# An Abstract of the Dissertation of

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In 1995, Barge and Ingram [4] investigated inverse limits on $[0, 1]$ using a single bonding map from the logistic family, $f_\lambda(x) = \lambda x(1 - x), 0 \leq \lambda \leq 4$. Many of their results rely upon the fact that maps from the logistic family have negative Schwarzian derivatives. Relying primarily on Feigenbaum's renormalization operator and results from kneading theory, we extend many of their results to inverse limits with a single bonding map from a Full family where a negative Schwarzian derivative is not assumed.

Apart from the logistic family, we are aware of only one other work [21] along these lines involving Full families. In fact, in [21], a negative Schwarzian derivative is still assumed and the only result involving Full families is the existence of a parameter value where the corresponding inverse limit is a topological ray limiting on the union of two copies of the Brower-Janiszewski-Knaster intersecting in a common endpoint.

Inverse Limits and Full Families

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Jonathon E. Fassett, Author

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INVERSE LIMITS AND FULL FAMILIES

# 1 Introduction

Inverse limits, besides being of intrinsic interest to topologists, can often be used to represent attractors of dynamical systems. For example, the inverse limit space with a single full unimodal bonding map is homeomorphic to the attracting set of Smale's horseshoe. Williams [36] and Block [8] were the first to address the relationship between inverse limits and attractors and many others have since followed (see, for example, [19], [34], [3],and [20]). These efforts have generated an increasing interest in the topological properties of inverse limit spaces with unimodal bonding maps.

Barge and Martin [1] were the first to show there is a strong relationship between the dynamics of the bonding map and the topology of the corresponding inverse limit. Holte [19] utilized kneading theory to show that two unimodal bonding maps with the same finite kneading sequence produce homeomorphic inverse limits. Work by Barge and Martin [1] showed that inverse limits corresponding to unimodal maps with finite kneading sequences of different lengths are not homeomorphic due to the fact that they have a different number of endpoints. Only recently has it been shown that two inverse limits with bonding maps having different kneading sequences of the same finite length are not homeomorphic [30] (see also [6] and [23]).

Many of these results have been concentrated on the *tent family,*

$$T_\lambda = \begin{cases} \lambda x & \text{if } 0 \le x \le \frac{1}{2} \\ \lambda(1-x) & \text{if } \frac{1}{2} \le x \le 1. \end{cases}, 0 \le \lambda \le 2$$

or the *logistic family*

$$f_\lambda(x) = \lambda x(1-x), 0 \le \lambda \le 4.$$

This is of no surprise as these two families are the most investigated and well understood examples of one-parameter families of interval maps. An important difference between the tent family and logistic family is that only the latter is an example of a *Full family* (see §3.5).

Our interest in Full families is two fold. First, Full families are the context in which Feigenbaum's celebrated Universality Theory was developed. Secondly, apart from the logistic family, there is little in the literature concerning the behavior of the corresponding inverse limit spaces as the parameter is varied.

We will see that in a Full family of $C^1$unimodal maps there exists a convergent sequence of parameter values

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \lambda_\infty$$

such that the critical point corresponding to $\lambda_n$ is periodic of period $2^n$. This is generally now referred to as the *period doubling route to chaos.* By investigating the logistic family, Feigenbaum [16], equipped with only a pocket calculator, made a remarkable discovery:

$$\lim_{n\to\infty} \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+2}-\lambda_{n+1}} = \delta = 4.6692106\cdots$$

and is identical for all such systems undergoing this period doubling. Feigenbaum went on to propose an explanation for the universality of $\delta$ which was inspired by the renormalization group theory in statistical mechanics. The main results of this work deal with Feigenbaum's Universality Theory in the topological setting of inverse limits.

As the dynamics of the logistic family are so well documented, it is only natural (and inevitable) that the behavior of the inverse limits as the parameter is varied be investigated. It was Barge and Ingram [4] who, relying on the fact that maps from the logistic family have negative Schwarzian derivative, revealed a number of striking features occur among the corresponding inverse limits. Using kneading theory and the renormalization operator introduced by Feigenbaum, we generalizes many of their results to Full families where a negative Schwarzian derivative is not assumed.

Denoting the inverse limit with unimodal bonding map $f$ by $\varprojlim(I,f)$ our main results can be summarized as follows: For parameter value $\lambda$ such that $f_\lambda$ has kneading sequence below $f_{\lambda_\infty}$, $\varprojlim(I,f_\lambda)$ is hereditarily decomposable with topological $\sin(\frac{1}{x})$-curves being the dominate subcontinua (Theorem 4.1). At $\lambda_\infty$, with a milder smoothness condition than a negative Schwarzian derivative, we show that $\varprojlim(I, f_{\lambda_\infty})$ is hereditarily decomposable and contains only three topologically different subcontinua (Theorem 4.6 and Theorem 4.7). For parameter value $\lambda$ such that $f_\lambda$ has kneading sequence above $f_{\lambda_\infty}$, we show $\varprojlim(I, f_\lambda)$ contains an indecomposable subcontinuum (Theorem 4.4). As a new result even for the logistic family we show the following: For each finite maximal sequence $AC$ there exists a sequence of parameter values

$$\lambda_\infty < \cdots < \mu_2 < \mu_1 < \mu_0$$

such that $f_{\mu_0}$ has kneading sequence $AC$ and $\varprojlim(I, f_{\mu_{n+1}})$ is a ray limiting on two homeomorphic copies of $\varprojlim(I, f_{\mu_n})$ intersecting in a common endpoint (Theorem 4.2).

In developing the necessary tools to prove the above results, we also obtain results applicable to inverse limits with unimodal bonding maps in general. In Corollary 3.3, we show that two unimodal maps with the same periodic kneading sequence produce homeomorphic inverse limit spaces if and only if the cardinality of the sets of accumulation points of forward orbit of the critical points are equal. In Theorem 3.9 we show that a decomposable core can always be decomposed as the union of two homeomorphic subcontinua.

# 2 Background

## 2.1 Dynamical Systems

We begin with a brief and elementary introduction to needed terminology and results in discrete dynamical systems. For a more detailed discussion the reader is directed to [12], [9],.

If $f : X \to X$ is a mapping of a topological space $X$, the *n-iterate* of $f$ is defined inductively by

$$\begin{aligned} f^0 &= id_X \\ f^1 &= f \\ f^{n+1} &= f^n \circ f,\, n \geq 1 \end{aligned}$$

If $f$ is a homeomorphism then, for $n \geq 0$, we can define $f^{-n} = (f^{-1})^n$. If $f$ is not invertible we define $f^{-n}(y) = \{x : f^n(x) = y\}$. The *forward orbit* of a point $x \in X$ is the set $\mathcal{O}_f^+(x) = \{f^n(x) : n \geq 0\}$ and the *backward orbit* is the set $\mathcal{O}_f^-(x) = \{f^{-n}(x) : n \geq 0\}$. The *orbit* of $x$ is $\mathcal{O}_f(x) = \{f^n(x) : n \in \mathbb{Z}\} = \mathcal{O}_f^-(x) \cup \mathcal{O}_f^+(x)$. A point $x$ is called a *periodic point of period* $n$ if $f^n(x) = x$ and $f^k(x) \neq x$ for $1 \leq k < n$. If $x$ has period one then it is called a *fixed point.* If $x$ is a periodic point of period $n$ , then the forward orbit of $x$, $\mathcal{O}_f^+(x) = \{x, f(x), f^2(x), \cdots, f^{n-1}(x)\}$, is called a *periodic orbit.* For

$C^1$ maps $f : \mathbb{R} \to \mathbb{R}$ with periodic point $p$ of period $n$, we make the following classification:

$$\begin{aligned} p \text{ is stable if } |(f^n)'(p)| &< 1, \\ p \text{ is unstable if } |(f^n)'(p)| &> 1, \\ p \text{ is neutral if } |(f^n)'(p)| &= 1, \end{aligned}$$

Notice that $|(f^n)'(p)| = \prod_{k=0}^{n-1} f'(f^k(p))$ so the stability of a periodic point is determined by the product of the derivatives along its orbit.

**Theorem 2.1** *[12, Proposition 4.4] Suppose p is a stable periodic point of period n. Then there exist an open interval U containing p such that, for all* $x \in U$, $\lim_{k\to\infty} f^{nk}(x) = p$.

**Proof.** Let $g = f^n$. Since $|g'(p)| < 1$ there exist an $\varepsilon > 0$ and a $0 < \lambda < 1$ such that $|g'(p)| < \lambda$ for all $x \in (p-\varepsilon, p+\varepsilon)$. By the Mean Value Theorem, for $x \in (p-\varepsilon, p+\varepsilon)$, there exist a $y$ between $x$ and $p$ such that

$$|g(x) - p| = |g'(y)||x - p| \le \lambda|x - p|.$$

An induction argument gives $|g^k(x) - p| \le \lambda^k|x - p|$ for $k \ge 0$. It follows that $f^{nk}(x) = g^k(x) \to p$ as $k \to \infty$. ■

**Theorem 2.2** *[12, Proposition 4.6] Suppose $p$ is an unstable periodic point of period $n$. Then there exist an open interval $U$ containing $p$ such that, for all $x \in U - \{p\}$, there exists a $k > 0$ such that $f^{nk}(x) \notin U$.*

**Proof.** Let $g = f^n$. Since $|g'(p)| > 1$ there exist an $\varepsilon > 0$ and a $\lambda > 1$ such that $|g'(p)| > \lambda$ for all $x \in (p - \varepsilon, p + \varepsilon)$. By the Mean Value Theorem there exist a $y$ between $x$ and $p$ $(x \neq p)$ such that

$$|g(x) - p| = |g'(y)||x - p| \geq \lambda|x - p| > |x - p|.$$

If $|g(x) - p| \geq \varepsilon$ we take $k = 1$. If $|g(x) - p| < \varepsilon$ we apply the mean value theorem again and find

$$|g^2(x) - p| \geq \lambda^2|x - p| > |x - p|.$$

If $|g^2(x) - p| \geq \varepsilon$ we take $k = 2$. Repeating the argument if necessary, we eventually find a $k > 0$ such that $f^{nk}(x) = g^k(x) \notin (p - \varepsilon, p + \varepsilon)$. ∎

If $X$ is a metric space and $f : X \to X$, then the *$\omega$-limit set* of the orbit of $x$ is the set

$$\omega(x) = \{y \in X : \exists \text{ a sequence } n_k \to \infty \text{ with } f^{n_k}(x) \to y\}.$$

If $x$ lies in a periodic orbit then $\mathcal{O}_f^+(x) = \omega(x)$.

Two maps $f : X \to X$ and $g : Y \to Y$ are *topologically conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. The homeomorphism $h$ is called a *topological conjugacy.* Note that $h \circ f = g \circ h$ implies $h \circ f^n = g^n \circ h$. Thus, $f^n$ and $g^n$ are also topologically conjugate. An important and well known feature of topologically conjugate maps is that they have the same dynamics. For our purposes we are mainly concerned with periodic points and limit sets in a metric space setting.

**Theorem 2.3** *Let $X$ and $Y$ be metric spaces with, $f : X \to X$, $g : Y \to Y$, and $h : X \to Y$ a topological conjugacy between $f$ and $g$. Then*

1. *$x$ is a periodic point of period $n$ for $f$ if and only if $h(x)$ is periodic point of period $n$ for $g$.*

2. *a sequence $\{x_n\}_{n \geq 1}$in $X$ converges to $x$ in $X$ if and only if the sequence $\{h(x_n)\}_{i \geq 1}$ in $Y$ converges to $h(x)$ in $Y$.*

**Proof.** (1) If $x$ is a periodic point of period $n$ for $f$, then $g^n(h(x)) = h(x)$ and $g^i(h(x)) = h(f^i(x)) \neq h(x)$ for all $0 \leq i < n$. Thus $h(x)$ is periodic of period $n$. Since $f \circ h^{-1} = h^{-1} \circ g$, we can use a similar argument to show $h^{-1}(y)$ is periodic of period $n$ if $y$ is periodic of period $n$. (2) is clear. ■

## 2.2 Continuum Theory

We present here only the concepts necessary for the discussion following. The reader is directed to [29] for more details. We begin with several definitions.

A *continuum* is a nonempty, compact, connected metric space. A *subcontinuum* is a continuum which is a subset of a space. When a continuum consists of more than one point it is called *nondegenerate.* The most well known nondegenerate continuum is an *arc*: any space homeomorphic to $I = [0, 1]$. A continuum is called *decomposable* if it can be expressed as the union of two proper subcontinua; otherwise it is *indecomposable.* An arc is clearly decomposable. A point $x$ in a continuum $X$ is an *endpoint* of $X$ if for subcontinua $H$ and $K$ both containing $x$ then $H$ is a subset of $K$ or $K$ is a subset of $H$.

For those new to continua theory, it may appear that all continua except one-point continua are decomposable. However, in a certain sense most continua are indecomposable [29]. Here are two important examples.

