THE ALGEBRA AND TOPOLOGY OF BINARY RELATIONS

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CHAPTER I

INTRODUCTION

From the time of Pythagoras' proclamation, "Everything is number," up to modern times the question, "What is number?" has evoked many and varied responses from mathematicians and philosophers. Perhaps the question never will be answered to the satisfaction of all philosophers. But an answer is now available that seems satisfactory to some philosophers and most mathematicians.

As a result of his inability to define "number", the early mathematician was forced to treat it as a primitive notion, undefinable in terms of simpler notions. The use of the word "notion" here throws some light on the difficulty. No attempt will be made here to define "notion".

A. DeMorgan, in the 1850's, was able to pin down the elusive notion that was causing the logical difficulty. It was what we call a relation. But DeMorgan did not possess an adequate apparatus for treating the subject, and was apparently unable to create such an apparatus.

The title of creator of the theory of relations was reserved for C. S. Peirce. In several papers published between 1870 and 1882, he introduced and made precise the fundamental concepts of the theory of relations and formulated and established its fundamental laws (2, pp. 1-117). Peirce's analysis amounted essentially to an inversion, or a turning inside out, of the notion of relation.

This inversion can perhaps be best illustrated by an example. For such an example we consider the statement:

1.1 Texas is bigger than Oregon.

As it stand this is simply a true statement describing a relationship of Texas to Oregon. Suppose now we remove the word "Texas" from 1.1, leaving only:

1.2 _____ is bigger than Oregon.

We no longer have a statement, but what is called a statement matrix; it is in need of a subject noun in order to become a statement again.

By inserting various nouns in the blank in 1.2 we find that some yield semantic gibberish for which it is impossible to decide on the truth or falsity of the statement. Such a statement will be called a meaningless statement, e.g., if we try "fire" in 1.2 we get a meaningless statement. Let us then restrict the class of permissible nouns to those for which 1.2 is determinable as either true or false. Obviously then we can use the matrix 1.2 as a means of partitioning this class of nouns into two disjoint sets of nouns, those which yield a true statement and those which yield a false statement.

Let us now carry this process one step further and remove "Oregon" from 1.2. We are left with only the matrix:

1.3 _____ is bigger than _____ .

We see that now we must supply two nouns in order that 1.3 be a statement, that is, we need to try nouns in 1.3 in pairs. The pairs will be special kinds of pairs in the sense that one noun will serve as a subject, the other as predicate. Given any two nouns we can form two pairs by interchanging the rôles of the two nouns. We call such pairs ordered pairs. Again we restrict our attention to the set of all such pairs which yield a meaningful statement upon substitution in 1.3. For the moment let us denote this set of ordered pairs by "U". Obviously U is partitioned into exactly two disjoint subsets, those pairs which make the matrix 1.3 a true statement and those which make it false.

We make the observation that U is a set of pairs of nouns. Each of these nouns denotes some thing, object, or concept, which we shall call an individual; many of the nouns may denote the same individual. Thus for two given individuals x and y for which it is meaningful to state, "x is bigger than y," we may find many different pairs of nouns in U which are names of x and y. To eliminate this possibility of duplication of pairs, we focus attention on the ordered pairs of individuals which are determined by U. Let us denote this set of ordered pairs of individuals by "l". In l then, there will be a subset R of ordered pairs of individuals corresponding to the subset of U for which the matrix 1.3 becomes a true statement.

The copula "is greater than" in the matrix 1.3 we ordinarily speak of as a relation. But this is a notion —or a concept—not submitting readily to rigorous logical analysis. On the other hand the sets l and R are concrete. They are definite sets of pairs of individuals. The notion "is greater than" is called the intension of the relation, and the set R is called the extension of the relation.

We have here started with the intension of a relation and (conceptually, at least) derived its extension. The extension of the relation can be easily dealt with by our formal logic, but the intension is quite elusive. The question then naturally arises as to whether or not if we are initially given a set R of ordered pairs of individuals, it determines a unique intensive relation. If so, we could define a relation by means of its extension and this would give us a powerful tool for logical analysis. This question is very difficult and to seek the answer would lead us back into the nebulous realm of words because of the very nature of the question. Thus we simply take the position (without defending it) that if a set, l, of ordered pairs of individuals is given, then every subset R of l will determine a binary relation. In case l and R happen to be such that two apparently different copulas, or intensive relations, can be used to derive the given sets, we say that the two intensions are equivalent.

In this way the concept of a relation is, so to speak, turned inside out and made the subject of a rigorous logical theory as well as a useful tool in the analysis and construction of a mathematical theory. This fact is brought out in the following sketch of the historical development and application of the theory of relations.

In his work on the logic of relations Peirce had the foundation laid by DeMorgan, as well as the algebra of propositions formulated by Boole. He modified Boole's algebra with respect to the exclusive "or", and eliminated division. He gave DeMorgan's calculus of relations a workable notation and as we remarked earlier, established the fundamental laws of the modern theory of relations. This work of Peirce's was enlarged upon extensively by E. Schröder between 1877 and 1900.

In 1895, Peano and his collaborators, in the <u>Formulaire de mathematiques</u>, began with mathematics as it was and by a process of analysis reached what seemed to be the very roots of arithmetic. They found that they could start with three undefined notions and five postulates stated in terms of these notions, and construct arithmetic. Basic in the whole construction was the undefined relation <u>successor</u>. The system thus constructed was purely formal, empty of any content except in so far as agreement could be found for the meaning of these undefined notions. Although Peano did not define the relation <u>successor</u>, he used it very successfuly (in extension) in his construction of the number system.

Thus the theory of relations was developed by Peirce and Schröder into a mathematical like symbolism as a purely logical device for treating statements while Peano, on the other hand, took mathematics apart in such a way that relations were his starting point. Whitehead and Russell, the their <u>Principia Mathematica</u>, bridged this gap completely. They defined all the ideas of arithmetic, the only undefined ideas being those of logic itself, such as "proposition", "negation", and "either-or". Relations play a dominational rôle in this development, but in the <u>Principia</u> the theory of relations is developed in a fairly narrow sense for the express purpose of connecting logic and mathematics, and not for its intrinsic worth (3, pp. 1-25).

We have seen that our number system depends heavily upon relations for its construction. At another level, the functions, or transformations, and operations of mathematics are nothing more than special kinds of relations. Clearly then, the theory of relations is properly a part of mathematics itself. The growth and expansion of the theory of relations, it would seem, cannot help but enhance mathematics.

CHAPTER II

THE VARIABLES AND THE NOTATION

In the introduction it was seen that when a set 1 of ordered pairs of individuals is given, and when a subset R of 1 is known, a binary relation is determined. Suppose that 1 is given and let "R" denote a variable subset of 1. By considering R as a variable, we see that we have as the subject of our investigation not the relation R itself, but the class of all binary relations in 1. It is this abstraction from a relation to a class of relations that gives rise to the algebra of relations. For the relation variables we use the capital letters R, S, T, and U.

Our discussion will require also another type of variable—the individuals in the pairs which constitute 1, and for which we use the small letters x, y, z, ... A given set of individuals will be called a space. In the following treatment of the theory of relations the spaces involved will be considered given at the outset, and will thus be treated as constants throughout. In several places we will need to consider subsets of the spaces of individuals. No symbols will be set forth here and reserved for this purpose. Rather, the notation for such subsets will be defined in context.

An ordered pair of individuals is denoted by "x;y", where the adjective "ordered" refers to position with respect to the semi-colon. If x and y are distinct individuals, then x;y and y;x are distinct ordered pairs.

The set 1 of ordered pairs discussed in the opening paragraph of this chapter is called a product space, and is constructed as follows: Given two non-empty spaces A and B of individuals, the product space $A \times B$ is the set of all ordered pairs x;y such that x is in A and y is in B.

We make the usual convention regarding the symbol "$\hat{}\{\ \}$", namely: If P(x) is a statement matrix involving no other variable than x, then "$\hat{x}\{P(x)\}$" stands for the class of all x such that P(x) is true. The Greek letter "ε" will be used to denote class membership. Thus if P(a) is true, we write $a \ \varepsilon\ \hat{x}\{P(x)\}$, which is read "a is an element of the set $\hat{x}\{P(x)\}$".

From the heuristic analysis of the example given in the introduction, it is seen that an algebra of relations will be interpretable as an algebra of statements. This suggests that our investigation might be facilitated by use of the notation of that branch of symbolic logic known as the restricted predicate calculus. We find that this is indeed the case. This section will be devoted to a discussion of the logical symbols which we shall use (4, pp. 82-162).

We have two primitive symbols which we describe in terms of words. All the other logical symbols used can be defined in terms of these two. The two primitive symbols are ".", read "and", and "~", read "not". These are called conjunction and negation respectively. Thus, for example, if P(x) and Q(x) denote statement matrices involving the variable x, the statement P(x).Q(x) is true for those and only those values of x for which P(x) and Q(x) are both true. The statement ~P(x) is true for precisely those values of x for which P(x) is false.

Disjunction "$_v$" is read "or", and is to be interpreted in the inclusive sense. Thus the compound statement $P_v Q$ is true if P is true or if Q is true, or if both are true together—otherwise it is false. We can define $_v$ by saying that $P_v Q$ is the same as ~(~P.~Q).

Implication "→" is read "implies" or "if _____ then _____". Thus the compound statement P → Q is read "if P then Q" or "P implies Q", and it is false if P is true and Q is false, otherwise it is true. We can define "→" by saying that the statement P → Q is the same as ~(P.~Q).

Equivalence "↔" between statements is read "_____ if and only if _____". The compound statement P ↔ Q is the same as (P→Q).(Q→P), or in terms of our primitive symbols, ~(P.~Q).~(Q.~P). Thus the statement P ↔ Q is true if both P → Q is true and Q → P is true, and it is otherwise false.

We call the four symbols ".", "v", "$\rightarrow$", and "$\leftrightarrow$" the logical connectives. Compound statements are formed by combining two simpler statements by means of one of the connectives, or else by prefixing the negation sign "$\sim$" to a statement.

