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In a discrete review inventory process, when the demand forms a stochastically convergent sequence of random variables, it seems reasonable that the optimal stationary (s,S) inventory policy will be a function of the limiting demand and cost structure only. The intent of this paper is to provide a rigorous justification of this conjecture under suitable restrictions. Assuming linear costs and integer valued demand, the problem is essentially reduced to showing the existence and finding an expression for the stationary inventory distribution.

The stationary inventory distribution, with an (s,S) policy in effect, is derived by applying renewal theory to the inventory process with renewals defined as those periods in which a positive amount is ordered. For this purpose a version of the key renewal theorem for stochastically convergent sequences is proved and

formulated in terms of integrals. The integral formulation is used to derive the stationary distribution of the excess variable and the stationary probability that a renewal will occur, or equivalently, that an order will be placed.

Optimal Stationary (s, S) Inventory Policies for Stochastically Convergent Demand Sequences

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I. INTRODUCTION

This paper extends key results in renewal theory to include sequences of stochastically convergent random variables and applies these results to find the optimal stationary (s,S) inventory policy, when demand forms a stochastically convergent sequence.

Two problems of considerable importance to those organizations whose operation requires the holding of stock are: deciding when to place an order for replenishment of their stock of items, and deciding how large an order to place. Uncertainty concerning the number of items which will be demanded during a given time period must be taken into account in making these decisions. If it were not for this uncertainty concerning demand, new stock might be ordered in such a manner that it would arrive precisely when needed and in exactly the right amount, thus eliminating the need for holding inventory.

1. The Inventory Problem

The inventory problem is a sequential decision problem whose general statement is embodied in the following notions. At the beginning of a certain number of equally spaced time periods a decision is made as to what amount, if any, should be added to present inventory in anticipation of future demands. This decision is made so as to minimize a cost associated with the ordering and holding of inventories.

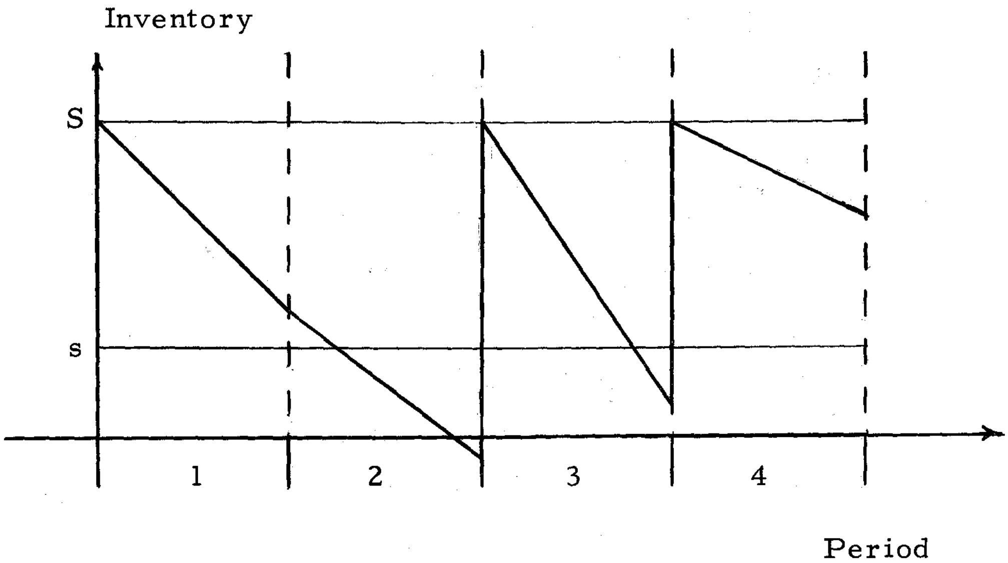

Figure 1. Four period inventory process.

Figure 1 represents a four period inventory process in which the demand is taken as uniform over the period. In each period the height of the left and right hand ends of the diagonal lines represent the beginning and ending inventories, respectively; the demand is the difference between the beginning and ending inventory. The vertical lines represent the stock delivered at the beginning of the period. This stock is added to the ending inventory of the previous period, resulting in the new starting inventory. In Figure 1 there are deliveries in periods three and four. There is excess demand in period two, represented by a negative inventory, and satisfied from the stock delivered at the beginning of period three.

From this basic formulation an inventory model is usually classified according to one or, more likely, a combination of the following: cost structure, nature of future demand, method used to satisfy orders, planning horizon, and/or the specific class of order policy under consideration. Each of these considerations is important in its own right and will be discussed separately.

The following three costs are usually taken into account each period: a holding cost $h(x)$ which is often taken as a function of the stock on hand at the end of the period, if that quantity is positive; a penalty cost $p(x)$ which is a function of the amount by which demand exceeds supply during the period; and an ordering cost $c(z)$ charged for ordering an amount z. In addition almost all models include a discount factor $\alpha (0 < \alpha \leq 1)$ which keeps the money values from period to period relevant.

There are a wide variety of assumptions about the form of these various costs of which the following seem to be the most popular. The holding and penalty costs are often assumed to be linear functions including a revenue term. Other assumptions concerning penalty cost run the gamut between two extremes. A priority order, with an accompanying high cost, is used to satisfy the excess demand immediately, or backlogging is allowed, in which case the excess demand is kept on the books and satisfied, in so far as possible, from the next shipment with little or no penalty. Also in this regard a common option is the lost sales case in which excess demand is lost. The ordering cost is often assumed to be of the form[1]

$$c(z) = \begin{cases} 0 & z = 0 \\ K+cz & z > 0 \end{cases} \tag{1}$$

with the analysis changing according as $K > 0$ or $K = 0$. For all three of these costs, in particular the order cost, some work has been done with more general functions. Referring to Figure 1 there would be an ordering cost in periods three and four, a penalty cost in period two, and a holding cost in periods one, three, and four.

Demand is either deterministic (known exactly in each period) or probabilistic, in which case demand per period is taken as a random variable satisfying a variety of conditions. Deterministic models generally serve as first approximations and are used to study the gross aspects of the inventory process. Probabilistic demand, with

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[1] Expression (equation, theorem, etc.) numbering will begin anew in each chapter. Reference in a chapter to an expression in that chapter will use just the expression number, while reference to an expression in a different chapter will be the chapter number followed by the expression number.

which we will be solely concerned, is often assumed to be independent and identically distributed from period to period, with many results available under these assumptions. Some work has been done for demands which are not identically distributed.

Lag in delivery, the time from the placing of an order till it arrives, is another factor which adds reality and complexity to inventory models. The lag is most often taken as a fixed number of periods, although it is sometimes more realistic to assume lag is a random variable with a known distribution. This latter interpretation is connected very closely with models where a decision to order is interpreted as a decision to produce. Even more generality may be allowed by admitting more than one kind of order, each with its own cost and lag, a longer lag usually implying a smaller cost.

Because of the complexity of general inventory problems it is quite often the case that the true optimal ordering policy is not feasible to calculate or apply. For this reason one often optimizes under the assumption that a particular type of policy will be used. There are two particular policies which lend themselves to this use: the (s, S) policy, and the single critical level policy. The reasons for their uses are: they have been shown to be optimal in a wide variety of situations, and they are easy to apply. The (s, S) policy, $s < S$, is effected by ordering up to the quantity S whenever the present stock falls below s. (See Figure 1.) The single critical level policy is the (s,S) policy with $s = S$, that is, whenever the inventory drops below S order up to S. Both of these policies are normally associated with the linear cost function given by (1), the single critical level policy being optimal when there is no set up cost, $K = 0$, while the (s,S) policy is normally optimal when $K > 0$. It must be mentioned here that although an (s,S) policy may be optimal for a particular inventory process the parameters s and S are generally difficult to evaluate and no generally feasible method has been forwarded for the calculation of these numbers.

The planning horizon is generally divided into the static and dynamic cases. Static models are concerned with a single period, while dynamic models have more than one period and are usually classified as finite or infinite according to the number of periods. Static models, aside from interest per se, provide insight into the structures and methods of solution of dynamic models. In static models the inventory at the end of the period is somehow salvaged or lost, while in dynamic models it serves as starting stock for the next period. This use of the ending inventory is one of the principal differences between the formulation of a static and dynamic model. The distinction in the formulation of a finite horizon model as compared to an infinite horizon model will be made apparent in the discussion of the mathematical model.

2. The Mathematical Model

We now turn to a formulation of mathematical models for dynamic inventory processes. Designate the demand in period i by ξ_i with distribution function Φ_i. The model to be used is essentially that of Arrow, Harris, and Marschak (1951) and has received extensive treatment in the literature. Let x be the size of the stock at the end of the $(i-1)^{th}$ period, then the expected cost for the i^{th} period if we order up to y, $y \geq x$, is

$$c(y-x) + L(y; \Phi_i) \tag{2}$$

where

$$L(y; \Phi_i) = \int_0^y h(y-\xi)d\Phi_i(\xi) + \int_y^\infty p(\xi - y)d\Phi_i(\xi) .$$

This is the expected value of the ordering, holding, and penalty costs. Let $f(x; \Phi_i, \Phi_{i+1}, \cdots)$ be the minimum discounted expected cost which will occur in an infinite number of time periods starting with the i^{th} period. Then the minimal expected cost for the i^{th} period and the $(i+1)^{th}$ period are related by the basic functional equation

$$(3)\quad f(x;\Phi_i,\Phi_{i+1},\cdots) = \inf_{y \geq x} \{c(y-x)+L(y;\Phi_i) + \alpha \int_0^\infty f(y-\xi\,;\Phi_{i+1},\Phi_{i+2},\cdots)d\Phi_i(\xi)\}\,.$$

Iglehart (1963a, p. 11-14) gives a rigorous justification of (3). In the special case where demand is identically distributed (3) becomes

$$f(x) = \inf_{y \geq x} \{c(y-x)+L(y) + \alpha \int_0^\infty f(y-\xi)d\Phi(\xi)\}\,.$$

If the planning horizon has only a finite number of periods then the basic functional equation is written

$$f_n(x;\Phi_i,\cdots,\Phi_{i+n-1}) = \inf_{y \geq x} \{c(y-x)+L(y;\Phi_i) + \alpha \int_0^\infty f_{n-1}(y-\xi\,;\Phi_{i+1},\cdots,\Phi_{i+n-1})d\Phi_i(\xi)\}$$

where the subscript on f refers to the number of periods remaining and f_0 is interpreted as the salvage value. Here, of course, f_n is the expected minimum discounted cost which will occur in n periods.

3. The (s,S) Policy

Turning now to one of the most widely discussed ordering policies, the (s,S) policy, we will summarize a particular selection of results concerned with this type of policy. The interest here is not so much the conditions themselves as their reasonableness as assumptions regarding the inventory process. The (s,S) policy is usually optimal, provided ordering costs are given by (1) with $K > 0$ or, more generally, when the ordering cost is concave. Additional restrictions, however, must be placed on the form of one or more of the following: demand distribution, holding costs, and/or penalty cost.

In the case of static probabilistic models Karlin (Arrow, Karlin, and Scarf, 1958, p. 109-134) has established that the (s,S) policy is optimal for $c(z)$ given by (1) with $K > 0$ under a variety of additional restrictions. If Karlin's restrictions are satisfied, the parameters in the optimal (s,S) policy are calculated as the solutions of the following set of equations

$$\frac{\partial}{\partial S} L(S, \Phi) = 0$$

and

$$L(s, \Phi) = L(S, \Phi)+K \qquad s < S.$$

Although no such considerations arise in a static model, the interpretation of what constitutes an optimal policy changes in dynamic models. To be more specific, if the parameters of the inventory process are stationary, then the optimal (s, S) parameters for a finite horizon model will be functions of the period (because of the changing length of the planning horizon). The optimal parameters for an infinite horizon model will be time independent because at the beginning of each period the future is exactly the same.

For the finite horizon dynamic model Scarf (1960) has shown that if $c(z)$ is given by (1) with $K > 0$, if the demands are identically distributed, and if $L(y; \Phi)$ is a convex function of y, having continuous second partials then the optimal policy in each period is (s, S) with the parameters being functions of the period. This result can easily be shown to hold for the case where the demands are not identically distributed as long as each $L(y; \Phi_i)$, $i = 1, 2, \cdots$, satisfies the conditions required by Scarf. There is also a version of this result which holds for discontinuous functions and therefore includes integer valued random variables (Zabel, 1962). To illustrate how the optimal parameters when m periods remain, s_m and S_m, would be calculated, consider an n period model with $m < n$ and future demand $\Phi_1, \cdots, \Phi_n$. Let

$$(4)\quad G_m(y;\Phi_{n-m+1},\cdots,\Phi_n) = cy + L(y;\Phi_{n-m+1}) + \alpha\int_0^\infty f_{m-1}(y-\xi;\Phi_{n-m+2},\cdots,\Phi_n)d\Phi_{n-m+1}(\xi)$$

then S_m is the universal minimum of $G_m(y)$ and s_m is the unique solution of

$$(5)\qquad G_m(s_m) = G_m(S_m) + K \qquad s_m < S_m .$$

Because of the generality of the assumptions, the function $G_m(y)$ may behave badly, and so far no practicable method has been forwarded for finding, in general, the minimum of $G_m(y)$.

Turning now to the infinite horizon dynamic model, Karlin (Arrow, 1958, p. 135-154) has shown for a cost function given by (1) with $K > 0$ that the (s,S) policy is optimal under the additional conditions: $h(x) = h\cdot x$, $p(x) = p\cdot x$, $p > c$, $c+h > \alpha p$, and identically distributed demand. This result holds for both the finite and infinite horizon models. In the finite horizon dynamic model the parameters are calculated using (4) and (5) as described. In the infinite horizon case the optimal (s,S) parameters are calculated using renewal theory to find the expected discounted cyclic cost and minimizing this cost to find the optimal (s,S) values. The cycle referred to is the natural inventory cycle defined by the interval from a period during which a positive amount is ordered to the next period in which a positive amount is ordered. Although the parameters are the same from period to period, the calculations can be done readily only in very special cases. Iglehart (1963b) has extended the work of Scarf (1960) to the infinite horizon dynamic inventory model with the result that, under essentially Scarf's conditions, the optimal policy is (s, S). Recently, Vienott (1966) has shown for nonidentically distributed demand and nonstationary costs that if the negatives of the one period expected costs, given by (2), are unimodal and their global minima are rising over time, then the (s, S) policy is still optimal. In the stationary case the unimodality is all that is needed, and therefore, in the stationary case, Vienott's conditions are weaker than Scarf's. Computation of these parameters is again prohibitive and most work is done with bounds.

4. Stationary Analysis

Thus we are led to the conclusion that the (s, S) policy is optimal in many circumstances. This policy, because of its form and simplicity, has strong intuitive appeal and is one of the few types of policies which lends itself to probabilistic study. We shall restrict ourselves to the study of stationary (s, S) policies when demand forms a stochastically convergent sequence. Specifically, it will be shown that under certain conditions, the optimal stationary

(s,S) policy is the same for both a sequence of demands (ξ_i), converging in distribution to ξ, and the sequence formed using ξ each period.