**Example 1** *The Brouwer-Janiszewski-Knaster (B-J-K) Continuum: Let $K$ be the classical middle-thirds Cantor set in $I = [0, 1]$. Let $X_0$ be the union of all semicircles in the upper half-plane with endpoints in $K$ such that the endpoints are symmetric with the line $x = \frac{1}{2}$. For $i = 1, 2, \cdots$, let $X_i$ be the union of all semicircles in the lower half-plane with endpoints in $K$ such that the endpoints symmetric with respect to the line $x = \frac{5}{2(3^i)}$. See Figure 1. Let $X = \bigcup_{i=0}^{\infty} X_i$. It can be shown that $X$ is indecomposable [25].*

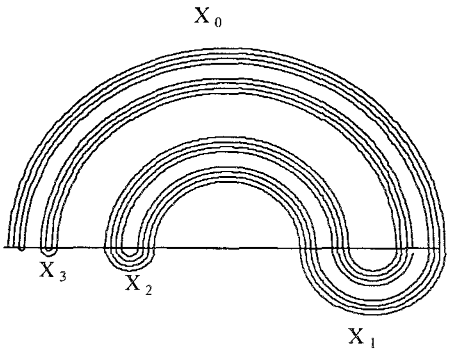


Figure 1: Contruction of the Brower-Janiszkowski-Knaster Continuum

**Example 2** *Three endpoint indecomposable continuum: Let $a, b$, and $c$ be three points in the plane. Construct simple chains $\mathcal{C}_1, \mathcal{C}_2, \cdots$ of open disks such that the sets in $\mathcal{C}_n$ have diameter less than $\frac{1}{n}$ and have closures contained in sets of $\mathcal{C}_{n-1}$, in the following manner: $\mathcal{C}_1$ is a simple chain from $a$ to $c$ passing through $b$, $\mathcal{C}_2$ is a simple chain from $b$ to $c$ passing through $a$, and $\mathcal{C}_3$ is a simple chain from $a$ to $b$ passing through $c$. See Figure 2. Now repeat this process. In general, $\mathcal{C}_{3n+1}$ is a simple chain from $a$ to $c$ passing through $b$ contained in $\mathcal{C}_{3n}$, $\mathcal{C}_{3n+2}$ is a simple chain from $b$ to $c$ passing through $a$ contained in $\mathcal{C}_{3n+1}$, and $\mathcal{C}_{3n+2}$ is a simple chain from $b$ to $c$ passing through $a$ contained in $\mathcal{C}_{3n+2}$ The intersection $\bigcap_{n=1}^{\infty}(\bigcup_{n=1}^{\infty} \mathcal{C}_n)$ is the desired continuum with endpoints $a, b$, and $c$.*

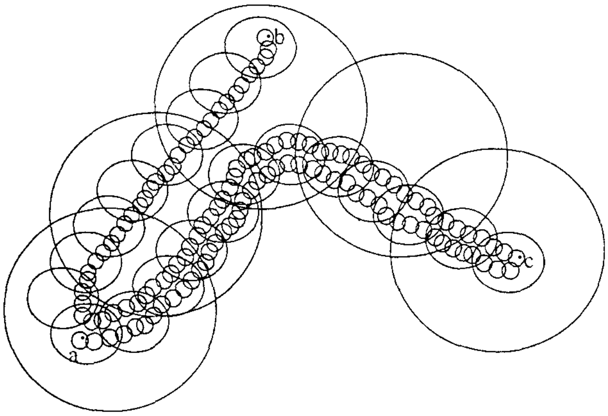


Figure 2: Construction of the Three-Endpoint Continuum

## 2.3 Inverse Limits

Let $X_0, X_1, \cdots$ be a sequence of topological spaces. Suppose for each $n \geq 0$ there exists a continuous function $f_n : X_{n+1} \to X_n$.

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} \cdots \leftarrow X_n \xleftarrow{f_n} X_{n+1} \leftarrow \cdots$$

The sequence of spaces and functions $\{X_n, f_n\}$ is called an *inverse sequence.*

The *inverse limit* of the inverse sequence $\{X_n, f_n\}$, denoted by $\varprojlim(X_n, f_n)$, is the subspace of $\prod_{n=0}^{\infty} X_n$ defined by

$$\varprojlim(X_n, f_n) = \{\underline{x} = (x_0, x_1, \cdots) \in \prod_{n=0}^{\infty} X_n : f_n(x_{n+1}) = x_n \text{ for all } n \geq 0\}.$$

The spaces $X_n$ are called *factor spaces* and the functions $f_n$ are called *bonding maps.* Our use of inverse limit theory will often be in the setting where, for each $n \geq 0$, $X_n$ is a specific metric space $X$ and, for each $n \geq 0$, $f_n$ is a specific continuous function $f$. In this case, we will denote the inverse sequence by $\{X, f\}$ and the inverse limit by $\varprojlim(X, f)$. For a more comprehensive treatment of inverse limits the reader is referred to [15] or [13].

**Example 3** *Suppose* $\{X_n, f_n\}$ *is an inverse sequence with* $X_0 \supset X_1 \supset \cdots$ *and* $f_n : X_{n+1} \to X_n$ *is the injection map. Then* $\varprojlim(X_n, f_n)$ *is homeomorphic to* $\bigcap_{n=0}^{\infty} X_n$. *The homeomorphism* $f : \bigcap_{n=0}^{\infty} X_n \to \varprojlim(X_n, f_n)$ *is given by* $f(x) = (x, x, \cdots)$.

It is clear from this example that the inverse limit may be empty even though the bonding maps are injections. To avoid this situation we have the following theorem.

**Theorem 2.4** *[35, Theorem 29.11] If* $\{X_n, f_n\}$ *is an inverse sequence of compact (nonempty) Hausdorff spaces, then the inverse limit* $\varprojlim(X_n, f_n)$ *is a compact (nonempty) Hausdorff space.*

The following well known theorem justifies our inclusion of continuum theory in the background information.

**Theorem 2.5** *[29, Theorem 2.4] If* $\{X_n, f_n\}$ *is an inverse sequence of continua, then the inverse limit* $\varprojlim(X_n, f_n)$ *is a continuum.*

**Theorem 2.6** *[22, Theorem 6.1] Suppose $\{X_n, f_n\}$ is an inverse sequence of continua. If for each $n$, $K_n$ is a subcontinuum of $X_n$ and $f_n(K_{n+1}) = K_n$, then $\varprojlim(K_n, f_n|K_{n+1})$ is a subcontinuum of $\varprojlim(X_n, f_n)$.*

**Proof.** Follows directly from Theorem 2.5 ■

For $n = 0, 1, \cdots$ let $\pi_n : \varprojlim(X_n, f_n) \to X_n$ denote the $n$th projection map. Suppose $K$ is a subcontinuum of a continuum $\varprojlim(X_n, f_n)$. Then $K_n = \pi_n(K)$ is a continuum ($\pi_n$ is continuous) and $f_n(K_{n+1}) = K_n$. We have therefore proved the following result that every subcontinuum is the inverse limit of its projections.

**Theorem 2.7** *If $K$ is a subcontinuum of an inverse limit of an inverse sequence $\{X_n, f_n\}$ of continua, then $K = \varprojlim(K_n, f_n|K_{n+1})$, where $K_n = \pi_n(K)$.*

We will later see that complicated spaces can be formed using inverse limits where the factor spaces and bonding maps are quite simple. As the next theorem points out, if all the factor spaces are homeomorphic to a given space $X$, then the inverse limit and $X$ might be homeomorphic.

**Theorem 2.8** *[32, Proposition 20] Suppose $\{X_n, f_n\}$ is an inverse sequence. If each $X_n$ is homeomorphic to a given space $X$ and each bonding map $f_n$ is a homeomorphism, then $\varprojlim(X_n, f_n)$ is homeomorphic to $X$.*

**Proof.** It suffices to show $\pi_0 : \varprojlim(X_n, f_n) \to X_0$ is a homeomorphism. Continuity of $\pi_0$ is clear. The bijectivity of $\pi_0$ follows from the assumption that each $f_n$ is a homeomorphism. It remains to show that $\pi_0$ is an open map (this is not true in general). Let $U \subseteq \varprojlim(X_n, f_n)$ be a basic open set. Then $U = \pi_n^{-1}(W)$ for some $n$ and some open set $W \subseteq X_n$ [10, Theorem 6.B.8]. Since each $f_n$ is a homeomorphism, $\pi_0 \circ \pi_n^{-1}(W) = f_0 \circ f_1 \circ \cdots \circ f_{n-1}(W)$ which is open in $X_0$. Thus, $\pi_0$ is an open map. ■

Given two inverse sequences $\{X_n, f_n\}$ and $\{Y_n, g_n\}$, there is a natural way to map $\{X_n, f_n\}$ to $\{Y_n, g_n\}$. For each $n \geq 1$ suppose $h_n : X_n \to Y_n$ is a continuous function such that

$$h_n \circ f_n = g_n \circ h_{n+1}.$$

The collection of functions $\{h_n\}$ induces a function

$$\widehat{h} : \varprojlim(X_n, f_n) \to \varprojlim(Y_n, g_n)$$

in the following manner. For each $\underline{x} = (x_0, x_1, \cdots)$ in $\varprojlim(X_n, f_n)$, define

$$\widehat{h}(\underline{x}) = (h_0(x_0), h_1(x_1), \cdots).$$

This can all be summarized in the following *commutative diagram.*

$$
\begin{tikzcd}
X_0 \arrow[d, "h_0"] & X_1 \arrow[l, "f_0"'] \arrow[d, "h_1"] & \cdots \arrow[l, "f_1"'] & X_n \arrow[l] \arrow[d, "h_n"] & X_{n+1} \arrow[l, "f_n"'] \arrow[d, "h_{n+1}"] & \cdots \arrow[l] & \varprojlim(X_n, f_n) \arrow[d, "\widehat{h}"] \\
Y_0 & Y_1 \arrow[l, "g_0"'] & \cdots \arrow[l, "g_1"'] & Y_n \arrow[l] & Y_{n+1} \arrow[l, "g_n"'] & \cdots \arrow[l] & \varprojlim(Y_n, g_n)
\end{tikzcd}
$$

The equations

$$g_n(h_{n+1}(x_{n+1})) = h_n(f_n(x_{n+1})) = h_n(x_n)$$

guarantee that $\widehat{h}(\underline{x})$ is an element of $\varprojlim(Y_n, g_n)$.

**Theorem 2.9** *[35, Theorem 29.13] Let $\{X_n, f_n\}$ and $\{Y_n, g_n\}$ be inverse sequences. If $h_n : X_n \longrightarrow Y_n$ is a continuous function satisfying $h_n \circ f_n = g_n \circ h_{n+1}$ for all $n \geq 0$, then the induced function $\widehat{h} : \varprojlim(X_n, f_n) \longrightarrow \varprojlim(Y_n, g_n)$ is continuous. Moreover, if each $h_n$ is a homeomorphism, then $\widehat{h}$ is a homeomorphism.*

**Proof.** Suppose each $h_n$ is continuous. Let $\pi_n$ and $\pi'_n$ denote the $n$th projection in $\prod_{n=0}^{\infty} X_n$ and $\prod_{n=0}^{\infty} Y_n$, respectively. We then have

$$\pi' \circ \widehat{h}(x_0, x_1, \cdots) = h_n(x_n) = h_n \circ \pi_n(x_0, x_1, \cdots).$$

Thus $\pi' \circ \widehat{h}$ is continuous from which it follows that $\widehat{h}$ is continuous. Now suppose that each $h_n$ is a homeomorphism. For each $\underline{y} = (y_0, y_1, \cdots) \in \varprojlim(Y_n, g_n)$, let

$$\underline{x} = (x_0, x_1, \cdots) = (h_0^{-1}(y_0), h_1^{-1}(y_1), \cdots).$$

From the equation

$$f_n(x_{n+1}) = f_n(h_{n+1}^{-1}(y_{n+1})) = h_n^{-1}(g_n(y_{n+1})) = h_n^{-1}(y_n) = x_n$$

we conclude that $\underline{x} \in \varprojlim(X_n, f_n)$. Clearly $\widehat{h}(\underline{x}) = \underline{y}$. Thus, $\widehat{h}$ is surjective. The injectivity of $\widehat{h}$ follows from the injectivity of each $h_n$. Since $h_n^{-1} \circ g_n = f_n \circ h_{n+1}^{-1}$ for all $n \geq 0$, $\widehat{h}^{-1} : \varprojlim(Y_n, g_n) \to \varprojlim(X_n, f_n)$ given by $\widehat{h}^{-1}(\underline{x}) = (h_0^{-1}(x_0), h_1^{-1}(x_1), \cdots)$ is also a continuous bijection. Furthermore, $\widehat{h}^{-1} \circ \widehat{h}(\underline{x}) = \underline{x}$ and $\widehat{h} \circ \widehat{h}^{-1}(\underline{y}) = \underline{y}$. Therefore, $\widehat{h}$ is a homeomorphism. ■

**Corollary 2.1** *Suppose $f : X \to X$ and $g : Y \to Y$ are topologically conjugate. Then $\varprojlim(X, f)$ is homeomorphic to $\varprojlim(Y, g)$.*

Now consider the case where $X_n = Y_n = X$ and $f_n = g_n = f$ for all $n \geq 0$, As suggested by the following diagram, the previous theorem states that the induced function $\widehat{f}$ is continuous.

$$\begin{array}{ccccc}
X & \xleftarrow{f} & X & \xleftarrow{f} \;\cdots & \varprojlim(X, f) \\
\downarrow f & & \downarrow f & & \downarrow \widehat{f} \\
X & \xleftarrow{f} & X & \xleftarrow{f} \;\cdots & \varprojlim(X, f)
\end{array}$$

We note that for $\underline{x} = (x_0, x_1, \cdots)$, $\widehat{f}(\underline{x}) = (f(x_0), x_0, x_1, \dots)$. In fact, we have the following well known and important feature of inverse limits.

