AN ABSTRACT OF THE THESIS OF

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Title   THE FOUR COLOR PROBLEM BEFORE 1890

Abstract approved ██████████████████████████████████████
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This work contains a brief history of the four color problem from 1840 to 1890. This includes Kempe's attempted proof of the problem as well as maps which illustrate Heawood's discussion of Kempe's error. The remaining part is a discussion of Kempe's and Story's work on patching out maps. Story modified Kempe's original work to make the formulas more rigorous. Unfortunately, Story's work also relied upon Kempe's error and so is also in part incorrect.

THE FOUR COLOR PROBLEM BEFORE 1890
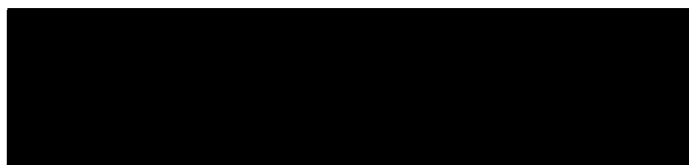
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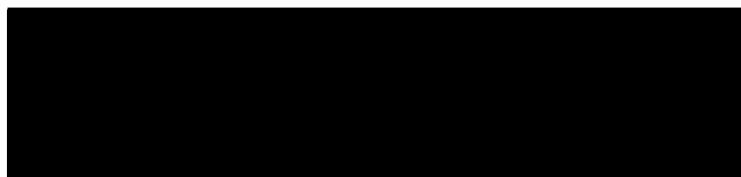the requirements for the
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TABLE OF CONTENTS

THE FOUR COLOR PROBLEM BEFORE 1890

I. AN EARLY HISTORY OF THE PROBLEM

A problem that has received much attention since it was first

mentioned in lectures in 1840 by A. F. Mobius is referred to as the

four color problem.   The question might be expressed by asking how

many colors are necessary to color a map of a country, divided into

regions, in such a way that no two contiguous regions are of the same

color.   Contiguous regions can be defined as districts that contain a

common boundary separating the two subdivisions in whole or part.

Also it is considered that a map be drawn on a simply-connected sur-

face, such as a plane or sphere.   The number of districts is finite

and each district will consist of a connected area.   Geographers had

long known from experience that the geographical maps of an area,

arranged by political subdivision, as of a continent into countries, or

a state into counties, could be colored using only four colors.

The formulated problem gained attention in 1850 when Francis

Guthrie communicated it to De Morgan, but it was not until 1878 that

the problem was presented before a wide mathematical society.   At

that time Cayley proposed it to the London Mathematical Society; in an

address published in the proceedings of that society he stated that he

could not obtain a rigorous proof of the proposed theorem [5] .   In the

next year there was published a solution to the proposed problem by

A. B. Kempe which was sent across the Atlantic to be recorded in the American Journal of Mathematics [12] and shortly thereafter communicated in simplified form to the London Mathematical Society Proceedings [11] and also to Nature [10]. William E. Story wrote a note which followed Kempe's work and modified the formula for special cases [14]. This helped in the clarification of Kempe's theory on patching out maps.

In 1880, Tait published a solution of a theorem which states that if a closed network of lines in a plane joining an even number of points is such that three and only three lines meet at each point, then three colors are sufficient to color the lines in such a way that no two lines meeting at a point are of the same color [2]. Tait's argument showed that some maps could be colored with four colors, but he was unable to extend this solution to include every map.

Thus, at this point, Kempe had presented the only work that seemed to be a solution to the proposed problem. In 1890, however, the problem was seen to be unsolved when P. J. Heawood pointed out an error in Kempe's reasoning. This work was published in articles appearing in the Quarterly Journal of Mathematics, London [9]. Although Kempe's work is not accepted as a proof, we find that Heawood was able to show that five colors are always sufficient for coloring ordinary maps.

It is easy to show that four colors are sometimes necessary to

color a map, but whether four colors are always sufficient to color

any ordinary map still remains an open question. It is agreeable,

today, to say that this problem is one of the simplest sounding un-

solved problems of mathematics.

II. KEMPE'S ATTEMPTED PROOF AND HEAWOOD'S COUNTER EXAMPLE

In 1879 Kempe attempted to solve the four color problem. But, later in 1890 Heawood presented an example that pointed out an error in Kempe's reasoning. The following is an account of the work leading up to Kempe's error along with maps that illustrate the account in detail. Heawood's discussion together with a map will be sufficient in pointing out Kempe's error.

We have recognized the fact that where it is important to have a distinction clearly marked between two areas, the method has been adopted by map-makers of painting districts with different colors so that boundaries are clearly defined. It has been pointed out that it is unnecessary that non adjacent districts be painted different colors. At the same time districts that meet at one or two points can be painted the same color. Kempe called a succession of contiguous districts that alternate in color a region. For example in Figure 6 the blue, green districts form a region.

Kempe confined his investigation primarily to simply or singly connected surfaces such as surfaces on a sphere or plane which are divided into two parts by a circuit. He explained that a simply connected surface divided in any manner, into districts, could be colored in such a way that no two adjacent districts would be of the same color.

Proceeding at random, coloring as many districts as possible with one color and then choosing another color, he sometimes found that many different colors were necessary. But, if a little care were used from the start, many times the number of colors could be reduced. He extended his discussion by stating that it was obvious that four colors are necessary to color a map where one considered the example of having one district surrounded by three others, Figure 1, but that four colors were sufficient was by no means obvious to the world.
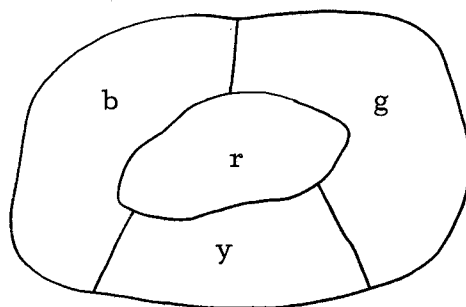

Figure 1

Kempe's work, then, is an attempt to show that four colors will suffice in all cases. When attempting to do this he found that a major difficulty arose when a very small alteration rendered it necessary to recolor the entire map.

Considering the difficulties, Kempe explored the idea of having a map that required four colors and directed his attention to those districts colored with two different colors. He found that they formed

one or more disjoint regions. These regions have no boundary in common, though possibly they could meet at a point or points. It was readily seen that he could interchange some pairs of colors and the map was still properly colored.

