation (1, pp.144-145). Frank and von Mises (2, p.516), and Hildebrand (4, p.432), included them as computational aids in their presentation of the Fredholm theory.

In the chapters that follow, we shall consider first the true solution as given by Fredholm and then the approximate solution. Following this will be the method of computation and the bound for the error in the approximate solution. The final chapter will contain a table for use in evaluating the bound and a numerical example.

The error bound is developed along the lines of Tricomi's derivation of an error bound for a similar problem (7, pp.483-486), and (8, pp.26-30). His approach to the problem is used and the error bound is given in terms of the functions that he used. The principal difference is that Tricomi used an approximate kernel $K^*(x,y)$, while here an approximation is made directly to the true resolvent kernel. To be explicit, Tricomi replaced the kernel $K(x,y)$ by $K^*(x,y)$ where $K^*(x,y)$ satisfied the condition

$$| K(x,y) - K^*(x,y)| < \epsilon$$

uniformly in x and y. He then developed an error bound for the solution obtained by using $K^*(x,y)$ as the kernel.

FREDHOLM'S SOLUTION

The integral equations under consideration are of the type

$$(2.1) \qquad u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt$$

Where $K(x,t)$ and $f(x)$ are known functions, λ is a known parameter, and $u(x)$ is the unknown function.

Fredholm solved this equation subject to the following conditions:

a. $K(x,y)$ is real and continuous on $R: a \leqq x,y \leqq b$.

b. $f(x)$ is real and continuous on $I: a \leqq x \leqq b$.

c. $D(\lambda) \neq 0$. (See equation 2.5)

Under these conditions, the unique, continuous solution is given by

$$(2.2) \qquad u(x) = f(x) + \int_a^b \frac{\Delta(x,t;\lambda)}{D(\lambda)} f(t)\, dt.$$

The functions $\Delta(x,y;\lambda)$ and $D(\lambda)$ are the integral power series referred to in the introduction and $\Delta(x,y;\lambda)$ is defined by

$$(2.3) \qquad \Delta(x,y;\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{n+1}}{n\,!} B_n(x,y)$$

with $B_o(x,y) = K(x,y)$ and for $n \geqq 1$,

(2.4)

$$B_n(x,y) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(x,y) & K(x,t_1) & \cdots & K(x,t_n) \\ K(t_1,y) & K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n,y) & K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt \ldots dt_n.$$

The series in the denominator is defined by

(2.5) $$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n\,!} A_n$$

with $A_o = 1$ and for $n \geqq 1$,

(2.6) $$A_n = \int_a^b \cdots \int_a^b \begin{vmatrix} K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \cdots\cdots & \cdots & \cdots\cdots \\ K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n .$$

Both series converge for $-\infty < \lambda < +\infty$ and $\Delta(x,y;\lambda)$ converges uniformly with respect to x and y on R: $a \leqq x,y \leqq b$.

THE APPROXIMATE SOLUTION

Our approximate solution $u_m(x)$ will be obtained by taking only a finite number m of the terms in the series for $\Delta(x,y;\lambda)$ and $D(\lambda)$. That is

(3.1) $$u_m(x) = f(x) + \int_a^b \frac{\Delta_m(x,t;\lambda)}{D_m(\lambda)} f(t)\, dt$$ where

(3.2) $$\Delta_m(x,y;\lambda) = \sum_{n=0}^{m} \frac{(-1)^n \lambda^{n+1}}{n\,!} B_n(x,y)$$ and

(3.3) $$D_m(\lambda) = \sum_{n=0}^{m} \frac{(-1)^n \lambda^n}{n\,!} A_n$$

with $B_n(x,y)$ and A_n defined by (2.4) and (2.6).

