A Partial Proof of Bertrand's Theorem

By
John Musgrove

A THESIS

submitted to

Oregon State University

University Honors College

in partial fulfillment of
the requirements for the
degree of

Honors Baccalaureate of Science in Mathematics
(Honors Scholar)

Presented May 26, 2020
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AN ABSTRACT OF THE THESIS OF

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Abstract approved:

Patrick De Leenheer

In this paper, we provide a thorough proof of most of Bertrand's Theorem. Using Arnold's book "Mathematical methods of classical mechanics" as a backbone and calculus methods demonstrated by Jovanović in his article "A note on the proof of Bertrand's theorem," we show that for masses in a central field with bounded orbits, closed orbits are restricted to central forces obeying a power law. This paper covers a subset of all possible power laws, leading to the conclusion that only Newton's Law of gravitation and Hooke's Law provide closed, bounded orbits.

Key Words: Mathematics, Calculus, Physics, Classical Mechanics, Two-Body Problem

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Honors Baccalaureate of Science in Mathematics project of John Musgrove presented on May 26, 2020

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I understand that my project will become part of the permanent collection of Oregon State University Honors College. My signature below authorizes release of my project to any reader upon request.

John Musgrove, Author

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1 Introduction

An old, basic problem in physics concerns the description of the orbits traced out by masses in a two-body system, particularly the case where a planet revolves around the sun. It is well-known from Kepler's laws that these orbits are closed and of elliptical shape. A key property underlying this type of motion is the Newtonian Law of gravitation which stipulates that the two bodies exert attractive forces on each other which are proportional to the inverse square of the distance between both bodies. In this project we investigate some consequences of how the motion would change if the force is not inversely proportional to the distance squared, but instead takes a more general functional form that only depends on the distance between the two bodies.

In 1873, Joseph Louis Francois Bertrand published a remarkable theorem stating that there are only two types of gravitational laws in which all bounded orbits are closed, namely under the mentioned Newtonian gravity, and when the force is proportional to the distance between the bodies (Hooke's Law). In other words, Bertrand established the astounding fact that we live in a universe governed by one of only 2 possible gravitational laws guaranteeing that all bounded orbits will be closed. While Bertrand's Theorem is accepted throughout mathematics and physics, a complete and exhaustive proof of the theorem does not exist in one piece. We aim to compile a partial proof of Bertrand's Theorem.

2 The Two-Body Problem

The two-body problem asks to find the positions of two point masses in space at any time t that experience forces caused only by each other. The key to solving this problem is to reduce it into two simpler one-body problems, one corresponding to the motion of the system's center of mass and the other corresponding to the displacement between the two masses.

2.1 Reducing the Problem

Let m_1 and m_2 denote the point masses of two bodies located at $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ at time t, respectively. According to Newton's laws of motion, we find that

$$\mathbf{F}_{12}(|\mathbf{x}_1 - \mathbf{x}_2|) = m_2\ddot{\mathbf{x}}_2 \tag{2.1}$$

$$\mathbf{F}_{21}(|\mathbf{x}_2 - \mathbf{x}_1|) = m_1\ddot{\mathbf{x}}_1 \tag{2.2}$$

where $\mathbf{F}_{ij}$ is a vector pointing from m_j to m_i whose magnitude depends only on the distance between them. We assume that this magnitude is sufficiently smooth for all positive distances. Because this system is isolated, by Newton's third law, we find

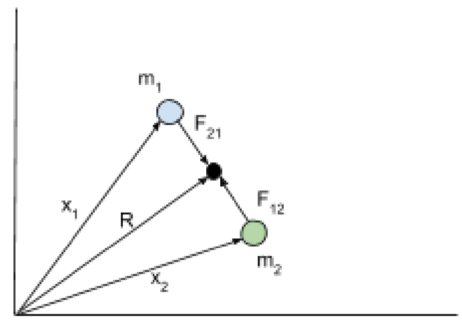

Figure 1: A depiction of masses m_1 and m_2 acting on each other by forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ at positions $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively.

that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. By adding these two equations, we find an equation describing the motion of the center of mass of the two bodies. Let $\mathbf{R}$ denote the position of the center of mass with respect to the origin depicted in Figure 1:

$$\mathbf{R} \equiv \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2} \tag{2.3}$$

Then adding the equations yields:

$$0 = \mathbf{F}_{12} + \mathbf{F}_{21} = m_2\ddot{\mathbf{x}}_2 + m_1\ddot{\mathbf{x}}_1 = (m_1 + m_2)\ddot{\mathbf{R}}$$

It follows directly that the velocity $\dot{\mathbf{R}}$ is constant and the position $\mathbf{R}(t)$ can be found for all t using initial conditions of the system:

$$\mathbf{R}(t) = \mathbf{R}(0) + \dot{\mathbf{R}}(0)t$$

where

$$\mathbf{R}(0) = \frac{m_1\mathbf{x}_1(0) + m_2\mathbf{x}_2(0)}{m_1 + m_2}$$

and

$$\dot{\mathbf{R}}(0) = \frac{m_1\dot{\mathbf{x}}_1(0) + m_2\dot{\mathbf{x}}_2(0)}{m_1 + m_2}$$

which can be determined from the initial positions and velocities of m_1 and m_2.

By subtracting these two force equations, we can find an equation of motion for $\mathbf{r}$, the vector that points from one of the two bodies to the other:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{x}}_1 - \ddot{\mathbf{x}}_2 = \frac{\mathbf{F}_{21}}{m_1} - \frac{\mathbf{F}_{12}}{m_2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F}_{21}$$

hence

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{21}(|\mathbf{r}|) \tag{2.4}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is called the reduced mass. This is the reduced problem that can be interpreted as the motion of a fictitious point mass in a central force field $\mathbf{F}_{12}(|\mathbf{r}|)$.

The crux of solving the two-body problem lies in finding $\mathbf{r}(t)$ using $\mathbf{F}_{12}(|\mathbf{r}|)$. Indeed, then $\mathbf{R}$ and $\mathbf{r}$ can be used to find the positions of the two bodies for future times with the following formulas:

$$\mathbf{x}_1(t) = \mathbf{R}(t) + \frac{\mu}{m_1}\mathbf{r}(t) \tag{2.5}$$

$$\mathbf{x}_2(t) = \mathbf{R}(t) - \frac{\mu}{m_2}\mathbf{r}(t) \tag{2.6}$$

2.2 Properties of the Reduced System

2.2.1 Conservation of Angular Momentum

One of the most important consequences of the reduced system is that the solution $\mathbf{r}(t)$ is planar. Let $\mathbf{L}$ be the angular momentum of the reduced system defined by

$$\mathbf{L} = \mathbf{r} \times \mu\frac{d\mathbf{r}}{dt}$$

where $\mathbf{r}$ is the relative position of the mass μ and its linear momentum is defined as the reduced mass times the velocity. The rate of change of $\mathbf{L}$ is:

$$\frac{d\mathbf{L}}{dt} = \dot{\mathbf{r}} \times \mu\dot{\mathbf{r}} + \mathbf{r} \times \mu\ddot{\mathbf{r}} = \dot{\mathbf{r}} \times \mu\dot{\mathbf{r}} + \mathbf{r} \times \mathbf{F}$$

But because the cross product of two vectors in the same direction is 0, and $\mathbf{r}$ and $\mathbf{F}$ are in opposite directions, it follows that $\frac{d\mathbf{L}}{dt} = 0$. Hence, $\mathbf{L}$ is constant and therefore $\mathbf{r}$ and $\dot{\mathbf{r}}$ lie in the plane orthogonal to $\mathbf{L}$. Let the magnitude of the angular momentum be denoted as l.

2.2.2 Change of Variables and Conservation of Energy

We return to the reduced one-body two-dimensional vector differential equation (2.4)

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{21}(|\mathbf{r}|) = -h(|\mathbf{r}|)\frac{\mathbf{r}}{|\mathbf{r}|}$$

where μ is the reduced mass and $h(|\mathbf{r}|)$ determines the magnitude of the force on the mass of interest. Because the forces of the system depend only on the distance between the masses, it is useful to define the one-body system in polar coordinates $\langle \rho, \theta \rangle$, where

$$\mathbf{r} = \langle x, y \rangle : \begin{cases} x = \rho\cos(\theta) \\ y = \rho\sin(\theta) \end{cases}$$

and

$$\dot{\mathbf{r}} = \langle \dot{x}, \dot{y} \rangle : \begin{cases} \dot{x} = \dot{\rho}\cos(\theta) - \rho\dot{\theta}\sin(\theta) \\ \dot{y} = \dot{\rho}\sin(\theta) + \rho\dot{\theta}\cos(\theta) \end{cases}$$

Note that

$$\dot{\mathbf{r}} = \dot{\rho}\langle \cos(\theta), \sin(\theta) \rangle + \rho\dot{\theta}\langle -\sin(\theta), \cos(\theta) \rangle$$

where the velocity is expressed in terms of unit vectors in the ρ and θ directions, respectively.

