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R. Charles Vars, Jr.

Theoretical work on the economics of information has traditionally treated information in a discrete optimization context. Because continuous differentiability is a dominant assumption in neoclassical economic theory, the modelling of information in discrete frameworks precludes the examination of some interesting problems involving information that are continuous in nature. These three essays present some theoretical constructs for the modelling of information in continuous frameworks.

The first essay defines the general case of information value in a "parameter preference" framework, where an individual's perceptions of uncertainty are represented by continuous subjective probability distributions. The analytical framework postulates that expected utility is an implicit function of information that induces changes in the mean and central moments of a distribution. A necessary and sufficient condition for information to have value is that it alter an individual's perceptions of uncertainty. Information will

have positive(negative) value if the net change in continuous distributional parameters is in a preferred(unpreferred) direction. These results constitute the conditions for defining a preference ordering over information.

The second essay extends the results of the first essay to the problem of defining a utility function for information. The essay argues that because a preference ordering for information depends upon the existence of preference directions for distributional parameters, restricted forms of expected utility serve as ideal candidates for utility functions for information. The restricted functional forms include two polynomial forms, exponential, constant elasticity and logarithmic utility. Each form is restated as a utility function for information by substituting functional relationships between information and distributional parameters into the restricted forms of expected utility. Comparative statics analyses are then conducted to compare the value of information for individuals that differ in tastes and beliefs.

The third essay applies the results of the first two essays to the derivation and comparative statics analyses of demand functions for securities and information about securities. These derivations are performed using a logarithmic utility function over securities and information.

Three Essays on the Theory of Information Valuation and Demand

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I. A PARAMETER PREFERENCE APPROACH TO THE VALUE OF INFORMATION

Introduction

Theoretical work on the economics of information[10] has treated the value of information in a decision-making-under-uncertainty context, where an individual seeks to take that action or set of actions that maximizes his expected utility. A major assertion found in the literature is that information has positive value to an individual if the information, by altering subjective probabilities assigned to possible perceived states of the world, leads the individual to expect gains in utility from altering his intended actions. Information never has negative value because an individual will not change his intended actions if he expects decreases in utility from doing so. This "state preference" approach assumes that the individual behaves as if he can (i) describe his uncertainty about the true value of a random variable by a discrete, state-specific subjective probability distribution, (ii) define specific alternative courses of action associated with each, and (iii) define his preferences for different action-specific consequences according to a utility function.

This essay defines the value of information in a "parameter preference" framework, where an individual's perceptions of uncertainty are represented by the distributional parameters (mean and central moments) of continuous subjective probability distributions. The analytical framework postulates that expected utility is an implicit function of information that induces changes in distributional parameters.

While the state preference approach to information value focuses on information as an input to decision-making under uncertainty, the parameter preference approach focuses on information as a commodity that, like any other commodity, is consumed because it is expected to generate positive utility. In the parameter preference case, the value of information is the information-induced change in expected utility over a random variable. If information changes perceptions of a random variable's true value, and if changes in those perceptions are valued nontrivially (given by the tastes embodied in the utility function), then information will have nontrivial value. Therefore, a necessary and sufficient condition for information to have value is that it alter an individual's perceptions. An individual's valuation of information does not depend upon the expected realization of improved consequences, reflected by higher levels of expected utility from improved actions. This is in contrast to the state preference case, where a change in perceptions (reflected by a change in the discrete state-specific probability vector) is a necessary condition and the existence of a matrix of competing alternative choices of action is a sufficient condition for information to have value.

In this essay parameter preference information valuation will be treated in the general case. The definition and analysis of parameter preference information valuation will be preceded by a comprehensive review of the Von Neumann-Morgenstern[25] Expected Utility Hypothesis, and brief reviews of state preference information valuation, probability theory pertinent to the case of integrable expected utility theory, and the theory of distributional parameter preference. The findings of the general case are interpreted to have significant theoretical and empirical implications for restricted forms of expected utility, cases from which specific testable propositions can be derived.

The Expected Utility Hypothesis and the Role of Information

The notion of information value and the Von Neumann-Morgenstern Expected Utility Hypothesis (EUH) are intimately related. EUH proceeds from traditionally deterministic utility theory by relaxing the assumption that a rational individual has perfect knowledge about the satisfaction obtainable from the consumption of a commodity. If U(X) is an individual's utility function over commodity X, then he is assumed to be able, with total certainty, to associate an index number U with a unique value for X. U(X) is therefore a <u>deterministic</u> function. When X is a random variable (such as claims to future income or commodities), U(X) is a <u>stochastic</u> function. A central assumption in EUH is that when U(X) is stochastic, an individual will have a set of perceptions about the uncertainty of the true value of X and these perceptions are representable in terms of a subjective probability distribution over X's possible values. Subjective probability distributions can be either <u>discrete</u>, where the individual can describe his uncertainty by assigning probabilities to possible values of X, or <u>continuous</u>, where he is assumed to formulate an estimate of the variable's true value and the corresponding error of that estimate. These continuous formulations are represented by distributional "parameters", where the estimate of the random variable's true value is the mean and perceived error of the estimate is a set of moments about the mean, more commonly referred to as "central" moments. Probabilistic models of behavior under uncertainty that assume continuous distributions are traditionally referred to as "parameter preference" models. Models assuming discrete distributions are traditionally referred to as "state preference" models.

Perceptions of uncertainty interact with an individual's attitudes toward uncertainty, these attitudes determined by structural characteristics of his utility function, to induce the individual to engage in maximization behavior. If the deterministic function U(X) states that an individual's preferences for totally certain values of X can be described by an index of utility U traced over the set X, then EUH states that preferences for uncertain values of X can be described by an index of mathematical expectations of utility E[U(X)], traced over X's subjective probability distribution. It follows that given the existence of the expectation of U(X), an individual will behave as if he maximizes his expected satisfaction E[U(X)].

The relationship between information and expected utility maximization is envisaged in Fig. 1. Information is defined to be any message that induces changes in perceptions of uncertainty, altering probabilities assigned to a random variable's possible values(discrete distribution) or distributional parameters(continuous distribution). The impact of information on subjective probability distributions is reflected by changes in the level(s) of maximum expected utility attainable.

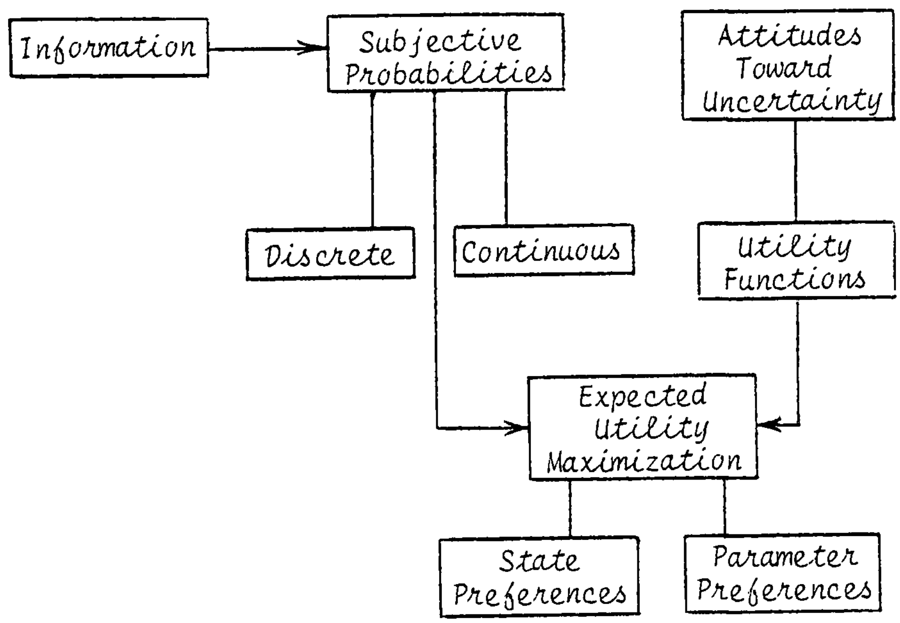

Figure 1. Information and expected utility maximization.

State and parameter preference approaches to behavior under uncertainty have distinctly different implications for information valuation. Although both proceed from expected utility maximization, the state preference approach assumes a discrete and cardinal expected utility function and the parameter preference approach an integrable function that can be either cardinal or ordinal. The definition of the expectation operator in E[U(X)] depends on whether the density function f(X) over the distribution function F(X) is continuous or discrete. In the state preference case, F(X) is discrete and the expectation of utility is given by the following formulation:

$$(1) \qquad E[U(X)] = \sum_{X} U(X)f(X)$$

In state preference expected utility maximization, E[U(X)] refers to the expected utility of an action X. The individual decision-maker is assumed to have formulated a matrix of competing alternative choices of action. The expected utility of any action is defined by (1) as the sum of the expected values of the possible utility-weighted consequences of that action. Given that the individual must choose between one of several competing actions, he is hypothesized to choose that action that yields the highest level of expected utility. By choice of this action, he is effectively choosing a probability distribution (one distribution corresponding to each action) that yields to him the highest level of expected utility.

The role of information in state preference expected utility maximization is to alter the expected values of the consequences of the actions.[1/] If information is available to the decision-maker, the processing of it will yield new values for f(X) and F(X), i.e. F(X) is converted from a prior to a posterior distribution. Previous work on state

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1/ It is conceivable that information could, in addition to altering probabilities of present perceived possible states of the world, add or delete states of the world to the individual's "awareness" of possible states, and add or delete alternative courses of action. From an empirical standpoint, proving that additions or deletions of states occur with information may be precluded by the possibility that the newly added states always existed in the individual's perceptions, but were assigned zero probabilities earlier.

preference information valuation [10] treats these distributional changes in a "Bayesian" framework. This framework assumes an a-priori distribution over states of the world and an a-priori conditional message probability distribution. The conditional message probability distribution states the probability that a message forecasting a particular state of the world will be received, given the individual's knowledge that the state will occur. The difference between the individual's subjective probability that the state will occur and the message's probability of occurrence is the individual's perception of message error.

A decision-maker is asserted to acquire information for the purpose of making decisions whose consequences are expected to generate higher levels of expected utility than would be realized without the information. Higher levels of expected utility for each alternative action are realized when new information effectively increases the expected values of positively valued consequences and lowers the expected values of negatively valued consequences. The new expected values are derived from the posterior probability distribution, stating the probability of a particular state of the world occurring, given that a message forecasting its occurrence has been received.2/ Allow a_0 to be an action that

2/ The information decision-making framework, given a Bayesian probability recalculation scheme summarized above, states that a posterior distribution, $Pr(s|m_s)$, the probability of state s occurring given a message m_s forecasting its occurrence, is given by:

$$Pr(s|m) = \frac{Pr(m|s)Pr(s)}{Pr(m)}$$

yields expected utility $E[U(a_0)]$, the maximum level of expected utility attainable, given a prior distribution. Let action a_1 generate $E[U(a_1)]$, the maximum level of expected utility attainable, given by a posterior distribution generated by new information. The value of that new information is the expected gain in utility from switching to a_1, or $\{E[U(a_1)] - E[U(a_0)]\}$.

In parameter preference models, the subjective probability distribution is continuous and describable over the mean and central moments, as given by the following definition of the expectation operator:[3/]

(2) $$E[U(X)] = \int_{-\infty}^{\infty} U(X)f(X)dX = \int_{-\infty}^{\infty} U(X)dF\{X:E(X),\ E[X-E(X)]^n\}$$
$$n = 0,1,2,3,\ldots..$$

The mean of F(X), E(X) is defined by the following familiar rule:

(3) $$E(X) = \int_{-\infty}^{\infty} Xf(X)dX$$

The nth central moment $E[X-E(X)]^n$ is given by the rule:

(4) $$E[X-E(X)]^n = \int_{-\infty}^{\infty} [X-E(X)]^n f(X)dX$$

The mean of a continuous subjective probability distribution is an estimate, an expectation of X's true value.

3/ F(X) is given by the following rule:

$$F(X) = \int_{-\infty}^{\infty} f(X)dX. \quad \text{Since } f(X) = \frac{dF(X)}{dX},$$

it follows that by multiplying both sides of the second equation by dX, $dF(X) = f(X)dX$.

Since the mean is the point of "central tendency" on the distribution, it represents in behavioral terms the value of X that the individual is most convinced will occur.

The central moments are measurements of an individual's perceptions of estimation error. Since $E[X-E(X)]^0 = 1$ and $E[X-E(X)] = 0$, the second central moment, variance, is the first appropriate measure of the individual's perceived error associated with the mean. Variance is a symmetrical measure of perceived error, as shown in Fig. 2's hypothetical distribution.

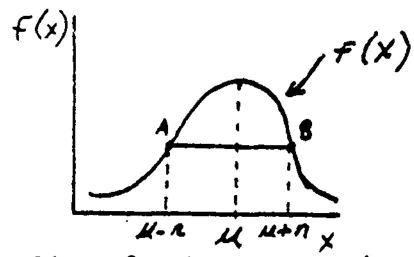

Fig. 2. A symmetric distribution.

Variance σ^2 about the mean μ is the horizontal distance between two points (for example, A and B) on the tails of the distribution. The horizontal distance from the mean out to a point on the distribution's tail is the standard deviation, the square root of variance. A distribution describable over the mean and variance is symmetric about the mean, i.e. the individual perceives that the probabilities that the estimated value of X is above or below the mean by some number n, are equal. From Fig. 2, the probability that the true value of X falls between μ and $(\mu+n)$ is:

(5) $$\Pr[\ \mu < X < (\mu+n)\ ] = \int_{\mu}^{\mu+n} f(X)\,dX$$

The probability that the true value of X falls between

μ and $(\mu-n)$ is given by:

(6) $$\Pr[(\mu-n) < X < \mu] = \int_{\mu-n}^{\mu} f(X)\,dX$$

Symmetry about the mean would be reflected by equality of the two previously specified probabilities, or:

(7) $$\Pr[\mu < X < (\mu+n)] - \Pr[(\mu-n) < X < \mu] = \int_{\mu}^{\mu+n} f(X)\,dX - \int_{\mu-n}^{\mu} f(X)\,dX = 0$$

The third central moment, commonly referred to as skewness, adds directional bias to the perceived error of an individual's estimate. A distribution with skewness is asymmetric about the mean, with positive skewness implying positive asymmetry and negative skewness implying negative asymmetry. Fig. 3 shows a hypothetical distribution with positive skewness.

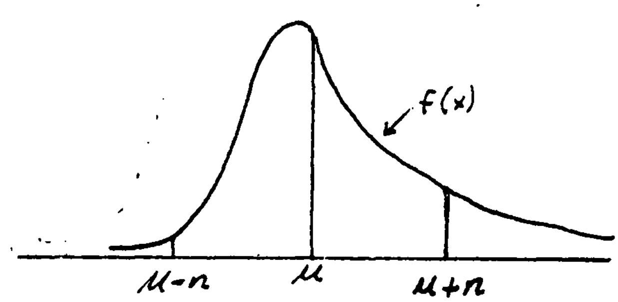

Fig. 3. A distribution with positive skewness.

With positive skewness, an individual perceives that there is a greater chance that his expectation of X, μ is an underestimate than an overestimate. The probability that the true value of X falls between μ and $(\mu+n)$ is, from Fig. 3, greater than the probability that X falls between μ and $(\mu-n)$, or:

(8) $$\Pr[\mu < X < (\mu+n)] - \Pr[(\mu-n) < X < \mu] = \int_{\mu}^{\mu+n} f(X)\,dX - \int_{\mu-n}^{\mu} f(X)\,dX,$$
>0

Negative skewness would be defined by the following example from Fig. 3:

(9) $\Pr[\mu < X < (\mu+n)] - \Pr[(\mu-n) < X < \mu] =$

$$\int_{\mu}^{\mu+n} f(X)\,dX - \int_{\mu-n}^{\mu} f(X)\,dX < 0$$

Much of the literature has not examined the behavioral implications of the fourth and higher central moments, largely because there is uncertainty as to what these moments measure, in a purely statistical sense.4/ The fourth central moment, commonly known as kurtosis, has been interpreted to be a measure of a distribution's peakedness, the slopes of the distribution's tails. Fig. 4 examines two hypothetical distributions, A and B, superimposed on one another.

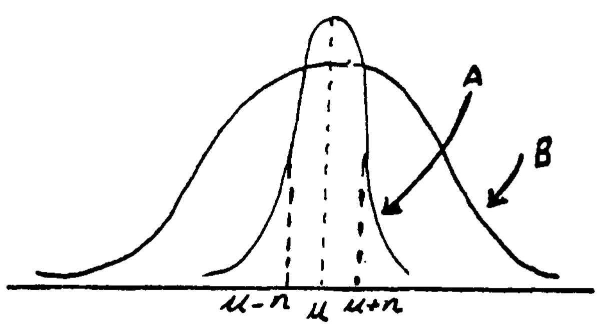

Figure 4. Comparison of kurtosis between two symmetric distributions.

In Fig. 4, both distributions have identical means μ. Distribution A has higher kurtosis than B in the area around the mean because the slopes of A's tails are greater than those of B's. An alternative interpretation with apparent behavioral implications can be made by considering the probabilities of underestimation or overestimation of the mean, as treated in earlier examples. $(\mu+n)$ and $(\mu-n)$

4/ See Kaplanski in [12].

are equal deviations from the mean. Distribution A has higher kurtosis than distribution B because there is higher probability that the true value of X will fall within the range of the interval $[(\mu-n), (\mu+n)]$ for A than for B.

There is obviously an inverse relationship between kurtosis and variance of a distribution. Distribution A's comparatively higher kurtosis over distribution B would imply a lower variance for A than for B. Therefore, kurtosis and all other existing higher even-numbered moments explain with progressive accuracy the nature of a distribution's symmetry. The fifth and higher odd-numbered moments explain with greater accuracy than skewness, the nature of a distribution's asymmetry. The fourth and higher moments may be useful in explaining an estimated distribution of a population of objects, but there is clear doubt as to the meaningfulness of depicting and justifying human perceptions of uncertainty by moments higher than skewness.

While F(X) in both the continuous and discrete cases summarizes perceptions as to the uncertainty of X's true value, the utility function U(X) determines the individual's attitudes toward these perceptions. These attitudes are describable over the nth-ordered derivatives of U(X):

$$\frac{d^n U(X)}{dX^n}$$

Any nth-ordered derivative of U(X) is defined to be a "direction of preference" for the corresponding nth central moment of the subjective probability distribution. Pre-

-ference directions are definable over the second and higher derivatives since perceptions of uncertainty are definable over the second and higher central moments. The first derivative of U(X) is the direction of preference for the mean of the distribution and has no implications for choices involving risk. If X is future wealth, then a positive first derivative implies that more wealth is preferred to less - a statement of increasing (or strictly increasing) utility of future wealth. The second derivative is the direction of preference for variance.5/ The third derivative is the direction of preference for skewness and the fourth derivative is the direction of preference for kurtosis.6/ A risk-

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5/In a historic pair of papers [6&7], Friedman and Savage show how directions of preference for the mean and variance can be derived by imposing the assumption of "certainty equivalence" on the individual's set of alternative choices. Friedman and Savage were primarily interested in proving the conditions for risk-aversion and preference for risk, as summarized by the following: Assume alternative A to be a chance $p(0<p<1)$ of wealth W_1 and chance $(1-p)$ of wealth W_2, $W_2>W_1$. Assume that the expected utility of the two alternatives are functions entirely of the wealth and corresponding probabilities. Let B be a riskless alternative where $B=U(I_0)$, I_0 a certain amount of income. The expected utility of A is given by the application of the Von Neumann-Morgenstern rule of expected utility given a discrete subjective probability distribution:

$$E[U(A)] = pU(W_1) + (1-p)U(W_2)$$

$E[U(A)] < U(I_0)$ implies a concave utility function and risk-aversion. $E[U(A)] > U(I_0)$ implies a convex utility function and preference for risk, while $E[U(A)] = U(I_0)$ implies a linear function and indifference towards risk.

6/ Levy [15] has derived the conditions for positive and negative skewness preference for a cubic utility function, given a-priori attitudes toward variance. Scott and Horvath [22] have examined directions of preference for higher moments beyond skewness, given a-priori attitudes toward the averse investor with an increasing utility function and preference for positive skewness is assumed to have a utility function with the following first three derivatives: 7/

$$\frac{dU(X)}{dX} > 0, \quad \frac{d^2U(X)}{dX^2} < 0, \quad \frac{d^3U(X)}{dX^3} > 0$$

6/ (cont.) second and third central moments. Levy proved that an individual investor's attitude towards variance will always change, both in magnitude and sign over the range of expected portfolio wealth. In addition, as long as the investor has a positive preference direction for the mean, he will always prefer positive skewness over negative skewness. Scott and Horvath demonstrated by use of the Mean Value Theorem that a risk-averse investor displaying positive skewness preference will have positive directions of preference for odd-numbered moments and negative directions of preference for even-numbered moments.

7/ It has been popular in the literature to normalize each derivative of the utility function by the first derivative, the resulting measures of parameter preference directions being invariant under linear transformations of U(X):

$$\frac{\frac{dU^n(X)}{dX^n}}{\frac{dU(X)}{dX}}$$

Pratt [18] and Arrow [2], working independently of each other, developed a measure of absolute risk-aversion, the absolute risk-aversion function defined by the following formulation:

$$r(X) = -\frac{d^2U(X)/dX^2}{dU(X)/dX} = -\frac{d}{dX}\log\frac{dU(X)}{dX}$$

r(X) = 0 implies constant absolute risk-aversion and r(X) < 0 implies decreasing absolute risk-aversion. Rubinstein [19] generalized the findings of Arrow and Pratt by normalizing the third and higher derivatives of U(X), the resulting measures allowing for global, cardinal and interpersonal comparisons of risk-aversion at different levels of wealth.