He then considered the state of things at a point where three or more boundaries and districts meet. He found it convenient to call such a point a point of concourse. It was then established that if three districts meet at the point, they must be colored with three different colors. Thus he found that when four districts meet, they could be colored with four colors, but then in some cases two or three colors were sufficient. He also stated that the same thing could be shown at a point of concourse where five boundaries meet. Under these conditions, the map would be colored with three colors or possibly four colors. If four colors were used, Figure 2 shows the only form which the coloring could take, that being where one color occurs twice.
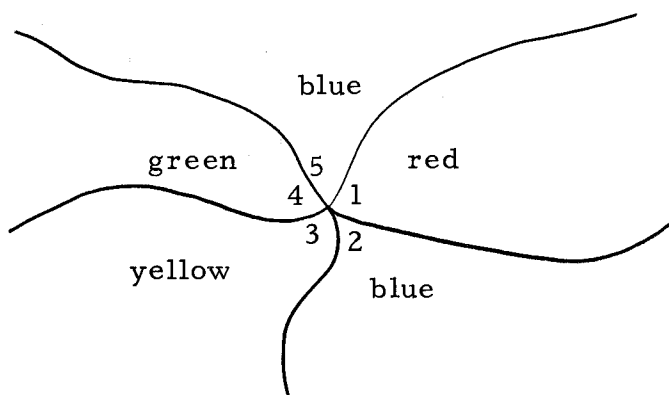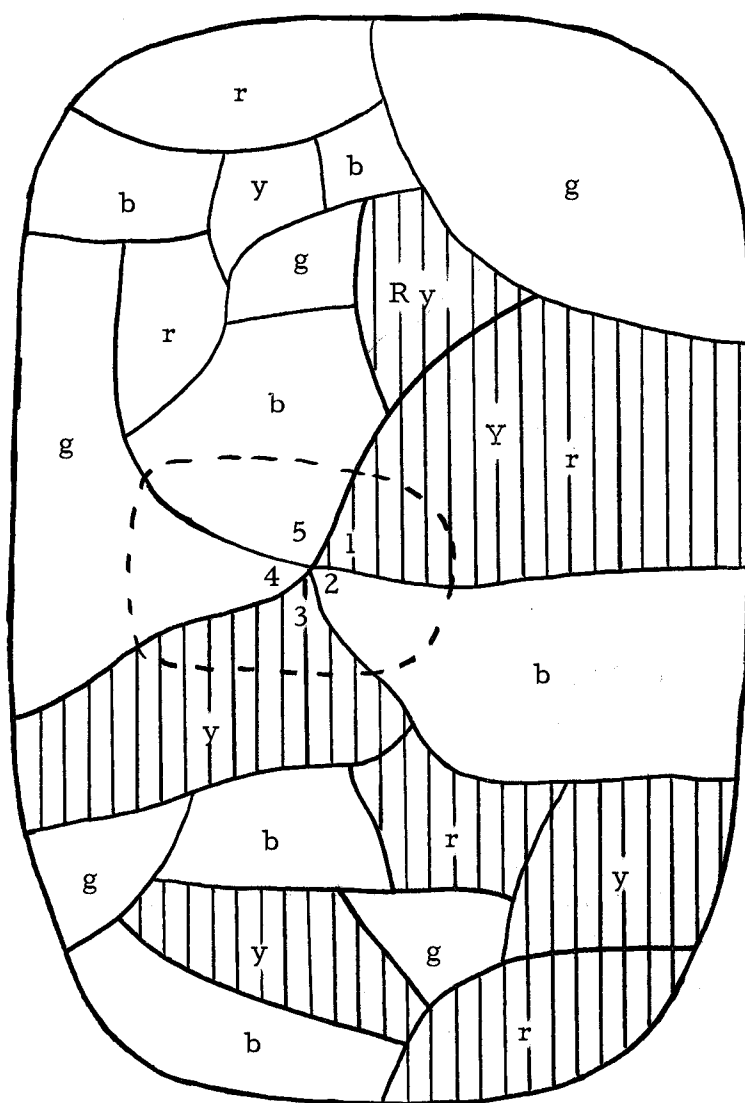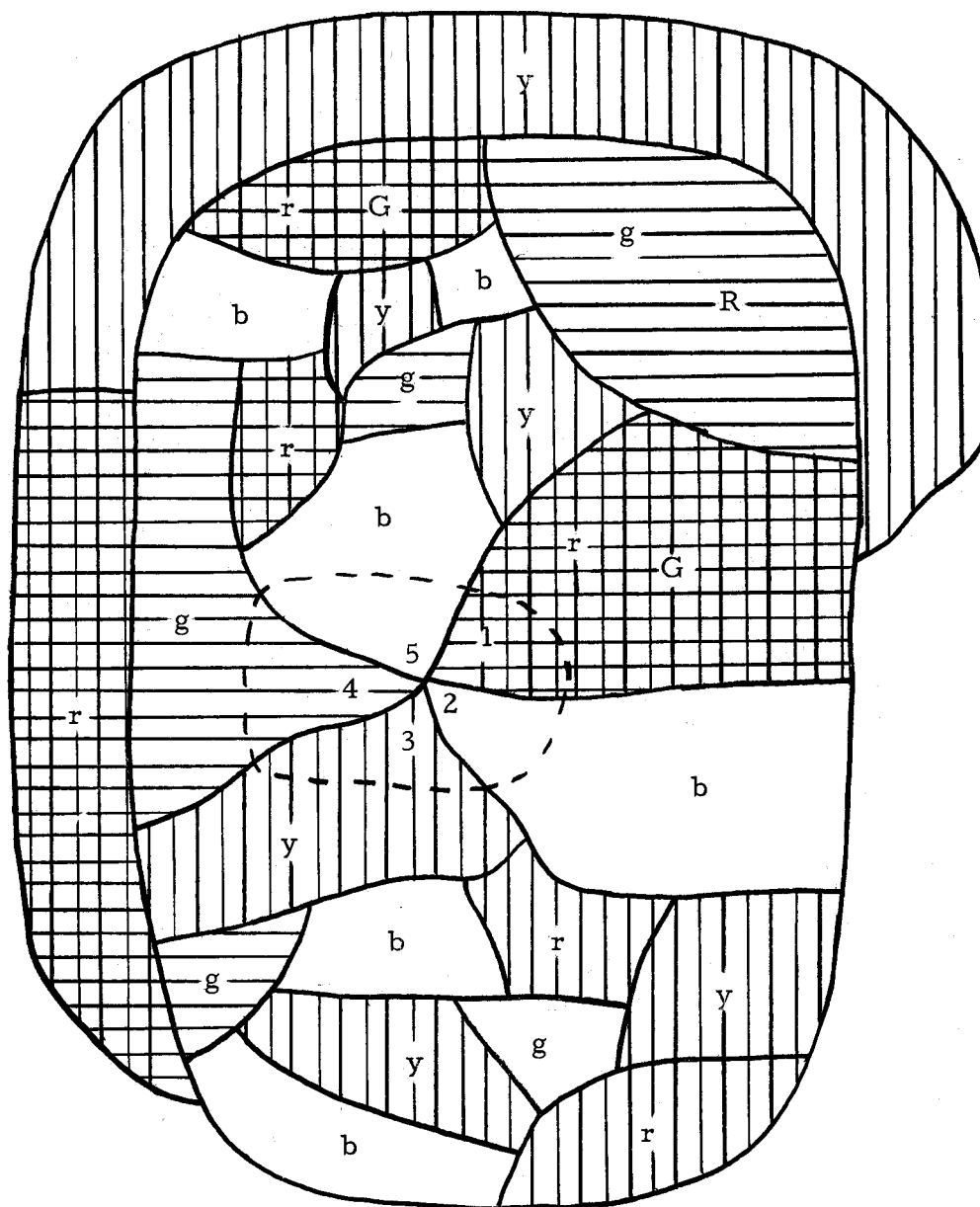

