AN ABSTRACT OF THE THESIS OF

ELIZABETH SIMPSON CURL for the degree of MASTER OF SCIENCE

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Title: ANALYTICAL FOUNDATIONS OF PLANE GEOMETRY

Abstract approved: Redacted for privacy

Harry Goheen

This thesis is a part of a supplement to "A Course In Plane Geometry" unpublished notes for an upper class undergraduate course written by Dr. P.D.Barry of the University College, Cork, Ireland. Presented here is the section on Circles and Arc Length .

The primary problem involved obtaining a measure for arc length. As Dr. Barry's definition of angle was in terms of vector rotation, a secondary problem was to obtain a measure of angle in terms of real numbers. Additionally I present a method of calculating cosines and sines.

Arc length is defined as the limit of a sum of chord (and tangent) lengths for a sequence of chords (and tangents) connecting a sequence of points of the arc. Several limit concepts from the Calculus are used.

No attempt to evaluate π is made but a few historical notes about this are included.

Analytical Foundations Of Plane Geometry

by

Elizabeth Simpson Curl

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Redacted for privacy

Chairman of Department of Mathematics

Redacted for privacy

Dean of Graduate School

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Typed by E.S.Curl for Elizabeth Simpson Curl

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ANALYTICAL FOUNDATIONS OF PLANE GEOMETRY

INTRODUCTION

"Acute reckoning - the entrance into the knowledge of all existing things and all obscure secrets."

Ahmes the Scribe
Egypt, 17^{th} Century ,B.C.

"What could be more beautiful than the order, symmetry and definiteness which mathematics exhibits in a very special degree?"

Aristotle (384-322 B.C.)

The early Greeks felt the use of mathematics was to train the mind and enable the soul to get a glimpse of truth; its practical application to every day problems was of no account in comparison.

To the Greeks the proofs of Mathematics yielded certain and eternal truths. In contrast knowledge derived from observation or experiment was merely approximate and ephemeral.

In his Elements Euclid sought to cast in axiomatic form all of the mathematics (geometry,algebra and the theory of numbers) known in his time.

Of the thirteen books of the elements, elementary plane geometry corresponds in one form or another to the first four of these. The great significance of the ELEMENTS is in their method rather than in what the Theorems, as such, say. He took five postulates which he believed so simple and obvious that everyone could accept them and used these as "foundation stones" on which he laid brick after brick making sure each brick was cemented with iron logic and firmly supported by the one previously laid. He thus built a "cathedral" set on a firm foundation and fathered not only geometry but a method of mathematical rigor as well.

The bricks for my humble "cottage" will be the ideas, method, definitions and theorems developed in the notes for "A Course In Plane Geometry" by P.D.Barry of the University College, Cork.

Dr. Barry introduces his work by writing, "...we plan to review the basic geometry we know. We suppose a knowledge of the set $\mathbb{R}$ of real numbers and our approach is essentially to cast our definitions and proofs in the form of coordinate geometry right from the start. Rather than use just coordinates we use vectors as these are very convenient, and start with an account of them."

I have supplemented his notes with illustrations, examples, exercises and several additional sections. However, for this work I will present only section 12.4 : Circles and Arc Lenght , which starts where Dr. Barry's notes stop.

The section numbers of this thesis follow those in Dr. Barry's notes. His last statement is :

12.4 Circles and Arc Length

Given ...and radius r.

My work begins:

12.4.1 Definations .

For clarity and continuity, I have included as appendices:

I) Table of Contents for Dr. Barry's notes.

II) Symbolism and Some Definitions from Barry.

III) Theorems from P.D.Barry used in the text with definitions fron Barry and theorems I added.

The primary problem involved obtaining a measure for arc length. As Dr. Barry's definition of angle was in terms of vector rotation, a secondary problem was to obtain a measure of angle in terms of real numbers. Additionally I present a method of calculating cosines (and sines).

Arc length is defined as the limit of a sum of chord (and tangent) lengths for a sequence of chords (and tangents) connecting a sequence of points of the arc. Several limit concepts from the Calculus are used.

No attempt to evaluate π is made but a few historical notes about this are included.

12.4 Circles and Arc - Lenght

Given $A \in \Pi$ and $r > 0$; the set

$\{ x \in \Pi : d(A,x) = r \}$ is called the circle with center A and radius r.

12.4.1 Definitions:

1) A <u>radius</u> is a segment of any ray with the circle center A as an end point. The lenght of the segment [A,x] is equal to a constant r and each ray has the form

$\overline{\lambda} = A + \ell[B]$.

2) <u>Equal circles</u> are all circles with the same center A and the same radial length r .

3) <u>Equivalent circles</u> are all circles with the same radial lenght but different centers : i.e.

$C_1 \sim C_2$ if and only if $r_1 = r_2$ and $A_1 \neq A_2$.

These can be "equalized" by the translation $t_k(x) = A_1 - A_2$.

4) <u>Concentric circles</u> are all circles with the same center but different radial lenght. These can be "equalized" by the use of a distance unit(section 10.2) $\overline{\mu} = \frac{r_1}{r_2}$.

5) A <u>central angle</u> is any angle whose vertex is at the center A and whose generating lines are circle radii. i.e. $\llcorner\overline{\lambda},\overline{\mu}$ has generating rays $\overline{\lambda} = A + [B]$;

$\overline{\mu} = A + \ell[B]$.

6) A <u>chord</u> is any line segment bounded by two x's of the circle set, i.e. $[x_1 , x_2]$.

7) <u>Chord length</u> is the distance along the chord from one circle point to another. i.e. $d[x_1, x_2]$. A method of computing this length will be discussed later.

8) A <u>diameter</u> is that line segment contained in two rays so related ($\overline{\lambda} = A + k[B],\ 0 < k \leq r$; $\overline{\mu} = A + k[-B],\ 0 < k \leq r$) that they generate a straight angle. If $\llcorner\, x_1 A x_2$ is a straight angle the diameter, $D = [X_1, x_2]$.

a) A diameter contains the center as its midpoint. $[x_1, x_2] = [x_1, A] \cup [A, x_2]$

b) The length of a diameter is 2r; $D = [x_1, x_2]$ and $d[x_1, x_2] = d[A, x_1] + d[A, x_2] = r + r = 2r$

c) A diameter is the longest chord of a circle. By Theorem 10.2.1 when $[x_1, x_2] \neq D$, $A \notin D$ and $d[x_1, x_2]$ is less than $d[A, x_1] + d[A, x_2]$ and the length of any other chord is less than 2r.

9) <u>Adjacent angles</u> are two angles with a common initial point and a common ray or half line.

10) <u>Equal angles</u> are equivalent angles (section 9.1) with the same initial point.

note: Corollary to Thm. 9.1.5 states that the bisecting half line with the same initial point as the angle, "divides" the angle into two equal adjacent angles with the bisecting ray as the common one.

12.4.2 The Unit Circle

Given a circle Γ where $\Gamma = \{ x \in \Pi : d(A,x) = r \}$ a translation $t_k = -A$ and a unit of distance $\mu = \frac{1}{r}$, Γ is transformed into a unit circle (radius =r=1) centered at the origin, $A_t = 0 = (0,0)$ and $r_\mu = 1$.

$$C = \{ x \in \Pi : d(0,x) = 1 \} .$$

Choose an initial point on the circle, $x_o = (1,0)$ and call it I. This can be done through the formalism : Given any half line $\overline{\lambda} = A + [B]$, $\| B \| = r$ use $t_k = -A$, $\mu = \frac{1}{r}$ now $\overline{I} = 0 + [B]$, $\| B \| = 1$ and $(b_1,b_2) = (1,0) = I$.

By section (12.1) with $I = (1,0)$, we can define $I^* = (0,1)$, $-I = (-1,0)$ and $(-I)^* = -(I^*) = (0,-1)$ and by the same section and the definition of diameter, the line segments $[I,-I]$ and $[I^*,-I^*]$ are both diameters, orthogonal to each other, and of lenght 2r with $r = 1$, $d[D] = 2$.

By section 12.2 $x = \ell(I)$ where $(I) = m_1 I + m_2 I^*$ which gives

$$C = \{ x \in \Pi : x = m_1 I + m_2 I^* \} ,$$

where

$$m_1^2 + m_2^2 = 1 \; ; \; -1 \le m_1 \le 1 \; , \quad -1 \le m_2 \le 1 \quad .$$

Also from section 12.2

$$m_1 = \cos \llcorner \overline{\lambda}, \overline{\mu} \qquad , \; m_2 = \sin \llcorner \overline{\lambda}, \overline{\mu} \; .$$

Under our transformation $\llcorner \overline{\lambda}, \overline{\mu}$ has become $\llcorner \overline{I}, \overline{\mu}$.

As $\overline{\lambda}$ is the radiating half line containing the radius $[0,x]$, $\llcorner\overline{\lambda},\overline{\mu}$ is now $\llcorner I,x$,(sec.7.1). This angle corresponds to the Angle of Inclination $I(\overline{\lambda})$ (section 12.2)

Thus

$$\cos \llcorner I,x = m_1 \ , \quad \sin \llcorner I,x = m_2 \ .$$

Specifically: (sections 12.1,12.2)

$$\cos \llcorner I, I = 1 \ , \quad \sin \llcorner I, I = 0 \ ;$$

$$\cos \llcorner I, I^* = 0 \ , \quad \sin \llcorner I, I^* = 1 \ ;$$

$$\cos \llcorner I,-I = -1 \ , \quad \sin \llcorner I,-I = 0 \ ;$$

$$\cos \llcorner I,-I^* = 0 \ , \quad \sin \llcorner I,-I^* = -1 \ .$$

Now we have a third definition for the unit circle :

$$C = \{x \in \Pi : x = (\cos \llcorner I,x \ , \ \sin \llcorner I,x)\}.$$

12.4.2.1 Quadrants

Theorem 12.4.2.1

1) The ray $\overline{I^*}$ and its complementary half line $-\overline{I^*}$ are the bisectors of angles $\llcorner I,-I$ and $\llcorner -I,I$.

2) These bisectors "divide" the circle into four equal adjacent angles.

proof:

1) By definition $\llcorner I,-I$ and its opposite angle

$\llcorner$ -I,I are straight angles.

As $I = k(B)$, $B = (1.0)$ and

$I^* = k(B)^*$, $B^* = (0,1)$

all conditions for the bisector are met, (Thms. 9.1.2,3)

2) $\overline{I}, , \overline{I^*}$ $\overline{-I}$, $\overline{-I^*}$ are all rays with the same initial point 0 = (0,0) therefore the conditions for equal adjacent angles are met. (Corallary to Theorem 9.1.5)

QED.

Each of these angles designate four parts of the circle. We will call these parts <u>Quadrants</u>.

Definition:

$$\text{Quadrant I} = Q_I = \{x \in C : x = m_1 I + m_2 I^*, \text{where } 0 < m_1 \leq 1 \ ; \ 0 \leq m_2 < 1\}.$$

$$\text{Quadrant}_{II} = Q_{II} = \{x \in C : x = m_1 I + m_2 I^*, \text{where } -1 < m_1 \leq 0 \ ; \ 0 < m_2 \leq 1\}.$$

$$\text{Quadrant}_{III} = Q_{III} = \{x \in C : x = m_1 I + m_2 I^*, \text{where } -1 \leq m_1 < 0 \ ; \ -1 < m_2 \leq 0\}.$$

$$\text{Quadrant}_{IV} = Q_{IV} = \{x \in C : x = m_1 I + m_2 I^*, \text{where } 0 \leq m_1 < 1 \ ; \ -1 \leq m_2 < 0\}.$$

Theorem 12.4.2.2

Every angle lies in one of the four quadrants.

proof:

By Theorem 1.4.1 Every vector in $\mathbb{R}^2$ can be expressed in just one way $C = k_1A + k_2B$ with A,B basis vectors and k_1, k_2 coordinates of the vector.

