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In this thesis we examine the approximation theory of the eigenvalue problem of bounded linear operators defined on a Banach space, and its applications to integral and differential equations. Special cases include the degenerate kernel method, projection method, collocation method, the Galerkin method, the method of moments, and the generalized Ritz method for solving integral or differential equations. Given a bounded linear operator, a sequence of bounded linear operator approximations is assumed to converge to it in the operator norm. We examine, among other things, the perturbation of the spectrum of the given operator; criteria for the existence and convergence of approximate eigenvectors and generalized eigenvectors; relations between the dimensions of the eigenmanifolds and generalized eigenmanifolds of the operator and those of the approximate operators.

Spectral Approximation Theory for Bounded Linear Operators

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SPECTRAL APPROXIMATION THEORY FOR BOUNDED LINEAR OPERATORS

I. INTRODUCTION

In this thesis we shall be concerned with the approximation theory of the eigenvalue problem for bounded linear operators defined on a Banach space and applications to integral and differential equations. To be more specific: given a bounded linear operator defined on a Banach space; a sequence of bounded linear approximate operators will be assumed to converge to it in the operator norm. This situation occurs quite often in practice, and yet the literature concerning it is relatively scarce. We will examine, among other things, the perturbation of the spectrum of the given operator; criteria for the existence and convergence of approximate eigenvectors and generalized eigenvectors; relations between the dimensions of the eigenmanifolds and generalized eigenmanifolds of the operator with those of the approximate operators. A related but different approach to the same subject is considered by Kato [13], Turner [24], and others given in their bibliographies. They often require the approximate operators to vary continuously or analytically with a parameter which is more restrictive than the approach used in this thesis.

Let X be a real or complex Banach space, and denote by $\mathcal{B} = \{x \in X: \|x\| \leq 1\}$ the closed unit ball in X. Let $[X]$ be the

Banach space of bounded linear operators $T: X \to X$ with the usual operator norm $\|T\| = \sup_{x \in \mathcal{B}} \|Tx\|$. Let T, T_n in $[X]$ be such that $\|T_n - T\| \to 0$ as $n \to \infty$. For T in $[X]$, let $\eta(T) = \{x \in X: Tx = 0\}$ be the null space of T. Scalars (real or complex) will be denoted by λ and μ, sometimes with subscripts. A scalar λ is an eigenvalue of T iff there exists x in X such that $x \neq 0$ and $Tx = \lambda x$, in which case $\eta(\lambda - T)$ is the corresponding eigenmanifold and $\eta(\lambda - T)^k$, $k = 1, 2, \ldots,$ are generalized eigenmanifolds. The resolvent set for T is defined by $\rho(T) = \{\lambda : (\lambda - T)^{-1} \in [X]\}$; the spectrum $\sigma(T)$ is the complement of $\rho(T)$. For T in $[X]$ and μ a scalar, the index $\nu = \nu(\mu, T)$ is defined to be the least positive integer ν, if it exists, satisfying $\eta(\mu - T)^{\nu} = \eta(\mu - T)^{\nu+1}$; otherwise $\nu = \nu(\mu, T)$ is defined to be ∞. Other concepts, such as spectral sets, spectral projections, spectral subspaces, and functions of operators are defined later in the chapter. The symbol $\mathbb{C}$ will denote the set of all complex numbers, and $[x_1, \ldots, x_n]$ denotes the subspace spanned by the vectors $x_1, \ldots, x_n$.

Since compact operators play an important role in this thesis, we now give the definition and summarize some of the properties of a compact operator. An operator T in $[X]$ is said to be compact (or completely continuous) if the closure of $T\mathcal{B}$ is compact. If T is compact and S in $[X]$, then TS and ST are compact. Moreover for any positive integer n and scalars $a_0, a_1, \ldots, a_n$, the operator $p(T) = a_0 = aT + \ldots + a_n T^n$ is compact. It is well known that the spectrum of a compact operator is at most countable and has no points of accumulation except possibly $\mu = 0$. For each non-zero μ in $\sigma(T)$, μ is an eigenvalue and the index $\nu = \nu(\mu, T)$ is a finite number. The generalized eigenmanifolds $\eta(\mu - T)^k$, $k = 1, 2, \ldots,$ are finite dimensional. Moreover the Fredholm Alternative states that if T is a compact operator and μ is non-zero, then $\mu I - T$ is one-to-one iff it is onto.

The main results of this thesis are Theorems 4, 9, 12, 19, and 20. Part of Theorem 4 was proved by Newburgh [15], using Banach algebra techniques. It states that for every neighborhood Ω of $\sigma(T)$ there exists an integer N such that $\sigma(T_n) \subset \Omega$ for all $n \geq N$. If in addition, we assume that T is compact and μ_n, $n = 1, 2, \ldots,$ are eigenvalues of T_n satisfying $\mu_n \to \mu \neq 0$, then Theorem 9 states that: (a) $\dim \eta(\mu_n - T_n) = \dim \eta(\mu - T)$ eventually iff (b) for every x in $\eta(\mu - T)$, $\|x\| = 1$, there is a sequences $\{x_n\}$ contained in $\eta(\mu_n - T_n)$ such that $x_n \to x$. This is a generalization of a theorem proved by Andrew and Elton [2]. In addition to the assumptions $\|T_n - T\| \to 0$ and T compact, they assumed that X is a Hilbert space and T_n is compact for $n = 1, 2, \ldots$. It is shown by Poskiy [18] by way of an example that for x in $\eta(\mu - T)$ there need not be x_n in $\eta(\mu_n - T_n)$ such that $x_n \to x$, even in the case that T_n is compact for all n. Theorem 12, which is a direct consequence of Theorem 9, gives a necessary and sufficient condition for the existence of a sequence of generalized eigenvectors x_n in $\eta(\mu_n - T_n)^r$, $n = 1, 2, \ldots$, such that x_n converges to an arbitrary but fixed generalized eigenvector x in $\eta(\mu - T)^r$, where r is any positive integer. Some geometric applications of Theorems 19 and 20 are given. This compares the Jordan canonical form of the restriction of T to a spectral subspace of T with the Jordan canonical form of the restriction of T_n to a spectral subspace of T_n, for n sufficiently large.

The theory presented in this paper can be applied to eigenvalue problems for integral or differential equations. In particular, the degenerate kernel method, the projection method, and the method of collocation for solving integral equation of the second kind are special cases of the approximation method discussed in this paper. Also the projection methods for solving certain partial differential equations are special cases of the method discussed in this paper. As special cases of the projection method we have the Riesz method, the generalized Riesz method, and the method of moments [16].

II. RESOLVENT AND SPECTRA

Our analysis begins with a well known result.

PROPOSITION 1. (a) If A belongs to $[X]$ and $\|A\| < 1$, then there exists $(I-A)^{-1}$ in $[X]$, and

$$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n, \qquad \|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}.$$

(b) Let S, T belong to $[X]$. Assume there exists S^{-1} in $[X]$ and set $\Delta = \|S^{-1}\| \, \|S-T\| < 1$. Then there exists T^{-1} in $[X]$ and

$$\|T^{-1}\| \leq \frac{\|S^{-1}\|}{1-\Delta}, \qquad \|T^{-1}-S^{-1}\| \leq \frac{\Delta\|S^{-1}\|}{1-\Delta}.$$

As an immediate consequence of Proposition 1, we have

PROPOSITION 2. Let $T, T_n \in [X]$ and $\|T_n - T\| \to 0$. Then λ belongs to $\rho(T)$ iff there exists N such that λ belongs to $\rho(T_n)$ for $n \geq N$ and $\{(\lambda-T_n)^{-1} : n \geq N\}$ is bounded, in which case

$$\|(\lambda-T_n)^{-1} - (\lambda-T)^{-1}\| \to 0.$$

<u>PROOF</u>. Note that $(\lambda-T_n) = [I-(T_n-T)(\lambda-T)^{-1}](\lambda-T) = (I-K_n(\lambda))(\lambda-T)$ where $K_n(\lambda) = (T_n-T)(\lambda-T)^{-1}$. Choose N such that for $n \geq N$, $\|K_n(\lambda)\| < 1$. Then by Proposition 1 there exist $(I-K_n(\lambda))^{-1}$ in

$[X]$ and hence there exist $(\lambda-T_n)^{-1} = (\lambda-T)^{-1}(I-K_n(\lambda))^{-1}$ in $[X]$. Moreover

$$\|(I-K_n(\lambda))^{-1}\| \leq \frac{1}{1-\|K_n(\lambda)\|}$$

and

$$\|(\lambda-T_n)^{-1}\| \leq \frac{\|(\lambda-T)^{-1}\|}{1-\|K_n(\lambda)\|},$$

and $\{(\lambda-T_n)^{-1} : n \geq N\}$ is bounded which implies λ belongs to $\rho(T_n)$ for $n \geq N$.