Figure 2

Kempe attempted to show that at such a point of concourse, the colors could always be reduced to three. For if 1 and 3 belonged to different yellow and red regions, he could interchange the colors so 1 and 3 were both yellow or both red (Figure 3). In the figures lower case letters represent the original colors and capital letters denote the colors after the interchanges. If 1 and 3 belong to the same yellow and red region, he checked on 1 and 4 and if they belonged to different green and red regions, they would be colored both red or both green (Figure 4). He then considered letting 1 and 3 belong to the same yellow and red region, along with letting 1 and 4 belong to the same green and red region (Figure 5), then he found that 2 and 5 were cut off by the region 1 and 3 and the region 1 and 4 (Figure 5). Thus Kempe claimed that the blue and green region to which 2 belongs is different from that to which 4 and 5 belong (Figure 6) and also that the blue yellow region to which 5 belongs is different from that to which 2 and 3 belong (Figure 7). Thus, Kempe interchanged the colors in the blue and green region to which 2 belonged, and in the blue yellow region to which 5 belonged, 2 was then green and 5 became yellow (Figure 8). Then Kempe was convinced that he had reduced the number of colors at the point of concourse to three for every possible case. Kempe was then convinced that his efforts allowed him to say that the number of colors at the point of concourse of five boundaries could always be reduced
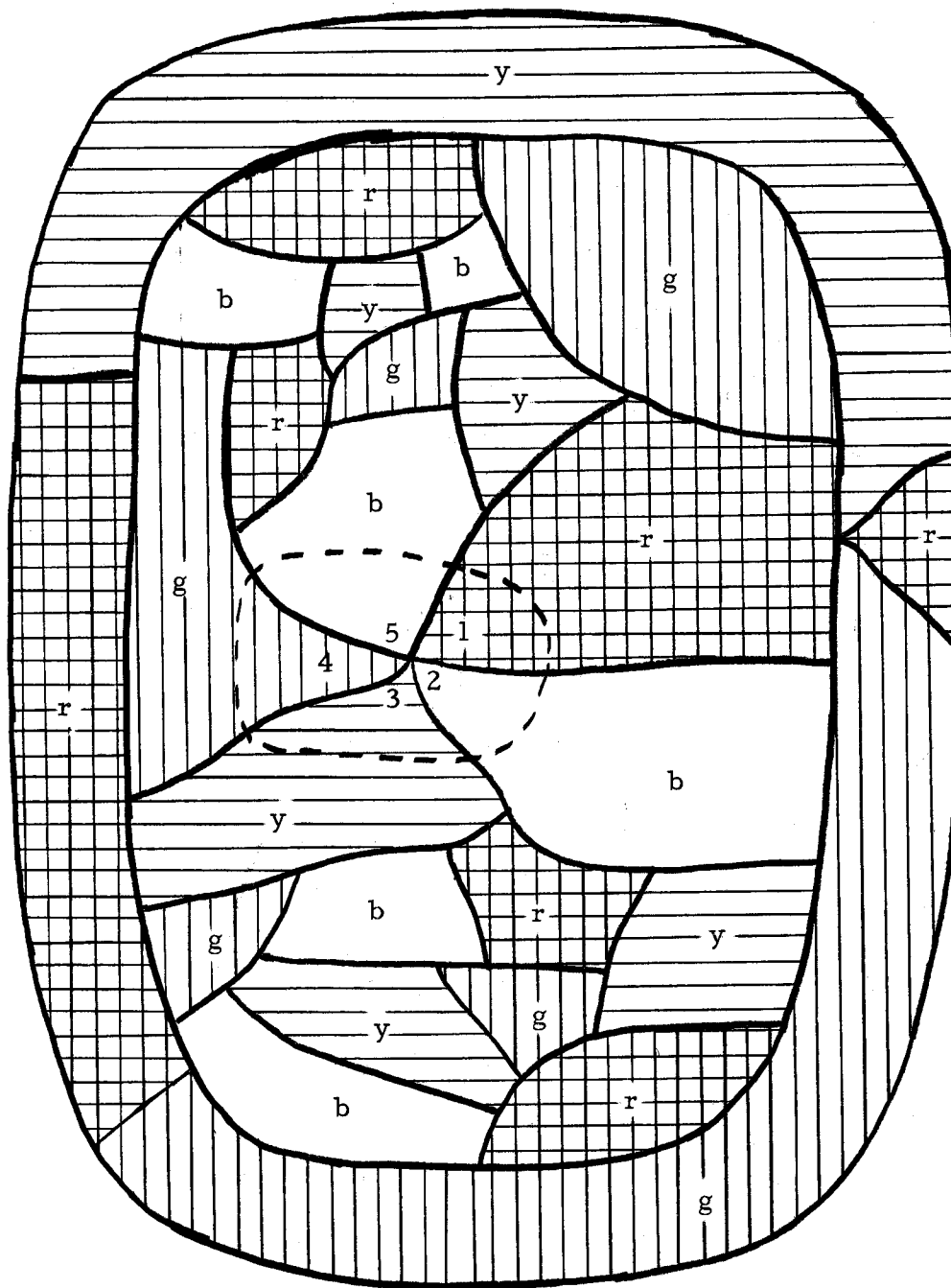
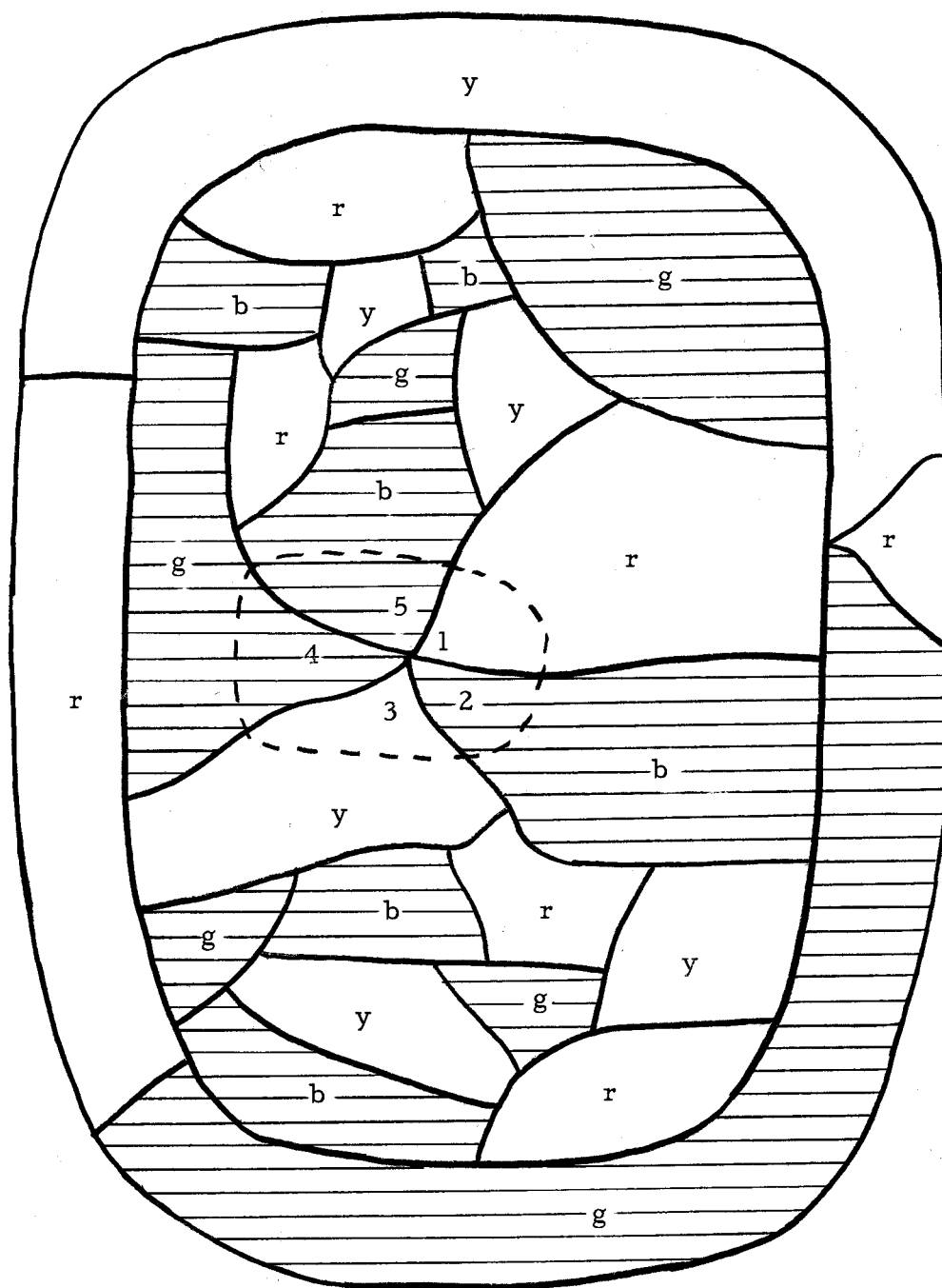Vertical lines denote the yellow and red regions of 1 and 3.

