Radio Telescope Resonator Design for Observation of the 21 cm Line

M. C. Atkins,
advised by Dr. William Hetherington
Department of Physics, Oregon State University

Contents

1 Abstract

Radio astronomy allows observations of unique objects and phenomena relating to the electric and magnetic fields of celestial objects. Radio astronomy can be preformed at any time of day and is less beholden to atmospheric conditions than optical astronomy. A radio telescope designed to receive radiation at 1.421 GHz, the 21 cm line, needs a resonator designed to maximize the signal to noise ratio at this frequency

Radiation reflected off of the parabolic dish of a radio telescope can be modeled as diffraction through a circular aperture. This leads to an Airy disk pattern of radiation at the focal plane. An integral describing this pattern was set up and evaluated using the scipy package in python, yielding a description of the Airy disk.

The diameter was determined by noting that the resonator should be large enough to encompass the bulk of the reflected radiation while being as small as possible to limit noise entering from other sources. This compromise is reached by having a diameter that extends to the first zero of the Airy disk pattern, yielding an ideal diameter of 11.235 cm. The ideal length of a resonator for a telescope observing the 21 cm line was found via interference calculations to be $\frac{5\lambda}{4} \approx 26.38$ cm.

2 Background Information

2.1 The 21 cm Line

Radio astronomy allows Earth based observations of objects emitting electromagnetic radiation within the 30 MHz to 100 GHz range [1]. The telescope in this project is designed to observe sources of radiation at 1.421 GHz (roughly 21 cm wavelength), including the Sun, Jupiter, and neutral hydrogen clouds.

Hydrogen, being the most common element in the universe, is of particular interest to this project. Determining the location and relative motion of hydrogen clouds gives astronomers a way to map the universe. Neutral hydrogen emits 1.421 GHz (wavelength $\approx$ 21 cm) radiation when an electron transitions from one of the higher energy hyperfine states which split the $1s$ energy level to the lower energy state.[2].

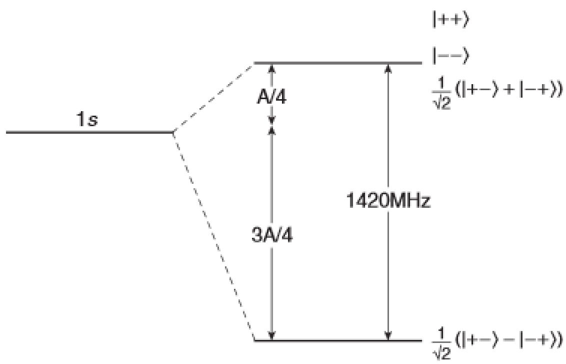

Figure 1: *Diagram displaying hyperfine splitting of hydrogen atom, figure 11.4 from "Quantum Mechanics, a Paradigms Approach", David McIntyre, page 365.*

The 21 cm line was first theorized to exist by H. C. van de Hulst in 1945 [3] and later it was found that the time it takes for any individual atom of hydrogen to make this transition is $4.4 * 10^7$ years [4]. Even with stimulation by collisions with other atoms, is only shortened to around 50 years. Due to the large amount of hydrogen in the galactic plane however, this transition occurs enough in bulk that the 21 cm line can be readily measured [5].

So how does one design a telescope to measure this radiation? A radio telescope is specialized to detect a certain frequency through the design of the resonator and antenna. The parameters of the parabolic reflecting dish affect the design of the resonator but, as long as the dish has a fine enough mesh surface that is close enough in shape to a parabola, the design of the dish is independent of wavelength [6]. The size of the dish is limited only by what can be structurally supported. A larger diameter dish has a larger aperture, and thus a greater angular resolution.