We also have

$$\ddot{\mathbf{r}} = \langle \ddot{x}, \ddot{y} \rangle : \begin{cases} \ddot{x} = \ddot{\rho}\cos(\theta) - 2\dot{\rho}\dot{\theta}\sin(\theta) - \rho\ddot{\theta}\sin(\theta) - \rho(\dot{\theta})^2\cos(\theta) \\ \ddot{y} = \ddot{\rho}\sin(\theta) + 2\dot{\rho}\dot{\theta}\cos(\theta) + \rho\ddot{\theta}\cos(\theta) - \rho(\dot{\theta})^2\sin(\theta) \end{cases}$$

where we denote again the acceleration in terms of unit vectors:

$$\ddot{\mathbf{r}} = (\ddot{\rho} - \rho(\dot{\theta})^2)\langle \cos(\theta), \sin(\theta) \rangle + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\langle -\sin(\theta), \cos(\theta) \rangle$$

We then find that equation (2.4) to be

$$\mu\Big((\ddot{\rho} - \rho(\dot{\theta})^2)\langle \cos(\theta), \sin(\theta) \rangle + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\langle -\sin(\theta), \cos(\theta) \rangle\Big) = -h(\rho)\langle \cos(\theta), \sin(\theta) \rangle$$

or the coupled equations

$$\begin{cases} \mu(\ddot{\rho} - \rho(\dot{\theta})^2) = -h(\rho) \\ \mu(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}) = 0 \end{cases} \tag{2.7}$$

Notice that the second equation is equivalent to

$$\mu\frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\theta}) = 0$$

thus $\rho^2\dot{\theta} = l$ is constant (notice also that this is equivalent to the conservation of angular momentum derived in Section 2.2.1). Solving for $\dot{\theta}$, we find

$$\dot{\theta} = \frac{l}{\mu\rho^2} \tag{2.8}$$

Inserting this into equation (2.7), we find

$$\mu\Big(\ddot{\rho} - \rho\Big(\frac{l}{\mu\rho^2}\Big)^2\Big) = \mu\ddot{\rho} - \frac{l^2}{\mu}\frac{1}{\rho^3} = -h(\rho) \tag{2.9}$$

We also see from equation (2.8) that $\dot{\theta}$ has fixed sign that depends on the value of l, the initial momentum. Because every monotone function has a monotone inverse function, we see that

$$\begin{cases} \theta(t(\theta)) = \theta \\ t(\theta(t)) = t \end{cases}$$

so we will use θ as the independent variable instead of t. We will then use the variable transformation $u(\theta) = \frac{1}{\rho(t(\theta))}$ to redefine equation (2.9) [2, 5]. We see that

$$\begin{aligned} \frac{du(\theta)}{d\theta} &= -\frac{1}{\rho^2(t(\theta))}\frac{d\rho(t(\theta))}{dt}\frac{dt(\theta)}{d\theta} \text{ by Chain Rule} \\ &= -\frac{\mu}{l}\frac{d\rho(t(\theta))}{dt} \end{aligned}$$

and

$$\begin{aligned} \frac{d^2u(\theta)}{d\theta^2} &= -\frac{\mu}{l}\frac{d}{d\theta}\Big(\frac{d\rho(t(\theta))}{dt}\Big) \\ &= -\frac{\mu}{l}\frac{d^2\rho(t(\theta))}{dt^2}\frac{dt}{d\theta} \\ &= -\Big(\frac{\mu}{l}\Big)^2\rho^2(t(\theta))\frac{d^2\rho(t(\theta))}{dt^2} \end{aligned}$$

We then multiply equation (2.9) by $-\frac{\mu}{l^2}\rho^2$ to obtain

$$-\frac{\mu^2}{l^2}\rho^2\ddot{\rho} + \frac{1}{\rho} = \frac{\mu}{l^2}\rho^2 h(\rho)$$

which we transform into

$$\frac{d^2u(\theta)}{d\theta^2} + u(\theta) = \frac{\mu}{l^2}\frac{1}{u^2(\theta)}h\Big(\frac{1}{u(\theta)}\Big)$$

We let

$$f(u) = \frac{\mu}{l^2}\frac{1}{u^2(\theta)}h\Big(\frac{1}{u(\theta)}\Big) \tag{2.10}$$

and find that

$$\frac{d^2u}{d\theta^2} + u = f(u) \tag{2.11}$$

where $f(u)$ is analogous to potential energy. We then define the (u, v) system of equations as

$$\begin{cases} \frac{du}{d\theta} = v \\ \frac{dv}{d\theta} = -u + f(u) \end{cases} \tag{2.12}$$

From this system we find the total energy of the system is

$$E = \frac{1}{2}(u^2 + v^2) - \int f(u)du \tag{2.13}$$

where energy E is defined to a constant. We let the constant of integration that arises from the integral of $f(u)$ be 0, as this constant can be absorbed into E. We see then that

$$\frac{dE}{du} = uv + v(-u + f(u)) - vf(u) = 0$$

so the total energy is conserved. Solving the energy equation for v, we find

$$v = \pm\sqrt{2\Big(E - G(u)\Big)} \tag{2.14}$$

where

$$G(u) = \frac{u^2}{2} - \int f(u)du \tag{2.15}$$

2.2.3 Phase Portraits of E in the (u, v)-plane

To aid in visualization, we plot phase portraits of E as expressed in (2.13). As we will prove in Lemma 3.3, it is only power laws of $f(u)$ of the form

$$f(u) = \kappa u^{1-c}, \text{ for some } c > 0 \text{ and some } \kappa > 0$$

that are relevant to Bertrand's Theorem. Hence, we plot level curves of E as given in (2.13) for $f(u)$ in the above form for $c = 0.5$, 1, 1.5, 2, 4, and 25 to show some range of power laws. As we will show in the conclusion of this paper, $c = 1$ and $c = 4$ correspond to Newton's and Hooke's Laws of gravitation, respectively.

Note that because u is the inverse of distance, solution curves where u becomes negative do not correspond to physically relevant solutions to the two body problem. Curves where u approaches 0 correspond to unbounded orbits as the distance between the bodies becomes infinite. A closed level curve entirely in the region where $u > 0$ corresponds to an orbit that is bounded in the two-body system, but the closure of an orbit cannot be determined by observation of the phase portraits. The closure requirement is explained below.

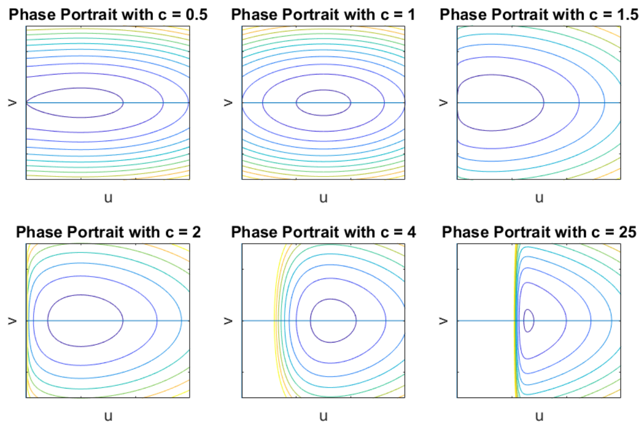

Figure 2: Phase portraits of E in the (u, v) plane.

3 Bertrand's Theorem

3.1 Context

Bertrand's theorem restricts to cases where the two-body problem has at least some bounded orbits. For a given bounded orbit, the pericenter and apocenter occur when the distance between the two bodies is minimal and maximal, respectively. When the pericenter and apocenter coincide, the orbit is in fact circular. Bounded orbits of the two body problem correspond to bounded solutions $(u(\theta), v(\theta))$ of the (u, v)-system defined above. But note that the pericenter and apocenter of the bounded solution in the two-body problem correspond to the maximal and minimal values for $u(\theta)$, respectively since ρ and u are related via $u = \frac{1}{\rho}$. Also note that the two-body problem has a circular orbit whenever the (u, v) system has a steady state. An orbit of the two-body system is said to be closed if and only if the corresponding solution $(u(\theta), v(\theta))$ of the (u, v) system is periodic with a period that is equal to a rational multiple of π.