Figure 3

Vertical lines denote the same yellow and red regions of 1 and 3.
Horizontal lines denote different green and red regions of 1 and 4.
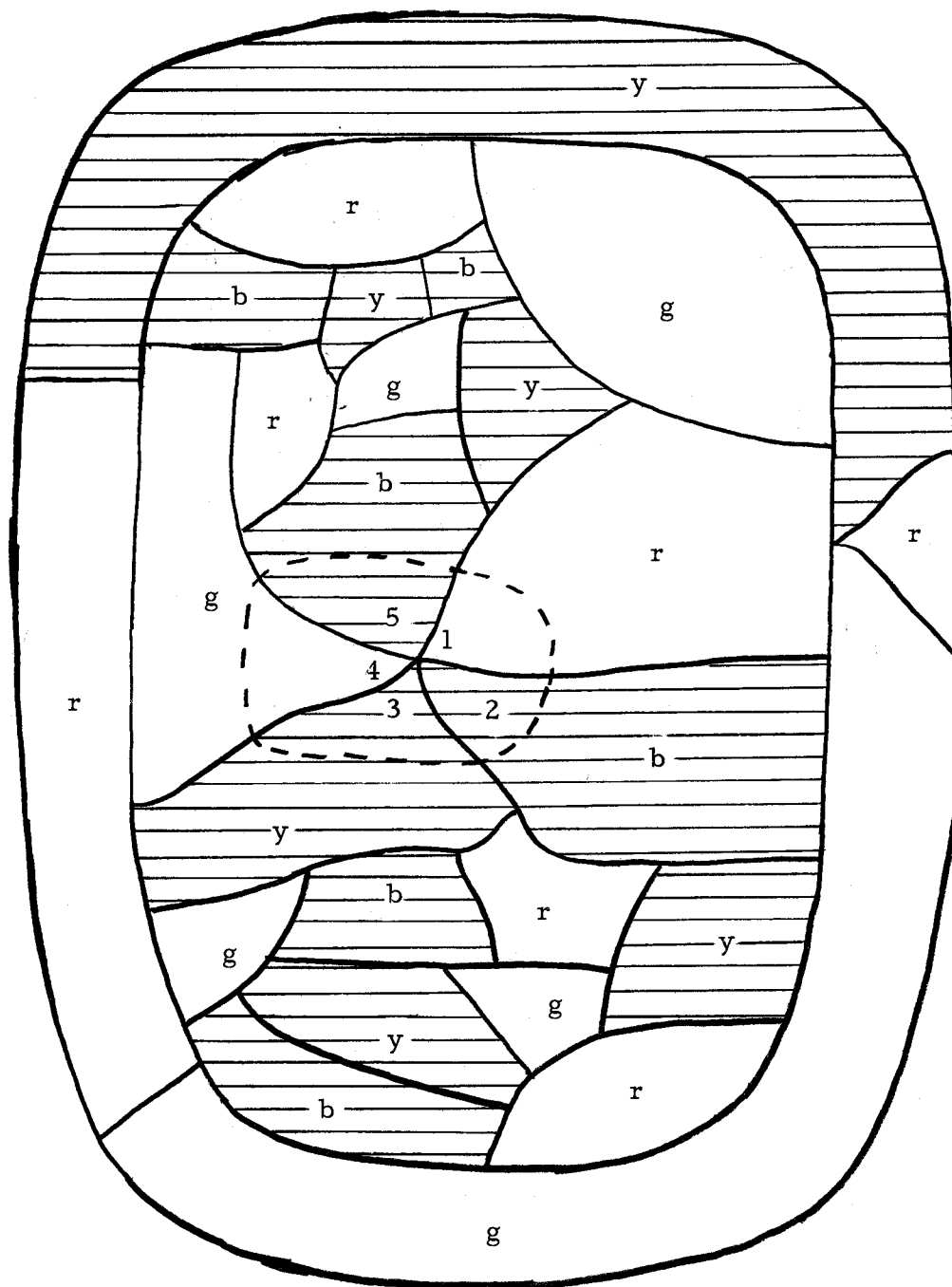
Figure 4

Vertical lines denote the same green and red region of 1 and 4.
Horizontal lines denote the same yellow and red region of 1 and 3.