The construction of compound statements described above could lead to ambiguity if we allowed more than two statements to be combined by connectives at one step—e.g.

2.1 $$P \rightarrow Q \leftrightarrow H$$

can be interpreted either as

2.2 $$(P \rightarrow Q) \leftrightarrow H$$

or as

2.3 $$P \rightarrow (Q \leftrightarrow H)$$

which are quite different. To avoid this ambiguity we could keep parentheses on the component parts of compound statements, but this would lead in many cases to cumbersome tangles of parentheses. Hence we shall follow the customary use of dots as punctuation in the more complicated compound statements. Using the system of dots, the statements 2.2 and 2.3 would be rendered as

2.4 $$P \rightarrow Q . \leftrightarrow . H$$

and as

2.5 $$P . \rightarrow . Q \leftrightarrow H$$

respectively. There is an order of strength of the connectives which helps to cut down the number of dots used. The strongest connectives are $\leftrightarrow$ and $\rightarrow$, which are of equal strength. Next in strength is the disjunction symbol "v". Finally, and weakest, are the conjunction without the dot PQ and negation ~. When there is a dot in a statement it is stronger than all of the other connectives. When standing by themselves between components of a compound statement, the dots are read as "and", but when standing by other connectives, they serve merely to strengthen that symbol. In more complicated statements several dots may be needed in one place; as many dots will be used as are necessary to render the statement unambiguous.

The quantifiers (x) and (Ex) are translated "for all x" and "there is at least one x" respectively. Thus, placing a quantifier in front of a statement matrix involving x forms a new statement which no longer depends on x. For example, the statement matrix "$x > 0$" is true for some values of x and is false for others. But "$(x).x>0$" is not a statement matrix, it is a complete statement, in no way depending on x, and it is irrevocably false. Also "$(Ex).x > 0$" is a complete statement, not depending on x; it simply states a (true) fact.

A statement matrix with two variables, x and y, say "$P(x,y)$" is converted by (x) into a matrix "$(x)P(x,y)$" of only one variable, y. Then this matrix can be made a complete statement by prefixing "(y)" to "$(x)P(x,y)$", obtaining "(y)(x)P(x,y)". This operation is commutative, and we use the symbol "(x,y)" to mean (x)(y) or (y)(x). Similarly we abbreviate (Ex)(Ey) to (Ex,y). Quantification is extended analogously to more than two variables. Thus (x,y,z,R)P(x,y,z,R) is read "for all x, y, z, and R, 'P(x,y,z,R)' is true".

In order to facilitate the symbolic presentation the headings "Postulate", "Definition", and "Theorem" will be abbreviated to "Pos", "Def", and "Thm" respectively throughout chapters three, four, and five.

CHAPTER III

THE GENERAL THEORY OF RELATIONS

In this chapter we shall formalize some of the ideas discussed in chapter one and lay the groundwork for the set-theoretic, or Boolean, theory of relations. We take as primitive notions the spaces A and B of individuals, the concept of ordered pair $x;y$, and the relation ε of elementhood. Since we are not interested in constructing a vacuous system we state

3.1 Pos. $(Ex,y).x\varepsilon A.y\varepsilon B.$

This insures that the space of ordered pairs defined next will possess at least one element.

3.2 Def. Put "1" for "$\widehat{x;y}\{x\varepsilon A.y\varepsilon B\}$".

3.3 Thm. $(x,y):x\varepsilon A.y\varepsilon B.\longleftrightarrow.x;y\ \varepsilon\ 1.$

3.4 Thm. $(Ex,y)x;y\ \varepsilon\ 1.$

We next define the class K of all relations in 1: Formally this is just the class of all subsets of the product space.

3.5 Def. Put "K" for "$\hat{R}\{(x,y).x;y\ \varepsilon\ R \rightarrow x;y\ \varepsilon\ 1\}$".

3.6 Thm. $(R):R\varepsilon K.\longleftrightarrow.(x,y).x;y\ \varepsilon\ R \rightarrow x;y\ \varepsilon\ x;y\ \varepsilon\ 1.$

3.7 Thm. $1\ \varepsilon\ K.$

This last theorem tells us that 1 is a relation. We call 1 the universal relation in K; it is the relation which every element x in A bears to each element y in B.

The next definition is not really a part of the theory of relations, we state it as a definition merely for the sake of formal completeness.

3.8 Def. Put "xRy" for "$x;y \varepsilon R$".

We interpret the symbol xRy as a statement matrix, read "x is in the relation R to y". However, as might be inferred from the statement of definition 3.5, our aim is to construct a theory of relations in which the relations are treated as just elements of a set K. The approach we take here insures though, that after the abstract theory is developed we will be able to interpret the final results in accordance with definition 3.8.

In the interest of economy of expression we shall make one more convention to be effective throughout the remaining chapters: The individual variables x, y, z, ... will be considered restricted to the ranges A and B. We will therefore omit and leave implicit the conditions of the form $x \varepsilon A$, $y \varepsilon B$, etc., which should appear immediately following every quantifier (x,y,...) or (Ex,y,...).

The following three theorems are immediate consequences of 3.3, 3.4, and 3.6 in view of definition 3.8.

3.9 Thm. (x,y)xly.

3.10 Thm. (Ex,y)xly.

3.11 Thm. $(R):R\varepsilon K. \longleftrightarrow .(x,y).xRy \rightarrow xly.$

We next define the null relation O in K, the relation which no element of A bears to any element of B.

3.12 Def. Put "O" for "$\widehat{x;y}\{x1y.\sim x1y\}$".

3.13 Thm. $(x,y)\sim xOy$.

3.14 Thm. $O\,\varepsilon\, K$.

We now proceed to define three operations—union, intersection, and complement on K in such a way that K will constitute a realization of the Boolean algebra postulates.

3.15 Def. Put "R+S" for "$\widehat{x;y}\{xRy_v xSy\}$".

3.16 Thm. $(R,S):R,S\varepsilon K.\rightarrow.(x,y).x(R+S)y\leftrightarrow xRy_v xSy$.

3.17 Thm. $(R,S).R,S\varepsilon K\rightarrow(R+S)\varepsilon K$.

3.18 Def. Put "RS" for "$\widehat{x;y}\{xRy.xSy\}$".

3.19 Thm. $(R,S):R,S\varepsilon K.\rightarrow.(x,y).x(RS)y \leftrightarrow xRyxSy$.

3.20 Thm. $(R,S).R,S\varepsilon K \rightarrow (RS)\varepsilon K$.

3.21 Def. Put "R'" for "$\widehat{x;y}\{\sim xRy\}$".

3.22 Thm. $(R):R\varepsilon K.\rightarrow.(x,y).xR'y \leftrightarrow \sim xRy$.

3.23 Thm. $(R).R\varepsilon K \rightarrow R'\varepsilon K$.

The next definition is the essential device that enables us to abstract relations and deal with them as elements of K without reference to the individuals x and y.

3.24 Def. Put "R=S" for "$(x,y).xRy \leftrightarrow xSy$".

3.25 Thm. $(R,S):.R,S\varepsilon K:\rightarrow:R=S.\leftrightarrow.(x,y).xRy\leftrightarrow xSy$.

3.26 Thm. $(R,S):R,S\varepsilon K.\rightarrow.R=S\rightarrow S=R$.

3.27 Thm. $(R,S,T):.R,S,T\varepsilon K:\rightarrow:R=T.S=T.\rightarrow.R=S$.

We shall now show that the system K that we have defined constitutes a Boolean algebra.

3.28 Thm. $(R,S,T):R,S,T\varepsilon K.\rightarrow.R=S\rightarrow R+T = S+T$.

Proof: Assume $R,S,T \varepsilon K$.

By 3.25, $R=S. \rightarrow .(x,y).xRy \leftrightarrow xSy$

$. \rightarrow .(x,y).xRy_{v}xTy \leftrightarrow xSy_{v}xTy$

By 3.16, $. \rightarrow .(x,y).x(R+T)y \leftrightarrow x(S+T)y$

by 3.25, $. \rightarrow .R+T = S+T.$

3.29 Thm. $(R,S,T):R,S,T\varepsilon K. \rightarrow .R=S \rightarrow RT=ST.$

Proof: Similar to the proof of theorem 3.28.

3.30 Thm. $(R,S).R,S\varepsilon K \rightarrow R+S = S+R.$

Proof: The theorem follows immediately from the equivalence $(x,y).xRy_{v}xSy \leftrightarrow xSy_{v}xRy$ and by 3.16 and 3.25.

3.31 Thm. $(R,S).R,S\varepsilon K \rightarrow RS=SR.$

Proof: Similar to the proof of 3.30.

3.32 Thm. $\sim(1=0).$

Proof:

By 3.25, $1=0. \rightarrow .(x,y).x1y \rightarrow x0y$

$. \rightarrow .(Ex,y)x1y \rightarrow (Ex,y)x0y$

$. \rightarrow . \sim\{(Ex,y)x1y. \sim(Ex,y)x0y\}$

$. \rightarrow . \sim\{(Ex,y)x1y.(x,y) \sim x0y\}$

Now taking the contrapositive of this implication we obtain

$$(Ex,y)x1y.(x,y)\sim x0y. \rightarrow . \sim(1=0).$$

The theorem now follows by 3.10 and 3.13.

3.33 Thm. $(R).R\varepsilon K \rightarrow R+0=R.$

Proof: Assume $R\varepsilon K$.

By 3.16, $(x,y):x(R+0)y. \leftrightarrow .xRy_{v}x0y$

$. \leftrightarrow . \sim x0y \rightarrow xRy$

by 3.13, $. \leftrightarrow .xRy.$

From this result the theorem now follows by 3.25.

3.34 Thm. $(R).R\varepsilon K \rightarrow R1=R.$

Proof: Similar to the proof of 3.33.