**Theorem 2.10** *Let $\{X, f\}$ be an inverse sequence. Then the induced function $\widehat{f}$ is a homeomorphism.*

**Proof.** It follows from Theorem 2.8 that $\widehat{f}$ is continuous. Let $\underline{x} = (x_0, x_1, \cdots) \in \varprojlim(X, f)$. Then $\underline{x}' = (x_1, x_2, \cdots) \in \varprojlim(X, f)$ and $\widehat{f}(\underline{x}') = \underline{x}$ so that $\widehat{f}$ is surjective. If $\widehat{f}(\underline{x}) = (f(x_0), x_0, x_1, \cdots) = (f(y_0), y_0, y_1, \cdots) = \widehat{f}(\underline{y})$, then $\underline{x} = \underline{y}$ so that $\widehat{f}$ is injective. Clearly $\widehat{f}^{-1}(\underline{x}) = (x_1, x_2, \cdots)$ is continuous. ■

Suppose $J$ and $K$ are two closed intervals with continuous functions $f : J \to J$ and $g : K \to K$. The following theorems are well known and will prove useful in what follows.

**Theorem 2.11** *$\varprojlim(J, f)$ is homeomorphic to $\varprojlim(J, f^n)$ for all $n > 1$, where $f^n = \underbrace{f \circ \ldots \circ f}_{n \text{ - } times}$.*

**Proof.** Let $n > 1$. Define $h : \varprojlim(J, f) \to \varprojlim(J, f^n)$ by $h(\underline{x}) = (x_0, x_n, x_{2n}, \cdots)$. It is clear that $h$ is continuous and bijective. Since $\varprojlim(J, f)$ and $\varprojlim(J, f^n)$ are compact metric spaces, $h$ is a homeomorphism [35, Theorem 17.14]. ■

The next theorem is actually a corollary to Theorem 2.8

**Theorem 2.12** *If $f$ is a homeomorphism, then $\varprojlim(J, f)$ is an arc.*

**Theorem 2.13** $\varprojlim(J, f) = \varprojlim(J', f)$, *where* $J' = \bigcap_{n=0}^{\infty} f^n(J)$.

**Proof.** Note first that $f_{|J'} : J' \twoheadrightarrow J'$. Clearly $\varprojlim(J', f) \subseteq \varprojlim(J, f)$. Let $(x_0, x_1, \cdots) \in \varprojlim(J, f)$. It follows that $x_k \in J' = \bigcap_{n=0}^{\infty} f^n(J)$ for all $k \geq 0$. Thus, $(x_0, x_1, \cdots) \in \varprojlim(J', f)$ and $\varprojlim(J, f) \subseteq \varprojlim(J', f)$. Therefore, $\varprojlim(J, f) = \varprojlim(J', f)$. ■

## 2.4 Unimodal Maps and Kneading Theory

A continuous function $f : I \to I$ is called a *unimodal map* if $f(0) = f(1) = 0$ and there exist a $c \in (0, 1)$ such that $f$ is strictly increasing on $[0, c]$ and strictly decreasing on $[c, 1]$. Obviously, we could use any closed interval $[a, b]$ in place of $I$. Let $x \in I$ and define the *itinerary of* $x$ by

$$I(x) = a_0 a_1 \ldots \text{ where } a_i = \begin{cases} L & \text{if } f^i(x) < c \\ C & \text{if } f^i(x) = c \\ R & \text{if } f^i(x) > c \end{cases}$$

with the convention that $I(x)$ is of finite length if $f^i(x) = c$ for some $i$. Notice that a unimodal map $f$ induces a shift map $\sigma$ on sequences by $I(f(x)) = \sigma(I(x))$, where $\sigma(a_0 a_1 \cdots) = a_1 a_2 \cdots$ (if $I(f(x)) = C$, $\sigma(I(x))$ is undefined). If we define an order on the symbols $L, R$, and $C$ by $L < C < R$, then it can be extended to an order on sequences as follows: If $A = a_0 a_1 \ldots$ and $B = b_0 b_1 \ldots$ are two different finite or infinite sequences, there is a smallest integer $i$ with $a_i \neq b_i$.

We call a finite sequence *odd* if it contains an odd number of $R$'s and *even* otherwise. We then define $A \prec B$ if, and only if, either $a_i < b_i$ and $a_0a_1 \ldots a_{i-1}$ is even or $a_i > b_i$ and $a_0a_1 \ldots a_{i-1}$ is odd.

**Theorem 2.14** *[11, Lemma II.1.2 and Lemma II.1.3] Let $f$ be unimodal with $x, y \in I$.*

1. *If $I(x) \prec I(y)$, then $x < y$.*

2. *If $x < y$, then $I(x) \preceq I(y)$.*

An itinerary $I(x)$ is said to be *maximal* if $\sigma^n(I(x)) \preceq I(x)$ for all $n \geq 0$ for which $\sigma^n(I(x))$ is defined. The fact that $I(f(c))$ is maximal and $\sigma^n(I(x)) \preceq I(f(c))$ for all $x \in I$ and all $n \geq 0$ leads to the following definition, which is a modification of the original ideas of Milnor and Thurston [28]. The *kneading sequence* of a unimodal map $f$ with critical point $c$, denoted $k(f)$, is defined as $k(f) = I(f(c))$. A sequence $a_0a_1 \ldots$ is *admissible* if it is infinite and contains only $L$'s and $R$'s or is a finite sequence of $L$'s and $R$'s ending with a $C$. A natural question is whether it is possible to identify which admissible sequences are realized as the itinerary of some $x \in I$. The answer is provided by the following theorem.

**Theorem 2.15** *[11, Theorem II.3.8] Let $f$ be unimodal and assume $A$ is an admissible sequence satisfying*

*(i)* *If $k(f)$ is infinite then $\sigma^n(A) \prec k(f)$ for all $n \geq 0$.*

*(ii)* *If $k(f) = DC$ is finite with $D$ even then $\sigma^n(A) \prec (DL)^\infty$ for all $n \geq 0$.*

*(iii)* *If $k(f) = DC$ is finite with $D$ odd then $\sigma^n(A) \prec (DR)^\infty$ for all $n \geq 0$.*

*Then there exists an $x \in I$ such that $I(x) = A$.*

Theorem 2.15 shows that the map $x \longrightarrow I(x)$ is in this sense surjective. However, two different points may have the same itinerary, as is implied by Theorem 2.14.

Let $\widehat{L} = R$, $\widehat{R} = L$, and $\widehat{C} = C$. For a finite sequence $A$ of $L$'s and $R$'s and admissible sequence $B = b_0 b_1 \cdots$ we define the $*$-operator as follows:

1. If $A$ is even and $B$ is infinite, $A * B = Ab_0Ab_1 \cdots$.

2. If $A$ is even and $B = b_0 b_1 \cdots b_{n-1}C$, $A * B = Ab_0Ab_1 \cdots Ab_{n-1}AC$.

3. If $A$ is odd and $B$ is infinite, $A * B = A\widehat{b_0}A\widehat{b_1} \cdots$.

4. If $A$ is odd and $B = b_0 b_1 \cdots b_{n-1}C$, $A * B = A\widehat{b_0}A\widehat{b_1} \cdots A\widehat{b_{n-1}}AC$.

**Lemma 2.16** *[11, Lemma II.2.6] If $A_1C$ and $A_2C$ and $B$ is admissible, then $A_1 * (A_2 * B) = A * B$, where $A = A_1 * A_2C$.*

As a result of this lemma we see there is no ambiguity in using the notation $(A*)^n B$.

**Theorem 2.17** *[11, Corollary II.2.4] If $AC$ and $B$ are maximal, then $A * B$ is maximal.*

**Theorem 2.18** *[11, Theorem II.2.5] If $AC$ is maximal then the map $A* : B \mapsto A * B$ is order preserving from the set of maximal itineraries to itself.*

**Proposition 2.1** *Let $B$ be infinite and admissible. If $R * B = B$ then $B$ is nonperiodic.*

**Proof.** Suppose $R*B = B$ and $B = (b_0 b_1 \cdots b_n)^\infty$. Then $(R\widehat{b}_0 R\widehat{b}_1 \cdots R\widehat{b}_n)^\infty = (b_0 b_1 \cdots b_n)^\infty$ so that $B = (RL)^\infty$. We arrive at a contradiction since $R * (RL)^\infty \neq (RL)^\infty$. Therefore, $B$ is nonperiodic. ■

The $*$-operator will be important when we consider the renormalization operator $\mathfrak{R}$ in §3.4.

## 2.5 Schwarzian Derivative

In this section we discuss an important tool introduced by Singer [33] in 1978 into the study of one-dimensional dynamical systems. In 1918, Julia asked the question of how many stable periodic orbits a unimodal map can posses. Julia was able to show for certain unimodal maps that are the restriction of analytic functions, there can be at most one stable periodic orbit. However, Singer's introduction of the negative Schwarzian provided the real breakthrough.

The *Schwarzian derivative* of a mapping $f$ is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

For a unimodal map $f$ with critical point $c$ we say $f$ has *negative Schwarzian derivative* denoted by $Sf < 0$ if $Sf(x) < 0$ for all $x \in I - \{c\}$. We will call the class of $C^3$ unimodal maps with negative Schwarzian derivative $S$-*unimodal.* This class includes the *logistic family* $f_\lambda(x) = \lambda x(1-x)$ for $0 < \lambda \leq 4$ and $g_\lambda(x) = \lambda \sin(\pi x)$ for $0 < \lambda < 1$. Singer was able to show that a unimodal map $f$ with negative Schwarzian derivative has at most one stable periodic orbit. One reason for the utility of the Schwarzian derivative is the following fundamental property:

$$S(f \circ g)(x) = Sf(g(x))(g'(x))^2 + Sg(x)$$

which can be verified by direct calculation and the chain rule.

**Lemma 2.19** *[33] If $Sf < 0$, then $Sf^n < 0$ for all $n \geq 1$.*

**Proof.** $Sf^n(x) = \sum_{i=o}^{n-1} Sf(f^i(x))[(f^i)'(x)]^2$. ■

Now suppose $f$ and $g$ are topologically conjugate unimodal maps with critical points $c$ and $c'$, respectively. Note that $h(c) = c'$. Since $h$ must be order preserving, $x < c$ if and only if $h(x) < c'$. This results in $I(x) = I(h(x))$.

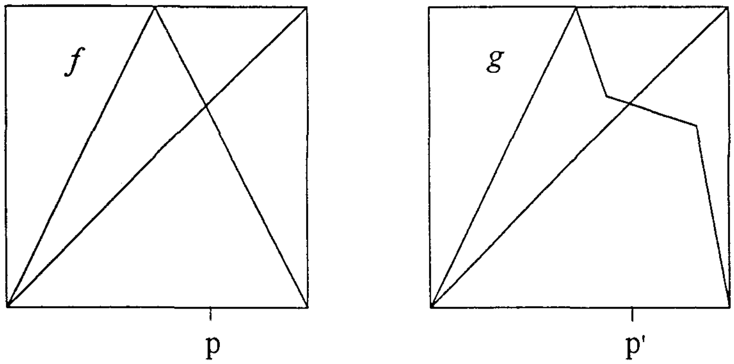


Figure 3: Non-conjugate unimodal maps with $k(f) = k(g)$

Therefore $k(f) = k(g)$. We see then that topologically conjugate unimodal maps have the same kneading sequence.

The converse is not true, however, as the following example shows.

**Example 4** *Figure 3 shows the two unimodal maps $f$ and $g$ with fixed points $p$ and $p'$, respectively. Both $f$ and $g$ are chosen (drawn) so that $k(f) = k(g) = RL^{\infty}$. However, $p$ is unstable ($|f'(p)| > 1$) while $p'$ is stable ($|g'(p')| < 1$). Thus, $f$ and $g$ cannot be topologically conjugate. However, $\varprojlim(I, f)$ is homeomorphic to $\varprojlim(I, g)$ (See §3.1).*

If we limit our consideration to $S$-unimodal maps, then the kneading sequence is very close to being a complete invariant of its topological conjugacy class. The theorem below is due to Guckenheimer [17], as stated in [11], and shows that $k(f)$ determines the topological conjugacy classes except for one case in which information about the stable periodic orbits is needed.