COMPUTATION OF A_n AND $B_n(x,y)$

As mentioned in the introduction, the direct computation of A_n and $B_n(x,y)$ for $n > 2$ is quite laborious. Therefore, we will present here two recursion formulas which are relatively easy to use. The recursion formulas are

$$(4.1) \qquad A_{n+1} = \int_a^b B_n(t,t)\,dt \qquad \text{and}$$

$$(4.2) \qquad B_{n+1}(x,y) = A_{n+1}K(x,y) - (n+1)\int_a^b B_n(x,t)\,K(t,y)\,dt.$$

Equations (4.1) and (4.2) are valid for all $n \geq 0$ and since $A_o = 1$ and $B_o(x,y) = K(x,y)$, the desired number of terms may be computed.

Equation (4.2) is essentially an expression of Fredholm's first fundamental relationship which is (3, p.373),

$$\Delta(x,y;\lambda) = \lambda D(\lambda)\,K(x,y) + \lambda\int_a^b \Delta(x,t;\lambda)\,K(t,y)\,dt$$

and may be derived from this relationship by associating like powers of the parameter λ. We will derive both equations directly from the definition in the following manner. In view of (2.4), we may write

$$\int_a^b B_n(t,t)\,dt = \int_a^b \left\{ \int_a^b \cdots \int_a^b \begin{vmatrix} K(t,t) & K(t,t_1) & \cdots & K(t,t_n) \\ K(t_1,t) & K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n,t) & K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n \right\} dt.$$

Replacing t by t_1, t_1 by t_2, ..., t_n by t_{n+1}, this may be written

$$\int_a^b B_n(t,t)dt = \int_a^b \cdots \int_a^b \begin{vmatrix} K(t_1,t_1) & \cdots & K(t_1,t_{n+1}) \\ K(t_{n+1},t_1) & \cdots & K(t_{n+1},t_{n+1}) \end{vmatrix} dt_1 \ldots dt_{n+1}$$

and considering (2.6) this becomes

$$\int_a^b B_n(t,t)\, dt = A_{n+1}$$

which is the desired result.

In order to derive equation (4.2) consider (2.4) which is

$$B_n(x,y) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(x,y) & K(x,t_1) & \cdots & K(x,t_n) \\ K(t_1,y) & K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n,y) & K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \ldots dt_n .$$

Developing the determinant in the integrand in terms of the elements of the first column, it is found that

$$B_n(x,y) = \int_a^b \cdots \int_a^b K(x,y) \begin{vmatrix} K(t_1,t_1) & \cdots & K(t_1,t_n) \\ K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \ldots dt_n$$

$$+ \sum_{i=1}^{n} (-1)^i \int_a^b \cdots \int_a^b K(t_i,y) \begin{vmatrix} K(x,t_1) & \cdots & K(x,t_n) \\ \cdots & \cdots & \cdots \\ K(t_{i-1},t_1) & \cdots & K(t_{i-1},t_n) \\ K(t_{i+1},t_1) & \cdots & K(t_{i+1},t_n) \\ \cdots & \cdots & \cdots \\ K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \ldots dt_n .$$

In view of (2.6), the first term reduces to $A_n K(x,y)$. In the terms of the summation, replace t_i by t, t_{i+1} by t_i, ... , t_n by t_{n-1}, and we obtain for that summation

$$\sum_{i=1}^{n}(-1)^i \int_a^b \cdots \int_a^b K(t,y)\, K\begin{Bmatrix} x, t_1, \ldots\ldots\ldots\ldots\ldots\ldots, t_{n-1} \\ t_1, t_2, \ldots, t_{i-1}, t, t_i, \ldots, t_{n-1}\end{Bmatrix} dt\, dt_1 \ldots dt_{n-1}$$

where
$$K\begin{Bmatrix} x, t_1, \ldots\ldots\ldots\ldots\ldots\ldots, t_{n-1} \\ t_1, t_2, \ldots, t_{i-1}, t, t_i, \ldots, t_{n-1}\end{Bmatrix} =$$

$$\begin{vmatrix} K(x,t_1) \ldots\ldots & K(x,t_{i-1}) & K(x,t) & K(x,t_i) \ldots & K(x,t_{n-1}) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ K(t_{i-1},t_1) \ldots\ldots & & & \ldots\ldots & K(t_{i-1},t_{n-1}) \\ K(t_i,t_1) \ldots\ldots & & K(t_i,t) & \ldots\ldots & K(t_i,t_{n-1}) \\ K(t_{i+1},t_1) \ldots\ldots & & & \ldots\ldots & K(t_{i+1},t_{n-1}) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ K(t_{n-1},t_1) \ldots & K(t_{n-1},t_{i-1}) & K(t_{n-1},t) & K(t_{n-1},t_i) \ldots & K(t_{n-1},t_{n-1}) \end{vmatrix}$$