Because the investor is risk- averse, an increase in variance generates disutility. With positive skewness preference, an increase in skewness generates utility. Risk-aversion thus implies a negative direction of preference for variance, and positive skewness preference implies a positive direction of preference for skewness.[8/]

Directions of preference for the mean and central moments provide the basis for the establishment of a preference ordering among continuous subjective probability distributions. The individual is assumed to be able to rank distributions (associate an index number E[U(X)]) according to the values of the distributions' parameters. With F(X) a probability distribution defined over the mean and existing central moments, the expected utility of X was given by the formulation:

$$(10)\quad E[U(X)] = \int_{-\infty}^{\infty} U(X)\,dF[X:E(X),\ \mu_0,\mu_1,\mu_2,\mu_3,\ldots\ldots\mu_n],$$

$$\text{where}\quad \mu_n = E[X-E(X)]^n$$

8/ The focal point of this discussion of moment preference directions is the uniformity that exists between the conditions guaranteeing a particular sign for the nth-ordered derivative of the deterministic function U(X) and the corresponding sign of the marginal expected utility of the nth central moment in E[U(X)]. For example if U(X) is quadratic, i.e. $U(X) = X + aX^2$, with a allowed to be negative, positive or trivial, then the condition guaranteeing the sign of the second derivative of U(X) is identical to the condition guaranteeing the same sign(s) for the marginal expected utility of variance. This can be seen by the following proof:

If $U(X) = X + aX^2$, then $\partial^2 U(X)/\partial X^2 = 2a \gtrless 0$ as $a \gtrless 0$.

$E[U(X)] = E(X) + a[E(X^2)]$ and $E(X^2) = \mu^2 + \sigma^2$,

where $\mu = E(X)$ and $\sigma^2 = E[X-E(X)]^2$.

Hence, $E[U(X)] = \mu + a(\mu^2 + \sigma^2)$

If F_1, F_2,.......,F_n are n different probability distributions defined on the set X, then the Expected Utility Hypothesis postulates that the most preferred distribution is the one that yields the highest value for (10). This postulate can, with great analytical convenience, be explicated by use of the expectation of a Taylor expansion of U(X) around the mean of X's distribution.9/ Allow U(X) to be approximated by an exact 10/ Taylor expansion around some arbitrary value U(X) = X_0:

$$U(X) = \sum_{n=0}^{\infty} \frac{U^n}{n!}(X-X_0)^n \tag{11}$$

8/(cont.) It follows that $\partial E[U(X)]/\partial\sigma^2 = a \gtrless 0$ as $a \gtrless 0$. The condition guaranteeing the sign of the marginal expected utility of variance is identical to that guaranteeing concavity or convexity of U(X).

9/The Taylor approximation is a generalized function that represents all the possible polynomials and non-polynomials that are restricted forms of U(X), e.g. quadratic, cubic, exponential, logarithmic, constant elasticity etc. The theoretical justification for using the Taylor approximation approach to generalizing expected utility has been discussed by Tsiang[24], Samuelson[21], Pratt[18] and others. Applications of the Taylor approximation to problems in parameter preference asset valuation have been demonstrated in a number of noteworthy papers: Rubinstein[19&20] derived efficiency conditions for parameter preference security valuation based on Taylor approximations of risk premiums. Levy [15], Arditti [2], Krauss & Litzenberger[14] and Jean [11], have used Taylor approximations for extensions of the Markowitz-Sharpe-Lintner Mean-Variance Capital Asset Pricing Model to the third moment through the derivation and testing of portfolio efficiency conditions. Scott and Horvath[22] also based their analyses on the use of Taylor approximations.

10/Exact Taylor expansions require only that the series is convergent, i.e. that the function can be treated over finite terms with a remainder. In some cases, as will be discussed later, if a polynomial is being approximated, a remainder may not be necessary. Since all restricted forms of expected utility are functions over finite central moments, From (11), U^n is the nth derivative of $U(X)$ evaluated at X_0. Allowing $X_0 = E(X)$, the expected value of (11) is given by:

(12) $$E[U(X)] = \sum_{n=0}^{\infty} \frac{U^n}{n!}\mu_n \quad , \quad \mu_n = \int_{-\infty}^{\infty}[X-E(X)]^n f(X)dX$$

Expected utility in (12) is a function of (i) the mean of X, (ii) the directions of preference for the central moments of the distribution of X, and (iii) the central moments themselves. Because (12) is additive (a property provided by the Taylor expansion), it is a cardinal function.11/

For the analysis that follows, we assume that the individual displays strict consistency in directional preferences for the mean and central moments. Allowing U^n to be the same as in (12), we may define strict consistency by the following conditions:

(13) $$\begin{aligned} &U^n > 0 \ \forall\ X \quad \text{or,} \\ &U^n < 0 \ \forall\ X \quad \text{or,} \\ &U^n = 0 \ \forall\ X \end{aligned}$$

10/ (cont.) approximations of them must guarantee convergence. Conditions for convergence of expected utility are explored in part by Loistl [16] and Tsiang [23]. As long as convergence is guaranteed, the finite Taylor expansion can represent any approximated polynomial (provided certain conditions are guaranteed) and the infinite expansion can represent any nonpolynomial.

11/If expected utility is assumed to be ordinal, the generalized case is given by:

$$E[U(X)] = E\{U[E(X), \mu_0, \mu_1, \mu_2, \ldots\ldots, \mu_n, \mu_{n+1}, \ldots]\},$$

where the mean and the central moments μ_n are the only arguments of the function. The directions of preference for the mean and the central moments are given by the derivatives of $E[U(X)]$ with respect to the arguments. For example, $\partial E[U(X)]/\partial\mu_3 > 0$ implies preference for positive skewness and $\partial E[U(X)]/\partial\mu_2 < 0$ implies risk-aversion.

Since the nth derivatives of U(X) in (11) and (12) are now implicitly defined to be coefficients, the mean and central moments are the only assumed differentiable arguments of E[U(X)]. Each term of (12) is the product of the moment preference direction and the corresponding nth central moment of the distribution.

Assume that an individual has a distribution definable over two moments, i.e. his distribution belongs to the family of two-parameter distributions. The individual's general expected utility function is represented by the expectation of the finite Taylor expansion of U(X) around E(X), truncated at the second term with a LaGrange remainder:[12/]

$$(14) \qquad E[U(X)] = U(X) + \frac{U^2(\beta)}{2}\mu_2 \quad , \ \mu_2 = E[X-E(X)]^2$$

$$\beta = X + \theta[X-E(X)], \ 0 \leq \theta \leq 1$$

The second term on the right side of (14) is the LaGrange remainder. The zeroth derivative (U itself) is evaluated at the mean of X, but by use of the Mean Value Theorem, the second derivative of U(X) is evaluated at a point β that falls within the interval [X,E(X)]. The theorem guarantees that β and θ assume values that allow the approximation error (the remainder) to equal the difference between the value of U(X), evaluated at any point X other than E(X), and U[E(X)].

12/ The formation of a remainder is justified if:(1) The Taylor series is approximating a polynomial of higher degree than the degree of the term at which the series is

With (14), assume that the individual is given a choice between two mean-variance distributions, F_1 and F_2. Distribution F_1 has mean $E(X)_1$ and variance $V(X)_1$, while distribution F_2 has mean $E(X)_2$ and variance $V(X)_2$. Allow $E(X)_1 < E(X)_2$ and $V(X)_1 > V(X)_2$. Assume risk-aversion, i.e. $U^2 < 0$ and increasing expected utility. The preferred distribution is clearly F_2, the one yielding the higher level of expected utility, $E[U(X)]_2$.

12/(cont.) truncated; (2) The series is approximating a non-polynomial. The Taylor approximation of a non-polynomial is a polynomial and the error of approximation (given by the LaGrange remainder) would reflect the difference between the polynomial approximation and the non-polynomial. In a vast majority of cases, (1) and (2) are the prevalent cases. When no special assumptions are made about either the utility function or underlying subjective probability distribution, truncation of the series at a particular term requires acknowledgement that higher-order terms may exist. The LaGrange remainder captures these higher terms and the use of it does not hinder the analysis that follows in subsequent sections, because we have invoked the assumption of strict consistency in preferences for directional changes in distributional parameters. This assumption implies that no matter where on X's range we evaluate U^n, the sign of U^n will always remain the same.

Truncation of the series without a remainder necessitates the very unrealistic assumptions that, (i) the central moment at the term of truncation is so small that higher moments are negligible, hence omittable, (ii) the series converges very quickly at the term of truncation and hence, higher moments are negligible and omittable. By the assumptions underlying the expectation of the Taylor series expansion, it is not valid to truncate a series and assume that higher moments are trivial- that automatically assumes a special distribution and utility function. Therefore, the decision to form a remainder requires recognition of the tradeoffs between adherence to mathematical consistency and the establishment of realistic behavioral assumptions. These issues have been addressed in Levy [15], Pratt [18] and Tsiang [24].

What constrains an individual from attaining expected utility level $E[U(X)]_2$ if he is presently at $E[U(X)]_1$? He is effectively constrained by his <u>perceptions</u> of X's true value, as represented by mean $E(X)_1$ and variance $V(X)_1$. It follows that the individual would <u>value</u> a change in his perceptions from $E(X)_1$ to $E(X)_2$ and from $V(X)_1$ to $V(X)_2$ by an amount equal to $\{E[U(X)]_2 - E[U(X)]_1\}$. This is the motivation for examining the role of information in parameter preference models.

Information as an Implicit Function of Expected Utility

Information in a parameter preference framework will be defined to be a message that induces changes in continuous distributional parameters. Specifically, a message can alter continuous distributions in three possible ways by: (1) inducing changes in the mean only; (2) inducing changes in central moments only; (3) inducing changes in both the mean and central moments.

The <u>content</u> of a message is defined to be the set of distributional values a message generates upon processing by an individual. For example, if an individual's distribution is definable over a mean and variance, then messages he processes can be distinguished in contents by the different sets of mean-variance values they generate upon processing. Consider mean-variance space in Fig. 5.

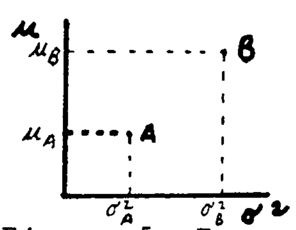

Figure 5. Two messages in mean-variance space.

Just as each point in mean-variance space in Fig. 5 can be shown to lie on an indifference curve, which defines the individual's desired tradeoffs between the mean and variance at fixed levels of expected utility, each set of mean-variance values can be associated with a unique message. In Fig. 5, point A corresponds to (μ_A, σ_A^2), parameter values generated by the processing of message A, while point B's set of mean-variance values (μ_B, σ_B^2) are generated by the processing of message B. The contents of messages A and B differ by the sets of mean-variance values they generate. The contents of the two messages are additive since both the mean and variance axes are common scales of measurement, i.e. μ and σ^2 are comparable.

In the preceding example, mean-variance space is also a two-dimensional "message space", since each point in the space can correspond to a unique message. In three-dimensional mean-variance-skewness space, a message can hypothetically be traced to each of the coordinates representing three points (mean, variance and skewness) in the space. In general, if all distributional parameters are subject to changes by information, (n+1)th-dimensional mean-central moment space are reflexive, i.e. message space and distributional parameter space are reflections of one another.

Consider three-dimensional mean-variance-skewness space in Fig. 6. Each point in this space can theoretically

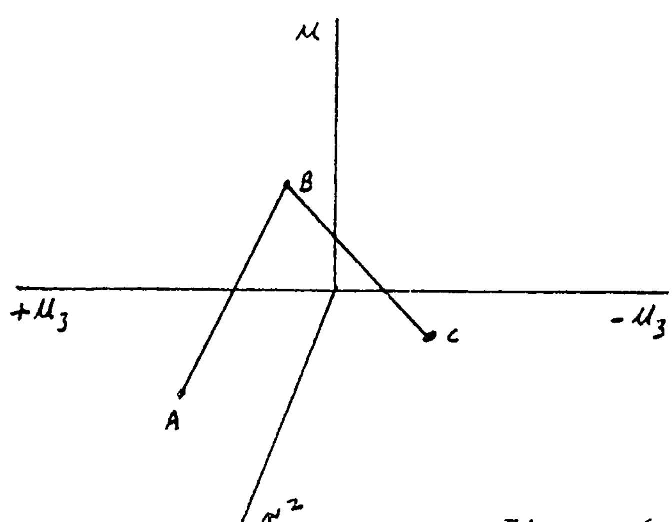

Figure 6. A discrete message path.

be traced to a unique message, therefore message space can also be three-dimensional. A unique message will correspond to a unique level of expected utility and this level of expected utility will constitute one unique static equilibrium in one time period. It follows that a unique message will correspond to a unique static equilibrium in one time period.

If an individual's consumption of information is considered in a multi-period framework, then over a sequence of periods, he will recieve and process a unique sequence of discrete messages. For example, in Fig. 6, the individual receives and processes three consecutive messages, A,B and C(shown as individual points) each in a total of three time periods. The individual can be considered to travel along a discrete message "path", given by the discontinuous function in Fig. 6.

The discrete message path in Fig. 6 is an example of how individuals realistically receive and process information in a multi-period framework. The receipt and processing of information is an inherently discrete process. For example, a securities investor receives a message in one period that increases his expectations and skewness but lowers his perceived risk(variance) of portfolio returns - reflected by a movement from message A to message B in Fig. 6. In the third period, a message is received that lowers expectations, but increases skewness and variance - reflected by a movement from message B to message C.

Under what conditions can we determine how the indi-

-vidual would assign a preference ordering over the three messages in Fig. 6. In other words, if the individual is assumed to be able to define his tastes for and perceptions of uncertainty, can we derive a framework that will state the conditions for a preference ordering over messages A, B and C and all other messages in the message space.

The notion of a preference ordering for messages implies a utility function for information. One of the crucial assumptions underlying any utility function is "continuity" of preferences, i.e. that an individual's preferences over different quantities of a commodity must be reducable to a continuum of quantities. In other words, no matter how infinitessimal the difference between the two quantities of a commodity, an individual must be able to state which quantity he prefers over the other.

In terms of a message space, an individual's preference ordering over different messages will be valid if the individual can state his preferences over a continuum of messages. A hypothetical continuum of messages I, which we will refer to as a continuous message path, is depicted in three-dimensional mean-variance-skewness space in Fig. 7. If a

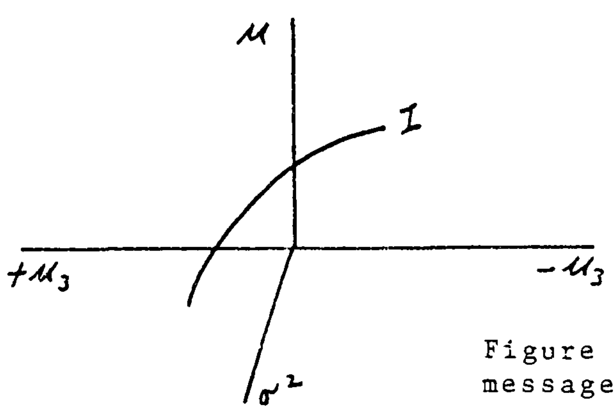

Figure 7. A continuous message path.

message path is continuous, then the mean μ and the n definable central moments μ_n of a random variable X's distribution are continuously differentiable over the path, which we refer to as I:

(15) $$\int_{-\infty}^{\infty} Xf(X)dX = \mu(I)$$

(16) $$\int_{-\infty}^{\infty} [X-\mu]^n f(X)dX = \mu_n(I)$$

The following partial derivatives represent changes in dsitributional parameters resulting from an infinitessimal change in information, i.e. an infinitessimal movement along the continuous message path I:

(17) $\frac{\partial\mu}{\partial I}$ (18) $\frac{\partial\mu_n}{\partial I}$

Relationships (17) and (18) can be differentiated with respect to I again, yielding expressions whose signs determine whether the individual experiences increasing, diminishing or constant information "potency":

(19) $\frac{\partial^2\mu}{\partial I^2}$ (20) $\frac{\partial^2\mu_n}{\partial I^2}$

If (19) and (20) are both positive, then both distributional parameters will change at increasing rates, i.e. the more information is consumed, the greater the impact of additional increments of information on the individual's perceptions. This would be a case of "increasing information potency". If (19) and (20) are both negative, then both distributional parameters will change at decreasing rates, i.e. the more information consumed, the less effective additional increments of information are in changing an indi-

-vidual's perceptions. If (19) and (20) are constant, then regardless of the quantity of information processed, the effectiveness of the information in changing perceptions will be constant.

In general, I is a message path in (n+1)th-dimensional mean-central moment space and can be discrete or continuous. In the case of a discrete message path, an individual receives and processes a sequence of messages whose contents vary discretely. In the case of a continuous message path, an individual receives and processes a sequence of messages whose contents vary infinitessimally and this sequence can thus be depicted as a continuum of messages. Based on relationships (17) and (18), it is clear that if the continuous distributional parameters are continuously differentiable ocer information, expected utility is continuously differentiable over information.

The dimensionality of message space depends entirely upon what distributional parameters are subject to changes by information. If an individual processes information in such a way that only his expectations are subject to change, then his message space is one-dimensional and information appears only in the expected utility function as an argument of the mean. If an individual processes information in such a way that only his perceived error of expectations is subject to change, then his message space is n-dimensional (n central moments) and information appears in the expected utility function as an argument of the central moments. In-

-formation can also be an argument of both the mean and central moments.

Relationships (15) - (20) and the concept of a discrete message path summarize the fundamental hypothesis of parameter preference information valuation. The general case of parameter preference information valuation posits that an individual's expectation of utility over a random variable is an <u>implicit</u> function of information. The functional relationship is implicit because the values of the explicit arguments of expected utility, the continuous distributional parameters, are themselves functions of information. Changes in expected utility are generated explicitly by changes in distributional parameters, which in turn are generated by movements along some discrete or continuous message path in message space. In terms of causation, it is changes in information that ultimately change expected utility. The role of information in parameter preference models is that of a <u>constraint</u> to the arguments of expected utility, analogous to the role of money income as a constraint to the quantities of commodities purchasable by a consumer.

The fundamental hypothesis of parameter preference information valuation determines an individual's preference ordering over any set of messages. Under the assumption that an individual's change in information is infinitessimal, the hypothesis will state whether a new message is preferred to an old message, or vice versa. While realistically, changes in information are discrete, the directional valu-

-ation of an infinitessimal change in information will have implications for the directional valuation of a discrete change in information. In the following sections, we will consider the hypothesis in terms of the implications of the valuation of infinitessimal changes in information on the valuation of discrete changes in information. We will explore these aspects of information valuation in terms of the dimensionality of message space. In the first section, we will consider the value of information when information is assumed only to change expectations. In the second section, we will consider the value of information when information is assumed to change only perceived error of expectations, and in the third section, we will consider the case where all distributional parameters are subject to information-induced changes.

Information and Expectations

An individual who processes information that affects only his expectations can be rationalized to be confident that the estimates he formulates have consistent accuracy. This could, for example, be the case of the oddsmaker or weatherman who through experience is thoroughly familiar with his "batting average" and perceives that information is only meaningful to him if, through the processing of it, it leads him to "alter" his estimates, not "improve" them. As his continous distribution changes with new information, only the mean changes, not the central moments. Therefore, the individual's distribution changes through parallel shifts only.

This hypothetical individual's message space is one-dimensional - the single dimension corresponding to the mean. In Fig. 7, where we consider for analytical simplicity mean-variance space, with constant variance σ_A^2, message space is given by ray I , which covers both positive and negative values for the mean. The processing of new information is reflected by the attainment of different points on the ray.

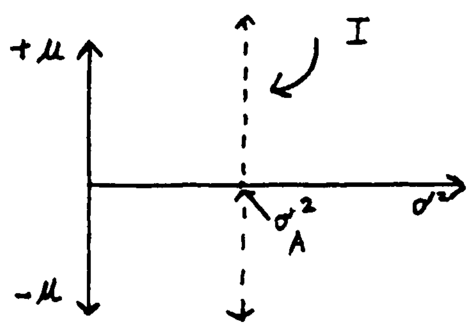

Figure 8. One-dimensional message space in two-dimensional mean-variance space.

From the generalization of expected utility by the expectation of the exact Taylor expansion of U(X) around E(X), only the mean is continuously differentiable over the single line I, which is also the message space in two-dimensional mean-variance space:

$$E[U(X)] = \sum_{n=0}^{\infty} \frac{U^n[\mu(I)]}{n!} \mu_n \tag{21}$$

From (21), U^n is the nth derivative of U(X) evaluated at $\mu(I)$, where information is an argument of the mean.

Since the mean is continuously differentiable over information, expected utility is differentiable over information. The first derivative of (21) with respect to information is defined to be the contribution of an infinitessimal change in information to expected utility. This contribution will be called the "marginal expected utility of information" and is given by the following formulation:

$$\frac{\partial E[U(X)]}{\partial I} = \frac{\partial \mu}{\partial I} \{ \sum_{n=0}^{\infty} \frac{U^{n+1}}{n!} \mu_n \} \tag{22}$$

The term to the left of the summation sign in (22) is the information-induced change in the mean. The term to the right of the summation sign is the expected marginal utility of X, the expectation of the exact Taylor series expansion of $\partial U(X)/\partial X$ around $\mu(I)$. A positive sign for this term would guarantee that expected utility is increasing, i.e. expected utility increases with an increase in the mean. From (22), if expected utility is increasing, then the marginal expected utility of information is positive if new information increased expectations. Conversely, the marginal expected utility of information is negative if new information reduced expectations.

We may now proceed to define the first version of information "value" in a parameter preference framework. The value of new information that changes the mean is the change in the mean, weighted by the expected marginal utility of wealth. If the change in information is infinitessimal, the value of information is given by the marginal expected utility of information. Based on the sign of the marginal expected utility of information, information that changes the mean is positively valued if expectations about a random variable's true value are raised, assuming that raised expectations are valued positively, reflected by a positive preference direction for the mean. The positive preference direction is guaranteed by a positive sign for the expected marginal utility of X. Information that changes the mean is valued negatively if expectations are lowered.