This happens when the independent angular variable θ between a consecutive minimum and maximum of the u-coordinate of the solution is equal to a rational multiple of π. We shall denote this angle by T. Given two distinct bounded solutions, it seems reasonable to expect that the values of T will be distinct as well. For each bounded orbit of the two body problem, the value of T depends on the specifics of the force field, the masses of the two bodies, and their initial conditions. That is, T will depend on $(u(0), v(0))$ as well as the masses m_1 and m_2. Bertrand's Theorem is the remarkable result that says that among all possible force fields, there are exactly two for for which all bounded solutions are closed, namely the force fields corresponding to those of Newton and of Hooke [2, 5]. In particular, in these two cases and only these two cases, the value of T of every bounded solution of the (u, v) system is a rational multiple of π, which is independent of the chosen bounded solution.

Although Bertrand's original paper does not contain a rigorous proof, Arnold's book *Mathematical methods of mechanical methods* [1] lays out a skeleton of a proof via a sequence of six problems. Arnold only provides short answers for these problems. Unfortunately, there are several errors in both the statements and solutions of these problems. Moreover, at least some of the solutions to these problems are far from trivial. In a recent paper which was published in 2015 for instance, Jovanović provides the solution for problem 4 in Arnold's list [4]. We shall include Jovanović's solution to this problem in Section 3.2.4. However, we believe that the solution of problem 5 in Arnold's book is not obvious either. The main result of this thesis is to provide a partial proof to problem 5. To make this paper more self-contained, we shall also include rigorous proofs for problems 1 through 4 from Arnold's list.

3.2 A Partial Proof of Bertrand's Theorem

3.2.1 Defining the Angle between the Pericenter and Apocenter

Lemma 3.1. *The angle $T(E)$ between the pericenter and apocenter of a solution of the one-body system with total energy E is*

$$T(E) = \int_{u_{min}(E)}^{u_{max}(E)} \frac{du}{\sqrt{2(E - G(u))}}$$

where $u_{min}(E) < u_{max}(E)$ are the solutions of the equation $G(u) = E$.

Proof. From equation (2.14), we see

$$v = \frac{du}{d\theta} = \sqrt{2\Big(E - G(u)\Big)}$$

hence

$$T(E) = \int_{u_{min}(E)}^{u_{max}(E)} \frac{du}{\sqrt{2(E - G(u))}}$$

Note that $T(E)$ is independent of the integration constant that arises from the function $G(u)$ because this constant can be absorbed into the constant E. □

3.2.2 Solving for Angle $T(E)$

In the next result, we obtain a formula for the angle $T(E)$ near a circular orbit.

Lemma 3.2. *Assume that $G : (0, +\infty) \to \mathbb{R}$ is a twice continuously differentiable function such that:*

1. For some $u_0 > 0$, $G(u_0) = E_0$

2. $G'(u_0) = 0$

3. $G''(u_0) > 0$

The angle T for an orbit close to the circle of radius $\frac{1}{u_0}$ approaches:

$$\lim_{E \to E_0} T(E) = T_{cir}(E) = \lim_{E \to E_0} \int_{u_{min}(E)}^{u_{max}(E)} \frac{du}{\sqrt{2(E - G(u))}} = \frac{\pi}{\sqrt{G''(u_0)}}$$

where $u_{max}(E) > u_{min}(E)$ are the two positive solutions to

$$G(u) = E \text{ for all } E > E_0$$

Note: We know that these are the only two solutions due to the restricted concavity of G.

Proof. Let $G : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable and assume that $u_0 = 0$ and $E_0 = 0$ so that we have:

1. $G(0) = G'(0) = 0$

2. $G''(0) > 0$

It suffices to assume that $u_0 = 0$ and $E_0 = 0$ because a trivial variable transformation can translate these parameters anywhere suitable on the (u, v) plane. First, we show this for the special case that

$$G(u) = \frac{G''(0)}{2} u^2$$

where $G''(0) > 0$. Note first that for this case,

$$u_{min}(E) = -\sqrt{\frac{2E}{G''(0)}}$$

and

$$u_{max}(E) = \sqrt{\frac{2E}{G''(0)}}$$

for all $E > 0$. Thus for all such $E > 0$, we have

$$\int_{u_{min}(E)}^{u_{max}(E)} \frac{du}{\sqrt{2(E-G(u))}} = \int_{-\sqrt{\frac{2E}{G''(0)}}}^{\sqrt{\frac{2E}{G''(0)}}} \frac{du}{\sqrt{2E-G''(0)u^2}}$$
$$= \frac{1}{\sqrt{2E}} \int_{-\sqrt{\frac{2E}{G''(0)}}}^{\sqrt{\frac{2E}{G''(0)}}} \frac{du}{\sqrt{1-\left(\frac{u}{\sqrt{2E/G''(0)}}\right)^2}}$$
$$= \frac{1}{\sqrt{G''(0)}} \int_{-1}^{1} \frac{dv}{\sqrt{1-v^2}}$$
$$= \frac{\pi}{\sqrt{G''(0)}}$$

Therefore the Lemma is proven for this particular case. To fully prove this result, it suffices to show that

$$\lim_{E\to 0^+} \int_0^{u_{max}(E)} \frac{du}{\sqrt{2(E-G(u))}} - \int_0^{\sqrt{2E/G''(0)}} \frac{du}{\sqrt{2E-G''(0)u^2}} = 0 \tag{3.1}$$

because a similar result can be obtained for the difference of the integrals between $u_{min}(E)$ and 0 and $-\sqrt{2E/G''(0)}$ and 0, respectively. By substitutions, we find that (3.1) is equivalent to

$$\lim_{E\to 0^+} \int_0^1 \left(\frac{u_{max}(E)}{\sqrt{2E}} \frac{1}{\sqrt{1-G(u_{max}(E)x)/E}} - \frac{1}{\sqrt{G''(0)}} \frac{1}{\sqrt{1-x^2}} \right) dx = 0 \tag{3.2}$$

Denote the integrand of (3.2) by $f(x,E)$. We shall prove (3.2) using the dominated convergence theorem by showing that:

1. $\lim_{E\to 0^+} f(x,E) = 0$ for all x in (0,1) and

2. There exists a function $g(x)$ defined for x in (0,1) such that $\int_0^1 g(x)dx$ is finite and $|f(x,E)| \le g(x)$ for all x in (0,1) and for all sufficiently small $E > 0$.

To prove item one, we choose $\bar{E} > 0$ sufficiently small so that $G(u) > 0$, $G'(u) > 0$ and $G''(u) \ge \alpha > 0$ for all $u \in (0, u_{max}(E))$ and all $E \in (0, \bar{E})$. Because $G(u_{max}(E)) = E$, it follows from Taylor's theorem that

$$E = G(u_{max}(E)) = G''(\zeta(E)) \frac{u_{max}^2(E)}{2}$$

for all $u \in (0, u_{max}(E))$ where $0 < \zeta(E) < u_{max}(E)$. Similarly, for all (x, E) in $(0,1) \times (0, \bar{E})$, Taylor's theorem implies that

$$G(u_{max}(E)x) = G''(\xi(x,E))\frac{(u_{max}(E)x)^2}{2}$$

for some $0 < \xi(x,E) < u_{max}(E)x$. These two equations imply that for all (x, E) in $(0,1) \times (0, \bar{E})$

$$\frac{u_{max}(E)}{\sqrt{2E}} \frac{1}{\sqrt{1 - G(u_{max}(E)x)/E}} = \frac{1}{\sqrt{G''(\zeta(E))}} \frac{1}{\sqrt{1 - \frac{G''(\xi(x,E))}{G''(\zeta(E))}x^2}}$$
$$\to \frac{1}{\sqrt{G''(0)}} \frac{1}{\sqrt{1-x^2}} \text{ as } E \to 0^+$$

because G is twice continuously differentiable and $\zeta(E) \to 0$ and $\xi(x,E) \to 0$ as $E \to 0$ for all $x \in (0,1)$. This proves item one needed to apply the dominated convergence theorem.