Figure 5

Horizontal lines denote the blue and green region to which 2 belongs and also the blue and green region to which 4 and 5 belong.
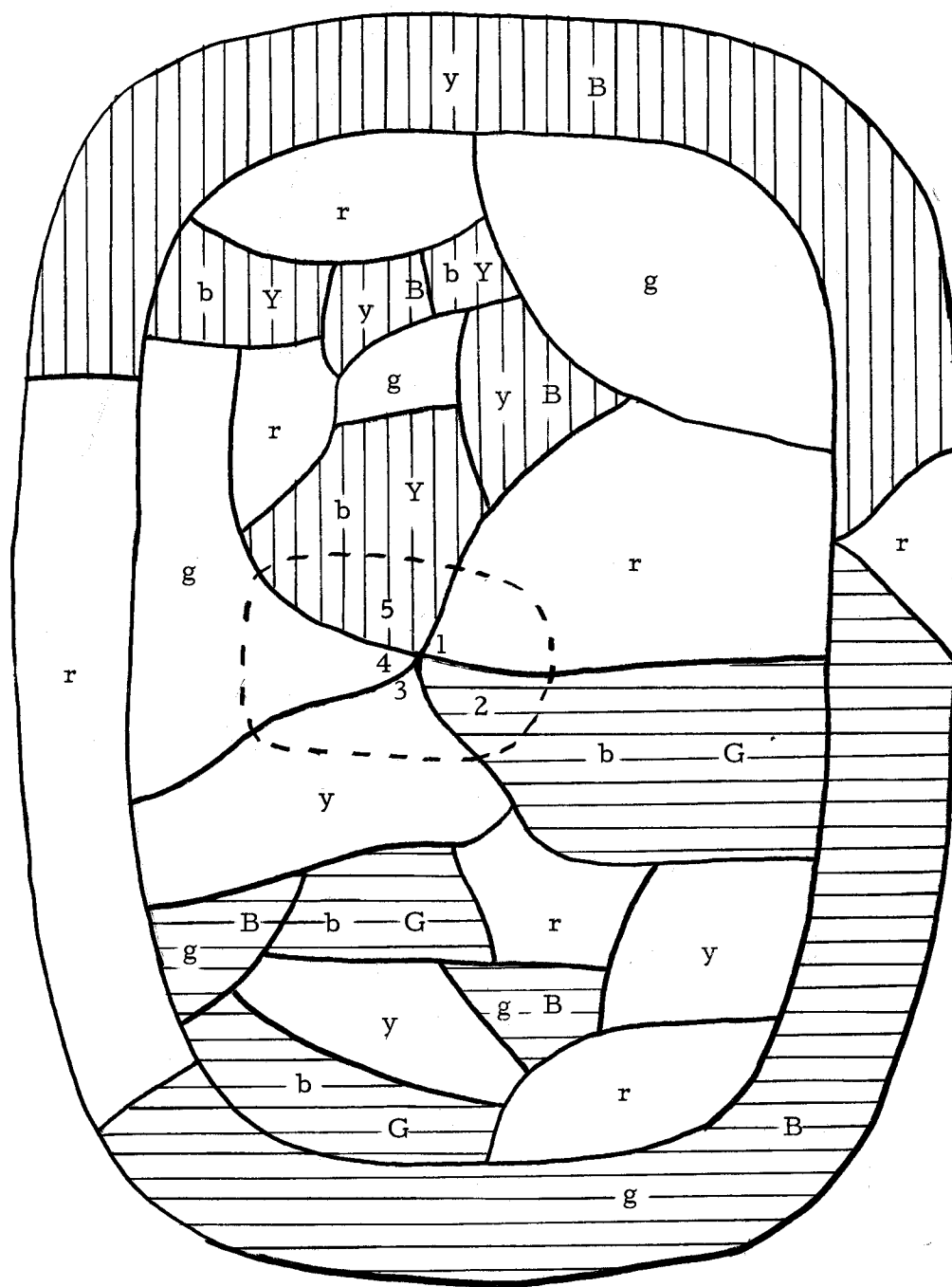
Figure 6

Horizontal lines denote the blue and green region to which 2 and 3
belong and also the blue and green region to which 5 belongs.
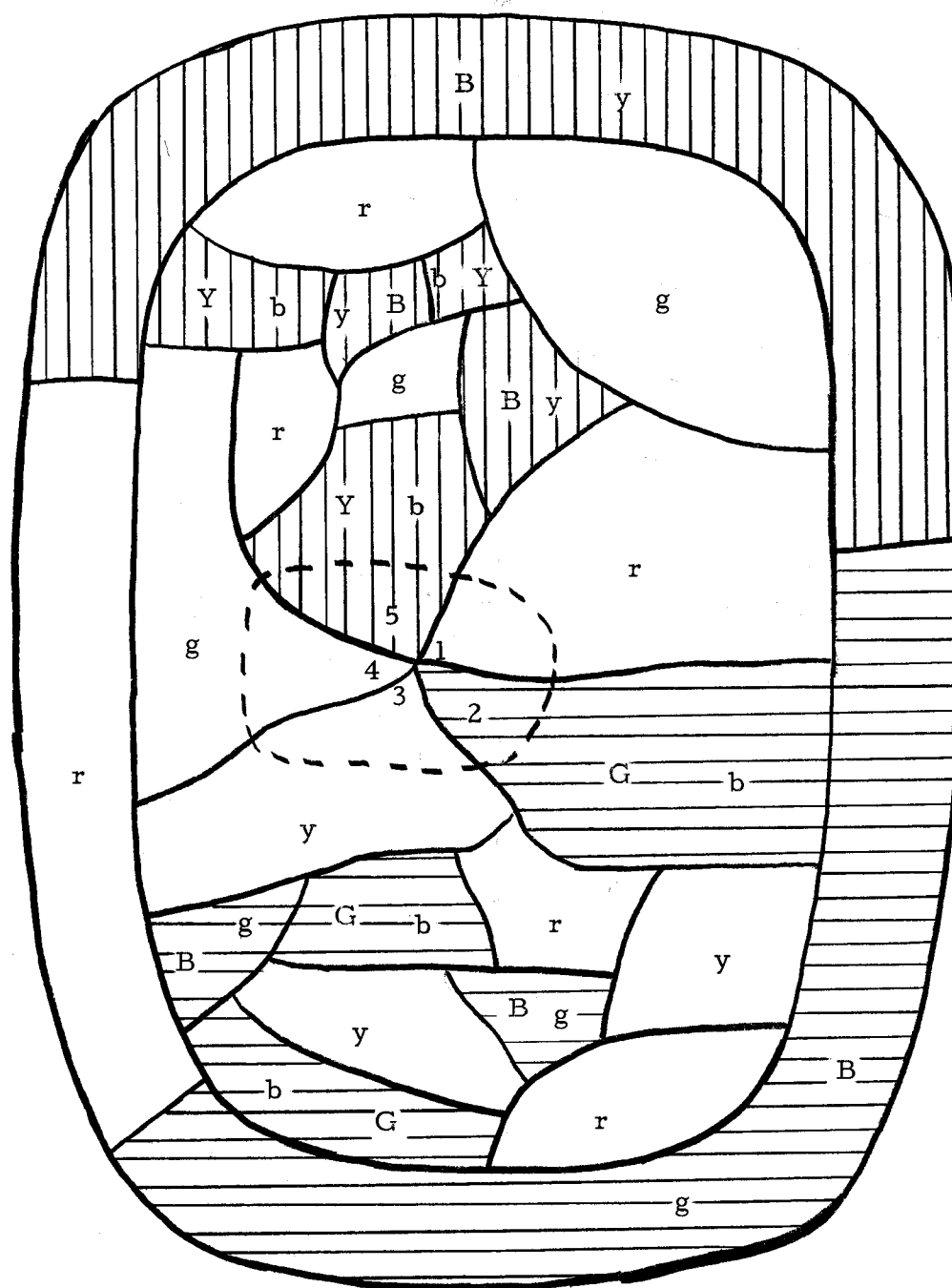
Figure 7

