Time-Dependent, Integrated Planetary Boundary Layer Flow

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ABSTRACT

The response of vertically averaged boundary layer flow to specified time-dependent pressure gradients is examined. The adjustment rate of the boundary layer flow to the pressure gradient field is found to be proportional to the strength of the coupling between the flow and boundary stresses. The angle between the steady flow and geostrophic wind is also proportional to the strength of this coupling.

The relationship of the transient component of the boundary layer wind to the "frictionless" transient wind component is found to be similar to the relationship of the steady boundary layer wind to the geostrophic wind. In flows characterized by transient Rossby number of order unity or greater, the production of cross-isobar mass transport is more sensitive to local accelerations than to typical variations of the surface drag coefficient. The transient portion of the vertical motion field, associated with the geostrophic vorticity tendency, is important when the time scale of the pressure gradient is comparable to or smaller than the boundary layer adjustment time scale.

1. Introduction

Local accelerations appear to be frequently important in the planetary boundary layer where large pressure tendencies may occur on synoptic scales as well as in diurnal and mesoscales of motion. Naistat and Young (1973) found that inclusion of local accelerations significantly improved the prediction of low-level winds associated with an intense synoptic-scale disturbance. Time-dependent Ekman solutions of Holton et al. (1971) and Chang (1973a) indicate that synoptic-scale local accelerations can completely change the character of the boundary layer when the transient Rossby number exceeds unity. A treatment of transient Ekman-Taylor flow based on an expansion in the Rossby number is presented in Young (1973). He found that the boundary layer isallobaric wind is reduced in magnitude and to the left of the frictionless isallobaric wind, and causes a general thickening of the boundary layer.

The application of the theory of transient Ekman flow to atmospheric flows is limited by the use of an eddy viscosity. A physically rigorous Reynolds stress parameterization is currently unavailable. The present study of transient boundary layer flow skirts such parameterization problems by treating the vertically integrated flow, resulting in equations of motion similar to those of Geisler and Kraus (1969) and Lavoie (1972). While this approach possesses appealing mathematical simplicity without requiring unjustified local shear-stress relationships, it also inherits a number of disadvantages. In particular, only layer-mean properties result so that the vertical structure of the boundary layer remains undetermined. Only in the case of a horizontally homogeneous, completely mixed layer do the layer mean equations approach those at a given level. Lenschow's (1970) case study of a convectively mixed layer illustrates atmospheric flow approaching these conditions.

In general, the shear may be significant throughout the boundary layer although the strongest shear is still expected to be in the surface layer and possibly near the top of the boundary layer. We will consider the boundary layer top to be a thermal inversion or condensation level. The two regions of possible strong shear will be parameterized as boundary stresses. Therefore, the layer average solution should be a reasonable indicator of flow behavior at a given level in the layer. In the subsequent sections, the layer equations with certain specified time-dependent pressure gradients will be solved to examine the influence of local accelerations on bulk characteristics of the boundary layer.

2. General solution

The vertically integrated equations of motion for horizontally homogeneous mean flow can be written as

\[ \frac{\partial \vec{u}}{\partial t} = f(\vec{u} - \vec{v}_g) - \left[ (\vec{u}'w')_x - (\vec{w}'w')_x \right]/h, \]  

\[ \frac{\partial \vec{v}}{\partial t} = -f(\vec{u} - \vec{v}_g) - \left[ (\vec{v}'w')_y - (\vec{w}'w')_y \right]/h, \]

where \( \vec{u}'w' \) and \( \vec{v}'w' \) are the Reynolds stresses, \( f \) the Coriolis parameter, \( \vec{u}_g \) and \( \vec{v}_g \) are the geostrophic wind components in the \( x \) and \( y \) directions, respectively, the
subscripts $s$ and $h$ refer to the top of the surface layer and the top of the planetary boundary layer, respectively, and other symbols are defined in the usual sense. The double bar operator refers to the operation

$$h^{-1} \int_z^h \rho \, dz,$$

where we define $z = 0$ to be the level $s$. The influence of the height variation of density on the stress divergence has been neglected.