To prove the second item, note that

$$|f(x,E)| \leq \frac{u_{max}(E)}{\sqrt{2E}} \frac{1}{\sqrt{1 - G(u_{max}(E)x)/E}} + \frac{1}{\sqrt{G''(0)}} \frac{1}{\sqrt{1-x^2}}$$

and that the second term is integrable over [0,1], so we focus our attention on the first term. We claim that there exists $\beta > 0$ such that for all $(x, E) \in (0,1) \times (0, \bar{E})$,

$$\frac{u_{max}(E)}{\sqrt{2E}} \frac{1}{\sqrt{1 - G(u_{max}(E)x)/E}} \leq \frac{\beta}{\sqrt{x(1-x)}} \tag{3.3}$$

If this is proven, then item two is proven, since

$$\int_0^1 \frac{\beta}{\sqrt{x(1-x)}} dx = 2\int_0^1 \frac{\beta}{\sqrt{1-v^2}} dv = \beta\pi$$

by the substitution $x = v^2$. To prove (3.3), note that for all $(x, E) \in (0,1) \times (0, \bar{E})$,

$$\frac{u_{max}(E)}{\sqrt{2E}} \frac{1}{\sqrt{1 - G(u_{max}(E)x)/E}} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\frac{G(u_{max}(E)) - G(u_{max}(E)x)}{u_{max}^2(E)}}} \tag{3.4}$$

Because $G(u)$ is strictly convex for $u \in (0, u_{max}(E))$ and $E \in (0, \bar{E})$ we have that

$$G(u_{max}(E)) > G(u_{max}(E)x) + G'(u_{max}(E)x)(u_{max}(E) - u_{max}(E)x)$$

for all $(x, E) \in (0,1) \times (0, \bar{E})$. We obtain from this that

$$0 < \frac{G'(u_{max}(E)x)(1-x)}{u_{max}(E)} < \frac{G(u_{max}(E)) - G(u_{max}(E)x)}{u_{max}^2(E)} \text{ for all } (x,E) \in (0,1)\times(0,\bar{E})$$

We insert this result into (3.4) and find that for all $(x, E) \in (0,1) \times (0, \bar{E})$,

$$\begin{aligned} \frac{u_{max}(E)}{\sqrt{2E}} \frac{1}{\sqrt{1 - G(u_{max}(E)x)/E}} &< \frac{1}{\sqrt{2}} \sqrt{\frac{u_{max}(E)}{G'(u_{max}(E)x)}} \frac{1}{\sqrt{1-x}} \\ &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{G''(\nu(x,E))}} \frac{1}{\sqrt{x(1-x)}} \end{aligned}$$

where $0 < \nu(x, E) < u_{max}(E)x$ by Taylor's theorem applied to $G'(u)$. Since $G''(u) \geq \alpha > 0$ for all $u \in (0, u_{max}(E))$ and all $E \in (0, \bar{E})$, we set

$$\beta = \frac{1}{\sqrt{2\alpha}}$$

We have then proved the second item needed to apply the dominated convergence theorem, so equation (3.2) is proven. We then have the desired result that

$$\lim_{E \to E_0} \int_{u_{min}(E)}^{u_{max}(E)} \frac{du}{\sqrt{2(E - G(u))}} = \frac{\pi}{\sqrt{G''(u_0)}}$$

□

3.2.3 Restriction to Power Laws

Here we show that the angle $T(E)$ near circular orbits is independent of the radius of the orbit only for specific power laws for $f(u)$.

Lemma 3.3. *The magnitude of*

$$T_{cir} = \frac{\pi}{\sqrt{G''(u_0)}}$$

is independent of u_0 *if in the* (u, v) *system,*

$$f(u) = \kappa u^{1-c}$$

where κ *and* c *are positive constants.*

Proof. Earlier, we defined $G(u) = \frac{u^2}{2} - \int f(u)du$. Then for $\frac{\pi}{\sqrt{G''(u)}}$ to be constant, we must have

$$G''(u) = 1 - f'(u) = c$$

for some $c > 0$, as assumed in the proof of Lemma 3.2. From that proof, we also have

$$G'(u) = u - f(u) = 0$$

so

$$1 - u\frac{f'(u)}{f(u)} = c$$

so

$$\frac{d}{du}(\ln f(u)) = \frac{1-c}{u} = (1-c)\frac{d}{du}\ln u = \begin{cases} \frac{d}{du}(\ln u^{1-c}) & \text{if } c \neq 1 \\ 0 & \text{if } c = 1 \end{cases}$$

By solving for $f(u)$, we find

$$f(u) = \kappa u^{1-c}$$

for some $\kappa > 0$ and some $c > 0$.

□

3.2.4 Angle $T(E)$ for $c \geq 2$

The following result was proved recently in [4].

Lemma 3.4. *Suppose that $f(u) = \kappa u^{1-c}$ for some $\kappa > 0$ and some $c \geq 2$. Then*

$$\lim_{E\to\infty} T(E) = \frac{\pi}{2}$$

Proof. Let

$$G(u) = \begin{cases} \frac{1}{2}u^2 - \kappa\frac{u^{2-c}}{2-c} \text{ for } c > 2 \text{ and some } \kappa > 0 \\ \frac{1}{2}u^2 - \kappa\ln(u) \text{ for } c = 2 \text{ for some } \kappa > 0 \end{cases}$$

We see that for $c \geq 2$ that

$$\lim_{u\to 0^+} G(u) = \infty$$

and

$$\lim_{u\to\infty} G(u) = \infty$$

as well as

$$\frac{dG(u)}{du} = u - \kappa u^{1-c} = 0 \text{ if and only if } u = u^* = \kappa^{\frac{1}{c}}$$

and

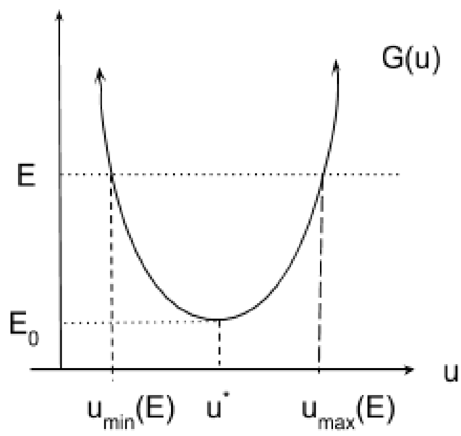

Figure 3: $G(u)$ for $c \geq 2$

$$\frac{d^2G(u)}{du^2} = 1 - \kappa(1-c)u^{-c} > 0 \text{ for } u > 0$$

In the integrand of $T(E)$, we make the transformation $y = \frac{u}{u_{max}(E)}$ and let $y_1(E) = \frac{u_{min}(E)}{u_{max}(E)}$. Then

$$T(E) = \int_{u_{min}(E)}^{u_{max}(E)} \frac{du}{\sqrt{2(E - G(u))}} = \int_{y_1(E)}^{1} \frac{dy}{\sqrt{H(y,E)}}$$

where

$$H(y,E) = \begin{cases} u_{max}^2(E)(1-y^2) + 2\kappa \ln(y), \text{ if } c = 2 \\ 1 - y^2 + \frac{2\kappa}{2-c}(u_{max}(E))^{-c}(y^{2-c} - 1), \text{ if } c > 2 \end{cases} \tag{3.5}$$

where we have used that $G(u_{max}(E)) = E$ to eliminate the parameter E.[1] We note that $\frac{\partial H(y,E)}{\partial y} = 0$ only for $y = y^*(E)$ where

$$y^*(E) = \frac{\kappa^{\frac{1}{c}}}{u_{max}(E)} \in (y_1(E), 1) \tag{3.6}$$

and

$$\frac{\partial^2 H(y,E)}{\partial y^2} = -2(1 - \kappa(u_{max}(E))^{-c}(1-c)y^{-c}) < 0$$

for all $y \in (y_1(E), 1)$ since κ is positive and $c \geq 2$.

We then see that for all $E > E_0$,

[1] The following steps apply to $c > 2$. A very similar argument can be used for $c = 2$ that leads to the same result.