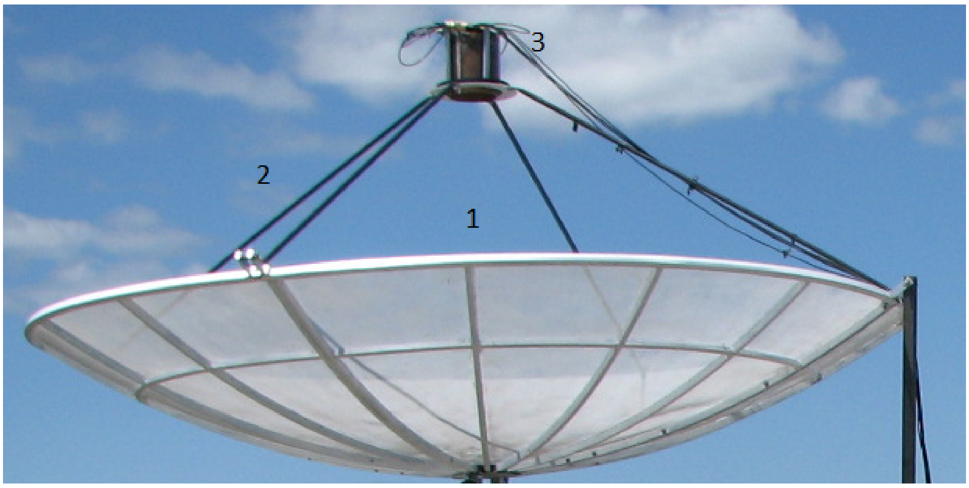

Figure 2: *Photo of the radio telescope used in this project.*
1) Parabolic reflecting dish, 3 meters in diameter.
2) Supports for resonator.
3)Current resonator housing antenna and electronic components.

The resonator takes radiation from the focal plane of the dish and directs it towards an antenna contained within the resonator. The wavelength of light that the telescope observes is determined by the size of the resonator and antenna. The size of the resonator is effected by the size of the dish. These interdependencies must be taken into account to build a geometrically ideal resonator.

2.2 Diffraction and Interference

The length of the resonating cavity is selected by accounting for the interference of incoming light in the resonating cavity. Electromagnetic waves vary between maximum amplitudes and nodes as they passes across a fixed point in space. It is optimal to receive the radiation at one of these maximum amplitudes to have a stronger signal. By taking advantage of constructive interference, a greater amplitude can be achieved at the same fixed point in space. If the peaks of two waves occur at the same point, they add to form a higher amplitude. If the back of the resonator can be set up to reflect radiation to add in amplitude to the incoming radiation at the antenna, the magnitude of the signal can be increased. See Figure 3

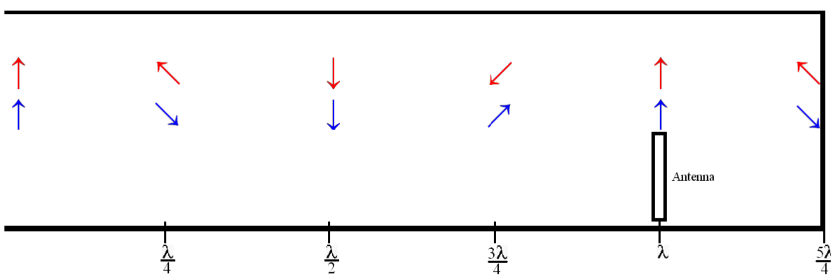

Figure 3: *The resonator, incident radiation in blue, reflected radiation from the back wall in red.*

The diameter of the resonator depends on the wavelength of the radiation, and the size of the dish. In a simple model, the diameter would be irrelevant; the light would reflect off of the parabolic dish and be focused to a single point at a fixed distance from the dish surface. In reality the diffraction of light serves to spread the focal point out into an intensity pattern. The act of a plane wave hitting the dish and reflecting off of it is analogous to the diffraction of light through a circular aperture[7], which creates an Airy disk pattern.[8]

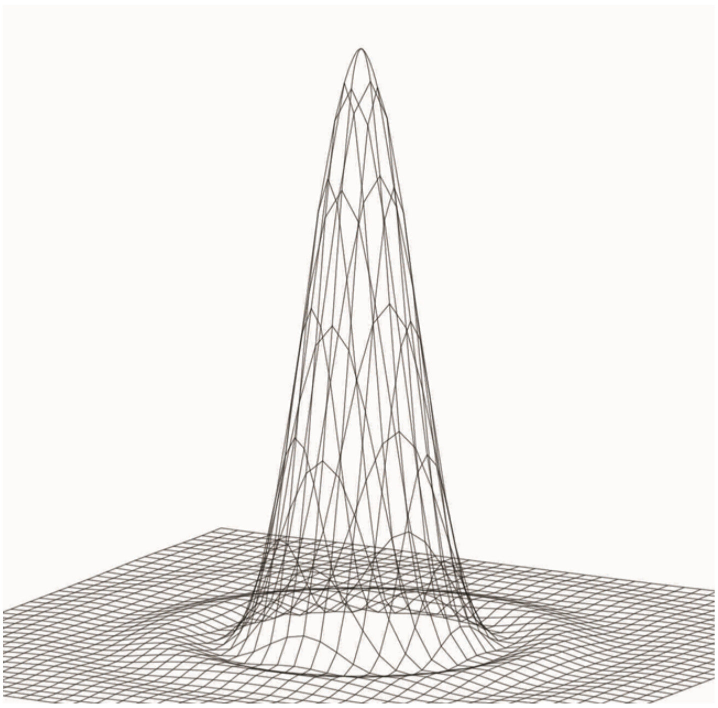

Figure 4: *A sample Airy disk pattern.*

The exact shape of the radiation pattern that arises from the dish is left as the main exercise of this thesis.

