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GRAPH FOLDING

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79-20-2

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ABSTRACT

A graph fold is the special case of a graph homomorphism where the two identified vertices are both adjacent to a common vertex. Like homomorphisms, folds are related to the chromatic number and we obtain an Interpolation Theorem for folds. If $\chi(G) = n$, then G is absolutely n-chromatic if every fold preserves the chromatic number. Every nontrivial bipartite graph is absolutely 2-chromatic. Given $m \geq 4$, we give a construction of a three chromatic graph that folds onto K_m and conjecture that this is the smallest such graph.

INTRODUCTION

In this paper we introduce the concept of graph folding, a special case of graph homomorphism, and show that it has properties similar and dissimilar to graph homomorphisms. If G has chromatic number n, then G folds onto K_n. The a-jointed number for folding is similar to the achromatic number of a homomorphism. The chromatic number and a-jointed number are bounds for an Interpolation Theorem for graph folding. We introduce the concept of a graph being absolutely chromatic, that is, every fold preserves its chromatic number. Bipartite graphs are absolutely chromatic whereas a homomorphism of a bipartite graph may have arbitrarily large chromatic number.

Definitions and examples of folding are given in Section 2. In Section 3 we show that just as for homomorphisms every graph G folds onto K_n where $n = \chi(G)$. If for every fold f, $\chi(G) = n = \chi(f(G))$ then G is absolutely n-chromatic. In Section 4 we show that every connected bipartite graph is absolutely n-chromatic. This is not true for homomorphisms. Characterizing absolutely m-chromatic graphs for $m \geq 3$ appears to be very difficult.

The a-jointed number of G, the maximum chromatic number of any graph H obtained from a fold of G, is defined in Section 4. The a-jointed number is less than or equal to the achromatic number, its homomorphism counterpart. An Interpolation Theorem nearly identical to the one for homomorphisms is obtained. Given an integer $m \geq 4$, Section 5 gives the construction of a three chromatic graph that folds onto K_m. It is conjectured that the construction gives the graph with the fewest number of vertices. Finally Section 6 lists some open problems on folding.

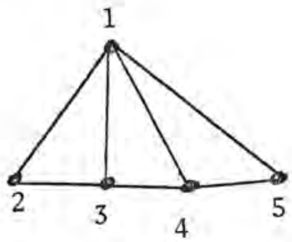
2. DEFINITIONS AND EXAMPLES

In this paper we consider only finite undirected connected graphs without loops or multiple edges. Two vertices of a graph are <u>adjacent</u> if they are joined by an edge. If x is the edge joining u and v, then x is said to be <u>incident</u> with u and with v. The <u>degree</u> of a vertex v is the number of edges incident with v. A set of vertices is <u>independent</u> if no two of them are adjacent. A <u>maximum independent set</u> is a largest independent of vertices. Two nonadjacent vertices are <u>jointed</u> if they are both adjacent to a common vertex. A <u>coloring</u> of a graph is an assignment of colors to the vertices of the graph such that no two adjacent vertices are assigned the same color. The <u>chromatic number</u> $\chi(G)$ of a graph G is the minimum number of colors needed in a coloring of the graph.

The <u>identification</u> of nonadjacent vertices u and v in a graph G is the graph resulting from removing vertices u and v and all edges incident with u and v from G and adding the vertex w and edges from w to all vertices that were adjacent to either u or v. An <u>elementary homomorphism</u> is an identification of two nonadjacent vertices. A <u>homomorphism</u> is a sequence of elementary homomorphisms. A <u>simple fold</u> is an elementary homomorphism in which the identified vertices are jointed. A <u>fold</u> is a sequence of simple folds. Hence a simple fold is merely the special case of an elementary homomorphism where the two identified vertices are both adjacent to a common vertex. A fold f is <u>complete of order n</u>, if $f(G) = K_n$.

Example 1.

1.



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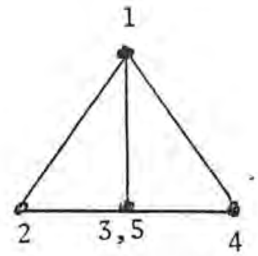
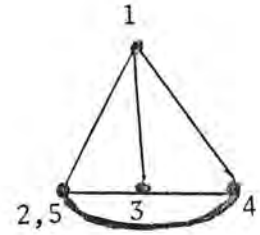
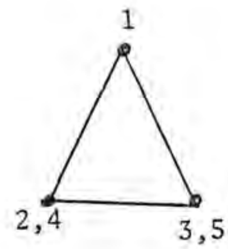


$f_{3,5}(G)$

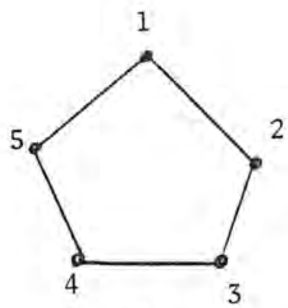


$f_{2,5}(G)$

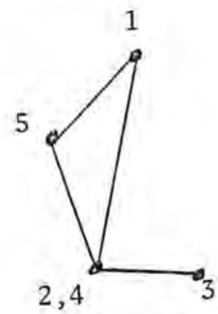


$f_{2,4}(f_{3,5}(G))$

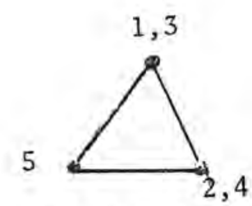
2.