To determine whether the marginal expected utility of information that changes the mean is decreasing or increasing, we differentiate (22) with respect to information:

(23) $$\frac{\partial^2 E[U(X)]}{\partial I^2} = [\frac{\partial \mu}{\partial I}]^2 \{ \sum_{n=0}^{\infty} \frac{U^{n+2}}{n!} \mu_n \} \gtrless 0 \text{ as}$$

$$\sum_{n=0}^{\infty} \frac{U^{n+2}}{n!} \mu_n \gtrless 0$$

The term to the right of the summation sign in (23) is the expectation of the exact Taylor series expansion of

$\partial^2 U(X)/\partial X^2$ around $\mu(I)$. The sign of (23) clearly depends upon the sign of this term. If the direction of preference for variance is negative, i.e. $\partial^2 U(X)/\partial X^2 < 0$, then the expectation of the exact Taylor series expansion of that derivative around $\mu(I)$ will be negative and the marginal expected utility of information will be decreasing. Hence, risk-aversion implies that the marginal expected utility of information will be decreasing and approach a maximum. Preference for risk, defined by $\partial^2 U(X)/\partial X^2 > 0$, implies that the marginal expected utility of information will be increasing. In the case of preference for risk, the value of information will have no maximum if the new information raised expectations. Risk-neutrality, defined by $\partial^2 U(X)/\partial X^2$ being trivial, will imply that the value of information that increases expectations will be constant and have no maximum.

Based on these results, it follows from (23) that an individual's marginal valuation of information will attain a global maximum if his utility function has a global maximum. Both global maxima occur at the same point, where expected marginal utility is zero, assuming the appropriate second-order conditions.

The conditions determining the valuation of information are effectively statements about an individual's preference ordering for messages that differ according to their contents, i.e. movements over the message path I. Because information-induced increases in expectations are positively valued and decreases in expectations are negatively valued, it is clear that an upward movement along I in Fig. 8 is "preferred" to a downward movement. Also, if A and B are both increases in the mean and $A > B$, then a shift to A on the message path is preferred to a shift to B. If A and B are both decreases in the mean and $A < B$, then while both movements involve decreases in expected utility, a movement to A would be preferred to a movement to B. These preference statements are provided by the existence of increasing expected utility.

Information and Perceived Error of Expectations

The second case of information valuation to be examined involves the individual who values information according to how it alters his perceptions of the error of his expectation μ of a random variable's true value. This hypothetical individual regards information that changes the central moments of his distribution to be meaningful if and only if it alters the confidence he attaches to an a-priori estimate. This could be the case of the conservative, highly risk-averse investor in securities who expects a low return on his portfolio, but is only concerned with minimizing the risk he attaches to his portfolio. Information that alters only the central moments of a distribution may also be consumed by the individual whose expectation is <u>qualitative</u> and the confidence he attaches to the expectation is <u>quantitative</u>. For example, the resident of a community in the close proximity of a nuclear power station may only be concerned about the <u>risk</u> of a nuclear waste leak (which he may easily be able to quantify in terms of probabilities) but his expectation of a disaster may be measured in terms of: (1) A disaster will occur, (2) A disaster will not occur. Knowledge of the risk of a disaster may allow the individual to also determine the expectation, but information was only consumed for the purpose of determining the risk of a disaster.

The value of information that induces changes in central moments will depend on the directions of changes in those moments and the directions of preferences for changes in those moments. We assume now, in the most general of cases, that message space is n-dimensional and the central moments are continuously differentiable over a message path I, $\mu_n(I)$. The expected utility of X is restated as:

$$(24) \qquad E[U(X)] = \sum_{n=0}^{\infty} \frac{U^n}{n!} \mu_n(I)$$

We obtain the marginal expected utility of information by differentiating (24) with respect to information:

$$(25) \qquad \frac{\partial E[U(X)]}{\partial I} = \sum_{n=2}^{\infty} \frac{U^n}{n!} \frac{\partial \mu_n}{\partial I} \gtrless 0 \text{ as both } U^n \text{ and } \frac{\partial \mu_n}{\partial I} \gtrless 0$$

In (25), the information-induced change in the nth central moment is given by the partial derivative of the central moment with respect to information. From (25), the marginal expected utility of information can be analyzed for (i) each term of the series, and(ii) the sum of the terms of the series. Any term will be positive/negative if the directions of change and preference for the relevant central moment are identical/opposite. Any term of the series will be zero if the direction of preference is non-existent. The sign of the marginal expected utility of information at each term of the series depends upon whether the relevant moment has changed in a direction that is preferred or not preferred.

Equation (25) will be positive if the net change in the terms of its series is positive, i.e. if the gain in expec-

-ted utility from changing central moments in preferred directions exceeds the loss in expected utility from changing central moments in unpreferred directions. For example, suppose an individual's distribution is defined over the first three moments. Assume that the individual displays a positive preference direction for μ and a negative direction of preference for variance ($U^2 < 0$) and preference for positive skewness ($U^3 > 0$). New information is assumed to reduce both variance and skewness. The marginal expected utility of information is given by the first derivative of the truncated expectation of the previous exact Taylor series expansion:

$$\frac{\partial E[U(X)]}{\partial I} = \frac{U^2}{2}\frac{\partial \mu_2}{\partial I} + \frac{U^3(\delta)}{3}\frac{\partial \mu_3}{\partial I} \tag{26}$$

$$\delta = X + \phi(X-\mu),\ 0 \leq \phi \leq 1$$

The LaGrange remainder is formed in (26) at the last term, which contains skewness. The first term on the right side of (26) is positive and the second term is negative. A negative value for (26) would imply that the loss in expected utility associated with the reduction in skewness was greater than the increase in expected utility from the reduction in variance.

A positive marginal expected utility of information would imply that the gain in expected utility from the reduction in variance was greater than the loss in expected utility from reduction in skewness. A zero value for (26) would imply that the gain in expected utility from the reduction in variance and the loss in expected utility from the reduction in skewness are equal.

As discussed earlier, variance, skewness and any other existing central moments of an individual's subjective probability distribution are measurements of his perceived error of estimation. A reduction in variance and higher even-numbered moments implies that the individual attaches more confidence to his estimate than without the new information. If the individual positively values this change in perceptions , reflected by the assumption of negative directions of preference for even-numbered moments, information that reduces these moments will be positively valued. Conversely, information that reduces the individual's confidence in his estimate will be valued negatively.

An information-induced change in skewness involves a change in an individual's perception that his estimate is an underestimate or overestimate. In the case of positive skewness preference, an individual will value positively information that increases his perception that his expectation is an underestimate. Conversely, he will value information negatively that lowers his perception that his estimate is an underestimate. If the individual displays preferences for variance and negative skewness, he will positively value information that <u>reduces</u> his confidence in his estimate and increases his perception that his estimate is an underestimate.

For many classes of probability distributions, such as the gamma and beta distributions, changes in skewness and higher odd-numbered moments can induce changes in the mean. In other words, when the asymmetry of a distribution changes, the point of "central tendency" can be altered. For the gamma and beta distributions, this connection between skewness and the mean is reflected by the following derivative: $\partial\mu/\partial\mu_3 > 0$.[13/]

The relationship between the mean and skewness given above implies that expectations of X's true value are raised when the individual's convictions that the mean is more of an underestimate than an overestimate (represented by positive skewness) are raised. Expectations of X's true value will fall when the individual's convictions that the mean is more of an overestimate than an underestimate (represented by negative skewness) are raised. The former case involves an increase in positive skewness and the latter case, an increase in negative skewness. The mean may also fall with a reduction in positive skewness and may increase with a reduction in negative skewness.

Skewness-induced changes in the mean, as opposed to direct changes in the mean (parallel shift of the distribution), can be rationalized to be cases where estimates undergo "corrections" due to changes in the directional bias of perceived estimation error. A correction in an estimate differs from a direct shift in the mean by the fact

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13/ The proof of this relationship is not given here because of the complexity of the integration.

that a direct shift has no effect on the shape of the distribution(perceived error of the estimate). The entire distribution shifts in a parallel fashion when there is a direct change in the mean. A change in the shape of a distribution, if it involves changes in odd-numbered moments, can change the mean. The distinction between the two types of changes in the mean is clearly one of causation.

When the odd-numbered moments of a distribution induce changes in the mean, the following relationship between the change in the mean and those moments is assumed:

(27) $$\mu = \mu(\mu_n(I)) \ , \ n = 3,5,7,9,\ldots\ldots$$

The marginal expected utility of information with relationship (27) is the following:

(28) $$\frac{\partial E[U(X)]}{\partial I} = \sum_{n=3}^{\infty} \frac{\partial \mu}{\partial \mu_n} \frac{\partial \mu_n}{\partial I} \frac{U^{n+1}}{n!} \mu_n(I) + \sum_{n=2}^{\infty} \frac{U^n}{n!} \frac{\partial \mu_n}{\partial I}$$

The first series on the right side of (28) is the contribution to the marginal expected utility of information of an odd-numbered central moments-induced change in the mean, weighted by the expected marginal utility of wealth. Assuming an increasing expected utility function, this series will be positive for increases in the mean and negative for reductions in the mean. The second series on the right side of (28) is the contribution to the marginal expected utility of information of changes in central moments, weighted by the corresponding directions of preferences for those moments.

For distributions where the mean does not change with changes in odd-numbered moments, message space is in the general case n-dimensional and fixed at some value for the mean. For geometric simplicity, the two and three moment cases are presented here. In mean-variance space, message space is represented by nonnegative ray I originating from constant mean μ_z, given in Fig. 9. Fig. 10 illustrates

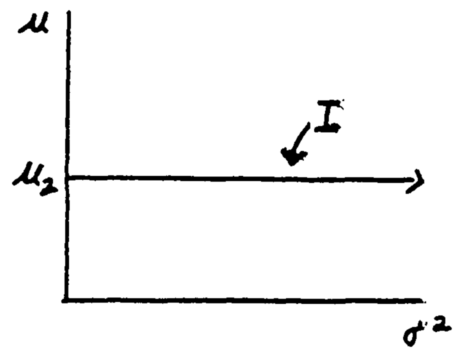

Figure 9. One-dimensional message space in two-dimensional mean-variance space.

case of discrete changes in information in three-dimensional mean-variance-skewness space with a two-dimensional message space and a fixed mean.

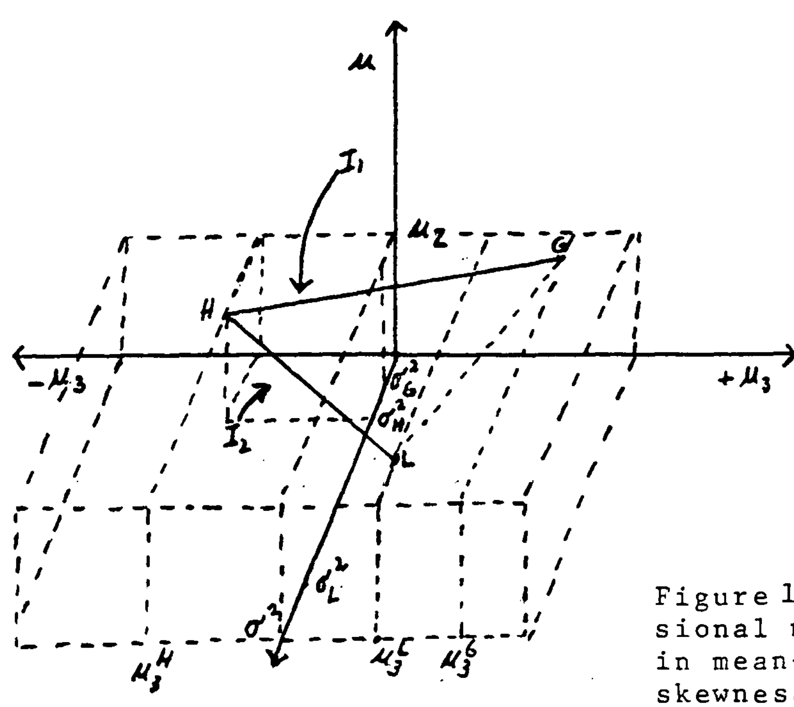

Figure 10. Two-dimensional message space in mean-variance-skewness space.

In Fig. 10, message space is a two-dimensional infinite horizontal plane bounded on the north side by the skewness axes (due to the nonnegativity of variance) and elevated above the origin (where $\mu = 0$). Allow the mean to be fixed at μ_z, hence there can be no direct and skewness-induced changes in the mean. Variance and skewness are the only distributional parameters that are continously differentiable over any message path I in message space.

Suppose an individual's initial position in mean-variance-skewness space in Fig. 10 to be at point H, which corresponds to μ_H, σ_H^2 and μ_3^H , where σ^2 is variance and μ_3 is skewness. Assume that a message G is received that upon processing, generates new values for variance and skewness - σ_G^2 and μ_3^G. The change in information equals the distance between messages H and G on message space, equal to the line $\overline{HG}$. Line $\overline{HG}$ is therefore a section of some message path I_1. From this message path, it can be seen by inspection that $\partial\sigma^2/\partial I_1 < 0$ and $\partial\mu_3/\partial I_1 > 0$.

Consider another message, L, which upon processing generates variance σ_L^2 and skewness μ_3^L. If the preceding position of the individual was at H, then the change in information would be line $\overline{HL}$ and this line would constitute a section of some other message path, I_2. It should be clear from inspection of Fig. 10 that, $\partial\sigma^2/\partial I_2 > 0$ and $\partial\mu_3/\partial I_2 > 0$. In a three period framework, where the individual processes two new messages each in sequential time periods, allow the individual's initial position (in the first time period) to be at H. If message G was processed in the second time period and message L was processed in the third time period, then angle HGL would be a message path over the three periods. H, G and L are in terms of levels of expected utility, three different static equilibria.

If H is the initial position of the individual, then what would be the preference ordering between messages G and L? Clearly, message G would be preferred to message L if the level of expected utility attainable with G exceeds the expected utility level attainable with L. This depends on the directions of preference that the individual has for variance and skewness. If the individual displays risk-aversion and positive skewness preference (and of course, increasing expected utility, implying a positive preference direction for the mean), then by inspection, message G is preferred to message L because skewness increases more with G than for L and G generates a reduction in variance while L generates an increase in variance. However, if the individual displays both preferences for variance and positive skewness, then message L would be preferred to G if the increase in expected utility from the increase in variance by L exceeded the difference in expected utilities between skewness levels μ_3^L and μ_3^G. Message G would be preferred to L if the difference in expected utility associated with μ_3^L and μ_3^G exceeded the loss in expected utility by moving from variance levels σ_H^2 to σ_L^2. These conclusions can be confirmed by superimposing the three-dimensional mean-variance-skewness indifference hypersurface onto message space and determining the expected utility levels corresponding to each message. This superimposition is not conducted because of graphical complexities.

If changes in skewness induce changes in the mean, still assuming that parallel shifts in the distribution are not allowed, then message space can be either a linear or curvilinear two-dimensional surface in mean-variance-skewness space. Fig. 11 shows the case of a linear surface.

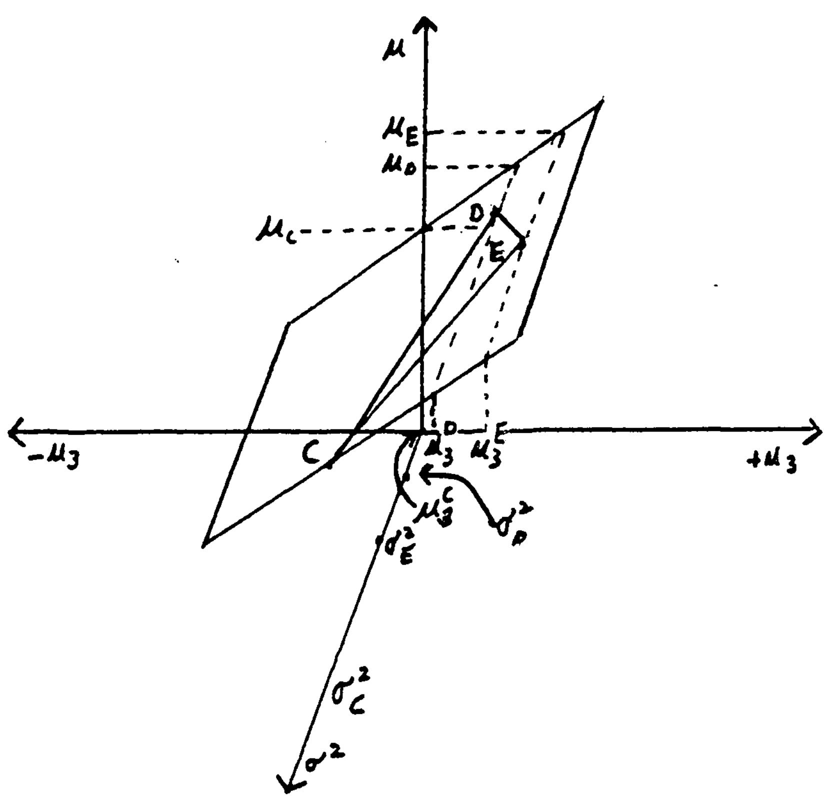

Figure 11. Two-dimensional message space in three- dimensional mean-variance- skewness space, when skewness- induced changes in the mean are allowed.

Given Fig. 11, we can analyze hypothetical message paths and preference orderings that now include changes in the mean(indirectly). Allow the individual's initial position to be at C, which corresponds to mean μ_C, variance σ_C^2 and skewness μ_3^C (which is zero). Assume the individual receives and processes message D, which generates mean μ_D, variance σ_D^2 and skewness μ_3^D. The mean has risen by the amount $(\mu_D - \mu_C)$ because skewness has increased from 0 to μ_3^D. Variance has been reduced from σ_C^2 to σ_D^2. Now assume that the individual receives and processes message E, hence he moves from point C to point E, his new perceptions represented by mean μ_E, σ_E^2 and μ_3^E.

Under what conditions will message D be preferred to message E? This of course depends on the individual's directional preferences for variance and skewness. Under the assumption of increasing expected utility and preference for positive skewness, message E will be preferred to message D if the gains in expected utility from increases in the mean and skewness, from processing message E rather than D, are sufficient to offset the loss in expected utility from accepting variance σ_E^2, as opposed to σ_D^2. If preference for risk is displayed, then message E will always be preferred to D.

Note that the linearity of message space in Fig. 11 is dependent on the relationship between the mean and skewness, i.e. $\partial\mu/\partial\mu_3$. With linearity, it is clear by inspection of Fig. 11 that $\partial\mu/\partial\mu_3 > 0$ and $\partial^2\mu/\partial\mu_3^2 = 0$, as is the case with some distributions. Curvilinearity of message space would of course imply that $\partial^2\mu/\partial\mu_3^2 \neq 0$.

Also on Fig. 11, note that $\overline{CD}$, $\overline{CE}$, $\overline{DE}$ and angle CDE are all hypothetical message paths. $\overline{CD}$, $\overline{CE}$ and $\overline{DE}$ correspond to two-period message processing frameworks, while angle CDE corresponds to a three-period framework.

Let us now briefly consider these cases of two and three moment distributions, using the truncated Taylor series expansions. First we consider the case of the individual with a distribution over the first two moments who values information that changes his perception of variance. Truncating expression (24) at the second term, forming a LaGrange remainder and differentiating with respect to information, we obtain the marginal expected utility of information that changes variance:

$$(29) \qquad \frac{\partial E[U(X)]}{\partial I} = \frac{U^2}{2}(\varepsilon)\frac{\partial \mu_2}{\partial I} \gtreqless 0 \text{ as } U^2 \text{ and } \frac{\partial \mu_2}{\partial I} \gtreqless 0$$

$$\varepsilon = X + \theta(X-\mu), \; 0 \leq \theta \leq 1$$

It is clear from (29) that an individual, whose distribution is a member of the family of two parameter distributions, will value information that changes his perception of risk positively if: (1) the individual is risk-averse and information reduces his variance, or (2) the individual has a preference for risk and information increases his perception of risk. The value of information will be negative if variance changes in a direction that is not preferred. If the individual is risk-neutral ($U^2 = 0$), then information that changes variance will be valueless.

When we add skewness to the individual's distribution, we find that positive marginal expected utility of information does not depend on both variance and skewness changing in preferred directions. For example, consider the truncation of (24) at the third moment with a LaGrange remainder, where no skewness-induced changes in the mean are assumed:

$$\text{(30)} \qquad \frac{\partial E[U(X)]}{\partial I} = \frac{U^2}{2}\frac{\partial \mu_2}{\partial I} + \frac{U^3(\psi)}{6}\frac{\partial \mu_3}{\partial I} \gtrless 0 \text{ as,}$$

$$\frac{\partial \mu_2}{\partial I} \lessgtr \frac{(-)U^3(\psi)}{3U^2}\frac{\partial \mu_3}{\partial I}$$

$$\psi = X + \phi(X-\mu), \quad 0 \leq \phi \leq 1$$

Equation (30) states that an individual whose distribution belongs to the family of three moment distributions, displaying risk-aversion and positive skewness preference, will tolerate a reduction in skewness if there is some minimum reduction in variance. The magnitude of the reduction in variance is determined by the degrees of risk-aversion, positive skewness preference, and the magnitude of the reduction in skewness. Also from (30), the individual will still value information positively if variance increases up to a limit and skewness increases. The upper limit on the increase in variance will depend on the increase in skewness and the magnitudes of U^2 and U^3. Therefore, information will still be valued positively if some central moment of the two moves in a preferred direction, while the movement of the other central moment in the unpreferred direction will be tolerated up to a threshold.

Let us briefly consider the conditions under which the marginal expected utility of information will be positive or negative when there are skewness-induced changes in the mean. Let the mean be a function of skewness in the truncation of (24) at the third moment, with the LaGrange remainder. Differentiating with respect to information, the value of information that changes variance, skewness and expected value through skewness is:

$$(31)\quad \frac{\partial E[U(X)]}{\partial I} = \sum_{n=0}^{3} \frac{U^{n+1}}{n!}\mu_n(I)\,\frac{\partial \mu}{\partial \mu_n}\frac{\partial \mu_n}{\partial I} + \frac{U^2}{2}\frac{\partial \mu_2}{\partial I} + \frac{U^3(\chi)}{6}\frac{\partial \mu_3}{\partial I},$$

$$\gtrless 0 \text{ as } \frac{\partial \mu_2}{\partial I} \lessgtr (-)\frac{2}{U^2}\sum_{n=0}^{3}\frac{\partial \mu}{\partial \mu_3}\frac{U^{n+1}}{n!}\mu_n(I) - \frac{U^3(\chi)}{3U^2}\frac{\partial \mu_3}{\partial I}$$

$$\chi = X + \phi(X-\mu),\ 0 \leq \phi \leq 1$$

From (31), assume that skewness and the mean both increase, and the individual displays aversion towards variance and positive preference directions for the mean and positive skewness. With an increase in the mean, the individual will accept a larger increase in variance than without increases in the mean. However, if both skewness and the mean are reduced, the minimum reduction in variance needed to maintain a positive value for information will be larger.