If the resonator diameter cannot be made very small to capture only a single point, why not make the diameter very large to encompass as much of the radiation pattern as possible?

There is a lot of radiation that is reflected from the ground or produced from terrestrial sources that is close in wavelength to the light the telescope is designed to detect. If this background radiation is within the scope of the resonator, it will create noise. This leads to some of the finer details the telescope could pick up being lost in the signal from unwanted radiation. The more noise that is within the scope of the resonator, the larger the details that can be washed out. If the radiation pattern at the focal plane is known, a compromise can be made to collect the bulk of the reflected radiation while minimizing unwanted background radiation.

3 Methods

3.1 Theoretical

The most involved portion of this research is determining the integral analgous to the diffraction of light through a circular aperture [9]

$$|E(\rho)| = \int_S |E_0'(\rho')| \frac{e^{ik|r|}}{|r|} dS$$

which will express the electric field of the dish at an arbitrary position. This integral is used to construct the magnitude of the electric field at the focal plane. $|E_0'(\rho')|$ describes the electric field at the dish and the other portion of the integrand accounts for how the field changes as it moves from the dish to the focal plane.

The first step is to assume that the surface of the dish can be expressed in cylindrical coordinates by $Z = a{\rho'}^2$, a parabola whose curvature is determined by a. The depth of the parabola at an arbitrary value of ρ' is

$$d(\rho') = a(R^2 - {\rho'}^2)$$

(See Figure 5) where R is the radius of the dish (meaning ρ' runs from 0 to R).

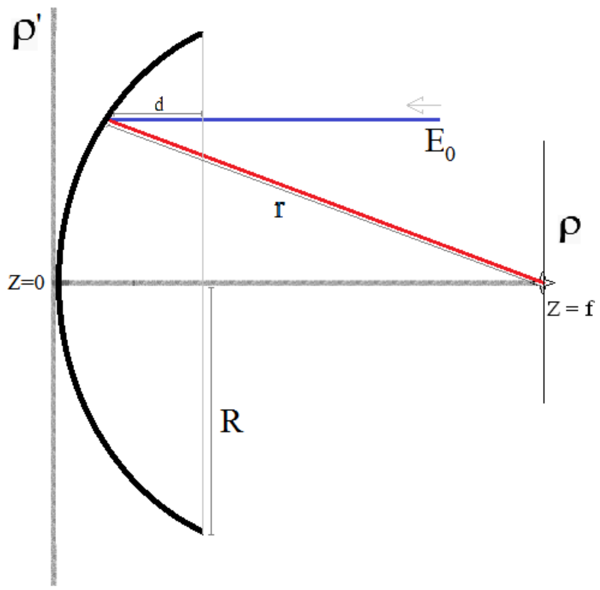

Figure 5: *Simple diagram of parabolic dish, incident radiation in blue, reflected radiation in red.*

The Z axis runs through the center of the dish to the focal point of the parabola. ρ' is the radius of the dish when it has a certain depth. This diagram shows incoming radiation focused to a point but we know that, due to diffraction and interference phenomena, there is a focal plane at a fixed value of Z (The focal length, $f = \frac{1}{4a}$) that extends to some radius ρ.

Incoming radiation is assumed to be plane waves propagating parallel the Z-axis. This assumption is defensible because the radiation is originating very far away and as the spherical wavefront increases in size, its curvature decreases as the inverse of the radius squared. The wavefront is in phase at $Z = aR^2$ (the 'face' of the dish) but as each $d\rho'$ section of the wave front travels a different distance to the surface of the dish and to the focal plane, the plane wave property is lost. Let $|E_0(\rho')|$ be the magnitude of the electric field of incoming radiation at the face of the dish. The accumulated phase difference between the incoming radiation at the face of the dish and that at the surface is $\theta = \frac{2\pi}{\lambda} a(R^2 - \rho'^2) = kd$, Where k is the wave number.