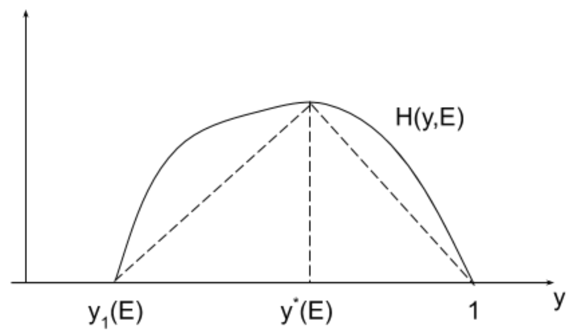

Figure 4: $H(y,E)$

$$H(y,E) \geq \frac{H(y^*(E),E) - H(y_1(E),E)}{y^*(E) - y_1(E)}(y - y_1(E)) + H(y_1(E),E) \quad \text{for } y \in (y_1(E), y^*(E))$$
$$H(y,E) \geq \frac{H(y^*(E),E) - H(1,E)}{y^*(E) - 1}(y - 1) + H(1,E), \quad \text{for } y \in (y^*(E), 1)$$

and $H(y_1,E) = H(1,E) = 0$ imply

$$\frac{1}{\sqrt{H(y,E)}} < \sqrt{\frac{y^*(E) - y_1(E)}{H(y^*(E),E)}} \frac{1}{\sqrt{y - y_1(E)}} \tag{3.7}$$

for $y \in (y_1(E), y^*(E))$ and

$$\frac{1}{\sqrt{H(y,E)}} < \sqrt{\frac{1 - y^*(E)}{H(y^*(E),E)}} \frac{1}{\sqrt{1 - y}} \tag{3.8}$$

for $y \in (y^*(E), 1)$. Note also that since $c \geq 2$, equation (3.5) implies that $H(y,E) < 1 - y^2$ for all $y \in (y_1(E), 1)$, and thus that

$$\frac{1}{\sqrt{1 - y^2}} < \frac{1}{\sqrt{H(y,E)}} \tag{3.9}$$

for all $y \in (y_1(E), 1)$.

The properties of $G(u)$ imply that $u_{min}(E) \to 0$ and $u_{max}(E) \to \infty$ as $E \to \infty$, therefore, we see that as $E \to \infty$ that $y_1(E) \to 0$, $y^*(E) \to 0$, and $H(y^*(E),E) \to 1$ as $E \to \infty$.

Note, first, importantly that as $E \to \infty$,

$$\int_{y_1(E)}^{1} \frac{dy}{\sqrt{1 - y^2}} \to \frac{\pi}{2}$$

It is then sufficient to show that as $E \to \infty$,

$$\left| \int_{y_1(E)}^{1} \frac{dy}{\sqrt{H(y,E)}} - \int_{y_1(E)}^{1} \frac{dy}{\sqrt{1-y^2}} \right| \to 0$$

We split up the integral to find the inequality

$$\left| \int_{y_1(E)}^{1} \frac{dy}{\sqrt{H(y,E)}} - \int_{y_1(E)}^{1} \frac{dy}{\sqrt{1-y^2}} \right| \leq \left| \int_{y^*(E)}^{1} \frac{dy}{\sqrt{H(y,E)}} - \int_{y^*(E)}^{1} \frac{dy}{\sqrt{1-y^2}} \right| + \int_{y_1(E)}^{y^*(E)} \frac{dy}{\sqrt{H(y,E)}} + \int_{y_1(E)}^{y^*(E)} \frac{dy}{\sqrt{1-y^2}} \quad (3.10)$$

We claim that the last two terms converge to 0 as $E \to \infty$. Indeed, by (3.10) and as $y_1(E), y^*(E) \to 0$ and $H(y^*(E), E) \to 1$ as $E \to \infty$, we see that

$$\begin{aligned} \int_{y_1(E)}^{y^*(E)} \frac{dy}{\sqrt{H(y,E)}} &\leq \int_{y_1(E)}^{y^*(E)} \sqrt{\frac{1-y^*(E)}{H(y^*(E),E)}} \frac{dy}{\sqrt{1-y}} \\ &= \sqrt{\frac{1-y^*(E)}{H(y^*(E),E)}} \Big(-2\sqrt{1-y} \Big) \Big|_{y=y_1(E)}^{y=y^*(E)} \to 0 \text{ as } E \to \infty \end{aligned}$$

as well as

$$\int_{y_1(E)}^{y^*(E)} \frac{dy}{\sqrt{1-y^2}} = \arcsin(y) \Big|_{y=y_1(E)}^{y=y^*(E)} \to 0 \text{ as } E \to 0$$

For the remaining two terms, we see that

$$\left| \int_{y^*(E)}^1 \frac{dy}{\sqrt{H(y,E)}} - \int_{y^*(E)}^1 \frac{dy}{\sqrt{1-y^2}} \right| = \int_{y_*(E)}^1 \frac{\sqrt{1-y^2} - \sqrt{H(y,E)}}{\sqrt{H(y,E)}\sqrt{1-y^2}} dy$$

$$= \int_{y_*(E)}^1 \frac{1-y^2-H(y,E)}{\sqrt{H(y,E)}\sqrt{1-y^2}(\sqrt{H(y,E)}+\sqrt{1-y^2})} dy$$

$$= \frac{2\kappa}{c-2}(u_{max}(E))^{-c} \int_{y^*(E)}^1 \frac{y^{2-c}-1}{\sqrt{H(y,E)}\sqrt{1-y^2}(\sqrt{H(y,E)}+\sqrt{1-y^2})} dy$$

$$\leq \frac{2\kappa}{c-2}(u_{max}(E))^{-c} \int_{y^*(E)}^1 \frac{y^{2-c}-1}{H(y,E)^{3/2}} dy \qquad \text{by (3.9)}$$

$$\leq \frac{2\kappa}{c-2}(u_{max}(E))^{-c} \left(\frac{1-y^*(E)}{H(y^*(E),E)} \right)^{3/2} \int_{y^*(E)}^1 \frac{y^{2-c}-1}{1-y} \frac{1}{\sqrt{1-y}} dy \qquad \text{by (3.8)}$$

Note that in the previous line, the factor

$$\frac{2\kappa}{c-2} \left(\frac{1-y^*}{H(y^*(E),E)} \right)^{3/2} \to \frac{2\kappa}{c-2} \text{ as } E \to \infty$$

Moreover, since $y^{2-c} - 1 < y^{2-c} \leq (y^*(E))^{2-c}$ for $y \geq y^*(E)$ and by the equality

$$u_{max}(E) y^*(E) = \kappa^{\frac{1}{c}}, \text{ from (3.6)}$$

it follows that the remaining factors in (3.8) can be expressed as follows, where we break up the integral from $y^*(E)$ to $\frac{1}{2}$ and $\frac{1}{2}$ to 1 (this is possible for sufficiently large E since $y^*(E) \to 0$ as $E \to \infty$).

$$(u_{max}(E))^{-c} \int_{y^*(E)}^1 \frac{y^{2-c}-1}{1-y} \frac{1}{\sqrt{1-y}} dy$$

$$= (u_{max}(E))^{-c} \int_{1/2}^1 \frac{y^{2-c}-1}{1-y} \frac{dy}{\sqrt{1-y}} + (u_{max}(E))^{-c} \int_{y^*(E)}^{1/2} \frac{y^{2-c}-1}{1-y} \frac{dy}{\sqrt{1-y}}$$

$$\leq (u_{max}(E))^{-c} \int_{1/2}^1 \frac{y^{2-c}-1}{1-y} \frac{dy}{\sqrt{1-y}} + (u_{max}(E))^{-c} \int_{y^*(E)}^{1/2} \frac{(y^*(E))^{2-c}}{(1-y)^{3/2}} dy$$

$$= (u_{max}(E))^{-c} \int_{1/2}^1 \frac{y^{2-c}-1}{1-y} \frac{dy}{\sqrt{1-y}} + \kappa^{\frac{2-c}{c}} (u_{max}(E))^{-2} \int_{y^*(E)}^{1/2} \frac{dy}{(1-y)^{3/2}}$$

We will show that both these terms tend to 0 as $E \to \infty$. Starting with the second term, we see that:

$$\begin{aligned}
\kappa^{\frac{2-c}{c}}(u_{max}(E))^{-2}\int_{y^*(E)}^{1/2}\frac{dy}{(1-y)^{3/2}} \\
&\leq \kappa^{\frac{2-c}{c}}(u_{max}(E))^{-2}\int_{0}^{1/2}\frac{dy}{(1-y)^{3/2}} \\
&= \kappa^{\frac{2-c}{c}}(u_{max}(E))^{-2}\left(2\sqrt{2}-1\right) \to 0 \text{ as } E\to\infty
\end{aligned}$$

Second, we see that the first factor in the first term, namely the function

$$y \to \frac{y^{2-c}-1}{1-y}$$

is bounded over $y \in [1/2, 1]$, because the singularity at $y = 1$ is removable by L'Hôpital's rule:

$$\lim_{y\to 1}\frac{y^{2-c}-1}{1-y} = c-2$$

Let $M > 0$ be an upper bound for the function $y \to (y^{2-c}-1)/(1-y)$. Then

$$\begin{aligned}
(u_{max}(E))^{-c}\int_{1/2}^{1}\frac{y^{2-c}-1}{1-y}\frac{dy}{\sqrt{1-y}} \\
&< (u_{max}(E))^{-c}\int_{1/2}^{1}\frac{M}{\sqrt{1-y}}dy \\
&= (u_{max}(E))^{-c}(M\sqrt{2}) \to 0 \text{ as } E\to 0
\end{aligned}$$

Thus the first term in the right hand side of (3.10) tends to 0 and so

$$\lim_{E\to\infty} T(E) = \frac{\pi}{2}$$

□

3.2.5 Angle $T(E)$ for $1 \leq c < 2$

We start with a relevant integral that can be computed with basic calculus tools:

Lemma 3.5. *For all $c > 0$, holds that*

$$\int_0^1 \frac{dy}{\sqrt{y^{2-c}-y^2}} = \frac{\pi}{c}$$

Proof. We first make the variable substitution $x = y^{c/2}$ and find

$$\begin{aligned}\int_0^1 \frac{dy}{\sqrt{y^{2-c}-y^2}} &= \int_0^1 \frac{y^{\frac{c}{2}}}{y\sqrt{1-y^c}}dy \\ &= \frac{2}{c}\int_0^1 \frac{yx}{xy\sqrt{1-x^2}}dx = \frac{2}{c}\arcsin(x)\Big|_0^1 \\ &= \frac{\pi}{c}\end{aligned}$$

□

The next result is the main result of this thesis.