H

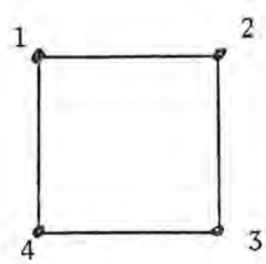


$f_{2,4}(H)$

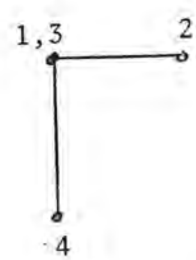


$f_{1,3}(f_{2,4}(H))$

3.



L



$f_{1,3}(L)$

$f_{2,4}(f_{1,3}(L))$

3. FOLDING

In this section we show that if the chromatic number of G is n, then there is a folding of G onto K_n, the complete graph with $n \geq 1$ vertices.

Theorem (Brooks). If the maximum degree of a vertex of G is n, then G can be colored using n colors unless i) n= 2 and G is an odd cycle, or ii) $n > 2$ and G is K_{n+1}.

Lemma 1. If $G \neq K_n$ has chromatic number n and if G is colored with n colors, then there exist two jointed vertices with the same color.

Proof:

Obviously true if $\chi(G) = 2$ and G is connected with $p > 2$ vertices. So assume that the chromatic number of G is greater than two. Suppose that G does not contain two jointed vertices of the same color. Then this implies that the maximum degree of any vertex in G is less than the chromatic number. But by Brook's Theorem, the chromatic number of G is less than or equal to the maximum degree unless G is an odd cycle or is a complete graph. If G is an odd cycle it contains two jointed vertices with the same color and if G is a complete graph it does not contain any jointed vertices.

Therefore G must contain two jointed vertices with the same color. ∎

Theorem 1. If $\chi(G) = n$, then there exists a complete fold of G onto K_n.

Proof:

By Lemma 1, G must contain two jointed vertices of the same color. If we identify these two vertices via a simple fold the resulting graph has chromatic number n. Hence by a sequence of simple folds in which the jointed vertices have the same color, we can fold G onto K_n. ∎

Since an elementary homomorphism increases the chromatic by at most one [2], we have the following.

Proposition 1. If f is a simple fold, then $\chi(G) \leq \chi(f(G)) \leq \chi(G) + 1$.

4. ABSOLUTELY n-CHROMATIC GRAPHS

As illustrated in Example 1, for a given graph some folds may increase the chromatic number while others do not. In this section we will consider those graphs whose chromatic number remains the same for all folds.

A graph G is said to be <u>absolutely n-chromatic</u> if $\chi(G) = n$ and $\chi(f(G)) = n$ for all folds f of G. A <u>bipartite graph</u> is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that every edge joins a vertex in V_1 with one in V_2. A <u>complete n-partite graph</u> is a graph whose vertices can be partitioned into n subsets such that there is an edge between every pair of vertices in distinct subsets.

<u>Theorem</u> 2. Al bipartite graphs are absolutely 2-chromatic.

<u>Proof</u>:

Let G be a bipartite graph with vertex set $V = V_1 \cup V_2$. If a and b are two jointed vertices of G, then both must be in either V_1 or V_2. Hence every simple fold must identify two vertices both of which are in either V_1 or V_2. Thus every bipartite graph is absolutely 2-chromatic. ∎

<u>Corollary</u> 2.1. All complete n-partite graphs are absolutely n-chromatic.

Even though folding is a special case of homomorphism, an elementary homomorphism of a bipartite graph may increase its chromatic number. For example, if G is a path of length three, then the identification of the two end vertices results in a K_3.

<u>Lemma</u> 2. If G is an odd cycle, then G is absolutely 3-chromatic.

<u>Proof</u>:

If G is an odd cycle of length m $(m > 3)$, then f(G) consists of an odd cycle of length m-2 and a vertex of degree one which is the vertex adjacent to the two vertices that were identified by f. In succeeding simple folds, either the odd cycle length is reduced by two and a vertex of degree one is added or the cycle remains the same and the vertex of degree one is one of the two vertices identified by the simple fold. ∎

5. A-JOINTED NUMBER

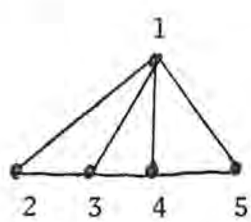
In this section we obtain bounds for $\chi(f(G))$ where f is a fold of G. We also show that for $n \geq 4$, we can construct a graph G with $\chi(G) = 3$ and G folds onto K_n. We conjecture that the graph given by the construction has the fewest number of vertices.