Horizontal lines denote the blue and green region of 2.

Vertical lines denote the blue and yellow region of 5.

Figure 8

to three. After more consideration Kempe stated that if less than six

boundaries meet at a point the colors of the district can always be

rearranged so that the number of colors at that point shall be only

three. He continued with the statement that the colors of any map

could be arranged so that not only will three colors meet at any given

point of concourse, however many boundaries meet there, but also

that at no point of concourse in the map will four colors appear.

However in 1890, Heawood pointed out an error in Kempe's

solution of the four color problem. Also at the same time, he was

able to show that five colors are always sufficient to color any map.

Heawood noted that much interest had been shown in the difficult if

not impossible problem of coloring a map with four colors, though no

case of failure had been found. He further stated that he did not pro-

fess to give a proof of the problem; in fact his aims were more de-

structive than constructive for he had found a defect in Kempe's work

which has just been presented. Heawood stated clearly that the re-

quired problem is to show that four colors are sufficient for the

coloring of any map and that this must be shown to follow from the

coloring of a reduced map, even where a five-contact point of con-

course was the lowest. He explained that Kempe claimed that if a

map could be colored with four colors and if there was a five-contact

point of concourse with four colors around it, he could reduce the

number of colors from four to three. Kempe did this by interchanging

colors so as to remove one color from the point, thus having a color to use if a district were formed at the point of concourse. Heawood presented, in his work, the possible cases that arose in coloring a figure and gave an example of a map that could not be colored by Kempe's method. Heawood found that Kempe's work was not correct because the blue, green region to which 2 belonged need not be disjoint from the blue, yellow region to which 5 belonged (Figure 9). In this case two contiguous districts would be colored the same by Kempe's method. Therefore Figure 9 illustrates that Kempe's proof is not valid for this case and the problem remains unsolved.

Vertical lines denote the blue and yellow region of 5.

Horizontal lines denote the blue and green region of 2.

Figure 9

III. KEMPE'S WORK ON PATCHING WITH STORY'S
MODIFICATIONS

The previous chapter expresses Kempe's work on reducing

the number of colors that may surround a district from five to three.

This chapter contains Kempe's further work of patching out districts

along with some modifications which are due to Story.

Disregarding for the moment the question of coloring, but

working with the case of the simply connected surface, Kempe con-

sidered the structure of a map on its surface. He expressed the dif-

ferent structures with the following: island districts having one

boundary (Figure 10); island regions composed of a number of dis-

tricts (Figure 11); peninsula districts having one boundary and one

point of concourse (Figure 12); peninsula regions composed of a num-

ber of districts (Figure 13); complex districts which have island and

peninsulas in them; simple districts which have as many boundaries

as points of concourse (Figure 14). Using these conditions Kempe

was able to note that with the exception of Figure 10 with an endless

boundary and Figure 12 which has one point of concourse, every

boundary will end in two points of concourse and also that every

boundary belonged to two districts. Kempe next introduced his idea

of covering a district of a map with a patch that was of the same gen-
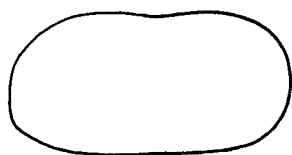
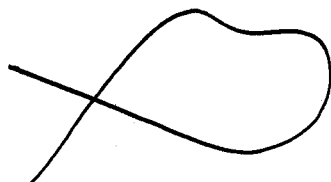eral shape as the district but just slightly larger so as to cover the
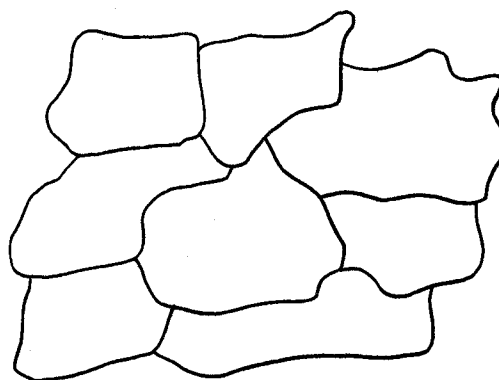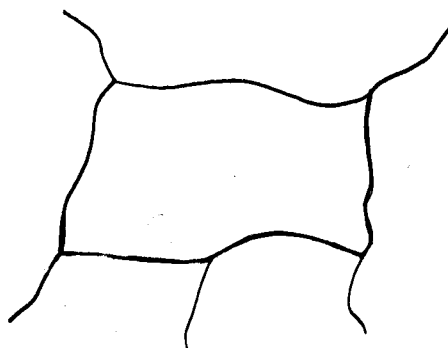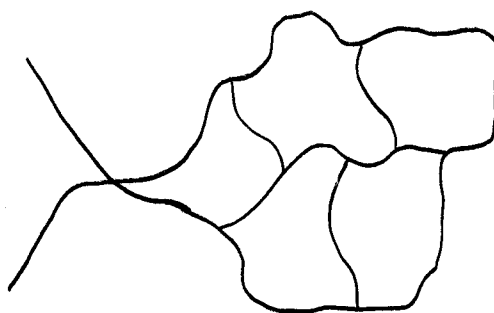
Figure 10

Figure 12

Figure 11

Figure 14

Figure 13

boundaries of the district. It was then seen that the map (Figure 15)

would have one less district and the boundaries would also be reduced
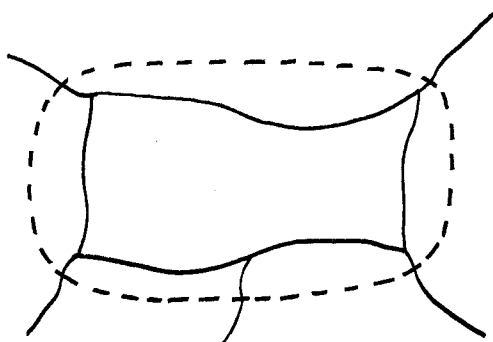
in number.