3.35 Thm. $(R,S,T).R,S,T\varepsilon K \rightarrow R(S+T)=RS+RT.$

Proof: Assume $R,S,T\varepsilon K$. By 3.17, we see that by 3.19, $x(R(S+T))y . \leftrightarrow . xRy . x(S+T)y$

by 3.16, $\qquad . \leftrightarrow . xRy . xSy \vee xTy$

$\qquad . \leftrightarrow . xRyxSy \vee xRyxTy$

by 3.19, $\qquad . \leftrightarrow . x(RS)y \vee x(RT)y$

by 3.16, $\qquad . \leftrightarrow . x(RS+RT)y.$

Using the generalization principle on this result, the conclusion follows by 3.25.

3.36 Thm. $(R,S,T).R,S,T\varepsilon K \rightarrow R+ST=(R+S)(R+T)$.

Proof: Similar to the proof of 3.35.

3.37 Thm. $(R).R\varepsilon K \rightarrow RR'=0$.

Proof: Assume $R\varepsilon K$. Then by 3.23 $R'\varepsilon K$, so

by 3.19, $x(RR')y . \rightarrow . xRy . xR'y$

by 3.22, $\qquad . \rightarrow . xRy . \sim xRy$

by 3.11, $\qquad . \rightarrow . x1y . \sim x1y$

by 3.12, $\qquad . \rightarrow . x0y.$

The contrapositive of this is

$$\sim x0y \rightarrow \sim x(RR')y$$

from which by 3.13 we conclude $(x,y) \sim x(RR')y$. Then since every statement implies a true statement, we have

$$(x,y) \sim x0y \leftrightarrow \sim x(RR')y.$$

The theorem follows from this and 3.25.

3.38 Thm. $(R).R\varepsilon K \rightarrow R+R'=1$.

Proof: Assume $R\varepsilon K$. Then by 3.23 $R'\varepsilon K$, so that

by 3.16, $x(R+R')y \leftrightarrow xRy \vee xR'y$

by 3.22, $\qquad \leftrightarrow xRy \vee \sim xRy$

from which we conclude that $(x,y)x(R+R')y$. Using this and 3.10, the rest of the proof is similar to the proof of 3.37.

Now theorems 3.7 and 3.14 together with theorems 3.17, 3.20, and theorems 3.28–3.38 are precisely the conditions which K must satisfy in order that it be a Boolean algebra. Since K has been shown to be a Boolean algebra, we can apply immediately any or all of the theorems of Boolean algebra to the elements of K—to the relations. We shall give here a list of the Boolean theorems which will be used in the development of the algebra of relations in a later section. For a reasonably complete treatment and an interesting historical account of Boolean algebra see Lewis and Langford, <u>Survey of symbolic logic</u>, Berkeley, 1918.

3.39 Thm. $(R,S):.R,S\varepsilon K:\rightarrow:R+S=1.RS=0.\rightarrow.S=R'$.

3.40 Thm. $(R).R\varepsilon K\rightarrow R''=R$.

3.41 Thm. $0'=1.1'=0$.

3.42 Thm. $(R).R\varepsilon K\rightarrow R+R=R$.

3.43 Thm. $(R).R\varepsilon K\rightarrow RR=R$.

3.44 Thm. $(R).R\varepsilon K\rightarrow R+1=1$.

3.45 Thm. $(R).R\varepsilon K\rightarrow R0=0$.

3.46 Thm. $(R,S).R,S\varepsilon K\rightarrow R+RS=R$.

3.47 Thm. $(R,S).R,S\varepsilon K\rightarrow R(R+S)=R$.

3.48 Thm. $(R,S).R,S\varepsilon K\rightarrow R+(R'+S)=1$.

3.49 Thm. $(R,S).R,S\varepsilon K\rightarrow R(R'S)=0$.

3.50 Thm. $(R,S).R,S\varepsilon K\rightarrow(R+S)'=R'S'$.

3.51 Thm. $(R,S).R,S\varepsilon K\rightarrow(RS)'=R'+S'$.

3.52 Thm. $(R,S):.R,S\varepsilon K:\rightarrow:RS'=0.R'S=0.\leftrightarrow.R=S$.

3.53 Thm. $(R,S,T):.R,S,T\varepsilon K:\rightarrow:RS'=0.RT'=0.\leftrightarrow.R(ST)'=0$.

3.54 Thm. $(R,S,T).R,S,T\varepsilon K\rightarrow(R+S)+T=R+(S+T)$.

3.55 Thm. $(R,S,T).R,S,T\varepsilon K\rightarrow(RS)T=R(ST)$.

3.56 Thm. $(R,S,T):.R,S,T\varepsilon K:\rightarrow:RT=0.ST'=0.\rightarrow.RS=0.$

3.57 Thm. $(R,S):.R,S\varepsilon K:\rightarrow:R=0.S=0.\leftrightarrow.R+S=0.$

3.58 Thm. $(R,S):R,S\varepsilon K.\rightarrow.RS+R'S'=1\leftrightarrow R=S.$

3.59 Thm. $(R,S):R,S\varepsilon K.\rightarrow.R+S=S\leftrightarrow RS=R.$

3.60 Thm. $(R,S):R,S\varepsilon K.\rightarrow.RS'=0\leftrightarrow RS=R.$

3.61 Thm. $(R,S):R,S\varepsilon K.\rightarrow.R+S=S\leftrightarrow R'+S=1.$

These last three theorems are all equivalent characterizations of the inclusion relation between relations. We shall define this relation here and list some of the properties of K in terms of it.

3.62 Def. Put "$R < S$" for "$RS'=0$".

3.63 Thm. $(R,S):R,S\varepsilon K.\rightarrow.R < S\leftrightarrow RS'=0.$

3.64 Thm. $(R).R\varepsilon K\rightarrow R<R.$

3.65 Thm. $(R):R\varepsilon K.\rightarrow.0<R.R<1.$

3.66 Thm. $(R,S):.R,S\varepsilon K:\rightarrow:R<S.S<R.\leftrightarrow.R=S.$

3.67 Thm. $(R,S,T):.R,S,T\varepsilon K:\rightarrow:R<T.T<S.\rightarrow.R<S.$

3.68 Thm. $(R,S).R,S\varepsilon K\rightarrow RS<R.$

3.69 Thm. $(R,S).R,S\varepsilon K\rightarrow R<R+S.$

3.70 Thm. $(R,S):R,S\varepsilon K.\rightarrow.R<S\rightarrow S'<R'.$

The next theorem gives an interpretation of "$<$" in the terms of our definition of K.

3.71 Thm. $(R,S):.R,S\varepsilon K:\rightarrow:R<S.\leftrightarrow.(x,y).xRy\rightarrow xSy.$

Proof: Assume $R,S\varepsilon K$. Then we have

by 3.63, $R<S.\leftrightarrow.RS'=0$

by 3.13, $\quad.\leftrightarrow.(x,y).\sim xRS'y$

by 3.19, $\quad.\leftrightarrow.(x,y).\sim(xRy.xS'y)$

$\quad.\leftrightarrow.(x,y).\sim xRy\vee\sim xS'y$

by 3.22, $R<S . \leftrightarrow . (x,y) . {-}xRy \vee xSy$
$. \leftrightarrow . (x,y) . xRy \rightarrow xSy.$

It may be remarked that K constitutes a complete atomic Boolean algebra. This fact would be of interest in the development of a convergence theory in the topological algebra of relations, but we shall not make use of it here; in the remaining chapters, only finite unions and intersections of relations will be considered.

In some cases it is helpful to have available a schematic representation of a relation. If we ignore the possibility of difficulties which may arise due to the cardinality of the spaces A and B, we can imagine the elements of A and B as represented uniquely by points on two different line segments. We draw these two segments perpendicular to one another forming two sides of a rectangle and we call them the A-axis and the B-axis. For every element x of A, draw through the point corresponding to x a line parallel to the B-axis; similarly, for every element y of B, draw a line parallel to the A-axis through the point representing y on the B-axis. This results in a cross-hatching of the rectangle defined by the A- and B-axis, the density of the cross-hatching corresponding to the density of the points used on the A- and the B-axis.

The totality of all the lattice points, or points of intersection of these lines will then represent the set of ordered pairs which constitute the product space 1. Any set of these points will be a relation in K. The set of points constituting a relation R is called the graph of R in $A \times B$.

For every relation R in $A \times B$ we have two projections, the domain of R and the converse domain of R. These are defined by "$\hat{x}\{(Ey).xRy\}$", and "$\hat{y}\{(Ex).xRy\}$" respectively.

The schematic representation of relations will be useful in the next chapter as an aid in the study of the relative operations in the homogeneous product space.

Let us step down momentarily from investigation of the properties of K for a look at some properties of a relation R. For the remainder of this chapter we shall consider R as given, and it will be treated as a constant.

We need three additional tools, which are given us by

3.78 Def. Put "$M<N$" for "$(z).z\varepsilon M \rightarrow z\varepsilon N$".

3.79 Def. Put "$M=N$" for "$M<N.N<M$".

3.80 Def. Put "$\bigcap_{\alpha \varepsilon M} N_\alpha$" for "$\hat{z}\{(\alpha).\alpha\varepsilon M \rightarrow z\varepsilon N_\alpha\}$".

Note that this use of "<" and "=" is consistent with our use of these same symbols in case M and N are relations. These definitions are intended to allow us to use corresponding symbols ambiguously between relations and arbitrary sets.

We shall now be considering two types of variables; for the individual variables we use "x" and "y" as before, and as set variables we use "X" and "Y". We have two unary operations $*$ and $^+$ on the subsets of A and B respectively. Accordingly we shall interpret "x^*" and "y^+" as $*$ and $^+$ operating respectively on the subsets of A and B whose sole members are the elements x and y. This interpretation will serve to simplify the notation without introducing ambiguity.