**Theorem 2.20** *[11, Theorem II.6.3] Let $f$ and $g$ be $S$-unimodal maps with $k(f) = k(g) = \alpha$.*

*1. If $\alpha$ is finite then $f$ and $g$ are topologically conjugate.*

*2. If $\alpha$ is infinite and periodic of period $n$ ($\alpha = A^\infty$, with $|A| = n$), then there are two possibilities:*

  **a.** *If $A$ is odd, then $f$ and $g$ are topologically conjugate if and only if their stable periodic orbits have the same period ($n$ or $2n$).*

  **b.** *If $A$ is even, then $f$ and $g$ are topologically conjugate if and only if their stable periodic orbits (of period $n$) are both stable from one side or stable from both sides.*

*3. If $\alpha$ is infinite and nonperiodic, then $f$ and $g$ are topologically conjugate.*

We will be mainly interested in the third claim of the statement when we consider infinitely renormalizable unimodal maps.

# 3 Inverse Limits with Unimodal Bonding Maps

## 3.1 Initial Results

**Lemma 3.1** *Let $L : I \to I$ be an order preserving homeomorphism with $L(f(c)) = g(c')$. Then there exists an order preserving homeomorphism $M : I \to I$ such that $L \circ f = g \circ M$.*

**Proof.** Let $g_0 = g_{|[0,c']}$ and $g_1 = g_{|[c',1]}$. and define

$$M(x) = \begin{cases} (g_0^{-1} \circ L \circ f)(x), & x \in [0, c] \\ (g_1^{-1} \circ L \circ f)(x), & x \in [c, 1] \end{cases}$$

Then $M : I \to I$ is a homeomorphism and

$$\begin{aligned} (g \circ M)(x) &= \begin{cases} g \circ (g_0^{-1} \circ L \circ f)(x), & x \in [0, c] \\ g \circ (g_1^{-1} \circ L \circ f)(x), & x \in [c, 1] \end{cases} \\ &= \begin{cases} (L \circ f)(x), & x \in [0, c] \\ (L \circ f)(x), & x \in [c, 1] \end{cases} \\ &= (L \circ f)(x). \end{aligned}$$

Since

$$M(0) = (g_0^{-1} \circ L \circ f)(0) = g_0^{-1} \circ L(0) = g_0^{-1}(0) = 0,$$

$M$ is order preserving. ■

**Theorem 3.2** *Let $f$ and $g$ be unimodal maps with critical points $c$ and $c'$, respectively. If there exists an order preserving homeomorphism $h : I \longrightarrow I$ such that $h(f^k(c)) = g^k(c')$ for all $k \geq 0$, then $\varprojlim(I, f)$ and $\varprojlim(I, g)$ are homeomorphic.*

**Proof.** Let $h_0 = h$. By the previous Lemma there exists a homeomorphism $h_1 : I \longrightarrow I$ such that the following diagram commutes

$$\begin{tikzcd} I \arrow[d, "h_0"] & I \arrow[l, "f"'] \arrow[d, "h_1"] \\ I & I \arrow[l, "g"] \end{tikzcd}$$

Since $h_1(f(c)) = g(c')$, we can again apply the previous Lemma to obtain a homeomorphism $h_2 : I \longrightarrow I$ and the commutative diagram

$$\begin{tikzcd} I \arrow[d, "h_0"] & I \arrow[l, "f"'] \arrow[d, "h_1"] & I \arrow[l, "f"'] \arrow[d, "h_2"] \\ I & I \arrow[l, "g"] & I \arrow[l, "g"] \end{tikzcd}$$

This process can be repeated indefinitely using the fact that $h(f^k(c)) = g^k(c')$ for all $k \geq 0$ to obtain homeomorphisms $h_0, h_1, h_2, \cdots$ and the commutative diagram

$$\begin{array}{ccccccccc}
I & \xleftarrow{f} & I & \leftarrow & \cdots & \leftarrow & I & \xleftarrow{f} & I & \leftarrow & \cdots \\
\downarrow h_0 & & \downarrow h_1 & & & & \downarrow h_{n-1} & & \downarrow h_n & & \\
I & \xleftarrow{g} & I & \leftarrow & \cdots & \leftarrow & I & \xleftarrow{g} & I & \leftarrow & \cdots
\end{array}$$

Theorem 2.9 shows that $\varprojlim(I, f)$ and $\varprojlim(I, g)$ are homeomorphic. ■

We note here that the above theorem also follows from the recent work of Barge and Diamond [6, Lemma 1.3].

Holte [19] has shown that if two unimodal maps $f$ and $g$ have the same finite kneading sequence then $\varprojlim(I, f)$ and $\varprojlim(I, g)$ are homeomorphic. As a corollary to the previous Theorem we obtain a different and easier proof of the same result.

**Corollary 3.1** *[19, Corollary 1] Let $f$ and $g$ be unimodal maps with the same finite kneading sequence. Then $\varprojlim(I, f)$ and $\varprojlim(I, g)$ are homeomorphic.*

**Proof.** Let $k(f) = k(g) = a_0 a_1 ... a_{n-1} C$. Then $I(f^i(c)) = I(g^i(c')) \neq I(g^j(c')) = I(f^j(c))$. It follows that the sets $\{c, f(c), f^2(c), \cdots, f^n(c)\}$ and $\{c', g(c'), g^2(c'), \cdots, g^n(c')\}$ have the same ordering in $I$. Therefore, there exist an order preserving homeomorphism $h : I \to I$ such that $h(f^i(c)) = g^i(c')$ for $i = 0, 1, \cdots, n$. It then follows from Theorem 3.2 that $\varprojlim(I, f)$ and $\varprojlim(I, g)$ are homeomorphic. ■

Using Theorem 3.2 we can obtain results when the orbit of the critical point is more complicated. First we need a couple of helpful tools.

**Theorem 3.3** *[11, Lemma II.3.2] If $f$ is unimodal and $I(x) = (a_0a_1 \cdots a_{n-1})^\infty$ then the sequence $\{f^i(x)\}_{i=0}^{\infty}$ converges to a periodic orbit of period $n$ provided $a_0a_1 \cdots a_{n-1}$ is even and period $n$ or $2n$ if $a_0a_1 \cdots a_{n-1}$ is odd.*

**Corollary 3.2** *Let $f$ be a unimodal map with critical point $c$ such that $k(f) = (a_1a_2 \cdots a_n)^\infty$. Then $|\omega_f(c)| = n$ if $a_1a_2 \cdots a_n$ is even and $|\omega_f(c)| = n$ or $2n$ if $a_1a_2 \cdots a_n$ is odd.*

**Lemma 3.4** *If $f$ is a unimodal map with critical point $c$ such that $k(f) = (a_0a_1 \cdots a_{n-1})^\infty$, then $f^i(c) \notin \omega_f(c)$ for all $i \geq 0$.*

**Proof.** First note that if $x \in \omega_f(c)$ then $x$ is on the periodic orbit of period $n$ or $2n$ of Theorem 3.3 above. Therefore $c \notin \omega_f(c)$ since $k(f)$ is not finite. Now suppose $f^i(c) \in \omega_f(c)$ for some $i > 0$. Then $f^i(c)$ is on a periodic orbit of period $n$ or $2n$. If $f^i(c) = f^{i+n}(c)$ then $I(f^i(c)) = I(f^{i+n}(c)) = (a_ia_{i+1} \cdots a_{i+n-1})^\infty$. It follows that $a_{i-j} = a_{i+n-j}$ for $0 \leq j \leq i$. In particular, $f^{i-1}(c)$ and $f^{i+n-1}(c)$ are on the same side of $c$ and $f(f^{i-1}(c)) = f(f^{i+n-1}(c)) = f^i(c)$. Therefore, $f^{i-1}(c) = f^{i+n-1}(c)$. Similarly, $f^{i-2}(c)$ and $f^{i+n-2}(c)$ are on the same side of $c$ with $f(f^{i-2}(c)) = f(f^{i+n-2}(c)) = f^{i-1}(c)$. Therefore, $f^{i-2}(c) = f^{i+n-2}(c)$. Inductively, we conclude that $f(c) = f^{n+1}(c)$. Since $c$ is the only value of $x$ for which $f(x) = f(c)$ and $f(f^n(c)) = f(c)$, it must be true that $f^n(c) = c$.

This is a contradiction of the assumption that $c$ is not periodic. If $f^i(c)$ is on a periodic orbit of period $2n$, a similar argument shows that $f^{2n}(c) = f(c)$; a contradiction. Thus, in either situation we arrive at a contradiction. Therefore $f^i(c) \notin \omega_f(c)$ for all $i \geq 0$. ■

**Theorem 3.5** *Let $f$ and $g$ be unimodal maps with critical points $c$ and $c'$, respectively. Suppose $k(f) = k(g) = (Ra_1 \cdots a_{n-1})^\infty$. Then there exists an order preserving homeomorphism $h : I \to I$ such that $h(f^k(c)) = g^k(c')$ for all $k \geq 0$ if and only if $|\omega_f(c)| = |\omega_g(c')|$ (which is $n$ or $2n$).*

**Proof.** First suppose that there exists an order preserving homeomorphism $h : I \to I$ such that $h(f^k(c)) = g^k(c')$ for all $k \geq 0$. Since $x \in \omega_f(c)$ if and only if $h(x) \in \omega_g(c')$, $|\omega_f(c)| = |\omega_g(c')|$.

Now suppose that $|\omega_f(c)| = |\omega_g(c')| \in \{n, 2n\}$. For $i = 1, 2, \cdots, n$ let $J_i$ and $J_i'$ be the smallest closed intervals containing $\{f^i(c), f^{n+i}(c), f^{2n+i}(c), \cdots\}$ and $\{g^i(c'), g^{n+i}(c'), g^{2n+i}(c'), \cdots\}$, respectively. Then $c$ is not in the interior of $J_i$ and $c'$ is not in the interior of $J_i'$. Therefore, $f_{|J_i}$ and $g_{|J_i'}$ are homeomorphisms into $J_{i \bmod(n)+1}$ and $J'_{i \bmod(n)+1}$. Furthermore, $f_{|J_i}$ is strictly increasing if and only if $g_{|J_i'}$ is strictly increasing. It follows that $f^n_{|J_i} : J_i \to J_i$ is strictly increasing if and only if $g^n_{|J_i'} : J_i' \to J_i'$ is strictly increasing. We also know that $f^n_{|J_1}$ is strictly increasing if and only if $Ra_1 \cdots a_{n-1}$ is even [11, page 68].

First suppose $Ra_1 \cdots a_{n-1}$ is even. Then $f^n_{|J_1}$ and$g^n_{|J_1'}$ are strictly increasing. Therefore,

$$p_1 < \cdots < f^{3n+1}(c) < f^{2n+1}(c) < f^{n+1}(c) < f(c)$$

and

$$p_1' < \cdots < g^{3n+1}(c') < g^{2n+1}(c') < g^{n+1}(c') < g(c')$$

Thus for $0 \leq j \neq k$, $f^{jn+1}(c) < f^{kn+1}(c)$ if and only if $g^{jn+1}(c') < g^{kn+1}(c'.)$. Since $f_{|J_i}$ is strictly increasing if and only if $g_{|J_i'}$ is strictly increasing, for $1 \leq i \leq n$ and $0 \leq j \neq k$, $f^{jn+i}(c) < f^{kn+i}(c)$ if and only if $g^{jn+i}(c') < g^{kn+i}(c'.)$. Observe also that

$$\lim_{j \to \infty} f^{jn+i}(c) = f^{i-1}(p_1) \text{ and } \lim_{j \to \infty} g^{jn+i}(c') = g^{i-1}(p_1').$$

so that $|\omega_f(c)| = |\omega_g(c')| = n$. Thus the supposition $|\omega_f(c)| = |\omega_g(c')|$ is not needed in this case. We conclude that the sets

$$\{c, f(c), f^2(c), \cdots, p_1, p_2, \cdots, p_n\} \text{ and } \{c', g(c'), g^2(c'), \cdots, p_1', p_2', \cdots, p_n'\}$$

have the same ordering in $I$ and there exists an order preserving homeomorphism $h : I \to I$ such that $h(f^k(c)) = g^k(c')$ for all $k \geq 0$.

Now suppose $Ra_1 \cdots a_{n-1}$ is odd. Then $f^n_{|J_1}$ and $g^n_{|J_1'}$ are strictly decreasing. Therefore,

$$f^{n+1}(c) < f^{3n+1}(c) < f^{5n+1}(c) < \cdots < f^{4n+1}(c) < f^{2n+1}(c) < f(c)$$

and

$$g^{n+1}(c') < g^{3n+1}(c') < g^{5n+1}(c') < \cdots < g^{4n+1}(c') < g^{2n+1}(c') < g(c')$$

Thus for $0 \leq j \neq k$, $f^{jn+1}(c) < f^{kn+1}(c)$ if and only if $g^{jn+1}(c') < g^{kn+1}(c'.)$. Note that either $\lim_{j\to\infty} f^{2jn+i}(c) = p_1 = \lim_{j\to\infty} f^{(2j+1)n+i}(c)$ with $p_1$ periodic of period $n$, or $\lim_{j\to\infty} f^{2jn+i}(c) = p_1 \neq p_2 = \lim_{j\to\infty} f^{(2j+1)n+i}(c)$ with $p_1$ periodic of period $2n$ and $f^n(p_1) = p_2$. Similarly for $g$. It follows that if $|\omega_f(c)| = |\omega_g(c')|$, then there exists an order preserving homeomorphism $h : I \to I$ such that $h(f^k(c)) = g^k(c')$ for all $k \geq 0$. ■

As a corollary, we can classify inverse limits with a single unimodal bonding map having periodic kneading sequence by looking at the forward orbit of the critical point.