Let x_n represent the stock at the end of period n when an (s,S) policy is applied. It will be shown that under certain restrictions the distribution of x_n approaches a limiting distribution as n becomes large, and that this limit distribution is the same for both types of demand sequences. This limiting or stationary distribution is a function of the limiting demand distribution and the (s,S) parameters. The benefits of such analysis, even though it does not necessarily give the parameters which minimize the expected discounted cost, are: it provides a policy which is easily computed; it reduces the inventory problems to a study of the associated renewal process; and it provides a method of studying the inventory process which is independent of the cost structures. Thus stationary analysis gives a practicable way of studying variation in optimal policy with changes in the cost structure, since a specific cost structure can be superimposed on the problem after the analysis.

It is the derivation of the stationary distribution of x_n which introduces a second field of interest, in which new results will be established. Two approaches to finding the stationary distribution are used by Karlin. One is to show that the distribution is a fixed point of the linear transformation which relates the final inventory in one period to that in the next and to solve the associated integral equation (Arrow, 1958, p. 223-269). The second method is to treat the problem as one in renewal theory and use renewal theoretic limit theorems to derive the desired limiting distribution (Arrow, 1958, p. 270-297). These methods overlap in that the integral equation is a renewal type equation. Both methods have been applied to problems where the demand from period to period is identically distributed. When the demand distributions form a convergent sequence, the first method appears formidable, but due to recent results in renewal theory, concerned with sequences of non-identically distributed random variables, renewal theory can be used to find this stationary distribution for convergent demand sequences. This approach to finding the stationary inventory distribution employs renewal theory to such a depth that a separate introduction to renewal theory, including a summary of pertinent results, will be given in Chapter 3.

Because we are primarily concerned with integer valued random variables, all integrals will signify Lebesgue-Stieltjes integration. Also, when dealing with a sequence of demand distributions $\{\Phi_i\}$, the notation $\Phi_i^{(n)}(x)$ denotes the convolution of $\Phi_i, \Phi_{i+1}, \cdots, \Phi_{i+n-1}$. In particular $\Phi_i^{(0)}(x)$ will be understood as the unit step function. Other notational conventions will be described where they are used. We will be concerned throughout with the comparison of properties of a convergent sequence of random variables to those of the sequence formed using the limiting random variable. These sequences will be described respectively as a renewal sequence and the associated renewal process.

When speaking of the convergence of a sequence of random variables, we mean convergence in distribution.

Definition: A sequence of random variables $\{\xi_i\}$, with distribution function $\{\Phi_i\}$, converges in distribution to the random variable ξ, with distribution function Φ, if $\Phi_i(x) \rightarrow \Phi(x)$ at continuity points of $\Phi(x)$.

For the integer valued random variables with which we will be concerned, convergence in distribution implies uniform convergence on every finite interval. Throughout this paper we make frequent reference to a "limiting" random vairable. This is to be interpreted as indicating any random variable which has the limiting distribution function.

II. OPTIMAL STATIONARY INVENTORY POLICY

As mentioned in the introduction, we are concerned with the analysis of the stationary behavior of the inventory level and costs associated with an (s,S) policy, when the future demands form a stochastically convergent sequence. If the expected cost in period n with stock level x_n at the end of the period is $V(x_n)$, we may wish (Arrow, 1958, p. 16-36) to choose the (s,S) policy which minimizes the expected average long run cost

$$\lim_{n \to \infty} \frac{V(x_n)}{n},$$

or equivalently minimizes a weighted discounted present cost

$$\lim_{a \to 1} (1-a)\left[\sum_{i=1}^{\infty} a^{(i-1)} V(x_i)\right].$$

The explicit regularity conditions to be imposed on the sequence of demand distributions are given in the following three definitions.

Definition: A sequence of nonnegative, independent, integer valued random variables $\{\xi_i\}$ will be said to satisfy condition (A) if it converges in distribution to a random variable ξ, and if the sequence $\mu_i = E(\xi_i)$ forms a positive, uniformly bounded sequence of real numbers converging to the positive real number $\mu = E(\xi)$.

Definition: A convergent sequence of independent, integer valued random variables is said to satisfy condition (B1) if the limiting random variable is of period no greater than one, and to satisfy condition (B2) if there exists a distribution function $\hat{\Phi}(x)$ such that

$\Phi_i(x) \geq \hat{\Phi}(x)$ for all x and i where $\int_{-\infty}^{\infty} |x| \, d\hat{\Phi}(x) < \infty$.

Sequences satisfying conditions (B1) and (B2) will be said to satisfy condition (B).

Definition: A convergent sequence of independent, integer valued random variables is said to satisfy condition (C) if the limit random variable has one of the properties: $P(\xi = 0) > 0$, or $P(\xi > n) > 0$ for all positive integers n.

The sequence of random variables $\{\xi_i\}$ defined by

$$P(\xi_i = n) = \begin{cases} 1/2^{n+1} & 0 \leq n \leq i \\ 1/2^{i+1} & n = i+1 \\ 0 & n > i+1 \end{cases}$$

converges in distribution to the random variable ξ defined by

$$P(\xi = n) = 1/2^{n+1}, \qquad n \geq 0 .$$

This is an example of a sequence of random variables satisfying conditions (A), (B), and (C). Examples of sequences of random variables violating these conditions will be given later in this chapter.

1. Optimal Stationary (s, S) Policy

As mentioned in the introduction it will be shown that, under certain restrictions, the stationary inventory levels are identical for both a convergent demand sequence and the associated demand process. The demands not being identically distributed cause the single period costs to be functions of the period; if the single period costs converge, then it seems logical that the optimal stationary (s, S) policies might be identical for both types of sequences.

We consider an order function given by (1.1), linear penalty cost, and linear holding cost. In the following analysis it becomes apparent that other assumptions concerning the holding and penalty costs would lead to the same result. We now state the principal theorem concerning the optimal stationary inventory policy.

Theorem 1: If a sequence $\{\Phi_i\}$ of integer valued demands satisfies conditions (A), (B), and (C); if the holding and penalty costs are linear; and if $c(z)$ is given by (1.1) then the optimal stationary

(s,S) policy for the inventory model

$$f(x;\Phi_1,\Phi_2,\cdots) = \inf_{y\geq x}\ \{c(y-x)+L(y;\Phi_1)+\alpha\int_0^\infty f(y-\xi;\Phi_2,\Phi_3,\cdots)d\Phi_1(\xi)\}$$

is the same as that for the model

$$f(x) = \inf_{y\geq x}\ \{c(y-x)+L(y)+\alpha\int_0^\infty f(y-\xi)d\Phi(\xi)\}.$$

The proof of this theorem is quite lengthy and depends on several preliminary theorems and lemmas. We now digress to develop these results, and will return to the proof of Theorem 1 in Chapter 4. The following gives a brief sketch of the steps which are taken.

Under the assumptions of Theorem 1 the optimal stationary (s,S) policy, when the demand is identically distributed, can be calculated in two steps. First, find the limiting distribution $F(x)$ of the stock x_n on hand at the end of the n^{th} period with an arbitrary (s,S) policy. Then, using this limiting distribution and the single period cost structure $C(x;\Phi)$, minimize

$$\int_{-\infty}^{S} C(x;\Phi)dF(x)$$

with respect to s and S, where

$$(1)\quad C(x;\Phi) = \begin{cases} K+c(S-x)+p\int_S^\infty (\xi-S)d\Phi(\xi) + h\int_0^S (S-\xi)d\Phi(\xi) & x < s \\ p\int_x^\infty (\xi-x)d\Phi(\xi) + h\int_0^x (x-\xi)d\Phi(\xi) & s \le x \le S \end{cases}.$$

We may calculate the average long run cost this way because

$$(2)\quad \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{S} C(x;\Phi)dF_i(x) = \int_{-\infty}^{S} C(x;\Phi) \lim_{n\to\infty}\left[\frac{\sum_{i=1}^{n} dF_i(x)}{n}\right]$$

$$= \int_{-\infty}^{S} C(x;\Phi)dF(x).$$

This sequence of equalities is easily justified and will follow from applying Theorem 1 to a sequence of identically distributed demands.

For the model where future demands are given by $\{\Phi_i\}$ we will show

$$(3)\quad \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{S} C(x;\Phi_i)dF_{i-1}(x) = \int_{-\infty}^{S} C(x;\Phi)dF(x).$$

This proof consists of showing that as $n\to\infty$ $F_n(x)\to F(x)$ and $C(x;\Phi_n)\to C(x;\Phi)$ uniformly in x on any finite interval.

2. Conditions (A), (B), and (C)

Turning to a discussion of conditions (A), (B), and (C) it will be shown that for general linear costs and uniformly bounded (s,S) policies, condition (A) implies $C(x; \Phi_n) \to C(x; \Phi)$ as $n \to \infty$. Linear costs simplify the exposition and are common assumptions concerning cost. Under these assumptions the only question which remains is what additional restrictions on the sequence $\{\xi_i\}$ are sufficient to insure that $\lim_{n\to\infty} F_n(x)$, for the sequence $\{\xi_i\}$, exists and is equal to the limiting distribution of the ending stock for the identically distributed sequence. Let $N(n)$ be the number of random variables that can be added in sequence, until adding one more would exceed n. Then the crux of the problem is finding sufficient conditions that

$$\lim_{n \to \infty} [\ E(N(n+1)) - E(N(n))\ ] = \frac{1}{\mu} . \tag{4}$$

This problem is the core of renewal theory. Smith (1961) and Williamson (1965) have considered (4) for sequences of nonidentically distributed random variables. The following will rely heavily on their arguments. We will consider only those results of Williamson's paper concerned with nonnegative, independent, integer valued random variables since they are our primary concern.

There are two main types of restrictions discussed in this context: one has to do with the lattice structure of the random variables; the other involves the uniform behavior of the tails of their distribution functions or the probability that they take on widely separated values. To guarantee the desired lattice structure, Williamson requires that there exist a subsequence $\{\xi_{i_k}\}$ of $\{\xi_i\}$, a sequence $\{a_k\}$ of real numbers, and positive constants M and ω such that: for all k,

$$(5\text{-}1) \qquad P(|\xi_{i_k} - a_k| < M) \geq \omega ;$$

for any d, $0 < d < \pi$, there exists $u > 0$, $u = u(d)$, for which

$$(5\text{-}2) \qquad \sup_{k \geq 1} |\rho_k(t)| \leq 1 - u$$

for all t satisfying $d \leq |t| < \pi$ where

$$\rho_k(t) = \int_0^\infty e^{itx} dP(\xi_{i_k} < x \mid |\xi_{i_k} - a_k| < M);$$

and there exists some positive constant B such that for all n and j

$$(5\text{-}3) \qquad \sum_{k=j}^{\infty} P(\sum_{i=j}^{k} \xi_i = n) \leq B.$$

For the random variables to have the desired uniform behavior of their distribution functions, Williamson requires that there exist positive constants α and N such that one of the following sets of conditions is satisfied:

(6-1) $$\lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} \mu_{n+j} = \alpha$$

uniformly for all $n \geq N$, for all x and all $n \geq N$ there exists $\hat{\Phi}(x)$ such that $\Phi_n(x) \geq \hat{\Phi}(x)$, and

(6-2) $$\int_0^{\infty} x \, d\hat{\Phi}(x) < \infty \; ;$$

or

(7-1) $$\lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} \mu_{n+j} = \alpha \; ,$$

(7-2) $$\lim_{b \to \infty} \int_b^{\infty} x \, d\Phi_n(x) = 0$$

uniformly for all $n > N$, and there exists positive constants c and z such that for all k,

(7-3) $$P(\xi_k \geq z) \leq \frac{c}{k} \; ;$$

or

$$\text{(8-1)} \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \int_0^N x \, d\,\Phi_{k+j}(x) = a$$

uniformly in k, and

$$\text{(8-2)} \qquad \sum_{k=1}^{\infty} P(\xi_k \geq N) < \infty .$$

Concerning the lattice structure of sequences of random variables satisfying (A), condition (B1) is more general than those of Williamson since it allows random variables of the form $m+kd$ where $d \geq 2$, g.c.d. $(m,d) = 1$, and $k = 1, 2, 3, \cdots$; such sequences would be ruled out by (5-2). Conditions (A) and (B2) constitute a special case of (6) — the first of Williamson's conditions for the uniform behavior of the distribution functions.

The determination of necessary conditions, on a sequence of nonidentically distributed random variables, in order that (3) holds, appears extremely complicated. From the following example it appears that some condition like condition (B1) is necessary. Let

$$P(\xi_n = 2) = 1, \quad n = 1, 2, 3, \cdots$$

then $\{\xi_n\}$ does not satisfy condition (B1) and $N(n) = [\frac{n}{2}]$. From which it follows that (4) is not satisfied although

$$\lim_{n \to \infty} \{E(N(n+2)) - E(N(n))\} = \frac{2}{E(\xi_1)} = 1.$$

Because of the complexity of the problem, even discussing the necessity of condition (B2) is difficult. We shall only consider whether some sequence exists which satisfies conditions (A) and (B1) and does not satisfy (4) or, at least, does not satisfy any of the three sets of sufficient conditions (equations (6), (7), and (8)) given by Williamson.

Given a sequence of random variables which satisfies conditions (A) and (B1), Williamson's results say that for (4) to hold it is sufficient that either (6-2), (7-3), or (8-2) hold, provided the random variables are not of the form $m+kd$, g.c.d. $(m,d) = 1$. Consider the random variable ξ, defined by

$$P(\xi = i) = 1/2^{i+1}, \quad i \geq 0 .$$

Let the sequence of random variables $\{\xi_n\}$ be defined by

$$(9) \qquad P(\xi_n = i) = \begin{cases} 1/2 - \dfrac{1}{(n+1)\ell n(n+3)} & i = 0 \\ 1/2^{i+1} & i \neq n, \quad i > 0 \\ 1/2^{n+1} + \dfrac{1}{(n+1)\ell n(n+3)} & i = n \end{cases}$$

The sequence $\{\xi_n\}$ converges in distribution to ξ and satisfies conditions (A) and (B1). Letting

$$\hat{\Phi}(x) = \inf_{i \geq 1} [P(\xi_i \leq x)]$$

it follows that

$$\int_0^{\infty} x d\hat{\Phi}(x) = \sum_{n=1}^{\infty} n\left[\frac{1}{2^{n+1}} + \left\{\frac{1}{(n+1)\ell n(n+3)} - \frac{1}{(n+2)\ell n(n+4)}\right\}\right]$$

$$\geq \sum_{n=1}^{\infty} \frac{1}{(n+2)\ell n(n+3)}$$

which diverges. Since any other uniform lower bound on the sequence $\{\xi_n\}$ would have a larger mean, it follows that (6-2) is not satisfied and it is easily seen that neither are (7-3) or (8-2). This, then, is an example of a sequence which satisfies conditions (A) and (B1) and does not satisfy any of the sufficient conditions given by Williamson. This still provides no answer as to what conditions are necessary for (4) or even whether conditions (A) and (B1) might not be sufficient. The sequences which might be constructed to look for counter examples to the latter questions, such as (9), appear impossible to analyze.