Finally, let us examine the conditions determining whether the marginal expected utility of information that changes central moments is increasing, constant or decreasing. In the following, assume that no indirect changes in the mean can occur. Differentiating (25) with respect to I, we obtain the following second-order conditions:

$$(32) \qquad \frac{\partial^2 E[\dot{U}(X)]}{\partial I^2} = \sum_{n=2}^{\infty} \frac{U^n}{n!} \frac{\partial^2 \mu_n}{\partial I^2} \gtreqless 0 \text{ as } U^n \text{ and } \frac{\partial^2 \mu_n}{\partial I^2} \gtreqless 0$$

As with (25), (32) must be first analyzed at each term of the series. At each term, if the nth central moment changes at a constant rate, then at that term, the value of information changes at a constant rate. If the sum of the terms is zero, then the value of information is neither increasing or decreasing. The expression $\partial^2\mu_n/\partial I^2 = 0$ implies constant information "potency", discussed in the section defining the value of information that changes expectations only. If at each term, the nth central moment changes at an increasing rate, then if the preference direction for that moment is positive, the value of information will be increasing. If the nth central moment changes at a decreasing rate (decreasing information potency), then at that term, if the preference direction for the moment is positive, the value of information will be decreasing. The value of information at each term will be increasing if the preference direction for that moment is negative and the moment changes at a decreasing rate. For example, if the individual is risk-averse, then the value of information that reduces variance will be increasing if variance decreases at a _decreasing_ rate, an interesting but puzzling finding. The value of information that increases skewness will, under the assumption of preference for positive skewness, be increasing at an _increasing_ rate, but will be decreasing if skewness increases at a decreasing rate.

Information, Expectations and Perceived Error of Expectations

The final case to be examined in this essay is that of information-induced changes in central moments and direct changes in the mean(in addition to changes induced by odd-numbered moments). This case involves both parallel shifts and changes in shape of the subjective probability distribution. It is also the case where information has the potential of being the strongest of the three cases in the altering of perceptions. The individual who processes information in such a way that both his expectations and perceived error of expectations can change both simultaneously and independently of each other, is capable with one message of altering his expectations without changing his convictions about expectation error and, with another message, is capable of altering his convictions about expectation error only. In essence, this individual views information in qualitative and quantitative dimensions, with respect to his expectations. By quantitative, we mean the actual change(s) in the individual's expectation(s). By qualitative, we mean the ability of the information to change his convictions about the "quality" of his expectations, i.e. perceptions of expectation error.

We consider once again the expectation of the exact Taylor series expansion of U(X) around μ. Allow the mean and central moments of the distribution to be continously differentiable over some message path I. We obtain the marginal expected utility of information by differentiating the expectation of utility with respect to I:

$$(33) \qquad \frac{\partial E[U(X)]}{\partial I} = \frac{\partial \mu}{\partial I}\left\{ \sum_{n=0}^{\infty} \frac{U^{n+1}}{n!}\mu_n(I) \right\} + \sum_{n=2}^{\infty} \frac{U^n}{n!} \frac{\partial \mu_n}{\partial I}$$

The distinction between the marginal expected utility of information given in (33) and the case of odd moments-induced changes in the mean, given in (28), is again one of causation. Recall that in the case of skewness-induced changes in the mean, an individual's directional valuation of information-induced changes in expectations is entirely dependent on directional changes in odd moments. In (33), changes in the mean can be independent of changes in odd moments. For example, a large increase in positive skewness may not guarantee a correspondingly large increase in the mean. In fact, the mean may fall independently. Thus the shape of the distribution can change with the increase in skewness, while the position of the distribution can change with the reduction in the mean. However, when changes in skewness (and in general, changes in odd moments) induce changes in the mean in the same direction as the independent change in the mean, then the indirect changes would reinforce direct changes in the mean.

From (33), we may derive a more robust statement of conditions for positive, negative or trivial information value. We do so by normalizing each nth-ordered derivative of U(X) by the expected marginal utility of future wealth.[14] Restating (33) as the following inequality, we find that the sign of the marginal expected utility of information depends on a threshold level of the change in the mean:

(34) $$\frac{\partial E[U(X)]}{\partial I} \gtreqless 0 \text{ as } \frac{\partial \mu}{\partial I} \gtreqless (-)\sum_{n=2}^{\infty} \theta_n \frac{\partial \mu_n}{\partial I},$$

$$\theta_n = \frac{\sum_{n=2}^{\infty} \frac{U^n}{n!}}{\sum_{n=0}^{\infty} \frac{U^{n+1}}{n!} \mu_n(I)}$$

The ratios θ_n are now invariant under linear transformations of U(X) and they allow for global, cardinal and interpersonal comparisons of central moment preference directions at different levels of wealth. In the context of information valuation, the ratios allow for global, cardinal and interpersonal comparisons of information valuation at different levels of wealth.

Interpreting (34), if information that changes central moments has a positive value(the term to the right of the summation sign is positive), then the individual can tolerate some reduction in the mean and still positively value information that changes all his distributional parameters. However, if information that changes the central moments has a negative value, then the individual must have some minimum increase in the mean to maintain positive information value. Of course, these conclusions assume that the expected marginal utility of wealth is

14/ See Footnote 7.

positive.

Let us first consider the case of the individual whose distribution belongs to the family of two-parameter distributions. The case of two moments is given by a truncated version of (33):

$$(35) \qquad \frac{\partial E[U(X)]}{\partial I} = \frac{\partial \mu}{\partial I}\left\{ \sum_{n=0}^{2} \frac{U^{n+1}}{n!}\mu_n(I)\right\} + \frac{1}{2}\left[U^2(\omega)\frac{\partial \mu_2}{\partial I}\right] \gtreqless 0$$

$$\text{as } \frac{\partial \mu}{\partial I} \gtreqless -\theta_2\frac{\partial \mu_2}{\partial I},$$

$$\theta_2 = \frac{1/2U^2(\omega)}{\sum_{n=0}^{2} \frac{U^{n+1}}{n!}\mu_n(I)} \qquad \text{and } \omega = X + \phi(X-\mu), \quad 0 \leq \phi \leq 1$$

The conditions for positive, negative or trivial information value are expressed as a version of (34). From (35), for positive information value, there must be a minimum increase in expectations. If we assume risk-aversion and increasing expected marginal utility of wealth, θ_2 will be negative and if variance is reduced, then it can be easily seen that the individual will be willing to accept a drop in the mean and still value the information positively. If variance is increased, then there must be a minimum <u>increase</u> in the mean to maintain positive information value. The conditions in (35) can be reexpressed in terms of a minumum reduction or maximum acceptable increase in variance, given a reduction or increase in expectations.

The conditions in (35) are depicted in Fig. 12., where two messages A and B are compared in mean-variance space. In Fig. 12 , if risk-aversion and increasing expected utility are assumed, then message B will always be preferred to A,

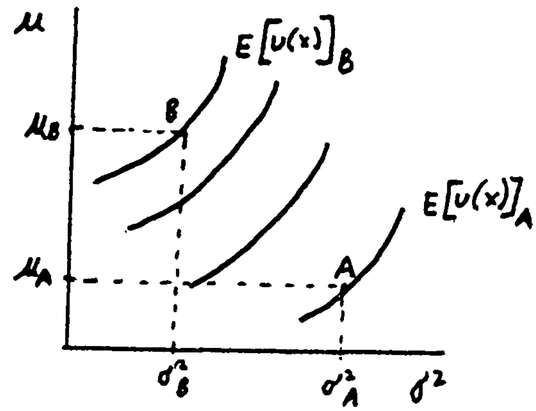

Figure 12. Two-dimensional message space in mean-variance space.

because of the reduction in variance and the increase in expectations. For graphical convenience, we superimpose the individual's indifference map onto mean-variance space. Message B allows the individual to attain a higher level of expected utility, $E[U(X)]_A$ than message A, where only an expected utility level of $E[U(X)]_B$ can be attained. Note that the preference direction for the individual is to the northwest on mean-variance space-by moving northwest, he attains higher levels of expected utility because of his willingness to sacrifice higher values of variance for higher values of the mean.

For an individual with a distribution over the first three moments, we state the conditions for the sign of information value by considering another truncated version of (33):

(36) $$\frac{\partial E[U(X)]}{\partial I} = \frac{\partial \mu}{\partial I}\{\sum_{n=0}^{3} \frac{U^{n+1}}{n!}\mu_n(I)\} + \frac{1}{2}[U^2 \frac{\partial \mu_2}{\partial I}] + \frac{1}{6}[U^3(\tau) \frac{\partial \mu_3}{\partial I}] \qquad \tau = X + \theta(X-\mu), \quad 0 \leq \theta \leq 1$$

Restating (36) in terms of the conditions expressed in (34):

$$(37)\quad \frac{\partial E[U(X)]}{\partial I} \gtreqless 0 \text{ as } \frac{\partial \mu}{\partial I} \gtreqless -[\theta_2 \frac{\partial \mu_2}{\partial I} + \theta_3 \frac{\partial \mu_3}{\partial I}] ,$$

$$\theta_3 = \frac{1/6U^3(\tau)}{\sum_{n=0}^{3} \frac{U^{n+1}}{n!} \mu_n(I)}$$

From (37), if variance and skewness have changed in unpreferred directions, there must be a minimum increase in the mean to maintain positive information value. If variance and skewness change in preferred directions, then the individual will be willing to accept a reduction in the mean to a threshold, and still maintain positive information value. Again as with (35), these conditions for positive information value can be reexpressed in terms of the two central moments.

Fig. 13 illustrates the case where no skewness-induced changes in the mean occur, but the mean changes independently of skewness. The shaded area on the figure is a subsec-

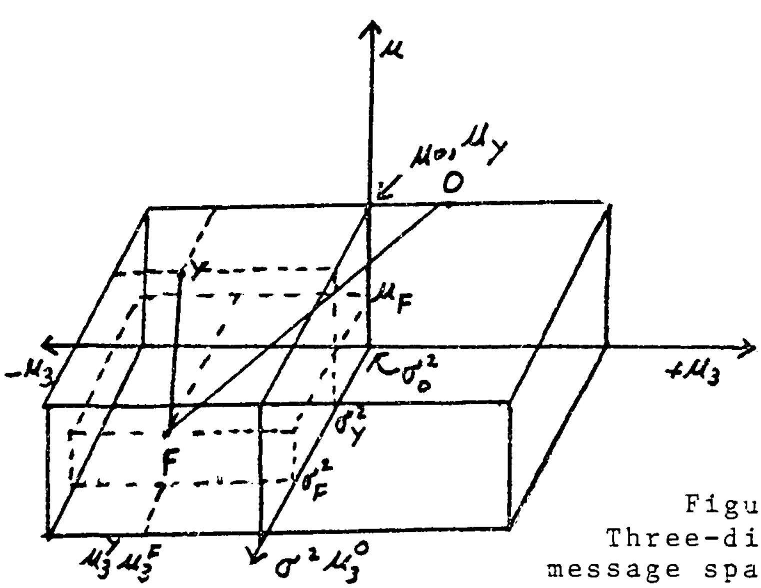

Figure 13. Three-dimensional message space in mean-variance-skewness space.

-tion of the three-dimensional message space. Since parallel shifts in the distribution are assumed, message paths within this message space can be vertical, as well as horizontal or a combination of the two. $\overline{FO}$ and $\overline{FY}$ are hypothetical message paths. Let F be the initial position of the individual with mean μ_F, variance σ_F^2 and skewness μ_3^F. Let Y be a message, which upon processing generates changes in all three distributional parameters, with new values of μ_Y, σ_Y^2 and μ_3^Y. With message Y, the mean increases, variance decreases, and the individual's distribution is negatively skewed. Consider message O, which upon processing generates the same value for the mean as message Y, μ_O, and variance σ_O^2 and skewness μ_3^O. With message O, variance decreases to zero and skewness increases.

If the expected utility function underlying Fig. 13 is increasing, then the values of messages Y and O at the <u>mean</u> only, are identical. If the individual is risk-averse and displays preference for positive skewness, then by inspection, message O is preferred to message Y. If preference for risk is displayed, then message Y would be preferred to message O if the loss in expected utility associated with moving from variance σ_Y^2 to zero variance (message O) exceeded the gain in expected utility from moving to skewness μ_3^O.

Since many distributions allow for skewness-induced changes in the mean, we illustrate that case in Fig. 14. Consider M and V to be two messages. Message M generates

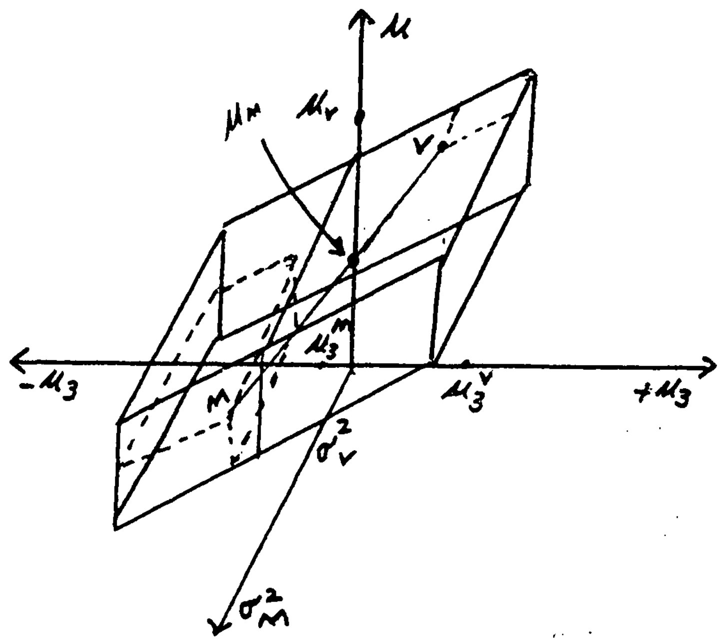

Figure 14. Three-dimensional message space in mean-variance-skewness space when direct and indirect changes in the mean are allowed.

upon processing, mean μ_M, variance σ_M^2 and skewness μ_3^M. Message V generates upon processing mean μ_V, variance σ_V^2 and skewness μ_3^V. If the individual is risk-averse and displays preferences for positive skewness and the mean, then clearly message V will be preferred to message M. However, if the individual displays preference for risk, then the comparatively large increase in variance with message M may be sufficient to generate a higher level of expected utility for M than for V.

Conclusion

The general case of parameter preference information valuation has posited that in parameter preference models of expected utility maximization, expected utility is an "implicit" function of information. The relationship is implicit because information determines the values of an individual's perceptions of the true value of a continuous random variable. These perceptions, represented by the mean and central moments of the subjective probability distribution, are the arguments of expected utility. Therefore, information acts as a constraint to the arguments of expected utility, a relationship that is analogous to that of the role of money income, which is a constraint to the maximization of utility over a commodity. Assuming the existence of deterministic and expected utility functions, changes in both types of constraints generate changes in satisfaction (or expected satisfaction), which vary according to the changes in the quantities of the arguments in the functions.

Assuming a continously differentiable utility function, the contribution to utility of a change in money income has traditionally been referred to as the marginal utility of money income. Analogously, the contribution to expected utility of an infinitessimal change in information is the "marginal expected utility of information". The marginal expected utility of information has been posited to be the appropriate basis for determining the value of information for an individual with a continuous subjective probability distribution. The value of information is the utility-weighted change in the individual's continuous distributional parameters.

The values of an individual's distributional parameters are constrained by his location during any period on "message space", defined here as the set of all points within mean-central moment space corresponding to all possible messages receivable and processable. The dimensions of message space depend upon what parameters in the individual's distribution are subject to change by information. In the case of a continuous message path, an individual's movement from one message to another is defined as the set of partial derivatives of each distributional parameter with respect to the movement along a message path I in message space - $\partial\mu/\partial I$ and/or $\partial\mu_n/\partial I$, where μ is the mean and μ_n is the nth central moment of the distribution. As an individual moves from one point on I to another, information-induced changes in distributional parameters interact with his directions of preference for changes in those parameters, to determine the value of a new message.

At one specific distributional parameter, information will have positive value if the change in that parameter is in a preferred direction. For all affected parameters, the value of information depends upon the <u>net</u> change in them, weighted by the directions of preference for each parameter.

If the net change is positive, a new message will be valued positively, and if the change is negative, the message will be valued negatively. If the net change is zero, the new message will be valueless.

Parameter preference information valuation does not depend upon a matrix of competing alternative choices of action. Rather, given that an individual's perceptions of uncertainty can be represented by a continuous subjective probability distribution, the value of information refers only to the change in expected utility from the processing of a message, reflected by changes in perceptions of uncertainty, weighted by the preference directions for those perceptions. Therefore, the only requirement for information to have value is that an individual has tastes for uncertainty and that information changes the true value of a random variable.

Since the general case of parameter preference information valuation provides results that are too general for empirical verification, the most promising theoretical extensions lie in applications to restricted forms of expected utility. These applications can involve the derivation of utility functions for information. These utility functions could be used in the analysis of certain types of constrained optimization problems, such as in the derivation of demand functions for information. Other interesting extensions include the implications of parameter preference information valuation on risky asset valuation and portfolio efficiency.

II. THE RESTATEMENT OF EXPECTED UTILITY FUNCTIONS AS UTILITY FUNCTIONS FOR INFORMATION

Introduction

In the first essay, a generalized cardinal expected utility function was used to examine the problem of information valuation for an individual with a continuous subjective probability distribution. While the relationship between expected utility over a random variable and information was posited to be implicit, it was argued that an individual could assign a preference ordering to messages that vary in content and potency. A message was found to correspond to a unique level of expected utility, and different messages could be ranked in the order of their corresponding levels of expected utility.

The objective of this essay is apply the general results of parameter preference information valuation to five restricted forms of expected utility by restating these functions as utility functions for information. While the general case of parameter preference information valuation showed that an implicit preference ordering for information could be derived from an explicit preference ordering for distributional parameters, a utility function for information explicitly states an individual's preference ordering for information. Based on an individual's preference directions for distributional parameters, he will have a set of "desired" message paths I, that set being the subset of message space within which the individual prefers to be on. This subset is referred to in the essay as "efficient" message space. A utility function for information would be a statement of the desired set of message paths. Following derivation of each function are comparative statics analyses of the implications of these utility functions for information valuation for individuals that differ according to their tastes, beliefs and information processing characteristics.

The restricted forms of expected utility analyzed include two polynomial forms - quadratic and cubic utility - and three nonpolynomial forms - exponential, constant elasticity, and logarithmic utility. The two classes of functions differ in their underlying assumptions concerning local and global risk-aversion. Restated as utility functions for information, the five forms are found to possess certain desirable characteristics that provide for the derivation of testable hypotheses such as demand functions for information.

From a General Preference Ordering For Messages to Special Cases

The expectation of a utility function $E[U(X)]$ is a statement of an individual's preference ordering for the parameters in his underlying subjective probability distribution. For example, if a random variable W = future wealth, $\partial E[U(X)]/\partial E(W) > 0$ is a statement that higher levels of expected wealth are preferred to lower levels. The expression $\partial E[U(W)]/\partial \sigma^2 < 0$, where σ^2 is variance, is a statement that lower levels of variance are preferred to higher levels. With information as an implicit function of the expected utility of future wealth, the following relationships are statements that information that changes perceptions of a random variable's true value in preferred directions is preferred to information that changes perceptions in unpreferred directions:

$$(38) \qquad \frac{\partial E[U(W)]}{\partial I} = \frac{\partial E(W)}{\partial I}\left\{\sum_{n=0}^{\infty} \frac{U^{n+1}}{n!}\mu_n\right\} > 0$$

$$(39) \qquad \frac{\partial E[U(W)]}{\partial I} = \sum_{n=2}^{\infty} \frac{U^n}{n!} \frac{\partial \mu_n}{\partial I} > 0$$

$$(40) \qquad \frac{\partial E[U(W)]}{\partial I} = \frac{\partial E(W)}{\partial I}\left\{\sum_{n=0}^{\infty} \frac{U^{n+1}}{n!}\mu_n(I)\right\} + \sum_{n=2}^{\infty} \frac{U^n}{n!} \frac{\partial \mu_n}{\partial I} > 0$$

Expressions (38) - (40) will be recognized as the marginal expected utilities of information for the three different cases explored in the first essay: (1) The individual who processes information that affects only his expectations; (2) The individual who processes information that affects only his perceived error of expectations; (3) The in-

-dividual who processes information that affects both his expectations and perceived error of expectations. Each of the three expressions states the condition for positive information value, and each expression is a statement of an individual's preference ordering over different messages.

By the definition of a utility function U(X) as a statement of an individual's preference ordering over a commodity X, a utility function for information must contain in some form one of the information preference statements, given by expressions (38) - (40). Viewed in terms of message spaces, the definition of a utility function for information is tantamount to stating that an individual prefers to be on that subset of message space containing all those possible messages which, upon receipt and processing, will generate increases in expected utility. This subset will be referred to as "efficient" message space. The individual prefers at any point in time to receive a message that yields a change in perceptions in preferred directions.

Restricted forms of expected utility are cases that assume special preference orderings for distributional parameters. Each case assigns: (1) Special directional preferences for underlying distributional parameters; (2) A general distribution describable over a finite number of parameters, or a special distribution (normal, lognormal, beta, etc.). Because the value of information depends upon the definitions and assumptions concerning distributional parameter preference directions and the underlying distribution, restricted forms of expected utility make ideal candidates for utility functions for information. As utility functions for information, they must be statements of an individual's preference ordering for messages over efficient message space. This requires that relationships between distributional parameters and information (given in the general case as $\mu(I)$ and $\mu_n(I)$, where μ is the mean and μ_n the nth central moment of the distribution) be explicitly defined in terms of signs and functional forms. Directional preferences for distribution parameters (defined as the derivatives of the utility function in the general case) are the <u>coefficients</u> of the arguments of expected utility. The specification of $\mu(I)$ and $\mu_n(I)$ involves replacing each distributional parameter with a special functional relationship that is consistent with an individual's preference ordering for messages. For example, if $\partial\mu/\partial I > 0$ is preferred over $\partial\mu/\partial I < 0$, then the utility function for information must honor that condition, in order for the function to be a valid statement of an individual's preference ordering over information.