To minimize the complexity of the solution, we assume a linearized stress law for the lower boundary condition

$$(\bar{u}' \bar{w}'), \bar{i} + (\bar{v}' \bar{w}'), \bar{j} = -C_D [\bar{V}] [(\bar{u} \cos \eta - \bar{v} \sin \eta) \bar{i} + (\bar{v} \cos \eta + \bar{u} \sin \eta) \bar{j}],$$

where $C_D$ is a drag coefficient, $[\bar{V}]$ is a scale velocity such as the geostrophic wind, and the negative of the surface stress direction is to the left of the layer mean wind direction by angle $\eta$. The value of $C_D$ will likely be smaller than normal drag coefficient values since it relates the stress near the surface to a boundary layer mean velocity. We do not relate the stress exclusively to the geostrophic wind since coupling between the surface stress and actual flow is desired in accelerated flow situations. If coupling between the wind and boundary stresses is not allowed, the adjustment time scale of the boundary layer flow is infinite and inertial oscillations would be undamped.

We allow for the importance of nonzero stress at the top of the boundary layer since we consider its top to be a thermal inversion or cloud base. Thus, in contrast to the classical concept of a boundary layer, the stress divergence will be nonzero above the boundary layer. It is hoped that this stress divergence will be much smaller than that in the boundary layer or at least only intermittently large. Deardorff (1973) found the stress to be quite large at the top of a growing boundary layer which was engulfing a low-level jet. Following Geisler and Kraus (1969) and Deardorff (1973), we parameterize the stress at the top of the boundary layer as

$$(\bar{u}' \bar{w}'), \bar{i} + (\bar{v}' \bar{w}'), \bar{j} = -w_s [(n_f - \bar{u}) \bar{i} + (v_f - \bar{v}) \bar{j}],$$

where $w_s$ is a coefficient with units of velocity and the subscript $f$ refers to conditions immediately above the boundary layer top. In the case of a growing turbulent boundary layer within a laminar fluid, $w_s$ becomes the rate of entrainment of free fluid into the boundary layer. In the case of zero net mass flux through the boundary layer top, $w_s$ can be thought of as the product of a drag coefficient and a scale velocity. Employing these boundary conditions, transforming (1) and (2) to complex space, and scaling time with $1/f$ and velocities with the geostrophic wind speed, we obtain

$$\frac{\partial \bar{V}}{\partial t} = -G(\bar{C}_D, \bar{w}_s, \eta) \bar{V} + i \bar{V}_o + \bar{w}_s \bar{V}_f$$

with

$$\bar{V} = (\bar{u} + i \bar{v})/(\bar{u}_s^2 + \bar{v}_s^2)^{1/2},$$

$$\bar{V}_o = (\bar{u}_o + i \bar{v}_o)/(\bar{u}_o^2 + \bar{v}_o^2)^{1/2},$$

$$i = i,$$

$$\bar{C}_D = (hf)^{-1} [\bar{V}] C_D,$$

$$\bar{w}_s = (hf)^{-1} \bar{w}_s,$$

$$G(\bar{C}_D, \bar{w}_s, \eta) = \bar{C}_D \cos(\eta) + \bar{w}_s + i [1 + \bar{C}_D \sin(\eta)].$$

(3)

All complex velocities are denoted by capital letters while all scaled variables are denoted by the caret. $Re[G(\bar{C}_D, \bar{w}_s, \eta)]$ is an indication of the strength of coupling between the layer wind and boundary stresses. In the special case $\eta = \bar{w}_s = 0$, the layer-average stress divergence is in the opposite direction of and proportional to the layer-mean wind. The system then becomes analogous to the one-level equations of motion developed by Guldberg and Mohr (see Hess, 1959, p. 179).