Condition (C) is needed in addition to (A) and (B) in order that convolutions of sequences of random variables satisfying conditions (A), (B), and (C) will again satisfy conditions (A) and (B). Some such condition is necessary since conditions (A) and (B) are not sufficient as is illustrated by the sequence $\{\xi_n\}$ defined by $P(\xi_n=1)=1$, for all n. This sequence satisfies conditions (A) and (B) while the sequence x_n; $x_n = \xi_{2n-1} + \xi_{2n}$, does not. It may be true that, taking into account the possible changes in the period of the convoluted random variables, a different interpretation of the limit theorems in Chapter 3 might eliminate the need for such a condition. For the present, however, we will dismiss such investigations and use condition (C).

III. RESULTS IN RENEWAL THEORY

A sequence of independent random variables is called a <u>renewal sequence</u>. If a renewal sequence consists of identically distributed random variables it is called a <u>renewal process</u>. We may specify a renewal sequence by its corresponding sequence of distribution functions $\{\Phi_n\}$. Since we are concerned with renewal sequences which constitute convergent sequences of random variables, we will define the <u>associated renewal process</u> as the renewal process formed using the limiting random variable. A renewal sequence is termed discrete if there exists a real number $d > 0$ such that $\sum_{k=-\infty}^{\infty} P(\xi_n = kd) = 1$ for all n, if there is no such d then the sequence is termed continuous. Since we are concerned with discrete sequences only we will assume (without loss of generality) $d = 1$ and will derive here those results concerned with discrete random variables. Finally, we denote the partial sums of $\{\xi_n\}$ by S_n, that is $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$.

Given a renewal sequence $\{\xi_n\}$ and a real number x, define

$$N(x) = \sum_{k=1}^{\infty} u(x-S_k)$$

where $u(x)$ is the unit step function. If the random variables $\{\xi_n\}$ are nonnegative then $N(x)$ is that integer for which $S_{N(x)} \leq x < S_{N(x)+1}$. Renewal theory is concerned with properties of the random variable $N(x)$ and random variables defined in terms of $N(x)$. In particular, the <u>renewal function</u> $H_1(x)$ for a renewal sequence $\{\Phi_n\}$ is defined by[2]

$$H_1(x) = E(N(x)).$$

Since, for nonnegative random variables,

$$E(N(x)) = \sum_{k=1}^{\infty} k[\Phi_1^{(k)}(x) - \Phi_1^{(k+1)}(x)]$$

it follows that

$$H_1(x) = \sum_{k=1}^{\infty} \Phi_1^{(k)}(x) . \tag{1}$$

Considering the random variables $\{\xi_n\}$ as observations of the life span of components which wear out and must be immediately replaced then $H_1(x)$ is the expected number of replacements in the time

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[1] Because we will be concerned with renewal sequences formed from $\{\Phi_n\}$ by dropping the first $(i-1)$ terms we will denote by $H_i(x)$ the renewal function associated with $\Phi_i, \Phi_{i+1}, \cdots$.

interval $(0, x]$, given that a replacement was made at time zero. If $\{\xi_n\}$ is a nonnegative integer valued renewal process, it follows from a theorem of Erdos, Feller, and Pollard (1949) that

$$\lim_{n \to \infty} [H(n+1) - H(n)] = \frac{1}{E(\xi_1)} \tag{2}$$

for suitably restricted $\{\xi_n\}$. This result is called the <u>key renewal theorem</u> and has been shown to hold for a wide variety of independent, identically distributed sequences of random variables. Another random variable associated with renewal theory and with which we will be concerned is the <u>excess variable</u> defined by

$$\eta(x) = S_{N(x)+1} - x \,.$$

Recalling the interpretation of the renewal function in terms of replacing worn out components, the excess variable is the amount by which the lifespan of the component in use at time x, exceeds time x. For a nonnegative renewal process $\{\xi_n\}$ with $0 < E(\xi_1) < \infty$ a well known (Smith, 1954) limit theorem states:

$$\lim_{x \to \infty} P(\eta(x) \leq y) = \int_0^y \frac{1 - \Phi(u)}{E(\xi_1)} \, du$$

where $\Phi(x)$ is the distribution of ξ_1. For integer valued random variables this becomes

(3) $$\lim_{n \to \infty} P(\eta(n) \leq k) = \int_0^k \frac{1-\Phi(u)}{E(\xi_1)}\, du = \frac{1}{E(\xi_1)} \sum_{i=0}^{k-1} (1-\Phi(i)).$$

Aside from the slight modification of allowing the first random variable to be distributed differently from the rest of the sequence, most results in renewal theory are concerned with renewal processes.

1. The Key Renewal Theorem

We now turn to the consideration of the key renewal theorem. Consider a nonnegative, integer valued renewal sequence $\{\xi_n\}$ which satisfies conditions (A) and (B). It will be shown that

(4) $$\lim_{n \to \infty} [H_i(n+1) - H_i(n)] = \frac{1}{\mu}$$

uniformly in i. Under these restrictions this result complements those of Williamson (1965). In part, the proof will use an argument based on the following theorem of Erdos, Feller, and Pollard (1949).

Theorem (EFP): Let p_k be a sequence of nonnegative numbers such that $\sum_{k=0}^{\infty} p_k = 1$ and let $m = \sum_{k=0}^{\infty} kp_k < \infty$. Suppose that

$P(x) = \sum_{k=0}^{\infty} p_k x^k$ is not a power series in x^t for any $t > 1$. Then $1-P(x)$ has no zeroes inside the circle $|x| < 1$ and the series $u(x) = \frac{1}{1-P(x)} = \sum_{k=0}^{\infty} u_k x^k$ has the property that $\lim_{k \to \infty} u_k = \frac{1}{m}$.

The key renewal theorem is derived from Theorem (EFP) by considering the renewal process whose random variable is such that $P(\xi = n) = p_n$ and observing that $H(n+1)-H(n) = u_{n+1}$ for $n \geq 0$.

We will prove a version of the key renewal theorem by showing that under conditions (A) and (B) a convergent integer valued renewal sequence has

$$\lim_{i \to \infty} H_i(n+1)-H_i(n) = u_{n+1},$$

where u_n is formed from the associated renewal process using

$u_0 = \frac{1}{1-p_0}$ and $u_n = \sum_{i=0}^{n} p_i u_{n-i}$. This definition of u_n is the same as that in Theorem (EFP). For a renewal sequence $\{\xi_n\}$ satisfying conditions (A) and (B) the following notational conventions will be used where $\Phi(x)$ is the distribution of the limit random variable and $\hat{\Phi}(x)$ is that distribution function guaranteed by condition (B2):

$$p_n = \Phi(n) - \Phi(n-1) \quad ,$$

$$p_n^i = \Phi_i(n) - \Phi_i(n-1) \; ,$$

$$\hat{p}_n = \hat{\Phi}(n) - \hat{\Phi}(n-1) \; .$$

Lemma 1: If $\{u_n^i\}$ is defined recursively by

$$u_0^i = 1 + \sum_{j=0}^{\infty} \prod_{\ell=0}^{j} p_0^{i+\ell}$$

and for $n \geq 1$

$$u_n^i = \sum_{j=0}^{n-1} p_{n-j}^i u_j^{i+1} + \sum_{\ell=0}^{\infty} \Big[\prod_{m=0}^{\ell} p_0^{i+m} \Big] \Big[\sum_{j=0}^{n-1} p_{n-j}^{i+\ell+1} u_j^{i+\ell+2} \Big] \; ,$$

and if conditions (A) and (B) hold, then for $n \geq 1$

(5) $$H_i(n) - H_i(n-1) = u_n^i \; .$$

Proof: From (1) it follows that

(6) $$H_i(n) - H_i(n-1) = \sum_{j=0}^{\infty} \Phi_i^{(j)}(n) - \sum_{j=0}^{\infty} \Phi_i^{(j)}(n-1) \; .$$

A consequence of conditions (A) and (B) being satisfied is $p_0 < 1$

and $p_0^i \to p_0$ as $i \to \infty$, which in turn implies $H_i(n)$ is finite for all finite n. Since the series in (6) have only positive terms they can be subtracted termwise so that

$$H_i(n) - H_i(n-1) = \sum_{j=0}^{\infty} [\Phi_i^{(j)}(n) - \Phi_i^{(j)}(n-1)] = \sum_{j=0}^{\infty} P(\sum_{\ell=0}^{j} \xi_{i+\ell} = n).$$

This latter expression for $H_i(n) - H_i(n-1)$ will be used many times without specific reference in what follows. When $n = 1$;

$$H_i(1)-H_i(0) = \sum_{j=0}^{\infty} P(\sum_{\ell=0}^{j} \xi_{i+\ell} = 1)$$

$$= P(\xi_i=1) + \sum_{j=1}^{\infty} \sum_{\ell=0}^{j} P(\xi_{i+\ell}=1)P(\sum_{m=0,\, m\neq\ell}^{j} \xi_{i+m} = 0)$$

$$= p_1^i[1+\sum_{j=1}^{\infty} P(\sum_{m=1}^{j} \xi_{i+m}=0)] + \sum_{\ell=1}^{\infty} \sum_{j=\ell}^{\infty} P(\xi_{i+\ell}=1)P(\sum_{m=0,m\neq\ell}^{j} \xi_{i+m}=0)$$

$$= p_1^i u_0^{i+1} + \sum_{\ell=1}^{\infty} p_1^{i+\ell} [\prod_{r=0}^{\ell-1} p_0^{i+r}] [1 + \sum_{j=\ell+1}^{\infty} [\prod_{m=\ell+1}^{j} p_0^{i+m}]]$$

$$= p_1^i u_0^{i+1} + \sum_{\ell=1}^{\infty} p_1^{i+\ell} [\prod_{r=0}^{\ell-1} p_0^{i+r}] u_0^{i+\ell+1}$$

$$= u_1^i .$$

Thus (5) holds when $n = 1$, uniformly in i. Suppose (5) is true for all n such that $1 \leq n \leq k$, uniformly in i, then

$$H_i(k+1)-H_i(k) = \sum_{j=0}^{\infty} P(\sum_{\ell=0}^{j} \xi_{i+\ell} = k+1)$$

$$= P(\xi_i=k+1)+\sum_{j=1}^{\infty}[\sum_{\ell=1}^{k+1} P(\xi_i=\ell)P(\sum_{m=1}^{j} \xi_{i+m}=k+1-\ell)]$$

$$+\sum_{j=1}^{\infty}\prod_{m=0}^{j-1} P(\xi_{i+m}=0)P(\xi_{i+j}=k+1)$$

$$+\sum_{j=2}^{\infty}\sum_{m=0}^{j-2}\prod_{r=0}^{m} P(\xi_{i+r}=0)[\sum_{\ell=1}^{k+1} P(\xi_{i+m+1}=\ell)P(\sum_{S=m+2}^{j}\xi_{i+S}=k+1-\ell)]$$

$$= p^i_{k+1}+\sum_{\ell=1}^{k+1} p^i_\ell[\sum_{j=1}^{\infty} P(\sum_{m=1}^{j} \xi_{i+m}=k+1-\ell)] +\sum_{j=1}^{\infty}[\prod_{m=0}^{j-1} p_0^{i+m}]p^{i+j}_{k+i}$$

$$+\sum_{m=0}^{\infty}\sum_{j=m+2}^{\infty}[\prod_{r=0}^{m} p_0^{i+r}][\sum_{\ell=1}^{k+1} p_\ell^{i+m+1}P(\sum_{S=m+2}^{j}\xi_{i+S}=k+1-\ell)]$$

$$= p^i_{k+1}[1+\sum_{j=1}^{\infty} P(\sum_{m=1}^{j}\xi_{i+m}=0] +\sum_{\ell=1}^{k} p^i_\ell u^{i+1}_{k+1-\ell}+\sum_{j=1}^{\infty}[\prod_{r=0}^{j-1} p_0^{i+r}]p^{i+j}_{k+1}$$

$$+\sum_{m=0}^{\infty}[\prod_{r=0}^{m} p_0^{i+r}][\sum_{j=m+2}^{\infty} p^{i+m+1}_{k+1}P(\sum_{S=m+2}^{j}\xi_{i+S}=0)]$$

$$+\sum_{m=0}^{\infty}\Big[\prod_{r=0}^{m} p_0^{i+r}\Big]\Big[\sum_{\ell=1}^{k} p_\ell^{i+m+1}\sum_{j=m+2}^{\infty} P\Big(\sum_{S=m+2}^{j}\xi_{i+S}=k+1-\ell\Big)\Big]$$

$$=\sum_{\ell=1}^{k+1} p_\ell^{i}\, u_{k+1-\ell}^{i+1}+\sum_{m=0}^{\infty}\Big[\prod_{r=0}^{m} p_0^{i+r}\Big]\, p_{k+1}^{i+m+1}\Big[1+\sum_{j=m+2}^{\infty} P\Big(\sum_{S=m+2}^{j}\xi_{i+S}=0\Big)\Big]$$

$$+\sum_{m=0}^{\infty}\Big[\prod_{r=0}^{m} p_0^{i+r}\Big]\Big[\sum_{\ell=1}^{k} p_\ell^{i+m+1}\; u_{k+1-\ell}^{i+m+2}\Big]$$

$$=\sum_{\ell=1}^{k+1} p_\ell^{i}\, u_{k+1-\ell}^{i+1}+\sum_{m=0}^{\infty}\Big[\prod_{r=0}^{m} p_0^{i+r}\Big]\Big[\sum_{\ell=1}^{k+1} p_\ell^{i+m+1}\; u_{k+1-\ell}^{i+m+2}\Big]$$

$$= u_{k+1}^{i}\,,$$

which completes an inductive proof of Lemma 1.

If a renewal function $H(x)$ is such that for any $\epsilon > 0$ there exists a finite $d(\epsilon) > 0$ with $H(x+\epsilon)-H(x) < d(\epsilon)$ uniformly in x, then $H(x)$ is said to be of uniformly bounded variation. Smith (1961) has established this property for quite general renewal sequences. Although Smith's result includes the special type of renewal sequences considered here, Lemma 2 gives a new proof for this special case and also establishes that under conditions (A) and (B) the bound on $H_i(n)-H_i(n-1)$ is uniform in i.

Lemma 2: If a renewal sequence satisfies conditions (A) and (B) then there exists β such that

$$H_i(n) - H_i(n-1) \le \beta \tag{7}$$

for all $n \ge 1$, $i \ge 1$.