**Corollary 3.3** *Let $f$ and $g$ be unimodal maps with critical points $c$ and $c'$, respectively. Suppose $k(f) = k(g) = (Ra_1 \cdots a_{n-1})^\infty$. Then $\varprojlim(I, f)$ and $\varprojlim(I, g)$ are homeomorphic if and only if $|\omega_f(c)| = |\omega_g(c')|$.*

**Proof.** Suppose $|\omega_f(c)| = |\omega_g(c')|$. By Theorem 3.5, there exists an order preserving homeomorphism $h : I \to I$ such that $h(f^k(c)) = g^k(c')$ for all $k \geq 0$. Therefore, $\varprojlim(I, f)$ and $\varprojlim(I, g)$ are homeomorphic by Theorem 3.2. Now suppose $|\omega_f(c)| \neq |\omega_g(c')|$. Then, using [3, Theorem 2.1], one can show that the number of endpoints of $\varprojlim(I, f)$ and $\varprojlim(I, g)$ are not the same. However, an endpoint is a topological invariant. Therefore, $\varprojlim(I, f)$ and $\varprojlim(I, g)$ are not homeomorphic. ■

We now provide an example showing why it is necessary to require $a_0 = R$. First note that $a_0 = L$ implies $k(f) = k(g) = L^\infty$ and $|\omega_f(c)| = |\omega_g(c')| = 1$. The example also shows that the converse is not true.

**Example 5** *Let $f(x) = x(1-x)$, $g(x) = \frac{3}{2}x(1-x)$, and $h(x) = 2(1-x)$. Since $f(\frac{1}{2}) < \frac{1}{2}$, $g(\frac{1}{2}) < \frac{1}{2}$, and $h(\frac{1}{2}) = \frac{1}{2}$, $k(f) = k(g) = L^\infty$ and $k(h) = C$. In fact, $|\omega_f(\frac{1}{2})| = |\omega_g(\frac{1}{2})| = |\omega_h(\frac{1}{2})|$ with $\omega_f(\frac{1}{2}) = \{0\}$, $\omega_g(\frac{1}{2}) = \{\frac{1}{3}\}$, and $\omega_h(\frac{1}{2}) = \{\frac{1}{2}\}$. We also have $\bigcap_{n=0}^{\infty} f^n(I) = \{0\}$, $\bigcap_{n=0}^{\infty} g^n(I) = [0, \frac{1}{3}]$, and $\bigcap_{n=0}^{\infty} h^n(I) = [0, \frac{1}{2}]$ It follows from Theorem 2.12 and Theorem 2.13 that $\varprojlim(I, f)$ is a point while both $\varprojlim(I, g)$ and $\varprojlim(I, h)$ are arcs.*

It is well known that for $f_4(x) = 4x(1-x)$, $\varprojlim(I, f)$ is homeomorphic to the B-J-K continuum of Example 1 in § 2.2. It follows from Theorem 3.2 that, for any *full unimodal* map $f$ $(f(c) = 1)$, $\varprojlim(I, f)$ is homeomorphic to the B-J-K continuum.

## 3.2 Bennett's Theorem and the Core

In this section we present a theorem by Ralph Bennett that has proven to be very valuable in the study of inverse limits on arcs. We begin with a definition.

A *topological ray* is a locally compact, connected set $R$ containing a point $p$ such that $R - p$ is connected and if $q$ is a point in $R$ distinct from $p$ then $R - q$ is the union of two mutually separated connected nonempty sets.

**Theorem 3.6** *[22, Theorem 2.15] If $\alpha_1, \alpha_2, \cdots$ is a sequence of arcs in a continuum $X$ with a common endpoint $p$ such that $\alpha_1 \subset \alpha_2 \subset \cdots$, $R = \bigcup_{n=1}^{\infty} \alpha_n$ and no point of $\alpha_n$ lies in $\overline{R - \alpha_{n+1}}$, then $R$ is a topological ray.*

Bennett's Theorem as stated here appears in [21] and is a slight generalization of his original theorem in [7].

**Theorem 3.7** ***(Bennett)*** *Suppose $f$ is a mapping of the interval $[a, b]$ onto itself and $d$ is a number between $a$ and $b$ such that*

1. $f([d, b]) \subseteq [d, b]$
2. $f_{|[a,d]}$ *is monotone and*
3. *there is a positive integer $k$ such that $f^k([a, d]) = [a, b]$.*

   *Then $\varprojlim(I, f)$ is the union of a topological ray $R$ and a continuum $K$ such that $\overline{R} - R = K$.*

Before exploring how Bennett's Theorem applies to inverse limits with unimodal bonding maps, we first consider the case where $f(c) \leq c$.

**Proposition 3.1** *If $f(c) \leq c$ then $\varprojlim(I, f)$ is either an arc or a point.*

**Proof.** Suppose $f(c) \leq c$. and let $p$ be the largest fixed point in $[0, c]$. If $p \neq 0$ then $J = \bigcap_{i=0}^{\infty} f^i(I) = [0, p]$, $f_{|J}$ is a homeomorphism, and $\varprojlim(I, f)$ is an arc by Theorem 2.12 and Theorem 2.13. If $p = 0$ then $J = \bigcap_{i=0}^{\infty} f^i(I) = \{0\}$ and $\varprojlim(I, f) = \{(0, 0, \ldots)\}$ by Theorem 2.13. ■

Now suppose $f(c) > c$. If $f^2(c) > c$ choose $d$ and $b$ of Bennett's Theorem to be equal to $c$ and $f(c)$, respectively. There are two possibilities: $J = \bigcap_{i=0}^{\infty} f^i([c, f(c)])$ is either an interval or a point. If $J$ is a point then $K = \varprojlim([f^2(c), f(c)], f)$ is a point and $\varprojlim(I, f)$ is an arc. If $J$ is an interval then $K = \varprojlim([f^2(c), f(c)], f)$ is an arc and $\varprojlim(I, f)$ is a topological ray limiting onto an arc, i.e. a *topological* $\sin(\frac{1}{x})$*-curve*. There are only two other cases which we need to consider: $f^2(c) < c$ with either $f^3(c) \leq f^2(c)$ or $f^2(c) < f^3(c)$. Bennett's Theorem does not apply when $f^3(c) < f^2(c)$ since $f([f^2(c), f(c)]) \not\subseteq [f^2(c), f(c)]$. If $f^3(c) = f^2(c)$, then the ray in Bennett's Theorem is replaced by an arc. When $f^2(c) < f^3(c)$ and there is a fixed point in $[f^3(c), c)$, then $f^k([0, f^2(c)]) \neq [0, f(c)]$ for all $k > 0$ and Bennett's Theorem does not apply.

This leaves us with the case when $f^2(c) < f^3(c)$ and there are no fixed points in $[f^3(c), c)$. We then have $f([f^2(c), f(c)]) = [f^2(c), f(c)]$ and can use Bennett's Theorem with $[a, b] = [0, f(c)]$ and $d = f^2(c)$. The continuum $K$ is equal to $\varprojlim([f^2(c), f(c)], f)$ and is called the *core* of $\varprojlim(I, f)$.

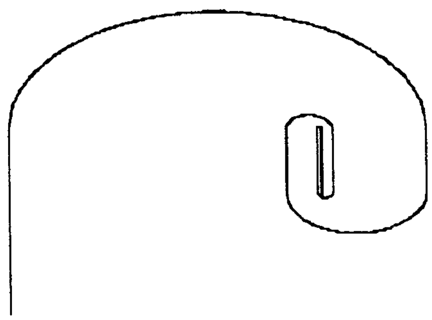

Figure 4: Topological ray limiting on an arc.

**Example 6** *Let $f$ be a unimodal maps with $k(f) = RC$. Then $f^2(c) = c$ and $f|_{[f^2(c),f(c)]}$ is a homeomorphism Thus, by Theorem 2.12 the core of $\varprojlim(I, f)$ is an arc and by Bennett's Theorem, $\varprojlim(I, f)$ is a topological $\sin(\frac{1}{x})$-curve. See Figure 4.*

Now that we have reduced the study of $\varprojlim(I, f)$ to a study of the core, we can focus our attention on the nature of the core.

**Theorem 3.8** *[21, Theorem 7] Suppose $f$ is a unimodal map and $q$ is the first fixed point for $f^2$ in $[c, f(c)]$. If $f([f^2(c), f(c)]) = [f^2(c), f(c)]$, then the core of $\varprojlim(I, f)$ is indecomposable if and only if $f^3(c) < q$.*

In the course of Ingram's proof it is observed that if $f(c) \geq q$ then

$$\varprojlim([f^2(c), f(c)], f) = \varprojlim([f^2(c), f(q)], f) \bigcup \varprojlim([q, f(c)], f).$$

More importantly to us is how the two pieces of the core are related. Recall that $p$ is the fixed point for $f$ in $(c, f(c))$.

**Theorem 3.9** *Suppose the core of* $\varprojlim(I, f)$ *is decomposable. Then the core is the union of two homeomorphic subcontinua intersecting in a point or an arc.*

**Proof.** Let $q$ be the first fixed point for $f^2$ in $[c, f(c)]$. Then $f^3(c) \geq q$ and $\varprojlim([f^2(c), f(c)], f)$ is the union of $\varprojlim([f^2(c), f(q)], f^2)$ and $\varprojlim([q, f(c)], f^2)$. Note that $f_{|[f(q),f(c)]}$ is a homeomorphism onto $[f^2(c), q]$. It follows that $f^2_{|[f(q),f(c)]}$ is topologically conjugate to $f^2_{|[f^2(c),q]}$ via $f_{|[f(q),f(c)]}$. Thus, $\varprojlim([f^2(c), f(q)], f^2)$ is homeomorphic to $\varprojlim([q, f(c)], f^2)$. Furthermore,

$$\varprojlim([f^2(c), f(q)], f^2) \cap \varprojlim([q, f(c)], f^2) = \varprojlim([f(q), q], f^2)$$

produces an arc if $q \neq p$ or a point otherwise. ■

**Example 7** *Let $f$ be as shown in Figure 5 with $f^3(c) = p$. Then according to Theorem 3.9, the core of $\varprojlim(I, f)$ is the union of the two homeomorphic subcontinua $\varprojlim([f^2(c), f(q)], f^2)$ and $\varprojlim([q, f(c)], f^2)$. However, because $f^3(c) = p$, we can also decompose the core as the union of $\varprojlim([f^2(c), p], f^2)$ and $\varprojlim([p, f(c)], f^2)$. $\varprojlim([f^2(c), p], f^2)$ and $\varprojlim([p, f(c)], f^2)$ are homeomorphic and, in fact, they are homeomorphic to the B-J-K continuum of Example 1 in § 2.2.*

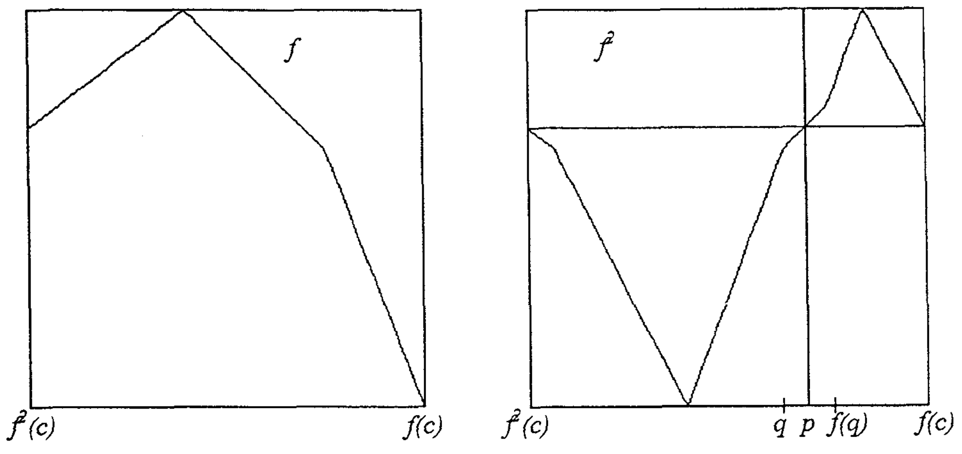


Figure 5: Decomposition of the core

## 3.3 Renormalization

As the previous example shows, if $f^3(c) \geq p$ then we can write the core as the union of $\varprojlim([f^2(c), p], f^2)$ and $\varprojlim([p, f(c)], f^2)$, where $\varprojlim([f^2(c), p], f^2)$ is homeomorphic to $\varprojlim([p, f(c)], f^2)$. In this section we introduce the renormalization operator on the space of unimodal maps that allows us to study the nature of inverse limits with unimodal bonding maps using kneading theory. The renormalization operator was used by Feigenbaum [16] to explain the universal transition of a class of one-parameter maps from simple to complicated dynamics.