Every point on the circle is in $\mathbb{R}^2$ with I,I* as basis vectors and the restriction $k_1^2 + k_2^2 = 1$.

Every angle can be represented as $\llcorner I,x$ where x is a point on the circle and

$$x = m_1I + m_2I^* \quad ; \quad m_1^2 + m_2^2 = 1 \quad .$$

Thus the restrictions

$$-1 \leq m_1 \leq 1 \quad , \quad -1 \leq m_2 \leq 1 \quad , \quad m_1 = \pm\sqrt{1 - m_2^2}$$

and the relationships of the m_1, m_2 in the four quadrants exhaust all the possibilities.

12.4.3 Measure of angles

As the quadrant designating angles are equal and there are four of them, each is one-forth of the whole circle. We will call the measure of the whole circle 2π, where π is some positive real number whose value has not yet been determined.

The measure of all angles will be represented in terms of π, and we can designate angles by their measure.

So far we have

$$m \llcorner I,I = 0 \; ; \quad m \llcorner I,I^* = \frac{\pi}{2} \; ; \quad m \llcorner I,-I = \pi \; ;$$

$$m \llcorner I,-I^* = \frac{3\pi}{2} \quad \text{and} \quad m(2 \llcorner I,-I) = 2\pi .$$

Theorem 12.4.3.1

By a sequence of repeated bisections the n^{th} bisection

1) divides the circle into 2^n equal adjacent angles.

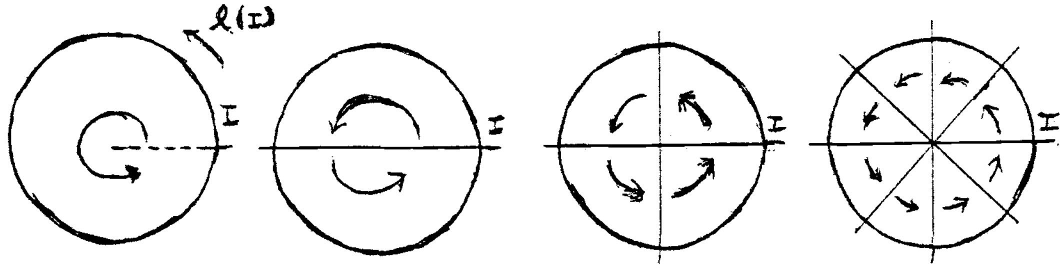
2) whose measure is $\frac{2\pi}{2^n}$ and 3) the angles designated by the $\llcorner I,x_i$, where $\overline{x_i}$ are the bisecting rays, have measures which are $\frac{1}{2^n}\pi, \frac{2}{2^n}\pi, \frac{3}{2^n}\pi, \ldots, \frac{2^{n+1}}{2^n}\pi$.

proof : By induction - By the corollary to Theorem 9.1.5. Each bisection doubles the number of bisectors and angles, and cuts the measure in half.

In tabular form the results are:

# of bisections	# of angles	measure of angles	ends of the x_i
n = 0	$2^0 = 1$	2π	2π
n = 1	$2(2^0)=2^1=2$	$\frac{1}{2}(\frac{2\pi}{2^0})=\frac{2\pi}{2^1}=\pi$	$\frac{1\pi}{2^0},\frac{2\pi}{2^0} = \pi, 2\pi$
n = 2	$2(2^1)=2^2=4$	$\frac{1}{2}(\frac{2\pi}{2^1})=\frac{2\pi}{2^2}=\frac{\pi}{2}$	$\frac{\pi}{2^1},\frac{2\pi}{2^1},\frac{3\pi}{2^1},\frac{4\pi}{2^1} = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$
now assume			
n = k	2^k	$\frac{2\pi}{2^k}$	$\frac{\pi}{2^{k-1}},\frac{2\pi}{2^{k-1}},\dots,\frac{2^k\pi}{2^{k-1}}$
let			
n = k+1	$2(2^k)=2^{k+1}$	$\frac{1}{2}(\frac{2\pi}{2^k})=\frac{2\pi}{2^{k+1}}$	$\frac{\pi}{2(2^{k-1})},\dots,\frac{2(2^k)\pi}{2(2^{k-1})}$ or $\frac{\pi}{2^k},\dots,\frac{2^{k+1}\pi}{2^k} = 2\pi$

n = 0 n = 1 n = 2 n = 3



An extension of the bisecting method gives a <u>measure for all angles</u> in the circle.

Consider the angle $\angle I, x_\theta$, where $x_\theta = m_1 I + m_2 I^*$, whose measure is unknown and two bisectors $\overline{x_i}$ and $\overline{x_j}$, whose angle measures are known. Comparing the chord lengths i.e. $d[x_i,x_\theta] \lesseqgtr d[x_\theta,x_j]$.

1) If = the next bisection gives the exact measure.

2) If $\neq$ the next bisection gives a bisection where one of the new chord lengths is less.

3) For any $\theta = \llcorner I, x_\theta$ and an $\varepsilon>0$ as n(the number of bisections) $\to\infty$ you can find an $\llcorner I, x_i$ so that $d[x_i,x_\theta] < \varepsilon$ no matter how small ε is chosen.

So any angle can be measured in terms of $k\pi$,a real number.

12.4.3.1 Order on the Circle

Since any angle can be represented by a real number, its measure, we can use the natural order of real numbers to order angles and the points (x_θ) of the circle.

Just as there are two natural orders on a sensed line. the reciprocal orders $\leq$ and $\geq$ (discussed in section 5.1,G_6); there are two natural orders on a circle $\ell(I)$ and $\ell^{-1}(I)$ which are reciprocal rotations:

$$\ell(I) = m_1 I + m_2 I^* \quad \text{(called positive rotation)} \quad \text{and}$$
$$\ell^{-1}(I) = m_1 I + m_2(-I^*) \quad \text{or}$$
$$= m_1 I - m_2 I^* \quad \text{(called negative rotation)}.$$

We will use positive rotation unless otherwise specified.

With θ_1 = any angle in Q_I ;

θ_2 = any angle in Q_{II} ;

θ_3 = any angle in Q_{III} and

θ_4 = any angle in Q_{IV} we now have

$\llcorner I,I \le \theta_1 < \llcorner I,I^* \le \theta_2 < \llcorner I,-I \le \theta_3 < \llcorner I,-I^* \le \theta_4 < 2\llcorner I,-I$.

Or in terms of π

$0 \le \theta_1 < \frac{\pi}{2} \le \theta_2 < \pi \le \theta_3 < \frac{3\pi}{2} \le \theta_4 < 2\pi$.

12.4.3.2 Sine and Cosine calculations for angles designated in $k\pi$ measure.

Theorem 12.4.3.2.1

If the values for the cosines and sines of all angles θ_r (the reference angle) where $0 \le \theta_r \le \frac{\pi}{4}$ are known, all other cosines and sines can be evaluated.

proof a) Given an angle θ_1 in Q_I there is an associated angle $\theta_1{}^c$ (the complementary angle) such that $\theta_1 + \theta_1{}^c = \frac{\pi}{2}$.

Then either: 1) $\theta_r = \theta_1 = \theta_1{}^c = \frac{\pi}{4}$; or

2) $\theta_1 = \theta_r$ and $\frac{\pi}{4} < \theta_1{}^c \le \frac{\pi}{2}$ or

3) $\frac{\pi}{4} < \theta_1 \le \frac{\pi}{2}$ and $\theta_1{}^c = \theta_r$.

If 1) or 2) by Theorem 12.2.1

$$\cos\theta_1 = \cos\theta_r \quad ; \quad \sin\theta_1 = \sin\theta_r .$$

If 3) by Theorem 12.2.6

$$\cos\theta_1 = \sin\theta_1{}^c = \sin\theta_r \quad \text{and}$$

$$\sin\theta_1 = \cos\theta_1{}^c = \cos\theta_r .$$

b) Given an angle θ_2 in Q_{II}, there is an associated angle $\theta_2{}^s$ (the supplementary angle) such that $\theta_2 + \theta_2{}^s = \pi$. Then $0 \leq \theta_2{}^s < \frac{\pi}{2}$, So $\theta_2{}^s = \theta_1$.

By Theorem 12.2.1

$$\cos\theta_2{}^s = \cos\theta_1 \quad , \sin\theta_2{}^s = \sin\theta_1 .$$

By Theorem 12.2.5

$$\cos\theta_2 = -\cos\theta_2{}^s = -\cos\theta_1 \quad \text{and}$$

$$\sin\theta_2 = \sin\theta_2{}^s = \sin\theta_1 .$$

c) Given an angle θ_3 in Q_{III}, there is an associated angle $0 \leq \theta_3{}^a < \frac{\pi}{2}$, or $\theta_3{}^a = \theta_1$ such that $\theta_3 = \pi + \theta_3{}^a$.

By Theorem 12.2.1

$$\cos\theta_3{}^a = \cos\theta_1 \quad \text{and} \quad \sin\theta_3{}^a = \sin\theta_1 .$$

By Theorem 12.2.4 and substitution

$$\cos\theta_3 = \cos(\pi + \theta_3{}^a) = \cos\pi\cos\theta_3{}^a - \sin\pi\sin\theta_3{}^a$$

$$= (-1)\cos\theta_1 - (0)\sin\theta_1 = -\cos\theta_1 ,$$

$$\sin\theta_3 = \sin(\pi + \theta_3{}^a) = \cos\pi\sin\theta_3{}^a + \sin\pi\cos\theta_3{}^a$$

$$= (-1)\sin\theta_1 + (0)\cos\theta_1 = -\sin\theta_1 .$$

d) Given an angle θ_4 in Q_{IV}, there is an associated angle θ_1 such that either $\theta_4 = \frac{3\pi}{2}$ or $\theta_4 = 2\pi - \theta_1$.
By Theorem 12.2.4 corollary 1 and substitution

$$\cos \theta_4 = \cos(2\pi - \theta_1) = \cos 2\pi \cos \theta_1 + \sin 2\pi \sin \theta_1 ,$$
$$= (1) \cos \theta_1 + (0) \sin \theta_1 = \cos \theta_1 ,$$
$$\sin \theta_4 = \sin(2\pi - \theta_1) = \sin 2\pi \cos \theta_1 - \cos 2\pi \sin \theta_1 ,$$
$$= (0) \cos \theta_1 - (1) \sin \theta_1 = -\sin \theta_1 .$$

By Theorem 12.4.2.2 : every angle lies in one of the four quadrants so all angles are in a),b),c),or d) above. QED.

To find the values of the cosine and sine of each θ_r, we will use the known values, $\cos 0 = 1, \sin 0 = 0$ and a bisection procedure to find the cosine values for a sequence of θ_r's, $0 \leq \theta_r \leq \frac{\pi}{4}$ where $\theta_r = \frac{\pi}{2(2^n)}$.
Starting with $\theta = \frac{\pi}{2}$, $\cos \frac{\pi}{2} = 0$, we bisect $\frac{\pi}{2}$ to get $\frac{\pi}{4}$ and apply Theorem 12.2.4 corollary 4 to get the value for $\cos \frac{\pi}{4}$. Then bisect $\frac{\pi}{4}$ to get $\frac{\pi}{8}$ and use $\cos \frac{\pi}{4}$ to get $\cos \frac{\pi}{8}$, continuing until we get $\cos \frac{\pi}{2(2^n)}$.

Because all the θ_r's are in Q_I, the positive square roots are always used.