Lemma 3.6. *Suppose that $f(u) = \kappa u^{1-c}$ for some $\kappa > 0$ and $1 \le c < 2$. Then*

$$\lim_{E\to 0^-} T(E) = \frac{\pi}{c}$$

Proof. First, notice that for $1 \le c < 2$, the profile of

$$G(u) = \frac{1}{2}u^2 - \kappa\frac{u^{2-c}}{2-c}$$

is significantly changed (Figure 5). We see that the limits of $G(u)$ have changed:

$$\lim_{u\to 0^+} G(u) = 0$$
$$\lim_{u\to\infty} G(u) = \infty$$

But we see that the properties of the derivatives of G hold as they did in the proof of Lemma 3.4:

$$\frac{dG(u)}{du} = u - \kappa^{1-c} = 0 \text{ if and only if } u = u^* = \kappa^{\frac{1}{c}}$$

and

$$\frac{d^2G(u)}{du^2} = 1 - \kappa(1-c)u^{-c} > 0 \text{ for } u > 0$$

Note yet again that $u_{min}(E)$ and $u_{max}(E)$ are the two solutions of the equation $G(u) = E$ that exist for $E_0 < E < 0$, where $E_0 = G(u^*)$. Moreover, $u_{min}(E) \to 0$ and $u_{max}(E) \to \left(\frac{2\kappa}{2-c}\right)^{1/c}$ as $E \to 0^-$.

We must show that for all $1 \le c < 2$ that

$$\lim_{E\to 0^-} T(E) = \lim_{E\to 0^-}\int_{u_{min}(E)}^{u_{max}(E)} \frac{du}{\sqrt{2(E-G(u))}} = \lim_{E\to 0^-}\int_{y_1(E)}^{1} \frac{dy}{\sqrt{H(y,E)}} = \frac{\pi}{c}$$

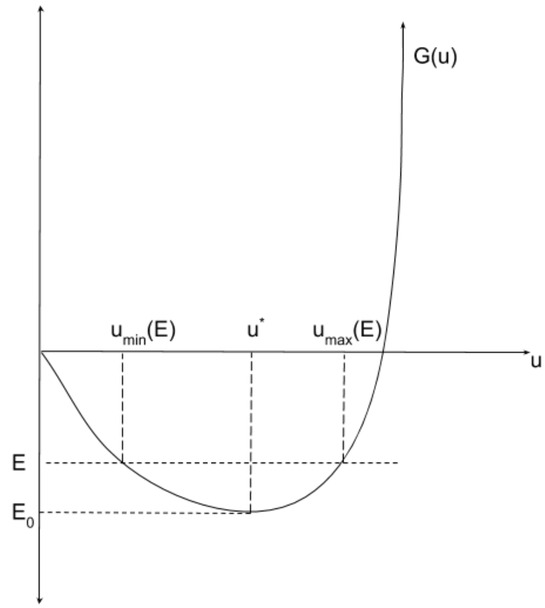

Figure 5: $G(u)$ for $1 \leq c < 2$

where we have made the variable transformation $y = u/u_{max}(E)$ and where $y_1(E) = u_{min}(E)/u_{max}(E)$ and

$$H(y,E) = 1 - y^2 + \frac{2\kappa}{2-c}(u_{max}(E))^{-c}(y^{2-c} - 1)$$

just as in the proof of Lemma 3.4. For future reference, note that

$$\frac{\partial H(y,E)}{\partial y} = 0 \text{ only for } y = y^*(E)$$

where

$$y^*(E) = \frac{\kappa^{1/c}}{u_{max}(E)} \in (y_1(E), 1)$$

and thus that

$$y^*(E) \to \left(\frac{2-c}{2}\right)^{1/c} \text{ as } E \to 0^-$$

Moreover, $H(y,E)$ is still concave because

$$\frac{\partial^2 H(y,E)}{\partial y^2} = -2(1-\kappa(u_{max}(E))^{-c}(1-c)y^{-c}) < 0 \text{ for all } E_0 < E < 0 \text{ and } y_1(E) \leq y \leq 1$$

By Lemma 3.5, we must therefore prove that for all $1 \leq c < 2$ holds that

$$\lim_{E\to 0^-} \int_{y_1(E)}^{1} \frac{dy}{\sqrt{H(y,E)}} = \int_0^1 \frac{dy}{\sqrt{y^{2-c} - y^2}} = \lim_{E\to 0^-} \int_{y_1(E)}^{1} \frac{dy}{\sqrt{y^{2-c} - y^2}}$$

since $y_1(E)$ is continuous and $y_1(E) \to 0$ as $E \to 0^-$. To achieve this, we shall prove that

$$\left| \int_{y_1(E)}^{1} \frac{dy}{\sqrt{H(y,E)}} - \int_{y_1(E)}^{1} \frac{dy}{\sqrt{y^{2-c} - y^2}} \right| \to 0 \text{ as } E \to 0^-$$

We claim the if $1 \leq c < 2$, then

$$H(y,E) \leq y^{2-c} - y^2 \text{ for all } E_0 < E < 0 \text{ and all } y_1(E) \leq y \leq 1$$

Indeed, note that $u_{max}(E) \to (\frac{2k}{2-c})^{1/c}$ from below as $E \to 0^-$ (that is, $u_{max}(E)$ increases to its limit). Consequently, $\frac{2k}{2-c}(u_{max}(E))^{-c} \to 1$ from above. Our claim then follows from subtracting the following two equalities:

$$\begin{aligned} y^{2-c} - y^2 &= 1 - y^2 + (y^{2-c} - 1) \\ H(y,E) &= 1 - y^2 + \frac{2k}{2-c}(u_{\max}(E))^{-c}(y^{2-c} - 1) \end{aligned}$$

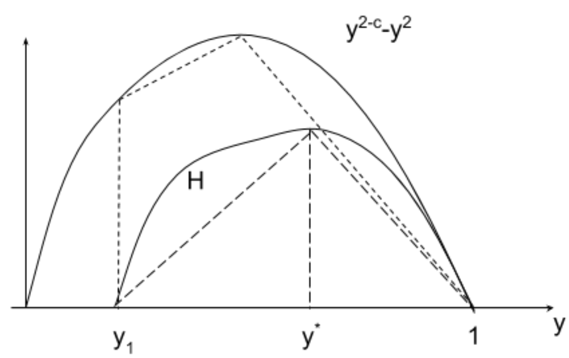

Figure 6: Profile of $H(y,E)$ and $y^{2-c} - y^2$

Breaking up the integrals from $y_1(E)$ to $y^*(E)$ and from $y^*(E)$ to 1 yields

$$\left| \int_{y_1(E)}^{1} \frac{dy}{\sqrt{H(y,E)}} - \int_{y_1(E)}^{1} \frac{dy}{\sqrt{y^{2-c}-y^2}} \right| \leq \left| \int_{y_1(E)}^{y^*(E)} \frac{dy}{\sqrt{H(y,E)}} - \int_{y_1(E)}^{y^*(E)} \frac{dy}{\sqrt{y^{2-c}-y^2}} \right| + \left| \int_{y^*(E)}^{1} \frac{dy}{\sqrt{H(y,E)}} - \int_{y^*(E)}^{1} \frac{dy}{\sqrt{y^{2-c}-y^2}} \right|$$