Proof: Condition (A) implies $\mu > 0$, and that there exists an M such that $p_0^i \le (1+p_0)/2 < 1$ for $i \ge M$. Therefore

$$u_0^i = 1 + \sum_{j=0}^{\infty} \prod_{m=0}^{j} p_0^{i+m} \le M + \frac{1}{\frac{1}{2} - \frac{p_0}{2}} .$$

Let $\beta = M + 1/(\frac{1}{2} - \frac{p_0}{2})$. We have already seen $u_0^i \le \beta$ for all $i \ge 1$. Suppose $u_n^i \le \beta$ uniformly in i for all $0 \le n \le k-1$. Then

$$u_k^i = \sum_{j=0}^{k-1} p_{k-j}^i u_j^{i+1} + \sum_{\ell=0}^{\infty} [\prod_{m=0}^{\ell} p_0^{i+m}] [\sum_{j=0}^{k-1} p_{k-j}^{i+\ell+1} u_j^{i+\ell+2}]$$

$$\le \beta [\sum_{j=0}^{k-1} p_{k-j}^i + \sum_{\ell=0}^{\infty} \{[\prod_{m=0}^{\ell} p_0^{i+m}] \sum_{j=0}^{k-1} p_{k-j}^{i+\ell+1} \}$$

$$\le \beta [1 - p_0^i + \sum_{\ell=0}^{\infty} [\prod_{m=0}^{\ell} p_0^{i+m}] (1-p_0^{i+\ell+1})$$

$$\le \beta [1 - p_0^i + \sum_{\ell=0}^{\infty} \prod_{m=0}^{\ell} p_0^{i+m} - \sum_{\ell=0}^{\infty} \prod_{m=0}^{\ell+1} p_0^{i+m}] \le \beta .$$

This completes an inductive proof of the lemma.

Next we prove a key renewal theorem essential to all that follows.

Theorem 1: If a renewal sequence $\{\xi_n\}$ satisfies conditions (A) and (B) then, uniformly in i

$$\lim_{n \to \infty} H_i(n) - H_i(n-1) = \frac{1}{\mu} .$$

Proof: By Lemma 2 there is a β such that $\frac{1}{\mu} < \beta$ and $u_n^i < \beta$ uniformly in i for all $n \geq 0$. Theorem (EFP) guarantees that given $\epsilon_1 > 0$ there is an $N_1(\epsilon_1)$ such that

$$(8) \qquad \left| u_n - \frac{1}{\mu} \right| < \frac{\epsilon_1}{4}$$

for $n \geq N_1$. From condition (B) there exists an $N_2(\epsilon_1)$ such that

$$(9) \qquad \sum_{n=N_2}^{\infty} n\widehat{p}_n < \frac{\epsilon_1}{4\beta^2} .$$

It is also possible to select $M(N_1 + N_2, \epsilon_1)$ such that

$$(10) \qquad |u_n^i - u_n| < \frac{\epsilon_1}{4}$$

for $i > M$, $0 \leq n \leq N_1 + N_2$. To justify this last statement observe that given any ϵ with $0 < \epsilon < (1-p_0)/2$ then for all sufficiently large i

(11) $$|p_n^i - p_n| < \epsilon ,$$

thus

(12) $$|u_0^i - u_0| \leq \max \left| \frac{1}{1-p_0 \pm \epsilon} - \frac{1}{1-p_0} \right| < \frac{\epsilon}{(1-p_0-\epsilon)^2}$$

and

(13) $$|u_n^i - u_n| \leq \max_{m \geq i} \left| \frac{p_n^m + p_{n-1}^m u_1^{m+1} + \cdots + p_1^n u_{n-1}^{m+1}}{1-p_0 \pm \epsilon} - \frac{p_n + p_{n-1} u_1 + \cdots + p_1 u_{n-1}}{1-p_0} \right| .$$

For any finite n, an inductive application of (11) and (12) to (13) shows that the right side of (13) can be made arbitrarily small by choosing i sufficiently large.

Letting $N = N_1 + N_2$ it follows inductively that

(14) $$\left| u_{N+n}^i - \frac{1}{\mu} \right| \leq \frac{\epsilon_1}{2} + 2\beta \sum_{j=0}^{n-1} u_j^i \, P(\hat{\xi} > N_2 + n - j + 1)$$

holds for all $n \geq 1$, $i > M$. To prove (14) let $n = 1$, then using Lemma 1, Lemma 2, and (10)

$$|u_{N+1}^{i} - \frac{1}{\mu}| \le |\sum_{j=0}^{N_1-1} p_{N+1-j}^{i} u_j^{i+1} + \sum_{\ell=0}^{\infty} (\prod_{m=0}^{\ell} p_0^{i+m})[\sum_{j=0}^{N_1-1} p_{N+1-j}^{i+\ell+1} u_j^{i+\ell+2}] |$$

$$+ |\sum_{j=N_1}^{N} p_{N+1-j}^{i} u_j^{i+1} + \sum_{\ell=0}^{\infty} (\prod_{m=0}^{\ell} p_0^{i+m})[\sum_{j=N_1}^{N} p_{N+1-j}^{i+\ell+1} u_j^{i+\ell+2}] - \frac{1}{\mu} |$$

$$\le \beta \; P(\hat{\xi} \ge N_2+2) \; [1 + \sum_{\ell=0}^{\infty} \prod_{m=0}^{\ell} p_0^{i+m}]$$

$$+ \max \{ (\frac{1}{\mu} + \frac{\epsilon_1}{2})[1 - p_0^{i} + \sum_{\ell=0}^{\infty} (\prod_{m=0}^{\ell} p_0^{i+m})(1 - p_0^{i+\ell+1})] - \frac{1}{\mu} \; ;$$

$$\frac{1}{\mu} - (\frac{1}{\mu} - \frac{\epsilon_1}{2})[1 - p_0^{i} - P(\hat{\xi} \ge N_2+2)$$

$$+ \sum_{\ell=0}^{\infty} (\prod_{m=0}^{\ell} p_0^{i+m})[1 - p_0^{i+\ell+1} - P(\hat{\xi} \ge N_2+2)] \;] \}$$

$$\le \beta \; P(\hat{\xi} \ge N_2+2) u_0^{i} + \max \{ \frac{\epsilon_1}{2} , \frac{\epsilon_1}{2} + \beta \; P(\hat{\xi} \ge N_2+2) u_0^{i} \}$$

$$\le \frac{\epsilon_1}{2} + 2\beta \; P(\hat{\xi} \ge N_2 + 2) \; u_0^{i} \; .$$

Suppose there exists $k > 1$ such that for all $n \in [1, k-1]$, $i > M$ (14) holds, then

$$|u^i_{N+k}-\frac{1}{\mu}| \le |\sum_{j=0}^{N_1-1} p^i_{N+k-j}\, u^{i+1}_j + \sum_{\ell=0}^{\infty} (\prod_{m=0}^{\ell} p_0^{i+m})[\sum_{j=0}^{N_1-1} p^{i+\ell+1}_{N+k-j}\, u^{i+\ell+2}_j] \,|$$

$$+ |\sum_{j=N_1}^{N+k-1} p^i_{N+k-j}\, u^{i+1}_j + \sum_{\ell=0}^{\infty} (\prod_{m=0}^{\ell} p_0^{i+m})[\sum_{j=N_1}^{N+k-1} p^{i+\ell+1}_{N+k-j}\, u^{i+\ell+2}_j] - \frac{1}{\mu}|.$$

Letting

$$(15)\qquad Q_k(i) = \sum_{n=1}^{k-1} \sum_{j=0}^{k-n-1} 2\beta\, [p^i_n\, u^{i+1}_j + \sum_{\ell=0}^{\infty} (\prod_{m=0}^{\ell} p_0^{i+m}) p_n^{i+\ell+1}]\, P(\widehat{\xi} \ge N_2+k-n-j+1)$$

then

$$|u^i_{N+k} - \frac{1}{\mu}| \le \beta\, u^i_0 P(\widehat{\xi} \ge N_2+k+1) + Q_k(i)$$

$$+ \max \begin{cases} (\frac{1}{\mu}+\frac{\epsilon_1}{2})[1-p^i_0 + \sum_{\ell=0}^{\infty} (\prod_{m=0}^{\ell} p^{i+m})[1-p_0^{i+\ell+1}]] - \frac{1}{\mu} \\ \\ \frac{1}{\mu} - (\frac{1}{\mu} - \frac{\epsilon_1}{2})[(1-p^i_0 - P(\widehat{\xi} \ge N_2+k+1))\, u^i_0] \end{cases}$$

$$\le \frac{\epsilon_1}{2} + 2\beta\, u^i_0\, P(\widehat{\xi} \ge N_2+k+1) + Q_k(i).$$

Letting $q = n+j$ in (15) we have

$$|u^i_{N+k} - \frac{1}{\mu}| \le \frac{\epsilon_1}{2} + 2\beta\, P(\hat{\xi} \ge N_2 + k + 1)\, u^i_0$$

$$+ 2\beta \sum_{q=1}^{k-1} P(\hat{\xi} \ge N_2 + k - q + 1) \sum_{r=1}^{q} \left(p^i_r u^{i+1}_{r-q} + \sum_{\ell=0}^{\infty} \left(\prod_{m=0}^{\ell} p^{i+m}_0\right) p^{i+\ell+1}_r u^{i+\ell+2}_{r-q}\right)$$

$$\le \frac{\epsilon_1}{2} + 2\beta \sum_{q=0}^{k-1} u^i_q\, P(\hat{\xi} \ge N_2 + k - q + 1)$$

which completes the inductive proof of (14).

Now since

$$|u^i_{N+n} - \frac{1}{\mu}| \le \frac{\epsilon_1}{2} + 2\beta \sum_{j=0}^{n-1} u^i_j\, P(\hat{\xi} > N_2 + n - j + 1)$$

$$\le \frac{\epsilon_1}{2} + 2\beta^2 \sum_{j=2}^{\infty} P(\hat{\xi} > N_2 + j)$$

$$\le \frac{\epsilon_1}{2} + 2\beta^2 \cdot \frac{\epsilon_1}{4\beta^2} \le \epsilon_1$$

as a consequence of (9); it follows that

$$|u^i_n - \frac{1}{\mu}| < \epsilon_1$$

for all $i > M$, $n \ge N$.

Letting $\epsilon = 2\epsilon_1$ then

$$(16) \qquad | u_n^i - \frac{1}{\mu} | < \frac{\epsilon}{2}$$

for $n \geq N$, $i > M$. Condition (B) guarantees the existence of N_3 such that

$$\sum_{n=N_3}^{\infty} n\widehat{p}_n < \epsilon/(2\beta),$$

and an N_4 such that

$$P(\widehat{\xi} > N_4) < \frac{\epsilon}{2M\beta^2} .$$

If $0 \leq u_n^1 - \frac{1}{\mu}$ then for $n \geq [N + M(N_3 + (M-1)/2)]$

$$u_n^1 - \frac{1}{\mu} < \beta \sum_{j=1}^{M} P(\widehat{\xi} \geq N_3 + j) + \max_{k \in [n - M(N_3 + \frac{M-1}{2}), n-M]} [u_k^{M+1} - \frac{1}{\mu}]$$

$$\leq \beta \sum_{j=1}^{\infty} P(\widehat{\xi} \geq N_3 + j) + (\frac{1}{\mu} + \frac{\epsilon}{2}) - \frac{1}{\mu}$$

$$\leq \beta \cdot \frac{\epsilon}{2\beta} + \frac{\epsilon}{2} \leq \epsilon .$$

This follows from

$$u_n^1 - \frac{1}{\mu} \leq \beta \; P(\widehat{\xi} \geq N_3 + 1) + \max_{k \in [n-N_3, n-1]} [u_k^2 - \frac{1}{\mu}]$$

and M applications of

$$\max_{k\in[n-m(N_3+\frac{m-1}{2}),n-m]} [u_k^{m+1}-\frac{1}{\mu}] \leq \beta\, P(\widehat{\xi}\geq N_3+m+1)+ \max_{k\in[n-(m+1)[N_3+\frac{m}{2}],n-m-1]} [u_k^{m+2}-\frac{1}{\mu}].$$

If $0 \leq \frac{1}{\mu} - u_n^1$ then for $n > N+MN_4$

$$\frac{1}{\mu} - u_n^1 \leq \frac{1}{\mu} - [1-\beta\, P(\widehat{\xi} > N_4)]^M \cdot \min_{k\in[n-MN_4,\, n-M]} [u_k^{M+1}]$$

$$\leq \frac{1}{\mu} - (1-\beta \frac{\epsilon}{2M\beta^2})^M (\frac{1}{\mu} - \frac{\epsilon}{2})$$

$$\leq \frac{1}{\mu} - (1 - \frac{\epsilon}{2\beta})(\frac{1}{\mu} - \frac{\epsilon}{2}) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon .$$

This follows from

$$\frac{1}{\mu} - u_n^1 \leq \frac{1}{\mu} - [1-\beta\, P(\widehat{\xi} \geq N_4)] \min_{k\in[n-N_4,\, n-1]} [u_k^2]$$

and repeated application of

$$\min_{k\in[n-(m-1)N_4,\, n-m-1]} u_k^m \geq [1-\beta\, P(\widehat{\xi}\geq N_4)] \min_{k\in[n-mN_4,\, n-m]} u_k^{m+1}.$$

It follows directly from the above arguments that

$$(17) \qquad |u_n^i - \frac{1}{\mu}| < \epsilon$$

for all $i \in [1, M]$ and all $n \geq N^* = \max\{N+MN_4, N+M(N_3+\frac{M-1}{2})\}$.

Since $N^* \geq N$ it follows from (16) and (17) that

$$|\bar{u}_n^i - \frac{1}{\mu}| < \epsilon$$

uniformly in i for all $n > N^*$, which proves the theorem.

2. An Integral Formulation

Smith (1954) has shown for continuous renewal processes that an alternate form of the key renewal theorem

$$\lim_{x \to \infty} [H(x+a) - H(x)] = \frac{a}{E(\xi_1)}$$

is

$$\lim_{x \to \infty} \int_{-\infty}^{\infty} k(x-z)dH(z) = \frac{1}{E(\xi_1)} \int_0^{\infty} k(z)dz ,$$

for every $k(z)$ which is zero for negative arguments, nonnegative, nonincreasing, and integrable on $(0, \infty)$. Because of our specialized considerations we will establish, for suitably restricted integer valued renewal sequences $\{\Phi_n\}$ and sequences of convergent step functions, that

$$\lim_{n\to\infty} \sum_{S=0}^{\infty} \int_0^n R_{S+j}(n-i)\,d\Phi_j^{(S)}(i) = \lim_{n\to\infty} \int_0^n R(n-i)\,dH(i) = \frac{1}{E(\xi_1)} \int_0^{\infty} R(z)\,dz$$

uniformly in j. The next three lemmas are preliminaries for the presentation of this result.

Lemma 3: If a renewal sequence $\{\Phi_n\}$ satisfies condition (A) then $H_k(\ell) \xrightarrow[k\to\infty]{} H(\ell)$ uniformly for all ℓ in a bounded interval $[0, L)$.