Assuming constant $\bar{V}_o$, $\bar{w}_s$, $\bar{C}_D$, and $\eta$, the general solution for $\bar{V}$ is obtained by using the integrating factor

$$\exp \left[ \int_0^t G(\bar{C}_D, \bar{w}_s, \eta) dt \right];$$

$$\bar{V} = i \exp \left[ -G(\bar{C}_D, \eta) \right] \int_0^t \{ \exp[\bar{G}(\bar{C}_D, \eta)] \bar{V}_o(t) \} dt$$

$$+ \bar{w}_s \bar{V}_o G^{-1}(\bar{C}_D, \eta) \{ 1 - \exp[ -G(\bar{C}_D, \eta) ] \}$$

$$+ \bar{V}_o \exp[ -G(\bar{C}_D, \eta) ],$$

(4)

where $\bar{V}_o$ is the initial layer average flow. The solution for $\bar{V}$ can be obtained for any time-dependent pressure field, provided the integral on the right-hand side of (4) is integrable. The solution can be generalized by allowing simple integrable time dependencies for $\bar{V}_o$, $\bar{w}_s$, $\bar{C}_D$ and $\eta$. In the case of most simple geostrophic wind tendencies, the solution for $\bar{V}$ can be obtained with $\bar{V}_o$ specified to be the frictionless solution of (4). These additional time dependencies, which result in additional terms on the right-hand side of (4), will not be considered in the present fundamental study. Discussion of the effects of nonzero $\bar{w}_s$ and $\eta$ will be confined to the case of boundary layer flow adjusting to a constant pressure gradient, which is the topic of the next section.

3. Constant pressure gradient

Consider the simple case of constant horizontal pressure gradient. Rotating the coordinate system so that
\( \dot{V}_e = 1 \) and integrating (4), we obtain
\[
\dot{V} = iG^{-1}(\dot{C}_D, \dot{\omega}_e, \eta) \{ 1 - \exp[-G(\dot{C}_D, \dot{\omega}_e, \eta)F] \} \\
\times (1 - i\dot{\omega}_e \dot{V}) + \dot{V}_e \exp[-G(\dot{C}_D, \dot{\omega}_e, \eta)F].
\]

The sum of the terms multiplied by the factor \( \exp[-G(\dot{C}_D, \dot{\omega}_e, \eta)] \) represents a frictionally damped inertial oscillation resulting from initial imbalances. The dimensional \( 1/e \) folding time scale of the damped oscillation is
\[
T = \left( \frac{f \Re(G(\dot{C}_D, \dot{\omega}_e, \eta))}{\eta} \right)^{-1} = \frac{1}{h} \frac{1}{C_D} [V] \cos(\gamma) + \omega^-1.
\]

Only the stress in the layer-mean wind direction influences the adjustment time scale. The adjustment time scale is to some extent inversely related to the strength of the coupling between the layer flow and boundary stresses. Note that \( T \) is not directly dependent on the Coriolis parameter. Choosing \( C_D = 1.5 \times 10^{-5} \), \([V] = 5 \text{ m sec}^{-1} \), \( h = 1 \text{ km} \) and \( \eta = \omega_e = 0 \), then \( T \approx 1\frac{1}{2} \text{ days} \). \( \dot{C}_D \) and \( T \) will generally be less variable than the parameters on which they depend, since \( C_D \) and \([V] \) are usually positively correlated with \( h \).

The damping time scale can also be estimated by regarding \( C_D [V] \) as \( u_f \), the friction velocity. Then if \( \eta = \omega_e = 0 \), \( T = C_D^{-1} (h f / u_f) (1 / f) \). Thus, the damping time scale relative to the Coriolis time scale is the ratio of the nondimensional layer depth \( h f / u_f \) to the square root of the drag coefficient. For a nondimensional layer depth of 0.25, \( C_D = 1.5 \times 10^{-5} \) and \( f = 10^{-4} \text{ sec}^{-1} \), the damping time scale is \( \sim 18 \text{ hr} \). These numerical examples indicate that a typical decay time scale for inertial oscillations is comparable to one day. The inverse dependence of this time scale on the drag coefficient replaces its inverse dependence on the square root of the viscosity or eddy viscosity present in the solutions of Stokes (see Schlichting, 1968), Pandolfo and Brown (1967), and Ching and Businger (1968).