Suppose $f$ is unimodal and satisfies $f^3(c) \geq p$. From Theorem 3.8 we conclude that the core of $\varprojlim(I, f)$ is decomposable. In fact, Theorem 3.9 shows that the core decomposes into two homeomorphic subcontinua intersecting at a

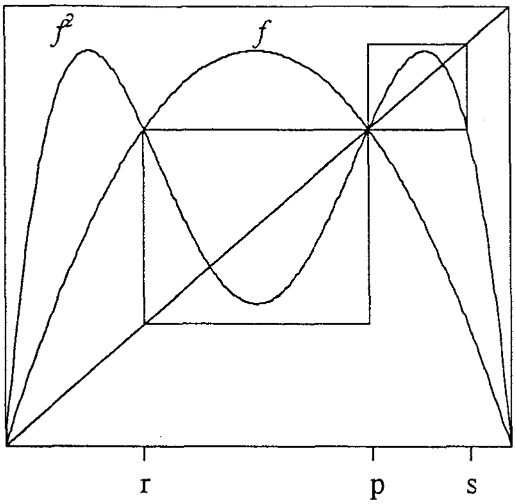


Figure 6: Renormalization of $f$.

common endpoint of a ray. For the discussion following refer to Figure 6. Let $p$ be the fixed point for $f$ greater then $c$, $r \in f^{-1}(p)$ with $r < c$, and $s \in f^{-1}(r)$ with $s > c$. By looking at $f^2$ we see that there is an interval on which $f^2$ is unimodal but upside-down. This observation motivates the following definition. The *renormalization of f*, denoted $\Re f$, is defined by $\Re f = h \circ f^2 \circ h^{-1}$ where $h : [r, p] \to I$ is a linear homeomorphism such that $h(r) = 1$ and $h(p) = 0$. In order for this definition to make sense we need $f(c) > c$ and $f^2([r, p]) \subseteq [r, p]$. For a more detailed discussion the reader is referred to [12].

**Theorem 3.10** *Let $f$ be a unimodal map with $\Re f$ defined. Then*

1. *$\Re f : I \to I$ is a unimodal map with critical point $h(c)$,*

2. *if $Sf < 0$, then $S\Re f < 0$.*

**Proof.** The proof of the first statement is clear. For the second statement recall that $\Re f = h \circ f^2 \circ h^{-1}$, where $h : [r,p] \to I$ is a linear homeomorphism such that $h(r) = 1$ and $h(p) = 0$. Since $h$ is linear, $Sh = Sh^{-1} \equiv 0$. Let $x \in I - \{h(c)\}$. A direct calculation shows that $S\Re f(x) = Sf^2(h^{-1}(x))(h^{-1}(x))^2$ which must be negative since $Sf^2(h^{-1}(x)) < 0$ and $(h^{-1}(x))^2 > 0$. ∎

Note that $\Re f : I \to I$ is topologically conjugate to $f^2$ restricted to $[r,p]$. It follows that $\varprojlim(I, \Re f)$ is homeomorphic to $\varprojlim([r,p], f^2_{[r,p]})$ by Corollary 2.1. The reason for studying the core using $\varprojlim(I, \Re f)$ instead of $\varprojlim([r,p], f^2_{[r,p]})$ is because we can use the power of kneading theory with $\Re f$. The following theorem is actually a corollary to Theorem 3.9.

**Theorem 3.11** *If $f$ is unimodal with $\Re f$ defined, then the core of $\varprojlim(I, f)$ is the union of two copies of $\varprojlim(I, \Re f)$ intersecting in a point.*

**Proof.** Note $f^2([r,q]) = [f^2(c), f(c)]$. Thus,

$$\varprojlim([f^2(c), f(c)], f) = \varprojlim([r,s], f_{[r,s]}) = \varprojlim([r,p], f^2_{[r,p]}) \cup \varprojlim([p,s], f^2_{[p,s]})$$

with $\varprojlim([r,p], f^2_{[r,p]})$ homeomorphic to $\varprojlim([p,s], f^2_{[p,s]})$. ∎

**Corollary 3.4** *If $f$ is unimodal with $\Re^n f$ defined for some $n > 0$, then, for $0 \leq i < n$, the core of $\varprojlim(I, \Re^i f)$ is a ray limiting on the union of two copies of $\varprojlim(I, \Re^{i+1} f)$ intersecting in a point for $0 \leq i < n$.*

**Proof.** Follows inductively from Theorem 3.11 ∎

The next result shows that the kneading sequence of a renormalizable map is restricted. Recall that $p$ is the fixed point for $f$ greater than $c$, $r \in f^{-1}(p)$ with $r < c$, and $s \in f^{-1}(r)$ with $s > c$ (see Figure 6).

**Proposition 3.2** *If $f$ is renormalizable then $k(f) = R * B$ for some admissible sequence $B$. If $k(f) = R * B$ for some admissible sequence $B < RL^\infty$ then $f$ is renormalizable.*

**Proof.** Suppose $f$ is renormalizable. Then $f(c) \in [p, s]$ and $f^2([p, s]) \subseteq [p, s]$. Thus, $f^{2i+1}(c) \in [p, s]$ for all $i \geq 0$. If $f^{2n+1}(c) = f(c)$ for some (smallest) $n$ then $f^{2n}(c) = c$. Therefore, either $f^{2i+1}(c) > c$ for all $i \geq 0$ or $f^{2i+1}(c) > c$ for $0 \leq i < n$ with $f^{2n}(c) = c$. Either $k(f) = Ra_2Ra_4\cdots$ is infinite, in which case $B = \widehat{a}_2\widehat{a}_4\widehat{a}_6\cdots$, or $k(f) = Ra_2Ra_4\cdots a_{2(n-1)}RC$, in which case $B = \widehat{a}_2\widehat{a}_4\cdots\widehat{a}_{2(n-1)}C$.

Now suppose $k(f) = R * B$ for some admissible sequence $B < RL^\infty$. We need to show that $f^2([r, p]) \subseteq [r, p]$. It suffices to show $f^2(c) \geq r$. We assume $f^2(c) < r$. Then $c < f^n(c)$ for all $n \geq 3$ so that $B = RL^\infty$; a contradiction. Therefore, $f^2(c) \geq r$. ■

If $\Re f$ is renormalizable, i.e. $\Re^2 f$ exists, then one can show that $f^{4i+2}(c) > c$ for all $i \geq 0$ or $f^{4i+2}(c) > c$ for $0 \leq i < n$ with $f^{4n}(c) = c$ for some $n > 0$. Inductively, if $f$ is *infinitely renormalizable* ($\Re^n f$ exists for all $n \geq 1$) then $k(f)$ is completely determined. The following proposition shows how the kneading sequences of $f$ and $\Re f$ are related.

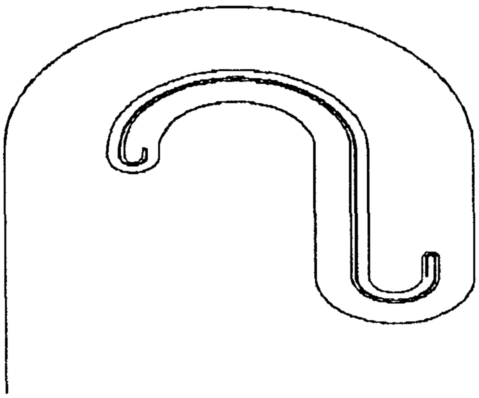

Figure 7: Topological ray limiting on two $\sin(\frac{1}{x})$-curves.

**Proposition 3.3** *[12] If $f$ is unimodal with $\Re f$ defined and $k(f) = a_1a_2\cdots$, then $k(\Re f) = \widehat{a}_2\widehat{a}_3\cdots$. In particular, $k(f) = R * k(\Re f)$.*

Knowing how the kneading sequence is changed by $\Re$ is useful in identifying inverse limits as the following examples demonstrate.

**Example 8** *If $f$ is unimodal with $k(f) = RLRC = R * RC$, then $k(\Re f) = RC$ by Proposition 1. Example 1 showed that $\varprojlim(I, \Re f)$ is a $\sin(\frac{1}{x})$-curve. Thus, $\varprojlim(I, f)$ is ray limiting on two $\sin(\frac{1}{x})$-curves intersecting at the endpoints of their rays. See Figure 7.*

**Example 9** *Suppose $k(f) = R^{*n} * R^{\infty}$ for some $n > 0$ and . Then $\Re^{n+1}f$ exists and $k(\Re^n f) = R^{\infty}$. If $|\omega_f(c)| = 2^n$ then $|\omega_{\Re^n f}(c)| = 1$ and $\varprojlim(I, \Re^n f)$ is* *an arc. By Theorem 3.11* $\varprojlim(I,f)$ *is the union of a ray limiting on a pair of rays intersecting a common endpoint each limiting on a pair of rays ... limiting on* $2^{n-1}$ *topological* $\sin(\frac{1}{x})$*-curves. If* $|\omega_f(c)| = 2^{n+1}$ *then* $|\omega_{\Re^n f}(c)| = 2$ *and* $\varprojlim(I, \Re^n f)$ *is a topological* $\sin(\frac{1}{x})$*-curve. Again by Theorem 3.11,* $\varprojlim(I,f)$ *is the union of a ray limiting on a pair of rays intersecting a common endpoint each limiting on a pair of rays ... limiting on* $2^n$ *topological* $\sin(\frac{1}{x})$*-curves. We will see later that this example is* **the** *model for all kneading sequences below the sequence* $(R*)^\infty$.

## 3.4 Full Families

Let **C** represent the class of $C^1$unimodal maps and let $\{f_\lambda : \alpha \leq \lambda \leq \beta\}$ represent a curve in **C** continuous in the $C^1$ topology. More precisely, the map $\lambda \mapsto f_\lambda$ is a map from $[\alpha, \beta]$ to **C** such that

$$\lim_{\lambda \to \lambda_0} \{ \sup_{0 \leq x \leq 1} (|f_\lambda(x) - f_{\lambda_0}(x)| + |f'_\lambda(x) - f'_{\lambda_0}(x)|)\} = 0.$$

**Theorem 3.12** *[11, Theorem III.1.1] Let* $\{f_\lambda : \alpha \leq \lambda \leq \beta\}$ *be a curve in* **C** *continuous in the* $C^1$ *topology. For every maximal sequence* $A$ *satisfying* $k(f_\alpha) < A < k(f_\beta)$ *there exists a* $\mu \in (\alpha, \beta)$ *such that* $k(f_\mu) = A$.

We say that $\{f_\lambda : \alpha \leq \lambda \leq \beta\}$ is a *Full family* if $k(f_\alpha) = L^\infty$ and $f_\beta(c_{f_\beta}) = 1$. If $S(f_\lambda) < 0$ for all $\alpha \leq \lambda \leq \beta$, we call $\{f_\lambda : \alpha \leq \lambda \leq \beta\}$ a $S$-Full family. For convenience we sometimes write $\{f_\lambda\}$ for $\{f_\lambda : \alpha \leq \lambda \leq \beta\}$. The logistic family

$\{f_\lambda(x) = \lambda x(1-x), 0 \leq \lambda \leq 4\}$ is an example of a $S$-Full family. For more details on Full families the reader is referred to [11]. To simplify the discussion that follows we assume there are no intervals on which $\lambda \mapsto k(f_\lambda)$ is constant.