Proof: As mentioned in Lemma 2, $H_k(\ell)$ for all $k \geq 1$ and $H(\ell)$ are finite for all finite ℓ. Since (1) has only nonnegative terms it follows that given any $\epsilon > 0$ and any $\ell \in [0, L)$ there exist integers M and M_1 such that

$$\Phi^{(M)}(\ell) < \frac{\epsilon}{2}$$

and

$$\sum_{n=M_1}^{\infty} \Phi^{(n)}(\ell) < \frac{\epsilon}{2M} .$$

From condition (A) there exists a K such that

$$|p_\ell^k - p_\ell| < \frac{\epsilon}{2LM^2}$$

for $k > K$, $\ell \geq 0$. Consequently

$$|\Phi_k(\ell) - \Phi(\ell)| \leq \sum_{i=0}^{\ell} |p_i^k - p_i| \leq \frac{\epsilon(\ell+1)}{2LM^2} \leq \frac{\epsilon}{2M^2}$$

for all $\ell \in [0, L)$. Suppose

$$|\Phi_k^{(n)}(\ell) - \Phi^{(n)}(\ell)| < \epsilon_1,$$

then

$$|\Phi_k^{(n+1)}(\ell) - \Phi^{(n+1)}(\ell)| \leq \int_0^\ell \Phi_k^{(n)}(m)|\, d\Phi_{k+n}(\ell-m) - d\Phi(\ell-m)|$$

$$+ \int_0^\ell |\Phi_k^{(n)}(m) - \Phi^{(n)}(m)|\, d\Phi(\ell-m)$$

$$\leq \epsilon_1 + \frac{\epsilon}{2M^2} .$$

Using this argument in an inductive manner it follows that

$$|\Phi_k^{(M)}(\ell) - \Phi^{(M)}(\ell)| < \frac{\epsilon}{2M}$$

and

$$\left| \sum_{i=1}^{M} \Phi_k^{(i)}(\ell) - \sum_{i=1}^{M} \Phi^{(i)}(\ell) \right| < M \cdot \frac{\epsilon}{2M} \leq \frac{\epsilon}{2} .$$

Using arguments similar to those just given it follows that there exists a K such that for $k > K$, $\ell \in [0, L)$ the following three inequalities hold:

$$\left| \Phi_k^{(M)}(\ell) - \Phi^{(M)}(\ell) \right| < \frac{\epsilon}{2} ;$$

$$\left| \Phi_k^{(M_1)}(\ell) - \Phi^{(M_1)}(\ell) \right| < \frac{\epsilon}{2M} ;$$

and

$$\left| \sum_{i=1}^{M_1+M} \Phi_k^{(i)}(\ell) - \sum_{i=1}^{M_1+M} \Phi^{(i)}(\ell) \right| < \frac{\epsilon}{2M} .$$

Now,

$$\sum_{i=M_1+M}^{\infty} \Phi_k^{i}(\ell) = \sum_{j=1}^{\infty} \sum_{i=M_1+jM}^{M_1+(j+1)M-1} \Phi_k^{(i)}(\ell)$$

and

$$\sum_{i=M_1+jM}^{M_1+(j+1)M-1} \Phi_k^{(i)}(\ell) \leq M \int_0^{\ell} \Phi_k^{(jM)}(m)\, d\Phi_{k+jM}^{(M_1)}(\ell - m) < M\epsilon^{j} \cdot \frac{\epsilon}{M} \leq \epsilon^{j+1} .$$

Hence, it follows that

$$\sum_{i=M_1+M}^{\infty} \Phi_k^{(i)}(\ell) < \sum_{j=1}^{\infty} \epsilon^{j+1} \leq \frac{\epsilon^2}{1-\epsilon} < \frac{\epsilon}{2}$$

for sufficiently small ϵ. Observing that

$$\left| \sum_{i=1}^{\infty} \Phi_k^{(i)}(\ell) - \sum_{i=1}^{\infty} \Phi^{(i)}(\ell) \right| \leq \left| \sum_{i=1}^{M_1+M} \Phi_k^{(i)}(\ell) - \sum_{i=1}^{M_1+M} \Phi^{(i)}(\ell) \right|$$

$$+ \sum_{i=M_1+M+1}^{\infty} (\Phi_k^{(i)}(\ell) + \Phi^{(i)}(\ell))$$

$$\leq \frac{\epsilon}{2M} + \frac{\epsilon}{2} + \frac{\epsilon}{2M} < \epsilon$$

completes the proof of Lemma 3.

<u>Lemma 4:</u> Let $\{\Phi_n\}$ be a renewal sequence satisfying conditions (A) and (B), and $\{R_n\}$ a sequence of step functions converging uniformly to the step function R in every finite interval. If each R_n and R are integrable, nonnegative, and bounded by a uniformly bounded integrable step function $\hat{R}$ such that $\sum_{n=0}^{\infty} \hat{R}(n) < \infty$ then

$$\lim_{j \to \infty} \left[\sum_{S=0}^{\infty} \int_0^n R_{S+j}(n-i)\, d\Phi_j^{(s)}(i) \right] = \int_0^n R(n-i)dH(i)$$

uniformly in n.

Proof: Let β be the uniform bound on $dH_j(n)$ guaranteed by Lemma 2 and A the uniform upper bound on $\hat{R}(n)$. Then, given $\epsilon > 0$, convergence of $\{R_n\}$ guarantees existence of a J_1 such that $|R_j(n)-R(n)| < \epsilon/(2N\beta)$ for $j > J_1$, $n \epsilon [0, N)$. Lemma 3 guarantees the existence of J_2 such that $|H_j(n)-H(n)| < \epsilon/(2A)$. Therefore for all $n \epsilon [0, N)$, $j \geq \max\{J_1, J_2\}$

$$(18) \qquad \left| \sum_{S=0}^{\infty} \int_0^n R_{S+j}(n-i)\, d\Phi_j^{(S)}(i) - \int_0^n R(n-i)dH(i) \right|$$

$$\leq \left| \sum_{S=0}^{\infty} \int_0^n [R_{S+j}(n-i)-R(n-i)]\, d\Phi_j^{(S)}(i) \right| + \left| \int_0^n R(n-i)[dH_j(i)-dH(i)] \right|$$

$$\leq H_j(n) \frac{\epsilon}{2N\beta} + A\,|H_j(n)-H(n)| \leq \epsilon .$$

Thus it need only be shown that there exists J and N such that (18) holds for all $j > J$, $n > N$.

Since $\sum_{i=0}^{\infty} \widehat{R}(i) < \infty$ and each $\widehat{R}(i) \geq 0$ there exists N_1 such that $\sum_{i=N_1}^{\infty} \widehat{R}(i) < \frac{\epsilon}{4\beta}$. From the convergence of $R_j(n)$, $n \epsilon [0, N_1]$ there exists a J_3 such that $|R_j(n)-R(n)| < \epsilon/(4\beta N_1)$ for $j > J_3$ and from Theorem 1 there exists an N_2 such that

$$|dH_j(n) - dH(n)| \leq |u_n^j - u_n| < \epsilon/(4AN_1)$$

for all $n \geq N_2$ uniformly in j. Therefore

$$|\sum_{s=0}^{\infty} \int_0^n R_{s+j}(n-i)d\Phi_j^{(s)}(i) - \int_0^n R(n-i)dH(i)|$$

$$\leq \sum_{s=0}^{\infty} \int_0^{n-N_1} R_{s+j}(n-i)\, d\Phi_j^{(s)}(i) + \int_0^{n-N_1} R(n-i)dH(i)$$

$$+ |\sum_{s=0}^{\infty} \int_{n-N_1}^{n} [R_{s+j}(n-i)-R(n-i)]\, d\Phi_j^{(s)}(i)| + |\int_{n-N_1}^{n} R(n-i)[dH_j(i)-dH(i)]|$$

$$\leq 2\beta \sum_{i=N_1}^{\infty} \widehat{R}(i) + \int_0^{N_1} \frac{\epsilon}{4\beta N_1}\, dH_j(n-i) + \int_0^{N_1} A|dH_j(n-i)-dH(n-i)|$$

$$\leq 2\beta \cdot \frac{\epsilon}{4\beta} + \frac{\epsilon}{4\beta N_1} \cdot \beta N_1 + \frac{AN_1\epsilon}{4AN_1} \leq \epsilon$$

for $n < N_1 + N_2$ and $j > J_3$ which proves the lemma.

<u>Lemma 5:</u> If $K(x)$ is a non-decreasing, bounded function of total variation $r > 0$ and $b(x)$ is a uniformly bounded function such that $B_1 \leq b(x) \leq B_2$ for x sufficiently large then

$$B_1 r \leq \int_{-\infty}^{\infty} b(x-z)dK(z) \leq B_2 r$$

for sufficiently large x.

<u>Proof:</u> It is sufficient to prove the lemma when $|b(x)| \leq B$ for x sufficiently large. Let A be a uniform bound on $|b(x)|$ then, if $A = B$, the lemma follows from

$$\left| \int_{-\infty}^{\infty} b(x-z)dK(z) \right| \leq \int_{-\infty}^{\infty} AdK(z) .$$

If $A > B$ then given $\epsilon > 0$ there is an $x_0(\epsilon)$ such that

$$|b(x)| \leq B + \frac{\epsilon}{2r}$$

and

$$\int_{x_0}^{\infty} dK(z) < \epsilon / 2(A-B)$$

for $x > x_0$. Thus $x > 2x_0$ implies

$$\left| \int_{-\infty}^{\infty} b(x-z)dK(z) \right| \leq \int_{-\infty}^{x_0} |b(x-z)| dK(z) + \int_{x_0}^{\infty} |b(x-z)| dK(z)$$

$$\leq B[K(x_0) - K(-\infty)] + A \int_{x_0}^{\infty} dK(z)$$

$$\leq Br + (A-B) \int_{x_0}^{\infty} dK(z) \leq Br + \epsilon$$

which proves the lemma.

This completes the necessary preliminaries for the second renewal theorem.

Theorem 2: If a renewal sequence $\{\Phi_n\}$ satisfies conditions (A) and (B), if a sequence of step functions $\{R_n\}$ converges uniformly to the step function R in every finite interval, and if each R_n and R are integrable, nonnegative, and bounded by a uniformly bounded step function $\hat{R}$ such that $\sum_{n=0}^{\infty} \hat{R}(n) < \infty$, then

$$(19) \quad \lim_{n \to \infty} \sum_{s=0}^{\infty} \int_0^n R_{s+j}(n-i) d\Phi_j^{(s)}(i) = \lim_{n \to \infty} \int_0^n R(n-i) dH(i) = \frac{1}{\mu} \sum_{i=0}^{\infty} R(i).$$

Proof: Given $\epsilon > 0$ the existence of N_1 such that

$$|dH(n) - \frac{1}{\mu}| < \epsilon/2 \sum_{i=0}^{\infty} \hat{R}(i)$$

for all $n \geq N_1$ follows from Theorem (EFP). Also there exists N_2 such that $\sum_{i=N_2}^{\infty} R(i) < \epsilon/(4\beta N_1)$ where β is the uniform bound guaranteed by Lemma 2. Therefore for $n > N_1+N_2$

$$\left| \int_0^n R(n-i)dH(i) - \frac{1}{\mu}\sum_{i=0}^{\infty} R(i) \right| \leq \left| \int_{N_1}^n R(n-i)dH(i) - \frac{1}{\mu}\sum_{i=0}^{n-N_1} R(i) \right|$$

$$+ \frac{1}{\mu}\sum_{i=n-N_1}^{\infty} R(i) + \int_0^{N_1} R(n-i)dH(i)$$

$$\leq \max \left| \left[\sum_{i=0}^{n-N_1} R(i) \right] \left(\frac{1}{\mu} \pm \frac{\epsilon}{2\sum_{i=0}^{\infty} \hat{R}(i)} \right) - \frac{1}{\mu}\sum_{i=0}^{n-N_1} R(i) \right|$$

$$+ \frac{1}{\mu}\frac{\epsilon}{4\beta N_1} + \int_0^{N_1} \frac{\epsilon}{4\beta N_1} dH(i)$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{4N_1} + \frac{\epsilon}{4} \leq \epsilon$$

which proves the second equality in (19).

Now, for a given $\epsilon > 0$, by Lemma 4 a J can be selected such that

$$\left| \sum_{s=0}^{\infty} \int_0^n R_{s+J}(n-i)d\Phi_J^{(s)}(i) - \int_0^n R(n-i)dH(i) \right| < \frac{\epsilon}{4}$$

uniformly in n. If

$$K(n) = \Phi_1^{(J-1)}(n)$$

and

$$b(n) = \sum_{s=0}^{\infty} \int_0^n R_{s+J}(n-i)d\Phi_J^{(s)}(i) \quad ,$$

then $K(n)$ is monotone increasing with variation 1, and $|b(n)| \leq \beta \sum_{i=0}^{\infty} \hat{R}(i)$. Also,

$$\frac{1}{\mu} \sum_{i=0}^{\infty} R(i) - \frac{\epsilon}{4} \leq b(n) \leq \frac{1}{\mu} \sum_{i=0}^{\infty} R(i) + \frac{\epsilon}{4}$$

for sufficiently large n. Thus by Lemma 5 there exists an $N_1(\epsilon)$ such that

$$(20) \qquad \left| \int_0^n d\Phi_1^{(J-1)}(j) \left[\sum_{s=0}^{\infty} \int_0^{n-j} R_{s+J}(n-j-\ell) d\Phi_J^{(s)}(\ell) \right] - \frac{1}{\mu} \sum_{i=0}^{\infty} R(i) \right| < \frac{\epsilon}{2}$$

for $n > N_1$. Also there exists an N_2 such that, for $n > N_2$

$$(21) \qquad \left| \sum_{s=0}^{J-1} \int_0^n R_{s+1}(n-i) d\Phi_1^{(s)}(i) \right| < \frac{\epsilon}{2} .$$

From (20) and (21) it follows that, for $n \geqslant N_1 + N_2$

$$\left| \sum_{s=0}^{\infty} \int_0^n R_{s+k}(n-i) d\Phi_k(i) - \frac{1}{\mu} \sum_{i=0}^{\infty} R(i) \right| < \epsilon$$

for $k = 1$, and since the same arguments hold for any $k > 1$ the theorem is proven.

3. Consequences of the Key Renewal Theorem

In concluding this chapter two renewal theoretic limits will be found for renewal sequences $\{\Phi_n\}$ satisfying conditions (A) and (B). The limits are the same as those of the corresponding renewal process. They are: (a) the limiting distribution of the excess variable as the renewal quantity becomes large; and (b) the probability with which a two state stochastic process will be in a particular one of the states, as the time for which the process has been observed approaches infinity.