3.81 Def. Put "x^*" for "$\hat{y}\{xRy\}$".

3.82 Thm. $(x,y).y\varepsilon x^* \leftrightarrow xRy$.

3.83 Def. Put "y^+" for "$\hat{x}\{xRy\}$".

3.84 Thm. $(x,y).x\varepsilon y^+ \leftrightarrow xRy$.

3.85 Def. Put "X^*" for "$\hat{y}\{(x).x\varepsilon X \rightarrow xRy\}$".

3.86 Thm. $(y):y\varepsilon X^*. \leftrightarrow .(x).x\varepsilon X \rightarrow xRy$.

3.87 Thm. $(x):x\varepsilon X. \rightarrow .(y).y\varepsilon X^* \rightarrow xRy$.

3.88 Def. Put "Y^+" for "$\hat{x}\{(y).y\varepsilon Y \rightarrow xRy\}$".

3.89 Thm. $(x):x\varepsilon Y^+. \leftrightarrow .(y).y\varepsilon Y \rightarrow xRy$.

3.90 Thm. $(y):y\varepsilon Y. \rightarrow .(x).x\varepsilon Y^+ \rightarrow xRy$.

We now shall state and prove some of the consequences of these definitions.

3.91 Thm. $X_1 < X_2 \rightarrow X_2^* < X_1^*$.

Proof: Assume $X_1 < X_2$. Then $x\varepsilon X_1 \rightarrow x\varepsilon X_2$.

By 3.86, $y\varepsilon X_2^* : \rightarrow : (x).x\varepsilon X_2 \rightarrow xRy$

$: \rightarrow : (x) : x\varepsilon X_1 \rightarrow x\varepsilon X_2 . x\varepsilon X_2 \rightarrow xRy$

$: \rightarrow : (x).x\varepsilon X_1 \rightarrow xRy$

by 3.86, $: \rightarrow : y\varepsilon X_1^*$.

The theorem follows now by 3.78.

3.92 Thm. $Y_1 < Y_2 \rightarrow Y_2^+ < Y_1^+$.

Proof: Similar to the proof of 3.91.

3.93 Thm. $X < X^{*+}$.

Proof:

By 3.87, $x \varepsilon X . \rightarrow . (y) . y \varepsilon X^* \rightarrow xRy$

by 3.89, $. \rightarrow . x \varepsilon X^{*+}$.

From this the theorem follows by 3.78.

3.94 Thm. $Y < Y^{+*}$.

Proof: Similar to the proof of 3.93.

3.95 Thm. $X^* = X^{*+*}$.

Proof: By 3.93 and 3.91, $X^{*+*} < X^*$. Also, by 3.94, $X^* < X^{*+*}$. The conclusion follows now by 3.79.

3.96 Thm. $Y^+ = Y^{+*+}$.

Proof: Similar to the proof of 3.95.

3.97 Thm. $X_1 < X_2 \rightarrow X_1^{*+} < X_2^{*+}$.

Proof: Use 3.91 and 3.92.

3.98 Thm. $Y_1 < Y_2 \rightarrow Y_1^{+*} < Y_2^{+*}$.

Proof: Use 3.92 and 3.91.

3.99 Thm. $X^* = \bigcap_{x \varepsilon X} x^*$.

Proof:

By 3.86, $y \varepsilon X^* : \leftrightarrow : (x) . x \varepsilon X \rightarrow xRy$

by 3.82, $: \leftrightarrow : (x) : x \varepsilon X \rightarrow xRy . xRy \rightarrow y \varepsilon x^*$

$: \leftrightarrow : (x) . x \varepsilon X \rightarrow y \varepsilon x^*$

by 3.80, $: \leftrightarrow : y \varepsilon \bigcap_{x \varepsilon X} x^*$.

The theorem follows now from 3.78 and 3.79.

3.100. Thm. $Y^+ = \bigcap_{y \varepsilon Y} y^+$.

Proof: Similar to the proof of 3.99.

Some of these properties will receive application in chapter five in the discussion of the topology of relations (1, p. 54).

CHAPTER IV

HOMOGENEOUS RELATIONS

The class K of relations introduced in chapter three was quite general, depending only on the nature of the two given spaces A and B. An important special case occurs when A and B are identical, that is, when they consist of exactly the same elements. In this case we say that the resulting product space 1 of ordered pairs is homogeneous, and that K is a class of homogeneous relations; this is signified by Kh.

For any class Kh of homogeneous relations we can introduce two new binary operations, the relative product R/S and the relative sum $R \not+ S$, and a unary operation $R^{\smile}$ called the converse of R. The algebra of relations resulting from considering these three operations in conjunction with the Boolean operations in Kh is a very rich deductive system.

We shall develop a basis for this algebra in a manner similar to the development in chapter three. The single additional postulate which we here require can be thought of as following directly after 3.1, as none of the subsequent statements in chapter three are affected in any way by this new postulate. Thus all of the definitions and theorems of chapter three will be applicable throughout the present chapter.

4.1 Pos. $(x).x\varepsilon A \longleftrightarrow x\varepsilon B$.

This postulate serves to make 1 homogeneous. It is this restriction which justifies our definitions 4.2, 4.5, and 4.8, and axioms 4.11 and 4.12 below. When both postulates 4.1 and 3.1 are satisfied we denote the resulting class of relations by "Kh", the "h" being intended to suggest "homogeneous".

Throughout this chapter, except where it is pertinent, the hypothesis $x,y,\ldots\varepsilon A$ will be omitted and left implicit as was done in most of chapter three. We consider the individual variables as restricted to the range A (or equivalently, to B) whereever they occur.

We introduce the converse operation by

4.2 Def. Put "$\breve{R}$" for "$\widehat{x;y}\{yRx\}$".

4.3 Thm. $(x,y,R).x\breve{R}y \longleftrightarrow yRx$.

4.4 Thm. $(R).R\varepsilon Kh \longrightarrow \breve{R}\varepsilon Kh$.

Proof: By 4.3, $x\breve{R}y. \longrightarrow .yRx$
by hypothesis, 3.11, $. \longrightarrow .y1x$
by 3.3, $. \longrightarrow .y\varepsilon A.x\varepsilon B$
by 4.1, $. \longrightarrow .x\varepsilon A.y\varepsilon B$
by 3.3, $. \longrightarrow .x1y.$
The conclusion follows from this and from 3.11.

Theorem 4.4 gives closure of Kh with respect to the converse operation. We next define the relative product and the relative sum operations, and demonstrate closure of Kh with respect to both.

4.5 Def. Put "R/S" for "$\widehat{x;y}\{(Ez).xRz.zSy\}$".

4.6 Thm. $(x,y,R,S):x(R/S)y.\leftrightarrow.(Ez).xRz.zSy.$

4.7 Thm. $(R,S).R,S\varepsilon Kh\rightarrow(R/S)\varepsilon Kh.$

Proof: Assume $R,S\varepsilon Kh$. Then

by 4.6, $x(R/S)y.\rightarrow.(Ez).xRz.zSy$

by 3.11, $.\rightarrow.(Ez).x1z.z1y$

by 3.3, $.\rightarrow.(Ez).x\varepsilon A.z\varepsilon B.z\varepsilon A.y\varepsilon B$

$.\rightarrow.x\varepsilon A.y\varepsilon B$

by 3.3, $.\rightarrow.x1y.$

The conclusion follows now by 3.11.

4.8 Def. Put "$R\not/S$" for "$\widehat{x;y}\{(z).xRz \vee zSy\}$".

4.9 Thm. $(x,y,R,S):x(R\not/S)y.\leftrightarrow.(z).xRz \vee zSy.$

4.10 Thm. $(R,S).R,S\varepsilon Kh\rightarrow(R\not/S)\varepsilon Kh.$

Proof: Assume $R,S\varepsilon Kh$. Then

by 4.9, $x(R\not/S)y.\rightarrow.(z).xRz \vee zSy$

by 3.11, $.\rightarrow.(z).x1z \vee z1y$

by choice of z, $.\rightarrow.x1x \vee x1y.x1y \vee y1y$

$.\rightarrow.x1y \vee x1x.y1y$

by 3.3, $.\rightarrow.x1y. \vee .x\varepsilon A.x\varepsilon B.y\varepsilon A.y\varepsilon B$

by 4.1, $.\rightarrow.x1y. \vee .x\varepsilon A.y\varepsilon B$

by 3.3, $.\rightarrow.x1y \vee x1y$

$.\rightarrow.x1y.$

The theorem follows from this and 3.11.

Peculiar to Kh, there is a very important constant relation I called the identity relation. This is just the relation which every element of A bears to itself. In order to avoid a cumbersom definition we depart from our usual defining scheme and define I axiomatically.

4.11 Axiom. $I\varepsilon Kh.$

4.12 Axiom. $(x)xIx.$

4.13 Axiom. $(x,y,z,R):.R\varepsilon Kh:\rightarrow:xRz.zIy.\rightarrow.xRy.$

We thus simply agree to call every element of Kh which satisfies 4.12 and 4.13 an identity relation. We shall see shortly that the identity relation has been well defined. The identity relation behaves exactly like ordinary equality, i.e., "xIy" will be interpretable as "x=y".

4.14 Thm. $\sim(I=0)$.

Proof: By 3.1, $(Ex).x\varepsilon A$. Hence by 4.1, $x\varepsilon B$. Then by 4.12, $(Ex).xIx$. The theorem follows now since

$$
\begin{array}{lll}
\text{by } 3.25, & I=0.\rightarrow.(x,y)xIy\rightarrow x0y \\
 & .\rightarrow.(x,y)\sim x0y\rightarrow\sim xIy \\
\text{by } 3.13, & .\rightarrow.(x,y)\sim xIy \\
\text{by choice of } y, & .\rightarrow.(x)\sim xIx \\
 & .\rightarrow.\sim(Ex).xIx,
\end{array}
$$

and the contrapositive of this is

$$(Ex).xIx.\rightarrow.\sim(I=0).$$

We proceed now to the development of the algebra of Kh based upon the results of chapter three and the definitions given in the present chapter.