Some restricted forms of expected utility are found to be advantageous over other forms as utility functions for information, according to their abilities to satisfy the following two criteria: (1) Analytical and empirical simplicity; (2) The ability of the expected utility function to accomodate a relatively wide range of behavioral assumptions, such as moment preference and information processing characteristics (such as $\partial^2\mu/\partial I^2 > 0$, i.e. increasing information potency). With (1), some utility-of-information counterparts of expected utility functions are much simpler to manipulate in certain constrained optimization problems. This factor has very important empirical ramifications. With (2), a function that can accomodate risk-aversion, preference for risk and risk-neutrality and also be analytically simple, is clearly to be preferred over functions that satisfy these criteria to lesser degrees.

In the following sections, we will consider the five restricted forms of expected utility by incorporating into each function special assumptions about directional preferences for the parameters of the underlying distribution, and the <u>desired</u> relationships between those parameters and information. We will then consider the resulting utility functions for information under assumptions concerning what parameters are subject to change by new information. For example, we will derive from one restricted form of expected utility, a utility function for information based on the assumption that an individual processes information in such a way that only the mean of his distribution is affected. In addition, we will consider functions where an individual is assumed to process information in such a way that all his parameters are subject to change.

Polynomial Forms of Utility

1. Quadratic Utility

The quadratic form of utility assumes an underlying subjective probability distribution that belongs to the family of two-parameter distributions. Because the underlying distribution is assumed to be describable by the individual over the mean and variance, a second-order polynomial is used.15/ No restriction is made on the shape of the probability density function other than symmetry about the mean. If W = future wealth, then the quadratic utility of future wealth is defined as;

(41) $$U(W) = W + aW^2 \quad , a \gtreqless 0$$

To guarantee that U(W) is increasing16/, the function must necessarily be bounded at some minimum level of future wealth:

(42) $$\frac{\partial U(W)}{\partial W} = 1 + 2aW, > 0 \text{ as } W > -\frac{1}{2}a$$

The sign of a, which is a statement of an individual's attitude towards risk, determines whether the utility function is concave, convex or linear:

(43) $$\frac{\partial^2 U(W)}{\partial W^2} = 2a \gtreqless 0, \text{ as } a \gtreqless 0$$

From (41) and (42), the risk-averse individual has a concave utility function with a positive wealth intercept,

15/ This is a special case of Borch's[3] general assertion that if an individual has a distribution describable over a mean and n central moments, a consistent preference ordering of a set of uncertain events can be represented by an nth-order polynomial

16/ Many scholars, including Arrow [2], Tsiang [24] and Tobin[23] have argued that it is meaningless to extend U(W) shown in Fig. 15. The individual who prefers risk has a convex utility function with a negative wealth intercept (Fig. 16) and the risk-neutral individual has a linear function with a zero wealth intercept (Fig. 17).

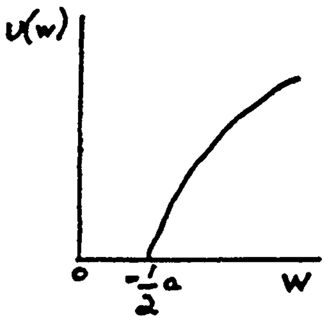

Fig. 15. Risk-aversion under quadratic utility.

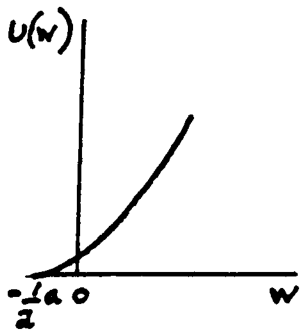

Fig. 16. Risk preference under quadratic utility.

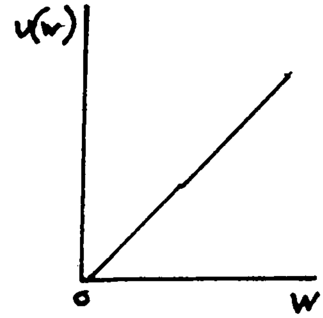

Fig. 17. Risk-neutrality under quadratic utility.

Carrying the analysis one step further, we can examine the individual's attitudes toward the mean and variance by the expected value of U(W):

$$(44) \qquad E[U(W)] = \mu + a(\sigma^2 + \mu^2) \;, \quad \mu = E(W), \quad \sigma^2 = E[W-E(W)]^2$$

The conditions for the marginal expected utility of expected wealth and variance are the same as the conditions for the first and second derivatives of the deterministic function U(W). For example, as long as the utility function is guaranteed to be strictly increasing, the marginal expected utility of the mean will always be positive:

$$(45) \qquad \frac{\partial E[U(W)]}{\partial \mu} = 1 + 2a\mu > 0 \text{ as } \mu > -\frac{1}{2}a$$

The conditions for concavity, convexity or linearity of U(W) are the same as those for the sign of the marginal

16/ (cont.) to be a statement of an individual's preferences over zero and negative wealth.

expected utility of variance:

(46) $$\frac{\partial E[U(W)]}{\partial \sigma^2} = a \gtrless 0 \text{ as } a \gtrless 0$$

It follows from (46) that the coefficient a is the appropriate measure of an individual's preference direction for variance.

Let us now assume that an individual with a quadratic utility function processes information in such a way that only his expectation μ is altered. We introduce the mean as a general function of information:

(47) $$E[U(W)] = \mu(I) + a\{\sigma^2 + [\mu(I)]^2\}$$

In (47), the mean is the only continously differentiable argument of expected utility, and σ^2 and a are constants. Differentiating (47) with respect to information, we find, as expected, that the marginal expected utility of information is positive only when the mean increases with information:

(48) $$\frac{\partial E[U(W)]}{\partial I} = \frac{\partial \mu}{\partial I}(1 + 2a\mu) \gtrless 0 \text{ as } \frac{\partial \mu}{\partial I} \gtrless 0$$

The marginal expected utility of information will be increasing, decreasing, or constant, depending upon whether the information is increasing, decreasing or constant in potency:

(49) $$\frac{\partial^2 E[U(W)]}{\partial I^2} = \frac{\partial^2 \mu}{\partial I^2}(1 + 2a\mu) \gtrless 0 \text{ as } \frac{\partial^2 \mu}{\partial I^2} \gtrless 0$$

Consider mean- variance space in Fig. 18. Because information is assumed to affect the mean only, message space is the ray I, with nontrivial constant variance σ_A^2. Assuming a hypothetical prior mean of μ_A, "efficient"

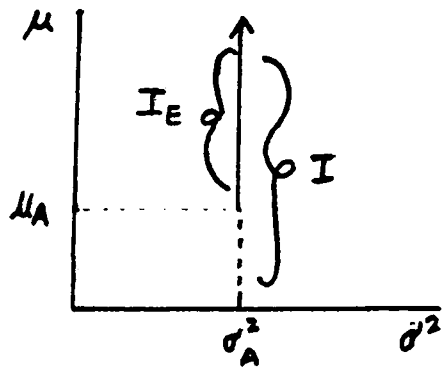

Figure 18. Efficient message space in two-dimensional mean-variance space.

message space is I_E, that portion of I that corresponds to values of the mean that are greater than or equal to μ_A.

If an individual's initial position on Fig. 18 is (μ_A, σ_A^2) and he displays a positive preference direction for the mean (increasing expected utility of future wealth), then regardless of his attitude towards variance, the attainment of any points along I_E will generate increases in expected utility. Equation (48) is then a statement that efficient message space is defined by the condition that $\partial\mu/\partial I > 0$ is desired. To show that I_E is efficient message space, regardless of the individual's attitude towards variance, let us investigate the possible slopes and curvatures of the indifference curves implied by each preference direction for variance. The properties of the indifference curves are given by the marginal rates of substitution between the mean and variance. Totally differentiating the expectation of utility in (44):

(50) $dE[U(W)] = (1 + 2a\mu)d\mu + (a)d\sigma^2$

Since by (45), $(1 + 2a\mu)$ is guaranteed to be positive, it follows that the sign of the marginal rate of substitution between the mean and variance depends on the sign of a:

(51) $\frac{d\mu}{d\sigma^2} = -\frac{a}{1 + 2a\mu} \gtrless 0$ as $a \lesseqgtr 0$

From (51), risk-aversion with positive marginal expected utility of the mean implies a positive marginal rate of substitution of the mean for variance, i.e. the individual is willing to sacrifice some expected wealth for a reduction in variance, and is willing to sacrifice a reduction in variance for a gain in expected wealth. Risk-preference implies that (51) is negative, i.e. the individual is willing to sacrifice some expected wealth for an <u>increase</u> in variance, while risk-neutrality implies that (51) is zero. While (51) gives the slopes of the level curves in the mean-variance indifference hypersurface, their curvatures are given by the derivative of (51) with respect to variance:

(52) $\frac{d[-a/(1 + 2a\mu)]}{d\sigma^2} = 0$

From (52), the second derivative vanishes, implying that the indifference curves are straight lines.[17/]

17/ This is in contrast to a mean-<u>standard deviation</u> indifference hypersurface, where the derivative of the marginal rate of substitution of expected wealth for standard deviation does not vanish, implying that the level curves are either convex(risk-aversion) or concave(preference for risk). It is well documented in the literature that, in

Figures (19) - (21) superimpose the indifference maps implied by equations (51) and (52) for hypothetical risk-averse, risk- preferential and risk- utral individuals over their corresponding efficient message spaces.

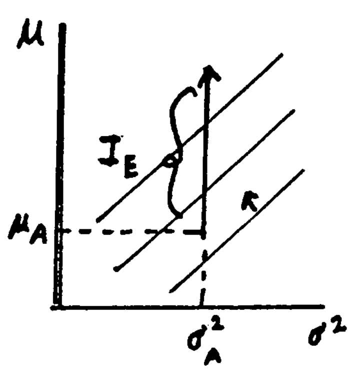

Figure Risk aversion and efficient message space under quadratic utility.

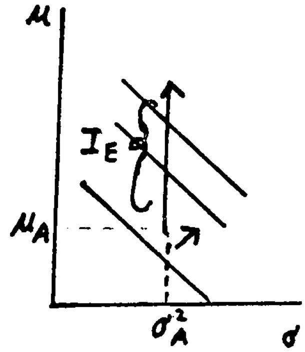

Figure Risk preference and efficient message space under quadratic utility.

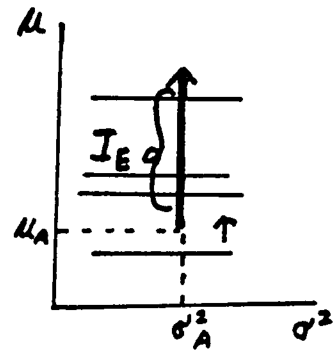

Figure Risk neutrality and efficient message space under quadratic utility.

In Fig. 19, the individual is assumed to be risk-averse, as shown by his indifference curves sloping upwards, reflecting his willingness to trade increases in expected wealth for reductions in variance. The arrow in the figure points in the direction of progressively

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17/(cont.) this context, concavity and convexity of indifference curves and their implications rest on the assumption that, while an individual's subjective probability distribution is describable over the mean and variance, his indifference hypersurface can validly be described over the mean and standard deviation. This involves the restatement of a few simple assumptions. The current assumption, that the indifference hypersurface is describable over the mean and variance,does not hinder the present discussion. See Borch [3], Tobin [23] and Feldstein [5] for discussions of level curve convexity, given mean-standard deviation hypersurfaces.

higher levels of expected utility. Assuming prior mean μ_A and variance σ_A^2, efficient message space is given by ray I_E, that ray intersecting progressively higher levels of expected utility. Fig. 20 superimposes the indifference map of an individual preferring risk over his corresponding efficient message space I_E - this ray also intersects progressively higher levels of expected utility. Fig. 21 superimposes a risk-neutral individual's indifference map over his corresponding efficient message space I_E. That message space also intersects progressively higher levels of expected utility.

From (51), assuming a positive preference direction for the mean, a utility function specifying $\partial\mu/\partial I > 0$ as the desired directional change in the mean would be a valid representation of an individual's preference ordering over messages that change the mean. Such a function would be an appropriate utility function for information. Specification of a restricted form of $\mu(I)$ with the proviso that the first derivative is positive, yields results such as the maxima, minima or nonsatiability of the expected utility of information.

Specification of $\mu(I)$ allows for the inclusion of special assumptions concerning the information processing characteristics of the individual, such as whether on efficient message space, he experiences increasing, decreasing or constant information potency.

For analytical simplicity, let us consider the case of constant information potency. We require, therefore, that not only must $\partial\mu/\partial I > 0$, but also that $\partial^2\mu/\partial I^2 = 0$. Any functional form satisfying these two assumptions can be substituted into (47). One function that is simple to use and easily satisfies these assumptions is a linear function with positive slope and intercept:

(53) $$\mu = h + dI \quad , \; d > 0$$

From (53), I refers to information that, upon an increase in the quantity processed of it, causes an individual to raise his expectations of future wealth. I is assumed to be a continuum of homogenous units and an increase in the processing of one unit of I raises expectations by a linearly proportional amount:

(56) $$\frac{\partial\mu}{\partial I} = d > 0$$

The coefficient d is a measure of the "strength" of the information, i.e. the larger the value of d, the more responsive the mean is to new information. The positive intercept h reflects the assumption that the individual's formulation of expected wealth <u>prior</u> to receiving new information is positive. This intercept could for equally valid reasons, be negative or trivial - analytical convenience is allowed with a positive intercept.

Substituting (56) into (47), we obtain the utility function for information U(I) that changes expectations of future wealth:

(57) $$U(I) = (h + dI) + a[\sigma^2 + (h + dI)^2]$$

In (57), I is the only argument of $U(I) - \sigma^2$, h, d, a are constants. Since increases in the mean are valued positively, so are increases in messages that increase the mean. We investigate these properties of valuation of information by differentiating (57) with respect to information, yielding the marginal utility of information:

$$\text{(58)} \qquad \frac{\partial U(I)}{\partial I} = d[1 + 2a(h + dI)] \gtrless 0, \quad \text{as } I \lessgtr d(1/2a - h) \quad \text{for } a < 0$$

$$\frac{\partial U(I)}{\partial I} \geq 0 \text{ as } a \geq 0$$

The marginal utility of information will increase at increasing, decreasing or constant rates, depending upon an individual's marginal utility for variance, i.e. upon the sign of a:

$$\text{(59)} \qquad \frac{\partial^2 U(I)}{\partial I^2} = 2ad \gtrless 0 \text{ as } a \gtrless 0$$

From the results in (58) and (59), the utility of information will have a maximum for the risk-averter, while it will increase at an increasing rate for the individual preferring risk, and will increase at a constant rate for the individual who is neutral to risk. These three individuals are depicted in Figs. (22) - (24).

For the risk-averter, when I = 0, the resulting intercept on the utility axis in Fig. 22 can be positive, negative or trivial, depending upon the magnitudes

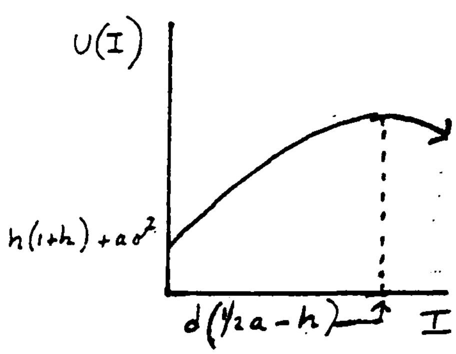

Figure 22. The risk-averter.

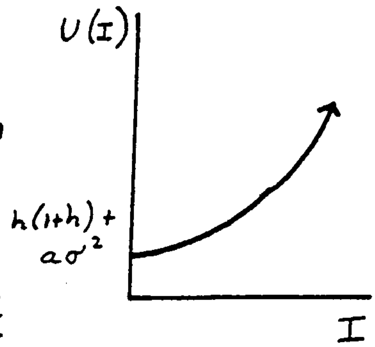

Figure 23. The risk-preferrer.

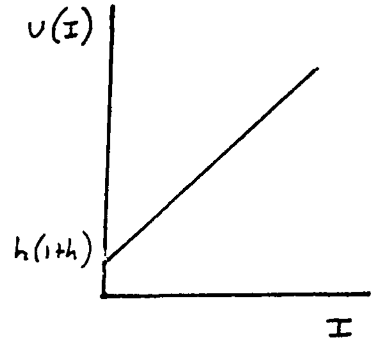

Figure 24. The risk-neutral individual.

Figures (22) - (24). The utility of information that raises expectations of future wealth for individuals with quadratic utility functions, classified according to their attitudes toward risk.

of the constants σ^2, a, h and d. The conditions for the sign of the intercept can be shown to depend on the condition that prior variance does not exceed a certain value, i.e. $\sigma^2 < h(1+h)/a$. We arbitrarily assume that the intercept is positive, as shown in Fig. 22. Note that the intercept is positive for individuals who prefer risk and who are indifferent towards risk. Therefore, for the risk-averter, if variance is sufficiently small, zero information will still allow for a positive information value. If variance exceeds the threshold, then the individual must have a <u>minimum</u> amount of information to obtain nonnegative utility for information. Individuals who prefer or are indifferent towards risk will value information positively at zero knowledge.

We now examine the case where information is assumed to alter perceptions of risk only. Allowing variance to be a generalized function of information, the expected utility function is restated as the following:

(60) $$E[U(W)] = \mu + a[\sigma^2(I) + \mu^2]$$

The marginal expected utility of information is given by:

(61) $$\frac{\partial E[U(W)]}{\partial I} = a\frac{\partial \sigma^2}{\partial I} \gtreqless 0 \text{ as } a \text{ and } \frac{\partial \sigma^2}{\partial I} \gtreqless 0$$

From (61), for the value of information that changes variance to be positive, the directional change in variance must be identical in sign to the direction of preference for variance, a, the marginal expected utility of variance. If the individual is risk-averse, variance must decrease for information to have positive value, while if the individual displays preference for risk, variance must increase for information to have positive value. These conditions imply that a utility function for information that changes variance only must specify that the desired relationship $\partial\sigma^2/\partial I$ and a are of identical signs.

It is clear from (61) that $\sigma^2(I)$ will differ for a risk-averter and a risk-preferrer. Let us treat the case of the risk-averter first. We assume again constant information potency, i.e. $\partial^2\sigma^2/\partial I^2 = 0$, for analytical convenience. For the risk-averter, a will be negative and $\partial\sigma^2/\partial I < 0$ must exist. A linear function stating that increases in information reduce variance would be a valid specification of $\sigma^2(I)$:

(62) $$\sigma^2 = b - kI \quad , \ k,b > 0$$

The coefficient k represents the strength of response of variance to new information, i.e. the higher the value of k, the greater the impact of new information on variance. Figure 25 shows the indifference map of a risk-averter superimposed over a hypothetical efficient message space. On Fig. 25, if μ_A is the prior constant mean and σ_A^2 prior variance, then efficient message space is given by ray I_E.

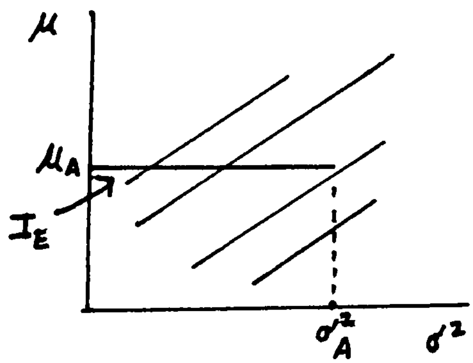

Figure 25. Efficient message space for information that reduces variance, in mean-variance space, for a risk-averter with quadratic utility.

Substituting (62) into (60), we obtain the utility function for information that reduces variance:

(63) $$U(I) = \mu + a[(b-kI) + \mu^2]$$

Observe with (63) that I is the only argument of the function - μ,b,k and a are constants. The marginal utility of information is given by:

(64) $$\frac{\partial U(I)}{\partial I} = -ak > 0 \text{ as } a < 0$$

Since a is negative, (64) will be increasing at a constant rate:

(65) $$\frac{\partial^2 U(I)}{\partial I^2} = 0$$

Figure 26 shows the utility of information that reduces variance for the risk-averter. The intercept on the utility axis will always be positive, implying that an individual will value information positively at zero knowledge.

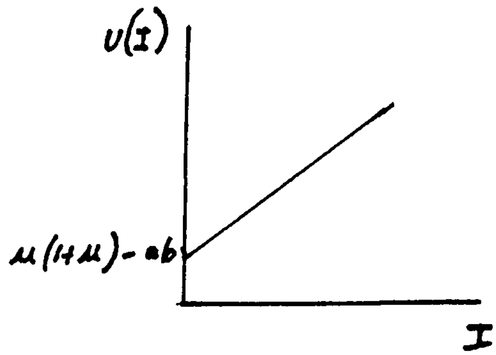

Figure 26. The utility of information that reduces variance for a risk-averter with quadratic utility.

For an individual that prefers risk, a functional form for $\sigma^2(I)$ with the proviso that $\partial\sigma^2/\partial I^2 > 0$ would be an appropriate form to use in the derivation of a utility function for information. A linear form again proves to be quite convenient:

(66) $$\sigma^2 = c + gI \qquad g,c > 0$$

Substituting (66) into (60), we obtain the utility function for information that increases variance:

(67) $$U(I) = \mu + a[(c+gI) + \mu^2]$$

The utility of information is increasing at an increasing rate, implying that no maximum for the utility of inforexists. Figure 27 graphs these results. Once again, there is a positive intercept on the utility axis, implying that the risk-preferrer will value information positively at zero knowledge.