An algebraic change in Theorem 12.2.4 gives the equation

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \frac{1}{2}\sqrt{2+2\cos\theta}\,.$$

Once all the cosines are known the sines can be found:

$$\sin\theta = \sqrt{1-\cos^2\theta}\,,$$

<u>Theorem 12.4.3.2.2</u> The value of the cosines for any

$$= \frac{\pi}{2(2^n)} \text{ is } \frac{1}{2}(2+\underbrace{(2+\ldots+(2)^{\frac12}\ldots)}_{\text{n times}}{}^{\frac12})^{\frac12} \text{ (n times)}$$

Proof: By induction:

$$n = 0,\ \theta_0 = \frac{\pi}{2}\qquad ,\cos\frac{\pi}{2} = 0\,.$$

$$n = 1,\ \theta_1 = \frac{\pi}{2(2^1)} = \frac{\pi}{4},\cos\frac{\pi}{4} = \frac{1}{2}\sqrt{2+2(0)} = \frac{1}{2}\sqrt{2}\,.$$

$$n = 2,\ \theta_2 = \frac{\pi}{2(2^2)} = \frac{\pi}{8},\cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2+2(\tfrac12\sqrt{2})} = \frac{1}{2}\sqrt{2+\sqrt{2}}\,.$$

Assume

$$n = k,\ \theta_k = \frac{\pi}{2(2^k)}\qquad ,\cos\frac{\pi}{2(2^k)} = \frac{1}{2}\sqrt{2+\sqrt{2}}\ \ldots \text{ or}$$

$$= \frac{1}{2}\underbrace{(2+(2+\ldots+(2)^{\frac12}\ldots)^{\frac12})^{\frac12}}_{\text{k times}}$$

Then

$$n = k+1,\theta_{k+1} = \frac{1}{2}\left(\frac{\pi}{2(2^k)}\right),\cos\frac{\pi}{2(2^{k+1})} = \frac{1}{2}(2+2(\tfrac12(\text{ k times}\ldots)^{\frac12})^{\frac12}$$

1 more time

$$= \frac{\pi}{2(2^{k+1})}\qquad\qquad = \frac{1}{2}(2+(2+\ldots+(2)^{\frac12}\ldots)^{\frac12})^{\frac12}\,.$$

QED.

Lemma: Any angle in Q_I can be written as a sum of angles of the preceding bisecting procedure.

Proof: Let B be the set of angles of the bisecting procedure.

For any angle $\alpha \in Q_I$, there is an angle $\beta \in B$ such that $\beta \leq \alpha$ and for any $\beta' \in B$, where $\beta' \neq \beta$, either $\beta' > \alpha$ or $\beta' < \beta$ (β is the largest angle of B which is smaller than α).

Similarly, there is an angle $\beta^* \in B$ which is the Largest angle of B such that $\alpha - \beta = \alpha^* \geq \beta^*$.

For any n: $\alpha^{(n-1)*} - \beta^{(n-1)*} = \alpha^{n*} \geq \beta^{n*}$ where β^{n*} is the largest angle of B which satisfies the inequality.

Then α can be represented as

$$\alpha = \sum_{n=1}^{\infty} \beta^{n*} \qquad \text{(see note on binary fractions)**}$$

Since all the β^{n*} are of the form $\frac{\pi}{2(2^n)}$,

$\sum_{n=1}^{\infty} \beta^{n*} \leq \pi \sum_{n=1}^{\infty} \frac{1}{2^n}$ and there is convergence as $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a well known convergent series. QED.

As we know the cosines (and sines) of our $\theta_{r_n} = \frac{\pi}{2(2^n)}$, we can use the lemma and Theorem 12.1.4 to get the cosine and sine values of all other angles $0 < \theta_r \leq \frac{\pi}{4}$.

Note on binary representation of fractions:** the next few pages discuss binary fractions and their relationship to decimal and proper fractions.

In the preceding bisecting procedure our angles

$$\theta_r = \frac{\pi}{2} \sum_{i=1}^{\infty} a_i \text{ , where } a_i = \frac{1}{2^i} \text{ or } a_i = 0 \text{ .}$$

In binary notation: (see table p.17b for examples)

Let $b_i = 1$ if $a_i = \frac{1}{2^i}$; $b_i = o$ if $a_i = 0$.

Then:

$$\frac{2\theta}{\pi}_r = .b_1 b_2 b_3 \ldots b_i b_{i+1} \ldots$$

Example I (see table p.17b for calculations)

Given : $\frac{2\theta}{\pi}_r$	$= .101011$	in binary
	$= \frac{1}{2^1}+\frac{0}{2^2}+\frac{1}{2^3}+\frac{0}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}$	$\frac{1}{2^n}$ form
	$= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{64}$	proper fraction
	$= .5 + .125 + .03125 + .015625$	
	$= .671875$	decimal fraction

and $\theta_r = \frac{.67185}{2} \pi = .335925\ \pi$.

Example II Given an angle α such that $\frac{\pi}{16} < \alpha < \frac{\pi}{8}$.

Once again,using the calculations on the table p.17b , we have the following : (PF = proper fractions,DF = decimal fractions,BF = binary fractions), as the bisections accur.

(contined p.17c)

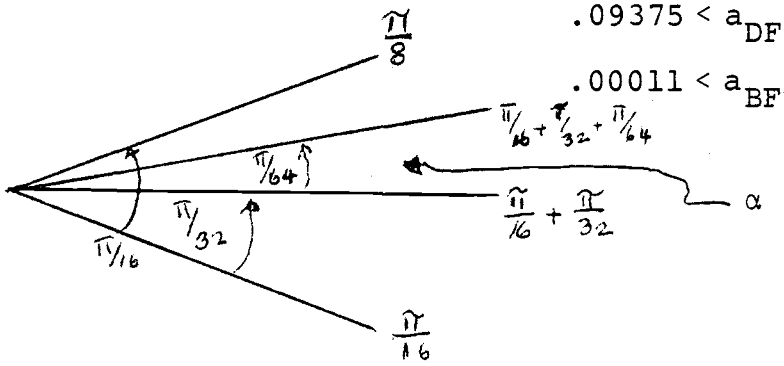
TABLE OF FRACTIONS $\frac{1}{2^n}$, $n \geq 1$, IN VARIOUS NOTATIONS

$\frac{1}{2^n}$	Proper Fraction	Decimal Fraction	Binary Fraction
		(all other places are zeros)	
$\frac{1}{2^1}$	$\frac{1}{2}$	.5	.1
$\frac{1}{2^2}$	$\frac{1}{4}$	.25	.01
$\frac{1}{2^3}$	$\frac{1}{8}$	.125	.001
$\frac{1}{2^4}$	$\frac{1}{16}$	.0625	.0001
$\frac{1}{2^5}$	$\frac{1}{32}$	.03125	.00001
$\frac{1}{2^6}$	$\frac{1}{64}$	.015625	.000001
$\frac{1}{2^7}$	$\frac{1}{128}$	.0078125	.0000001
$\frac{1}{2^8}$	$\frac{1}{256}$	.00390625	.00000001
$\frac{1}{2^9}$	$\frac{1}{512}$	.001953125	.000000001
$\frac{1}{2^{10}}$	$\frac{1}{1024}$	.0009765625	.0000000001

After two bisections ($\alpha = a\pi$): $\frac{1}{16} + \frac{1}{32} < a_{PF} < \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$

$.09375 < a_{DF} < .109375$

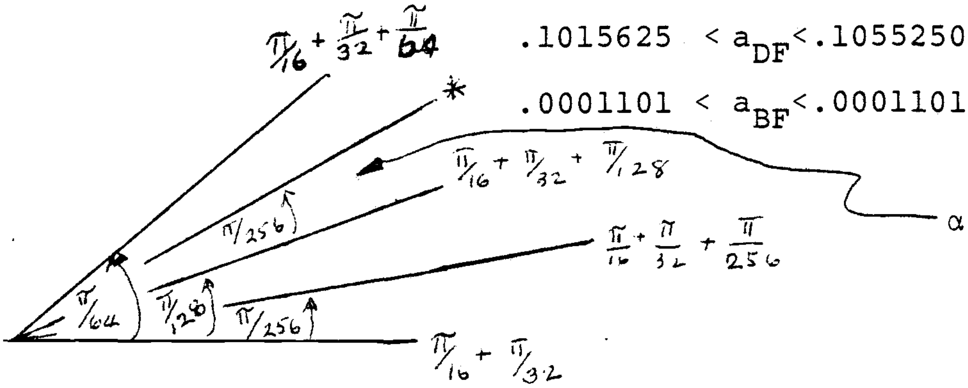
$.00011 < a_{BF} < .000111$



After two more bisections: $\frac{1}{16} + \frac{1}{32} + \frac{1}{128} < a_{PF} < \frac{1}{16} + \frac{1}{32} + \frac{1}{128} + \frac{1}{256}$ *

$.1015625 < a_{DF} < .1055250$

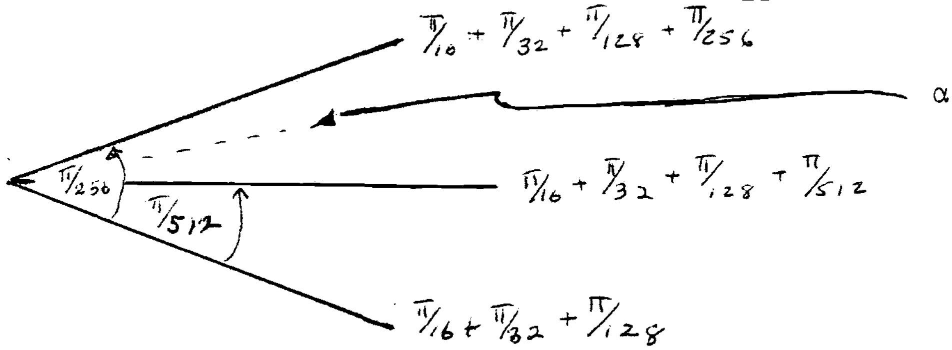
$.0001101 < a_{BF} < .00011011$



One more gives: $\frac{1}{16} + \frac{1}{32} + \frac{1}{128} + \frac{1}{512} < a_{PF} < \frac{1}{16} + \frac{1}{32} + \frac{1}{128} + \frac{1}{256}$

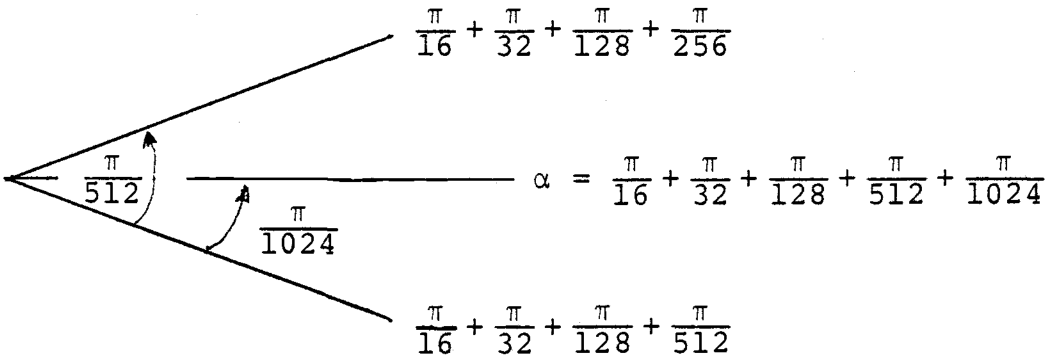
$.10351562 < a_{DF} < .1055250$

$.000110101 < a_{BF} < .000110110$



Checking the chord lenghts shows one more bisection will give us α exactly. (If not we would continue.)