4.15 Thm. $(R).R\varepsilon Kh\rightarrow R^{\smile\smile}=R.$

Proof: By 4.4, $xR^{\smile\smile}y\leftrightarrow yR^{\smile}x$

and again $yR^{\smile}x\leftrightarrow xRy.$

The theorem follows from these and 3.25.

4.16 Thm. $(R,S).R,S\varepsilon Kh\rightarrow(R/S)^{\smile}=(S^{\smile}/R^{\smile}).$

Proof: Assume $R,S\varepsilon Kh$.

$$
\begin{array}{ll}
\text{By } 4.4, & x(R/S)^{\smile}y.\leftrightarrow.y(R/S)x \\
\text{by } 4.6, & .\leftrightarrow.(Ez).yRz.zSx \\
\text{by } 4.4, & .\leftrightarrow.(Ez).xS^{\smile}z.zR^{\smile}y \\
\text{by } 4.6, & .\leftrightarrow.x(S^{\smile}/R^{\smile})y.
\end{array}
$$

The conclusion follows now from 3.25.

4.17 Thm. $(R,S,T).R,S,T\varepsilon Kh\rightarrow R/(S/T)=(R/S)/T.$

Proof: $x(R/(S/T))y:\leftrightarrow:(Eu):xRu.u(S/T)y$

by 4.6, $:\leftrightarrow:(Eu):xRu.(Ev).uSv.vTy$

$:\leftrightarrow:(Eu,v):xRu.uSv.vTy$

by 4.6, $:\leftrightarrow:(Ev):x(R/S)v.vTy$

by 4.6, $:\leftrightarrow:x((R/S)/T)y.$

The result follows now from 3.25.

4.18 Thm. $(R).R\varepsilon Kh\rightarrow R/I=R.$

Proof: Assume $R\varepsilon Kh$. First we show $(R/I)<R$.

By 4.6, $x(R/I)y.\rightarrow.(Ez).xRzzIy$

by 4.13, $.\rightarrow.xRy.$

Hence, by 3.71, this yields $(R/I)<R$. To show that $R<(R/I)$, we have that

by 4.16, $\sim x(R/I)y.\rightarrow.\sim(Ez).xRz.zIy$

$.\rightarrow.(z)\sim(xRz.zIy)$

$.\rightarrow.(z).zIy\rightarrow\sim xRz$

by choice of z, $.\rightarrow.yIy\rightarrow\sim xRy$

by 4.12, $.\rightarrow.\sim xRy,$

the contrapositive of which is "$xRy\rightarrow x(R/I)y$". Then by 3.71, $R<(R/I)$. By 3.66 the theorem now follows.

4.19 Thm. $(R).R\varepsilon Kh\rightarrow R/1=1_{\vee}1/R'=1.$

Proof: In view of 3.9 and 3.25, we want to prove $(x,y)x(R/1)y_{\vee}(x,y)x(1/R')y$. Or, using 4.6,

$$(x,y):(Ez).xRzz1y:_{\vee}:(x,y):(Ez).x1z.zR'y,$$

which, by 3.9 is equivalent to

$$(x,y).(Ez)xRz._{\vee}.(x,y).(Ez)zR'y.$$

By 3.22, this is just

$$(x).(Ez)xRz._{\vee}.(y).(Ez)\sim zRy$$

or $$(x).(Ez)xRz._{\vee}.\sim\{(Ey).(z)zRy\},$$

or $$(Ey).(z)zRy.\rightarrow.(x).(Ez)xRz.$$

Since the variables on both ends of this implication are quantified we can change the letters, obtaining:

$$(Ey).(x)xRy.\rightarrow.(x).(Ey)xRy,$$

a theorem in the restricted predicate calculus.

4.20 Thm. $(R,S,T):R,S,T\varepsilon Kh.\rightarrow.(R/S)T=0\leftrightarrow(S/T^{\smile})R^{\smile}=0.$

Proof: By 3.13 and 3.25 we write $(R/S)=0$ as

$(R/S)T=0:\leftrightarrow:(x,y).\sim x(R/S)Ty$

by 3.19, $:\leftrightarrow:(x,y).\sim\{x(R/S)y.xTy\}$

by 4.6, $:\leftrightarrow:(x,y):\sim\{(Ez).xRz.zSy\}\vee\sim xTy$

$:\leftrightarrow:(x,y):\{(z).\sim xRz\vee\sim zSy\}\vee\sim xTy$

$:\leftrightarrow:(x,y,z).\sim xRz\vee\sim zSy\vee\sim xTy$

by 4.3, $:\leftrightarrow:(x,y,z).\sim zSy\vee\sim yT^{\smile}x\vee\sim zR^{\smile}x$

$:\leftrightarrow:(z,x,y,).\sim\{zSy.yT^{\smile}x\}\vee\sim zR^{\smile}x$

$:\leftrightarrow:(z,x).\sim\{(Ey).zSy.yT^{\smile}x\}\vee\sim zR^{\smile}x$

by 4.6, $:\leftrightarrow:(z,x).\sim z(S/T^{\smile})x\vee\sim zR^{\smile}x$

$:\leftrightarrow:(z,x).\sim\{z(S/T^{\smile})x.zR^{\smile}x\}$

by 3.19, $:\leftrightarrow:(z,x).\sim z(S/T^{\smile})R^{\smile}x$

by 3.13,3.25, $:\leftrightarrow:(S/T^{\smile})R^{\smile}=0.$

4.21 Thm. $(R,S).R,S\varepsilon Kh\rightarrow(R\not/S)'=R'/S'.$

Proof: Assume $R,S\varepsilon Kh$. Then

by 3.22, $x(R\not/S)'y.\leftrightarrow.\sim x(R\not/S)y$

by 4.9, $.\leftrightarrow.\sim\{(z).xRz\vee zSy\}$

$.\leftrightarrow.(Ez).\sim xRz.\sim zSy$

by 3.22, $.\leftrightarrow.(Ez).xR'zzS'y$

by 4.6, $.\leftrightarrow.x(R'/S')y.$

The theorem now follows by generalization and by 3.25.

4.22 Thm. $(R,S):R,S\varepsilon Kh.\rightarrow.RS'=0\rightarrow R^{\smile}S^{\smile}{}'=0.$

Proof: Assume $R,S\varepsilon Kh$. Then by 3.13 and

by 3.25, $RS'=0.\rightarrow.(x,y)\sim x(RS')y$

by 3.19, $.\rightarrow.(x,y)\sim\{xRy.xS'y\}$

by 4.3,3.22, $.\rightarrow.(x,y).\sim\{yR^{\smile}x.\sim yS^{\smile}x\}$

by 3.22, $.\rightarrow.(x,y).\sim\{yR^{\smile}x.yS^{\smile}{}'x\}$

by 3.19, $.\rightarrow.(x,y).\sim y(R^{\smile}S^{\smile}{}')x$

by 3.13,3.25, $.\rightarrow.R^{\smile}S^{\smile}{}'=0.$

4.23 Thm. $(R,S):R,S\varepsilon Kh.\longrightarrow.R=S\rightarrow R^{\smile}=S^{\smile}.$

Proof: Assume $R,S\varepsilon Kh.$ Then

by 3.52, $R=S.\rightarrow.RS'=0.R'S=0$

by 4.22, $\quad.\rightarrow.R^{\smile}S^{\smile}{}'=0.R^{\smile}{}'S^{\smile}=0$

by 3.52, $\quad.\rightarrow.R^{\smile}=S^{\smile}.$

4.24 Thm. $(R,S).R,S\varepsilon Kh\rightarrow(R+S)^{\smile}=R^{\smile}+S^{\smile}.$

Proof: Assume $R,S\varepsilon Kh.$ Then

by 3.50 and 3.49, $R^{\smile}(R^{\smile}+S^{\smile})'=R^{\smile}(R^{\smile}{}'S^{\smile}{}')=0.$

Similarly, $\quad S^{\smile}(R^{\smile}+S^{\smile})'=S^{\smile}(R^{\smile}{}'S^{\smile}{}')=0.$

By 4.22,4.15, $\quad R(R^{\smile}+S^{\smile})^{\smile}{}'=0.S(R^{\smile}+S^{\smile})^{\smile}{}'=0$

by 3.57,3.35, $\quad(R+S)(R^{\smile}+S^{\smile})^{\smile}{}'=0$

by 4.22,4.15, $\quad(R+S)^{\smile}(R^{\smile}+S^{\smile})'=0.$

The above argument with R and S replaced by $R^{\smile}$ and $S^{\smile}$ respectively yields

$$(R+S)^{\smile}{}'(R^{\smile}+S^{\smile})=0.$$

The theorem now follows by 3.52 from these last two results.

4.25 Thm. $0^{\smile}=0.$

Proof: By 3.45, $00^{\smile}{}'=0.$ Hence, by 4.22, $0^{\smile}0^{\smile\smile}{}'=0.$ Then by 4.15 and 3.41, $0^{\smile}1=0.$ The conclusion now follows from 3.34.

4.26 Thm. $1^{\smile}=1.$

Proof: Similar to the proof of theorem 4.25.

4.27 Thm. $(R,S,T):R,S,T\varepsilon Kh.\rightarrow.RS'=0\rightarrow(T/R)(T/S)'=0.$

Proof: Assume $R,S,T\varepsilon Kh.$ Then by 3.37 $(T/S)(T/S)'=0,$ hence by 4.20 $((T/S)'^{\smile}/T)S^{\smile}=0.$ Also, by 4.22, $RS'=0\rightarrow R^{\smile}S^{\smile}{}'=0.$ Hence by 3.56, $((T/S)'^{\smile}/T)R^{\smile}=0.$ Then by 4.20, $(T/R^{\smile\smile})(T/S)'=0.$ The theorem now follows by 4.15.

4.28 Thm. $(R,S,T)R,S,T\varepsilon Kh.\rightarrow.R=S\rightarrow T/R=T/S.$

Proof: Assume $R,S,T\varepsilon Kh.$ By 3.52 then, $R=S.\rightarrow.RS'=0.R'S=0.$ The theorem now follows from 4.27

and another application of 3.52.