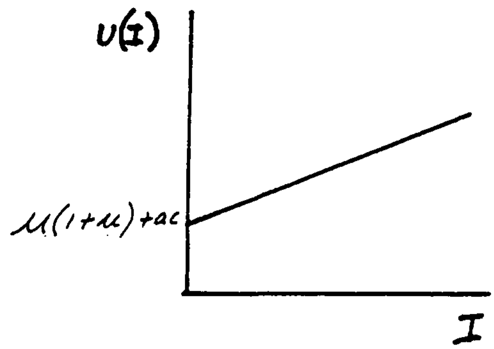

Figure 27. The utility of information that increases variance for a risk-preferrer with quadratic utility.

The final derivation involving quadratic utility is that of information that changes both the mean and variance. We assume initially that the individual displays positive marginal expected utility for the mean and risk-aversion. We incorporate the two previously used relationships between the mean, variance and information. Substituting these two relationships into (60), we obtain the utility function for information that increases the mean and reduces variance:

(68) $$U(I) = (h+dI) + a[(b-kI) + (h+dI)^2]$$

The marginal utility of information is given by:

(69) $$\frac{\partial U(I)}{\partial I} = d[1 + 2a(h+dI)] + ak \gtrless 0 \text{ as}$$

$$I \lessgtr \frac{1}{d}(1/2k - 1/2a - h) \text{ for } a < 0$$

From (69), the utility of information that increases expectations of future wealth and reduces variance is the sum of the marginal utility of information that increases the mean (holding variance constant) and the marginal utility of information that reduces variance (holding the mean constant. This sum attains a maximum at a finite level of information, as given by the derivative of (69) and depicted in Fig. 28:

$$\frac{\partial^2 U(I)}{\partial I^2} = 2ad^2 < 0 \text{ for } a < 0 \tag{70}$$

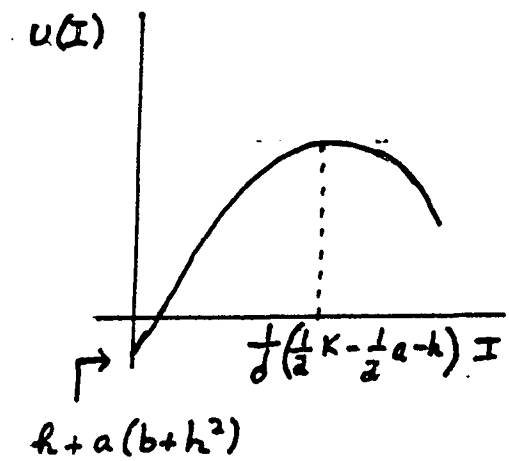

Figure 28. The utility of information that increases expectations and reduces variance for a risk-averter with quadratic utility .

On Fig. 28, the intercept on the utility axis is always negative, implying that at zero knowledge, the marginal utility of information will always be negative.

Fig. 29 shows efficient message space for the risk-averse individual, under the assumption that information increases the mean and reduces variance. Assuming prior mean-variance set (μ_A, σ_A^2), efficient message space consists of all those points northwest of the solid boundary I_E.

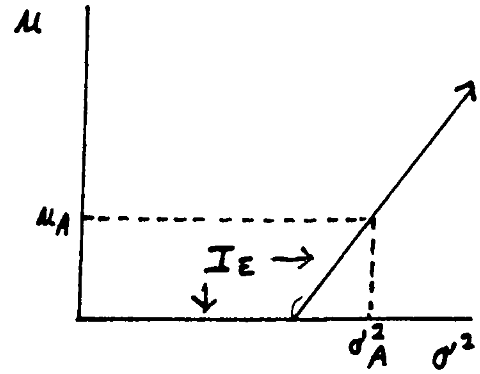

Figure 29. Efficient message space for the risk-averter who processes information that increases his expectations and reduces variance, with quadratic utility.

For the individual who prefers risk, we incorporate the previously used relationships $\sigma^2 = (c+gI)$ and $\mu = (h+dI)$ into (60) and obtain the utility function for information that increases both expectations and variance:

$$U(I) = (h+dI) + a[(c+gI) + (h+dI)^2] \tag{71}$$

The utility of information that increases both variance and expectations increases at an increasing rate, implying utility is insatiable:

$$\frac{\partial U(I)}{\partial I} = d[1 + 2(h+dI)] + ag > 0 \tag{72}$$

$$\frac{\partial^2 U(I)}{\partial I^2} = 2d^2 > 0 \tag{73}$$

Fig. 30 depicts the results in (72) and (73), and Fig. 31 depicts the corresponding efficient message space, which consists of all those points to the right of the boundary I_E.

Compare Fig.30 with Fig. 31. An individual preferring risk values information that changes his distribu-

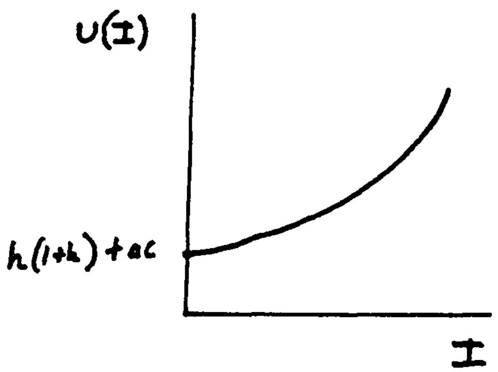

Figure 30. The utility of information that increases the mean and variance for a risk-preferrer.

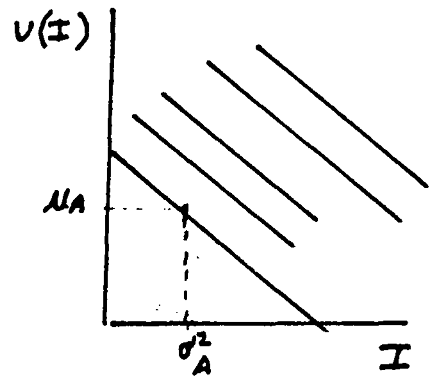

Figure 31. Efficient message space for a risk-preferrer who processes information that increases the mean and variance.

-tion in preferred directions at a greater rate than does a risk-averter. This is clearly because the marginal utility of future wealth for the risk-preferrer is increasing while for the risk-averter, it is diminishing.

It should be noted that the linear relationships between the two parameters and information used in these comparative statics analyses, represent only one fashion in which individuals process information. Alternative functional relationships between distribution parameters and information could be nonlinear, if $\partial^2\mu/\partial I^2 \neq 0$ and/or $\partial^2\mu_n/\partial I^2 \neq 0$, where μ is the mean and μ_n the nth central moment of the distribution. Linear or nonlinear functional relationships can be validly used, provided that the appropriate sign of the first derivative is maintained. The choice of linearity or nonlinearity depends on the type of information and perhaps the previous information processing characteristics of the individual. These questions are clearly empirical in nature and their signif-

-igance become apparent when these utility functions are used in the derivation of testable propositions.

The expected value of (74) is given by the following:

(77) $$E[U(W)] = \mu + b\mu^2 + c\mu^3 + (b+3c\mu)\sigma^2 + c\mu_3$$

$$\mu = E(W),\ \sigma^2 = E[W-E(W)]^2,$$

$$\mu_3 = E[W-E(W)]^3$$

In (77), μ is expected wealth, σ^2 is variance and μ_3 is skewness.

The conditions for a positive preference direction for expected future wealth are the same as those for an increasing deterministic U(W):

(78) $$\frac{\partial E[U(W)]}{\partial \mu} = 1 + 2b\mu + 3cE(W^2)$$

Since $b^2 < 3c$, (78) is guaranteed to be positive. c is the marginal expected utility of skewness. Since c is guaranteed to have a minimum positive value, we find that for (78) to be positive, the individual will have a minimum positive preference direction for positive skewness:

(79) $$\frac{\partial E[U(W)]}{\partial \mu_3} = c > b^2/3$$

The individual's preference direction for variance is given by:

(80) $$\frac{\partial E[U(W)]}{\partial \sigma^2} = 3c\mu + b \gtrless 0 \text{ as } \mu \gtrless -b/3c$$

Clearly, since c is positive and expected wealth is assumed to be positive throughout, b can be either positive or negative. If we assume that b is negative, we find that the individual is risk-averse for low values of expected wealth and a risk-preferrer for high values of expected wealth. If b is positive, then the individual is a risk-preferrer for all positive values of expected wealth. Fig. 32 depicts the cubic utility function under the more realistic assumption that b is negative.

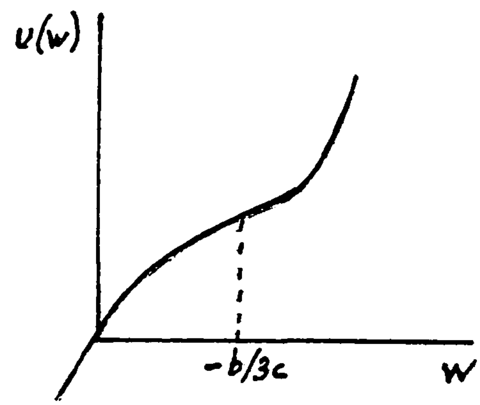

Figure 32. The cubic utility function.

If the marginal expected utility of the mean is always positive, then the marginal expected utility of information that increases the mean will always be positive. Allowing the mean to be a generalized function of information in the expectation of cubic utility:

$$E[U(W)] = \mu(I) + b[\mu(I)]^2 + c[\mu(I)]^3 + [b+3c\mu(I)]\sigma^2 + c\mu_3 \tag{81}$$

The marginal expected utility of information will always be increasing as long as the mean increases with information:

$$\frac{\partial E[U(W)]}{\partial I} = \frac{\partial \mu}{\partial I}\{1 + 2b\mu(I) + 3c[\mu(I)]^2 + 3c\sigma^2\} > 0 \text{ as } \frac{\partial \mu}{\partial I} > 0 \tag{82}$$

From (82), we must incorporate a restricted form of $\mu(I)$ that has a positive first derivative in order to allow the resulting utility function for information to

2. Cubic Utility

With the introduction of skewness, a cubic utility function assumes an underlying subjective probability distribution that belongs to the family of three-parameter distributions. Based on a third order polynomial, the cubic utility function over future wealth is defined as the following:

(74) $$U(W) = W + bW^2 + cW^3$$

If U(W) in (74) is guaranteed to be strictly increasing, the coefficient c will have a minimum positive value.[18]

(75) $$\text{If } \frac{\partial U(W)}{\partial W} = 1 + 2bW + 3cW^2 > 0, \text{ then } c > b^2/3$$

The coefficient b may be negative, as well as positive, as the second derivative of U(W) implies:

(76) $$\frac{\partial^2 U(W)}{\partial W^2} = 6cW + 2b \quad \begin{array}{l} \leq 0 \text{ if } W < -b/3c \\ \geq 0 \text{ if } W > -b/3c \end{array}$$

It follows from (76) that if b is negative, the utility function is concave for some positive W. If b is positive, the function displays concavity only for negative W.

These conditions for concavity and convexity become particularly meaningful when we examine an individual's attitudes toward the mean, variance and skewness.

18/ In order for the cubic utility function to be strictly increasing, the roots of the first derivative have to be imaginary. This implies in the quadratic solution for W in (75) that $b^2 < 3c$ and hence, $c > b^2/3$.

be a valid statement of an individual's preference ordering over information that changes the mean. Substitutung the familiar relationship, $\mu = h+dI$ into (81), we obtain the appropriate utility function for information:

$$(83) \quad U(I) = (h+dI) + b(h+dI)^2 + c(h+dI)^3 + [b + 3c(h+dI)]\sigma^2 + c\mu_3$$

In (83), information is the only argument of the function and variance and skewness are constants.

The marginal utility of information is given by the first derivative of (83):

$$(84) \quad \frac{\partial U(I)}{\partial I} = d[1+2b(h+dI)+3c(h+dI)^2 + 3c\sigma^2] > 0$$

If $b < 0$, then $2b < 3c$ and (84) will always be positive. In addition, (84) is increasing, implying that the utility of information that raises the mean has no maximum:

$$(85) \quad \frac{\partial^2 U(I)}{\partial I^2} = 2b + 6c(d/h + I) > 0$$

If b is negative, then $2b < 6c$ and (85) will always be positive. Figure 33 provides a graphical depiction of these results, under the assumption that b is negative:

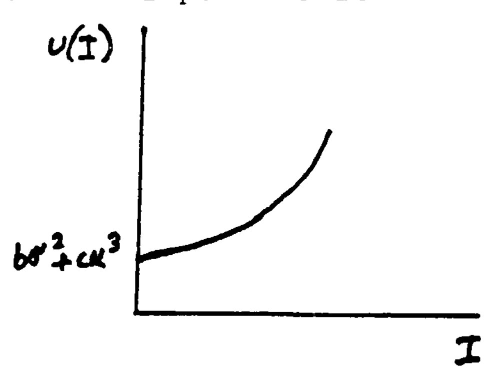

Figure 33. The utility of information that increases expectations for an individual with cubic utility.

Assume now that an individual's marginal expected utility of variance varies according to the condition set forth in (80) - namely, that b is negative. If this condition is assumed, the individual will be risk-averse for relatively low levels of future wealth and will display preference for risk at relatively high levels of future wealth. Information that reduces variance in the region of the expected utility function that is concave and increases variance for the region of the function that is convex, will clearly be valued positively. This is depicted by a graph of the parabolic relationship implied by the preceding discussion. This parabola is plotted in Fig. 34 and its source of derivation, the cubic utility function itself , is plotted directly underneath Fig. 34 in Fig. 35.

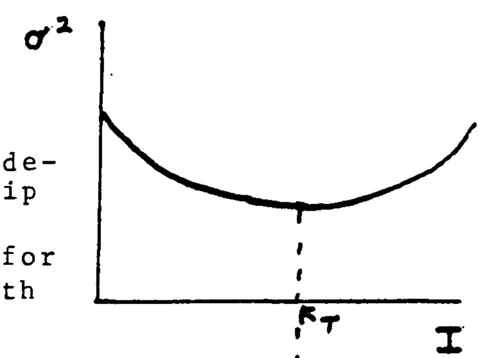

Figure 34. The desired relationship between variance and information for an individual with cubic utility.

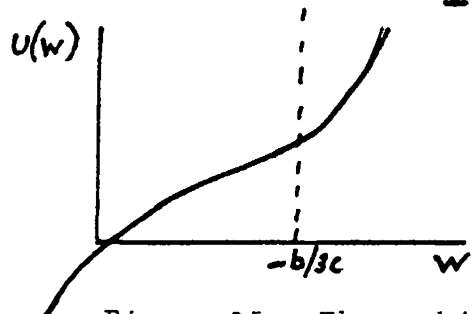

Figure 35. The cubic utility function.

Note that the point where the parabola reaches its minimum (zero first derivative, point T on Fig. 33) corresponds to the inflection point on the utility function (where $W = -b/3c$).

An equation for the parabolic relationship implied by Fig. 33 is given by the following:[19]

$$\sigma^2 = \frac{(I + b/3c)^2}{P} + L \quad , \quad L,P > 0 \tag{86}$$

L is the distance which the parabola is elevated above the information axis I. In addition, a positive value for L implies that the individual faces a minimum variance - L itself. To prove that L is minimum variance, we investigate the necessary and sufficient conditions for a minimum:

$$\frac{\partial \sigma^2}{\partial I} = \frac{2(I+b/3c)}{P} = 0 \text{ when } I = -b/3c \tag{87}$$

$$\frac{\partial^2 \sigma^2}{\partial I^2} = 2/p > 0 \tag{88}$$

Substituting $I=-b/3c$, the condition for minimum variance into (86), L is confirmed to be minimum variance.

The expression $I=-b/3c$ is that quantity of information at which the individual's marginal utility for variance changes, i.e. the inflection point on the cubic utility

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19/ This is derived from the familiar equation for a parabola, expressed in terms of the coefficients and arguments used in (86):

$$(I + b/3c)^2 = P(\sigma^2 - L)$$

Recall that b is negative, hence $I-(-)b/3c$ implies $I+b/3c$.

function.

Assuming that information processed by an individual changes subjective variance in the manner given by (86), we substitute (86) into (77) to obtain the utility function for information that changes variance:

$$(89)\quad U(I) = \mu + b\mu^2 + c\mu^3 + (b+3c\mu)[(I+b/3c)^2/P + L] + c\mu_3$$

The marginal utility of information is given by:

$$(90)\quad \frac{\partial U(I)}{\partial I} = \frac{2(b+3c\mu)(I+b/3c)}{P} > 0 \text{ as } \mu, I \gtrless -b/3c$$

Equation (90) states that information that reduces variance will generate positive utility over the concave region of the cubic utility function, and information that increases variance will be positively over the convex region of the cubic utility function.

The derivative of (90) implies that information that reduces variance (for $\mu<-b/3c$) will be valued positively at a <u>decreasing</u> rate, while information that increases variance (for $\mu>-b/3c$) will be valued positively at an increasing rate.[20] These results are given by the following equa- and Fig. 36.

$$(91)\quad \frac{\partial^2 U(I)}{\partial I^2} = \frac{2(b+3c\mu)}{P} \gtrless 0 \text{ as } \mu \gtrless -b/3c$$

20/ If b is positive, i.e. the individual is a risk lover throughout, then a function for $\sigma^2(I)$ with positive first derivative would be used in the derivation of the utility function for information. With a positive first derivative, the utility function is convex throughout.

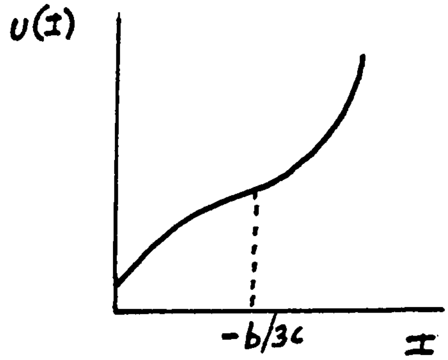

Figure 36. The utility of information that changes variance for an individual with cubic utility.

We now add the assumption that information increases the mean by the familiar relationship $\mu = h+dI$, to the utility function with the previously discussed relationship between variance and information. The utility for information that increases the mean and changes variance is now stated as:

$$(92)\quad U(I) = (h+dI) + b(h+dI)^2 + c(h+dI)^3 + \{b + 3c(h+dI)[(I+b/3c)^2/P + L\} + c\mu_3$$

The marginal utility of information is given by:

$$(93)\quad \frac{\partial U(I)}{\partial I} = d[1+2b(h+dI) + 3c(h+dI)^2] + \frac{3c}{P}[d(I+b/3c)^2 + 2(I+b/3c)(h+dI)] > 0$$

Equation (93) will always be positive. In addition, the utility of information will be increasing:

$$(94)\quad \frac{\partial^2 U(I)}{\partial I^2} = d^2[2+6c(h+dI)] + \{\frac{6cd}{P}[I(1+d) + b/3c + h]\} > 0$$

These results are depicted in Fig. 37.

By the result that c is positive, implying positive skewness preference, any relationship $\mu_3(I)$ with a positive first derivative can be incorporated into the ex-

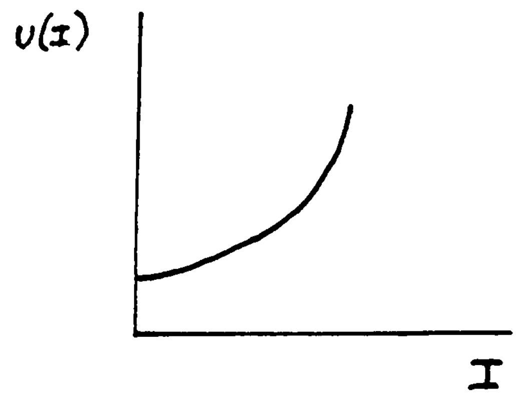

Figure 37. The utility of information that increases expectations and changes variance for an individual with cubic utility.

-pectation of cubic utility. Assume that skewness cannot exist without information. This means that positive or negative bias of an individual's perceived error of expectations depends entirely on the existence of information, i.e. I must be positive for an asymmetric distribution to exist. In effect, information not only changes skewness, but <u>creates</u> it. This is not to say that an individual under this assumption experiences a change in the way that he formulates perceptions of uncertainty (by now being aware that the third moment exists) but only that the third moment changes in value from zero to a nontrivial value (he experiences a change in his perceptions).

The following relationship between skewness and information proves to be quite convenient:

$$\mu_3 = xI \qquad , \; x>0 \tag{95}$$

Let us now substitute (95) into (92), obtaining the utility function for information that changes all three moments:

$$(96)\quad U(I) = (h+dI) + b(h+dI)^2 + c(h+dI)^3 + (b + 3c(h+dI)[(I+b/3c)^2/P + L] + cxI$$

The marginal utility of information is given by:

$$(97)\quad \frac{\partial U(I)}{\partial I} = d[1 + 2b\ (h+dI) + 3c(h+dI)^2 + 3c(I+b/3c)^2] + \frac{6c(h+dI)(I+b/3c)}{P} + cx$$

Expression (97) is guaranteed to be positive. 21/ The last term in (97), cx, is the marginal expected utility of skewness and is clearly positive, by our previous assumptions. The derivative of (97) is positive, implying that utility of information that changes all three moments has no maximum:

$$(98)\quad \frac{\partial^2 U(I)}{\partial I^2} = d^2(h+dI)[2b+6c(1 + 1/Pd^2)] + \underbrace{12cd(I+b/3c)}_{>0},$$

These results are depicted in Fig. 38.

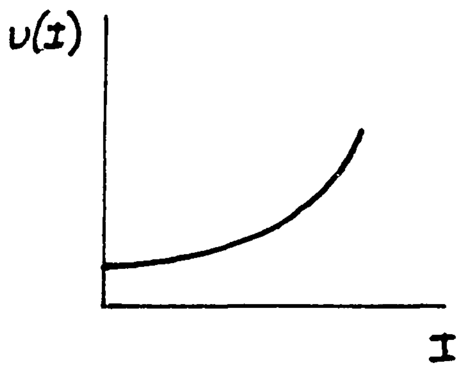

Figure 38. The utility of information that increases expectations, changes variance and increases skewness for an individual with cubic utility.