Conversely, let b be a bound for $\{(\lambda-T_n)^{-1} : n \geq N\}$. With λ and n fixed, let $\lambda \in \rho(T_n)$. Let $L_n(\lambda) = (T-T_n)(\lambda-T_n)^{-1}$; then $(\lambda-T) = [I-(T-T_n)(\lambda-T_n)^{-1}](\lambda-T_n) = (I-L_n(\lambda))(\lambda-T_n)$, and there exists N such that $\|L_n(\lambda)\| < 1$ for all $n \geq N$. By Proposition 1 there exist $(I-L_n(\lambda))^{-1}$ in $[X]$ and

$$\|(I-L_n(\lambda))^{-1}\| \leq \frac{1}{1-\|L_n(\lambda)\|},$$

and hence there exists $(\lambda-T)^{-1}$ in $[X]$ and

$$\|(\lambda-T)^{-1}\| \leq \|(\lambda-T_n)^{-1}\| \; \|(I-L_n(\lambda))^{-1}\|$$

$$\leq \frac{b}{1-\|L_n(\lambda)\|}.$$

Since $(\lambda-T_n)^{-1} - (\lambda-T)^{-1} = (\lambda-T_n)^{-1}(T_n-T)(\lambda-T)^{-1}$, we have

$$\|(\lambda-T_n)^{-1} - (\lambda-T)^{-1}\| \leq \|(\lambda-T_n)^{-1}(T_n-T)(\lambda-T)^{-1}\| \to 0 .$$

In Proposition 2 we have considered a fixed λ. We now let λ vary in a closed set Λ contained in the extended resolvent set $\tilde{\rho}(T)$ of T, by which we mean the set consisting of the resolvent set $\rho(T)$ and the compactification point(s). We will define $(\lambda-T)^{-1} = 0$ for $|\lambda| = \infty$.

THEOREM 3. Let $T, T_n \in [X]$, $\|T_n - T\| \to 0$. Then:

(a) For each closed set $\Lambda \subset \tilde{\rho}(T)$ there exists N such that $\Lambda \subset \tilde{\rho}(T_n)$ for $n \geq N$ and $\{(\lambda-T_n)^{-1} : \lambda \in \Lambda, n \geq N\}$ is bounded.

(b) For any closed set $\Lambda \subset \tilde{\rho}(T_n)$, $n \geq N$; if $\{(\lambda-T_n)^{-1} : \lambda \in \Lambda, n \geq N\}$ is bounded then $\Lambda \subset \tilde{\rho}(T)$.

(c) If either (a) or (b) holds then $\|(\lambda-T_n)^{-1} - (\lambda-T)^{-1}\| \to 0$ as $n \to \infty$, uniformly for $\lambda \in \Lambda$.

<u>PROOF</u>. It follows from the identity $(\lambda-T)^{-1} - (\mu-T)^{-1} = (\lambda-T)^{-1}(\mu-\lambda)(\mu-T)^{-1}$ and Proposition 1(b) that the function $\lambda \to \|(\lambda-T)^{-1}\|$ is a real valued continuous function, which is therefore bounded on the compact subset $\Lambda \subset \tilde{\rho}(T)$. For $K_n(\lambda)$ of Proposition 2 $\|K_n(\lambda)\| \leq \|T_n - T\| \, \|(\lambda-T)^{-1}\| \to 0$ uniformly for λ in Λ. Now (a) follows from the proof of Proposition 2.

(b) This follows from the fact that $\|L_n(\lambda)\| \to 0$ uniformly for λ in Λ as $n \to \infty$, and the proof of Proposition 2.

The proof of (c) is also evident from the proof of Proposition 2.

By taking the complement of the set Λ in Theorem 3 we obtain

THEOREM 4. Let $T, T_n \in [X]$, $\|T_n - T\| \to 0$. Then:

(a) For each open set $\Omega \supset \sigma(T)$ there exists N such that $\Omega \supset \sigma(T_n)$ for all $n \geq N$.

(b) For any open set $\Omega \supset \sigma(T_n)$, $n \geq N$, if $\{(\lambda - T_n)^{-1} : \lambda \notin \Omega,\ n \geq N\}$ is bounded then $\Omega \supset \sigma(T)$.

Part (a) of the above theorem was proved by Newburgh [15] using Banach algebra techniques.

III. EIGENVECTORS AND EIGENMANIFOLDS

In this chapter we examine eigenvectors and eigenmanifolds of the operators T and T_n. The main theorem (Theorem 7) of this section is an essential tool in obtaining some of the later results. We start with a well known lemma by Riesz.

PROPOSITION 5. Let M be a closed proper subspace of X. For each $\epsilon > 0$ there exists x_ϵ in X such that $\|x_\epsilon\| = 1$ and $\|x_\epsilon - y\| \geq 1 - \epsilon$ for each y in M. If $\dim M < \infty$, there exists x in X such that $\|x\| = 1$ and $\|x-y\| \geq 1$ for all y in M.

From this we immediately obtain

COROLLARY 6. If M is a m-dimensional subspace of X, then M has a basis $\{x_1, x_2, \ldots, x_m\}$ such that

$$\|x_k\| = 1, \quad \left\|x_k - \sum_{j=1}^{k-1} c_j x_j\right\| \geq 1$$

for $1 \leq k \leq m$ and all choices of c_j.

THEOREM 7. Assume $T, T_n \in [X]$ and $\|T_n - T\| \to 0$. Let μ_n in $\sigma(T_n)$ be such that $\mu_n \to \mu$. Then μ belongs to $\sigma(T)$. If T is compact, $\mu \neq 0$, $x_n \in \eta(\mu_n - T_n)$ and $\|x_n\| = 1$, then there exists sequences $\{T_{n_i}\}$, $\{x_{n_i}\}$, and x in X such that

$x_{n_i} \to x \in \eta(\mu - T)$ as $i \to \infty$.

For n sufficiently large we have $\dim \eta(\mu_n - T_n) \leq \dim \eta(\mu - T)$.

Let $\mathcal{M} \subset \eta(\mu - T)$ and $\mathcal{M}_n \subset \eta(\mu_n - T_n)$ be subspaces such that $x_n \in \mathcal{M}_n$, $x_n \to x \Rightarrow x \in \mathcal{M}$. Then $\dim \mathcal{M}_n \leq \dim \mathcal{M}$ eventually.

PROOF. The first part is well known and can be proved as follows: for $\mu \notin \sigma(T)$, $\Lambda = \{\mu\}$ is a closed set in $\tilde{\rho}(T)$. By Theorem 3 $\Lambda \subset \tilde{\rho}(T_n)$ for n sufficiently large, thus contradicting the fact that $\mu_n \to \mu$.

To prove the second part, let us consider the sequence $\{Tx_n\}$. Now T is compact implies there exists a subsequence $\{Tx_{n_i}\}$ and $x \in X$ such that $Tx_{n_i} \to \mu x$ as $i \to \infty$. Since $T_{n_i} x_{n_i} = \mu_{n_i} x_{n_i}$ we have

$$\|\mu_{n_i} x_{n_i} - \mu x\| \leq \|T_{n_i} - T\| \, \|x_{n_i}\| + \|Tx_{n_i} - \mu x\|.$$

Hence $\|\mu_{n_i} x_{n_i} - \mu x\| \to 0$ as $i \to \infty$. Now $\mu_{n_i} \neq 0$ eventually and

$$\|x_{n_i} - x\| = \left\|\frac{1}{\mu_{n_i}} [\mu_{n_i} x_{n_i} - \mu x - x(\mu_{n_i} - \mu)]\right\|$$

implies $\|x_{n_i} - x\| \to 0$ as $i \to \infty$. It follows that

$$\|Tx-\mu x\| \leq \|Tx-T_{n_i}x\| + \|T_{n_i}x-T_{n_i}x_{n_i}\|$$

$$+ \|T_{n_i}x_{n_i}-\mu_{n_i}x_{n_i}\| + \|\mu_{n_i}x_{n_i}-\mu x\| .$$

But each term on the right hand side of the inequality tend to zero, so $Tx = \mu x$ and x is an eigenvalue of T corresponding to μ.

Note that special cases of $\mathfrak{M}$ and $\mathfrak{M}_n$ are $\mathfrak{M} = \eta(\mu-T)$, $\mathfrak{M}_n = \eta(\mu_n-T_n)$. It remains to prove that $\dim \mathfrak{M}_n \leq \dim \mathfrak{M}$ for n sufficiently large. Suppose that $\dim \mathfrak{M}_n \geq m$ for all n in an infinite set J. Then there exists x_{n_k} in $\mathfrak{M}_n$ such that

$$\|x_{n_k}\| = 1, \qquad \|x_{n_k} - \sum_{j=1}^{k-1} c_j x_{nj}\| \geq 1$$

for $n \in J$, $k = 1, \ldots, m$, and all choices of c_j. Hence by the hypotheses on $\mathfrak{M}$ and $\mathfrak{M}_n$ and the part of the theorem already proved there exists $\{T_{n_i}\}$, $\{x_{n_i k}\}$, and x_k, $k = 1, \ldots, m$, in $\mathfrak{M}$ with $x_{n_i k} \to x_k$ as $i \to \infty$, $n \in J$. Therefore $\|x_k\| = 1$ and

$$\|x_k - \sum_{i=1}^{k-1} c_i x_i\| \geq 1$$

for $k = 1, \ldots, m$, and all choices of c_j, so that $\dim \mathfrak{M} \geq m$. Contrapositively, if $\dim \mathfrak{M} < m$ then $\dim \mathfrak{M}_n <$ for all n sufficiently large. The proof of the last part is similar to

Theorem 4.11 of [3].