4.29 Thm. $(R,S,T).R,S,T\varepsilon Kh\rightarrow R/(S+T)=(R/S)+(R/T)$.

Proof: Assume $R,S,T\varepsilon Kh$. For convenience, set $(R/S)+$ $+(R/T)=P$. Then by 3.50, $(R/S)P'=0$ and $(R/T)P'=0$. By 4.20, $(P'^{\smile}/R)S^{\smile}=0$ and $(P'^{\smile}/R)T^{\smile}=0$. Hence by 3.57 and 3.35, $(P'^{\smile}/R)(S^{\smile}+T^{\smile})=0$, and by 4.24, $(P'^{\smile}/R)(S+T)^{\smile}=0$. Using 4.20 then, $(R/(S+T))P'=0$. On the other hand, by 3.50 $S(S+T)'=0$ and $T(S+T)'=0$, thus by 4.27, $(R/S)(R/(S+T))'=0$ and $(R/T)(R/(S+T))'=0$. Then by 3.57 and 3.35, $P(R/(S+T))'=0$. Therefore, by 3.52, $R/(S+T)=P$.

4.30 Thm. $(R,S,T):R,S,T\varepsilon Kh.\rightarrow.RS'=0\rightarrow(R/T)(S/T)'=0$.

Proof: Similar to the proof of theorem 4.27.

4.31 Thm. $(R,S,T):R,S,T\varepsilon Kh.\rightarrow.R=S\rightarrow(R/T)=(S/T)$.

Proof: Similar to the proof of theorem 4.28.

4.32 Thm. $(R,S,T).R,S,T\varepsilon Kh\rightarrow(R+S)/T=(R/T)+(S/T)$.

Proof: Similar to the proof of theorem 4.29.

4.33 Thm. $(R).R\varepsilon Kh\rightarrow R/0=0$.

Proof: Assume $R\varepsilon Kh$. By 3.45, $(1/R)0=0$. Then by 4.20, $(R/0^{\smile})1^{\smile}=0$, so by 4.25 and 4.26, $(R/0)1=0$. Then by 3.34 $R/0=0$.

4.34 Thm. $(R).R\varepsilon Kh\rightarrow 0/R=0$.

Proof: Similar to the proof of theorem 4.33.

4.35 Thm. $I^{\smile}=I$.

Proof: By 4.16 and 4.15, $(R^{\smile}/I)^{\smile}=(I^{\smile}/R^{\smile\smile})=(I^{\smile}/R)$. Now by 4.18 $(R^{\smile}/I)=R^{\smile}$, hence $(R^{\smile}/I)^{\smile}=R^{\smile\smile}=R$. Therefore $(R).(I^{\smile}/R)=R$. By choosing R as I in this, $I^{\smile}/I=I$. But by 4.18, $I^{\smile}/I=I^{\smile}$. The conclusion now follows by 3.27.

4.36 Thm. $(R).R\varepsilon Kh\rightarrow I/R=R$.

Proof: Assume $R\varepsilon Kh$. By 4.35 and 4.31, $I/R=I^{\smile}/R$. Then by 4.15, 4.16, and 4.18,

$I/R=(I^{\smile}/R)^{\smile\smile}=(R^{\smile}/I^{\smile\smile})^{\smile}=(R^{\smile}/I)^{\smile}=R^{\smile\smile}=R.$

4.37 Thm. $(R):R/I_1=R.R/I_2=R:\rightarrow:I_1=I_2.$

Proof: By separate choice of R in the hypothesis, $I_2/I_1=I_2$ and $I_1/I_2=I_1$. Then by 4.35, and 4.16, $I_1=I_1^{\smile}=(I_1/I_2)^{\smile}=I_2^{\smile}/I_1^{\smile}=I_2/I_1=I_2.$

4.38 Thm. $(R,S,T,U):.R,S,T,U\varepsilon Kh:\rightarrow:R=S.T=U.\rightarrow.R/T=S/U.$

Proof: Follows from 4.31 and 4.28.

4.39 Thm. $(R,S,T,U):.R,S,T,U\varepsilon Kh:\rightarrow:R=S.T=U.\rightarrow.R\not+T=S\not+U.$

Proof: Follows from 4.38 and 4.21.

4.40 Thm. $(R,S,T):R,S,T\varepsilon Kh.\rightarrow.R\not+(S\not+T)=(R\not+S)\not+T.$

Proof: Follows from 4.17 and 4.21.

We have now established an associative, non-commutative algebra on Kh with the operations "/", "$\not+$", and "$\smile$". That is, the operations are well defined and Kh is closed with respect to all of these operations.

Theorems 4.7, 4.38, and 4.17 show that Kh constitutes a semi-group with respect to relative multiplication; moreover theorems 4.18 and 4.36 show that this is a semi-group with a unit.

The rest of this section will be devoted to the development of further useful and interesting results, and to the characterization of special subclasses of Kh which are important in mathematics.

Commutativity of the converse and complement operations in Kh is given by

4.41 Thm. $(R).R\varepsilon Kh\rightarrow R^{\smile}{}'=R'^{\smile}.$

Proof: Assume $R\varepsilon Kh$. First, we have $R^{\smile}{}'R'^{\smile}{}'=(R^{\smile}+R'^{\smile})'$,

$(R^{\smile}+R'^{\smile})'=(R+R')^{\smile\prime}=1^{\smile\prime}=1'=0$. Also, since $RR'=0$, $(R/I)R'=0$. Hence, by 4.20, $(I/R'^{\smile})R^{\smile}=0$. Then $R'^{\smile}R^{\smile}=0$, so that $R'^{\smile}R^{\smile\prime\prime}=0$. The theorem now follows from 3.52.

4.42 Thm. $(R,S).R,S\varepsilon Kh\rightarrow(RS)^{\smile}=R^{\smile}S^{\smile}$.

Proof: Follows from 3.50, 3.51, and 4.41.

The following three theorems are easy consequences of 4.21 with 4.29 and 4.32.

4.43 Thm. $(R,S,T).R,S,T\varepsilon Kh\rightarrow R\not+(ST)=(R\not+S)(R\not+T)$.

4.44 Thm. $(R,S,T).R,S,T\varepsilon Kh\rightarrow(RS)\not+T=(R\not+T)(S\not+T)$.

4.45 Thm. $(R,S,T).R,S,T\varepsilon Kh\rightarrow R/(S\not+T)=(R\not+S)/T$.

We have also the relative addition analogues of 4.16, 4.27, and 4.30 in

4.46 Thm. $(R,S).R,S\varepsilon Kh\rightarrow(R\not+S)^{\smile}=S^{\smile}\not+R^{\smile}$.

4.47 Thm. $(R,S,T):R,S,T\varepsilon Kh.\rightarrow.RS'=0\rightarrow(R\not+T)(S\not+T)'=0$.

4.48 Thm. $(R,S,T):R,S,T\varepsilon Kh.\rightarrow.RS'=0\rightarrow(T\not+R)(T\not+S)'=0$.

In view of theorem 4.21 we see that Kh is a semi-group with respect to relative addition. As the next theorem shows, the diversity relation I' is a unit element for this semi-group.

4.49 Thm. $(R).R\varepsilon Kh\rightarrow R\not+I'=I'\not+R=R$.

The next is a useful special case of 4.20.

4.50 Thm. $(R,S):R,S\varepsilon Kh.\rightarrow.R^{\smile}S=0\leftrightarrow(R/S)I=0$.

We now define a subset of Kh which is called the class of many-one relations. This is just the class of all functions, or maps, of modern mathematical analysis which map a given space onto or into itself.

4.51 Def. Put "F" for "$\hat{R}\{(x,y,z):xRy.xRz.\rightarrow.yIz\}$".

4.52 Thm. $(R):.R\varepsilon F:\leftrightarrow:(x,y,z):xRy.xRz.\rightarrow.yIz.$

The next theorem gives a characterization of the class F in our algebra.

4.53. Thm. $(R):R\varepsilon Kh.\rightarrow.R\varepsilon F\leftrightarrow R<R'\dagger I.$

Proof: For the left to the right implication, by 4.52,

$R\varepsilon F:\rightarrow:(x,y,z):xRy.xRz.\rightarrow.yIz$

$:\rightarrow:(x,y,z):xRy.xRz._{\vee}.xRy.xR'z.\rightarrow.yIz._{\vee}.xRy.xR'z$

$:\rightarrow:(x,y,z):xRy.xRz_{\vee}xR'z.\rightarrow.yIz_{\vee}xRy.yIz_{\vee}xR'z$

$:\rightarrow:(x,y,z):xRy.\rightarrow.xR'z_{\vee}zIy$

$:\rightarrow:(x,y).xRy\rightarrow x(R'\dagger I)y.$

The implication now follows from 3.71. For the right to left implication, by 3.71,

$R<R'\dagger I:\rightarrow:(x,y,z):xRy.\rightarrow.xR'z_{\vee}zIy$

$:\rightarrow:(x,y,z):xRy.\rightarrow.-xRz_{\vee}zIy$

$:\rightarrow:(x,y,z):xRy.\rightarrow.xRz\rightarrow zIy$

$:\rightarrow:(x,y,z):xRy.xRz.\rightarrow.zIy$

$:\rightarrow:R\varepsilon F.$

4.54 Thm. $I'=I'^{\smile}.$

Proof: Follows from 4.35 and 4.41.