The magnitude of the electric field at the surface of the dish can be described by

$$|E_0'|(\rho') = |E_0|e^{i\theta} = |E_0|e^{ikd} = |E_0|e^{ika(R^2-\rho'^2)}$$

This is the $|E_0'(\rho')|$ in the original integral. The dependence on ρ' is very important because ρ' is one of the integration variables. The integral is now

$$|E(\rho)| = \int_S |E_0|e^{ika(R^2-\rho'^2)} \frac{e^{ik|r|}}{|r|} dS.$$

The next factor is an expression for r. It can be easily derived that the distance between two points in cylindrical coordinates is

$$r = \sqrt{\rho'^2 + \rho^2 - 2\rho\rho' \cos(\phi' - \phi) + (Z - Z')^2}.$$

The solution will have cylindrical symmetry so let $\phi = 0$. In this description $Z' = a\rho'^2$, the depth at an arbitrary value of ρ', and $Z = \frac{1}{4a}$, the focal length. If a focal point were being considered $\rho = 0$ would be a further simplification, but for a focal plane, the coordinate ρ is needed to describe the Airy disk. The integral is now

$$|E(\rho)| = \int_S |E_0|e^{ika(R^2-\rho'^2)} \frac{e^{ik\sqrt{\rho'^2+\rho^2-2\rho\rho'\cos(\phi')+(\frac{1}{4a}-a\rho'^2)^2}}}{\sqrt{\rho'^2 + \rho^2 - 2\rho\rho'\cos(\phi') + (\frac{1}{4a} - a\rho'^2)^2}} dS.$$

When evaluating surface integrals, one must construct a description for dS, a patch of your surface. If your surface happens to be functionally described as $f(x, y, g(x, y))$ where $g(x, y)$ is a differentiable function on some domain, a further step can be made to simplify integration over the surface. This changes the integral over the surface to an area integral over the projection of the surface onto a plane [10]. The statement of this is

$$\int_S f(x, y, z) dS = \int_A f(x, y, g(x, y)) \sqrt{\frac{\partial g}{\partial x}^2 + \frac{\partial g}{\partial y}^2 + 1} \; dA.$$

For a parabola

$$g(x, y) = Z(x, y) = a\rho'^2 = a(x'^2 + y'^2)$$

so the scaling factor added is

$$\sqrt{4a^2x'^2 + 4a^2y'^2 + 1} = \sqrt{4a^2(x'^2 + y'^2) + 1} = \sqrt{4a^2\rho'^2 + 1}$$

The full form of the integral which describes the electric field at the focal plane is then:

$$\int_0^{2\pi} \int_0^R e^{ika(R^2 - \rho'^2)} \frac{e^{ik\sqrt{\rho'^2 + \rho^2 - 2\rho\rho' \cos(\phi') + (\frac{1}{4a} - a\rho'^2)^2}}}{\sqrt{\rho'^2 + \rho^2 - 2\rho\rho' \cos(\phi') + (\frac{1}{4a} - aR^2)^2}} \sqrt{4a^2\rho'^2 + 1} \;\; \rho' d\rho' d\phi'$$

The above integral is very unwieldy and has no analytic solution without making approximations based on the focal plane being relatively far away compared to the radius of the dish, which isn't true ($R \approx 3f$). Using python, we can reach a numerical solution. The method for solving this is first to break the integral into two distinct portions: a real part and an imaginary one. Each of these is integrated separately using the integration function in the scipy package for python, which evaluates the integral on a point by point basis. For each point the real and imaginary integrals are used to find the intensity ($|E(\rho)|^2$) there. This is the radiation pattern we are looking for.

An empty array is set up with each element corresponding to a distinct value of ρ. The array is then filled with the results from the integral. Finally a plot of Intensity versus Radius is created from the array which shows the behavior of the radiation at the focal plane. The pattern that arises has Bessel Function behavior (called an Airy Disk [11]) and the first zero of intensity corresponds to the intended radius of the resonator.

4 Results

4.1 Theoretical Results

4.1.1 Length

The ideal length of the resonator considering only elementary wave reflection and interference phenomena is $\frac{5\lambda}{4}$. This point was made clearer via Figure 3.

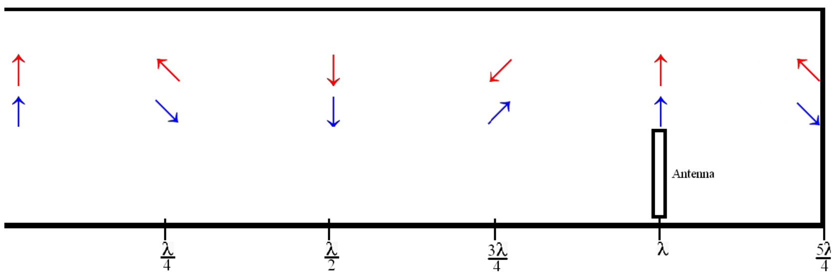

Figure 6: *The resonator, incident radiation in blue, reflected radiation from the back wall in red.*

This result is purely theoretical and derived from the behavior of the wave as it reflects from the back wall and the principle of interference.