LEMMA 8. Let M and M_n, $n = 1, 2, \ldots,$ be subspaces of X, and $\dim M < \infty$. If for every x in M there exists x_n in M_n such that $\|x_n - x\| \to 0$ then there exists an integer N such that $\dim M_n \geq \dim M$ for all $n \geq N$.

PROOF. Without loss of generality assume $\dim M = m$. Let $\{x_i : i = 1, \ldots, m\}$ be a basis for M. Suppose for each $i = 1, \ldots, m$, there exist x_{ni} in M_n such that $\|x_{ni} - x_i\| \to 0$ as $n \to \infty$. Let $E^m = \{(c_1, \ldots, c_m) : c_i$ is a scalar for $1 \leq i \leq m\}$. Define the compact set $D \subset E^m$ by

$$D = \{(c_1, \ldots, c_m) : \max |c_i| = 1\}.$$

Define functions f and f_n on D:

$$f(c_1, \ldots, c_m) = \| \sum_{i=1}^{m} c_i x_i \| ,$$

and

$$f_n(c_1, \ldots, c_m) = \| \sum_{i=1}^{m} c_i x_{ni} \| .$$

Note that f is continuous, and by the triangle inequality $f_n \to f$ uniformly on D. Now it follows from the linear independence of $\{x_i : i = 1, \ldots, m\}$ that

$\min_D f > 0$. Therefore there exists an integer N such that $\min_D f_n > 0$ for all $n \geq N$, which implies that $\{x_{ni} : i = 1, \ldots, m\}$ is linearly independent and $\dim M_n \geq \dim M$ for all $n \geq N$.

The following theorem gives a necessary and sufficient condition for the existence of a sequence $x_n \in \eta(\mu_n - T_n)$, $n = 1, 2, \ldots,$ such that x_n converges to an arbitrary but fixed element x in $\eta(\mu - T)$. Polskiy [18] showed by way of an example that when $\dim \eta(\mu - T) > 1$ there may be vectors in $\eta(\mu - T)$ which can not be obtained as the limit of any sequence of eigenvectors of T_n, even with T_n compact for $n = 1, 2, \ldots$.

THEOREM 9. Let T be compact and $\|T_n - T\| \to 0$. Let $\mu \neq 0$ and μ_n be eigenvalues of T and T_n respectively such that $\mu_n \to \mu$. Then the following are equivalent:

(a) $\dim \eta(\mu_n - T_n) = \dim \eta(\mu - T)$ eventually;

(b) for every x in $\eta(\mu - T)$, $\|x\| = 1$, there is a sequence $\{x_n\}$ contained in $\eta(\mu_n - T_n)$ such that $x_n \to x$.

<u>PROOF</u>. (a) => (b). Since T is compact $\dim \eta(\mu - T) = m < \infty$. Suppose (a) $\not\Rightarrow$ (b). Then there exist: an x in $\eta(\mu - T)$, a strictly increasing sequence of positive integers S, and a number $d > 0$ such that

$$\|x_n - x\| > d \text{ for all } n \text{ in } S, \text{ and for all } x_n \text{ in } \eta(\mu_n - T_n) \cap \mathcal{B}.$$

By (a) for each n sufficiently large, n in S, there is ψ_{ni} in $\eta(\mu_n - T_n)$ such that

$$\|\psi_{ni}\| = 1, \quad \|\psi_{ni} - \sum_{j=1}^{i-1} c_j \psi_{nj}\| \geq 1 \quad \text{for} \quad 1 \leq i \leq m,$$

and for any choices of c_j. By Theorem 7 there exist subsequence of positive integers $S_0 \subset S$ and ψ_i in $\eta(\mu - T)$ with $\psi_{ni} \to \psi_i$ as $n \to \infty$, for $1 \leq i \leq m$, $n \in S_0$. It follows that $\|\psi_i\| = 1$ for $1 \leq i \leq m$ and

$$\|\psi_i - \sum_{j=1}^{i-1} c_j \psi_j\| \geq 1$$

for any choices of c_j. Therefore $\psi_1, \ldots, \psi_m$ are linearly independent, and $\eta(\mu - T) = [\psi_i, \ldots, \psi_m]$. Hence there exist a_i, $1 \leq i \leq m$ such that $x = \sum_{i=1}^{m} a_i \psi_i$. Let $x_n = \sum_{i=1}^{m} a_i \psi_{ni}$ for n in S_0. Then $x_n \in \eta(\mu_n - T_n)$ and $\|x_n - x\| \to 0$ as $n \to \infty$, n in S_0. A contradiction.

(b) => (a) follows from Theorem 7 and Lemma 8.

REMARKS.

1. In the above theorem if we assume in addition that the T_n, $n = 1, 2, \ldots,$ are compact, then the existence of eigenvalues μ_n such that $\mu_n \to \mu$ is guaranteed by a corollary to a theorem proved by Putnam [20].

2. Theorem 9 is a generalization and an improvement of a

theorem proved by Andrew and Elton [2]. In addition to the hypotheses in Theorem 9, they assumed that X is a Hilbert space and the operators T_n, $n = 1, 2, \ldots,$ are compact. As a consequence they obtained a dimensional inequality $\dim \eta(\mu_n - T_n) \geq \dim \eta(\mu - T)$ in (a) instead of the dimensional equality $\dim \eta(\mu_n - T_n) = \dim \eta(\mu_n - T_n)$, for n sufficiently large.

3. If in Theorem 9 we assume in addition that X is a Hilbert space, T_n is compact for each n and μ is a non-zero simple eigenvalue of T, (i.e., $\dim \eta(\mu - T) = 1$), then a result of Andrew [1] states that for x in $\eta(\mu - T)$, $\|x\| = 1$, there exist x_n in $\eta(\mu_n - T_n)$ such that $x_n \to x$. Therefore with these additional assumptions we have

COROLLARY 10. $\dim \eta(\mu_n - T_n) = \dim \eta(\mu - T)$ eventually.

4. If, in Theorem 9, the condition $\|T_n - T\| \to 0$ is replaced by $\|T_n x - Tx\| \to 0$ for every $x \in X$, and if we assume in addition that $\{T_n\}$ is collectively compact; that is $\cup\{T_n B\}$ has compact closure; then Theorem 4.11 of [3] applies, and we have the following theorem.

THEOREM. Let $T_n x \to Tx$ for every x in X, and let $\{T_n\}$ be a collectively compact set of operators, and let T be compact.

If μ_n and $\mu \neq 0$ are eigenvalues of T_n and T respectively with $\mu_n \to \mu$, then the following are equivalent:

(a) $\dim \eta(\mu_n - T_n) = \dim \eta(\mu - T)$ eventually;

(b) for every x in $\eta(\mu - T)$, $\|x\| = 1$, there is a sequence x_n in $\eta(\mu_n - T_n)$ such that $x_n \to x$.

For a detailed discussion of collectively compact operator approximation theory see [3, 4, 5, 6], and [8].

IV. GENERALIZED EIGENVECTORS AND GENERALIZED EIGENMANIFOLDS

Assume T is compact, and μ is a non-zero eigenvalue of T. Let $D(\mu, \epsilon)$ be a disc (or interval in the real case) centered at μ with radius ϵ. Choose ϵ so small that $D(\mu, \epsilon) \cap D(\mu', \epsilon) = \emptyset$ for μ' any eigenvalue of T other than μ. For $k_n < \infty$, let μ_{nj} in $D(\mu, \epsilon)$, for $j = 1, \ldots, k_n$, be eigenvalues of T_n. We note that for a fixed n there may be no eigenvalue of T_n in $D(\mu, \epsilon)$; on the other hand there may be an infinite number of eigenvalues of T_n in $D(\mu, \epsilon)$. We will see, by Theorem 17, that when X is a complex Banach space and for n sufficiently large, the set of eigenvalues of T_n in $D(\mu, \epsilon)$ is a non-empty finite set, $\{\mu_{nj} : j = 1, \ldots, k_n\}$, such that $\max_{1 \leq j \leq k_n} |\mu - \mu_n| \to 0$.

The following theorem compares the dimensions of the generalized eigenmanifolds of T with those of T_n.

THEOREM 11. Let $T, T_n \in [X]$, $n = 1, 2, \ldots$. Assume $\|T_n - T\| \to 0$, T compact and μ an non-zero eigenvalue of T. Suppose for each n, μ_{nk} is an eigenvalue of T_n for $k = 1, \ldots, k_n$, and $\max_{1 \leq k \leq k_n} |\mu_{nk} - \mu| \to 0$. Choose any non-negative integers r and r_{nk}, $k = 1, \ldots, k_n$, such that $\sum_{k=1}^{k_n} r_{nk} \leq r$. Then, for all n sufficiently large,

$$\sum_{k=1}^{k_n} \dim \eta[(\mu_{nk} - T_n)^{r_{nk}}] \leq \dim \eta[(\mu - T)^r] .$$

PROOF. Without loss of generality,

$$\sum_{k=1}^{k_n} r_{nk} = r \quad \text{for all} \quad n.$$

It follows from [23, p. 317] that

$$\eta[\prod_{k=1}^{k_n} (\mu_{nk} - T_n)^{r_{nk}}] = \bigoplus_{k=1}^{k_n} \eta[(\mu_{nk} - T_n)^{r_{nk}}] ,$$

and

$$\dim \eta[\prod_{k=1}^{k_n} (\mu_{nk} - T_n)^{r_{nk}}] = \sum_{k=1}^{k_n} \dim \eta[(\mu_{nk} - T_n)^{r_{nk}}] .$$

Define μ_n, $\widetilde{T}_n$ and $\widetilde{T}$ by

$$\prod_{k=1}^{k_n} (\mu_{nk} - T_n)^{r_{nk}} = \mu_n - \widetilde{T}_n, \quad \mu_n = \prod_{k=1}^{k_n} (\mu_{nk})^{r_{nk}}, \quad (\mu - T)^r = \mu^r - \widetilde{T}.$$

Then $\widetilde{T}_n \rightarrow \widetilde{T}$ and $\mu_n \rightarrow \mu^r$. Since $\widetilde{T}$ is compact, Theorem 7 implies that

$$\dim \eta(\mu_n - \widetilde{T}_n) \leq \dim \eta(\mu^r - \widetilde{T})$$

eventually. The assertion follows.