Property (a) is a new result which, aside from its renewal theoretic interest, is an indispensable tool in the proof of Theorem 2.1. The result closest to property (a) which the author has encountered is a theorem by Smith (1954) which says; if, in a renewal process, the first variable is allowed to be distributed differently from the others, the limiting distribution of the excess variable remains the same.

Let n be the renewal quantity and $\{\xi_i\}$ be observations of the renewal sequence. The excess variables can be pictured and expressed from Figure 2.

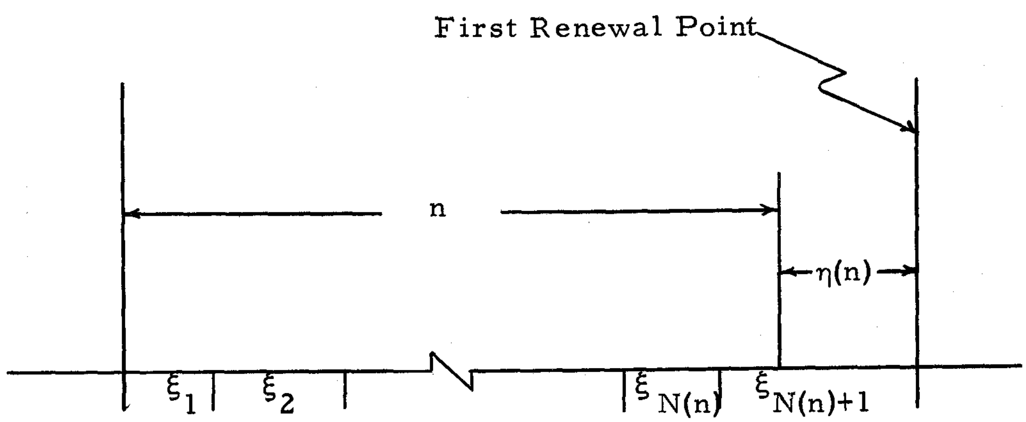

Figure 2. Excess variable (large renewal quantity).

From Figure 2, it is apparent that if $\eta(n)$ is the excess variable of $\{\xi_i\}$ then

$$P(\eta(n) \leq k) = \sum_{s=0}^{\infty} \int_0^n [\Phi_{s+1}(n+k-i) - \Phi_{s+1}(n-i)] \, d\Phi_1^{(s)}(i).$$

For the excess variable of the associated renewal process

$$P(\eta(n) \leq k) = \int_0^n [\Phi(n+k-i) - \Phi(n-i)] \, dH(i) + \Phi(n+k) - \Phi(n)$$

(Smith, 1954, p. 25). We therefore have

Theorem 3: If a renewal sequence $\{\Phi_i\}$ satisfies conditions (A) and (B), then for both $\{\Phi_i\}$ and its associated renewal process

$$\lim_{n \to \infty} P(\eta(n) \leq k) = \frac{1}{\mu} \sum_{i=0}^{k} [1 - \Phi(i)] .$$

Proof: Let $R_i(n) = \Phi_i(n+k) - \Phi_i(n)$ then: $R_i(n) \xrightarrow[i \to \infty]{} R(n)$, $(R(n) = \Phi(n+k) - \Phi(n))$ uniformly in n, $R_i(n) \leq 1$ for all i and n, and $\Phi_i(n+k) - \Phi_i(n) \leq 1 - \widehat{\Phi}(n)$. Since $\sum_{n=0}^{\infty} (1 - \widehat{\Phi}(n)) = E(\widehat{\xi}) < \infty$, Theorem 2 applies and noting that $\sum_{i=0}^{\infty} [\Phi(i+k) - \Phi(i)] = \sum_{i=0}^{k-1} [1 - \Phi(i)]$ proves the theorem.

Turning now to property (b), consider two positive real numbers u and v, and a renewal sequence $\{\xi_n(u)\}$ satisfying conditions (A) and (B) which, for reasons which will become apparent, we think of as functions of the real number u. Form the sequence $v, \xi_1(u), v, \xi_2(u), \cdots$, and from this sequence form the new sequence of random variables $\{L_i\}$ defined by

$$L_1 = v, \quad L_2 = v+\xi_1(u), \cdots, \quad L_{2m} = vm + \sum_{j=1}^{m} \xi_j(u), \quad L_{2m+1} = L_{2m}+v, \cdots.$$

<u>Definition:</u> A real number t is said to be covered by the sequence $\{L_i\}$ if there exists n such that $L_{2n} < t \leq L_{2n+1}$. Karlin (Arrow, 1958, p. 276) has shown, for restricted renewal processes, that

$$\lim_{t \to \infty} P(t \text{ not covered by } \{L_i\}) = \frac{E(\xi(u))}{v+E(\xi(u))} .$$

Before showing that this same result holds for suitably restricted convergent renewal sequences we will develop some quantities used in the proof.

Letting $z_i = v + \xi_i(u)$, it follows that

$$F_i(\alpha) = P(z_i \leq \alpha) = P(\xi_i(u) \leq \alpha - v).$$

Given any positive real t let $\ell(t)$ be such that

$z_1+z_2+\cdots+z_{\ell(t)} < t$ and $z_1+z_2+\cdots+z_{\ell(t)+1} \geq t$. It follows that

$$P(\{L_i\} \text{ does not cover } t) = \sum_{n=1}^{\infty} P(\{L_i\} \text{ does not cover } t/\ell(t)=n)P(\ell(t)=n).$$

Now

$$P(\ell(t)=n) = F_1^{(n)}(t) - F_1^{(n+1)}(t)$$

and

$$P(z_1+\cdots+z_m \leq s/\ell(t)=n) = \int_0^s \frac{[F_{m+1}^{(n-m)}(t-x) - F_{m+1}^{(n-m+1)}(t-x)]}{[F_1^{(n)}(t) - F_1^{(n+1)}(t)]} \, dF_1^{(m)}(x)$$

which implies

$$dP(z_1+\cdots+z_n \leq s/\ell(t)=n) = \frac{[1-F_{n+1}(t-s)]}{[F_1^{(n)}(t) - F_1^{(n+1)}(t)]} \, dF_1^{(n)}(s) \quad .$$

Thus

$$\sum_{n=1}^{\infty} P(\{L_i\} \text{ does not cover } t/\ell(t)=n)P(\ell(t)=n)$$

$$= \sum_{n=1}^{\infty} P(\ell(t)=n) \int_0^t P(\{L_i\} \text{ does not cover } t/\ell(t)=n;$$

$$z_1+\cdots+z_n=s)dP(z_1+\cdots+z_n \leq s/\ell(t)=n)$$

and

$$P(\{L_i\} \text{ does not cover } t/\ell(t)=n; z_1+\cdots+z_n=s)$$

$$= P(v< t-s/ v+\xi_{n+1}(u) \geq t-s)$$

$$= P(v<t-s;\ v+\xi_{n+1}(u) > t-s)/ P(v+\xi_{n+1}(u)> t-s)$$

$$= P(v<t-s;\ v+\xi_{n+1}(u)> t-s)/ [1-F_{n+1}(t-s)] .$$

Summarizing these statements we have

(22) $$P(\{L_i\} \text{ does not cover } t)$$

$$= \sum_{n=1}^{\infty} \int_0^t [\frac{P(v<t-s; v+\xi_{n+1}(u)>t-s)}{1-F_{n+1}(t-s)}] \frac{(1-F_{n+1}(t-s))dF_1^{(n)}(t)}{F_1^{(n)}(t)-F_1^{(n+1)}(t)} \cdot [F_1^{(n)}(t)-F_1^{(n+1)}(t)]$$

$$= \sum_{n=1}^{\infty} \int_0^t P(v < t-s;\ v+\xi_{n+1}(u) > t-s)dF_1^{(n)}(s)$$

$$= \sum_{n=1}^{\infty} \int_0^{t-v} P(\xi_{n+1}(u) > t-v-s)\ dF_1^{(n)}(s)$$

$$= \sum_{n=1}^{\infty} \int_0^{x} [1-P(\xi_{n+1}(u) \leq x-s)]\ dF_1^{(n)}(s).$$

With this expression for $P(\{L_i\}$ does not cover $t)$ we are prepared to prove

Theorem 4: Given two positive real numbers u and v, if the renewal sequence $\{v+\xi_n(u)\}$ satisfies conditions (A) and (B) then

$$\lim_{t \to \infty} P(\{L_i\} \text{ does not cover } t) = \frac{E(\xi(u))}{v+E(\xi(u))} .$$

Proof: Since

$$\lim_{n \to \infty} [1-P(\xi_n(u) \leq x)] = 1-P(\xi(u) \leq x)$$

and

$$\sum_{i=0}^{\infty} [1-P(\xi_{n+1}(u) \leq i)] = E(\xi_{n+1}(u)) < E(\hat{\xi}(u)) < \infty$$

it follows that the sequence of functions $(1-P(\xi_n(u) \leq x)$ satisfy the conditions on $R_n(x)$ in Theorem 2. Since the sequence of distributions $\{F_n\}$ is assumed to satisfy conditions (A) and (B) with $\mu = v+E(\xi(u))$, Theorem 2 can be applied to (22). This implies

$$\lim_{n\to\infty} P(\{L_i\} \text{ does not cover } t) = \frac{1}{E(v+\xi(u))} \sum_{i=0}^{\infty} [1-P(\xi(u) \leq i)]$$

$$= \frac{E(\xi(u))}{v+E(\xi(u))} .$$

This proves the theorem.

IV. PROOF OF PRINCIPAL INVENTORY THEOREM

The proof of Theorem 2. 1 was sketched in Chapter 2. We proceed to fill in the details of this proof by deriving the stationary distribution of the inventory level for demand sequences satisfying conditions (A), (B), and (C). We will parallel the method used by Karlin (Arrow, 1958, p. 270-297) for the case of identically distributed variables. The stock distribution at the end of period n before ordering, x_n, is written in two segments

(1) $$P(S-\alpha \leq x_n \leq S/ s \leq x_n \leq S)$$

and

(2) $$P(s-\alpha \leq x_n < s/ x_n < s).$$

Thus, to find the stationary distribution of the inventory level for a sequence of demands $\{\xi_i\}$ satisfying conditions (A), (B), and (C) we must find the limiting values of:

$$P(s \leq x_n \leq S),$$

$$P(x_n < s) ,$$

and the probabilities given by (1) and (2).

1. Stationary Inventory Distribution

We first consider

$$\lim_{n \to \infty} P(x_n < s)$$

and

$$\lim_{n \to \infty} P(s \leq x_n \leq S) .$$

Let $u = S - s$ and form the sequence of integer valued random variables $N_i(u)$ defined by

$$\sum_{j=1}^{N_i(u)} \xi_{i+j} \leq u \tag{3-1}$$

and

$$\sum_{j=1}^{N_i(u)+1} \xi_{i+j} > u . \tag{3-2}$$

Then letting

$$i_n = \begin{cases} 0 & n = 1 \\ n-1+\sum_{j=1}^{n-1} N_{i_j}(u) & n > 1 \end{cases} \tag{4}$$

form the sequence $N_{i_1}(u)$, 1, $N_{i_2}(u)$, 1, $\cdots$, $N_{i_n}(u)$, 1, $\cdots$, and let

$$F_n(k) = P(N_{i_n}(u)+1 \le k) = P(N_{i_n}(u) \le k-1) = \sum_{r=0}^{k-1} [\Phi^{(r)}_{i_n+1}(u) - \Phi^{(r+1)}_{i_n+1}(u)$$

$$= 1-\Phi^{(k)}_{i_n+1}(u).$$

From Lemma 3.3 it follows that

$$\lim_{n\to\infty} F_n(k) = 1-\Phi^{(k)}(u).$$

Also

$$\lim_{n\to\infty} \int_0^\infty x dF_n(x) = \lim_{n\to\infty} \sum_{i=1}^{\infty} i\,[\Phi^{(i-1)}_{i_n+1}(u) - \Phi^{(i)}_{i_n+1}(u)]$$

$$= \lim_{n\to\infty} (1+H_{i_n+1}(u)) = 1+H(u)$$

and for all n

$$1 \le \int_0^\infty x\, dF_n(x) \le 1+\beta u,$$

which imply the sequence $\{F_n\}$ satisfies condition (A).

Since the sequence $\{\Phi_n\}$ satisfies condition (A) it follows that there exists a distribution function $\bar{\Phi}(x)$ such that $\bar{\Phi}(x) \geq \Phi_n(x)$ for all x and n, and $0 < \int_0^\infty x\, d\bar{\Phi}(x) < \infty$. Thus, for the renewal process formed using $\bar{\Phi}(x)$, it follows from Lemma 3.3 that $E(\bar{N}(u)) = \bar{H}(u) < \infty$. Therefore

$$F_n(k) = 1 - \Phi^{(k)}_{i_n+1}(u) \geq 1 - \bar{\Phi}^{(k)}(u)$$

and $\{F_n\}$ saitsfies condition (B2). Unfortunately, the fact that $\{\Phi_n\}$ satisfies conditions (A) and (B) is not sufficient to guarantee $\{F_n\}$ satisfies condition (B1). As an example, let $u = 4$ and

$$P(\xi_n = 2) = P(\xi_n = 3) = 1/2,$$

then $\{\xi_n\}$ satisfies conditions (A) and (B) but $P(N(4)+1=3) = 1$ which shows that $\{F_n\}$ does not satisfy condition (B1).

If $\{\Phi_n\}$ also satisfies condition (C), then $\{F_n(k)\}$ satisfies condition (B1). This follows by observing that if $p_n > 0$ for $n > u$ then $P(N(u)+1 = 1) > 0$. If, however, there does not exist $p_n > 0$ for some $n > u$, but $p_0 > 0$ then let n_0, $0 < n_0 \leq u$, be the largest n such that $p_n > 0$. Let k_0 be the integer such that $k_0 n_0 \leq u$ and $(k_0+1)n_0 > u$ then

$$P(N(u)+1 = k_0+1) > 0$$

and

$$P(N(u)+1=k_0+2) = F(k_0+2) - F(k_0+1) = \Phi^{(k_0+1)}(u) - \Phi^{(k_0+2)}(u)$$

$$= \Phi^{(k_0+1)}(u) - [p_0\Phi^{(k_0+1)}(u)+p_1\Phi^{(k_0+1)}(u-1)+\cdots+p_{n_0}\Phi^{(k_0+1)}(u-n_0)]$$

$$= \sum_{i=1}^{n_0} p_i[\Phi^{(k_0+1)}(u) - \Phi^{(k_0+1)}(u-i)] .$$

Now $j = u-k_0n_0+1$ is such that $j\epsilon[1,n_0]$ and $\Phi^{(k_0+1)}(u-j) = \Phi^{(k_0+1)}(k_0n_0-1)$ which is less than $\Phi^{(k_0+1)}(u)$ by at least $p_0p_{n_0}^{k_0}$. Therefore

$$P(N(u)+1=k_0+2) > p_0p_{n_0}^{k_0} > 0$$

which implies, since two consecutive integers have positive probabilities, that condition (B1) is satisfied.