**Theorem 3.13** *[11, Proposition III.1.2] In a Full family $\{f_\lambda : \alpha \leq \lambda \leq \beta\}$ every maximal sequence of the form $R\cdots$ occurs as the kneading sequence for some $\lambda \in [\alpha, \beta]$.*

**Theorem 3.14** *[31, Theorem 1.1] If $f_\lambda$ is a Full family then there exists parameter values*

$$\alpha < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_3' < \lambda_2' < \lambda_1' < \beta$$

*such that $c_{\alpha_n}$ is periodic of period $2^n$, $\Re^n f_{\lambda_n'}$ is full unimodal, and $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \lambda_n'$.*

The limiting parameter value $\lambda_\infty = \lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \lambda_n'$ is called the *Feigenbaum value.*

**Lemma 3.15** *Let $\{f_\lambda\}$ be a Full family with sequences $\{\lambda_n\}$ and $\{\lambda_n'\}$ as above. Then*

1. *$k(f_{\lambda_1}) = RC$ and $k(f_{\lambda_n}) = (R*)^{n-1}RC = R * k(f_{\lambda_{n-1}})$ for all $n > 1$,*

2. *$k(f_{\lambda_1'}) = RLR^\infty$ and $k(f_{\lambda_n'}) = (R*)^{n-1}RLR^\infty = R*k(f_{\lambda_{n-1}'})$ for all $n > 1$,*

3. *$k(f_{\lambda_\infty}) = \lim_{n\to\infty} k(f_{\lambda_n}) = (R*)^\infty RC$ is nonperiodic.*

**Proof.** (1) Follows directly from [11, Lemma.2.12]. (2) Let $n \geq 1$ and suppose $\Re^n f_{\lambda'_n}$ is full unimodal. Then $k(\Re^n f_{\lambda'_n}) = RL^\infty$. Repeatedly applying Proposition 3.4 results in

$$\begin{aligned} k(f_{\lambda'_n}) &= R * k(\Re f_{\lambda'_n}) = (R*)^2 k(\Re^2 f_{\lambda'_n}) = \cdots = (R*)^n k(\Re^n f_{\lambda'_n}) \\ &= (R*)^n RL^\infty = (R*)^{n-1} RLR^\infty = R * k(f_{\lambda'_{n-1}}). \end{aligned}$$

(3) $R * k(f_{\lambda_\infty}) = k(f_{\lambda_\infty})$ so that $k(f_{\lambda_\infty})$ is nonperiodic by Proposition 2.1. ■

# 4 Main Results

## 4.1 Introduction

The main results of this paper deal with the inverse limits using bonding maps at the above parameter values. For all $\lambda$ with $k(f_\lambda) \leq k(f_{\lambda_\infty})$ we show $\varprojlim(I, f_\lambda)$ is hereditarily decomposable. If $k(f_\lambda) > k(f_{\lambda_\infty})$ then $\varprojlim(I, f_\lambda)$ contains an indecomposable subcontinuum. At the Feigenbaum value we show that, under some smoothness conditions, $\varprojlim(I, f_{\lambda_\infty})$ contains only three topologically different subcontinua. See [4] for similar results involving the logistic family.

## 4.2 Below the Feigenbaum Value

**Proposition 4.1** *Suppose $f$ is unimodal with $k(f) = BC$ with $|BC| = 2^n$ for some $n \geq 1$. If $g$ is unimodal with $k(g) \in \{(BL)^\infty, (BR)^\infty\}$ and $|\omega_g(c_g)| = 2^n$, then $\varprojlim(I, g)$ and $\varprojlim(I, f)$ are homeomorphic.*

**Proof.** We use induction on the length of $B$. For $n = 1$, we must have $k(f) = RC$ so that $\varprojlim(I, f)$ is a topological $\sin(\frac{1}{x})$-curve. If $k(g) = R^\infty$ with $|\omega_g(c_g)| = 2$, then the core of $\varprojlim(I, g)$ is an arc so that $\varprojlim(I, g)$ is a topological $\sin(\frac{1}{x})$-curve. If $k(g) = (RL)^\infty$ with $|\omega_g(c_g)| = 2$ then $k(\Re g) = R^\infty$ with $|\omega_{\Re g}(c_g)| = 1$ so that $\varprojlim(I, \Re g)$ is an arc. Thus, the core of $\varprojlim(I, g)$ is the union of two arcs intersecting in an endpoint. It follows that the core of $\varprojlim(I, g)$ is an arc and $\varprojlim(I, g)$ is a topological $\sin(\frac{1}{x})$-curve. Therefore the claim is true for $n = 1$. Suppose the claim is true for $n \geq 1$. Let $|k(f)| = |BC| = 2^{n+1}$ with $B = b_1 b_2 \cdots b_{2^{n+1}-1}$. Suppose also that $k(g) \in \{(BL)^\infty, (BR)^\infty\}$ with $|\omega_g(c_g)| = 2^{n+1}$. It follows from [11, Theorem II.2.9 and Lemma II.2.12] and Proposition 3.4 that $\Re f$ and $\Re g$ exist with $|k(\Re f)| = |\widehat{b_2}\widehat{b_4} \cdots \widehat{b}_{2^{n+1}-2}C| = 2^n$ and $k(\Re g) \in \{(\widehat{b_2}\widehat{b_4} \cdots \widehat{b}_{2^{n+1}-2}L)^\infty, (\widehat{b_2}\widehat{b_4} \cdots \widehat{b}_{2^{n+1}-2}R)^\infty\}$. Furthermore, $|\omega_{\Re g}(c_g)| = 2^n$ since it was assumed that $|\omega_g(c_g)| = 2^{n+1}$. By the induction hypothesis, $\varprojlim(I, \Re g)$ and $\varprojlim(I, \Re f)$ are homeomorphic. From Bennett's Theorem and Theorem 3.11 we conclude that $\varprojlim(I, g)$ and $\varprojlim(I, f)$ are homeomorphic. Therefore, the claim is true for $n + 1$. ■

**Theorem 4.1** *Let $\{f_\lambda\}$ be a Full family with sequence $\{\lambda_n\}$ as in Theorem 3.14. Then $\varprojlim(I, f_{\lambda_1})$ is a topological $\sin(\frac{1}{x})$-curve and the core of $\varprojlim(I, f_{\lambda_{n+1}})$ is the union of two copies of $\varprojlim(I, f_{\lambda_n})$ intersecting in a point. Furthermore, for each $n$ there exist an $\varepsilon_n$ such that for all $\lambda \in (\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n)$ $\varprojlim(I, f_\lambda)$ and $\varprojlim(I, f_{\lambda_n})$ are homeomorphic.*

**Proof.** Let $n > 1$. Since $\Re f_{\lambda_{n+1}}$ exists, it follows by Theorem 3.11 that the core of $\varprojlim(I, f_{\lambda_{n+1}})$ is the union of two copies of $\varprojlim(I, \Re f_{\lambda_{n+1}})$ intersecting in a point. Noting that $k(\Re f_{\lambda_{n+1}}) = k(f_{\lambda_n})$ and is finite, we then conclude using Corollary 3.1 that $\varprojlim(I, \Re f_{\lambda_{n+1}})$ is homeomorphic to $\varprojlim(I, f_{\lambda_n})$. That $\varprojlim(I, f_{\lambda_1})$ is a topological $\sin(\frac{1}{x})$-curve follows from Example 6.

We now prove the second claim using an argument similar to [11, Lemma III.1.3]. Let $n \geq 1$ and $k(f_{\lambda_n}) = BC$. From Lemma 3.15 (1) we conclude

$|BC| = 2^n$. By the argument of [24, Theorem 16.4.2], there exists $\varepsilon_n > 0$ such that, for all $\lambda \in (\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n)$, there exists $\delta_\lambda > 0$ with, for all $x \in [c_\lambda - \delta_\lambda, c_\lambda + \delta_\lambda]$,

1. $I_{f_\lambda}(f_\lambda(x)) = B\cdots$;

2. $|(f_\lambda^{2^n})'(x)| < \frac{1}{2}$;

3. $|f_\lambda^{2^n}(x) - c_\lambda| < \frac{\delta_\lambda}{2}$.

Let $\lambda \in (\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n)$. If $f_\lambda^{2^n}(c_\lambda) = c_\lambda$ then $k(f_\lambda) = BC$. So now suppose $f_\lambda^{2^n}(c_\lambda) - c_\lambda = a > 0$. Let $x \in [c_\lambda, 2a]$. Since $2a < \delta_\lambda$, we have $|f_\lambda^{2^n}(x) - f_\lambda^{2^n}(c_\lambda)| \leq \frac{|x-c_\lambda|}{2}$ by (2). Therefore,

$$f_\lambda^{2^n}(x) \geq f_\lambda^{2^n}(c_\lambda) - \frac{1}{2}|x - c_\lambda| > a + c_\lambda - \frac{1}{2}(2a) = c_\lambda$$

and

$$f_\lambda^{2^n}(x) \leq f_\lambda^{2^n}(c_\lambda) + \frac{1}{2}|x - c_\lambda| < a + c_\lambda + \frac{1}{2}(2a) = c_\lambda + 2a.$$

It follows that $f_\lambda^{2^n}([c_\lambda, 2a]) \subseteq (c_\lambda, 2a)$. Thus $k(f_\lambda) = (BR)^\infty$. A similar argument shows $k(f_\lambda) = (BL)^\infty$ if $f_\lambda^{2^n}(c_\lambda) - c_\lambda = a < 0$.

By Proposition 4.1, it remains to show that $|\omega_{f_\lambda}(c_\lambda)| = 2^n$. This follows directly from the above argument since, in either case, $f_\lambda^{2^n} : [c_\lambda, 2a] \to [c_\lambda, 2a]$ and $f_\lambda^{2^n} : [2a, c_\lambda] \to [2a, c_\lambda]$ are contraction mappings by (2). ■

It follows that $\varprojlim(I, f_{\lambda_n})$ is a ray limiting on the union of two rays intersecting in a common endpoint each limiting on the union of two rays ... limiting on $2^{n-1}$ $\sin(\frac{1}{x})$ curves. We should also point out that for any parameter value $\lambda$ with $k(f_\lambda) < k(f_{\lambda_\infty})$ there exist an $n$ such that $\varprojlim(I, f_\lambda)$ is homeomorphic to $\varprojlim(I, f_{\lambda_n})$. This is true because the possible kneading sequences that can occur below $k(f_{\lambda_\infty})$ are limited [11, Lemma II.2.12]. Thus we have a complete classification for $\varprojlim(I, f_\lambda)$ with $k(f_\lambda) < k(f_{\lambda_\infty})$

If we restrict ourselves to $S$-Full families, then not only is $\varprojlim(I, \Re f_{\lambda_{n+1}})$ homeomorphic to $\varprojlim(I, f_{\lambda_n})$, but the dynamics of $\widehat{f_{\lambda_n}}$ acting on $\varprojlim(I, f_{\lambda_n})$ is identical to the dynamics of $\widehat{\Re f}_{\lambda_{n+1}}$ acting on $\varprojlim(I, \Re f_{\lambda_{n+1}})$. This is not generally the case for Full families.

In Theorem 3 of [4], Barge and Ingram were able to identify all possible inverse limits occurring with a single bonding map chosen from the logistic family and below the Feigenbaum value. As the next example shows, their results follow from Proposition 4.1 and Theorem 4.1.

**Example 10** *Consider $f_\lambda(x) = \lambda x(1-x)$ for $0 \leq \lambda \leq 4$. Note that $c_{f_\lambda} = \frac{1}{2}$ for all $\lambda$. As we have already discovered, there is an increasing sequence of parameter values*

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

*at which the logistic map $f_{\lambda_n}$has periodic critical point of period $2^n$. These* *values are in fact unique since the map* $\lambda \longmapsto k(f_\lambda)$ *is monotone [27, Theorem 10.1 and Corollary 1]. It is well documented that there also exists an unique sequence of period-doubling bifurcation values* $\{\lambda'_n\}_{n\geq 0}$ *such that* $\lambda_n < \lambda'_n < \lambda_{n+1}$ *(see [11]). This sequence is such that, for all* $\lambda \in (\lambda'_n, \lambda'_{n+1}]$, $|\omega_{f_\lambda}(\frac{1}{2})| = 2^n$. *Furthermore, for* $k(f_{\lambda_n}) = BC$, $k(f_\lambda) = (BL)^\infty$ *for* $\lambda \in (\lambda_{n-1}, \lambda_n)$ *where* $\lambda_{-1} = 0$ *and* $k(f_\lambda) = (BR)^\infty$ *for* $\lambda \in (\lambda_n, \lambda_{n+1})$. *By Proposition 4.1,* $\varprojlim(I, f_\lambda)$ *and* $\varprojlim(I, f_\mu)$ *are homeomorphic for all* $\lambda, \mu \in (\lambda'_n, \lambda'_n]$. *By Theorem 4.1,* $\varprojlim(I, f_\lambda)$ *is a topological* $\sin(\frac{1}{x})$*-curve for all* $\lambda \in (\lambda_0, \lambda'_0]$ *and, for all* $\lambda \in (\lambda'_n, \lambda'_{n+1}]$, *the core of* $\varprojlim(I, f_\lambda)$ *is the union of two copies of* $\varprojlim(I, f_\mu)$ *intersecting at a common endpoint of a ray for any* $\mu \in (\lambda'_{n-1}, \lambda'_n]$.

## 4.3 Above the Feigenbaum Value

**Theorem 4.2** *Let* $\{f_\lambda\}$ *be a Full family with the sequence* $\{\lambda'_n\}$ *from Theorem 3.14. Then the core of* $\varprojlim(I, f_{\lambda'_1})$ *is the union of two B-J-K continua intersecting in a common endpoint of a ray and the core of* $\varprojlim(I, f_{\lambda'_n})$ *is the union of two copies of* $\varprojlim(I, f_{\lambda'_{n-1}})$ *intersecting in a common endpoint of a ray.*

**Proof.** Recall from Theorem 3.14 that $\Re^n f_{\lambda'_n}$ is full-unimodal. By Theorem 3.11, the core of $\varprojlim(I, f_{\lambda'_1})$ is the union of two B-J-K continua intersecting at a common endpoint of a ray. Corollary 3.4 completes the proof. ■

We note that since $\lambda'_n$ is on the boundary of where $\Re^n f_{\lambda'_n}$ exists, we cannot hope to get results equivalent to the second part of Theorem 3.14.

The following result is known. We include it for completeness.