4.1.2 Diameter

In the methods section the theoretical integral that gives electric field pattern at the focal plane was determined to be:

$$\int_0^{2\pi}\int_0^R e^{ika(R^2-\rho'^2)}\frac{e^{ik\sqrt{\rho'^2+\rho^2-2\rho\rho'\cos(\phi')+(\frac{1}{4a}-a\rho'^2)^2}}}{\sqrt{\rho'^2+\rho^2-2\rho\rho'\cos(\phi')+(\frac{1}{4a}-aR^2)^2}}\sqrt{4a^2\rho'^2+1}\quad \rho' d\rho' d\phi'$$

which was noted to have no analytic solution. Using python (and the scipy.integrate package specifically) the integral was solved numerically and yielded the following intensity pattern...

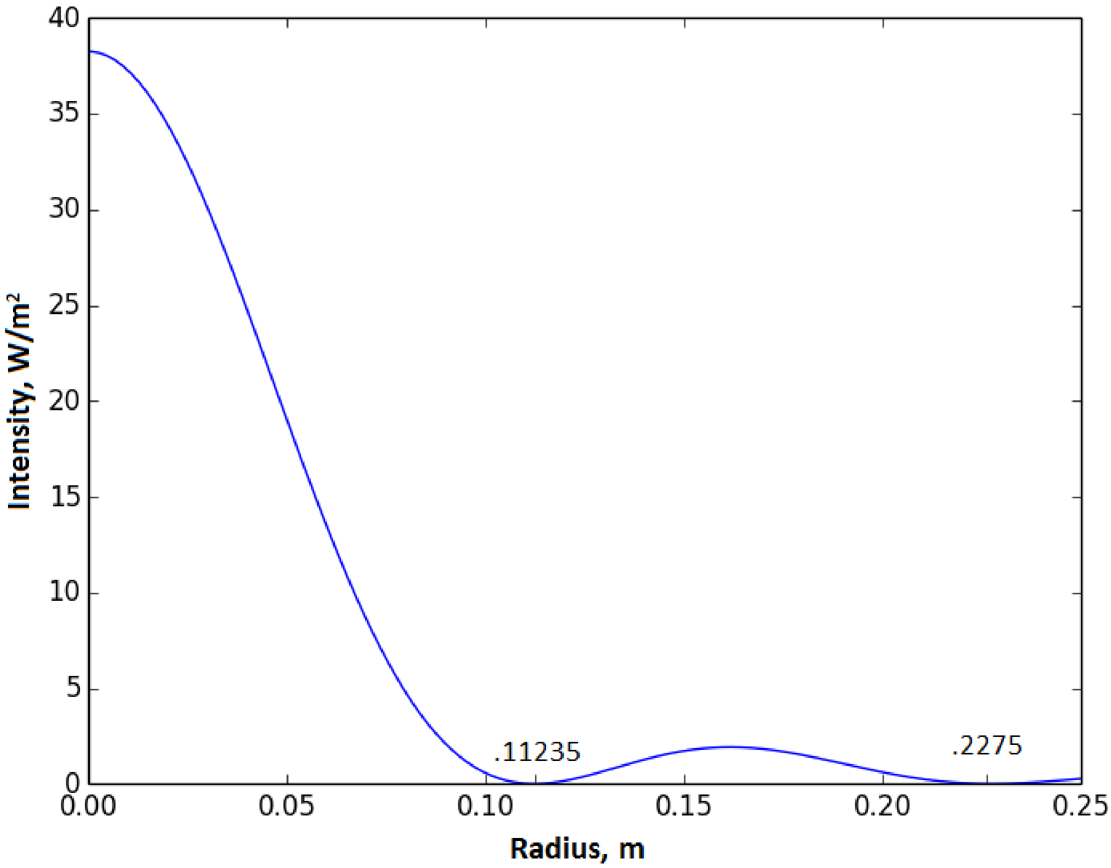

Figure 7: *The result of the code determining the Airy Disk pattern, note Bessel function like behavior.*