This last bisection



gives:

$$a_{PF} = \frac{1}{16} + \frac{1}{32} + \frac{1}{128} + \frac{1}{512} + \frac{1}{1024} = \frac{107}{1024}$$

$$a_{DF} = .0625 + .03125 + .0078125 + .001953125 + .0009765625 = .10449218$$

a_{BF} = .0001101011 (this is the number the lemma would give and the other forms follow) .

Thus the value for α we found is

$$\alpha = .10449218\,\pi = \frac{107}{1024}\pi \cdot$$

12.4.4 Arcs

12.4.4.1 Definition of Arc

An <u>arc</u> is a circle segment.

The arc from x_0 to x_t (positive rotation) is defined:

$$x_o @ x_t = \{x \in \Gamma : x = (a_1 + r \cos \theta, a_2 + r \sin \theta) \text{ where } \theta_0 \leq \theta \leq \theta_t\}$$

for any circle of center A, $r > 0$. More simply for the unit circle

$$x_0 @ x_t = \{x \in C : x = (\cos \theta, \sin \theta); \theta_0 \leq \theta \leq \theta_t\}.$$

The length of this arc, if it exists, will be defined as the limit of a sum of chord lengths for a sequence of chords connecting a sequence of points of the arc. But before making the arc length definition precise, additional ideas need to be developed.

<u>Theorem 12.4.4.1</u> The length of the chord $[x_1, x_2]$ is equal to two times the radius times the absolute value of the sine of one half the central angle subtended by the radii $0x_1$ and $0x_2$.

$$d[x_1, x_2] = 2r \left| \sin \tfrac{1}{2} \angle x_1, x_2 \right| .$$

proof: For simplicity let us use a unit circle with a central angle whose measure is less than one half π . Refer to Figure 12.4.4.1.

1) Draw the radii $0,x_1$ and $0,x_2$ and the chord $[x_1,x_2]$

2) Drop the altitude (unique by Thm. 2.3.1) from x_2 to $[0,x_1]$ and call the foot of the perpendicular, P.

3) We now have 2 triangles $P0x_2$ and x_1Px_2 with Px_2 perpendicular to 0P and Px_1 at P. So by Theorem 12.3.2

$$\sin \angle P0x_2 = \frac{d(P,\ x_2)}{d(0,\ x_2)} = \frac{d(P,\ x_2)}{r} \quad \text{and}$$

$$\cos \angle P0x_2 = \frac{d(P,\ 0\ )}{d(0,\ x_2)} = \frac{d(P,\ 0\ )}{r}$$. Call the angle θ.

4) $d(x_1,0) = d(x_1,P) + d(P,0)$ by construction ,

then $d(x_1,P) = d(x_1,0) - d(P,0) = r(1 - \cos\theta)$.

5) In triangle x_1Px_2 by Theorem 10.2.2

$d(x_1,x_2)^2 = d(x_1,P)^2 + d(P,x_2)^2$. By the above and substitution

$$\begin{aligned} d(x_1,x_2)^2 &= r^2(1 - \cos\theta)^2 + r^2(\ \sin\ \theta\ )^2 \\ &= r^2[\ 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta] \\ &= r^2[\ 2 - 2\cos\theta\ ] = 4r^2[\frac{1 - \cos\theta}{2}] . \end{aligned}$$

6) By corollary 4 to Theorem 12.2.4

$d(x_1,x_2)^2 = 2^2r^2(\sin\frac{1}{2}\theta)^2$ or $d[x_1,x_2] = 2r\,|\sin\frac{1}{2}\theta|$.

In the other quadrants, the sine values are the same if you consider the absolute values. QED .

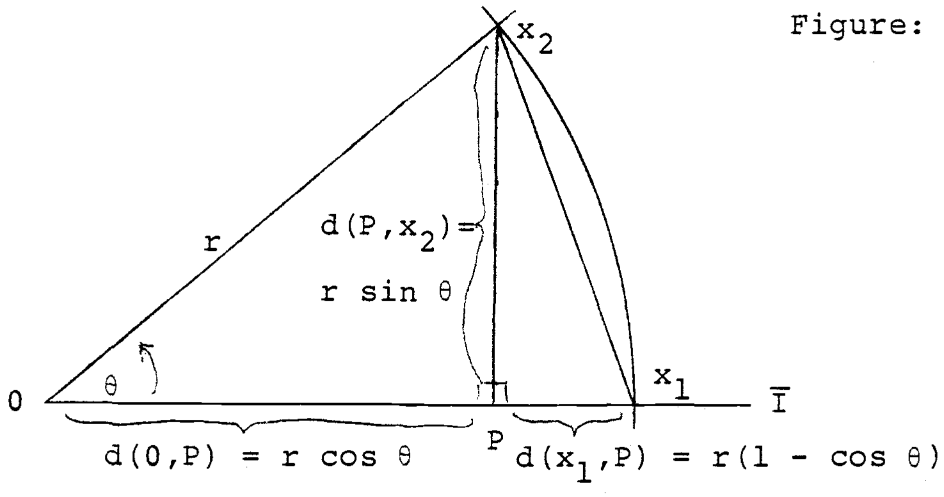


Figure: 12.4.4.1

<u>Theorem 12.4.4.2</u> All points on a chord are interior. That is at a distance from the center which is less than or equal to the radius, with equality only at the bounding points.

Proof: Refer to Figure 12.4.4.2.

1) Given the unit circle, radii and chord $[x_1, x_2]$. Choose an arbitrary point D on the chord $[x_1, x_2]$.

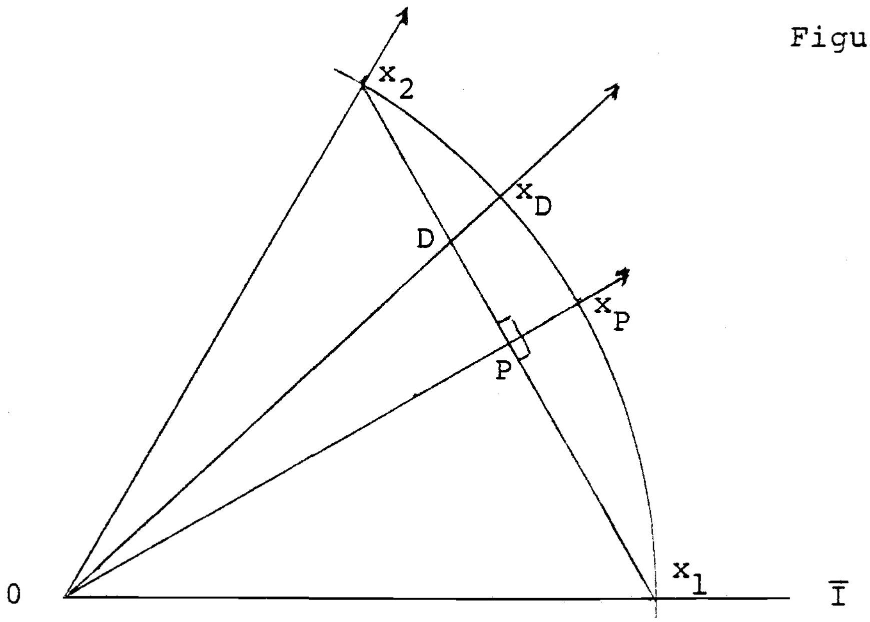
2) If $D = x_1$ or x_2 it lies on both chord and circle, is a bounding point and at a distance equal to the radius.

3) If $D \neq x_1$ or x_2 , construct the ray containing D and the radius $[0, x_D]$

4) $d[D, x_1] > 0$; $d[D, x_2] > 0$ by construction and Theorem 10.2.1 (ii) .

5) $d[x_1, D] + d[D, x_2] = d[x_1, x_2]$, Theorem 10.2.1 (iii).

Figure 12.4.4.2



6) If $d[x_1,D] = d[D,x_2]$, D is the midpoint and $[0,D]$ is the altitude of triangle $x_1 0 x_2$ and perpendicular to the chord. $d[0,x_1]^2 = d[0,D]^2 + d[D,x_1]^2$ and

$d[0,D] < d[0,x_1] = r$.

7) If $d[x_1,D] \neq d[D,x_2]$, construct the ray containing the perpendicular from 0 to the chord. Call the foot of the perpendicular, P.

8) By Theorem of Pythogoras again

$d[0,x_2]^2 = d[0,P]^2 + d[P,x_2]^2$ and

$d[0, D]^2 = d[0,P]^2 + d[P, D]^2$.

Thus: $d[P,x_2] = d[P,D] + [d[D,x_2] \Rightarrow d[P,D] < d[P,x_2]$

By substitution:

$$d[0,D]^2 < d[0,x_2]^2 \Rightarrow d[0,D] < d[0,x_2] \text{ .}$$ QED.

Def. 12) The unique line perpendicular to a radius $[0,x_T]$ at the point x_T on the circle is called the <u>tangent to the circle</u> at x_T.

Def. 13) The <u>point of tangency</u> is the unique point, x_T, contained in the circle set, radius $[0,x_T]$ and the line tangent to the circle.

<u>Theorem 12.4.4.3</u> A tangent to a circle intersects the circle only at the point of tangency. All points on the tangent are exterior (at a distance from the center equal to or greater than the radius, with equality only at the point of tangency. (See figure 12.4.4.3 below).

Proof: 1) given the circle, rays $\bar{I}$ and $\bar{\mu}$ as shown, construct $\bar{T} = x_1 + kI^*$, <u>the</u> perpendicular to $[0,x_1]$ (Thm. 2.3.1)

2) Choose $S \in \bar{T}$, construct $\bar{\alpha}$ containing $[0,S]$.

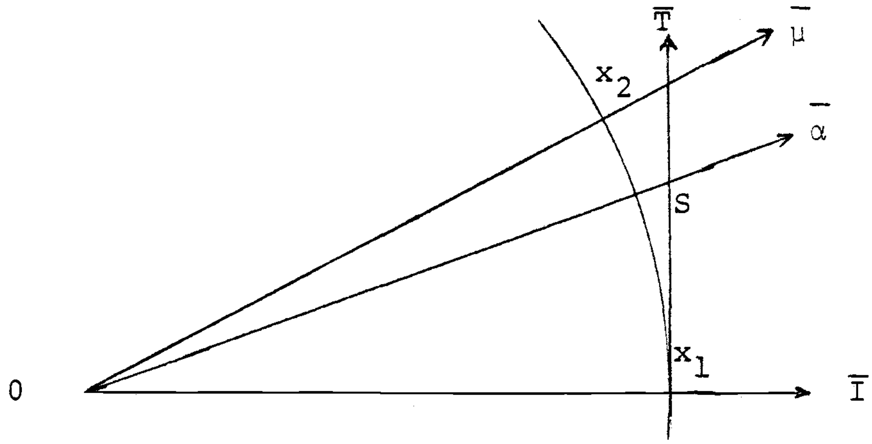


Figure 12.4.4.3

3) If $S = x_1$ then the $d[0,S] = d[0,x_1] = r$.

4) If $S \neq x_1$; $d[x_1,S] > 0$ and by Thm. of Phythogoras $d[0,S]^2 = d[0,x_1]^2 + d[x_1,S]^2 \Rightarrow d[0,S]^2 > d[0,x_1]^2 = r$.

Corollary to Theorems 12.4.2.2 and 12.4.2.3

Points in the circle set are both interior and exterior.

Theorem 12.4.4.4 Tangents to a circle at any two points not on a diameter (see Thm 2.3.2 iii), intersect at a point which is equidistant from both points of tangency.