The two preceeding theorems lead to a more suggestive characterization of F given by

4.55 Thm. $(R):R\varepsilon Kh.\rightarrow.R\varepsilon F\leftrightarrow(R^{\smile}/R)I'=0.$

Proof: Assume $R\varepsilon Kh$. Then

by 4.53, $R\varepsilon F\leftrightarrow R<R'\dagger I$

by 3.63, $\leftrightarrow R(R'\dagger I)'=0$

by 4.21, $\leftrightarrow R(R/I')=0$

by 4.15, $\leftrightarrow(R/I'^{\smile\smile})R^{\smile\smile}=0$

by 4.20, $\leftrightarrow(R^{\smile}/R)I'=0.$

For an interpretation of R as a function in the usual notation of mathematics we write "$y = R(x)$" for "xRy" when $R \varepsilon F$. With this notation 4.51 states the usual condition of single valuedness of a function; if $y = R(x)$ and $z = R(x)$, then $y = z$. From 4.6 we have the interpretation of R/S in function notation where $R, S \varepsilon F$, $y=(R/S)(x)$ means "there is a z such that $y = S(z)$ and $z = R(x)$, that is, $y = S(R(x))$." If R is a function, the inverse function, or inverse map R^{-1}, is just what we have been denoting by "$\breve{R}$". With these conventions 4.55 becomes, "R is a function in A to A if and only if for all x and y in A, $x = R(z)$ and $y = R(z)$ together imply that $x = y$."

We shall now define a class of relations which will lead us one step nearer to the previously mentioned group with respect to relative multiplication.

4.56 Def. Put""Onto" for "$\hat{R}\{(y)(Ex)xRy\}$".

4.57 Thm. $(R):R\varepsilon Kh. \rightarrow .R\varepsilon \mathrm{Onto} \leftrightarrow (y)(Ex)xRy$.

The next two theorems give characterizations of Onto in the algebra of relations.

4.58 Thm. $(R):R\varepsilon Kh. \rightarrow .R\varepsilon \mathrm{Onto} \leftrightarrow (\breve{R}/R)'I=0$.

Proof: Assume $R\varepsilon Kh$. Then

by 3.13, 3.25, $(\breve{R}/R)'I=0. \leftrightarrow .(x,y). -x((\breve{R}/R)'I)y$

$. \leftrightarrow .(x,y). -x(\breve{R}/R)'y \lor -xIy$

$. \leftrightarrow .(x,y).xIy \rightarrow x(\breve{R}/R)y$

$. \leftrightarrow .(x)x(\breve{R}/R)x$

$. \leftrightarrow .(x)(Ez)x\breve{R}z zRx$

$. \leftrightarrow .(x)(Ez)zRx.$

The theorem now follows from 4.57.

4.59 Thm. $(R):R\varepsilon Kh.\rightarrow.R\varepsilon Onto\leftrightarrow 1/R=1.$

Proof: Assume $R\varepsilon Kh$. Then

by 3.9, 3.25, $1/R=1\leftrightarrow(x,y)x(1/R)y$

by 4.6, $\leftrightarrow(x,y)(Ez)x1zzRy$

by 3.10, $\leftrightarrow(y)(Ez)zRy$

by 4.57, $\leftrightarrow R\varepsilon Onto.$

4.60 Thm. $(R):R\varepsilon Kh\rightarrow R\varepsilon Onto_{v}\breve{R}'\varepsilon Onto.$

Proof: Using 4.16, 4.15, and 4.41, we have

$$1/\breve{R}'=1\leftrightarrow R'/1=1.$$

The theorem now follows from 4.59 and 4.19.

4.61 Thm. $(R,S):R,S\varepsilon Kh.\rightarrow.R,S\varepsilon F\rightarrow R/S\varepsilon F.$

Proof: Assume $R,S\varepsilon Kh$. Then

by 4.55, $R,S\varepsilon F.\rightarrow.(\breve{R}/R)I'=0.(\breve{S}/S)I'=0$

by 4.30, $.\rightarrow.((\breve{R}/R)/S)(I/S)'=0.(\breve{S}/S)I'=0$

by 4.36, $.\rightarrow.(\breve{R}/(R/S))S'=0.(\breve{S}/S)I'=0$

by 4.27, $.\rightarrow.(\breve{S}/(\breve{R}/(R/S)))(\breve{S}/S)'=0.(\breve{S}/S)I'=0$

by 3.56, $.\rightarrow.((\breve{S}/\breve{R})/(R/S))I'=0$

by 4.16, $.\rightarrow.((R/S)^{\smile}/(R/S))I'=0$

by 4.58, $.\rightarrow.R/S\varepsilon F.$

4.62 Thm. $(R,S):R,S\varepsilon Kh.\rightarrow.R,S\varepsilon Onto\rightarrow R/S\varepsilon Onto.$

Proof: Similar to the proof of theorem 4.61.

We next define a class of relations in Kh which is very useful in mathematics. It is the class of all functions which map the space A into a subset of itself. These relations are sometimes called transformations.

4.63 Def. Put "Tr" for "$\hat{R}\{R\varepsilon F.\breve{R}\varepsilon Onto\}$".

4.64 Thm. $(R):.R\varepsilon Kh:\rightarrow:R\varepsilon Tr.\leftrightarrow.R\varepsilon F.\breve{R}\varepsilon Onto.$

4.65 Thm. $(R,S):R,S\varepsilon Kh.\rightarrow.R,S\varepsilon Tr.\Rightarrow R/S\varepsilon Tr.$

Proof: Follows from 4.64, 4.61, and 4.62.

4.66 Thm. $I\varepsilon Tr.$

Proof: By 4.18, 4.35, and 4.36, $(\breve{I}/I)I'=II'=0$ and $(I/\breve{I})'I=I'I=0$. The theorem now follows from 4.55, 4.58, and 4.64.

We are now in a position to define a subset of Kh constituting a group with respect to relative multiplication. This class of relations is called a transformation group in Kh.

4.67 Def. Put "TG" for "$\hat{R}\{R,\breve{R}\varepsilon Tr\}$".

4.68 Thm. $(R):.R\varepsilon Kh:\rightarrow:R\varepsilon TG.\leftrightarrow.R\varepsilon Tr.\breve{R}\varepsilon Tr.$

4.69 Thm. $I\varepsilon TG.$

Proof: Follows from 4.66, 4.35, and 4.68.

It is seen from theorem 4.7 that the elements of TG obey the associative law with respect to relative multiplication. Also, by theorems 4.18, 4.36, and 4.69, we see that the identity relation I can be taken as a unit element of TG for relative multiplication. The next three theorems show that TG is closed under relative multiplication and that each element in TG has a right and a left inverse in TG, and thus that TG does, in fact, constitute a group.

4.70 Thm. $(R):R\varepsilon Kh.\rightarrow.R\varepsilon TG\rightarrow\breve{R}\varepsilon TG.$

4.71 Thm. $(R,S):R,S\varepsilon Kh.\rightarrow.R,S\varepsilon TG\rightarrow R/S\varepsilon TG.$

Proof: Follows from 4.68 and 4.65, using 4.16.

4.72 Thm. $(R):R\varepsilon Kh.\rightarrow.R\varepsilon TG\leftrightarrow\breve{R}/R = R/\breve{R}= I.$

Proof: Assume RεKh. Then by 4.68, 4.64, and 3.52,

$R\varepsilon TG.\leftrightarrow.R\varepsilon Tr.\breve{R}\varepsilon Tr$

$.\leftrightarrow.R\varepsilon F.R\varepsilon Onto.\breve{R}\varepsilon F.\breve{R}\varepsilon Onto$

$$R\varepsilon TG. \leftrightarrow .(\breve{R}/R)I'=0.(\breve{R}/R)'I=0.(R/\breve{R})I'=0.(R/\breve{R})'I=0 \\ .\leftrightarrow .\breve{R}/R=I.R/\breve{R}=I.$$

In TG then we see that we may take the converse of a relation as its multiplication inverse. Theorem 4.23 shows that this inverse is unique. It may be of interest to note that theorems 4.55 and 4.58 together give a necessary and sufficient condition for the existence of a left inverse. This is given explicitly in

4.73 Thm. $(R):.R\varepsilon Kh:\rightarrow:\breve{R}/R=I.\leftrightarrow .R\varepsilon F.R\varepsilon Onto.$

By taking R to be $\breve{R}$ in theorem 4.73 we have the corresponding condition for the existence of a right inverse.

There are many interesting relations and classes of relations in Kh, such as the ordering relations and equivalence relations. Since our interest here is primarily with the basis of the formal algebra of relations, we shall not examine further subclasses of Kh, but shall look now at a geometric interpretation of the three relative operations.

In chapter three we gave a method of construction for a picture of the product space for the graphical representation of a relation. For the homogeneous relations it will be helpful to extend this technique to three dimensions. We use the ordinary cartesian representation where now all of the axes are identical. For our purpose it will be convenient to distinguish between these axes;

accordingly, we label them with subscripts 1, 2, and 3, with x_1, x_2, and x_3 used to denote variables on the A_1-, A_2-, and A_3-axes.

We have three square product spaces, or planes, $A_1 \times A_2$, $A_2 \times A_3$, and $A_1 \times A_3$, which determine a cube. We may think of the $A_1 \times A_2$ plane as our principle product space and the other two planes as auxiliary spaces.

The identity relation I is just the diagonal of $A_1 \times A_2$ through the point of intersection of the three axes.

For any relation R in $A_1 \times A_2$, $R^{\smile}$ is just a rigid rotation of R through 180° about I as an axis of rotation. This rotation is equivalent to a rotation through 90° about the A_1-axis into the $A_1 \times A_3$ plane, followed by a rotation about the A_3-axis into the $A_2 \times A_3$ plane, and then a rotation about the A_2-axis into the $A_2 \times A_1$ plane. This gives the relation R in the $A_2 \times A_1$ plane which is $R^{\smile}$ in the $A_1 \times A_2$ plane.

The graph of the relative product of R and S in $A_1 \times A_2$ can be obtained in the following way. Rotate the graphs of R and S rigidly through 90° about the A_1-axis and the A_2-axis respectively. Then R is in $A_1 \times A_3$ and S is in $A_3 \times A_2$. Now draw lines parallel to the A_2-axis through every point in R, and parallel to the A_1-axis through every point in S. The projection back into the $A_1 \times A_2$ plane of every point of intersection of these lines will constitute the relation R/S in $A_1 \times A_2$.