Which we can analyze more closely near the first zero

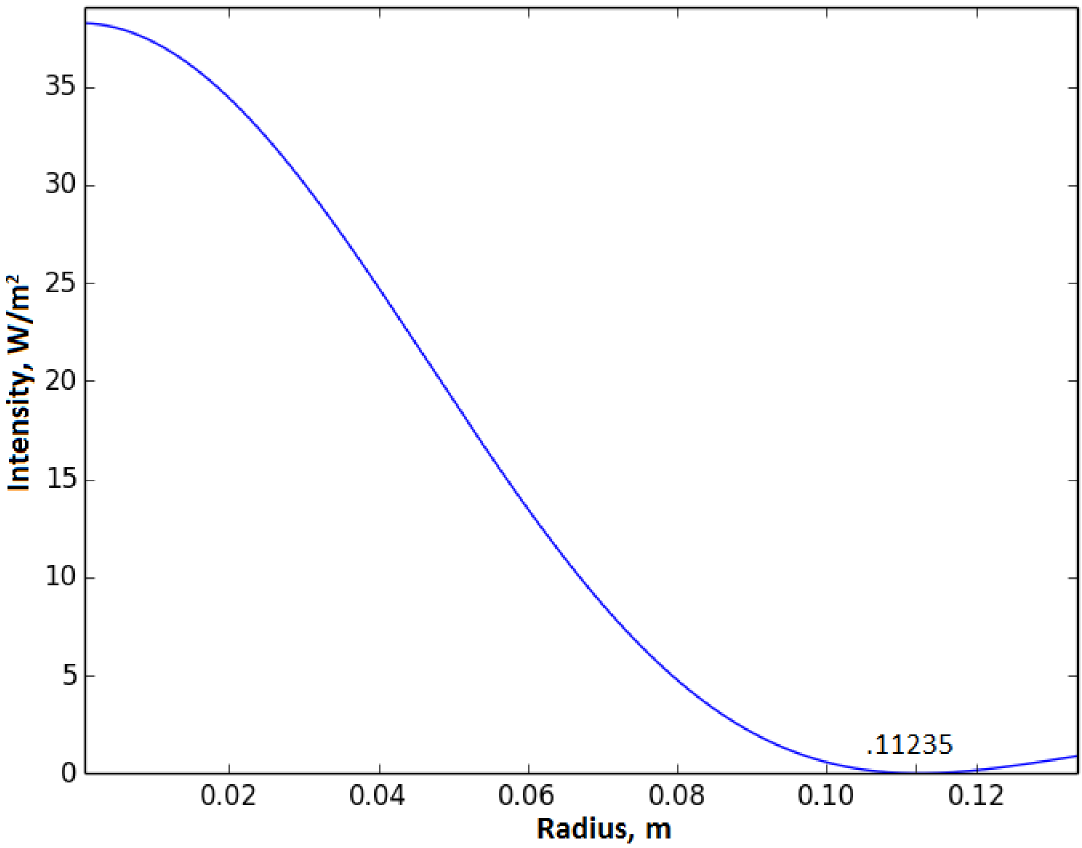

Figure 8: *The result of the code determining the Airy Disk pattern focusing on a narrower range for the radius.*

The first zero of the graph is at 11.235 cm and corresponds to the desired radius. This graph shows the intensity pattern from $\rho = 0$ cm to $\rho = 15$ cm in steps of 0.01 cm. The code is provided in an appendix.

5 Conclusions

A constant goal throughout all facets of the radio telescope project is to increase the signal to noise ratio. This issue has been addressed on two fronts in this paper: Ensuring the highest amplitude of radiation possible arrives at the antenna by idealizing the length of the resonating cavity, and finding a compromise between including more signal and introducing more noise by choosing an appropriate resonator diameter. The choice of our ideal length depended on the simplest of wave principles, showing even a concept learned at the 212 level can be a source of gains later on. Determining the ideal diameter was a more involved task.

The first logical step to determining what an ideal diameter might be came from associating the reflection of light off of the dish with the diffraction of light through a circular aperture. This is conceptually affirmed when one thinks of the calculus manipulation used to change the surface integral over a paraboloid into an area integral over the projection of the parabolic surface onto a plane. At this point light coming through a circular aperture is indistinguishable with what the integral is describing. This led to the ability to predict Airy disk like patterns of intensity that were confirmed via numeric integration.

5.1 Further Work

Experimentally confirming the results found through numerical integration will be an involved process. First the dish of the radio telescope must be erected. This task requires the assistance of five or more other people. The dish is lifted into position and then held there while someone on a ladder fixes the screws on the front and back to keep it in place.