The above proof is similar to Theorem 4.14 of [3].

An immediate consequence of Theorems 9 and 11 is the following generalized version of Theorem 9.

THEOREM 12. Let $T, T_n \in [X]$, $n = 1, 2, \ldots$. Assume $\|T_n - T\| \to 0$, T compact and μ a non-zero eigenvalue of T. For $k = 1, \ldots, k_n$, let μ_{nk} be eigenvalues of T_n such that $\max_{1 \le k \le k_n} |\mu_{nk} - \mu| \to 0$. Choose any non-negative integers r and r_{nk}, $k = 1, \ldots, k_n$ satisfying $\sum_{k=1}^{k_n} r_{nk} \le r$. Then the following are equivalent:

(a) $\sum_{k=1}^{k_n} \dim \eta(\mu_{nk} - T_n)^{r_{nk}} = \dim \eta(\mu - T)^r$ eventually.

(b) for every x in $\eta(\mu - T)^r$, $\|x\| = 1$, there exists a sequence x_n in $\eta[\prod_{k=1}^{k_n} (\mu_{nk} - T_n)^{r_{nk}}]$ such that $x_n \to x$.

PROOF. Define μ_n, $\tilde{T}_n$ and $\tilde{T}$ as in the proof of Theorem 11, and apply Theorem 9.

Apply Theorem 12 to the case in which $\mu_n \to \mu$, $T_n x_n = \mu_n x_n$ and $Tx = \mu x$ with $\mu \ne 0$ we then obtain a necessary and sufficient condition for the existence of a sequence of generalized eigenvectors $\{x_n\}$ of $\{T_n\}$ converging to an arbitrary but fixed generalized eigenvector x of T.

COROLLARY 13. Let μ, and μ_n, $n = 1, 2, \ldots$, be eigenvalues of T and T_n respectively, such that $\mu \neq 0$ and $\mu_n \to \mu$. Then for any positive integer r the following are equivalent:

(a) $\dim \eta(\mu_n - T_n)^r = \dim \eta(\mu - T)^r$ eventually.

(b) for every x in $\eta(\mu - T)^r$, $\|x\| = 1$, there exists a sequence $\{x_n\}$ such that $x_n \in \eta[(\mu_n - T)^r]$ and $x_n \to x$.

REMARK. If, in Theorem 12 and Corollary 13, the hypothesis $\|T_n - T\| \to 0$ is replaced by $\|T_n x - Tx\| \to 0$ for all x in X and the family $\{T_n\}$ is collectively compact, a similar theorem and a similar corollary can be obtained.

V. FUNCTIONS OF OPERATORS

For the remainder of this thesis, unless otherwise stated, X will be a complex Banach space. For each T in $[X]$, let $\mathcal{F}(T)$ denote the family of all complex functions f which are analytic on open, but not necessarily connected, domains $\mathcal{D}(f)$ containing the spectrum $\sigma(T)$ of T. Given T and f, there is an open set Ω such that $\sigma(T) \subset \Omega \subset \overline{\Omega} \subset \mathcal{D}(f)$ and the boundary Γ of Ω consists of a finite number of circular arcs. The operator $f(T)$ in $[X]$ is defined, for each $f \in \mathcal{F}(T)$, by

$$f(T) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda - T)^{-1} d\lambda ,$$

where Γ has positive orientation and the integral is the limit in the operator norm of the usual approximating sums. If f, g belong to $\mathcal{F}(T)$, and α, β are complex numbers then $\alpha f + \beta g$ and (fg) both belong to $\mathcal{F}(T)$ and

$$(\alpha f+\beta g)(T) = \alpha f(T) + \beta g(T),$$

$$(fg)(T) = f(T)g(T).$$

If f has a power series expansion $f(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$, valid in a neighborhood of $\sigma(T)$, then $f(T) = \sum_{k=0}^{\infty} a_k T^k$.

For derivations of these and other properties of functions of operators see [10] and [23].

THEOREM 14. Assume T, T_n belong to $[X]$, $\|T_n - T\| \to 0$, and f belongs to $\mathcal{F}(T)$. Then there exists N such that f belongs to $\mathcal{F}(T_n)$ for $n \geq N$ and

$$\|f(T_n) - f(T)\| \to 0, \quad \text{as} \quad n \to \infty.$$

PROOF. By Theorem 4 there exists N such that f belongs to $\mathcal{F}(T_n)$ for $n \geq N$. Whereas Theorem 3 implies that

$$\|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\| \to 0 \quad \text{uniformly for } \lambda \text{ in } \Gamma .$$

Therefore

$$\|f(T_n) - f(T)\| \leq \frac{1}{2\pi} \int_\Gamma |f(\lambda)| \; \|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\| \, d\lambda ,$$

and

$$\|f(T_n) - f(T)\| \to 0, \quad \text{as} \quad n \to \infty.$$

We will use Theorem 14 to obtain results concerning spectral projections in the next chapter.

VI. SPECTRAL PROJECTIONS AND SPECTRAL SUBSPACES

A bounded linear operator E in $[X]$ is a projection iff $E^2 = E$, in which case $I - E$ is the complementary projection. The following proposition, which was proved by Sz-Nagy [22], will be needed in the proof of Theorem 17.

PROPOSITION 15. Let P and Q be two projections, projecting X onto the subspaces $M_1 = PX$ and $M_2 = QX$. If $\|P-Q\| < 1$, then the subspaces M_1 and M_2 are isomorphic and hence $\dim M_1 = \dim M_2$ (finite or infinite).

As an immediate consequence we have the following corollary.

COROLLARY 16. Let E, E_n in $[X]$ be projections such that $\|E_n - E\| \to 0$. Then $\dim E_n X = \dim EX$ for n sufficiently large.

For T in $[X]$, a spectral set σ of T is a subset of $\sigma(T)$ which is both closed and open in $\sigma(T)$. Given T and σ there exists e in $\mathcal{F}(T)$ such that

$$e(\lambda) = \begin{cases} 0 & \text{for } \lambda \in \sigma(T) - \sigma \\ 1 & \text{for } \lambda \in \sigma \end{cases} .$$

Let $E = E(\sigma, T) = e(T)$. Then

$$E = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - T)^{-1} d\lambda$$

for any contour Γ with σ inside and $\sigma(T) - \sigma$ outside. Since $e^2 = e$ we have $E^2 = E$. Furthermore $E = 0$ iff $\sigma = \phi$; $E = I$ iff $\sigma = \sigma(T)$; and $T(EX) \subset EX$. If $T_{EX} = T|_{EX}$ then $\sigma(T_{EX}) = \sigma$ and $\sigma_p(T_{EX}) = \sigma \cap \sigma_p(T)$, where $\sigma_p(T)$ denotes the set of eigenvalues of T.

Recall that, for T in $[X]$ and λ in $\mathbb{C}$, the index $\nu = \nu(\lambda, T)$ is the least positive integer satisfying $\eta(\lambda - T)^{\nu+1} = \eta(\lambda - T)^{\nu}$, or $\nu = +\infty$.

Let $\sigma = \{\mu_1, \ldots, \mu_m\}$ where each μ_k is an eigenvalue of T with finite index $\nu_k = \nu(\mu_k, T)$. Let $E_k = E(\mu_k, T)$, $k = 1, \ldots, m$. Then $E_j E_k = 0$ for $j \neq k$ and $E = \sum_{k=1}^{m} E_k$ where $E = E(\sigma, T)$. These in turn imply

$$EX = \bigoplus_{k=1}^{m} E_k X = \bigoplus_{k=1}^{m} \eta(\mu_k - T)^{\nu_k} = \eta\Big[\prod_{k=1}^{m} (\mu_k - T)^{\nu_k}\Big] \quad \underline{1}/$$

For the proofs of the above statements and additional information concerning spectral projections see [10] and [23].

$\underline{1}/$Equation follows from [23, p. 317]. Note that the compactness of T is not needed here.

THEOREM 17. Let T, T_n belong to $[X]$. Assume $\|T_n - T\| \to 0$. Let σ be a spectral set for T, $E = E(\sigma, T)$, and let Γ be a contour with σ inside and $\sigma(T) - \sigma$ outside. Then there exists N such that for $n \geq N$:

$\Gamma \subset \rho(T_n)$,

$\sigma_n = \{\lambda \in \sigma(T_n) : \lambda \text{ inside } \Gamma\}$ is a spectral set for T_n,

the spectral projections $E_n = E(\sigma_n, T_n)$ are defined,

$\|E_n - E\| \to 0$, $\dim E_n X = \dim EX$,

$\sigma = \phi$ iff $\sigma_n = \phi$.