Now bringing the column containing $K(x,t)$ into the first column, the summation becomes

$$\sum_{i=1}^{n}(-1)^{2i-1} \int_a^b \cdots \int_a^b K(t,y) \begin{vmatrix} K(x,t) & K(x,t_1) \ldots\ldots & K(x,t_{n-1}) \\ K(t_1,t) & Kt_1,t_1) \ldots\ldots & K(t_1,t_{n-1}) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ K(t_{n-1},t) & K(t_{n-1},t_1) \ldots & K(t_{n-1},t_{n-1}) \end{vmatrix}$$
$$dt\, dt_1 \ldots dt_{n-1} .$$

This form of the integrand shows that the terms of the summation are all equal. In addition, we may integrate first with respect to $t_1 \ldots t_{n-1}$ and consider $K(t,y)$ constant for these integrations. $K(t,y)$ may be taken before the $(n-1)$-fold integral and we have

$$-n\int_a^b K(t,y)\left\{\int_a^b \cdots \int_a^b \begin{vmatrix} K(x,t) & K(x,t_1) & \cdots\cdots & K(x,t_{n-1}) \\ K(t_1,t) & K(t_1,t_1) & \cdots\cdots & K(t_1,t_{n-1}) \\ \cdots & \cdots & \cdots & \cdots \\ K(t_{n-1}t) & K(t_{n-1},t_1) & \cdots & K(t_{n-1},t_{n-1}) \end{vmatrix}\right.$$

$$\left. dt_1 \cdots dt_{n-1} \right\} dt.$$

Considering (2.4), this may be written

$$-n\int_a^b K(t,y)\, B_{n-1}(x,t)\, dt$$

and the final relationship is

$$B_n(x,y) = A_n K(x,y) - n\int_a^b B_{n-1}(x,t)\, K(t,y)\, dt.$$

THE BOUND FOR THE ERROR

We will now develop a bound for the error $|u(x) - u_m(x)|$ with $u(x)$ as the true solution and $u_m(x)$ as the approximate solution. Subtracting (3.1) from (2.1) we obtain

$$(5.1) \qquad u(x) - u_m(x) = \int_a^b \left[\frac{\Delta(x,t;\lambda)}{D(\lambda)} - \frac{\Delta_m(x,t;\lambda)}{D_m(\lambda)} \right] f(t)\, dt.$$

Omitting the arguments for the present, (5.1) may be written

$$u - u_m = \int_a^b \left[\frac{\Delta D_m - \Delta_m D}{D\, D_m} \right] f\, dt$$

or

$$u - u_m = \int_a^b \left[\frac{\Delta(D_m - D) + D(\Delta - \Delta_m)}{D\, D_m} \right] f\, dt.$$

Taking absolute values, we arrive at the inequality

$$(5.2) \qquad |u - u_m| \leq \int_a^b \left[\frac{|\Delta|\,|D_m - D| + |D|\,|\Delta - \Delta_m|}{|D|\,|D_m|} \right] |f|\, dt.$$

In order to get (5.2) into a usable form, upper bounds must be found for $|D|$, $|\Delta|$, $|D_m - D|$, and $|\Delta - \Delta_m|$. We will get these bounds in terms of the functions;

$$(5.3) \qquad \Omega(x) = \sum_{n=0}^{\infty} \frac{n^{\frac{n}{2}}}{n!} x^n,$$

$$\Omega'(x) = \sum_{n=1}^{\infty} \frac{n^{\frac{n}{2}}}{(n-1)!} x^{n-1},$$

$$\Omega_m(x) = \sum_{n=m+1}^{\infty} \frac{n^{\frac{n}{2}}}{n!} x^n,$$

and
$$\Omega_m'(x) = \sum_{n=m+1}^{\infty} \frac{n^{\frac{n}{2}}}{(n-1)!} x^{n-1} .$$

In deriving the bounds, frequent use will be made of Hadamard's theorem concerning determinants. It is stated here briefly.