Figure 9: *Photo of the dish being fixed in place.*

To confirm the theoretical results from code, several resonators will need to be constructed. Each is essentially a copper cylinder with one closed end and a few modifications to support an antenna and electronic components...

The signal leaves the antenna and immediately enters a Low Noise Amplifier (LNA). This increases the signal strength while introducing relatively

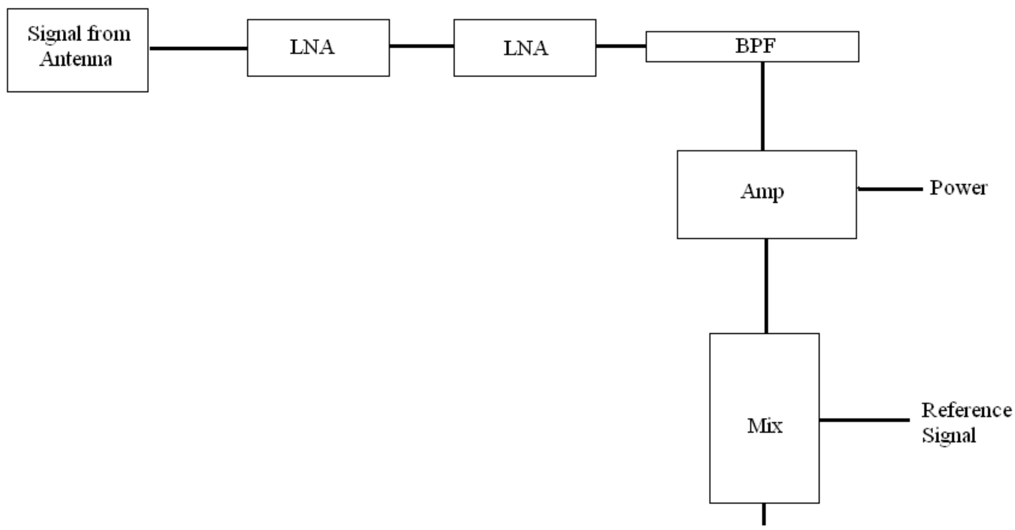

Figure 10: *Simple progression of electronics on the resonator.*

little noise, but that comes at the cost of relatively lower amplification. After passing through another LNA, the signal is passed through a Band Pass Filter which allows certain frequencies to pass through while attenuating others. The narrower the band of allowed frequencies can be the better, we are concerned with only one frequency, not a range. The signal, now slightly more refined, enters another amplifier and is finally passed through a mixer where the signal can be mixed with various reference signals before moving on to the computer for analysis.

After the test resonators have been constructed and affixed to the dish, a source of 1.421 GHz radiation will be needed to test the resonators' performance. This can be done by powering a Voltage Controlled Oscillator (VCO) connected to a quarter wavelength antenna affixed to a grounded plane. The VCO oscillates at a frequency determined by the applied voltage driving current up and down through the antenna which radiates at that frequency. The source configuration will be held in place on the roof as far as possible from the telescope to ensure that the radiation arriving at the dish is as near to planar wave fronts as possible. The telescope will be directed towards the source and receive the signal. After testing several times for each of the resonators, there will be an experimental statement of which resonator produced the best results. I have personally never acquired and analyzed signal from the radio telescope before so my knowledge of the process after manufacturing a signal for the dish is sparse.

While constructing test resonators it may also be gainful to test how much extra noise is introduced by extending the diameter out to the second zero of the intensity pattern, see Figure 7 .

If the level of noise picked up by this resonator is deemed tolerable (that is, the new resonator increases the signal to noise ratio), a small but still appreciable amount of the intensity pattern can be added to the signal.

6 Acknowledgements

Dr. William Hetherington; my research advisor who provided the concept for the question and tools to pursue it.

Dr. Janet Tate; the instructor of the PH403 thesis writing who provided advice for writing the thesis in general, as well as edits once sections were complete.

7 References

[1] Radio Astronomy Frequency Allocations, (http://www.ukaranet.org.uk/basics/frequency_allocation.html), accessed on 5-30-14.

[2] David McIntyre, *Quantum Mechanics: a Paradigms Approach* (Addison Wesley, San Francisco, 2012), Chap. 11, Sec. 4, p. 364.

[3] Woodruff Turner Sullivan, *Classics in Radio Astronomy* (D. Reidel Publishing Company, Dordrecht, Holland), Chap. 5, Paper 34, p. 311-312.

[4] Woodruff Turner Sullivan, *Classics in Radio Astronomy* (D. Reidel Publishing Company, Dordrecht, Holland), Chap. 5, Paper 35, p. 319.