Filtering out inertial oscillations, or allowing \( \dot{\omega}_e \to T \), the solution for constant pressure gradient becomes
\[
\dot{V} = (1 - \dot{C}_D \sin(\gamma)) \left[ \dot{C}_D \cos(\gamma) + \dot{\omega}_e \right]
\times \left[ \left( 1 - \dot{C}_D \sin(\gamma) \right)^2 + \left( \dot{C}_D \cos(\gamma) + \dot{\omega}_e \right)^2 \right]^{-1}
\times (1 - i\dot{\omega}_e \dot{V}).
\]

We will now briefly consider three special cases: \( \eta = \omega_e = 0 \), \( \eta \neq 0 \) and \( \omega_e \neq 0 \). In the case of \( \eta = \omega_e = 0 \), the steady solution becomes simply \( \dot{V} = (1 + i\dot{C}_D) / (1 + i\dot{C}_D') \), which is of the same mathematical form as the solution of Guldberg and Mohn. The effect of the surface stress is to rotate the layer-average wind vector away from the geostrophic wind toward the horizontal pressure gradient by an angle of \( \alpha = \tan^{-1}(\dot{C}_D) \) and to reduce the wind speed from the geostrophic wind speed by a factor of \( [1 + \dot{C}_D'^2]^{-1} \) (see Fig. 1). This corresponds to a total cross-isobar flow of \( h \dot{C}_D / (1 + \dot{C}_D') \), which is only weakly dependent on \( h \) since \( \dot{C}_D = C_D [V] / (h f) \).

Although the steady-state angle \( \alpha \) is proportional to the frictional damping parameter \( \dot{C}_D \), it does not necessarily vanish as \( \dot{C}_D \) vanishes. This is due to the fact that the damping time scale \( (\dot{C}_D')^{-1} \) becomes infinite as \( \dot{C}_D \) vanishes so that the inertial part of the flow never dissipates. Deviations from geostrophy and generation of boundary layer cross-isobar flow may be quite sensitive to the presence of local accelerations associated with adjustment to a constant pressure gradient. For example, evaluation of (4) with zero initial flow indicates that \( \alpha_e \), during the adjustment to steady state, may become nearly double its steady-state value.

Now consider the cases where \( \eta \) is nonzero. Except with strong baroclinicity, the surface stress will generally
be to the left of the layer-average wind ($u > 0$). Examination of (5) indicates that inclusion of nonzero positive $\eta$ results in a decreased $\alpha$. That is, as $\eta$ increases, the component of the stress divergence perpendicular to the pressure gradient decreases so that both the component of the Coriolis term perpendicular to the pressure gradient and $\alpha$ decrease.

Nonzero stress at the top of the boundary layer increases the rate at which the flow adjusts to the pressure gradient. This adjustment speed-up is due to increased coupling between the layer flow and its boundaries, and occurs independently of the flow above the boundary layer. To estimate the possible importance of this stress, we employ the following free-convection relationships for entrainment rate (Deardorf, 1973; Tennekes, 1973):

$$\dot{w}_e = 1.2 \left( \bar{w}' \theta'_e \right)_0 \left[ \frac{\partial \theta'\dot{}}{\partial z} \right]_f,$$