Proof: This follows from Thm. 11.2.1 which also says the intersection point is on the bisector of the central angle. Or by construction and Pythagoran Theorem ,

$d[0,x_1] = d[0,x_2] = r$, $d[0,V] = d[0,V]$ (reflexive) ,

tangents perpendicular by definition and construction,

$d[0,V]^2 = d[0,x_1]^2 + d[V,x_1]^2$, and

$d[0,V]^2 = d[0,x_2]^2 + d[V,x_2]^2$, then by substitution

$$d[V,x_1]^2 = d[V,x_2]^2 \text{ and } d[V,x_1] + d[V,x_2] .$$

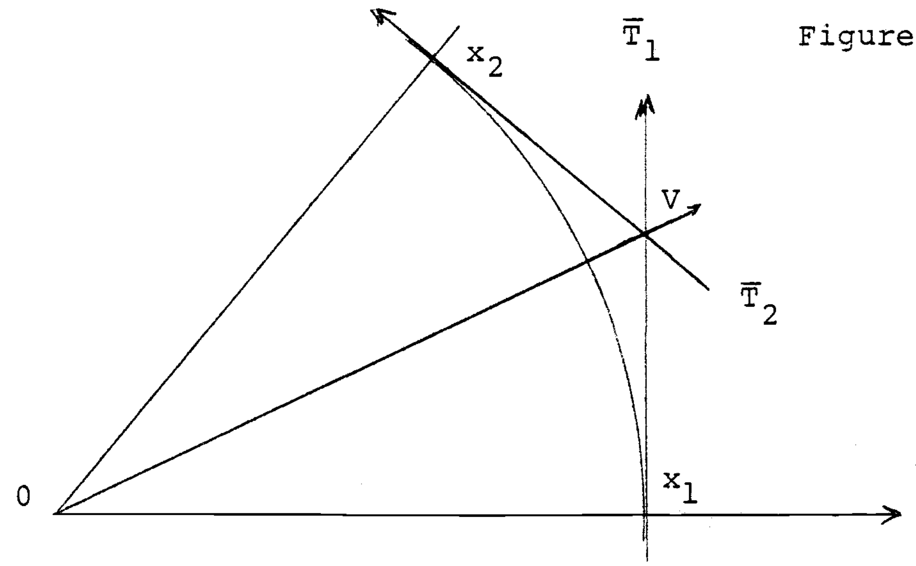
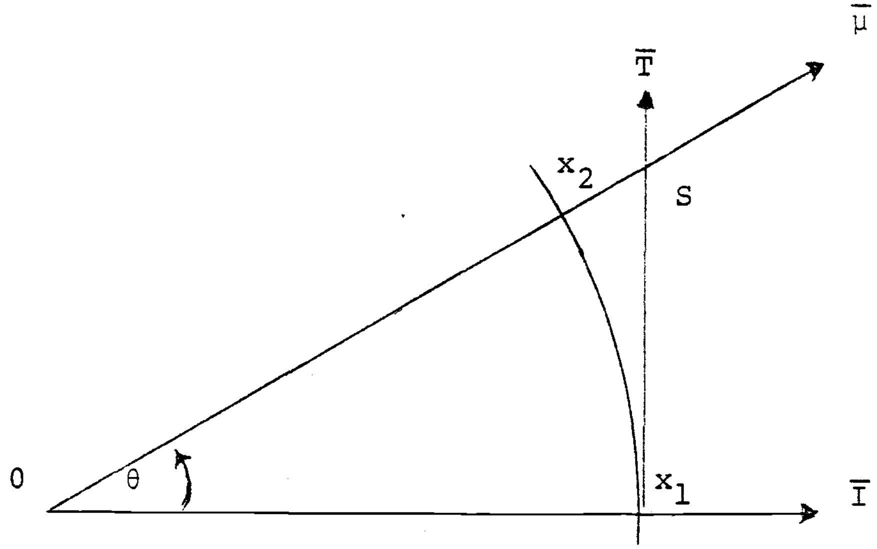


Figure 12.4.4.4

Definition 14) The <u>tangent function</u> of the angle θ

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{m_2}{m_1}.$$

The name "tangent" is an appropriate one for the tangent function since it is a segment of the tangent line to the circle. Refer to Figure 12.4.4.5.



1) Given $\overline{I}$ containing $[0,x_1] = r$ and $\overline{\mu}$ containing $[0,x_2] = r$, construct the tangent to $\overline{I}$ at x_1. Call the point of intersection of $\overline{T}$ and $\overline{\mu}$, S.

2) By Thm 12.3.2 $\cos\theta = \dfrac{d[0,x_1]}{d[0,S]}$, $\sin\theta = \dfrac{d[x_1,S]}{d[0,S]}$.

3) The ratio $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\dfrac{d[x_1,S]}{d[0,S]}}{\dfrac{d[0,x_1]}{d[0,S]}} = \dfrac{d[x_1,S]}{d[0,x_1]}$ and for a unit circle,

where $d[0,x_1] = r = 1$ $\quad \dfrac{\sin\theta}{\cos\theta} = \tan\theta = d[x_1,S]$.

Definition 15) <u>Polygonal arc</u>

Given points of the circle $(x_1, x_2, x_3, \ldots, x_{n+1})$, where $\ell_1(x_1) = x_2$, $\ell_2(x_2) = x_3$, ... ,$\ell_n(x_n) = x_{n+1}$.

A one sided polygonal arc is the chord $[x_1, x_2]$.

A two sided polygonal are is the union of the chords : $[x_1, x_2] \cup [x_2, x_3]$.

An n sided polygonal arc is the union $[x_1, x_2] \cup [x_2, x_3] \cup \ldots \cup [x_n, x_{n+1}]$.

Definition 16) Length of a polygonal arc:

$H = \sum_{i=1}^{n} d[x_i, x_{i+1}]$, where n is the number of sides and n + 1 is the number of points.

12.4.5 Arc Length

12.4.5.1 Approximating the arc by chordal polygonal arcs.

Consider an arc $x_1@x_k$. We will construct a sequence of chordal polygonal arcs i.e. $[x_1,x_k]$; $[x_1,x_2]\cup[x_2,x_k]$;... $[x_1,x_2]\cup[x_2,x_3]\cup \ldots \cup[x_{k-1},x_k]$ and call each sucessive polygonal arc a refinement of the previous one.

The following are true:

1) The chordal polygonal arcs all consist of interior points (Thm. 12.4.2.2 and Def.15).

2) Each refinement will have more chords (and more bounding points) than the previous arc (Construction).

3) The length of each refinement is greater than the previous one (Refer to Figure 12.4.5.1).

$d[x_1,x_2] + d[x_2,x_k] > d[x_1,x_k]$ by the triangle inequality Thm 10.2.1 iii; and inductively the length of one chord is less than the lenght of two chords. When at least one more point is chosen the refinement has at least one more chord and the length increases.

12.4.5.2 Approximating the arc by tangental polygonal arcs

Consider an arc $x_1 @ x_k$. Construct a sequence of tangental arcs : $[x_1, V_1] \cup [V_1, x_k]$; $[x_1, V_{12}] \cup [V_{12}, x_2] \cup [x_2, V_{2k}] \cup [V_{2k}, x_k]$; ... and call each successive polygonal arc a refinement of the previous arc.

The following are true:

1) The tangential polygonal arcs all consist of exterior points (Thm 12.4.2.3 and Def. 15).

2) Each refinement will have more points of tangency, more tangents, and more intersections of tangents. (by construction - see Figure 12.4.5.2).

3) The length of each refinement is less than the length of the previous refinement (see Fig.12.4.5.2).

By Thm 10.2.1 iii the triangle inequality $d[V_{12}, V_{2k}] < d[V_{12}, V_{1k}] + d[V_{1k}, V_{2k}]$, so by substitution

$$d[x_1, V_{12}] + d[V_{12}, V_{2k}] + d[V_{2k}, x_k] <$$

$$d[x_1, V_{12}] + d[V_{12} V_{1k}] + d[V_{1k} V_{2k}] + d[V_{2k}, x_k].$$

Inductively, the addition of at least one new point shortens the path between the adjacent points thus shortening the total arc.

Figure 12.4.5.1

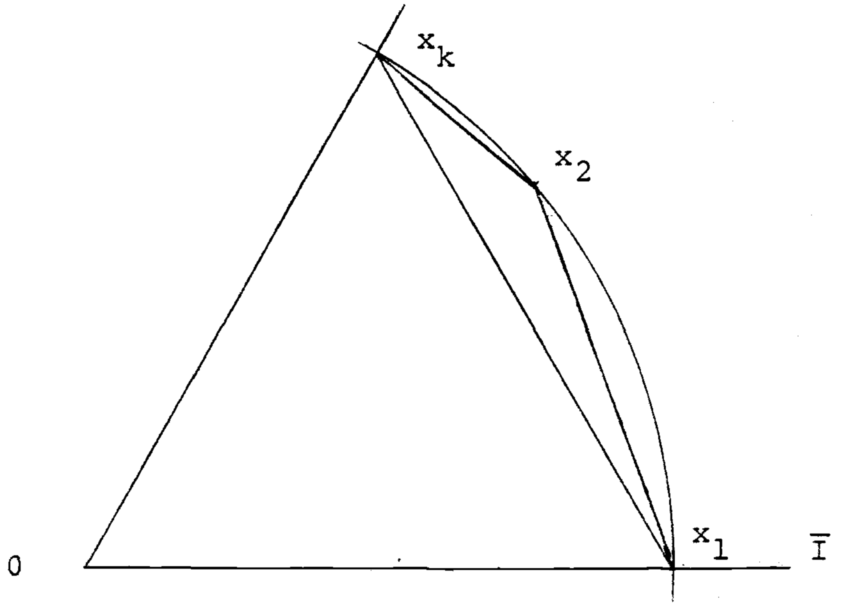
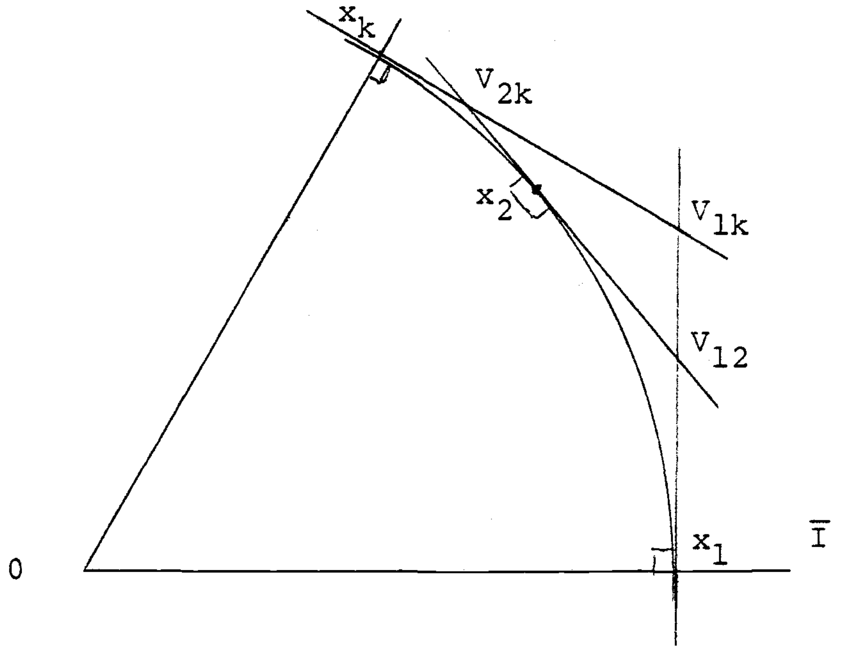