HADAMARD'S THEOREM. If the elements a_{ij} of the determinant

$$A = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

are real and satisfy the inequality

$$|a_{ij}| \leqq M ,$$

then

$$|A| \leqq M^n n^{\frac{n}{2}} .$$

AN UPPER BOUND FOR $|D(\lambda)|$. Restating (2.5), we write

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} A_n$$

with $A_0 = 1$ and for $n \geqq 1$,

$$A_n = \int_a^b \cdots \int_a^b \begin{vmatrix} K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \cdots\cdots & \cdots & \cdots\cdots \\ K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n .$$

Since $K(x,y)$ is continuous in the closed rectangle R, it is also bounded there and we may assume that $|K(x,y)| \leqq M$ in R. Therefore, Hadamard's theorem may be applied to the absolute value of the determinant in the integrand and the resulting inequality is

$$|A_n| \leqq \int_a^b \cdots \int_a^b n^{\frac{n}{2}} M^n \, dt_1 \cdots dt_n$$

or

(5.4) $$|A_n| \leqq n^{\frac{n}{2}} M^n \ (b-a)^n .$$

Now $$|D(\lambda)| \leqq \sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!} |A_n|$$

and if we substitute (5.4) into this expression, we will have the inequality

$$|D(\lambda)| \leqq \sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!} n^{\frac{n}{2}} M^n (b-a)^n .$$

If we let $N = |\lambda| \ M \ (b-a)$ and consider (5.3), the result is

(5.5) $$|D(\lambda)| \leqq \Omega \ (N) .$$

AN UPPER BOUND FOR $|\Delta(x,y;\lambda)|$. We will start with equation (2.3) which is

$$\Delta(x,y;\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{n+1}}{n!} B_n(x,y)$$

where $B_o(x,y) = K(x,y)$ and for $n \geqq 1$,

$$B_n(x,y) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(x,y) & K(x,t_1) & \cdots & K(x,t_n) \\ K(t_1,y) & K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n,y) & K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n .$$

Again we may apply Hadamard's theorem to the absolute value of the determinant and it is found that

$$|B_n(x,y)| \leqq \int_a^b \cdots \int_a^b (n+1)^{\frac{n+1}{2}} M^{n+1} \ dt_1 \cdots dt_n$$

or

(5.6) $$|B_n(x,y)| \leqq (n+1)^{\frac{n+1}{2}} M^{n+1} (b-a)^n .$$

Now $$|\Delta(x,y;\lambda)| \leqq \sum_{n=0}^{\infty} \frac{|\lambda|^{n+1}}{n!} |B_n(x,y)|$$

and if we substitute (5.6) into this inequality, it will become

$$|\Delta(x,y;\lambda)| \leqq \sum_{n=0}^{\infty} \frac{|\lambda|^{n+1}}{n!} (n+1)^{\frac{n+1}{2}} M^{n+1} (b-a)^n .$$

Letting $N = |\lambda| M (b-a)$ and changing the index of summation this may be written

$$|\Delta(x,y;\lambda)| \leqq |\lambda| M \sum_{n=1}^{\infty} \frac{n^{\frac{n}{2}}}{(n-1)!} N^{n-1}$$

and considering (5.3), the final form is

$$(5.7) \qquad |\Delta(x,y;\lambda)| \leqq |\lambda| M \Omega'(N).$$

AN UPPER BOUND FOR $|D_m(\lambda) - D(\lambda)|$. Subtracting equation (2.5) from (3.3) we get the difference

$$D_m(\lambda) - D(\lambda) = -\sum_{n=m+1}^{\infty} \frac{(-1)^n \lambda^n}{n!} A_n .$$