21/ To see this, note that the first expression in brackets in (97) is positive, while I+b/3c is negative when I<-b/3c. However, the expression 6c(h+dI)(I+b/3c)/P > 0. Proving that this expression is positive is simplified to proving that P(d+3chd) + 3ch > -b. Since $b^2<3c$, 3ch < -b.

Nonpolynomial Forms of Utility

It has come to be widely recognized that there are signifigant limitations to the use of polynomials, such as the preceding quadratic and cubic forms, as utility functions. For example, the quadratic form cannot describe the utility function for the whole range of future wealth on the grounds that marginal utility cannot be negative.22/ Hence, the function is bounded above and below by values for the risk attitude coefficient. In addition, within the function's range of applicability, there is the prevailing assumption of increasing absolute risk-aversion, which many scholars23/ have argued is unrealistic. Increasing absolute risk-aversion is defined by the following derivative:

(99) $$\frac{d\left[-\frac{U^2(W)}{U^1(W)}\right]}{dW} \geq 0 \ , \quad U^2 = \frac{\partial^2 U(W)}{\partial W^2} \ , \quad U^1 = \frac{\partial U(W)}{\partial W}$$

In (99), W refers, as before, to future wealth.

Arrow [2] posits that for a risk-averse individual, a utility function should have, in addition to a positive first derivative and negative second derivative, the following two essential properties:

(100) $$d\left[-\frac{U^2(W)}{U^1(W)}\right]/dW \leq 0$$

(101) $$d\left[-\frac{WU^2(W)}{U^1(W)}\right]/dW \geq 0$$

22/ Tobin [23] was not only the first to employ the function, but the first to point out its inherent limitations. See Borch [3] and Tsiang [24] on this issue.

23/ Among these, Hicks [9] and Arrow [2].

Expression (100) states that the utility function should provide for decreasing marginal absolute risk-aversion, i.e. an individual's risk- aversion falls with an increase in wealth. Expression (101) states that the utility function should provide for increasing marginal relative risk-aversion, i.e. an individual's wealth-weighted risk-aversion increases with wealth.

Polynomial forms of utility satisfy the first two requirements of a positive first derivative and, in most cases, a negative second derivative, but not properties (100) and (101). Functions that do satisfy all of Arrow's desired properties are nonpolynomials, which include (i) the exponential form, (ii) the constant elasticity form, (iii) the logarithmic form. We will examine each of these and their utility-of-information counterparts in subsequent sections.

1. The Exponential Function

The exponential utility function is based on the familiar exponential function, e.g. $y = e^x$. Allowing W to be future wealth, the exponential form of U(W) is given by: 24/

(102) $$U(W) = t^2 W + \frac{1}{t} e^{tW}$$

24/ This is identical to a function derived by Glustoff and Nigro [8]. Another form is employed by Tsiang [24] and is given by $U(W) = B(1-e^{-zW})$, were z is the preference direction for variance. This function is commonly referred to as the "negative" exponential function, which displays constant absolute risk-aversion. The same properties exist for (102) if t is assumed to be negative.

The coefficient t is the parameter whose sign determines the individual's preference direction for variance and higher moments.

From the differentiation of (102), we find that the marginal utility of future wealth is always positive:

(103) $$\frac{\partial U(W)}{\partial W} = t^2 + e^{tW} > 0$$

Marginal utility will diminish only if t is negative, implying risk-aversion, while marginal utility will be increasing if t is positive, implying preference for risk:

(104) $$\frac{\partial^2 U(W)}{\partial W^2} = We^{tW} \gtreqless 0 \text{ as } W \gtreqless 0$$

Assuming the expectation of (102) exists, allow the underlying subjective probability distribution in E[U(W)] to be describable over the mean and variance.25/ We make no other assumptions about the distribution's properties. If the exact properties of the density function f(W) are known, e.g. normality, then the expectation of U(W) would be calculated by the following familiar rule:

(105 $$E[U(W) = \int_{-\infty}^{\infty} U(W)f(W)dW$$
$$= \int_{-\infty}^{\infty} [t^2W + \frac{e^{tW}}{t}]f(W)dW$$

However, if the exact properties of the density function are not known, but only that the distribution belongs to the family of n-parameter distributions, then E[U(W)]must be

25/ The exponential form of utility and the other two non-polynomial forms discussed in this essay have the advantageous property that they can accomodate any special distribution or any general distribution of any order.

approximated. This approximation can be conducted by considering the expectation of an exact Taylor series expansion of U(W) around the mean. Depending on how many parameters over the probability distribution we wish to accomodate, the expected Taylor series would be truncated at the term containing the highest describable central moment.

Under the assumption that the individual's distribution belongs to the family of two-parameter distributions, let us approximate E[U(W)] by expanding U(W) around the mean μ in a Taylor series and calculating its expected value.

$$(106) \qquad E[U(W)] = t^2\mu + \frac{e^{t\mu}}{t} + \frac{te^{t\beta}\sigma^2}{2}$$

$$\beta = W + \theta(W-\mu),\ 0 \leq \theta \leq 1$$

A LaGrange remainder is formed at the final term of)
(106).

Differentiating (106) with respect to the mean, we find that the marginal expected utility of the mean is always positive:

$$(107) \qquad \frac{\partial E[U(W)]}{\partial \mu} = t^2 + e^{t\mu} > 0$$

As with the conditions for concavity and convexity of U(W), the marginal expected utility of variance depends upon the sign of t. If t is negative, the individual is a risk-averter, and if t is positive, the individual pre-

-fers risk:

(108) $$\frac{\partial E[U(W)]}{\partial \sigma^2} = \frac{1}{2} te^{t\beta} \gtrless 0 \text{ as } t \gtrless 0$$

Assuming that t is negative, implying risk-aversion, let $\sigma^2 = a-kI$ and $\mu = h+dI$, the familiar relationships employed in the cases of quadratic and cubic utility. The utility function for information is given as:

(109) $$U(I) = t^2(h+dI) + \frac{e^{t(h+dI)}}{t} + \frac{te^{t\beta}(a-kI)}{2}$$

Differentiating (109) with respect to information, we obtain the marginal utility of information that increases expectations and reduces variance:

(110) $$\frac{\partial U(I)}{\partial I} = d(1+t^2) - \frac{tke^{t\beta}}{2} > 0$$

Since t is assumed to be negative, (110) is always positive. The derivative of (110) is always positive, implying that the risk-averter will have a convex utility function for information:

(111) $$\frac{\partial^2 U(I)}{\partial I^2} = d^2 e^{(h+dI)} > 0$$

Now let us assume that the individual prefers risk, reflected by a positive value for t. Substituting the familiar relationships $\sigma^2 = c+gI$ and $\mu = h+dI$, into (106), the utility function for information that increases the mean and variance is given by:

(112) $$U(I) = t^2(h+dI) + \frac{e^{t(h+dI)}}{t} + \frac{te^{t\beta}(c+gI)}{2}$$

The marginal utility of information is positive and increasing, implying that the risk-averter's utility function is convex to the origin:

$$(113)\qquad \frac{\partial U(I)}{\partial I} = d[t^2 + e^{t(h+dI)}] + \frac{t^2 g e^{t\beta}}{2} > 0$$

$$(114)\qquad \frac{\partial^2 U(I)}{\partial I^2} = d^2 t e^{t(h+dI)} > 0$$

Therefore, when information is posited to affect both parameters of the risk-averter's and risk-preferrer's subjective probability distributions, both types of individuals will have utility functions for information that are convex to the origin.

Let us now examine the case of an individual with exponential utility whose distribution falls within the family of three-parameter distributions. The expectation of the Taylor series expansion of U(W) around the mean, truncated at the third term with LaGrange form remainder is given by the following:

$$(115)\qquad E[U(W)] = t^2\mu + \frac{1}{t}e^{t\mu} + \frac{te^{t\mu}\sigma^2}{2} + \frac{t^2 e^{t\psi}\mu_3}{6}$$

$$\psi = W + \theta(W-\mu)$$

$$0 \le \theta \le 1$$

The conditions guaranteeing positive marginal expected utility of the mean and negative marginal expected utility of variance are the same as for the two moment case. The marginal expected utility of skewness in (115) is always positive, since $t^2e^{t\mu} > 0$.

Let us examine the case of a utility function for information, under the assumption that information changes all three parameters of the distribution of a risk-averter. Substituting $\mu = h + dI$, $\sigma^2 = a - kI$ and $\mu_3 = xI$ into (115), the utility function is given by:

$$(116)\quad U(I) = t^2(h+dI) + \frac{1}{t}e^{t(h+dI)} + \frac{te^{t(h+dI)}(a-kI)}{2} + \frac{t^2e^{t\psi}xI}{6}$$

The marginal utility of information is positive and decreasing, implying that the utility function is concave to the origin:

$$(117)\quad \frac{\partial U(I)}{\partial I} = \frac{t^2(d+e^{t\psi}x)}{6} + t^2d + e^{t(h+dI)}[d(1+1/2a-kI/2)-tk/2] > 0$$

$$(118)\quad \frac{\partial^2 U(I)}{\partial I^2} = dte^{t(h+dI)}[d(1+a/2 + k/2) - tk/2 + kI/2] < 0$$

2. The Constant Elasticity Function

The constant elasticity form of utility is more restrictive in its underlying assumptions than the exponential form. For example, the constant elasticity function assumes local risk-aversion and decreasing absolute risk-aversion over wealth. The function is defined by the following:

$$(119)\quad U(W) = \frac{W^{(1-a)}}{(1-a)}, \quad a > 0 \text{ and } \neq 1$$

The coefficient a is always assumed to be positive. This implies that the function is concave and the individual is risk-averse:

(120) $$\frac{\partial U(W)}{\partial W} = W^{(-a)} \quad > 0$$

(121) $$\frac{\partial^2 U(W)}{\partial W^2} = -aW^{-(a+1)} \quad < 0$$

Let us first derive a utility function for information, under the assumption that the individual's distribution belongs to the family of two-parameter distributions. The expectation of the exact Taylor series expansion of U(W) around the mean μ, truncated at the second term with LaGrange remainder form, is given by:

(122) $$E[U(W)] = \frac{\mu^{(1-a)}}{(1-a)} - a\beta^{-(a+1)}\sigma^2$$

$$\beta = W + \theta(W-\mu) \;, \; 0 \leq \theta \leq 1$$

The marginal expected utility of the mean is positive and decreasing, and the marginal expected utility of variance is negative and constant:

(123) $$\frac{\partial E[U(W)]}{\partial \mu} = \mu^{-a} \quad > 0$$

(124) $$\frac{\partial^2 E[U(W)]}{\partial \mu^2} = -a\mu^{-(a+1)} \quad < 0$$

(125) $$\frac{\partial E[U(W)]}{\partial \sigma^2} = -a\beta^{-(a+1)} \quad < 0$$

(126) $$\frac{\partial^2 E[U(W)]}{\partial (\sigma^2)^2} = 0$$

Allowing μ = h+dI and σ^2 = a-kI, the utility function for information is given by:

(127) $$U(I) = \frac{(h+dI)^{(1-a)}}{(1-a)} - a\beta^{-(a+1)}(b-kI)$$

The marginal utility of information is found to be positive and decreasing, implying that the utility function is concave to the origin:

(128) $$\frac{\partial U(I)}{\partial I} = d(h+dI)^{-(a+1)} + ak\beta^{-(a+1)} > 0$$

(129) $$\frac{\partial^2 U(I)}{\partial I^2} = -d^2(a+1)(h+dI)^{-(a+1)} < 0$$

Now assume that the individual's distribution is a member of the family of three-parameter distributions. The expectation of the exact Taylor series expansion of U(W) around the mean, truncated at the third term with LaGrange remainder form is given by the following:

(130) $$E[U(W)] = \frac{\mu^{(1-a)}}{(1-a)} - \frac{a\mu^{-(a+1)}\sigma^2}{2} + \frac{a(a+1)\phi^{-(a+2)}\mu_3}{6}$$

$$\phi = W + \theta(W-\mu),\ 0 \leq \theta \leq 1$$

In (130), μ_3 is skewness and the marginal expected utility of skewness is positive, implying positive skewness preference.

Substituting $\mu = h+dI$, $\sigma^2 = a-kI$ and $\mu_3 = xI$ into (130), we obtain the utility function for information that changes all three distribution parameters:

(131) $$U(I) = \frac{(h+dI)^{(1-a)}}{(1-a)} - \frac{a(h+dI)^{-(a+1)}(b-kI)}{2} + \frac{a(a+1)\phi^{-(a+2)}xI}{6}$$

The marginal utility of information is positive and decreasing, implying that the utility function is concave to the the origin:

(132) $$\frac{\partial U(I)}{\partial I} = d(h+dI)^{-(a+1)} + \frac{da(a+1)(h+dI)^{-(a+2)}(b-kI)}{2} + \frac{ak(h+dI)^{-(a+1)}}{2} + \frac{a(a+1)\phi^{-(a+2)}x}{6} > 0$$

(133) $$\frac{\partial^2 U(I)}{\partial I^2} = -d^4(a+1)(h+dI)^{-(a+2)} - \frac{da(a+1)(a+2)(h+dI)^{-(a+3)}(b-kI)}{2} - \frac{dak(a+1)(h+dI)^{-(a+2)}}{2} - \frac{ak(a+1)(h+dI)^{-(a+2)}}{2} < 0$$

3. The Logarithmic Utility Function

The logarithmic form of utility, like the constant elasticity form, applies only to individuals with local risk-aversion and decreasing risk-aversion over different levels of wealth. The utility function is defined as the following:

(134) $$U(W) = \mathrm{Log}(W)$$

Differentiating (134) twice, we find the utility function to be concave, implying risk-aversion throughout:

(135) $$\frac{\partial U(W)}{\partial W} = \frac{1}{W} > 0$$

(136) $$\frac{\partial^2 U(W)}{\partial W^2} = -\frac{1}{W^2} < 0$$

Let us first derive a utility function for information, under the assumption that the risk-averter's probability distribution belongs to the family of two-parameter distributions. The expectation of the exact Taylor series expansion of U(W) around the mean, truncated at the second term with LaGrange form remainder, is given by the following:

(137) $$E[U(W)] = \text{Log}(\mu) - \frac{\sigma^2}{2\varepsilon^2}$$

$$\varepsilon = W + \theta(W-\mu) \ , \ 0 \leq \theta \leq 1$$

Allowing $\mu = h+dI$ and $\sigma^2 = a-kI$ into (137), we obtain the utility function for information that increases expectations and reduces variance:

(138) $$U(I) = \text{Log}(h+dI) - \frac{(a-kI)}{2\varepsilon^2}$$

Differentiating (138) with respect to information, we find that the utility function is concave to the origin, implying that the utility of information has a maximum:

(139) $$\frac{\partial U(I)}{\partial I} = \frac{d}{(h+dI)} + \frac{k}{2\varepsilon^2} > 0$$

(140) $$\frac{\partial^2 U(I)}{\partial I^2} = -\frac{d^2}{(h+dI)^2} < 0$$

Finally, we examine the case of the risk-averter's probability distribution belonging to the family of three-parameter distributions. The expectation of the exact Taylor series expansion of U(W) around the mean, truncated at the third term with LaGrange remainder form, is given by the following:

(141) $$E[U(W)] = \text{Log}(\mu) - \frac{\sigma^2}{2\mu^2} + \frac{\mu_3}{6\delta^3}$$

$$\delta = W + \theta(W-\mu), \ 0 \leq \theta \leq 1$$

From (141), the marginal expected utility of skewness is found to be always positive, implying positive skewness preference.

Substituting $\mu = h+dI$, $\sigma^2 = a-kI$ and $\mu_3 = xI$ into (141), we obtain the utility function for information that increases expectations, reduces variance and increases skewness:

$$(142) \qquad U(I) = \mathrm{Log}(h+dI) - \frac{(a-kI)}{2(h+dI)^2} + \frac{xI}{6\delta^3}$$

The utility function is found to be concave to the origin:

$$(143) \qquad \frac{\partial U(I)}{\partial I} = \frac{d}{(h+dI)} + \frac{k}{2(h+dI)^2} + \frac{d(a-kI)}{(h+dI)^3} + \frac{x}{6\delta^3} > 0$$

$$(144) \qquad \frac{\partial^2 U(I)}{\partial I^2} = \frac{-d^2}{(h+dI)^2} - \frac{dk}{(h+dI)^3} - \frac{3d^2(a-kI)}{(h+dI)^4} < 0$$

Conclusion

From the standpoint of theoretical completeness, the findings of this essay are only <u>examples</u> of how the five restricted forms of expected utility can effectively be restated as deterministic utility functions for information. For analytical convenience, we have examined exclusively cases where individuals are assumed to process information whose effectiveness in changing perceptions of uncertainty is constant. We have been led to the conclusion that in most cases where risk-aversion is assumed, the utility of information will be a function that is concave to the origin. In most cases where preference for risk is assumed, the utility function will be convex to the origin. These results will vary as we impose different assumptions concerning information potency. There are found to be many variations of a utility function for information for each case of expected utility. This is because the general case of parameter preference information valuation allows us to be quite flexible in imposing assumptions concerning the properties of the utility function and subjective probability distribution.

The five restricted forms of expected utility, restated as utility functions for information, clearly differ on the basis of how they satisfy the two fundamental criteria discussed in the early part of this essay. For extensions of the results of this essay to further theo-

-retical and empirical problems in the economics of information, we must have at our disposal a utility function that is as simple as possible in functional form and yet can accomodate as many combinations of assumptions concerning the underlying expected utility function and subjective probability distribution. The advantage of polynomial forms of utility is that they can accomodate all types of assumptions concerning preference directions for variance and skewness. The disadvantages are: (1) They can only be applied to certain, narrow classes of distributions; (2) They assume constant absolute risk-aversion; (3) They can only apply to certain ranges of wealth. Nonpolynomials, except for the exponential utility function, have the disadvantage of being able to accomodate only the assumption of risk-aversion. However, they do not share the disadvantages listed above for polynomials.

If one is willing to accept the assumption of risk-aversion, then it appears from the results of this essay that the logarithmic form of utility will come closest to satisfying the two fundamental criteria discussed on page 64. Therefore, that function's utility-of-information counterpart probably is the most optimal function to use in analyzing specific theoretical and empirical implications of the general case of parameter preference information valuation.

III. A COMPARATIVE STATICS ANALYSIS OF THE DEMAND FOR INFORMATION ON SECURITIES

Introduction

Under the assumption that securities investors are single-period maximizers of the expected utility of their future portfolio wealth, this essay applies the general case of parameter preference information valuation and the special case of the logarithmic utility of information, to the derivation and comparative statics analyses of individual demand functions for securities and information on securities. The demand function for information is a statement of an investor's willingness and ability to pay for information that increases his expectations and reduces his perceived risk of future portfolio returns. Both demand functions are outcomes of the same optimization process involving a logarithmic utility function over securities and information, and a single equality wealth constraint over securities and information expenditures.

Although some studies, including[10], have treated information as a commodity commanding a unit price, very little theoretical work and no empirical work has been conducted on the demand for information. Kihlstrom[13] derived the general case and one special case involving constant elasticity utility, of the demand for information about product quality by the use of Bayesian "preposterior" analysis. Kihlstrom's model differs from the investor be--havior model developed in this essay by the latter's as-

-sumption that investors demand information if and only if that information changes their perceptions of future portfolio returns in preferred directions.

Following derivation and comparative statics analyses of the demand functions, the essay concludes by suggesting the extension of the results to certain avenues of future theoretical and empirical work, and commenting on the possibilities for empirical verification of the theoretical demand functions presented.

A Model Of Investor Behavior

Assume that a securities investor has a utility function U over future portfolio wealth W of the logarithmic form:

$$U(W) = \text{Log}(W) \tag{145}$$

This function was demonstrated to be concave to the origin, as given by its positive first derivative and negative second derivative, implying risk-aversion.

Future portfolio wealth is defined to be the product of the dollar investment S in securities and the future percentage return R on the investment, with R a random variable:

$$W = SR \tag{146}$$

Allow S to equal the product of the total shares N of J different securities and their average price P:

$$S = PN\ , \tag{147}$$

$$N = \sum_{j=1}^{J} N_j \quad \text{and} \quad P = \frac{\sum_{j=1}^{J} P_j N_j}{\sum_{j=1}^{J} N_j}$$

The random variable R refers to the percentage return of the <u>entire</u> portfolio at the end of some future time period - which could be twelve months, six months, etc. The investor is assumed to be able to describe his subjective probability distribution over R, hence future portfolio wealth, in terms of a mean μ and variance σ^2 of portfolio wealth, i.e. the distribution is symmetric about the mean. Based on a general two-moment distribution, the expectation of $U(W)$ over the mean and variance is defined as the expectation of the finite Taylor series expansion of $U(W)$ around μ, truncated at the second term with the remainder in LaGrange form:

$$E[U(W)] = \text{Log}(\mu) - \frac{\sigma^2}{2\beta^2} , \tag{148}$$

$$\mu = E(W),\ \sigma^2 = E[W-E(W)]^2$$

$$\beta = W + \theta(W-\mu),\ 0 \leq \theta \leq 1$$

The mean and variance of future portfolio wealth are the products of the total dollar investment and the mean μ_R and variance σ_R^2 of percentage future portfolio returns, respectively:

$$E[U(W)] = \text{Log}(PN\mu_R) - \frac{PN\sigma_R^2}{2\beta^2} \tag{149}$$

Assume that there are J+Ø securities available in the marketplace, where Ø is significantly greater than J. Since it is the rational investor's objective to hold at any point in time a portfolio of securities that offers the highest expected return μ_R and lowest perceived risk σ_R^2 , he consults a security analysis firm which is willing to evaluate his portfolio at a cost. The firm will evaluate each security for a price i. When each security is evaluated, the firm provides the investor with a data "package" I, summarizing the firm's research findings on the security, and a recommendation to "buy", "sell" or "hold" the security. Upon receipt of each unit of I, the investor ascertains the possible impact of the addition, deletion or retention of the security on the expected return and perceived risk of his portfolio. Therefore, the mean and variance of portfolio returns are posited to be functions of information I.