Figure 12.4.5.2



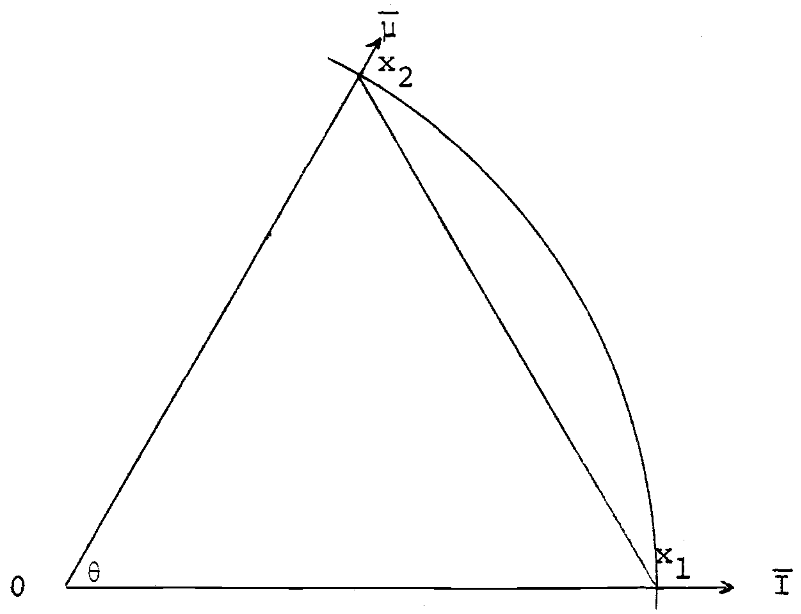
12.4.5.3 Closer approximations

<u>Theorem 12.4.5.3.1</u> The length of a chordal polygonal arc of 2^{n-1} chords and n-1 refinements is:

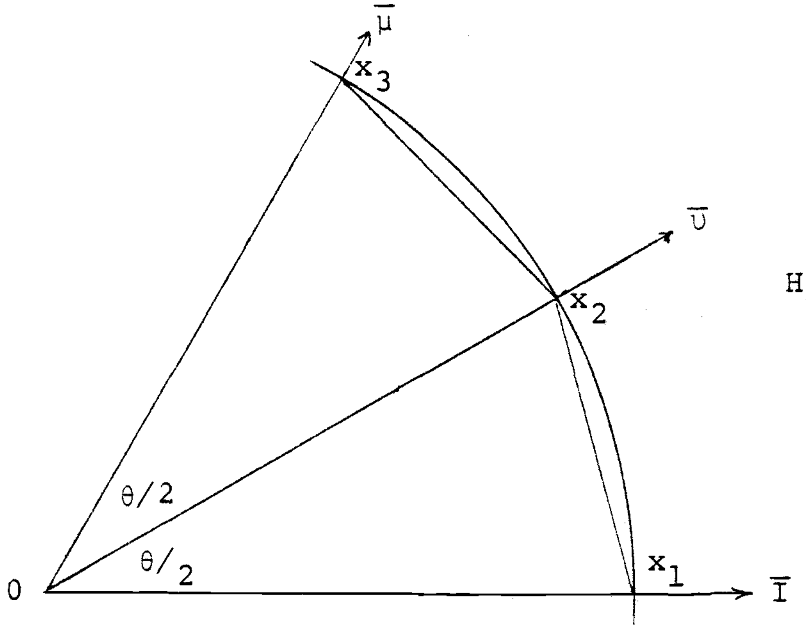
$$H_{c_n} = 2^n \sin \frac{\theta}{2^n} .$$

Proof: We will pick our points so that for each successive refinement new points are picked that lie on the bisector(s) of each of the previous central angles. This gives chords of uniform length.

1) Consider the unit circle, $x_1 @ x_2, [x_1, x_2]$ and central angle θ as shown in Figure 12.4.5.3.1 below. Here n = 0, no refinements and by Theorem 12.4.2.1 $H_{c_1} = d[x_1, x_2] = 2 \sin \frac{\theta}{2}$.



2) Make the 1st refinement by constructing $\bar{\upsilon}$ the bisector of θ ,select the point where it intersects the circle, renumber the points, continue the refinements as shown in the figures below, Figures 12.4.3.3.2-3.

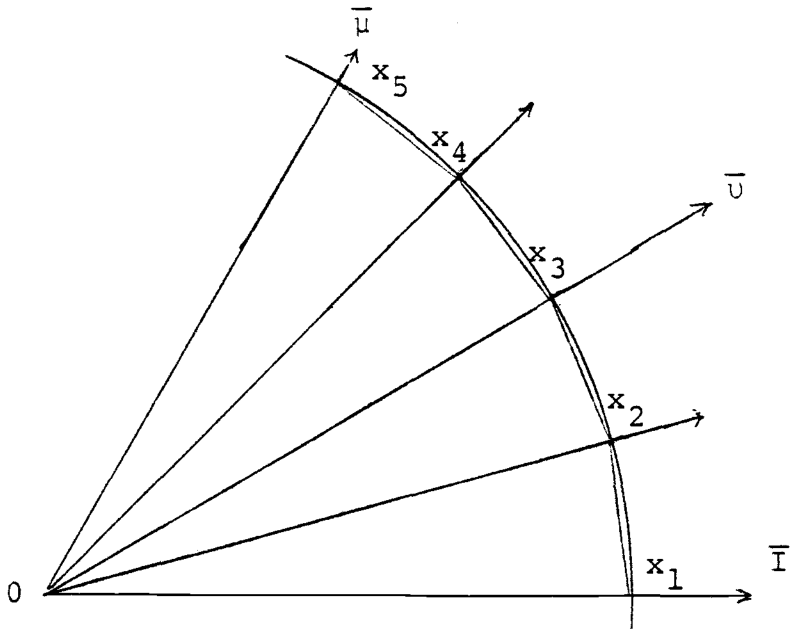


1[st] refin.

$n = 2$

2 chords

$$H_{c_2} = d[x_1, x_2] + d[x_2, x_3] = 2 \sin \frac{\theta/2}{2} + 2 \sin \frac{\theta/2}{2} = 2^2 \sin \frac{\theta}{4} .$$



2[nd] refin.

$n = 3$, 4chords and 4 angles each $1/4\ \theta$ so

$$H_{c_3} = 8 \sin \frac{\theta}{8} .$$

Continuing by induction:

Assume $n = k$ $\quad H_{C_k} = 2^k \sin \frac{\theta}{2^k}$. At the next refinement

$n = k+1$, # chords $= 2(2^k) = 2^{k+1}$, angle $\frac{1}{2}(\frac{\theta}{2^k}) = \frac{\theta}{2^{k+1}}$

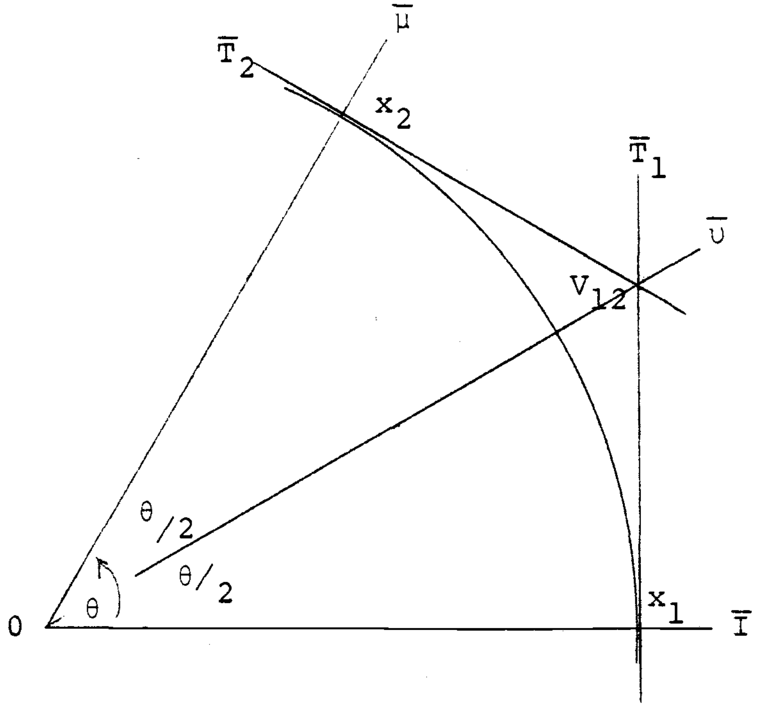
Thus $H_{C_k} = 2^k \sin \frac{\theta}{2^k}$; $H_{C_{k+1}} = 2^{k+1} \sin \frac{\theta}{2^{k+1}}$. QED.

<u>Theorem 12.4.5.3.2</u> The length of a tangental polygonal arc of n-1 refinements is:

$$H_{T_n} = 2^n \tan \frac{\theta}{2^n} .$$

Proof: Using the same point selection method as in the preceding theorem, n = 1, no refinements, 2 equal tangents. and by Def.14,15,16 and Thm. 12.4.2.4.

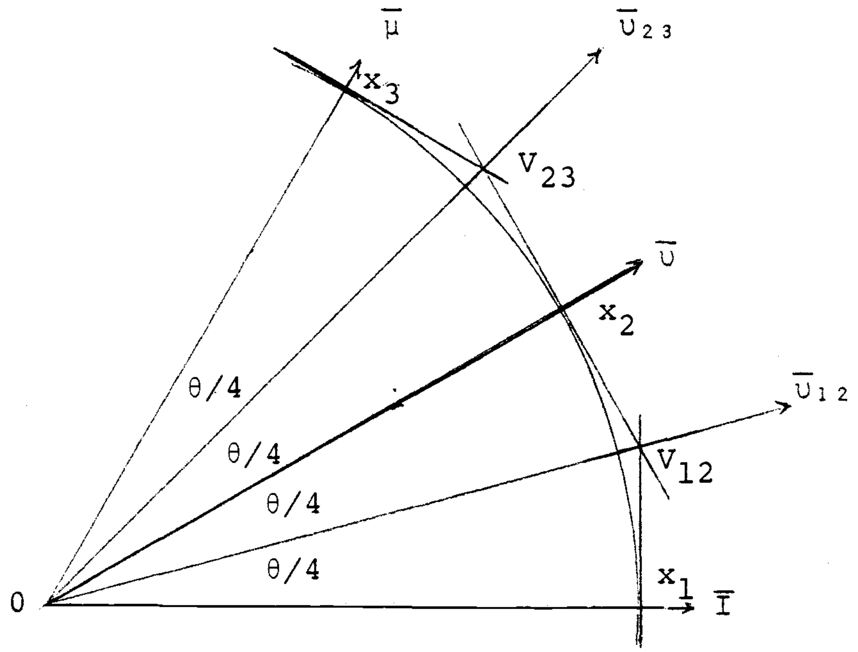
$H_{T_1} = d[x_1, V_{12}] + d[V_{12}, x_2] = \tan \frac{\theta}{2} + \tan \frac{\theta}{2} = 2 \tan \frac{\theta}{2}$.