For the absolute value, we get the inequality

$$|D_m(\lambda) - D(\lambda)| \leqq \sum_{n=m+1}^{\infty} \frac{|\lambda|^n}{n!} |A_n| .$$

Using (5.4) with $N = |\lambda| M (b-a)$ and in view of (5.3), this may be written as

$$(5.8) \qquad |D_m(\lambda) - D(\lambda)| \leqq \sum_{n=m+1}^{\infty} \frac{n^{\frac{n}{2}}}{n!} N^n = \Omega_m(N) .$$

AN UPPER BOUND FOR $|\Delta(x,y;\lambda) - \Delta_m(x,y;\lambda)|$. With $B_n(x,y)$ as previously defined, the difference between (2.3) and (3.2) is

$$\Delta(x,y;\lambda) - \Delta_m(x,y;\lambda) = \sum_{n=m+1}^{\infty} \frac{(-1)^n \lambda^{n+1}}{n!} B_n(x,y) .$$

Using (5.6), we get the inequality

$$|\Delta(x,y;\lambda) - \Delta_m(x,y;\lambda)| \leqq \sum_{n=m+1}^{\infty} \frac{|\lambda|^{n+1}}{n!} (n+1)^{\frac{n+1}{2}} M^{n+1} (b-a)^n .$$

With N as previously defined, this may be written

$$|\Delta(x,y;\lambda) - \Delta_m(x,y;\lambda)| \leqq |\lambda| M \sum_{n=m+2}^{\infty} \frac{n^{\frac{n}{2}}}{(n-1)!} N^{n-1} .$$

Considering (5.3), this becomes

$$(5.9) \qquad |\Delta(x,y;\lambda) - \Delta_m(x,y;\lambda)| \leqq |\lambda| M \Omega'_{m+1}(N) .$$

SUBSTITUTING THE BOUNDS. We will now substitute the expressions just found for the corresponding quantities in (5.2) which is

$$|u(x) - u_m(x)| \leqq \int_a^b \frac{|\Delta| |D_m - D| + |D| |\Delta - \Delta_m|}{|D| |D_m|} |f(t)| \, dt .$$

Let $F = \max |f(x)|$ for $a \leqq x \leqq b$. Since $D(\lambda)$ and $D_m(\lambda)$ are constants, we may bring them from under the integral sign. Now substituting the bounds we have just found, (5.5), (5.7), (5.8), and (5.9), the inequality becomes

$$|u(x) - u_m(x)| \leqq \left[\frac{|\lambda| M \Omega'(N) \Omega_m(N) + \Omega(N) |\lambda| M \Omega'_{m+1}(N)}{|D| |D_m|} \right] F (b-a)$$

or

$$(5.10) \quad |u(x) - u_m(x)| \leqq N F \left[\frac{\Omega'(N) \Omega_m(N) + \Omega(N) \Omega'_{m+1}(N)}{|D| |D_m|} \right] .$$

The quantity $|D_m|$ may be left in this expression because it will be computed as part of the solution but a lower bound must be found for $|D|$. In order to get this lower bound, consider (5.8) which is

$$|D_m - D| \leq \Omega_m(N) .$$

Since $|D_m| - |D| \leq |D_m - D|$, we may write

$$|D_m| - |D| \leq \Omega_m(N) \quad \text{or}$$

(5.11) $$|D_m| - \Omega_m(N) \leq |D| .$$

Providing that the quantity $|D_m| - \Omega_m(N)$ is positive, it may be substituted for $|D|$ in (5.10) without disturbing the inequality. There is no reason to believe that for a particular value of m this will be true. However, $\Omega_m(N)$ is positive, monotone decreasing with m, and its limit is zero as m becomes infinite. On the other hand, $|D_m|$ will converge to a fixed constant $|D|$. Since we have assumed that $D(\lambda) \neq 0$, there will exist a positive integer p such that

$$|D_m| > \Omega_m(N) \quad \text{for} \quad m > p .$$

This means that in order for this substitution to be valid, m must be taken large enough to insure that $|D_m| > \Omega_m(N)$. If $D(\lambda)$ is very close to zero, this would require that m be very large. In practice, the computation will indicate the point at which the expression is valid.