where $\left( \bar{w}' \theta'_e \right)_0$ is the flux of virtual potential temperature near the surface and $(\partial \theta'/\partial z)_f$ the gradient of virtual potential temperature above the boundary layer. For $(\partial \theta'/\partial z)_f = 3.5 \text{ K km}^{-1}$, $(\bar{w}' \theta'_e)_0 = 2 \times 10^{-2} \text{ m sec}^{-1}$ ($^\circ \text{K}$), $h = 1 \text{ km}$, $C_D = 1.5 \times 10^{-4}$ and $|V| = 5 \text{ m sec}^{-1}$, $\dot{w}_e = 7 \times 10^{-3} \text{ m sec}^{-1}$ and $\dot{w}_e/C_D = 0.9$. Therefore, even in the case of a mildly heated boundary layer growing due to entrainment of mass from above, the stress at the top of the boundary layer could be significant. Since $(\bar{w}' \theta'_e)_0$ and $h$ are generally positively correlated, $\dot{w}_e$ may be expected to be less variable than $(\bar{w}' \theta'_e)_0$ and $h$. If the growth of the boundary layer is rapid, (4) would have to be modified to include a time-dependent $h$.

The effect of $\dot{w}_e$ through the frictional factor $[1 + i(C_D + \dot{w}_e)]^{-1}$ is to increase the frictional reduction of wind speed and to increase the frictional rotation of the wind vector toward the pressure gradient. The effect of the nonzero $\dot{w}_e$ in the forcing factor $(1 - i \dot{w}_e V_f)$ is to rotate the wind toward a direction $90^\circ$ to the right of the free atmosphere flow $V_f$. This rotation is a reflection of the fact that the steady Coriolis term must partially balance the flux of $V_f$ momentum. In the case of barotropic unaccelerated flow where $V_f = V_\phi = 1$, the effect of nonzero $\dot{w}_e$ via the forcing factor is to increase the wind speed and decrease $\alpha$. This opposes the effect of $\dot{w}_e$ through the frictional factor. As a result, the total effect of nonzero $\dot{w}_e$ on $V_f$ is small, as is numerically shown in Fig. 2. In the more general case of baroclinic or accelerated flow where $V_f \neq V_\phi$, 1) a positive (negative) imaginary $V_f$, which results from a free flow component along (against) the boundary layer pressure gradient, increases (decreases) the forcing factor magnitude $|1 - i \dot{w}_e V_f|$ and therefore increases $|V_f|$ and 2) a positive (negative) real $V_f$ rotates the forcing factor such that $\alpha$ decreases (increases).

4. Linear time dependence

To examine the response of the layer-average flow to changes in the horizontal pressure gradient, con-
consider the simple case of constant "acceleration" of the 
geostrophic wind:

$$\hat{V}_\theta(t) = 1 + \frac{\partial \hat{V}_\theta}{\partial t},$$

where $\partial \hat{V}_\theta / \partial t$ is a complex constant. Substituting this 
geostrophic wind dependence into (4), integrating by 
parts, neglecting entrainment stress and rearranging 
terms, we obtain

$$\hat{V} = (\hat{V})_{\theta=0} + \frac{\partial \hat{V}_\theta}{\partial t} [iG^{-1}(\hat{C}_D, \eta)]$$

$$\times \{ t - G^{-1}(\hat{C}_D, \eta)(1 - \exp[-G(\hat{C}_D, \eta)t]) \}$$

$$+ \hat{V}_\theta \exp[-G(\hat{C}_D, \eta)t].$$

We consider the simplified case where $\eta=0$ and inertial 
oscillations are filtered out. After rearranging terms, the 
above expression becomes

$$\hat{V} = (1 + i\hat{C}_D)(1 + \hat{C}_D^2)^{-1}$$

$$\times \{ 1 + i\text{Ro}(1 + i\hat{C}_D)(1 + \hat{C}_D^2)^{-1} \},$$

(6)

where

$$\text{Ro} = \frac{\partial \hat{V}_\theta}{\partial t} \left[ fV_\theta(t) \right]$$