Making the first refinement $n = 2$, 4 tangent segments

$$H_{T_2} = d[x_1,V_{12}] + d[V_{12},x_2] + d[x_2,V_{23}] + d[V_{23},x_3]$$

$$= \tan\frac{\theta}{4} + \tan\frac{\theta}{4} + \tan\frac{\theta}{4} + \tan\frac{\theta}{4} = 4\tan\frac{\theta}{4}\ .$$



When $n = 3$, 2^{nd} refinement, 8 tangent sections $H_{T_3} = 8\tan\frac{\theta}{8}$

Assume $H_{T_k} = 2^k \tan\frac{\theta}{2^k}$ then $H_{T_{k+1}} = 2(2^k)\tan\frac{1}{2}(\frac{\theta}{2^k})$

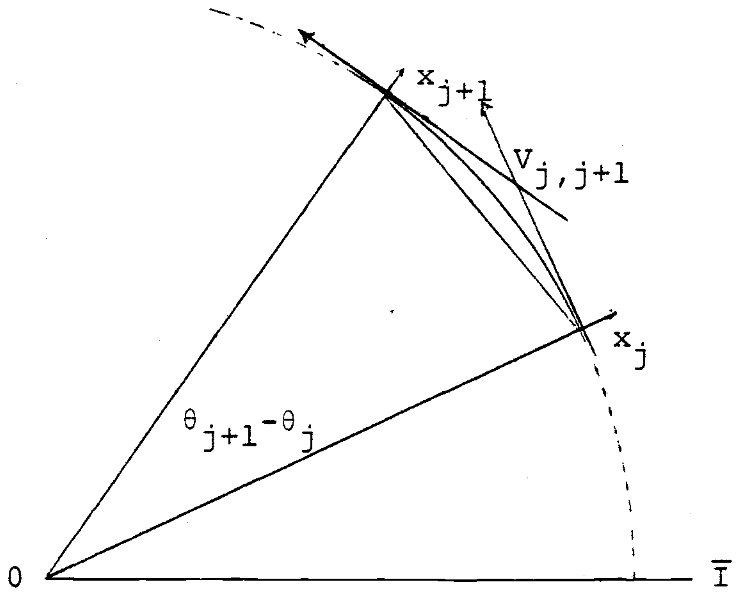
So $H_{T_{k+1}} = 2^{k+1}\tan\frac{\theta}{2^{k+1}}$.

QED .

12.4.5.4 Evaluation of arc length.

Comparing chordal polygonal arcs and tangental polygonal arcs. Examine a small section from x_j to x_{j+1}. See Figure 12.4.5.4.1 below.

Figure 12.4.5.4.1



By Theorems 12.4.4.2 and 3, $V_{j,j+1}$, the tangent intersection point $\notin$ chord $[x_j, x_{j+1}]$.

By the triangle inequality (Thm. 10.2.1 iii)

$d[x_j, x_{j+1}] \le d[x_j, V_{j,j+1}] + d[V_{j,j+1}, x_{j+1}] \Rightarrow$

Σ chord lengths $\le$ Σ tangent lengths $\Rightarrow$

length chordal polygonal arc $\le$ length tangental polygonal arc and by substitution with n-1 refinements

$$2^n \sin \frac{\theta}{2^n} \le 2^n \tan \frac{\theta}{2^n} .$$

12.4.5.4.1 Definition

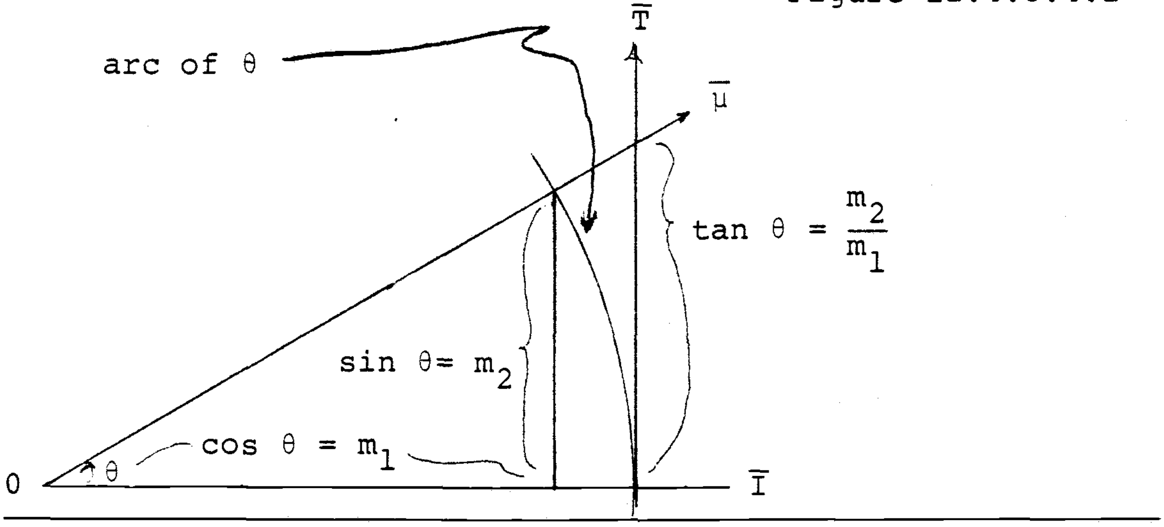
The <u>length of arc</u> for the unit circle is given by

$\lim_{n\to\infty} H_{C_n} = \lim_{n\to\infty} H_{T_n}$, provided that these limits exist and are equal.

Lemma:

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 .$$

Figure 12.4.5.4.2



$\sin\theta \le \theta \le \tan\theta$, (Def. of functions, Thm.12.4.4.3, and Def. 12.4.5.1)

$$\Rightarrow \quad 1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta} \qquad \text{(Def. of functions)}$$

$$\Rightarrow \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Theorem 12.4.5.4.1

1) The limit of the sequences of tangental polygonal arcs $\lim_{n\to\infty} H_{T_n}$ exists,

2) The limit of the sequences of chordal polygonal arcs $\lim_{n\to\infty} H_{C_n}$ exists, and

3) These limits are equal.

Proof:

1) $\lim_{n\to\infty} H_{T_n}$ exists because the sequence is monotonically decreasing and bounded by the H_{C_n} sequence (sections directly preceding).

2) $\lim_{n\to\infty} H_{C_n}$ exists because the sequence is monotonically increasing and bounded by the H_{T_n} sequence (sections directly preceding) .

3) The limits are equal because the limit of their difference is zero. i.e.

$$\lim_{n\to\infty} \left| 2^n \tan \frac{\theta}{2^n} - 2^n \sin \frac{\theta}{2^n} \right| = \lim_{n} 2^n \left|\tan \frac{\theta}{2^n}\right| \left| 1 - \cos \frac{\theta}{2^n} \right| .$$

As
$$\left| \frac{\sin \frac{\Theta}{2^k}}{\cos \frac{\Theta}{2^k}} \right| \left| 1 - \cos \frac{\Theta}{2^k} \right| \approx \left| \frac{\frac{\Theta}{2^k}}{\cos \frac{\Theta}{2^k}} \right| \left| 1 - \cos \frac{\Theta}{2^k} \right| ,$$

by lemma 12.4.5.4.1 when Θ is small, and since we know from calculus that $\lim_{n\to\infty} \cos \frac{\Theta}{2^n} = 1$,

$$\lim_{n\to\infty} \left| 2^n \tan \frac{\Theta}{2^n} - 2^n \sin \frac{\Theta}{2^n} \right| = \lim_{n\to\infty} \left| \frac{\Theta}{\cos \frac{\Theta}{2^n}} \right| \left| 1 - \cos \frac{\Theta}{2^n} \right| = 0.$$

QED.

<u>Theorem 12.4.5.4.2</u> For a unit circle, the length of an arc is given by the measure of the angle whose rays subtend it.

$$d(x_0 @ x_t) = \theta$$

Proof: By the previous section

$$\lim_{n\to\infty} 2^n \sin \frac{\theta}{2^n} = \lim_{n\to\infty} 2^n \frac{\theta}{2^n} = \theta \ .$$

QED.

<u>Theorem 12.4.5.4.3</u> For a circle of radius r, the length of an arc is given by the radius length times the measure of the angle whose rays subtend it.

$$d(x_0 @ x_t) = r\theta$$

Proof: By Theorem 12.4.4.1

$$d[x_1,x_2] = 2r \left| \sin \frac{1}{2} \llcorner x_1,x_2 \right|$$

Let $\llcorner x_1,x_2 = \theta$,then by Theorem 12.4.5.3.1 the length of the chordal polygonal arc from x_1 to x_2 is given by

$2^n \, r \sin \frac{\theta}{2^n}$ and $\lim_{n\to\infty} 2^n \, r \sin \frac{\theta}{2^n} = \lim_{n\to\infty} 2^n \, r \frac{\theta}{2^n} = r\theta$. QED.

12.4.6 Historical notes

I have made no attempt here to evaluate Pi (π). Estimates of the value of π have occupied both mathematicians and non-mathematicians alike since ancient times. One interesting and informative book on the subjuct is Petr Beckman's A History of Pi. A few of the many "circle squarers" and their estimates of π mentioned are:

From Babylonian tablets $\pi = 3\frac{1}{8} = 3.125$.

Archimedes of Syracuse (ca. 287-212 BC) used a "method of exhaustion" to approximate circle (area) by a sequence of regular polygons. Thru this process the number π was found to be the area of a unit circle, and the estimate $3.14163 < \pi < 3.142857$. Toeplitz (1963) says "Archimedes ...(wished) to find fractions with lowest possible numerators and denominators which should equal the ratio between the circle and the square over its radius as closely as possible. ... $3\,\frac{10}{71} < \pi < 3\,\frac{10}{70}$.

The Bible gives the integer 3 as the ratio of radius divided into circumference.

5th Century chinese scholars gave the value

$$3.1415926 < \pi < 3.1415927 .$$

Legendre in his Elements de Geometrie (1794) proved the irrationality of π more vigorously than had Lambert.

Legendre also joined Euler and many others in supposing it was also not algebraic but transcendental.

In 1882 F. Lindemann finally succeeded in proving π was transcendental.

It is interesting to compare two essentially different conceptions of the number π ,that employed by Archimedes and the concept employed in this thesis.

To Archimedes π was the ratio of the area of a circle to the square of the radius. In this thesis π is the angular measure of a straight angle. The fact that these two conceptions lead to the same number, π , is a discovery and must be formulated as a theorem and proved. The theorem takes us beyond the subject of this thesis.

12.4.7 Suggestions for additional work

Within the formalism of the original notes the following would make a logical follow up.

1) Stating and proving the classic circle, triangle theorems - particulary: Any triangle inscribed in a circle with the diameter as one side is a right triangle.

2) Extending the ideas to areas of triangles,other straight sided geometric figures and circles.

3) Comparing π as a measure of area with the angle measurement use of π in this thesis (backwards Archimedes).

4) Compilation of the notes of Dr. Barry , with additions ,into a study guide or text for the upper undergraduate level.

BIBLIOGRAPHY

Beckmann, Petr. The History of π (Pi). St. Martin's Press, New york, New York, 1971.

Toeplitz, Otto. The Calculus, A Genetic Approach. Edited by Gottfried Köthe. Translated from the German by Luise Lange. The University of Chicago Press, Chicago, Illinois, 1963.

APPENDICES

APPENDIX I

Table of Contents for "A Course In Plane Geometry" unpublished class notes by P.D.Barry

APPENDIX II

Symbolism and Some Definitions (P.D.Barry)

I. General use of letters:

A. Arabic Capitols: Vectors and Points in $\mathbb{R}^2$

B. Arabic small: scalors (real numbers) in $\mathbb{R}^2$

C. Greek letters - special uses

Π = plane = $\mathbb{R}^2$; $\bar{\lambda}, \bar{\mu}$, ... = half lines or rays ;

λ, μ, ... = lines ; $\angle \bar{\lambda}, \bar{\mu}$ = angle from λ to μ

II. Vectors: Given A,B vectors in $\mathbb{R}^2$, not linearly dependent: any vector in $\mathbb{R}^2$ can be written $C = k_1 A + k_2 B$; A and B basis vectors.