Figure 15

He further indicated that the boundaries meeting the patch (Figure 15)

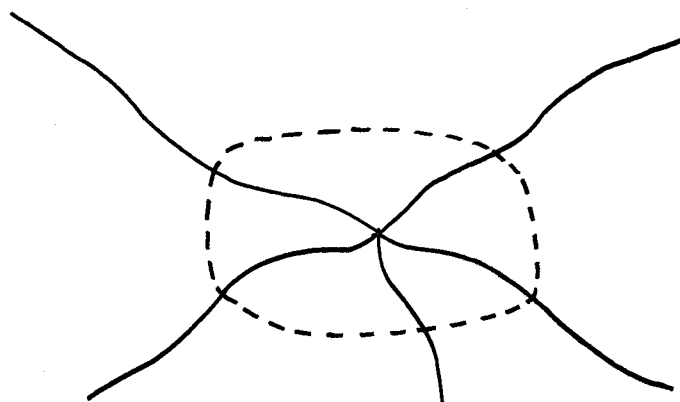should be extended to meet at a point of concourse (Figure 16).

Figure 16

In this way he found the patching process could be continued as long as there were simple districts left to operate on. He also found that if he confined his attention to patching island and peninsula districts first, the more complex districts were reduced to simple ones which could be patched by the same process. He continued this process until every district of the map was patched out, patches sometimes over-lapping other districts. His patching process then resulted in a single district with no boundaries and no points of concourse. Kempe next introduced what he called a reverse process, that being to strip off the patches in reverse order, taking off first that which was put on last so that as each patch was removed a new district appeared and the new map was developed by degrees.

Kempe developed some expressions which would give the rela-tion on the surface at any stage of the development depending on the nature of the next district to appear. He supposed that at any stage there would be on the surface

D   districts

B   boundaries

P   points of concourse,

and that at the next stage or after the next patch was removed there would be

D'   districts

B'   boundaries

P'   points of concourse .

If a patch had no point of concourse on it, i. e. if when it was stripped off an island appeared, he could then write the equations

$$D' = D + 1,$$

$$B' = B + 1,$$

$$P' = P.$$

Kempe wrote that if a patch has no point of concourse but rather a single line then a peninsula would appear and the equations would be

$$D' = D + 1,$$

$$B' = B + 2 \qquad\qquad (a)$$

$$P' = P + 1.$$

At this point Story pointed out, in his article, that another case could occur if no point of concourse appeared on the patch to be removed,

that being when a district with two boundaries appeared as in Figure 17.
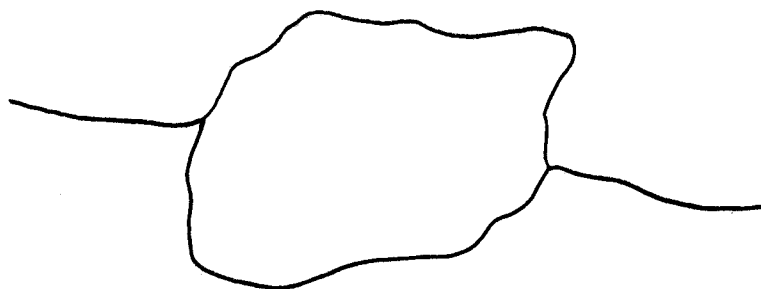
Figure 17

He then gave the following equations for this case. They are

$$D' = D + 1,$$

$$B' = B + 3, \qquad \text{(b)}$$

$$P' = P + 2 .$$

Story pointed out that these relations would hold only if the boundaries joined by the line on the patch were counted as two (and not one as in Figure 18 and Figure 19) before the patch was put on. Kempe continued by expressing the equations when the patch has a point of concourse on it where σ boundaries meet as

$$D' = D + 1$$

$$B' = B + \sigma \qquad \text{(c)}$$

$$P' = P + \sigma - 1$$
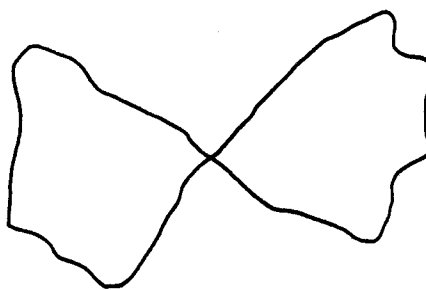
Figure 18   Figure 19

In each case Kempe concluded that

$$P' + D' - B' - 1 = P + \sigma - 1 + D + 1 - (B + \sigma) - 1$$

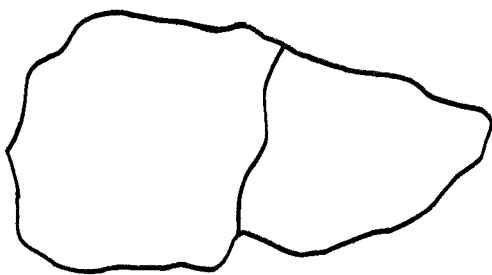$$P' + D' - B' - 1 = P + D - B - 1 \qquad\qquad (d)$$

by which he concluded that   P + D - B - 1   had the same value before

and after the patch was removed. He also stated that since   P = 0,

D = 0, B = 0   at the first stage we would always have

$$P + D - B - 1 = 0 \ . \qquad\qquad (e)$$

Story stated that the equations of (c) are valid only when three and only

three boundaries meet in each point of concourse about the district

patched out, when σ is the number of boundaries meeting the

boundaries of the district in question. He further considered the con-

dition when a patch had no point of concourse but only a single line

forming part of the boundary of an island-district on the patched map,

so that when the patch was removed a form like Figure 18 or Figure

19 appeared. He claimed in this case

$$D' = D + 1$$

$$B' = B + 1$$

$$P' = P + 2$$

and in the case where the point in question was a point of concourse he claimed

$$D' = D + 1$$

$$B' = B + 1$$

$$P' = P + 1$$

and therefore he concluded for either of these two cases

$P' + D' - B' - 1 = P + D - B$ instead of equation (d). Story continued defining a contour as an aggregate of boundaries such that any two could be mutually connected, either directly or by means of other boundaries of the same contour, but could not be connected with any other boundaries in the map. He remarked that each contour could be a map itself and that a contour would be simple if it consisted of one boundary or complex if it consisted of more than one boundary. Story wrote that using Kempe's method of patching out a map formed by any complex-contour, the map would at some time take the form of Figure 18 or Figure 19, then form an island, and then be hidden under a final patch. This leads to the question of what are the values