For $y \in (y^*(E), 1)$, we see

$$H(y,E) \geq \frac{H(y^*(E),E)}{1-y^*(E)}(1-y)$$

by the concavity of $H(y,E)$, and thus as $H(y,E) \leq y^{2-c} - y^2$, we find that

$$\begin{aligned}
0 \leq \int_{y^*(E)}^{1} \frac{dy}{\sqrt{H(y,E)}} - \int_{y^*(E)}^{1} \frac{dy}{\sqrt{y^{2-c}-y^2}} \\
&= \int_{y^*(E)}^{1} \frac{\left(\frac{2\kappa}{2-c}(u_{max}(E))^{-c}-1\right)(1-y^{2-c})dy}{\sqrt{H(y,E)}\sqrt{y^{2-c}-y^2}\left(\sqrt{H(y,E)}+\sqrt{y^{2-c}-y^2}\right)} \\
&\leq \left(\frac{2\kappa}{2-c}(u_{max}(E))^{-c}-1\right) \int_{y^*(E)}^{1} \frac{1-y^{2-c}}{(H(y,E))^{\frac{3}{2}}} dy \\
&\leq \left(\frac{2\kappa}{2-c}(u_{max}(E))^{-c}-1\right) \left(\frac{1-y^*(E)}{H(y^*(E),E)}\right)^{\frac{3}{2}} \int_{y^*(E)}^{1} \frac{1-y^{2-c}}{(1-y)^{\frac{3}{2}}} dy \\
&= \left(\frac{2\kappa}{2-c}(u_{max}(E))^{-c}-1\right) \left(\frac{1-y^*(E)}{H(y^*(E),E)}\right)^{\frac{3}{2}} \int_{y^*(E)}^{1} \frac{1-y^{2-c}}{1-y} \frac{dy}{\sqrt{1-y}}
\end{aligned}$$

We see that the function $y \to (1-y^{2-c})/(1-y)$ in the integrand is bounded since

$$\lim_{y\to 1} \frac{1-y^{2-c}}{1-y} = \lim_{y\to 1} \frac{-(2-c)y^{1-c}}{-1} = 2-c$$

and also that

$$\int_{y^*(E)}^{1} \frac{dy}{\sqrt{1-y}} = 2\sqrt{1-y^*(E)}$$

and $\left(\frac{1-y^*(E)}{H(y^*(E),E)}\right)^{\frac{3}{2}}$ have finite limits as $E \to 0^-$, but because $\frac{2\kappa}{2-c}(u_{max}(E))^{-c} \to 1$ as $E \to 0^-$, there follows that

$$\left| \int_{y^*(E)}^{1} \frac{dy}{\sqrt{H(y,E)}} - \int_{y^*(E)}^{1} \frac{dy}{\sqrt{y^{2-c}-y^2}} \right| \to 0 \text{ as } E \to 0^-$$

We see that the concavity of H implies that

$$H(y,E) \geq \frac{H(y^*(E),E)}{y^*(E)-y_1(E)}(y-y_1(E)) \text{ for all } E_0 < E < 0 \text{ and } y_1(E) \leq y \leq y^*(E)$$

and similarly that the concavity of the map $y \to f^*(y) = y^{2-c} - y^2$ on $[0,1]$ implies that

$$f^*(y) \geq \frac{f^*(y^*(E)) - f^*(y_1(E))}{y^*(E)-y_1(E)}(y-y_1(E)) + f^*(y_1(E))$$

We then find that

$$\begin{aligned}
0 &\leq \int_{y_1(E)}^{y^*(E)} \frac{dy}{\sqrt{H(y,E)}} - \int_{y_1(E)}^{y^*(E)} \frac{dy}{\sqrt{y^{2-c}-y^2}} \\
&= \int_{y_1(E)}^{y^*(E)} \frac{\left(\frac{2\kappa}{2-c}(u_{max}(E))^{-c} - 1\right)(1-y^{2-c})}{\sqrt{H(y,E)}\sqrt{y^{2-c}-y^2}\left(\sqrt{H(y,E)}+\sqrt{y^{2-c}-y^2}\right)} dy \\
&\leq \left(\frac{2\kappa}{2-c}(u_{max}(E))^{-c} - 1\right) \int_{y_1(E)}^{y^*(E)} \frac{dy}{\sqrt{H(y,E)}(y^{2-c}-y^2)} \\
&\leq \frac{\frac{2\kappa}{2-c}(u_{max}(E))^{-c} - 1}{\sqrt{\frac{H(y^*(E),E)}{y^*(E)-y_1(E)}}\left(\frac{f^*(y^*(E))-f^*(y_1(E))}{y^*(E)-y_1(E)}\right)} \\
&\quad * \int_{y_1(E)}^{y^*(E)} \frac{dy}{\sqrt{y-y_1(E)}\left((y-y_1(E)) + \frac{f^*(y_1(E))}{f^*(y^*(E))-f^*(y_1(E))}(y^*(E)-y_1(E))\right)}
\end{aligned}$$

Let

$$\alpha(E) = \frac{f^*(y_1(E))}{f^*(y^*(E)) - f^*(y_1(E))}(y^*(E)-y_1(E))$$

a positive continuous function that approaches 0 as $E \to 0^-$. We see then that the integral evaluates to

$$
\begin{aligned}
\int_{y_1(E)}^{y^*(E)} \frac{dy}{\sqrt{y-y_1(E)}(y-y_1(E)+\alpha(E))} &= \int_0^{y^*(E)-y_1(E)} \frac{dx}{\sqrt{x}(x+\alpha(E))} \\
&= \int_0^{\sqrt{y^*(E)-y_1(E)}} \frac{2du}{u^2+\alpha(E)} \\
&= \frac{2}{\sqrt{\alpha(E)}} \int_0^{\sqrt{\frac{y^*(E)-y_1(E)}{\alpha(E)}}} \frac{dv}{v^2+1} \\
&= \frac{2}{\sqrt{\alpha(E)}} \arctan\left(\sqrt{\frac{y^*(E)-y_1(E)}{\alpha(E)}}\right)
\end{aligned}
$$

through a series of variable changes: $x = y - y_1(E)$, $u = \sqrt{x}$, and $v = \frac{u}{\sqrt{\alpha(E)}}$. By definition, $H(y_1(E), E) = 0$, for all $E_0 < E < 0$, hence

$$
\frac{2\kappa}{2-c}(u_{max}(E))^{-c} - 1 = \frac{2\kappa}{2-c}(u_{max}(E))^{-c}(y_1(E))^{2-c} - (y_1(E))^2
$$

Let

$$
\sqrt{\frac{H(y^*(E),E)}{y^*(E)-y_1(E)}\left(\frac{f^*(y^*(E))-f^*(y_1(E))}{y^*(E)-y_1(E)}\right)} = \beta(E)
$$

which is a bounded positive continuous function that converges to a positive value β^* as $E \to 0^-$. So together we have

$$
\begin{aligned}
&\frac{\frac{2\kappa}{2-c}(u_{max}(E))^{-c}-1}{\sqrt{\frac{H(y^*(E),E)}{y^*(E)-y_1(E)}\left(\frac{f^*(y^*(E))-f^*(y_1(E))}{y^*(E)-y_1(E)}\right)}}\left(\frac{2}{\sqrt{\alpha(E)}}\right)\arctan\left(\sqrt{\frac{y^*(E)-y_1(E)}{\alpha(E)}}\right) \\
&= \frac{2}{\beta(E)}\frac{\frac{2\kappa}{2-c}(u_{max}(E))^{-c}(y_1(E))^{2-c}-(y_1(E))^2}{\sqrt{(y_1(E))^{2-c}-(y_1(E))^2}}\arctan\left(\sqrt{\frac{y^*(E)-y_1(E)}{\alpha(E)}}\right) \\
&= \frac{2}{\beta(E)}\frac{(y_1(E))^{2-c}\left(\frac{2\kappa}{2-c}(u_{max}(E))^{-c}-(y_1(E))^c\right)}{(y_1(E))^{\frac{2-c}{2}}\sqrt{1-(y_1(E))^c}}\arctan\left(\sqrt{\frac{y^*(E)-y_1(E)}{\alpha(E)}}\right) \\
&= \frac{2}{\beta(E)}\frac{(y_1(E))^{\frac{2-c}{2}}\left(\frac{2\kappa}{2-c}(u_{max}(E))^{-c}-(y_1(E))^c\right)}{\sqrt{1-(y_1(E))^c}}\arctan\left(\sqrt{\frac{y^*(E)-y_1(E)}{\alpha(E)}}\right)
\end{aligned}
$$

Because $1 \le c < 2$ and since $u_{max}(E) \to \left(\frac{2\kappa}{2-c}\right)^{1/c} > 0$, $y_1(E) \to 0$, $y^*(E) \to$

$\left(\frac{2-c}{2}\right)^{1/c} > 0$, $\alpha(E) \to 0$, and $\beta(E) \to \beta^* > 0$ as $E \to 0^-$, we find that

$$\lim_{E\to 0} \frac{2}{\beta(E)} \frac{(y_1(E))^{\frac{2-c}{2}}\left(\frac{2\kappa}{2-c}(u_{max}(E))^{-c} - (y_1(E))^c\right)}{\sqrt{1-(y_1(E))^c}} \arctan\left(\sqrt{\frac{y^*(E)-y_1(E)}{\alpha(E)}}\right) = 0$$

and therefore

$$\lim_{E\to 0}\left|\int_{y_1(E)}^{1} \frac{dy}{\sqrt{H(y,E)}} - \int_{y_1(E)}^{1} \frac{dy}{\sqrt{y^{2-c}-y^2}}\right| = 0$$

hence also

$$\lim_{E\to 0} T(E) = \frac{\pi}{c}$$

□

4 Conclusion

We establish first that for two specific power laws for $f(u)$, the angle $T(E)$ is, in fact, independent of the value of E.