This method of treating the converse and relative multiplication operations will be found quite useful in the next chapter, where we discuss the topology of relations.

CHAPTER V

THE TOPOLOGY OF RELATIONS

The study of relations in extension has been seen in chapters three and four to be the study of a class of subsets of a given set 1. This consideration of relations as sets suggests that the ideas of topology should be applicable. R. Vaidyanathaswamy has remarked (5, p. 189) that the topological calculus of relations ought to be rich in content, but that it had not been systematically developed. He then states and proves two theorems (our theorems 5.5 and 5.11) in the topology of relations. In this chapter we shall give connections between the three relative operations defined in chapter four and topology. These connections can serve as a nucleus or a basis for a systematic development of the topological calculus of relations.

We begin with relations in Kh. In order to obtain our results in the product space $A \times A$ we shall find it convenient to work in the space $A \times A \times A$ of ordered triples. Referring to the discussion of the geometry of relations in chapter four, we consider the space $A \times A \times A$ as determined by three axes, the A_1-, A_2-, and A_3-axes. The subscripts here do not indicate a distinction between the spaces but serve only to distinguish between the three axes, on each of which the space A is represented. We denote by R_{ij} the relation R on the $A_i \times A_j$ plane, $i,j = 1,2,3$.

We define three kinds of operators in this space; the permutator P_{ij} indicates replacement of the subscript i by the subscript j wherever i occurs in the operand, the plane projections Π_{ij} which project a set of ordered triples into the $A_i \times A_j$ plane, and the axis projections Π_i which project from $A_1 \times A_2 \times A_3$ into the A_i-axis.

Referring to definitions 4.2 and 4.5, and the above mentioned section on the geometry of the relative operations in Kh, we see that the relative product and the converse can be expressed in terms of the permutation and projection operators. These expressions are given in the next two theorems.

5.1 Thm. $\breve{R}_{12} = P_{32}P_{21}P_{13}(R_{12})$.

5.2 Thm. $R_{12}/S_{12} = \Pi_{12}\{(\Pi_{13}^{-1}P_{23}(R_{12}))(\Pi_{32}^{-1}P_{13}(S_{12}))\}$.

Suppose that the space A is a topological space. Then let $A_1 \times A_2 \times A_3$ be the corresponding topological product. With this topology, the projection operators we have defined are open, continuous maps. Also, since the six permutators constitute a finite group of automorphisms of $A \times A \times A$ onto itself, they are each open, closed, and continuous maps.

We say that a relation R is open, closed, or compact according as the graph of R is an open, closed, or compact subset of the topological product $A \times A$. We denote the closure of R by R^-.

With these tools at hand we turn now to the topological calculus of relations. As implicit hypotheses for theorems 5.3 to 5.10 we shall understand $R, S \varepsilon Kh$.

5.3 Thm. If R is open, so is $R^\smile$.

Proof: In $A_1 \times A_2 \times A_3$ take $A_1 \times A_2 = 1$. Then $R = R_{12}$. Since the P_{ij} are open maps for $i,j=1,2,3$, it follows from 5.1 that $R^\smile_{12}$ is open if R_{12} is open.

5.4 Thm. If R is compact, so is $R^\smile$.

Proof: Use 5.1 and continuity of the P_{ij}.

5.5 Thm. If R is closed, so is $R^\smile$.

Proof: Use 5.1 and the fact that the P_{ij} are closed maps.

A somewhat stronger form of theorem 5.5 is given by

5.6 Thm. $R^{-\smile} = R^{\smile -}$.

Proof: Since $R_{mn} < R^-_{mn}$, $P_{ij}(R_{mn}) < P_{ij}(R^-_{mn})$. But P_{ij} is a closed map so $P_{ij}(R^-_{mn})$ is closed. Hence $\{P_{ij}(R_{mn})\}^- < P_{ij}(R^-_{mn})$. Also, since P_{ij} is continuous, $P_{ij}(R^-_{mn}) < \{P_{ij}(R_{mn})\}^-$. Then by 3.78, $P_{ij}(R^-_{mn}) = \{P_{ij}(R_{mn})\}^-$. The theorem now follows from 5.1 and iteration of this process.

5.7 Thm. If R and S are open, so is R/S.

Proof: Take $R = R_{12}$ and $S = S_{12}$. Then since the P_{ij}

are open maps and since the Π_{ij} are continuous, the sets $\Pi_{13}^{-1}P_{23}(R_{12})$ and $\Pi_{32}^{-1}P_{13}(S_{12})$ are open in $A_1 \times A_2 \times A_3$, and their intersection is therefore open. By 5.2, R_{12}/S_{12} is then open since Π_{12} is an open map.

In view of theorem 4.21, the topological character of relative addition is given as the dual of theorem 5.7

5.8 Thm. If R and S are closed, so is $R \dagger S$.

Proof: Follows from 5.8 and 4.21.

It will be noticed that theorem 5.8 does not admit a strengthing as was done in the case of theorem 5.5, that is, we cannot say that $R^{-} \dagger S^{-} = (R \dagger S)^{-}$ in general. Consider, for example, $R = \widehat{x;y}\{x \text{ is real}.y \text{ is rational}\}$ and $S = \widehat{x;y}\{x \text{ is rational}.y \text{ is real}\}$. Then $R^{-} = S^{-} = 1$ so that $R^{-} \dagger S^{-} = 1 \dagger 1 = 1$. However, it is easily seen that $(R \dagger S)^{-} = (0)^{-} = 0$. The same example will serve to show that theorem 5.7 cannot be strengthened to the form "intR/intS = int(R/S)" where "intR" is the interior of R.

We may note also, that R and S being closed is not sufficient to imply that R/S is also closed. This can be shown by an example. Let $R = \widehat{x;y}\{xy=1\}$ in E^2. The graph of R is closed in E^2, but the graph of $R/R = \widehat{x;y}\{(Ez).xz=1.zy=1\}$ is not closed. This defect can be partially remedied by the imposition of more severe conditions on both the topology of the space A and on the relations R and S. Up to this point no stipulation was made as to the strength of the topology on the space A; the only requirement being that the topology of the product space be the naturally induced topology of the component spaces. In the next two theorems, however, we shall require that A be at least a T_2 (Hausdorff) space.

5.9 Thm. If R and S are compact, so is R/S.

Proof: Take $R = R_{12}$ and $S = S_{12}$ in $A_1 \times A_2 \times A_3$. Since the P_{ij} are continuous maps, both $P_{23}(R_{12})$ and $P_{13}(S_{12})$ are compact, and hence closed since A is T_2. Therefore, by continuity of the Π_{ij} projections, both $\Pi_{13}^{-1}P_{23}(R_{12})$ and $\Pi_{32}^{-1}P_{13}(S_{12})$ are closed. Now by continuity of the Π_i and the P_{ij}, it follows that $\Pi_2 P_{13}(S_{12})$ is compact. Then the product set $P_{23}(R_{12}) \times \Pi_2 P_{13}(S_{12})$ is compact. But we have $(\Pi_{13}^{-1}P_{23}(R_{12}))(\Pi_{32}^{-1}P_{13}(S_{12})) < P_{23}(R_{12}) \times \Pi_2 P_{13}(S_{12})$. Thus $(\Pi_{13}^{-1}P_{23}(R_{12}))(\Pi_{32}^{-1}P_{13}(S_{12}))$ is a closed subset of a compact set in $A_1 \times A_2 \times A_3$, and is therefore compact. The theorem follows from this by 5.2 and the continuity of Π_{12}.

S Since the identity relation I plays an important rôle in the algebra of relations, its topological character should be of interest in the topology of relations. We find that here too, we need the Hausdorff separation postulate T_2 for the space A in order to demonstrate closure of I.

5.10 Thm. I is closed.

Proof: Suppose $xI'y$. Then since A is T_2, and by 4.12, there exist in A neighborhoods N_x and N_y of x and y such that N_xN_y = empty set. For convenience, write $R = N_x \times N_y$. Then $(x)\sim xRx$. From 4.35 and 4.13 we get $(x,y):xRy.xIy.\rightarrow.xRx$; the contrapositive of which is $(x,y):\sim xRx.\rightarrow.\sim x(RI)y$. We conclude $(x,y)\sim x(RI)y$, from which $RI=0$. It follows from this that I is closed.

We leave Kh now for the more general class of relations K. For this we shall require two topological spaces A and B, and we shall suppose the topology of the corresponding product space 1 to be the natural topology induced by A and B. For the next three theorems we need the stipulation that the topology of the space A satisfy the weak separation postulate T_1.

5.11 Thm. If R is closed, so is x^*.

Proof: Since A is T_1, x is closed. Then since B is closed, $x \times B$ is closed. Since the intersection of two closed sets is closed, $(x \times B)R = x \times x^*$ is closed. Hence $(x \times x^*)^- = x^- \times x^{*-} = x \times x^*$, from which we conclude $x^* = x^{*-}$.

5.12 Thm. If R is closed, for any $X<A$, X^* is closed.

Proof: By theorem 5.11, x^* is closed. Since the intersection of any number of closed sets is a closed set, and by 3.101, $X^* = \bigcap_{x\varepsilon X} x^*$ is closed.

5.13 Thm. If R is open, so is x^*.

Proof: Since R is open, R' is closed. Now $y\varepsilon x^*_{R'} \leftrightarrow xR'y \leftrightarrow \sim xRy \leftrightarrow \sim y\varepsilon x^*_R \leftrightarrow y\varepsilon x^{*\prime}_R$, hence $x^*_{R'} = x^{*\prime}_R$.

By 5.11, $x_{R'}^*$ is closed, so that $x_R^{*\prime}$ is also closed. Thus x_R^* is open.

The material presented in this chapter does not constitute a systematic development of the topological calculus of relations as suggested by Vaidyanathaswamy. It does, however, establish a basis for such an undertaking by setting forth some very general and fundamental connections between the topology of 1 and the three relative operations in Kh.

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