is the Rossby number. $\hat{V}$ is now nondimensionalized 
with respect to the instantaneous geostrophic wind 
vector. Since the equations describing the flow are 
linear, it has been possible to divide the flow into a 
component in equilibrium with the geostrophic wind 
[first term in the brackets on the right-hand side of 
(6)], and a transient component (second term). The 
transient term will hereafter be referred to as the 
boundary layer isallobaric wind. This boundary layer 
isallobaric wind is equal to the frictionless isallobaric 
wind $i\text{Ro}$ plus a smaller correction of $O(\hat{C}_D)$. This is 
qualitatively similar to transient Ekman-Taylor flow 
(Young, 1973), where the correction terms are 
dependent on a surface drag coefficient and square root of 
the eddy viscosity. Eq. (6) indicates that the boundary 
layer isallobaric wind can also be written in the form

$$i\text{Ro}[\hat{V}]_{\text{Ro=0}} F$$

where $[\hat{V}]_{\text{Ro=0}}$ is the equilibrium component 
corresponding to the instantaneous geostrophic wind. In other words, the boundary layer isallobaric wind is to the left of the frictionless isallobaric wind 
by an angle of $2(\alpha)_{\text{Ro=0}}$ and is reduced in magnitude by 
a factor of $[\hat{V}]_{\text{Ro=0}}^2$. Thus, the relationship of the 
boundary layer isallobaric wind to the frictionless 
isallobaric wind is somewhat analogous to the relationship 
of the equilibrium layer average wind to the 
geostrophic wind. This is in qualitative agreement with 
Young (1973).

It can be shown from (6) that the general relationship 
for the angle between the accelerated wind and the 
hypothetical steady-state wind corresponding to the 
instantaneous geostrophic wind is

$$\tan^{-1} \left\{ \frac{[\text{Re}(\text{Ro}) - \hat{C}_D \text{Im}(\text{Ro})]}{[1 + \hat{C}_D^2 - \text{Im}(\text{Ro}) - \hat{C}_D \text{Re}(\text{Ro})]} \right\}$$

and the ratio of the accelerated wind speed to this 
steady wind speed is

$$(1 + \hat{C}_D^2)^{-1} \{ [1 + \hat{C}_D^2 - \text{Im}(\text{Ro}) - \hat{C}_D \text{Re}(\text{Ro})]^2$$

$$+ [\text{Re}(\text{Ro}) - \hat{C}_D \text{Im}(\text{Ro})]^2 \}.$$
sensitive to accelerations than to surface conditions for many flow situations. For example, the angle between the layer-average wind and the geostrophic wind increases by approximately a factor of 10 from the steady case to the case of unity Rossby number. The surface drag coefficient $C_D$ must increase by more than a factor of 10 to cause a comparable increase in $\alpha$.

To determine the influence of local accelerations on $w_a$, the dimensional vertical motion at the top of the boundary layer, we take the divergence of the layer average flow (6) and multiply by $h$. After rearranging terms, we have

$$w_a = h(1 + C_D)^{-1} \times \left[ C_D \tilde{\zeta} \left\{ (1 - C_D) \left[ f(1 + C_D) \right]^{-1} \frac{\partial}{\partial t} \left( \tilde{\zeta}_s \right) \right\} \right],$$

where $\tilde{\zeta}_s$ is the layer-average geostrophic relative vorticity. The first term on the right-hand side is the equilibrium part of the vertical motion which is linearly proportional to the geostrophic vorticity (Charney and Eliassen, 1949; Priestly, 1967). Since $C_D$ is expected to be small compared to 1, the equilibrium contribution is essentially linearly dependent on the drag coefficient and only weakly dependent on the boundary layer depth. The weak dependence on $h$ is due to the fact that for a fixed $C_D$ increasing $h$ results in decreasing stress divergence and therefore decreasing cross-isobar flow speed.