Assuming that the substitution is valid, (5.10) becomes

(5.12) $$|u(x) - u_m(x)| \leq N F \frac{\Omega'(N)\Omega_m(N) + \Omega(N)\Omega'_{m+1}(N)}{|D_m|\{|D_m| - \Omega_m(N)\}} .$$

This is the final form for the error bound.

TABULATIONS AND A NUMERICAL EXAMPLE

The function $\Omega(x)$ has been tabulated for $0 \leqq x \leqq 1$ at intervals of 0.05 . It was tabulated by Tricomi for use in his error bound which was mentioned in the introduction (8, p. 28).

The functions $\Omega_m(x)$ and $\Omega_m'(x)$ are not convenient to tabulate. That is, they must be computed for each value of m. For this reason, the coefficients in the series expressions for $\Omega(x)$ and $\Omega'(x)$ are tabulated below. The series for $\Omega(x)$ may be used to determine the value of $\Omega_m(x)$ by omitting the term containing x^m and all preceding terms. Similarly, to obtain $\Omega_m'(x)$, from $\Omega'(x)$, omit the term containing x^{m-1} and all preceding terms.

The second table is a tabulation of $\Omega(x)$, $\Omega'(x)$, $\Omega_2(x)$, $\Omega_4(x)$, $\Omega_6(x)$, $\Omega'_2(x)$, $\Omega'_4(x)$, and $\Omega'_6(x)$ for $0 \leqq x \leqq 1$ at intervals of 0.1 .

COEFFICIENTS OF THE SERIES FOR

$\Omega(x)$ AND $\Omega'(x)$

	$\Omega(x)$	$\Omega'(x)$		$\Omega(x)$	$\Omega'(x)$
x^0	1.000000	1.000000	x^{12}	0.006234	0.036333
x	1.000000	2.000000	x^{13}	0.002795	0.016928
x^2	1.000000	2.598077	x^{14}	0.001209	0.007591
x^3	0.866026	2.666667	x^{15}	0.000506	0.003284
x^4	0.666667	2.329238	x^{16}	0.000205	0.001375
x^5	0.465848	1.800000	x^{17}	0.000081	0.000558
x^6	0.300000	1.260406	x^{18}	0.000031	0.000220
x^7	0.180058	0.812698	x^{19}	0.000011	0.000084
x^8	0.101587	0.488170	x^{20}	0.000004	0.000031
x^9	0.054241	0.275573	x^{21}	0.000001	0.000011
x^{10}	0.027557	0.147196	x^{22}		0.000004
x^{11}	0.013382	0.074805			

TABLE OF NUMERICAL VALUES

	x = 0	x = 0.1	x = 0.2	x = 0.3	x = 0.4	x = 0.5	x = 0.6	x = 0.7	x = 0.8	x = 0.9	x = 1.0
$\Omega(x)$	1.000000	1.110938	1.248166	1.420181	1.638871	1.921110	2.291210	2.784745	3.454618	4.380827	5.686444
$\Omega_2(x)$	0.000000	0.000938	0.008166	0.030181	0.078871	0.171104	0.331210	0.594745	1.014618	1.670827	2.686444
$\Omega_4(x)$	0.000000	0.000005	0.000171	0.001398	0.006379	0.021184	0.057748	0.137631	0.298146	0.602094	1.153752
$\Omega_6(x)$	0.000000	0.000000	0.000003	0.000047	0.000379	0.001939	0.007527	0.024042	0.066854	0.167584	0.387904
$\Omega'(x)$	1.000000	1.228901	1.529652	1.930203	2.471325	3.213360	4.246948	5.710354	7.817878	10.907159	15.519250
$\Omega'_2(x)$	0.000000	0.028901	0.129652	0.330203	0.671325	1.213360	2.046948	3.310354	5.217878	8.107159	12.519250
$\Omega'_4(x)$	0.000000	0.000253	0.004395	0.024376	0.084966	0.230507	0.535640	1.122630	2.189775	4.058717	7.254507
$\Omega'_6(x)$	0.000000	0.000002	0.000092	0.001133	0.006905	0.028680	0.093803	0.260854	0.645896	1.467622	3.125269