PROOF. By Theorem 3 there exists N Such that $\Gamma \subset \rho(T_n)$ for $n \geq N$. Therefore σ_n is a spectral set for T_n, $n \geq N$. Let Ω and Ω' be open subsets of $\mathbb{C}$ such that $\Omega \cap \Omega' = \phi$ and $\Omega \supset \sigma$, $\Omega' \supset \sigma(T) - \sigma$. Define

$$f(\lambda) = \begin{cases} 1 & \lambda \in \Omega \\ 0 & \lambda \in \Omega' . \end{cases}$$

Then $E = f(T)$ and $E_n = f(T_n)$. By Theorem 14 we have $\|E_n - E\| \to 0$. Now Corollary 16 implies $\dim E_n X = \dim EX$ eventually. Finally if $\sigma = \phi$ then Γ together with the points inside Γ, denoted by $\widetilde{\Gamma}$, is a closed subset of $\rho(T)$ and by Theorem 3 $\Gamma \subset \rho(T_n)$ for $n \geq N$, which implies $\sigma_n = \phi$ for $n \geq N$. On the other hand if $\sigma_n = \phi$ for $n \geq N$ then $E_n = 0$ for $n \geq N$ and hence $E = 0$ which is equivalent to $\sigma = \phi$.

VII. EIGENVECTORS AND GENERALIZED EIGENMANIFOLDS IN THE COMPLEX CASE

Throughout this section the hypotheses and notations of Theorem 17 prevail. Assume T is compact. We will specialize Theorem 17 to the case where the spectral set σ consists of an isolated point of $\sigma(T)$. These assumptions will not be repeated in the statements of theorems.

A variant of the following theorem is proved by Andrew [1]. In addition to the assumptions given below, he assumed X to be a Hilbert space and the operators T_n are compact for $n = 1, 2, \ldots$. His theorem holds in both the real and the complex case, while the following theorem holds only in the complex case.

THEOREM 18. Let $\sigma = \{\mu\}$ and $\dim EX = 1$. Then there is a non-zero x in X such that $Tx = \mu x$, $Ex = x$, and $EX = [x]$. There exists μ_n such that $\sigma_n = \{\mu_n\}$, $\mu_n \to \mu$. Let $x_n = E_n x$; then $T_n x_n = \mu_n x_n$ and $x_n \to x$. For n sufficiently large, we have $x_n \neq 0$ and $E_n X = [x_n]$.

<u>PROOF</u>. Now T is compact and μ an isolated point of $\sigma(T)$ imply that μ is an eigenvalue of T. So there exists x in X such that $x \neq 0$, $Tx = \mu x$. It is easy to see that $Ex = x$. Now $\dim EX = 1$ implies $EX = [x]$.

Let Γ be a contour as in Theorem 17. Then there exists N

such that for all $n \geq N$, $\sigma_n = \{\lambda \in \sigma(T_n) : \lambda \text{ inside } \Gamma\}$ is a spectral set of $\sigma(T_n)$. Let E_n be the corresponding projections as in Theorem 17. Then $\|E_n - E\| \to 0$, $\dim E_n X = 1$, and $\sigma_n \neq \phi$ for all $n \geq N$. Let $T_n' = T_n|_{E_n X}$. Then $\dim E_n X = 1$ implies σ_n consists of eigenvalues of T_n' and hence σ_n consists of eigenvalues of T_n. If σ_n contains more than one element then eigenvectors corresponding to different eigenvalues are linearly independent which would imply $\dim E_n X > 1$; a contradiction. Therefore $\sigma_n = \{\mu_n\}$ for all $n \geq N$.

By Theorem 17 with Γ replaced by smaller and smaller circles about μ, we have $\mu_n \to \mu$.

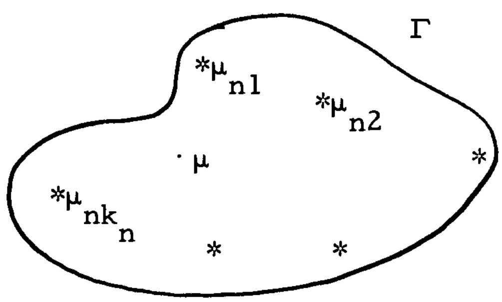
Let $x_n = E_n x$. Then $x_n = E_n x \to Ex = x$ which implies $x_n \to x$ and $x_n \neq 0$ for n sufficiently large. Finally $E_n x_n = E_n E_n x = E_n x = x_n$ and $E_n X = [x_n]$.

In general we would not have $\dim EX = 1$ as assumed in Theorem 18.

If T is compact and σ is a spectral set of T which consists of a single element μ, then it is well known [10, p. 579, Th 5] that for $\mu \neq 0$, μ is an eigenvalue of T, the index $\nu = \nu(\mu, T)$ is finite, and $EX = \eta(\mu - T)^{\nu}$ with $\dim EX < \infty$. It follows from Theorem 17 that $\dim E_n X = \dim EX < \infty$ eventually. Therefore $T_n' = T_n|_{E_n X}$ is an operator defined on a finite dimensional subspace and $\sigma(T_n') = \sigma_n$ consists of finite number of eigenvalues μ_{nk},

$k = 1, \ldots, k_n$ each with finite indices $\nu_{nk} = \nu(\mu_{nk}, T'_n)$, $k = 1, \ldots, k_n$. But another theorem in [10, p. 574, Th 20] implies that the index $\nu_{nk} = \nu(\mu_{nk}, T'_n)$ of T'_n is the same as the index $\nu(\mu_{nk}, T_n)$ of T_n for each $k = 1, \ldots, k_n$. Therefore σ_n consists of finite number of eigenvalues of T_n each of which has finite index.

This situation can best be illustrated by the following picture:



where Γ is a contour with $\sigma = \{\mu\}$ inside and $\sigma(T) - \sigma$ outside, and μ_{nk}, $k = 1, \ldots, k_n$ are eigenvalues of T_n inside Γ. It follows from Theorems 17 and 3 that $\sigma_n \neq 0$ eventually and

$$\max_{1 \leq k \leq k_n} |\mu_{nk} - \mu| \to 0.$$

We can choose $\epsilon > 0$ such that the closed disc $D(\mu_{nk}, \epsilon)$, with center μ_{nk}, radius ϵ satisfying

1. $D(\mu_{nk}, \epsilon)$ lies entirely inside Γ, $k = 1, \ldots, k_n$,
2. $D(\mu_{nk}, \epsilon) \cap D(\mu_{nj}, \epsilon) = \phi$ for $k \neq j$.

Let C_k be the boundary of $D(\mu_{nk}, \epsilon)$. Then we have

$$E_n x = \sum_{k=1}^{k_n} E_{nk} x, \quad \text{and} \quad E_{nk} E_{nj} = 0 \quad \text{for} \quad k \neq j. \quad \text{Therefore}$$

$$E_nX = \bigoplus_{k=1}^{k_n} E_{nk}X = \bigoplus_{k=1}^{k_n} \eta(\mu_{nk}-T_n)^{\nu_{nk}}$$

$$= \eta\left[\prod_{k=1}^{k_n} (\mu_{nk}-T_n)^{\nu_{nk}}\right].$$

Define $P(\lambda) = (\mu-\lambda)^{\nu}$ and $P_n(\lambda) = \prod_{k=1}^{k_n} (\mu_{nk}-\lambda)^{\nu_{nk}}$. Then $EX = \eta(P(T))$ and $E_nX = \eta(P_n(T_n))$. Furthermore by a well known result [10, p. 573, Th 18] P and P_n have minimal degrees among all polynomials satisfying

$$EP(T) = P(T)E = 0$$

and

$$E_nP_n(T_n) = P_n(T_n)E_n = 0$$

respectively. The degree of P_n is

$$\nu_n = \sum_{k=1}^{k_n} \nu_{nk}.$$

THEOREM 19. $\nu_n \geq \nu$ eventually.

<u>PROOF</u>. Suppose that $\nu_n = \alpha$ for all n in an infinite set J. Since $\|T_n-T\| \to 0$ and $\|E_n-E\| \to 0$ it follows that, as $n \to \infty$ through J, $\|P_n(T_n)E_n-(\mu-T)^{\alpha}E\| \to 0$. Since $P_n(T_n)E_n = 0$ and $\|E_n-E\| \to 0$, $(\mu-T)^{\alpha}E = 0$. By the minimality of ν we have $\alpha \geq \nu$.

THEOREM 20. Choose any non-negative integer r and r_{nk}, $k = 1, 2, \ldots, k_n$ such that

$$\sum_{k=1}^{k_n} r_{nk} \leq r.$$

Then

$$\sum_{k=1}^{k_n} \dim \eta(\mu_{nk} - T_n)^{r_{nk}} \leq \dim \eta(\mu - T)^r$$

eventually.