As in Young (1973), the transient part of the vertical motion is equal to the transient vertical motion that would occur without friction, $(h/f)(\partial / \partial t) \tilde{\zeta}_s$, minus a smaller boundary layer correction, in this case of $O(C_D^2)$. Therefore, the transient vertical motion is essentially linearly proportional to the geostrophic vorticity tendency and boundary layer depth and depends only weakly on the drag coefficient. Approximating $(\partial / \partial t) \tilde{\zeta}_s$ as $\tilde{\zeta}_s / \tau$, where $\tau$ is the time scale of the circulation system maintaining $\tilde{\zeta}_s$, the ratio of the transient vertical motion to the equilibrium vertical motion is $T / \tau$ where $T$ is again the frictional damping time scale $h / (C_D \| \tilde{V} \|^2)$. Since $T$ is expected to be of the order of one day, the production of vertical motion by the vorticity tendency will be significant in mesoscale and fast-moving synoptic-scale systems.

In conclusion, the bulk tendencies of the behavior of low Rossby number, Ekman-Taylor flow (Young, 1973) qualitatively predict the fundamental layer-average flow behavior for all values of the Rossby number.

5. Rotating pressure gradient

The Ekman solution to a rotating geostrophic wind of the form $U_{\phi} = e^{i\omega t}$, where $U_{\phi}$ is a constant, is developed in Ching and Businger (1968). The analogous layer solution is obtained by substituting $\tilde{V}_s = e^{i\omega t}$ into (4) and integrating. After assuming $\eta = \bar{w}_s = 0$ and filtering out inertial oscillations, the layer solution becomes

$$\tilde{V} = (1 + \Delta + i C_D) [(1 + \Delta)^2 + C_D^2],$$

where the wind is nondimensionalized with respect to the instantaneous geostrophic wind vector. The complex right-hand side of (7) rotates the layer wind vector away from the geostrophic wind by an angle of $\alpha = \tan^{-1} [(C_D / (1 + \Delta))]$ and reduces the boundary layer wind speed by a factor of $[(1 + \Delta)^2 + C_D^2]^{-1}$. Thus, the angle between the boundary layer wind and geostrophic wind, as well as the scaled wind and cross-isobar flow speeds, are inversely proportional to the absolute vorticity $1 + \Delta$ (Fig. 3). Therefore, the angle $\alpha$ is largest when the geostrophic wind rotates away from the boundary layer wind. The positive correlation between $\alpha$ and $|\tilde{V}|$ for varying $\Delta$ can be related to the balance between pressure gradient generation of kinetic energy and frictional dissipation. This balance requires that $\sin(\alpha)(C_D / |\tilde{V}|)$ be unity for all values of $\Delta$.

The boundary layer isallobaric wind $\tilde{V}_b$ is obtained by subtracting the equilibrium solution $\tilde{V}_b$ (corresponding to the instantaneous pressure gradient (5) from the actual time-dependent solution (7). After rearranging terms and reusing expressions for $\tilde{V}$ and $\tilde{V}_b$, we obtain

$$V_i = -\Delta \tilde{V}_b \tilde{V}_b = -i \text{Ro} \tilde{V}_b V,$$  \tag{8}

where $\text{Ro} = \omega / f$. Eq. (8) indicates that the boundary layer isallobaric wind deviates from the frictionless isallobaric wind $-i \text{Ro}$ by an angle of $\alpha + \Delta$ and is reduced in magnitude by a factor of $|\tilde{V}_b|$. Thus, as in the case of constant geostrophic wind tendency, the frictional influence on the transient component of the flow is qualitatively similar to the frictional influence on the equilibrium flow.

6. The low-latitude solution

The behavior of the layer-average flow is quite different for low-latitude values of the Coriolis parameter. While the boundary layer damping time scale is independent of the Coriolis parameter for fixed drag coefficient and boundary layer depth, the steady-state values of $\alpha$ and $|\tilde{V}|$ increase respectively with decreasing $f$ (Fig. 4). For example, for the case $\eta = \bar{w}_s = 0$, steady-state $\alpha$ is 90° at the equator (flow directly across the isobars). The fact that $|\tilde{V}|$ vanishes is merely a reflection of the fact that $|V|$ remains finite even though $|V_s|$ approaches infinity for non-vanishing pressure gradient.