For an example of the application of this error bound, we will consider the boundary value problem

$$\frac{d^2u}{dx^2} + \lambda u + g(x) = 0, \quad u(0) = u(1) = 0 .$$

The corresponding integral equation is

$$u(x) = f(x) + \lambda \int_0^1 K(x,t)\, u(t)\, dt$$

where

$$K(x,t) = \begin{cases} x(1-t) & \text{for } 0 \leqq x \leqq t \leqq 1 \\ t(1-x) & \text{for } 0 \leqq t \leqq x \leqq 1 \end{cases}$$

and

$$f(x) = \int_0^1 K(x,t)\, g(t)\, dt .$$

For this kernel, we find that to satisfy $|K(x,y)| \leqq M$ in R, we must take $M = 0.25$. It will be assumed that $\lambda = 1$ and $\max |f(x)| = F$ for $0 \leqq x \leqq 1$. Therefore $N = |\lambda|\, m(b-a)=0.25$ and the error expression becomes

(6.1)

$$|u(x) - u_m(x)| \leqq 0.25\, F \left[\frac{\Omega'(0.25)\,\Omega_m(0.25) + \Omega(0.25)\,\Omega'_{m+1}(0.25)}{|D_m(1)|\{|D_m(1)| - \Omega_m(0.25)\}} \right]$$

This expression will be evaluated for $m = 2, 4, 6$ and for this purpose we will need the following quantities:

(6.2)

$$\Omega(0.25) = 1.329177 \qquad \Omega'(0.25) = 1.715269$$

$$\Omega_2(0.25) = 0.016677 \qquad \Omega'_3(0.25) = 0.052890$$

$$\Omega_4(0.25) = 0.000541 \qquad \Omega'_5(0.25) = 0.002125$$

$$\Omega_6(0.25) = 0.000013 \qquad \Omega'_7(0.25) = 0.000059$$

$$D_2(1) = 0.841661$$

$$D_4(1) = 0.841471$$

$$D_6(1) = 0.841471$$

Considering these values, it is seen that the error expression is valid for $m \geqq 2$. That is,

$$|D_2(1)| > \Omega_2(0.25) .$$

Substituting the values as given by (6.2) into (6.1), we obtain

$$|u(x) - u_2(x)| \leqq 0.0356 F$$

$$|u(x) - u_4(x)| \leqq 0.00133 F$$

$$|u(x) - u_6(x)| \leqq 0.000035 F .$$

BIBLIOGRAPHY

1. Evans, G. C. A review of the book, Linear Integral Equations, by W. V. Lovitt. The American mathematical monthly 34:142-150. 1927.

2. Frank, Phillip and Richard von Mises. Die Differential und Integralgleichungen der Mechanik und Physik. 2d. ed. Vol. 1. Braunschweig, Friedrich Vieweg und Sohn, 1930. 916p.

3. Fredholm, Ivar. Sur une classe d'équations fonctionnelles. Acta mathematica 27:365-390. 1903.

4. Hildebrand, F. B. Methods of applied mathematics. New York, Prentice-Hall, 1952. 432p.

5. Lalesco, Trajan. Introduction à la théorie des équations intégrales. Paris, A. Herman et fils, 1912. 152p.

6. Lovitt, William Vernon. Linear integral equations. New York. McGraw-Hill, 1924. 253p.

7. Tricomi, Franceso. Sulla risoluzione numerica delle equazioni integrali di Fredholm. Atti della reale accademia nazionale dei lincei, Rendiconti 33, 1° Semestre: 483-486. 1924.

8. Tricomi, Francesco. Ancora sulla risoluzione numerica delle equazioni integrali di Fredholm. Atti della reale accademia nazionale dei lincei, Rendiconti 33, 2° Semestre:26-30. 1924.