[5] F. Graham Smith, *Radio Astronomy* (Penguin Books, Baltimore, 1960), Chap. 8, p. 92.

[6] F. Graham Smith, *Radio Astronomy* (Penguin Books, Baltimore, 1960), Chap. 18, p. 216-217.

[7] Dr. William Hetherington (Private contact).

[8] Eugene Hecht, *Optics* 4th Ed. (Addison Wesley, San Francisco, 2002), Chap. 10, Sec. 2.5, p. 469.

[9] Eugene Hecht, *Optics* 4th Ed. (Addison Wesley, San Francisco, 2002), Chap. 10, Sec. 2.4, p. 464, eqn. 10.37.

[10] Mary L. Boas, *Mathematical Methods in the Physical Sciences* 3rd Ed. (John Wiley and Sons, Inc, Hoboken, 2006), Chap. 5, Sec. 4, p. 271.

[11] Eugene Hecht, *Optics* 4th Ed. (Addison Wesley, San Francisco, 2002), Chap. 10, Sec. 2.5, p. 469.

8 Appendices

8.1 Code

```
from __future__ import division
import scipy as sp
import numpy as np
import pylab as pl
from scipy.integrate import quad, dblquad

'''
Important Numbers
W = 0.21112       # Wavelength, meters
R = 3             # Dish Radius, meters
f = 1.02          #Focal length, meters
a = 1/(4*f)       #Scale facor of the parabola, inverse meters
```

```
k = (2*sp.pi)/W   #Wavenumber, inverse meters

'''

def DishPropagatorImaginary(rho_p, phi_p, k, E, a, f, R, rho):
#rho and phi are locations of a point on the image plane.
#rho_p and phi_p are locations of a point 'on' the 2-dimensional projection
#of the dish onto a circular plane, the origin is at the center of the plane,
#p = 0.

d = a*(R**2 - rho_p**2)
E_0 = rho_p*E*(sp.exp(k*d*1.0j))*sp.sqrt(4*a**2*rho_p**2+1)
#Including R scaling factor from polar cylindrical surface integral
# and projection scaling factor.
r = sp.sqrt(rho_p**2+rho**2-2*rho*rho_p*sp.cos(phi_p)+(f-a*rho_p**2)**2)
g = (E_0*sp.exp((1.0j*k*r)))/r
return sp.imag(g)

def DishPropagatorReal(rho_p, phi_p, k, E, a, f, R, rho):

d = a*(R**2 - rho_p**2)
E_0 = rho_p*E*(sp.exp(k*d*1.0j))*sp.sqrt(4*a**2*rho_p**2+1)
r = sp.sqrt(rho_p**2+rho**2-2*rho*rho_p*sp.cos(phi_p)+(f-a*rho_p**2)**2)
g = (E_0*sp.exp((1.0j*k*r)))/r
return sp.real(g)
```

```
def DishIntegral(k, E, a, f, R, rho) :

real_part = sp.integrate.dblquad(DishPropagatorReal, 0, 2.0*sp.pi, lambda x:0,
lambda x: R, args=(k, E, a, f, R, rho), epsabs=1.0e-02, epsrel=1.0e-02)[0]
imaginary_part = sp.integrate.dblquad(DishPropagatorImaginary, 0, 2.0*sp.pi,
 lambda x:0, lambda x: R, args=(k, E, a, f, R, rho),
 epsabs=1.0e-02, epsrel=1.0e-02)[0]
return sp.absolute(real_part + 1.0j * imaginary_part)
*sp.absolute(real_part + 1.0j * imaginary_part)


def DishIntensity() :
R = 1.5                        #Radius, meters
k = (2.0 * sp.pi)/(.21112)     # Wavenumber, m^-1
f = 1.02                       #focal length, in meters.
a = (1/(4*f))                  # parabola factor, m^-1
phi = 0                        #answer has axial symmetry

E = 1                          #Magnitude of the electric field at the dish.

rho = sp.arange(0.0, .15, 0.0001) # Observe from rho = 0 to
                                  # rho = 15 cm in steps of 0.01 cm.
```

```
i = 0

v = np.zeros(1500) #Array of zeros to be filled with values


while i < len(rho):

d = DishIntegral(k, E, a, f, R, rho[i])

v[i] = v[i] + d

i = i+1




print(np.argmin(v))

pl.plot(rho,v)

pl.show()

#-----------------------------------------------------------------------


if __name__ == "__main__" :

    DishIntensity()
```