**Theorem 4.3** *Let $\{f_\lambda\}$ be a Full family with $BC > k(f_{\lambda_\infty})$ a maximal sequence. Then there exists a decreasing sequence $\{\mu_n\}$ converging to $\lambda_\infty$ such that $k(f_{\mu_n}) = (R*)^n BC$. Moreover, there exists an $m \geq 1$ such that $\lambda'_{m+n+1} < \mu_n < \lambda'_{m+n}$ for all $n \geq 1$.*

**Proof.** First note that $(R*)^n BC$ is maximal [11, Corollary II.2.4] and $k(f_{\lambda_\infty}) = (R*)^\infty BC$. Since $(R*)^\infty BC < R * BC < R * (RL)^\infty = k(f_{\lambda'_1})$, there exists an $m \geq 1$ such that

$$k(f_{\lambda'_{m+1}}) = (R*)^{m+1}(RL)^\infty < R * BC < (R*)^m (RL)^\infty = k(f_{\lambda'_m}).$$

It follows from Theorem 3.12 that there exists $\mu_1 \in (\lambda'_{m+1}, \lambda'_m)$ such that $k(f_{\mu_1}) = R * BC$. (Note that $\mu_1$ is not necessarily unique. If we would like a unique prescription we could take $\mu_1$ to be the infimum or supremum over all $\mu \in (\lambda'_{m+1}, \lambda'_m)$ such that $k(f_{\mu_1}) = R * BC$.) By [11, Theorem II.2.5],

$$(R*)^{m+2}(RL)^\infty < (R*)^2 BC < (R*)^{m+1}(RL)^\infty.$$

Again by Theorem 3.12, there exists $\mu_2 \in (\lambda'_{m+2}, \lambda'_{m+1})$ such that $k(f_{\mu_2}) = (R*)^2 BC$. Continuing in this manner we obtain a sequence $\{\mu_n\}$ with the desired properties. ■

This leads to an interesting observation: Given any maximal sequence $BC$ with sequence $\{\mu_n\}$ guaranteed by Theorem 4.3, we have a similar situation as in Theorem 4.2 with the B-J-K continua replaced by $\varprojlim(I, f_\mu)$ where $k(f_\mu) = BC$.

**Example 11** *Let $\{f_\lambda\}$ be a Full family with $BC = RLC$. Theorem 4.3 guarantees a decreasing sequence $\{\mu_n\} \to \lambda_\infty$. Let $\mu > \lambda_1'$ be such that $k(f_\mu) = RLC$. By Corollary 3.1, $\varprojlim(I, f_\mu)$ is the three endpoint continuum of Example 2 in §2.2. Therefore, $\varprojlim(I, f_{\mu_n})$ is a ray limiting on the union of two rays intersecting an endpoint each limiting on two rays ... limiting on $2^n$ copies of the three endpoint continuum.*

Unlike the case where every inverse limit with bonding map having kneading sequence less than $k(f_{\lambda_\infty})$ contains copies of $\sin(\frac{1}{x})$-curves, we are unable to classify all inverse limits occurring with bonding map from above the Feigenbaum value. Perhaps the broadest statement we can make is the following.

**Theorem 4.4** *Let $\{f_\lambda\}$ be a Full family with Feigenbaum value $\lambda_\infty$. Then for all $\lambda$ with $k(f_\lambda) > k(f_{\lambda_\infty})$, $\varprojlim(I, f_\lambda)$ contains an indecomposable continuum.*

**Proof.** If $k(f_\lambda) > k(f_{\lambda_\infty})$ then there exists a smallest $n$ such that $\Re^n f$ does not exist. Thus, there exists an interval $[a, b]$ containing the critical in its interior such that $[a, b] \subsetneq f_\lambda^{2^n}([a, b])$. The result follows from [9, Lemma 3] and [1, Corollary 11]. ■

## 4.4 The Feigenbaum Value

In this section we identify the inverse limit occurring at the Feigenbaum value under certain smoothness conditions. Noting that $k(\Re f_{\lambda\infty}) = k(f_{\lambda_\infty})$, we would like to conclude $\varprojlim(I, f_{\lambda_\infty})$ is homeomorphic to $\varprojlim(I, \Re f_{\lambda_\infty})$. However, $k(f_{\lambda_\infty})$ is infinite and nonperiodic so we cannot use Corollary 3.1. By limiting ourselves to $S$-Full families we can apply Theorem 2.20 to conclude $\Re f_{\lambda\infty}$ is topologically conjugate to $f_{\lambda\infty}$. In addition, if $\{f_\lambda\}$ and $\{g_\mu\}$are $S$-Full families with Feigenbaum values $\lambda_\infty$ and $\mu_\infty$, respectively, then $k(f_{\lambda_\infty}) = k(g_{\mu_\infty})$ so that $f_{\lambda_\infty}$ and $g_{\mu\infty}$ are topologically conjugate. We arrive at the following conclusions.

**Proposition 4.2** *If $\{f_\lambda\}$ is a $S$-Full family with Feigenbaum value $\lambda_\infty$, then $\varprojlim(I, f_{\lambda_\infty})$ is homeomorphic to $\varprojlim(I, \Re^n f_{\lambda_\infty})$ for all $n \geq 1$.*

**Proof.** Let $n \geq 1$. By Proposition 3.2 and Lemma 3.15, $k(\Re^n f_{\lambda\infty}) = k(f_{\lambda_\infty})$ is infinite and nonperiodic with $S\Re^n f_{\lambda_\infty} < 0$. It follows from Theorem 2.20 and Corollary 2.1 that $\varprojlim(I, f_{\lambda_\infty})$ is homeomorphic to $\varprojlim(I, \Re^n f_{\lambda_\infty})$. ■

**Theorem 4.5** *Suppose $\{f_\lambda\}$ and $\{g_\mu\}$are $S$-Full families with Feigenbaum values $\lambda_\infty$ and $\mu_\infty$, respectively, then $\varprojlim(I, f_{\lambda_\infty})$ is homeomorphic to $\varprojlim(I, g_{\mu_\infty})$.*

**Proof.** Follows from Theorem 2.20, Corollary 2.1 and Lemma 3.15. ■

In considering only $S$-unimodal maps we are also forcing the dynamics of $\widehat{f}_{\lambda\infty} : \varprojlim(I, f_{\lambda\infty}) \to \varprojlim(I, f_{\lambda\infty})$ and $\widehat{g}_{\mu_\infty} : \varprojlim(I, g_{\mu_\infty}) \to \varprojlim(I, g_{\mu_\infty})$ to be identical. Of course two bonding maps need not be topologically conjugate in order to produce homeomorphic inverse limits (see Example 4). Also, requiring a negative Schwarzian derivative is very restrictive and unnatural: it implies that $|f'|$ cannot have a local maximum and is not preserved under smooth coordinate changes. It is well known that for an infinitely renormalizable unimodal map $f$ with $Sf < 0$, $\omega(c)$ is a Cantor set. However, $\omega(c)$ is a Cantor set even if $Sf \not< 0$ provided there are some smoothness requirements in a neighborhood of the critical point. This is what is needed to extend the results of Theorem 4.5 to a larger class of functions.

Following [27], we call a critical point $c$ for a $C^2$ map $f$ *non-flat* if there exists a $C^2$ local diffeomorphism $\phi$ with $\phi(c) = 0$ such that $f(x) = \pm|\phi(x)|^\alpha + f(c)$ for some $\alpha \geq 2$. For example, any $C^2$ map $f$ which is $C^{k+1}$ in a neighborhood of $c$ with $f^{(k)}(c) \neq 0$ for some $k \geq 2$ implies $c$ is non-flat [26].

**Theorem 4.6** *Suppose $f$ and $g$ are infinitely renormalizable, $C^2$ unimodal maps with non-flat critical points. Then $\varprojlim(I, f)$ is homeomorphic to $\varprojlim(I, g)$.*

**Proof.** Let $c$ and $c'$ be the critical points for $f$ and $g$, respectively. For $a, b \in I$, $[a, b]$ will denote the smallest closed interval containing $a$ and $b$, while $(a, b)$ will denote the interior of $[a, b]$. The sets

$$\omega_f(c) = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{n} [f^i(c), f^{i+2^n}(c)] \quad \text{and} \quad \omega_g(c') = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{n} [g^i(c'), g^{i+2^n}(c')]$$

are Cantor sets [27, Theorem 6.2, p. 156, and Proposition 4.5, p. 242], [18, page 346]. Because $k(f) = k(g)$ is infinite and nonperiodic, $I(f^i(c)) = I(g^i(c')) \neq$

$I(f^j(c)) = I(g^j(c'))$ for all $i \neq j \geq 0$. It follows that the sets $\{c, f(c), f^2(c), \cdots\}$ and $\{c', g(c'), g^2(c'), \cdots\}$ have the same ordering in $I$. We construct an order preserving homeomorphism $h : I \to I$ with $h(f^i(c)) = g^i(c')$ for all $i \geq 0$ as follows.

Let

$$A_n = \bigcup_{i=1}^{n} [f^i(c), f^{i+2^n}(c)] \qquad \text{and} \qquad B_n = \bigcup_{i=1}^{n} [g^i(c'), g^{i+2^n}(c')]$$

and define $h_1 : I \to I$ as the (unique) order preserving piecewise-linear homeomorphism such that $h(A_1) = B_1$. Since $\cdots \subseteq A_3 \subseteq A_2 \subseteq A_1$ and $\cdots \subseteq B_3 \subseteq B_2 \subseteq B_1$ are such that $A_{n-1} - A_n$ and $B_{n-1} - B_n$ consist of $2^n$ open intervals, one interval from each of the interiors of $[f^i(c), f^{i+2^n}(c)]$ and $[g^i(c'), g^{i+2^n}(c')]$ $1 \leq i \leq 2^n$, we can inductively define $h_n : I \to I$ as $h_n(x) = h_{n-1}(x)$ for all $x \notin A_{n-1} - A_n$ and $h_n$ is order preserving piecewise-linear on $A_{n-1} - A_n$ such that $h_n(A_{n-1} - A_n) = B_{n-1} - B_n$. Each $h_n$ is continuous and $h_n(f^i(c)) = g^i(c')$ for $0 \leq i \leq 2^n$. Using the fact that the diameters of $[f^i(c), f^{i+2^n}(c)]$ and $[g^i(c'), g^{i+2^n}(c')] \to 0$ as $n \to \infty$, we conclude $h_n$ converges uniformly to a continuous function $h$, which is onto and $h(f^i(c)) = g^i(c')$ for all $0 \leq i$. It remains to show that $h$ is one-to-one. If $h$ is not one-to-one it must be monotone on some interval $[x, y]$. Since $\omega(c)$ is perfect and $h(f^i(c)) \neq h(f^j(c))$ for all $i \neq j$, $f^i(c) \notin (x, y)$ for all $0 \leq i$. Therefore, there exist some $i \neq j$ and $k \geq 0$ such that $[x, y] \subseteq [f^i(c), f^j(c)]$ and $[f^i(c), f^j(c)] \cap A_k = \{f^i(c), f^j(c)\}$. But $h$

was constructed so that $h$ restricted to $[f^i(c), f^j(c)]$ is equal to $h_k$ restricted $[f^i(c), f^j(c)]$. Since $h_k$ is a homeomorphism $h(x) \neq h(y)$. It follows that $h$ is an order preserving homeomorphism. Therefore, by Theorem 3.2, $\varprojlim(I, f)$ is homeomorphic to $\varprojlim(I, g)$. ■

Since every member $f_\lambda$ from the logistic family has $Sf < 0$ and non-flat critical point, we see that the inverse limits considered in Theorem 4.5 and Theorem 4.6 are homeomorphic. This is an important observation since Theorem 4.5 does not follow as a corollary to Theorem 4.6. (We do not know whether $Sf < 0$ implies $f$ has non-flat critical point). We now turn our attention to the topological properties of the inverse limits of the previous theorem. Recall that if $f$ is infinitely renormalizable with $Sf < 0$, then $\varprojlim(I, f)$ and $\varprojlim(I, \Re^n f_{\lambda_\infty})$ are homeomorphic for all $n \geq 1$. Thus the core of $\varprojlim(I, \Re^n f)$ consists of two copies of $\varprojlim(I, f)$ intersecting at a common endpoint of a ray. This suggests some restrictions on the types of topologically different subcontinua of $\varprojlim(I, f)$. That this is the case was observed by Barge and Ingram [4, Theorem 7] for the logistic family. Thus, the result also applies to the inverse limits considered in Theorem 4.5 and Theorem 4.6.

**Theorem 4.7** *[4, Theorem 7] For the logistic family $\{f_\lambda(x) = \lambda x(1-x), 0 \leq \lambda \leq 4\}$ with Feigenbaum value $\lambda_\infty$, $\varprojlim(I, f_{\lambda_\infty})$ is hereditarily decomposable and contains only three topologically different subcontinua: arcs, copies of $\varprojlim(I, f_{\lambda_\infty})$, or the union of two copies of $\varprojlim(I, f_{\lambda_\infty})$ intersecting in a point.*

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