of D, B, and P   when the patches are removed. These were ex-

pressed at the first stage when   D = 1,   B = 0, and P = 0   to be

$$P + D - B - 1 = 0.$$

At the second stage Figure 19 gives   D = D' + 2,   B = B' + 2 and

P = P' + 1   so equation (d) becomes

$$P' + D' - B' - 1 = 0 \qquad\qquad (f)$$

or Figure 18 gives D = D' + 2, B = B' + 3 and P = P' + 2 and equa-

tion (d) becomes

$$P' + D' - B' - 1 = 0 .$$

At the third stage when   D = D' + 3,   B = B' + 5 and P = P' + 1

equation (d) will be

$$P' + D' - B' - 1 = 1,$$

and at every stage thereafter Story claimed that

$$P + D - B - 1 = 1 \qquad\qquad (g)$$

i.e. the right member of equation (d) would have the value one for

every case. It was pointed out that, in the case of a map formed by

a simple-contour only the first and second stages exist, and for this

equation (e) holds. If there are x complex-contours formed by the

boundaries of any map then Story wrote

$$P + D - B - 1 = x .$$ (h)

This theorem which was rewritten by Story was expressed as follows: In every map drawn on a simply-connected surface the number of points of concourse and the number of districts are together one greater than the number of boundaries and number of complex-contours together [14] .

Kempe next considered the expressions

$$D = d_1 + d_2 + d_3 + \cdots$$

and

$$P = p_3 + p_4 + p_5 + \cdots$$

where d_1, d_2, d_3, etc. denote the number of districts at any stage with one, two, three, etc. boundaries and where p_3, p_4, etc. denote the numbers of points of concourse at the same stage of development at which three, four, etc. boundaries meet. Story claimed that when considering Figure 19 where two such boundaries form a point of concourse that p_2 should be added to the expression listed by Kempe so that

$$P = p_2 + p_3 + p_4 + \cdots$$

where p_2, p_3, p_4, etc. denote the number of points of concourse where

two, three, etc. boundaries meet. Since every boundary belongs to

two districts, Kempe formulated the equation

$$2B = d_1 + 2d_2 + 3d_3 + \cdots .$$

When Story introduced p_2, Kempe's original expression of 2B

became

$$2B = 2\beta_0 + \beta_1 + 2p_2 + 3p_3 + 4p_4 + \cdots$$

where β_0 is the number of continuous boundaries which have no

points of concourse, and where β_1 is the number of boundaries

around peninsula districts which have one point of concourse. It was

also noted that a point of concourse in which two boundaries meet

would be counted once in p_2 and twice in β_1. Thus Story ex-

pressed (h) as $(6D-2B)+(6P-4B)-6(x+1)=0$ where $(x+1)$ in the

last term of the equation is due to the complex-contours that may be

contained in the map. The equation $(6D-2B)+(6P-4B)-6(x+1)=0$

can be written as $(6d_1+6d_2+6d_3+\cdots)-(d_1+2d_2+3d_3+\cdots)+$

$(6p_2+6p_3+6p_4+\cdots)-(4\beta_0+2\beta_1+4p_2+6p_3+8p_4+10p_5+\cdots)-6(x+1)=0$. When

this equation is simplified it becomes $(5d_1+4d_2+3d_3+2d_4+1d_5+2p_2)-$

$(1d_7+2d_8+3d_9+\cdots)-(2p_4+4p_5+6p_6+\cdots)-6(x+1)=0$. Then the equation

can be rewritten as $5d_1+4d_2+3d_3+2d_4+1d_5+2p_2=(1d_7+2d_8+3d_9+\cdots)+$

$(2p_4+4p_5+6p_6+\cdots)+6(x+1)$. Then

$(1d_7+2d_8+3d_9+\cdots)+(2p_4+4p_5+6p_6+\cdots)+6(x+1)>0$ since $6(x+1)>0$

for x is the number of complex-contours which will not be less than

zero. The first two terms $(1d_7+2d_8+3d_9+\cdots)$ and

$(2p_4+4p_5+6p_6+\cdots)$ will either have the value zero or be positive. In

either case $(1d_7+2d_8+3d_9+\cdots)+(2p_4+4p_5+6p_6+\cdots)+6(x+1)>0$.

Therefore $(5d_1+4d_2+3d_3+2d_4+1d_5)+(2p_2)>0$. If $2p_2=0$ then

$(5d_1+4d_2+3d_3+2d_4+1d_5)>0$. If $2p_2\neq 0$ then $5d_1\neq 0$ since d_1

represents the number of peninsula districts and p_2 represents

the number of points of concourse where two boundaries meet which

only occurs with peninsula districts. Therefore

$(5d_1+4d_2+3d_3+2d_4+1d_5)>0$ since $2p_2$ does not affect the inequality.

Then at least one of the quantities d_1,d_2,d_3,d_4,d_5 must not vanish

i.e. every map drawn on a simply-connected surface must have a dis-

trict with less than six boundaries [14] .

It was observed by Story that the number of boundaries meet-

ing in the point of concourse on a patch was equal to the number of

boundaries of the district covered by the patch only when the number

of boundaries meeting in each point of concourse about the district did

not exceed three. When the number of boundaries meeting a point of

concourse exceeded three a small patch, with a line around it, was

placed on the point of concourse thus creating a number of new points

of concourse where only three boundaries meet. This particular

patch was designated an auxiliary-patch by Story. It should be

understood that all the line segments between the patch and the line around the patch be erased to form one district. Also, the number of districts of any map was not changed by the use of an auxiliary-patch since this formed only an extension of a district over a point of concourse. Story continued by saying that after having modified a map by placing auxiliary-patches on each point of concourse with more than three boundaries, the process of patching out could then be continued. Thus he finally arrived at a map containing one district and no boundaries which he could color with any of the four colors. Kempe explained that if each patch was numbered as it was placed on the map, there would be no confusion when the patches were to be stripped off. Then as Story developed the map by stripping off the patches and auxiliary-patches in the inverse order to which they were put on, he colored each district as it reappeared.

Story considered the situation at some stage of the development in which a certain district was disclosed that required the fourth color. He described the situation as follows: the patch next to be stripped off would be either an ordinary-patch with no lines, or with one line and no point of concourse, or with a point of concourse in which not more than three boundaries meet; or it would be an auxiliary-patch. Story explained that if an auxiliary-patch is stripped off a point of concourse would appear, in which four or five boundaries meet. The colors of these districts were then to be extended over their uncovered

portions, and the number of colors at the point of concourse reduced to not more than three by Kempe's method. However, this cannot always be done as Heawood pointed out. Thus Story's work is also not acceptable.

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