PROOF. Without loss of generality assume

$$\sum_{k=1}^{k_n} r_{nk} = r \qquad \text{for all} \quad n.$$

Let

$$q(T) = (\mu - T)^r, \quad q_n(T_n) = \prod_{k=1}^{k_n} (\mu_{nk} - T_n)^{r_{nk}}.$$

Note that

$$\| q_n(T_n) - q(T) \| \to 0 ,$$

$$\eta(q(T)) \subset EX = \eta(I-E),$$

and

$$\eta(q_n(T_n)) \subset E_n X = \eta(I - E_n) .$$

Therefore, for x_n in $\eta(q_n(T_n))$ such that $x_n \to x$ we have

$$\| q_n(T_n)x_n - q(T)x \| \leq \| q_n(T_n)x_n - q(T)x_n \| + \| q(T)x_n - q(T)x \|$$

which implies $q(T)x = 0$; in other words, $x \in \eta(q(T))$. Recall that $\dim EX < \infty$, which implies E is compact. By Theorem 17 $\|E_n - E\| \to 0$. Now we can apply Theorem 7 with $\mu - T$ and $\mu_n - T_n$ replaced by $I-E$ and $I-E_n$ and conclude

$$\dim \eta(q_n(T_n)) \leq \dim \eta(q(T)).$$

Since $\eta(q_n(T_n)) = \bigoplus_{k=1}^{k_n} \eta(\mu_{nk} - T_n)^{r_{nk}}$, the desired result follows.

The proofs of Theorems 19 and 20 are similar to those in Theorems 4.17 and 4.18 of [3].

If in Theorem 20 we assume $0 \leq r \leq \nu$, then Theorem 19 indicates that there exist r_{nk} such that $0 \leq r_{nk} \leq \nu_{nk}$, $\sum_{k=1}^{k_n} r_{nk} = r$ for n sufficiently large.

We can now restate Theorem 12 in the complex case. Here we apply Theorem 20 instead of Theorem 11 to obtain the dimensional inequality $\sum_{k=1}^{k_n} \dim \eta(\mu_{nk} - T_n)^{r_{nk}} \leq \dim \eta(\mu - T)^r$, and we are able to drop the hypothesis $\max_{1 \leq k \leq k_n} |\mu_{nk} - \mu| \to 0$ as $n \to \infty$.

THEOREM 21. Choose any non-negative integer r, and r_{nj}, $j = 1, \ldots, k_n$, such that $\sum_{k=1}^{k_n} r_{nk} \leq r$. Then the following are equivalent:

(a) $\sum_{k=1}^{k_n} \dim \eta(\mu_{nk} - T_n)^{r_{nk}} = \dim \eta(\mu - T)^r$ eventually;

(b) for every x in $\eta(\mu-T)^r$, $\|x\| = 1$, there exists a sequence x_n in $\eta[\prod_{k=1}^{k_n} (\mu_{nk} - T_n)^{r_{nk}}]$ such that $x_n \to x$.

REMARKS. (1) If $r \geq \nu$ and $r_{nk} \geq \nu_{nk}$, then we have by Theorem 17, $\dim E_n X = \dim EX$ eventually. It follows from the above theorem that for every x in $\eta(\mu-T)^\nu = EX$ there exists a sequence $\{x_n\}$ such that x_n belongs to $\eta[\prod_{k=1}^{k_n} (\mu_{nk} - T_n)^{\nu_{nk}}] = E_n X$ and $x_n \to x$.

(2) The same theorem will hold if in the hypothesis we replace $\|T_n - T\| \to 0$ by $\|T_n x - Tx\| \to 0$ for every $x \in X$ and $\{T_n\}$ collectively compact.

VIII. SOME GEOMETRIC INTERPRETATIONS OF THEOREMS 19 AND 20

In this chapter we will apply Theorems 19 and 20 to obtain results that compare the Jordan canonical form of $T|_{EX}$ with those of $T_n|_{E_nX}$. These compare the lengths of the longest string of 1's and the number of 0's in the super diagonal of the Jordan form of $T|_{EX}$ with those of $T_n|_{E_nX}$.

Suppose $\dim EX < \infty$ and μ is an isolated eigenvalue with index ν. Then $EX = \eta(\mu - T)^{\nu}$ and there exists x_1 in X such that $(\mu-T)^{\nu-1}x_1 \neq 0$. Let $A = \mu - T$; then the vectors $A^{\nu-1}x_1, A^{\nu-2}x_1, \ldots, x_1$ are linearly independent. Let

$$M_1 = [A^{\nu-1}x_1, A^{\nu-2}x_1, \ldots, x_1] \tag{1}$$

and N_1 be a subspace of EX satisfying $EX = M_1 \oplus N_1$, $TN_1 \subset N_1$. Then $\dim M_1 = \nu$ and the matrix representation of $A|_{M_1}$ with respect to the given basis has the form

$$A|_{M_1} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \tag{2}$$

with ones on the super diagonal and zeros elsewhere. Note that the rank of $A|_{M_1}$ is $\nu-1$. We will call $\nu-1$ the size of the Jordan matrix of $A|_{M_1}$, which corresponds to the number of one's in the super diagonal. Let m_2 be the smallest positive integer satisfying $A^{m_2} = 0$ and $A^{m_2-1} \neq 0$ on N_1. Then clearly $m_2 \leq \nu$. Choose x_2 in N_1 such that $A^{m_2-1} x_2 \neq 0$. Then $A^{m_2-1} x_2, \ldots, x_2$ are linearly independent. Define

$$M_2 = [A^{m_2-1} x_2, A^{m_2-2} x_2, \ldots, x_2],$$

and let N_2 be a subspace of N_1 satisfying $N_1 = M_2 \oplus N_2$, $TN_2 \subset N_2$.

$$A|_{M_2} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$$

with ones on the super diagonal and zeros elsewhere. The size of the Jordan matrix of $A|_{M_2}$ is $m_2 - 1$. Now we have

$$EX = M_1 \oplus M_2 \oplus N_2.$$

Continue this process on N_2. In a finite number of steps, we obtain a direct sum decomposition of EX such that

$$EX = M_1 \oplus M_2 \oplus \cdots \oplus M_s,$$

and $\nu = \dim M_1 \geq \dim M_2 \geq \cdots \geq \dim M_s$. A Jordan matrix corresponding to $A|_{M_i}$ is

$$A|_{M_i} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}, \quad i = 1, \ldots, s. \tag{3}$$

where the size of the matrix $A|_{M_i}$ is $m_i - 1$. Furthermore

$$EX = [A^{\nu-1}x_1, A^{\nu-2}x_1, \ldots, x_1, \ldots, A^{m_s-1}x_s, \ldots, x_s].$$

Corresponding to this basis, the matrix representation of A on EX is

$$A|_{EX} = \begin{pmatrix} 0 & 1 & & & & & & & & \\ & 0 & \ddots & & & & & & & \\ & & \ddots & 1 & & & & & & \\ & & & 0 & 0 & & & & & \\ & & & & 0 & 1 & & & & \\ & & & & & 0 & \ddots & & & \\ & & & & & & \ddots & 0 & & \\ & & & & & & & \ddots & 1 & \\ & & & & & & & & 0 & \ddots \\ & & & & & & & & & \ddots & 1 \\ & & & & & & & & & & 0 \end{pmatrix}.$$

The matrix is composed of s subblocks, with the ith subblock being the matrix representation of $A|_{M_i}$ as in (3). The largest subblock corresponds to the Jordan matrix of $A|_{M_1}$ as in (2). Consequently a matrix representation of $T|_{EX}$ is

$$T|_{EX} = \begin{pmatrix} \mu & 1 & & & & & & & & & \\ & \mu & \ddots & & & & & & & & \\ & & \ddots & 1 & & & & & & & \\ & & & \mu & 0 & & & & & & \\ & & & & \mu & 1 & & & & & \\ & & & & & \mu & \ddots & & & & \\ & & & & & & \ddots & 0 & & & \\ & & & & & & & \mu & 1 & & \\ & & & & & & & & \mu & \ddots & \\ & & & & & & & & & \ddots & 1 \\ & & & & & & & & & & \mu \end{pmatrix} . \tag{4}$$

For reference on the Jordan form of an operator in finite dimensional spaces see [9].

Two important observations:

1. $\dim \eta(\mu-T)$ equals to the number of subblocks in the matrix representation of $\mu-T$, since

$$\eta(\mu-T) = [A^{\nu-1}x_1, A^{m_2-1}x_2, \ldots, A^{m_s-1}x_s]$$

and $\dim \eta(\mu-T) = s$,

2. $\nu-1$ is the size of the largest subblock in the matrix representation of $\mu-T$.