Letting

$$L_1=N_{i_1}(u),\ L_2=N_{i_1}(u)+1,\cdots,\ L_{2m}=m+\sum_{j=1}^{m} N_{i_j}(u),\ \ L_{2m+1}=L_{2m}+N_{i_{n+1}}(u),\cdots$$

we see that all the conditions of Theorem 3.4 are satisfied, and that

$$\lim_{t\to\infty} P(\{L_i\} \text{ does not cover } t) = \lim_{n\to\infty} P(x_n< s).$$

Since

$$\lim_{n \to \infty} P(s \leq x_n \leq S) = 1 - \lim_{n \to \infty} P(\{L_i\} \text{ does not cover } t)$$

it follows that

$$\lim_{n \to \infty} P(x_n < s) = \frac{1}{1+E(N(u))} = \frac{1}{1+H(u)}$$

and

$$\lim_{n \to \infty} P(s \leq x_n \leq S) = 1 - \frac{1}{1+H(u)} = \frac{H(u)}{1+H(u)} .$$

The limits obtained in Theorem 3.4 are interchanged because of the new significance of $\{L_i\}$.

Now $\lim_{n \to \infty} P(s-\alpha \leq x_n < s/x_n < s)$ is exactly the limiting distribution of the excess variable as the number of renewals becomes large. Before deriving an expression for this limit we will need the following extension of the Helly-Bray Theorem;

Lemma 1: Let $\{f_n(x)\}$ be a sequence of uniformly bounded, integrable functions defined on the interval $[a,b]$ and converging uniformly on that interval to $f(x)$. Let $\{g_n(x)\}$ be a sequence of functions which converges uniformly to a bounded function $g(x)$ on $[a,b]$. If the total variation of each $g_n(x)$ on $[a,b]$ is uniformly bounded then

$$\lim_{n \to \infty} \int_a^b f_n(x)dg_n(x) = \int_a^b f(x)dg(x) .$$

Proof: The proof of this lemma follows, with some modification, from the usual proof of the Helly-Bray theorem (Gnedenko, 1963, p. 264).

We are now ready to show that as the number of renewals becomes large, the distribution of the excess variable for a renewal sequence satisfying condition (A) is the same as that of the associated renewal process. The distribution of the excess variable for the associated renewal process is given by

$$(5) \quad P(\eta(z) \leq r) = \Phi(z+r) - \Phi(z) + \int_0^z (\Phi(z+r-x) - \Phi(z-x))\, dH(x)$$

(Smith, 1954, p. 25). That renewals will occur with probability one is a consequence of the Borel-Cantelli Lemma (Gnedenko, 1963, p. 246). Let N be a period during which a renewal occurs when the renewal quality is z; then Figure 3 defines the quantities in the following discussion.

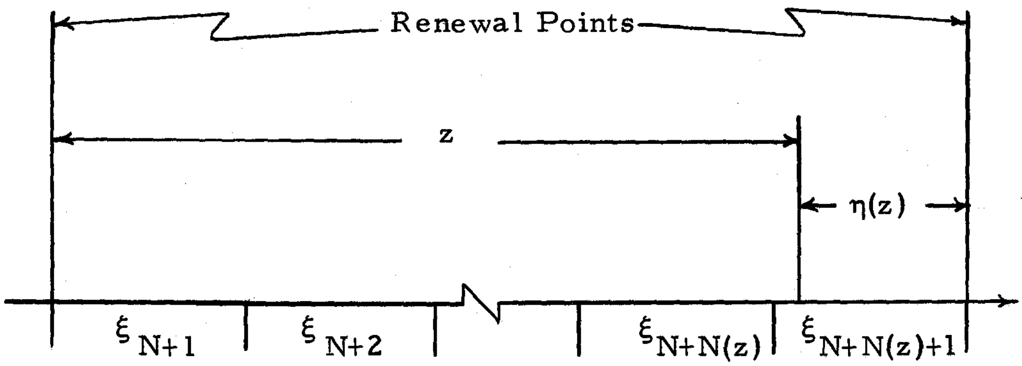

Figure 3. Excess variable (large number of renewals).

Thus it can be seen that

$$P(\eta(z) \leq r, \quad N(z) = 0) = \Phi_{N+1}(z+r) - \Phi_{N+1}(z)$$

and

$$P(\eta(z) \leq r; \quad N(z) = L > 0) = \int_0^z (\Phi_{N+L+1}(z+r-x) - \Phi_{N+L+1}(z-x)) d\Phi_{N+1}^{(L)}(x).$$

Therefore

$$(6) \quad P(\eta(z) \leq r) = \Phi_{N+1}(z+r) - \Phi_{N+1}(z)$$

$$+ \sum_{L=1}^{\infty} \int_0^z (\Phi_{N+L+1}(z+r-x) - \Phi_{N+L+1}(z-x)) d\Phi_{N+1}^{(L)}(x).$$

From Lemma 3.3, the uniform convergence on $[0, z)$ of the functions involved, and the fact that for M sufficiently large the term $\sum_{L=M}^{\infty} \Phi_{N+1}^{(L)}(z)$ is arbitrarily small uniformly in N, it follows that Lemma 1 applies and the summation and integration in (6) may be interchanged. Thus

$$\lim_{n \to \infty} P(\eta(z) \leq r) = \Phi(z+r) - \Phi(z) + \int_0^z (\Phi(z+r-x) - \Phi(z-x))dH(x)$$

which agrees with (5).

To calculate

$$\lim_{n \to \infty} P(S-\alpha \leq x_n \leq S / s \leq x_n \leq S)$$

let $u = S-s$ and consider the sequence of random variables defined by (3) and (4). Given an integer $j > 1$, let r be the integer such that $\sum_{n=1}^{r} N_{i_n}(u) < j$ and $\sum_{n=1}^{r+1} N_{i_n}(u) \geq j$. If for some j no such r exists then $\lim_{n \to \infty} P(s \leq x_n \leq S) = 0$ which implies $\lim_{n \to \infty} P(S - \alpha \leq x_n \leq S) = 0$, $\alpha \leq u$. The existence of such an r is guaranteed by condition (C). Let $w_m(u) = \sum_{n=1}^{m} N_{i_n}(u)$ and define $\psi_j = \xi_{w_r(u)+1} + \xi_{w_r(u)+2} + \cdots + \xi_j$. Then

$$\lim_{j \to \infty} P(\psi_j \leq \alpha) = \lim_{n \to \infty} P(S-\alpha \leq x_n \leq S/s \leq x_n \leq S).$$

The sequence of random variables $\{N_{i_n}(u)\}$ form an integer valued renewal sequence and letting $G_n(k) = P(N_{i_n}(u) \leq k)$ we have

$$G_n(k) = \sum_{m=0}^{k} (\Phi^{(m)}_{w_{n-1}(u)+1}(u) - \Phi^{(m+1)}_{w_{n-1}(u)+1}(u)) = 1 - \Phi^{(k+1)}_{w_{n-1}(u)+1}(u).$$

If $\lim_{n \to \infty} P(s \leq x_n \leq S) \neq 0$ then a modification of the arguments concerning $\{F_n(k)\}$ shows that $\{G_n(k)\}$ satisfies conditions (A) and (B).

To calculate $\lim_{j \to \infty} P(\psi_j \leq \alpha)$, we first consider the amount θ_j by which the sum of the random variables $N_{i_n}(u)$, previous to j being exceeded, fall short of j. That is, we are interested in $\theta_j = j - w_r(u)$. The distribution of θ_j is given by

$$P(\theta_j \leq k) = 1 - P(\theta_j > k) = 1 - \sum_{s=0}^{\infty} \int_0^{j-k-1} [1 - G_{s+1}(j-1-i)]\, dG_1^{(s)}(i)$$

$$= 1 - \sum_{s=0}^{\infty} \int_0^{m} [1 - G_{s+1}(m+k-i)]\, dG_1^{(s)}(i).$$

Since

$$\sum_{m=0}^{\infty} (1 - G_{s+1}(m+k)) = \sum_{m=0}^{\infty} [1 - (1 - \Phi^{(m+k+1)}_{i_{s+1}}(u))] = \sum_{\ell=k+1}^{\infty} \Phi^{(\ell)}_{i_{s+1}}(u) \leq \sum_{i=0}^{\infty} \hat{\Phi}^{(i)}(u) < \infty,$$

the sequence of functions $\{1-G_{s+1}(m+k)\}$ satisfies the conditions on the sequence $\{R_s\}$ in Lemma 3.4 with

$$\lim_{s \to \infty} (1-G_{s+1}(m+k)) = \Phi^{(m+k+1)}(u).$$

Thus Theorem 3.2 applies and

$$\lim_{j \to \infty} P(\theta_j \leq k) = 1 - \sum_{m=0}^{\infty} \Phi^{(m+k+1)}(u)/H(u)$$

$$= (\sum_{m=1}^{\infty} \Phi^{(m)}(u) - \sum_{m=k+1}^{\infty} \Phi^{(m)}(u))/H(u) = \sum_{m=1}^{k} \Phi^{(m)}(u)/H(u)$$

or

(7) $$P(\theta_j = k) = \Phi^{(k)}(u)/H(u).$$

Now

(8) $$P(\psi_j \leq \alpha / \psi_j \text{ has } k \text{ terms})$$

$$= \sum_{i=0}^{\infty} P(\psi_j \leq \alpha / \psi_j \text{ has } k \text{ terms; } N_{i_{r+1}}(u)=k+i) P(N_{i_{r+1}}(u)=k+i/\psi_j \text{ has } k \text{ terms}).$$

An evaluation of conditional probabilities gives

(9) $$P(\psi_j \le \alpha / \psi_j \text{ has } k \text{ terms; } N_{i_{r+1}}(u) = k+i)$$

$$= \int_0^{\alpha} \frac{[\Phi^{(i)}_{i_{r+1}+k+1}(u-x) - \Phi^{(i+1)}_{i_{r+1}+k+1}(u-x)]\, d\Phi^{(k)}_{i_{r+1}+1}(x)}{\Phi^{(k+i)}_{i_{r+1}+1}(u) - \Phi^{(k+i+1)}_{i_{r+1}+1}(u)},$$

and

$$P(N_{i_{r+1}}(u) \le k+i / \psi_j \text{ has } k \text{ terms}) = \frac{P(k \le N_{i_{r+1}}(u) \le k+i)}{P(N_{i_{r+1}}(u) \ge k)}$$

$$= \frac{\Phi^{(k)}_{i_{r+1}+1}(u) - \Phi^{(k+i+1)}_{i_{r+1}+1}(u)}{\Phi^{(k)}_{i_{r+1}+1}(u)}$$

which implies

(10) $$P(N_{i_{r+1}}(u) = k+i/\psi_j \text{ has } k \text{ terms}) = \frac{\Phi^{(k+i)}_{i_{r+1}+1}(u) - \Phi^{(k+i+1)}_{i_{r+1}+1}(u)}{\Phi^{(k)}_{i_{r+1}+1}(u)}.$$

Substituting (10) and (9) in (8) and interchanging summation and integration, which is easily justified, it follows that

(11) $$P(\psi_j \leq \alpha / \psi_j \text{ has } k \text{ terms})$$

$$= \int_0^{\alpha} \sum_{i=0}^{\infty} \frac{[\Phi^{(i)}_{i_{r+1}+k+1}(u-x) - \Phi^{(i+1)}_{i_{r+1}+k+1}(u-x)]\, d\Phi^{(k)}_{i_{r+1}+1}(x)}{\Phi^{(k)}_{i_{r+1}+1}(u)}$$

$$= \int_0^{\alpha} \frac{d\Phi^{(k)}_{i_{r+1}+1}(x)}{\Phi^{(k)}_{i_{r+1}+1}(u)} = \Phi^{(k)}_{i_{r+1}+1}(\alpha) / \Phi^{(k)}_{i_{r+1}+1}(u).$$

We now show that

$$\lim_{j \to \infty} P(\psi_j \leq \alpha) = H(\alpha)/H(u).$$

Given $\epsilon > 0$, from (7) and Lemma 3.3, there exists K_1 and J_1 such that

$$\sum_{k=K_1}^{\infty} \Phi^{(k)}(u)/H(u) < \epsilon/8$$

and

$$|P(\theta_j > K_1) - \lim_{j \to \infty} P(\theta_j > K_1)| < \epsilon/8$$

for all $j > J_1$. Also from (11) and Lemma 3.3 there exists J_2 such that for all $j > J_1$, $k \epsilon [0, K_1]$

$$\left| \frac{\Phi_j^{(k)}(a)}{\Phi_j^{(k)}(u)} - \frac{\Phi^{(k)}(a)}{\Phi^{(k)}(u)} \right| < \frac{\epsilon}{4K_1}$$

and

$$\left| P(\theta_j = k) - \frac{\Phi^{(k)}(u)}{H(u)} \right| < \frac{\epsilon}{4K_1} .$$

If $J = \max\{J_1, J_2 + K_1\}$ then

$$|P(\psi_j \leq a) - H(a)/H(u)|$$

$$= \left| \sum_{k=1}^{j} P(\psi_j \leq a / \psi_j \text{ has } k \text{ terms}) P(\theta_j = k) - \frac{H(a)}{H(u)} \right|$$

$$\leq \sum_{k=1}^{K_1} \left| \frac{\Phi_{j-k}^{(k)}(a)}{\Phi_{j-k}^{(k)}(u)} - \frac{\Phi^{(k)}(a)}{\Phi^{(k)}(u)} \right| \frac{\Phi^{(k)}(u)}{H(u)} + \sum_{k=1}^{K_1} \frac{\Phi^{(k)}(a)}{\Phi^{(k)}(u)} \left[P(\theta_j = k) - \frac{\Phi^{(k)}(u)}{H(u)} \right]$$

$$+ \sum_{k=K_1+1}^{j} \frac{\Phi_{j-k}^{(k)}(a)}{\Phi_{j-k}^{(k)}(u)} P(\theta_j = k) + \sum_{k=K_1+1}^{\infty} \frac{\Phi^{(k)}(a)}{H(u)}$$

$$\leq K_1 \cdot \frac{\epsilon}{4K_1} + K_1 \cdot \frac{\epsilon}{4K_1} + P(\theta_j > K_1) + \epsilon/8 < \epsilon$$

for all $j > J$.

Thus we have shown that

$$\lim_{n \to \infty} P(S-\alpha \leq x_n \leq S) = \frac{H(u)}{[1+H(u)]} \cdot \frac{H(\alpha)}{H(u)} = \frac{H(\alpha)}{1+H(u)}$$

for $0 \leq \alpha \leq S-s$ and

$$\lim_{n \to \infty} P(s-\alpha \leq x_n \leq s) = \frac{1}{1+H(u)} [\Phi(u+\alpha) - \Phi(u) + \int_0^u [\Phi(u+\alpha-x) - \Phi(u-x)] \, dH(x)] .$$

Since these arguments hold without change for the associated demand process we have proven the following ;

<u>Theorem 1:</u> If a sequence of demands $\{\Phi_i\}$ satisfies conditions (A), (B), and (C) then the stationary inventory distribution is the same for $\{\Phi_i\}$ and the associated demand process.