Of particular interest is the latitude where $f + \omega = 0$. Holton et al. (1971) found that Ekman flow generated by pressure fields corresponding to free-wave solutions on an equatorial $\beta$-plane (Matsumo, 1966) exhibited large vertical motions and a phase shift at the "critical latitude" where $f + \omega = 0$. Both the wind speed and
phase angle of the present layer solution is continuous at the critical latitude. This is partly due to the presence of a frictional damping time scale $h/C_D[V]$ in addition to the Coriolis time scale and time scale of the pressure gradient oscillation. Thus, the factor \( \left[ \frac{f + \omega}{C_D[V]} \right]^{-\frac{1}{2}} \) occurs in place of the inverse dependence on \( f + \omega \) which resulted in the theories of Pandolfo and Brown (1967), Ching and Businger (1968), and Holton et al. (1971). The singularity in the boundary layer depth at the critical latitude reported by Holton et al. (1971) does not occur in the layer solution because of the unfortunate specification of the depth independently of the flow. Chang (1973a) and Mahrt (1972a) have found that the large boundary layer production of vertical motion at a "critical" or "transition" latitude is largely due to a local increase in the depth of the boundary layer. Young (1973) has shown that local accelerations may, in general, lead to a thickening of the boundary layer.

Although continuous, the layer solution exhibits unique features near the latitude \( f + \omega = 0 \). For example, at this latitude, the boundary layer adjustment is slow enough so that the boundary layer wind vector is \( 90^\circ \) out of phase with the geostrophic wind, resulting in flow directly across the isobars and maximum production of the cross-isobar flow. Of course, in actual flow situations, coupling between the cross-isobar flow and the pressure field (Chang, 1973b; Mahrt 1972b) and the importance of \( \beta \)-plane-induced advective accelerations (Mahrt 1972a) will significantly modify the cross-isobar flow distribution. In conclusion, the present study agrees with previous studies in that the maximum efficiency of boundary layer pumping, associated with local accelerations resulting from periodicity of the pressure gradient, appears to occur near the latitude at which \( f + \omega = 0 \).

7. Summary and further discussion

The above theory indicates that boundary layer flow adjusts to the pressure gradient field at a rate which is proportional to the strength of the coupling between the flow and boundary stresses. This coupling is of the mathematical form \( (C_D[V] + \omega_*)/h \), where \( C_D \) is a surface drag coefficient and \( \omega_* \) a stress coefficient at the top of the boundary layer.

The influence of local accelerations depends crucially on the orientation of the change of the pressure gradient with respect to the existing pressure gradient. For example, accelerations will increase (decrease) the cross-isobar flow relative to the geostrophic wind when the pressure gradient is intensifying (weakening), or is changing direction so that the geostrophic wind rotates away from (toward) the boundary layer wind. For mesoscale flows, where the Rossby number is of order unity or greater, accelerations may increase the cross-isobar flow relative to the geostrophic wind by as much as an order of magnitude.

The flow solution can be divided into an equilibrium component and a transient component (isallobaric wind). The stress divergence influences the transient component in a manner similar to its influence on the equilibrium component. The equilibrium boundary layer production of vertical motion is essentially linearly proportional to the geostrophic relative vorticity and drag coefficient, and depends only weakly on the boundary layer depth. The transient portion of the boundary layer vertical motion is essentially linearly proportional to the geostrophic vorticity tendency and boundary layer depth and depends only weakly on the drag coefficient. The transient part of the vertical motion field is important when the time scale of the pressure gradient is comparable to or smaller than the frictional damping time scale.
The above simplified theory allows for precise relationships between local accelerations and fundamental boundary layer properties. However, application of these relationships to the more complicated atmosphere must be made with caution. For example, in most atmospheric flows where local accelerations are important, advective accelerations are also important. In fact, advective and local accelerations may often oppose each other such that the total acceleration remains small. Furthermore, accelerations may significantly modify the depth of the boundary layer. Finally, specification of the pressure field independently of the flow, when cross-isobar mass transports are large, is somewhat hypothetical.

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