From Theorem 17 we know that $\dim E_nX = \dim EX$ eventually. If we denote $\eta(\mu_{ni}-T_n)^{\nu_{ni}}$ by $E_{ni}X$ for $1 \leq i \leq k_n$ then

$$E_nX = E_{n1}X \oplus \ldots \oplus E_{nk_n}X .$$

Applying the previous results to $A_{ni} = \mu_{ni} - T_n$ and $\eta(A_{ni})$, we find there exist subspaces $M_1^{ni}, \ldots, M_{s(ni)}^{ni}$ such that

$$E_{ni}X = \eta(A_{ni}^{\nu_{ni}}) = M_1^{ni} \oplus \cdots \oplus M_{s(ni)}^{ni}, \quad 1 \leq i \leq k_n,$$

where

$$\dim M_1^{ni} \geq \dim M_2^{ni} \geq \cdots \geq \dim M_{s(ni)}^{ni},$$

$$M_j^{ni} = [A_{ni}^{m_j^{ni}-1} y_j^{ni}, A_{ni}^{m_j^{ni}-2} y_j^{ni}, \ldots, y_j^{ni}], \quad 2 \leq j \leq s(ni),$$

and

$$M_1^{ni} = [A_{ni}^{\nu_{ni}-1} y_1^{ni}, A_{ni}^{\nu_{ni}-2} y_1^{ni}, \ldots, y_1^{ni}].$$

Moreover,

$$E_nX = E_{n1}X \oplus \cdots \oplus E_{nk_n}X$$

$$= \sum_{i=1}^{k_n} (M_1^{ni} \oplus \cdots \oplus M_{s(ni)}^{ni}),$$

where $s(ni)$, $1 \leq i \leq k_n$, corresponds to the number of subblocks in the matrix representation of $T_n|_{E_{ni}}$. Finally, as a matrix representation of $T_n|_{E_nX}$ we have

$$T_n|_{E_n X} = \begin{pmatrix} \mu_{n1} & 1 & & & & & & & & \\ & \mu_{n1} & \ddots & & & & & & & \\ & & \ddots & 1 & & & & & & \\ & & & \mu_{n1} & 0 & & & & & \\ & & & & \mu_{n2} & 1 & & & & \\ & & & & & \ddots & 0 & & & \\ & & & & & & \mu_{nk_n} & 1 & & \\ & & & & & & & \ddots & \ddots & \\ & & & & & & & & \ddots & 1 \\ & & & & & & & & & \mu_{nk_n} \end{pmatrix} \tag{5}$$

The matrix consists of k_n subblocks, with the ith subblock being a matrix representation of $T_n|_{E_{ni} X}$ where

$$T_n|_{E_{ni} X} = \begin{pmatrix} \mu_{ni} & 1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & \ddots & 1 & & & & & \\ & & & \ddots & 0 & & & & \\ & & & & \mu_{ni} & 1 & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & \ddots & 0 & \\ & & & & & & & \ddots & 1 \\ & & & & & & & & \ddots & 1 \\ & & & & & & & & & \mu_{ni} \end{pmatrix} . \tag{6}$$

The matrix representation of $T_n|_{E_{ni} X}$ consists of $s(ni)$ subblocks, where the size of the jth subblock is m_j^{ni} and the size of the largest subblock is $\nu_{ni}-1$. On the super diagonal is a string of one's followed by a zero then followed by another string of one's then followed by a zero and so on. On the main diagonal is a string of μ_{ni}'s.

We now apply Theorems 19 and 20 to the matrix representations of $T|_{EX}$ as in (4) and $T_n|_{E_nX}$ as in (5) to obtain the following comparisons of the total number of subblocks in the two matrices and the size of the largest subblock in the two matrices.

A. If we let $r = 1$, $r_{nj} = 1$ and $r_{ni} = 0$ for $i \neq j$ for all n sufficiently large in Theorem 20, then

$$\dim \eta(\mu_{nj} - T_n) \leq \dim \eta(\mu - T)$$

implies the number of subblocks in the Jordan form of $T_n|_{E_{nj}X}$ is less than or equal to the number of subblocks in the Jordan form of $T|_{EX}$, for $1 \leq j \leq k_n$.

B. If we let $r_{nj} = 1$ for $1 \leq j \leq k_n$ and $r = \sum_{j=1}^{k_n} r_{nj}$, then it follows from Theorem 20 that

$$\sum_{j=1}^{k_n} \dim \eta(\mu_{nj} - T_n) \leq \dim \eta(\mu - T)^r \leq \dim \eta(\mu - T)^\nu ,$$

which implies that the total number of subblocks in the Jordan form of $T_n|_{E_nX}$ is less than or equal to $\dim EX = \dim \eta(\mu - T)^\nu$ eventually.

C. From Theorem 19 we have $\sum_{j=1}^{k_n} \nu_{nj} \geq \nu$ eventually which implies that

$$\sum_{j=1}^{k_n} \dim M_1^{nj} \geq \dim M_1$$

eventually. In other words the sum of the dimensions of the largest subblocks in the Jordan forms of $T_n|_{E_{nj}X}$, $1 \le j \le k_n$, is greater than or equal to the dimension of the largest subblock in the Jordan form of $T|_{EX}$ for n sufficiently large.

Let μ, μ_n, $n = 1, 2, 3, \ldots$ be eigenvalues of T and T_n respectively such that $\mu \neq 0$ and $\mu_n \to \mu$. In Theorem 21, if we assume $\sigma_n = \{\mu_n\}$ and either (a) or (b) holds for $r = 1, \ldots, \nu\text{-}1$, then it follows from Theorem 17 that for any integer $r > 0$

$$\dim \eta(\mu_n - T_n)^r = \dim \eta(\mu - T)^r$$

eventually. Here we will make a double usage of the symbol ν_n, and let $\nu_n = \nu(\mu_n, T_n)$ be the index of μ_n.

From previous discussions we see that there exist non-zero vectors $x_1, x_2, \ldots, x_s$ and $y_1, \ldots, y_t$ such that

$$\eta(\mu - T) = [A^{\nu-1}x_1, A^{m_2-1}x_2, \ldots, A^{m_s-1}x_s],$$

and

$$\eta(\mu_n - T_n) = [A_n^{\nu_n-1}y_1, A_n^{m_2'-1}y_2, \ldots, A_n^{m_t'-1}y_t].$$

Applying Theorem 21 with $r = 1$, we have $t = 2$ for n sufficiently large. Now for $r = 2$, we have by the same theorem

$$\eta(\mu-T)^2 = [A^{\nu-2}x_1, A^{\nu-1}x_1, A^{m_2-2}x_2, A^{m_2-1}x_2, \ldots, A^{m_s-2}x_s, A^{m_s-1}x_s],$$

where $A^{m_2-2}x_i = 0$ whenever $m_2 \leq 2$ for $1 \leq i \leq s$, and

$$\eta(\mu_n-T_n)^2 = [A_n^{\nu-2}y_1, A_n^{\nu-1}y_1, A_n^{m_2'-2}y_2, A_n^{m_2'-1}y_1, \ldots, A_n^{m_s'-2}y_s, A_n^{m_s'-1}y_s].$$

Since

$$\nu = m_1 \geq m_2 \geq \cdots \geq m_s,$$

$$\nu_n = m_1' \geq m_2' \geq \cdots \geq m_s'$$

and

$$\dim \eta(\mu_n - T_n)^2 = \dim (\mu - T)^2$$

eventually, we have $m_s = 1$ iff $m_s' = 1$, and $m_i = 1$, $1 \leq i \leq s$, iff $m_i' = 1$, $1 \leq i \leq s$. Proceed in this manner inductively for $r \leq \nu$ we obtain

$$\eta(\mu-T)^r = [A^{\nu-r}x_1, A^{\nu-r+1}x_1, \ldots, A^{\nu-1}x_1, \ldots, A^{m_s-r}x_s, \ldots, A^{m_s-1}x_s],$$

$$\eta(\mu_n-T_n)^r =]A_n^{\nu-r}y_1, A_n^{\nu-r+1}y_1, \ldots, A_n^{\nu-1}y_1, \ldots, A_n^{m_s'-r}y_s, \ldots, A_n^{m_s'-1}y_s],$$

and $m_i = m_i'$ for $1 \leq i \leq \nu$ for n sufficiently large. In particular we have $\nu_n = \nu$ eventually.

Now by the observations (1) and (2) we have: (3) The number of subblocks in the Jordan form of $T_n|_{E_nX}$ eventually equals to the number of subblocks in the Jordan form of $T|_{EX}$. (4) The size of the ith subblock, $m_i'-1$, in the Jordan form of $T_n|_{E_n X}$ equals to the size of the ith subblock, m_i-1, in the Jordan form of $T|_{EX}$ for n sufficiently large.

IX. ERROR ESTIMATES FOR GENERALIZED EIGENVECTORS

Let $T_n, T \in [X]$, $n = 1, 2, \ldots$. Assume $\|T_n - T\| \to 0$. Let E and E_n be defined as in Theorem 17. Let x_n belong to E_nX, $\|x_n\| = 1$, for $n \geq 1$. The following theorem estimates the distance, $d(x_n, EX) = \inf_{x \in EX} \|x_n - x\|$, from x_n to EX.

THEOREM 22. Let $T_n, T \in [X]$, $n = 1, 2, \ldots$. Assume $\|T_n - T\| \to 0$. Let E be a spectral projection for T, and E_n, $n \geq N$ be related spectral projections for T_n as defined in Theorem 17. Let $x_n = E_nX$, $\|x_n\| = 1$ for $n \geq N$. Then

$$\|x_n - Ex_n\| \leq c\|T_n - T\|$$

for some constant c, and $\|Ex_n\| \to 1$.