If the mean and variance are functions of information, then what appropriate desired message path for I should be incorporated into the expected utility function? Since the utility function is concave to the origin, the following relationships must be used in some special form, as statements of the investor's preference ordering over information:

(150) $$\frac{\partial \mu_R}{\partial I} > 0 \ , \ \frac{\partial \sigma_R^2}{\partial I} < 0$$

Any functional relationships satisfying (150) can be substituted for the distribution parameters in the expected utility function. We assume that the investor experiences increasing information "potency" with respect to his expectations of future returns and constant potency with respect to his perceived risk of future returns. These assumptions are reflected by the following properties of the desired message path:

(151) $$\frac{\partial^2 \mu_R}{\partial I} > 0 \ , \ \frac{\partial^2 \sigma_R^2}{\partial I^2} = 0$$

Two functional relationships that validly satisfy these assumptions are the exponential function between the mean and information, and the linear function between variance and information. Allow the mean to exponentially increase with information and the variance to be linearly inversely related to information:

(152) $\mu_R = e^{dI}$, $d > 0$, $\sigma_R^2 = a-kI$, $a,k > 0$

Given (7), the original expected utility function over future portfolio wealth is now restated as the utility of securities and information:

(153) $$U(N,I) = \text{Log}(PNe^{dI}) - \frac{PN(a-kI)}{2\beta^2}$$

The marginal utility of securities is found to have a maximum, shown in (154) and Fig. 39.

(154) $$\frac{\partial U(N,I)}{\partial N} = \frac{1}{N} - \frac{P(a-kI)}{2\beta^2} \gtreqless 0 \text{ as } N \lesseqgtr \frac{2\beta^2}{P(a-kI)} \quad \text{for}$$

$$N = (1,\infty)$$

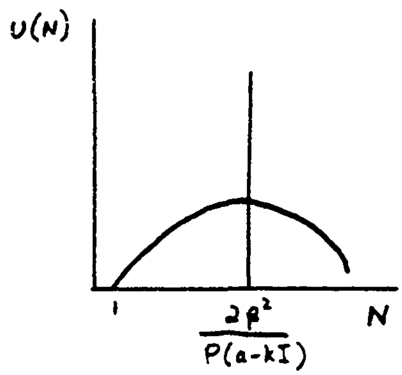

Figure 39. The utility of securities.

The marginal utility of information that increases expectations and reduces variance is positive and constant, as defined by (155) and (156), and depicted in Fig. 40:

(155) $$\frac{\partial U(I)}{\partial I} = d + \frac{PNk}{2\beta^2} > 0$$

(156) $$\frac{\partial^2 U(I)}{\partial I^2} = 0$$

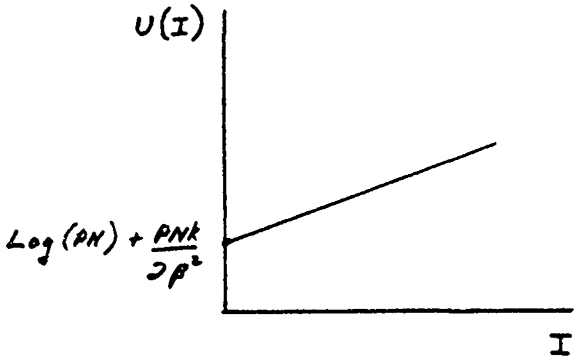

Figure 40. The utility of information about securities.

The marginal utility of securities increases with an increase in information, holding the number of securities in the portfolio constant. This implies that securities and information are complements in the investor's portfolio. However, there is a positive relationship between the marginal utility of information and a change in the number of securities held, holding the investor's knowledge constant. These results are given in (157) and (158). The first result is depicted in Fig. 41.

(157) $$\frac{\partial(\partial U/\partial N)}{\partial I} = \frac{Pk}{2\beta^2} > 0$$

(158) $$\frac{\partial(\partial U/\partial I)}{\partial N} = \frac{Pk}{2\beta^2} > 0$$

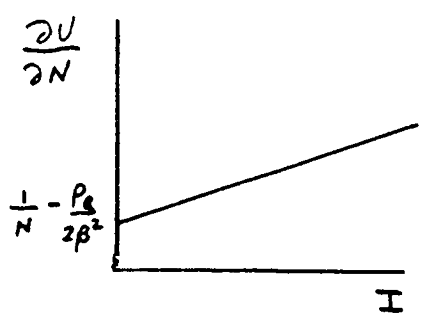

Figure 41. The complementarity of securities and information.

In Fig. 41 and equation (154), observe that the positive intercept on the marginal utility axis applies to the following condition for N:

(159) $$\frac{\partial U(N,I)}{\partial N} \gtreqless 0 \text{ when } N \lesseqgtr \frac{Pa}{2\beta^2} \quad \text{and } I = 0.$$

Clearly, when the investor holds very few securities, zero knowledge will not preclude him from experiencing a positive level of utility for securities. However, when the investor holds relatively many securities, zero knowledge will cause him to experience a negative utility for securities. Therefore, the investor must have a minimum amount of information to maintain positive utility for a large portfolio.

The investor is assumed to have present positive wealth W_0, allocatable between expenditures on securities and information about securities:

(160) $$W_0 = PN + iI$$

By choice of N shares of securities at average market price P and I units of information at price i, the investor is assumed to maximize the utility of securities and information (153) subject to the present wealth constraint (160). Forming a LaGrangian L, the investor's choice problem is

(161) $$\underset{\{N,I,\lambda\}}{\text{Max}} \quad L = \text{Log}(PNe^{dI}) - \frac{PN(a-kI)}{2\beta^2} + \lambda(W_0 - PN - iI)$$

with first-order necessary conditions:

(162) $$\frac{\partial L}{\partial N} = \frac{1}{N} - \frac{P(a-kI)}{2\beta^2} = \lambda P$$

(163) $$\frac{\partial L}{\partial I} = d + \frac{PNk}{2\beta^2} = \lambda i$$

(164) $$\frac{\partial L}{\partial \lambda} = W_0 - PN - iI = 0$$

λ is a LaGrange multiplier and represents the investor's marginal utility of present wealth W_0. At a constrained equilibrium, the marginal utility of present wealth, along with the marginal utilities of securities and information, will be at constrained maxima.

The second-order conditions for a constrained maximum require that the relevant Hessian determinant is positive:

(165) $$\begin{vmatrix} \frac{\partial^2 L}{\partial N^2} & \frac{\partial^2 L}{\partial I \partial N} & \frac{\partial^2 L}{\partial \lambda \partial N} \\ \frac{\partial^2 L}{\partial N \partial I} & \frac{\partial^2 L}{\partial I^2} & \frac{\partial^2 L}{\partial \lambda \partial I} \\ \frac{\partial^2 L}{\partial N \partial \lambda} & \frac{\partial^2 L}{\partial I \partial \lambda} & \frac{\partial^2 L}{\partial \lambda^2} \end{vmatrix} = D_3 > 0$$

Performing the necessary differentiation and calculation to obtain the determinant in (165):

(166) $$\frac{\partial^2 L}{\partial N^2} = -\frac{1}{N^2}$$

(167) $$\frac{\partial^2 L}{\partial I \partial N} = \frac{\partial^2 L}{\partial N \partial I} = \frac{Pk}{2\beta^2}$$

(168) $$\frac{\partial^2 L}{\partial \lambda \partial N} = \frac{\partial^2 L}{\partial N \partial \lambda} = -P$$

(169) $$\frac{\partial^2 L}{\partial I^2} = 0$$

(170) $$\frac{\partial^2 L}{\partial \lambda \partial I} = \frac{\partial^2 L}{\partial I \partial \lambda} = -i$$

(171) $$\frac{\partial^2 L}{\partial \lambda^2} = 0$$

The determinant will be positive for a range of N:

(172) $$D_3 = N^2 + \frac{2\beta^2 i}{P^2 k} - \frac{2\beta^2 i^2}{Pk} \gtrless 0 \text{ as}$$

$$N \gtrless \sqrt{\frac{2\beta^2 i^2}{Pk} - \frac{2\beta^2 i}{P^2 k}}$$

A positive value for (172) will exist up to a threshold value for N, which can easily be shown to be much smaller in magnitude than the threshold value for N where the marginal utility of securities changes sign:

(173) $$N = \frac{2\beta^2}{P(a-kI)} > \sqrt{\frac{2\beta^2 i^2}{P} - \frac{2\beta^2 i}{P^2 k}}$$

We conclude that in the interval $N = \left\{ \sqrt{\frac{2\beta^2 i^2}{P} - \frac{2\beta^2 i}{P^2 k}}, \frac{2\beta^2}{P(a-kI)} \right\}$ the investor is at a constrained maximum.

The investor's constrained equilibrium in securities-information space is given by the following equality:

(174) $$-\frac{N(2\beta^2 d + PNk)}{2\beta^2 - PN(a-kI)} = \frac{P}{i}$$

To determine the slopes and curvatures of the underlying indifference curves in security-information space, the expected utility function is totally differentiated to obtain the marginal rate of substitution between securities and information:

(175) $$\frac{dN}{dI} = -\frac{N(2\beta^2 d + PNk)}{2\beta^2 - PN(a-kI)} \lesseqgtr 0 \text{ as } N \lesseqgtr \frac{2\beta}{P(a-kI)}$$

For the range of N yielding positive marginal utility for securities, the marginal rate of substitution of securities for information will be negative. Beyond that range, the marginal rate of substitution will be positive. The derivative of (175) is positive, implying that the indifference curves are convex to the origin:

(176) $$\frac{d^2N}{dI^2} = \frac{PN^2k(2\beta^2 d + PNk)}{\left(2\beta^2 - PN(a-kI)\right)^2} > 0$$

Based on expressions (175) and (176), the investor's indifference map is depicted in Fig. 42 . Since the rational

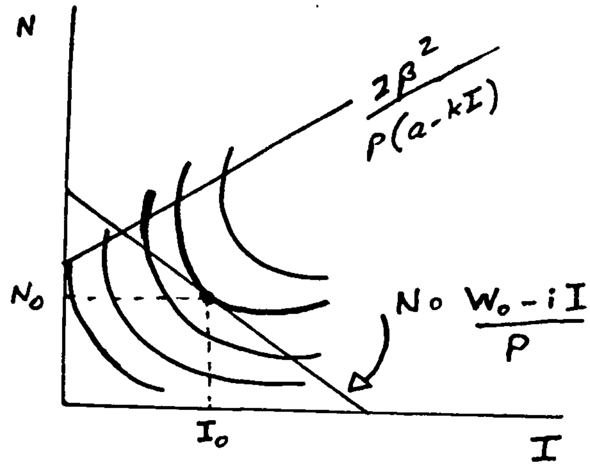

Figure 42. The investor's indifference map and constrained equilibrium in securities-information space.

investor will never acquire that level of securities at which he experiences negative marginal utility, the preference map is the series of curves which lie beneath the line intersecting the securities axis at the value for N at which the marginal utility for securities becomes negative. Beneath that line, the indifference curves are concave to the origin with higher levels of utility attainable to the northeast of the origin, in the direction of the arrow. Given present wealth W_0, a hypothetical constrained equilibrium at N_0 securities and I_0 units of information is depicted in Fig. 41 by the superimposition of the wealth constraint W_0 on the indifference map.

With the second-order conditions for a constrained maximum for (161) guaranteed over most of N yielding positive marginal utility, sufficient conditions now exist for the derivation of demand functions for securities and informa-

-tion. Solving the simultaneous equations (162) - (164), the demand function for securities is given by the following positive root of the quadratic solution for N:

$$(177)\quad N = \frac{1}{4P}\left[-\left(\frac{2\beta^2 d+ai}{k} - W_0\right)+\sqrt{\left(\frac{2\beta^2 d+ai}{k}-W_0\right)^2 + \frac{16\beta^2 i}{k}}\right]$$

The positive root for N was selected because the negative root applied only to negative values for N and the positive root to positive values for N.

The demand function for securities is convex to the origin, implying that the quantity demanded of securities and the average price of securities are inversely related, and as price falls, the quantity demanded of securities increases at an increasing rate:

$$(178)\quad \frac{\partial N}{\partial P} = -\frac{1}{4P^2}\left[-\left(\frac{2\beta^2 d+ai}{k} - W_0\right)+\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}\right] < 0$$

$$(179)\quad \frac{\partial^2 N}{\partial P^2} = \frac{1}{8P^3}\left[-\left(\frac{2\beta^2 d+ai}{k} - W_0\right)+\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}\right] > 0$$

The demand for securities increases with present wealth for relatively high levels of wealth and decreases for relatively low levels of wealth:

$$(180)\quad \frac{\partial N}{\partial W_0} = \frac{1}{4P} - \frac{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)}{2P\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}} \gtrless 0 \text{ as,}$$

$$W_0 \gtrless -\frac{(2\beta^2 d+ai)/k}{2\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}}$$

Regardless of the investor's present wealth position, the demand for securities will increase and decrease with present wealth , both at <u>increasing</u> rates:

(181)
$$\frac{\partial^2 N}{\partial W_0^2} = \frac{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2}{P\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}} > 0$$

The relationship between the demand for securities and the present wealth of the investor is depicted in Fig. 43. In that figure, W_T is the threshold level of present wealth given in (180). The demand for securities will al-

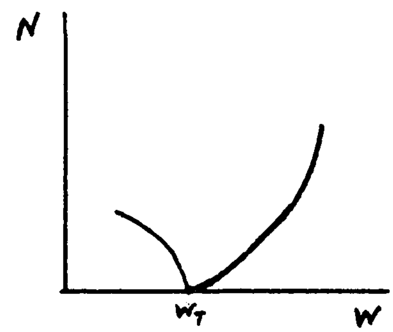

Figure 43. The demand for securities and present positive wealth.

-ways increase(decrease) at an increasing rate with an increase(decrease) in the price of information. Therefore, as the price of information rises(falls), the investor will substitute securities(information) for information (securities):

(182)
$$\frac{\partial N}{\partial i} = -\frac{a}{4P} + a\left(\frac{2\beta^2 d+ai}{k} - W_0\right) + \frac{16\beta^2/k^2}{\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}} > 0$$

(183)
$$\frac{\partial^2 N}{\partial i^2} = -\frac{a}{\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}} + \frac{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2}{k}}{\sqrt[3]{\left((2\beta^2 d+ai)/k - W_0\right)^2 + 16\beta^2 i/k}} > 0$$

Substituting the demand function for securities into the first-order condition in (164), we obtain the demand function for information that increases expectations and reduces perceived risk of portfolio returns:

$$(184) \quad I = \frac{1}{i}\left[W_0 + \frac{1}{4}\left(\frac{2\beta^2 d+ai}{k} - W_0\right) - \frac{1}{4}\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}\right]$$

From (184), positive values for I are guaranteed by condition (164), confirming that I is a nonnegative function. To see this, note the following condition:

$$(185) \quad I > 0 \text{ as } W_0 > \frac{1}{4}\sqrt{\frac{2\beta^2 d+ai}{k} - W_0{}^2 + \frac{16\beta^2 i}{k}} - \left(\frac{2\beta^2 d+ai}{k} - W_0\right)$$

The expression to the right of the inequality sign in (185) is equivalent to PN, hence the condition may be restated as:

$$(186) \qquad I > 0 \text{ as } W_0 > PN$$

Therefore, I is guaranteed to be nonnegative by condition (164).

The demand function for information is convex to the origin, implying that the quantity demanded and the price of information are inversely related, and that as price falls, the quantity demanded increases at an increasing rate:

$$(187) \quad \frac{\partial I}{\partial i} = -\frac{1}{i^2}\, W_0 + \left(\frac{1}{4}\,\frac{2\beta^2 d+ai}{k} - W_0\right) - \frac{1}{4}\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}$$
$$+ \frac{1}{i}\left[\frac{a}{4k} - \frac{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)\frac{a}{k} + \frac{16\beta^2 i}{k}}{\sqrt{4\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}}\right] < 0$$

$$(188)\quad \frac{\partial^2 I}{\partial i^2} = \frac{1}{i^3}\left[W_0 + \frac{1}{4}\left(\frac{2\beta^2 d+ai}{k} - W_0\right) - \frac{1}{4}\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}\right]$$

$$- \frac{1}{i^2}\left[\frac{a}{4k} - \frac{\left(\frac{4a}{k}\right)\left(\frac{2\beta^2 d+ai}{k} - W_0\right) + \frac{16\beta^2 i}{k}}{\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}}\right]$$

$$- \frac{1}{i^2}\left[\frac{a}{4k} - \frac{\left(\frac{a}{4k}\right)\left(\frac{2\beta^2 d+ai}{k} - W_0\right) + \frac{16\beta^2 i}{k}}{\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}}\right]$$

$$- \frac{1}{4}\left[\frac{\left(\frac{a}{k}^2\right)\left(\frac{2\beta^2 d+ai}{k} - W_0\right) + \frac{16\beta^2 i}{k}}{\sqrt{\frac{2\beta^2 d+ai}{k} - W_0^{\ 2} + \frac{16\beta^2 i}{k}}}\right]$$

$$+ \left[\frac{\left(\frac{a}{k}\right)\left(\frac{2\beta^2 d+ai}{k} - W_0\right) + \frac{16\beta^2 i}{k}}{8\sqrt[3]{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}}\right] > 0$$

In contrast to the demand for securities, the demand for information increases at an increasing rate over relatively <u>low</u> levels of wealth and <u>decreases</u> at an increasing rate over relatively high levels of wealth:

$$(189)\quad \frac{\partial I}{\partial W_0} = \frac{3}{4}i + \frac{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)}{\sqrt[2]{\left(\frac{2\beta^2 d+ai}{} - W_0\right)^2 + \frac{16\beta^2 i}{k}}} \gtreqless 0 \text{ as}$$

$$W \lesseqgtr \frac{1}{2}\left(\frac{2\beta^2 d+ai}{k}\right) + \frac{3}{4}i\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}$$

$$(190)\quad \frac{\partial^2 I}{\partial W^2} = -\frac{2}{\sqrt{\frac{2\beta^2 d+ai}{k} - W_0^{\ 2} + \frac{16\beta^2 i}{k}}} + \frac{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2}{\sqrt[3]{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}} > 0$$

The threshold wealth level guaranteeing a positive, negative or trivial sign for (189) is larger than the threshold wealth level for the securities demand function. This implies that over some range of present wealth, an investor will devote all of the increase in his wealth to the purchase of information. That range is given by the following interval:

$$(191)\quad W = \left\{ 2\beta^2 d + ai - \frac{\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}}{2},\ \frac{3i\sqrt{\left(\frac{2\beta^2 d+ai}{k} - W_0\right)^2 + \frac{16\beta^2 i}{k}}}{4} \right\}$$

A decrease in the demand for information with increases in wealth beyond the threshold level in (189) can possibly be justified on the grounds that: (1) The investor is assumed, by an important property of the logarithmic utility function, to experience decreasing marginal absolute risk-aversion over increases in wealth; (2) At the threshold level for present wealth in (189), the individual will desire to hold less securities than he would desire at lower wealth levels. With (1), decreasing marginal absolute risk-aversion implies that the investor's marginal valuation of information that reduces variance will be lower (a shift in the marginal utility of information function) at higher levels of present wealth. With (2), at higher levels of present wealth, the investor will be holding fewer securities and will hence require less evaluation of his portfolio. These possible explanations deserve further inquiry.

A second interesting result is the disappearence of the average security price $\dot{P}$ as an argument of the demand function for information, implying that the demand for information and securities prices are completely independent. This finding runs counter to intuition, which would suggest that as securities prices change, investors will gain interest in obtaining information on those securities whose prices have fallen (assuming that there is no short selling), and lose interest in those securities whose prices have risen. One justification for this finding is that current security prices are not considered information about the securities by the investor. In other words, current prices do not alter expected returns and perceived risk of the portfolio. These parameters depend on past and expected future prices and the investor will demand information about them. Hence, when P changes, the investor perceives that he has the same amount of information as before and he will not be motivated to increase or decrease his knowledge of securities.

Conclusion

The preceding analysis has demonstrated that a risk-averse investor's demand function for information that increases his expectations and lowers his perceived risk of future portfolio returns, has properties quite similar to that of a conventional demand function for a commodity: (1) Convexity to the origin; (2) A relationship between the demand for information and the investor's present positive wealth position that implies that information is a normal good for relatively low levels of present wealth and an inferior good for relatively high levels of wealth. The demand function differs from many conventional functions by the complete independence between the demand for information and prices of other goods (securities) in the investor's expenditure plans.

Perhaps the central implicit feature of the demand function for information is that the function is a statement of an investor's willingness and ability to pay for information that changes his perceptions about future portfolio wealth in preferred directions. The investor is hypothesized to purchase information with the expectation that the information will increase his expectations and reduce the risk of future portfolio returns. If the information's effects on the investor's perceptions match his expectations of its effects, then the price paid for that information will reflect his true valuation of it. If, however, the investor purchases information that changes his perceptions in <u>unpreferred</u> directions, then the price paid for that information will not truly reflect his underlying negative valuation of it. Clearly, for an investor with a demand function for information to experience a net gain in utility from the purchase of information, the information induced change in his perceptions must increase utility sufficiently enough to offset the loss in utility from the income expended on the information.

With the demand functions derived and analyzed in the preceding section, the most promising empirical extension of these results would be the statistical estimation of the demand functions themselves. This could involve the examination of a cross-section of investors and their transactions with securities brokerage and consulting firms, these firms presumably being the best sources of data. With the prudent use of non-linear estimation techniques, one suggested approach would be to regress purchases of stock recommendations on the effective prices of securities and the wealth positions of the investors surveyed. The same cross-section of investors could be used to regress security transactions on security prices, investor wealth and stock recommendation prices.

Theoretical extensions of these results might include the development of a market demand function for information under the assumption of heterogenous beliefs and tastes. With the integration of a theory of individual and market supply of information, this could be used to explain the structure, efficiency and other characteristics of the information industry. Since demand functions for information and securities are both outcomes of the same optimization process, the results of this essay could provide a foundation for the theoretical analyses of the interaction between the information and securities industries. For example, both industries could be viewed as components of a general equilibrium model along with households and firms that engage in the buying and selling of securities. One result in this essay stated that when information prices rise(fall), the demand for securities falls(rises). This result could be refined and extended to the investigation of how shifts in information industry equilibrium affect securities market equilibrium.

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