A complete partition of the vertices of a graph G is a partition of the vertices into independent sets with the property that for each pair of sets there is an edge of G joining a vertex in one set with a vertex in the other. The achromatic number, denoted $\Psi(G)$, is the maximum number of blocks in a complete partition. An independent set of vertices has the jointed property if it cannot be partitioned into two non-empty subsets A and B such that no vertex in A is jointed to a vertex in B. An a-jointed partition of a graph is a complete partition of its vertices each block of which has the jointed property. The a-jointed number is the maximum number of blocks in an a-jointed partition.

Lemma 3. The a-jointed number of a graph is less than or equal to the achromatic number.

Lemma 4. Let π be an a-jointed partition of G and let f be a simple fold of a pair of vertices in the same block of π. Then the a-jointed partition of f(G) induced by π is an a-jointed partition.

Example 2.



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Two a-jointed partitions of G are:

1. {1} , {2,4} , {3,5}
2. {1} , {2,5} , {3} , {4}

The a-jointed number of G is 4.

Theorem 3. If n is the a-jointed number of G, then G folds onto K_m, $m \leq n$.

Proof:

If G folds onto K_m, then the partition of the vertices of G into m sets corresponding to the vertices of K_m is a complete partition. Furthermore, the partition must have the a-jointed property and hence is an a-jointed partition. ∎

Theorem 4. (Interpolation Theorem). For any graph G, if $\chi(G) = m$ and n is the a-jointed number of G, then G folds onto K_p for $m \leq p \leq n$.

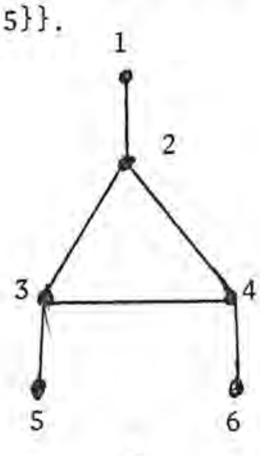
Proof:

Proof identical to the proof of the Interpolation Theorem for graph homomorphisms [3]. ∎

Since the a-jointed number of a bipartite graph with two or more vertices is two, we obtain Theorem 2 as a corollary of Theorem 4.

Corollary 4.1. Every bipartite graph with more than one vertex is absolutely 2-chromatic.

Example 3. The graph G below has achromatic number 4 and a-jointed number 3. The achromatic partition of G is {{1,5,6}, {2}, {3}, {4}} and the a-jointed partition is {{1,3}, {2,6}, {4,5}}.



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Next we will show how to construct a graph with chromatic number three that folds onto K_n, $n \geq 4$. We conjecture that the construction yields the smallest graph.

Theorem 5. For any $n \geq 4$, there is a graph G with chromatic number three that folds onto K_n.

Proof:

We will construct a three chromatic graph G that folds onto K_n by "unfolding" K_n. Label the vertices of K_n with the integers 1 to n. Assign color a to vertex 1, color b to all even numbered vertices, and color c to all odd numbered vertices greater than 1. For each edge (i,j), $i < j$, between two vertices of the same color, remove the edge (i,j) and add a vertex labeled (i,j) and edges from vertex (i,j) to vertex 1 and to vertex i. The resulting graph G has chromatic number 3. By folding each vertex (i,j) of G onto vertex j we obtain K_n. ∎

The graph G constructed in Theorem 5 has $\frac{n^2+3}{4}$ vertices and $\frac{3n^2-6n+3}{4}$ edges if n is odd and $\frac{n^2+4}{4}$ vertices and $\frac{3n^2-6n+4}{4}$ edges if n is even.

Conjecture 1. The graph constructed in the proof of Theorem 5 has the minimum number of vertices and edges of any graph that satisfies the conditions of the theorem.

6. OPEN PROBLEMS

1. Nontrivial bipartite graphs constitute the class of absolutely 2-chromatic graphs. The second author [1] has obtained a characterization of absolutely 3-chromatic graphs. Characterize the class of absolutely m-chromatic graphs for $m > 3$.

2. Prove or disprove Conjecture 1.

3. It is obvious that given $n \geq 4$, we can find a path with a homomorphism onto K_n. For each n find the smallest path. For each n find the smallest tree that is homomorphic to K_n. Does the unfolding construction used in the proof of Theorem 5 yield the smallest tree? Find the smallest bipartite graph homomorphic to K_n.

4. Characterize the class of graphs whose a-jointed number is strictly less than (equal to) its achromatic number.

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3. F. Harary, S. Hedetniemi, and G. Prins, An Interpolation Theorem for Graphical Homomorphisms, Portugalie Mathematica 20 (1967), pages 453-462.