<u>PROOF.</u>

$$\|x_n - Ex_n\| = \|(E_n - E)x_n\| \leq \|E_n - E\|$$

$$\leq \frac{1}{2\pi} \int_\Gamma \|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\| d\lambda$$

By Theorem 3 there exists $b > 0$ such that $\|(\lambda - T_n)^{-1}\| \leq b$ for all λ in Γ, for all $n \geq N$ and $\|(\lambda - T\ )^{-1}\| \leq b$ for all λ in Γ. If we let the length of Γ be ℓ then, by means of $(\lambda - T_n)^{-1} - (\lambda - T)^{-1} = (\lambda - T_n)^{-1} (T_n - T) (\lambda - T)^{-1}$, we obtain

$$\|x_n - Ex_n\| \leq \frac{b^2 \ell}{2\pi} \|T_n - T\| = c\|T_n - T\| \to 0 ,$$

$$\|x_n\| = 1 \Rightarrow \|Ex_n\| \to 1.$$

The following theorem is an improvement of Remark 1 following Theorem 21.

THEOREM 23. Let $T_n, T \in [X]$, $n = 1, 2, \ldots$. Assume $\|T_n - T\| \to 0$. Let E and E_n be defined as in Theorem 22. Let $x \in EX \cap \mathcal{B}$. Then there exist x_n in $E_n X \cap \mathcal{B}$ such that

$$\|x_n - x\| \leq 2c\|T_n - T\|$$

for c as in the proof of Theorem 22.

<u>PROOF</u>. $\|Ex\| = \|x\| = 1$ and $\|E_n x - Ex\| \leq \|E_n - E\| \to 0$ implies $\|E_n x\| \to 1$. Define

$$x_n = \frac{E_n x}{\|E_n x\|} .$$

Then

$$\|x - x_n\| \leq \|x - E_n x\| + \|E_n x - x_n\|$$

$$\leq \|Ex - E_n x\| + \|E_n x - \frac{E_n x}{\|E_n x\|}$$

$$\leq \frac{1}{2\pi} \int_T \|(\lambda - T)^{-1} - (\lambda_n - T_n)^{-1}\| \; \|x\| d\lambda$$

$$+ \|E_n x - \frac{E_n x}{\|E_n x\|}\|$$

$$\leq c\|T_n - T\| + \left\| \frac{(\|E_n x\| - 1)E_n x}{\|E_n x\|} \right\|$$

$$\leq c\|T_n - T\| + \big| \|E_n x\| - 1 \big|$$

$$\leq 2c\|T_n - T\|$$

since

$$\|E_n x\| \to 1.$$

X. APPLICATIONS

1. Let X = space $C[a,b]$ with max norm, $K(s,t)$ continuous (on closed square). Consider a Fredholm integral equation of the second kind

$$\lambda x(s) - \int_a^b K(s,t)x(t)dt = y(s), \quad a \leq s \leq b. \tag{1}$$

If we let $(\mathcal{K}x)(s) = \int_a^b K(s,t)x(t)dt$ then the above equation becomes

$$(\lambda - \mathcal{K})x = y, \tag{2}$$

where the operator $\mathcal{K}$ is compact.

A. A common method of solving (1) approximately is the degenerate kernel method [12]. In this method the kernel $K(s,t)$ is approximated by a sequence of kernels

$$K_n(s,t) = \sum_{j=1}^{n} a_{j,n}(s)b_{j,n}(t), \quad n = 1,2,\ldots .$$

Instead of solving (1), for each n the approximate equation

$$\lambda x_n(s) - \int_a^b K_n(s,t)x_n(t)dt = y(s) \tag{3}$$

is solved. If we let

$$\mathcal{K}_n x = \int_a^b K_n(s,t)x(t)dt ,$$

then Equation (3) becomes

$$(\lambda - \mathcal{K}_n)x_n = y \tag{4}$$

Assume $K_n(s,t) \to K(s,t)$ uniformly. Then

$$\|\mathcal{K}_n - \mathcal{K}\| = \max_{a \le s \le b} \int_a^b |K(s,t) - K_n(s,t)|\,dt \to 0 ,$$

and the theory developed earlier applies.

B. Another useful method of solving (1) is the projection method. Let X_n be a finite dimensional subspace of X and P_n be a projection from X onto X_n satisfying $\|P_n x - x\| \to 0$ for $x \in X$. We approximate Equation (2) by solving

$$P_n(\lambda - \mathcal{K})x_n = P_n y, \quad x_n \in X_n$$

or equivalently

$$(\lambda - P_n\mathcal{K})x_n = P_n y, \quad x_n \in X_n .$$

It is shown [11] that $\|P_n\mathcal{K} - \mathcal{K}\| \to 0$, and hence the results in the earlier chapters apply. For further discussion of this method see [11].

As a special case of the projection method we have the collocation method. For discussions concerning this method see [14, 17, 19] and others listed in their bibliographies. A thorough discussion of methods in solving integral equations of the second kind can be found in [7].

2. Projection methods are often used in solving eigenvalue problems associated with certain ordinary or partial differential equations. In many cases we have an operator equation

$$Tx = \mu x \tag{5}$$

in a Banach space X. Again let $\{X_n\}$ and P_n be defined as in 1.B. Instead of solving (5) the approximate equations

$$P_n T x_n = \mu_n x_n, \quad P_n x_n = x_n$$

are solved for $x_n \in X_n$. Again we have $\|P_n T - T\| \to 0$, so that the theory in earlier chapters applies.

A. We now apply the projection method to the case that X is a Hilbert space. Let $\{\phi_i\}$ be a sequence of linearly independent vectors in X, and X_n be the subspace of X spanned by the first n vectors in the sequence. Then as special cases of the projection method we have the Ritz method, the generalized Ritz method, the Galerkin method and the method of moments. For a brief discussion of these methods see [16].

As an example, let T^{-1} be a differential operator with compact inverse T. The eigenvalue problem

$$T^{-1}x = \mu x$$

may be solved by solving the corresponding eigenvalue problem

$$T(T^{-1}x) = \mu Tx \quad \text{or} \quad \lambda x = Tx$$

where $\lambda = \frac{1}{\mu}$, since μ is an eigenvalue of T^{-1} iff $\lambda = \frac{1}{\mu}$ is an eigenvalue of T.

B. Let X be a real Hilbert space. Consider the operator equation

$$(I+T)x = \mu Mx \tag{6}$$

where T and M are assumed to be compact, and $(I+T)^{-1} \in [X]$. Reiden [21] has considered this problem by looking at the approximate equation

$$(I+P_nT)x_n = \mu_n P_n Mx_n, \quad x_n \in X_n, \tag{7}$$

with $\lim \inf \{\|x-x_n\| : x_n \in P_nX\} = 0$ for all $x \in X$ and $\|P_n\| \leq M$ for all n. In his paper he gives error estimates for the approximated eigenvalues, and he also gives an estimation of $d(x_n, X_0) = \inf\limits_{x \in X_0} \|x_n - x\|$, where $X_0 = \eta(I+T-\mu_0 M)$. As an application, he considered the real valued partial differential equation

$$u^m(s) + \sum_{j=0}^{m-1} e_j(s)u^j(s) = \lambda \sum_{j=0}^{\ell} f_j(s)u^j(s),$$

for $a < s < b$, $0 \leq \ell \leq m-1$ with boundary conditions

$$\sum_{j=0}^{m-1} [\alpha_{ij}u^j(a)+\beta_{ij}u^j(b)] = 0, \quad i = 1, \ldots m.$$

If we let $\lambda = \frac{1}{\mu}$ and $A = (I+T)^{-1}M$ then A is compact and $(I+T)x = \mu Mx$ iff $\lambda x = Ax$. And therefore $X_0 = \eta(I+T-\mu_0 M) = \eta(\lambda_0 - A)$. By Lemma 1.1 of [21] $(I + P_nT)^{-1}$ exist for n sufficiently large and are bounded uniformly in n. Now let $\lambda_n = \frac{1}{\mu_n}$ and $A_n = (I+P_nT)^{-1}P_nM$. Then $(I+P_nT)x_n = \mu_n P_n Mx_n$ iff $\lambda_n x_n = A_n x_n$. It follows that x is an eigenvector of (6) iff x is an eigenvector of A corresponding to the eigenvalue $\lambda = \frac{1}{\mu}$; and x_n is an eigenvector of (7) iff x_n is an eigenvector of A_n corresponding to the eigenvalue $\lambda_n = \frac{1}{\mu_n}$. It is clear that $\mu_n \to \mu \neq 0$ iff $\lambda_n \to \lambda \neq 0$. Finally we have $\|A_n - A\| \to 0$. To see this we note that

$$
\begin{aligned}
\|A_n - A\| &= \|(I+P_nT)^{-1}P_nM - (I+T)^{-1}M\| \\
&\leq \|(I+P_nT)^{-1}P_nM - (I+P_nT)^{-1}M\| \\
&\quad + \|(I+P_nT)^{-1}M - (I+T)^{-1}M\| \\
&\leq \|(I+P_nT)^{-1}\| \, \|P_nM - M\| \\
&\quad + \|(I+P_nT)^{-1}(T-P_nT)(I+T)^{-1}\| \, \|M\| .
\end{aligned}
$$

By Lemma 1.1 of [21] we have $\|P_nM - M\| \to 0$, $\|(I+P_nT)^{-1}\|$ bounded uniformly in n, and $\|T - P_nT\| \to 0$. Thus $\|A_n - A\| \to 0$ and the theory discussed in this thesis applies.

The theory developed in this thesis has many other applications. The author has made no attempts to exhaust them.

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