Lemma 4.1. *Suppose that*

$$T(E) = \int_{u_{min}(E)}^{u_{max}(E)} \frac{du}{\sqrt{2(E-G(u))}}, \text{ where } G(u) = \frac{1}{2}u^2 - \kappa\frac{u^{2-c}}{2-c}$$

for some $\kappa > 0$ and some $c > 0$, but $c \neq 2$. Let $u_{min}(E) < u_{max}(E)$ be the two solutions to the equation $G(u) = E$, for those values of E for which these exist.

Let $E_0 = G(u_0)$ where $u_0 = \kappa^{1/c}$ is the unique, positive value of u where $G'(u) = 0$. Then

1. *If $c = 1$, then $T(E) = \pi$, for all $E_0 < E < 0$.*

2. *If $c = 4$, then $T(E) = \pi/2$, for all $E > E_0$.*

In particular, in both these cases, $T(E)$ is independent of E.

Proof. 1. Assume that $c = 1$. Then

$$u_{min}(E) = \sqrt{2}\left(\kappa - \sqrt{E + \left(\frac{\sqrt{2}\kappa}{2}\right)^2}\right), \text{ and } u_{max}(E) = \sqrt{2}\left(\kappa + \sqrt{E + \left(\frac{\sqrt{2}\kappa}{2}\right)^2}\right)$$

for all $E > E_0$, and then by completing the square in $G(u) = u^2/2 - \kappa u$, we find that

$$T(E) = \int_{u_{min}(E)}^{u_{max}(E)} \frac{du}{\sqrt{2\Big(E + \big(\frac{\sqrt{2}\kappa}{2}\big)^2 - \big(\frac{u-\sqrt{2}\kappa}{\sqrt{2}}\big)^2\Big)}}, \text{ for all } E > E_0$$

and by the substitution

$$v = \frac{u - \sqrt{2}\kappa}{\sqrt{2}\sqrt{E + \big(\frac{\sqrt{2}\kappa}{2}\big)^2}}$$

$T(E)$ simplifies to

$$T(E) = \int_{-1}^{1} \frac{dv}{\sqrt{1-v^2}} = \pi, \text{ for all } E > E_0$$

which is independent of the value E.

2. Assume that $c = 4$. Then

$$u_{min}(E) = \frac{\sqrt{2(E+\sqrt{\kappa})} - \sqrt{2(E-\sqrt{\kappa})}}{2}, \text{ and } u_{max}(E) = \frac{\sqrt{2(E+\sqrt{\kappa})} + \sqrt{2(E-\sqrt{\kappa})}}{2}$$

for all $E_0 < E < 0$, so we find that

$$T(E) = \int_{u_{min}(E)}^{u_{max}(E)} \frac{udu}{\sqrt{2Eu^2 - u^4 - \kappa}}, \text{ for all } E_0 < E < 0$$

By completing the square of the polynomial in the denominator, this leads to

$$T(E) = \int_{u_{min}(E)}^{u_{max}(E)} \frac{udu}{\sqrt{(E^2-\kappa) - (u^2-E)^2}}, \text{ for all } E_0 < E < 0$$

Then the substitution

$$v = \frac{u^2 - E}{\sqrt{E^2 - \kappa}}$$

yields that

$$T(E) = \frac{1}{2}\int_{-1}^{1} \frac{dv}{\sqrt{1-v^2}} = \frac{\pi}{2}$$

which is independent of the value E.

□

To conclude, we show why the previous results (almost) imply Bertrand's Theorem.

First, Lemma 3.3 shows that the only gravitational fields having a circular orbit, and such that T_{cir} is independent of the radius $\rho_0 = 1/u_0$, are those for which the corresponding function $f(u)$ in the (u, v)-system is a specific power law $f(u) = \kappa u^{1-c}$, for some $\kappa > 0$ and some $c > 0$. In this case, we find that

$$T_{cir} = \lim_{E \to E_0^-} T(E) = \frac{\pi}{\sqrt{c}}$$

Moreover, the theory of ODE's implies that for these specific power laws $f(u)$, the function $T(E)$ is a continuous function of E, as long as E belongs to the interval where $T(E)$ is well-defined (this is for $E_0 < E < 0$ in case $1 \leq c < 2$, and for $E > E_0$ in case $c \geq 2$). Bertrand's Theorem identifies exactly those power laws $f(u) = \kappa u^{1-c}$ for which the function $T(E)$ is a constant that equals a rational multiple of π.

- When assuming that $c \geq 2$, it follows from Lemma 3.4 that $\lim_{E\to\infty} T(E) = \pi/2$. In this case, constancy of the function $T(E)$ implies that there must hold that

  $$\lim_{E \to E_0^-} T(E) = \lim_{E \to \infty} T(E)$$

  or equivalently, that

  $$\frac{\pi}{\sqrt{c}} = \frac{\pi}{2}$$

  Solving this equation for c yields that $c = 4$. Recall from equation (2.10) that

  $$f(u) = \frac{\mu}{l^2} \frac{1}{u^2(\theta)} h\Big(\frac{1}{u(\theta)}\Big)$$

  Solving for $h(\rho)$, we find

  $$h(\rho) = \frac{\kappa l^2}{\mu} \rho$$

  hence

  $$\mu \ddot{\mathbf{r}} = -h(\rho)\hat{\boldsymbol{\rho}} = -\frac{\kappa l^2}{\mu} \rho \hat{\boldsymbol{\rho}}$$

  where $\hat{\boldsymbol{\rho}} = \frac{\mathbf{r}}{|\mathbf{r}|}$. This is precisely of the form of Hooke's Law of Gravitation. Lemma 4.1 above then implies that in this case $T(E)$ is indeed a constant function, and equals $\pi/2$.

- When assuming that $1 \leq c < 2$, it follows from Lemma 3.6 that $\lim_{E\to 0^-} T(E) = \pi/c$. In this case, constancy of the function $T(E)$ implies that there must hold that

$$\lim_{E\to E_0^-} T(E) = \lim_{E\to 0^-} T(E)$$

or equivalently, that

$$\frac{\pi}{\sqrt{c}} = \frac{\pi}{c}$$

Solving this equation for c yields that $c = 1$. From (2.10), we again have that

$$f(u) = \frac{\mu}{l^2}\frac{1}{u^2(\theta)}h\Big(\frac{1}{u(\theta)}\Big)$$

Solving for $h(\rho)$, we find

$$h(\rho) = \frac{\kappa l^2}{\mu}\rho^{-2}$$

hence

$$\mu\ddot{\mathbf{r}} = -h(\rho)\hat{\boldsymbol{\rho}} = -\frac{\kappa l^2}{\mu}\rho^{-2}\hat{\boldsymbol{\rho}}$$

which is precisely of the form of Newton's Law of Gravitation. Lemma 4.1 above then implies that in this case $T(E)$ is indeed a constant function, and equals π.

Remark: We have mentioned earlier that these results *almost* imply Bertrand's Theorem. This is because we have not yet proved that the conclusion of Lemma 3.6, which is that

$$\lim_{E\to 0^-} T(E) = \frac{\pi}{c}$$

remains valid for power laws $f(u) = ku^{1-c}$ where $0 < c < 1$.

Provided that this can indeed be proved, Bertrand's Theorem will follow because we shall be able to conclude that if $0 < c < 2$ (and not just when $1 \leq c < 2$), the constancy of $T(E)$ would force $c = 1$ as shown in the second item above.

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