2. Proof of Theorem 2.1

As mentioned in Chapter 2, the proof of Theorem 2.1 consists of showing that: $\lim_{n \to \infty} C(x; \Phi_n) = C(x; \Phi)$; the limiting stock level distribution exists and is the same for a demand sequence satisfying conditions (A), (B), and (C) as for the associated demand sequence; and the various interchanges of operations employed as well as the existence of the various quantities can be justified. Theorem 1 has established that $F_n(x) \to F(x)$ as $n \to \infty$ where $F_n(x)$ is the distribution of the ending stock in the n^{th} period. The proof that

$C(x;\Phi_n) \rightarrow C(x;\Phi)$ as $n \rightarrow \infty$ is contained in

<u>Lemma 2:</u> If a sequence of demands satisfies condition (A) then $C(x;\Phi_n) \rightarrow C(x;\Phi)$ as $n \rightarrow \infty$ uniformly in x for all s and S such that $S-s$ is bounded.

<u>Proof:</u> From (2.1)

$$|C(x;\Phi)-C(x;\Phi_i)| = \begin{cases} \left| p\int_S^\infty (\xi-S)(d\Phi(\xi)-d\Phi(\xi))+h\int_0^S (S-\xi)(d\Phi(\xi)-d\Phi_i(\xi))\right| & x<s \\ \left| p\int_x^\infty (\xi-x)(d\Phi(\xi)-d\Phi(\xi))+h\int_0^x (x-\xi)(d\Phi(\xi)-d\Phi_i(\xi))\right| & s\le x\le S \end{cases}$$

$$\le \begin{cases} \left|(h+p)\int_0^S (S-\xi)(d\Phi(\xi)-d\Phi_i(\xi))+\mu-\mu_i\right| & x<s \\ \left|(h+p)\int_0^x (x-\xi)(d\Phi(\xi)-d\Phi_i(\xi))+\mu-\mu_i\right| & s\le x\le S \end{cases}$$

Now for $x\in[s,S]$

$$\left|(h+p)\int_0^x (x-\xi)(d\Phi(\xi)-d\Phi_i(\xi))+\mu-\mu_i\right| \le (h+p)\int_0^S (S-\xi)\,|d\Phi(\xi)-d\Phi_i(\xi)|+|\mu-\mu_i|\ .$$

Therefore

$$|C(x;\Phi_i)-C(x;\Phi)| \leq (h+p)\int_0^S S\,|d\Phi(\xi)-d\Phi_i(\xi)| + |\mu-\mu_i| .$$

From condition (A), there exists an N such that $|\mu-\mu_i| < \frac{\epsilon}{2}$ and $|d\Phi(\xi)-d\Phi_i(\xi)| < \epsilon/2S^2(h+p)$ for all $i > N$. Thus

$$|C(x;\Phi_i)-C(x;\Phi)| \leq (h+p)\int_0^S S\,\frac{\epsilon}{2S^2(h+p)}\,d\xi + \frac{\epsilon}{2} < \epsilon$$

which proves the lemma.

All that remains to be shown, in order to prove Theorem 2.1, is

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{S} C(x;\Phi_i)dF_i(x) = \int_{-\infty}^{S}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} C(x;\Phi_i)dF_i(x)$$

$$= \int_{-\infty}^{S} C(x;\Phi)dF(x)$$

given $C(x;\Phi_i) \to C(x;\Phi)$ and $F_i(x) \to F(x)$ as $i \to \infty$. We first note that the uniform integrability of the means μ_i of the distributions $\{\Phi_i\}$ (that is, given $\epsilon > 0$ there exists M such that $\int_M^\infty x\,d\Phi_i(x) < \epsilon$ uniformly in i) follows immediately from conditions (A) and (B2). The following lemmas provide the necessary uniform behavior of $\int_{-\infty}^{S} C(x;\Phi)\, dF_{i-1}(x)$ in order that Lemma 3.3 apply.

Lemma 3: If a demand sequence $\{\Phi_i\}$ satisfies condition (A) then $\int_{-\infty}^{S} C(x;\Phi_{i+1})\, dF_i(x)$, $i \geq 0$ and $\int_{-\infty}^{S} C(x;\Phi)\, dF(x)$ are finite.

Proof: The relation between stock levels at the end of two consecutive periods gives

$$(12)\quad dF_{k+1}(x) = \begin{cases} d\Phi_k(S-x)\displaystyle\int_{-\infty}^{S} dF_k(t) + \int_{x}^{S} d\Phi_k(t-x)\, dF_k(t) & s \leq x \leq S \\[2ex] d\Phi_k(S-x)\displaystyle\int_{-\infty}^{S} dF_k(t) + \int_{s}^{S} d\Phi_k(t-x)\, dF_k(t) & x < s \end{cases}$$

The structure of $C(x;\Phi_i)$ and (12) imply we need only show $\int_{-\infty}^{S} x dF_k(x)$ and $\int_{-\infty}^{S} x dF(x)$ are finite. Equation (12) implies

$$\int_{-\infty}^{S} x dF_k(x) \leq \int_{-\infty}^{S} x d\Phi_k(S-x) + \int_{-\infty}^{S} x \int_{s}^{S} d\Phi_k(t-x) dF_k(t).$$

The first integral on the right is bounded. Interchanging the order of integration in the second and letting $t-x=u$ we have

$$\int_{-\infty}^{S} x dF_k(x) \le \mu_k + S + \int_s^S dF_{k-1}(t) \int_{t-S}^{\infty} (t-u) d\Phi_k(u) \le \mu_k + S + \int_s^S (t+\mu_k) dF_{k-1}(t)$$

which is bounded for bounded s and S. Karlin (Arrow, 1958, p. 234-237) has shown that $F(x)$ is a stationary point of (12). Thus the above arguments show that $\int_{-\infty}^{S} C(x;\Phi) dF(x)$ is finite by letting $\Phi_k(x) = \Phi(x)$ and $F_k(x) = F_{k-1}(x) = F(x)$, which proves the lemma.

Lemma 4: If a demand sequence $\{\Phi_i\}$ satisfies condition (A) then the expected one period costs are uniformly integrable.

Proof: Since all the one period expected costs are finite (Lemma 3) we need only show that given $\epsilon > 0$ there exists M and N such that

$$\int_{-\infty}^{-M} C(x;\Phi_{k+1}) dF_k(x) < \epsilon \tag{13}$$

for all $k > N$. Given ϵ in $(0, 1/4)$ from Lemma 2 there exists $N_1 > 0$ such that $|C(x;\Phi_k) - C(x;\Phi)| < \epsilon^2$ for all $k > N_1$. Since the form of $C(x;\Phi_k)$ for $x < s$ is $C - cx$, it follows from the uniform integrability of the Φ_k and their means that there exists M_1 such that $\int_{-\infty}^{-M_1} C(x;\Phi_{k+1}) d\Phi_k(t-x) < \epsilon^2$ uniformly in k for all $t \in [s,S]$.

From Lemma 3 and the Cauchy criteria for the convergence of improper integrals (Olmsted, 1959, p. 495) there exists $M_2(\epsilon, N_1)$ such that $\int_{-\infty}^{-M_3} C(x; \Phi_{N_1+2})dF_{N_1+1}(x) < \epsilon$. Letting $M = \max\{M_2, M_1, -s\}$ we have demonstrated (13) for $k = N_1+1$. For $k = m > N_1+1$ using (12) we have

$$\int_{-\infty}^{-M} C(x; \Phi_{m+1})dF_m(x) \leq \left| \int_{-\infty}^{-M} [C(x; \Phi_{m+1}) - C(x; \Phi_m)]\, dF_m(x) \right| + \int_{-\infty}^{-M} C(x; \Phi_m)dF_m(x)$$

$$\leq \epsilon^2 + \int_{-\infty}^{-M} C(x; \Phi_m)[d\Phi_m(S-x) + \int_{s}^{S} d\Phi_m(t-x)dF_{m-1}(t)]$$

$$\leq \epsilon^2 + \epsilon^2 + \int_{s}^{S} dF_{m-1}(t) \int_{-\infty}^{-M} C(x; \Phi_m)d\Phi_m(t-x) < \epsilon$$

which proves the lemma.

We now have all the material to show that (2.3) holds. Given $\epsilon > 0$ by Lemma 3 there exists M_1 such that

$$\int_{-\infty}^{-M_1} C(x; \Phi)dF(x) < \frac{\epsilon}{6}, \tag{14}$$

by Lemma 4 there exist N_2 and M_2 such that

$$\int_{-\infty}^{-M_2} C(x; \Phi_{n+1})dF_n(x) < \frac{\epsilon}{6} \tag{15}$$

for all $n > N_2$. Also from Lemma 1, Theorem 1, and Lemma 3.3 there exists N_1 such that, for $n > N_1$, $M = \max\{M_1, M_2\}$

(16) $$\left| \int_{-M}^{S} (C(x; \Phi_{n+1}) dF_n(x) - C(x; \Phi_n) dF(x)) \right| < \frac{\epsilon}{6} .$$

Letting $N = \max\{N_1, N_2\}$ we have from (14), (15), and (16) that for $n > N$

$$\left| \int_{-\infty}^{S} C(x; \Phi_{n+1}) dF_n(x) - \int_{-\infty}^{S} C(x; \Phi) dF(x) \right| < \frac{\epsilon}{2} .$$

Thus $n > N$ implies

$$\left| \sum_{i=1}^{n} \frac{1}{n} \int_{-\infty}^{S} C(x; \Phi_i) dF_{i-1}(x) - \int_{-\infty}^{S} C(x; \Phi) dF(x) \right|$$

$$\leq \left| \frac{1}{n} \sum_{i=N+1}^{n} \left[\int_{-\infty}^{S} C(x; \Phi_i) dF_{i-1}(x) - \int_{-\infty}^{S} C(x; \Phi) dF(x) \right] \right|$$

$$+ \left| \frac{1}{n} \sum_{i=1}^{N} \left[\int_{-\infty}^{S} C(x; \Phi_i) dF_{i-1}(x) - \int_{-\infty}^{S} C(x; \Phi) dF(x) \right] \right|$$

$$\leq \frac{\epsilon}{2} + \frac{N}{n} \max_{i \epsilon [1, N]} \left| \int_{-\infty}^{S} C(x; \Phi_i) dF_{i-1}(x) - \int_{-\infty}^{S} C(x; \Phi) dF(x) \right| .$$

Since

$$\frac{N}{n} \max_{i \in [1, N]} \left| \int_{-\infty}^{S} C(x; \Phi_i) dF_{i-1}(x) - \int_{-\infty}^{S} C(x; \Phi) dF(x) \right|$$

becomes arbitrarily small by taking n large, Theorem 2.1 is proven.

V. DISCUSSION

A primary justification for a paper such as this is the interest in non-identically distributed random variables as shown by articles on stochastically increasing sequences of random variables (Karlin, 1960a and Vienott, 1963), stochastically convergent estimates of the distributions of a sequence of random variables (Scarf, 1959), and periodic demand sequences (Karlin, 1960b). Secondly, these results provide an improved and practicable way to calculate optimal long-run policies based on a predicted stationary demand distribution. In some cases one would suspect that the optimal long-run policy might not deviate much from the true optimal policy for a finite model. Justification of such use would surely require that bounds on the differences between the policies be found. Once again, the true optimal parameters are not used because they vary from period to period and generally can not be calculated.

Because of the complicated relations between the theorems in this paper and results obtained by others, it is necessary that the contributions of these new results be put in a proper perspective. For this reason the objective of this chapter is a discussion of how the theorems in the previous chapters contain or are contained in similar results and to what extent they generalize the usual results in renewal theory and stationary inventory analysis.

1. Renewal Theory

Because of their key importance it seems logical to first discuss the renewal theoretic results. The principal source of results for the discrete, nonidentically distributed random variables, with which we are concerned, is Williamson (1965), although he was only concerned with proving a key renewal theorem. Of Williamson's three sets of conditions, given by (2. 5), (2. 6), and (2. 7), only (2. 5) would necessarily be satisfied by a nonnegative renewal process with a finite mean. For this reason condition (B2) seems reasonable and sufficiently general to justify its use, with no immediate probing into other conditions sufficient for Theorem 3. 1.

Williamson's proof of the key renewal theorem is based on showing the existence of a subsequence of the renewal sequence such that

$$\lim_{n \to \infty} \{ [H_i(n+1) - H_i(n)] - [H_i(n) - H_i(n-1)] \} = 0$$

and

$$\lim_{k \to \infty} \left[\frac{H_i(n+k) - H_i(n)}{k} - \frac{1}{\mu} \right] = 0$$

uniformly in n; thus Theorem 3. 1 is a new proof of the key renewal theorem for discrete, nonnegative, nonidentically distributed stochastically convergent sequences of random variables. Besides being a new proof of a key renewal theorem, Theorem 3.1 holds for random variables whose values are of the form $m+kd$; g.c.d$(m,d)=1$, a situation which Williamson was not able to handle. Thus Theorem 3.1 is not implied by Williamson's result.

An integral formulation of the key renewal theorem was done for discrete and continuous renewal processes by Smith (1954), and he attained an approximation to this result for suitably restricted continuous renewal sequences (Smith, 1961). His results seem to summarize what has been done along these lines and thus Theorem 3.2 should be a strong result. The reason for interest in writing the key renewal theorem in the form of an integral is that the form lends itself to application, a point which is very evident in Theorem 3.3 and Theorem 3.4.

Next to the renewal function the most widely discussed quantity in renewal theory is the excess variable. Theorem 3.3 is a generalization of the usual result concerning the stationary distribution of the excess variable (Smith, 1954). For discrete random variables Theorem 3.3 includes the distribution of the excess variable for renewal processes as a special case.

Theorem 3.4 is another consequence of the key renewal theorem. The significance of this result is discussed previous to the proof of the theorem. Other uses and users are mentioned by

Karlin (Arrow, 1958, p. 277).

2. Inventory Theory

The results in inventory theory, although new, are exactly what would be expected. Since the arguments are based on the results in renewal theory, we expect the same type of generalizations. Thus the derivation of the stationary stock distribution, Theorem 4. 1, includes the same result for renewal processes as a special case. Theorem 2. 1 provides a rigorous derivation of an expression defining the optimal stationary (s, S) parameters. Finally condition (C) is a restriction which guarantees that the convolution of the random variables in a demand sequence satisfying conditions (A), (B) and (C) will again satisfy conditions (A) and (B), a property which other authors assume. An interesting sidelight would be further investigations of conditions like (C), if bounds on the quantity $(S-s)$ were known. We mention in conclusion that the extension of the results to convergent sequences of nonnegative continuous random variables, for which a key renewal theorem holds, should follow with slight modifications.

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