A. Scalor or Dot Product

Given vectors $A = (a_1, a_2)$ and $B = (b_1, b_2)$

$A \cdot B = a_1 b_1 + a_2 b_2$: Iff $A \cdot B = 0$, orthogonal

B. Length or Norm of a vector A

$\| A \| = \sqrt{a_1^2 + a_2^2} = \sqrt{A \cdot A}$

C. Orthogonality:

If $A = (a_1, a_2)$ its orthogonal vector $A^* = (-a_2, a_1)$

III. Planes,Points and Lines:

A. Plane: Set of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ the Cartesian produst symbolized by $\Pi = \{(a_1, a_2) : a_1 \in \mathbb{R},\ a_2 \in \mathbb{R}\}$

B. Point: each vector $A=(a_1, a_2)$ in $\mathbb{R}^2$ = point A

C. Line: Given E a vector in $\mathbb{R}^2$, F a non-zero vector in $\mathbb{R}^2$,and k a parameter

1) $\lambda = \{E + kF,\ k \in \mathbb{R}\}$

III.C. 2) $\lambda = E + [F]$, note if $E = 0$ then
$\lambda = 0 + [F] = [F] = \{kF, k \; \mathbb{R}\}$

3) AB (if A and B are distinct points of Π then
AB is the unique line containing A and B)

D. Special lines : Given $\lambda = A + [B]$; $\mu = C + [D]$

1) Same line : $\lambda = \mu$ iff $A = C$; $[B] = [D]$

2) Parallel lines : $\lambda \parallel \mu$ iff $A \neq C$; $[B] = [D]$

3) Perpendicular lines: $\lambda \perp \mu$ iff $B.C = 0$

4) Concurrent lines : one common point on all

E. Sensed lines and parts of lines
(Given P_1 and P_2 , points in $\mathbb{R}^2$)

1) With $P_1 < P_2$: $\overrightarrow{P_1P_2}$

2) Segment of λ

a) $[P_1,P_2] = \{P_1 + k(P_2 - P_1) \; ; \; 0 \leq k \leq 1\}$

b) $[P_1,P_2] = \{P \in \lambda : P_1 \leq P \leq P_2\}$

3) Half-line or ray (of line λ)

a) $\overline{\lambda} = A + k[B]$; $k \geq 0$ or $k \leq 0$ (A=end pt.)

b) (with end point P_1)

$\overline{\lambda} = \{P \in \lambda : P_1 \leq P\} = \{P_1 + kB; \; k \geq 0\}$

one sense $\rightarrow$ or the opposite sense $\leftarrow$

$\overline{\lambda} = \{P \in \lambda : P \leq P_1\} = \{P_1 + k(-B); k \geq 0\}$

IV. Special Functions on Π

A. Linear translation : t

$t_k(X) = k + X$; for all X in $\mathbb{R}^2$; k fixed

IV.B. Linear transformations : ℓ

1) $\ell(\mu + \lambda) = \ell(\mu) + \ell(\lambda)$

2) $\ell(k\mu) = k\,\ell(\mu)$

C. Parallel Projection:

$$\mathbb{T}(X) = \mathbb{T}(x_1, x_2) = A + \frac{d_2(x_1 - a_1) - d_1(x_2 - a_2)}{b_1 d_2 - b_2 d_1} B$$

(the line ∥ to C + [D] thru X a point on A + [B])

D. Central Symmetry :

$s_A(X)$ = symmetry in the point A

for each $X \in \Pi$ there is a unique point Y such that A is the midpoint of [XY] or

$$A = \frac{1}{2}(X + Y) \quad ; \; Y = 2A - X$$

∴ $s_A(X) = 2A - X$ note $(s_B \circ s_A = t_X(B-A))$

E. Axial Symmetry :

$s_\lambda(X) = 2\mathbb{T}(X) - X$ (in the line λ relative to X)
where $\mathbb{T}(X) = \frac{1}{2}(X + Y)$

F. Rotations :

$$\gamma = r = s \circ s$$

any transformation of the form $Z = s_\mu \{ s_\lambda(X) \}$

G. Affine transformations :

$$a = t \circ \ell$$

(any composition where t is a translation and ℓ is a non-singular linear transformation)

H. Distance :

$d_\mu(A,B) = {}_\mu\|B - A\|$ or iff $\mu = \frac{1}{\|B-A\|}$;

d (A,B) = 1 and just d(A,B) means distance

IV.I. Euclidean Transformation : $e = t_k \circ \ell$

an affive transformation for which distance is invarient ex:(translations, axial symmetries, and rotations)

V. Angles and Trigonometry

A. Angle: Given half-lines $\overline{\lambda} = A + kB$; $\overline{\mu} = A + kC$

A the common initial point

$\angle \overline{\lambda}, \overline{\mu}$ or $\angle EAF$ if $\overline{\lambda} = \{A + k(E-A); k \geq 0\}$ and

$$\overline{\mu} = \{A + k(F-A); k \geq 0\}$$

B. Angle Bisector : υ (a special function)

$$s_\upsilon(\overline{\lambda}) = \overline{\mu} \quad ; \quad s_\upsilon(\overline{\mu}) = \overline{\lambda}$$

C. Angle of Inclination:

$$\angle IOB \text{ of } \overline{\lambda}$$

where $\overline{\lambda} = 0 + [B]$ and $0 = (0,0); I = (1,0) = I(\overline{\lambda})$

D.Trigonometic functions :

$$\cos \angle \overline{\lambda}, \overline{\mu} = m_1 \quad ; \quad \sin \angle \overline{\lambda}, \overline{\mu} = m_2$$

given $\overline{\lambda} = \{A + kB : k \geq 0\}$; $\overline{\mu} = \{A + kC : k \geq 0\}$

$\ell(B)$ maps $\overline{\lambda} \to \overline{\mu}$

$\ell(B) = m_1 B = m_2 B^*$

APPENDIX III

Theorems,Definitions (P.D.Barry) and*ones added by me

THEOREM 1.4.1 If A and B are vectors in $\mathbb{R}^2$, not linearly dependent, then any vector C in $\mathbb{R}^2$ can be expressed, and in just one way, in the form $C = k_1A + k_2B$ ... A and B are a basis for $\mathbb{R}^2$ ... we call k_1 and k_2 the coordinates of the vector C.

THEOREM 2.3.1 If $\lambda = A + [B]$ is a line in Π and C is a point in Π, there is a unique line μ in Π, $C + [(-b_2, b_1)]$ in fact, such that (i) $C \in \mu$ and (ii) $\lambda \perp \mu$.

Definitions in section 3.1.1

Given a triangle A,B,C ,there is a unique line λ through A which is perpendicular to the line BC. ... it intersects it in a unique point which we will denote by D. We call D the foot of the perpendicular from A onto BC, we also call the line AD an altitude of the triangle.

THEOREM 2.3.2 For all lines $\lambda, \mu, \nu, \omega$ in Π we have:

(i) $\lambda \not\perp \lambda$,

(ii) $\lambda \perp \mu$ implies $\mu \perp \lambda$

(iii) $\lambda \perp \mu$ and $\mu \perp \nu$ implies $\lambda \parallel \nu$

(iv) $\lambda \perp \mu$, $\lambda \parallel \nu$ and $\mu \parallel \omega$ imply $\nu \perp \omega$.

Definition section 9.1 Equivalence of Angles Given two angles $\llcorner \bar{\lambda}, \bar{\mu}$ and $\llcorner \bar{\rho}, \bar{\sigma}$, we say that they are equivalent written,

$$\llcorner \bar{\lambda}, \bar{\mu} \sim \llcorner \bar{\rho}, \bar{\sigma},$$

if there is a translation t_k and a rotation γ such that

$$t_k\{\gamma(\bar{\lambda})\} = \bar{\rho} \quad , \quad t_k\{\gamma(\bar{\mu})\} = \bar{\sigma} \,.$$

<u>THEOREM 9.1.2</u> Equivalence of angles has the following properties:

(i) $\angle\bar{\lambda},\bar{\mu} \sim \angle\bar{\lambda},\bar{\mu}$ in all cases; (reflexive)

(ii) $\angle\bar{\lambda},\bar{\mu} \sim \angle\bar{\rho},\bar{\sigma}$ implies $\angle\bar{\rho},\bar{\sigma} \sim \angle\bar{\lambda},\bar{\mu}$ (symmetric)

(iii) $\angle\bar{\lambda},\bar{\mu} \sim \angle\bar{\rho},\bar{\sigma}$ and $\angle\bar{\rho},\bar{\sigma} \sim \angle\bar{\theta},\bar{\phi}$ implies $\angle\bar{\lambda},\bar{\mu} \sim \angle\bar{\theta},\bar{\phi}$. (transitive)

<u>Definition</u> Given an angle $\angle\bar{\lambda},\bar{\mu}$, if $\bar{\rho}$ is the complementary half-line of $\bar{\lambda}$ and $\bar{\sigma}$ is the complementary half-line of $\bar{\mu}$, then $\angle\bar{\rho},\bar{\sigma}$ is called the <u>opposite angle</u> of $\angle\bar{\lambda},\bar{\mu}$.

<u>Definition</u> If $\bar{\mu}$ is the complementary half-line of $\bar{\lambda}$, we call the angle $\angle\bar{\lambda},\bar{\mu}$ a <u>straight angle</u>.

<u>THEOREM 9.1.3</u> Equivalence of angles has the properties that:

(i) $\angle\bar{\lambda},\bar{\mu} \sim \angle\bar{\rho},\bar{\sigma}$ implies $\angle\bar{\mu},\bar{\lambda} \sim \angle\bar{\sigma},\bar{\rho}$;

(ii) opposite angles are equivalent;

(iii) each two straight angles are equivalent

(iv) if $\angle\bar{\lambda},\bar{\mu}$ is such that λ is perpendicular to μ, and $\angle\bar{\rho},\bar{\sigma}$ is such that ρ is perpendicular to σ, then either $\angle\bar{\lambda},\bar{\mu} \sim \angle\bar{\rho},\bar{\sigma}$ or $\angle\bar{\lambda},\bar{\mu} \sim \angle\bar{\sigma},\bar{\rho}$.

<u>THEOREM 9.1.4</u> If $\angle\bar{\lambda},\bar{\mu} \sim \angle\bar{\mu},\bar{\lambda}$ then either $\bar{\lambda} = \bar{\mu}$ or $\angle\bar{\lambda},\bar{\mu}$ is a straight angle.

<u>Theorem 9.1.5</u> For any angle $\angle\bar{\lambda},\bar{\mu}$ and line ν,

$$\angle\bar{\lambda},\bar{\mu} \sim \angle s_\nu(\bar{\mu}),\ s_\nu(\bar{\lambda}),$$

s_ν being axial symmetry in ν.

<u>Corollary to Thm 9.1.5</u> : Let ν be the bisector of the angle $\angle\bar{\lambda},\bar{\mu}$ and $\bar{\nu}$ have the same initial point as $\bar{\lambda}$ and $\bar{\mu}$. Then $\angle\bar{\lambda},\bar{\nu} = \angle\bar{\nu}.\bar{\mu}$.

THEOREM 9.2.1 Given equivalent pairs of angles

$$\angle\bar{\alpha},\bar{\beta} \sim \angle\bar{\gamma},\bar{\delta} \; ; \quad \angle\bar{\lambda},\bar{\mu} \sim \angle\bar{\rho},\bar{\sigma}$$

let $\bar{\theta}$ and $\bar{\phi}$ be arbitrary half-lines. Let f_1, f_2, f_3, f_4 be functions, each of the form $t_k \circ \gamma$, where γ is a rotation and t_k a translation, chosen so that, first $f_1(\bar{\alpha}) = \bar{\theta}$, $f_2(\bar{\gamma}) = \bar{\phi}$, and then

$f_3(\bar{\lambda}) = f_1(\bar{\beta})$, $f_4(\bar{\rho}) = f_2(\bar{\delta})$.

Then the angles

$\angle\bar{\theta}, f_3(\bar{\mu})$, $\angle\bar{\phi}, f_4(\bar{\sigma})$ are equivalent.

Definitions and Theorems following show addition of angles is possible (section 9.2)

1) $\angle\bar{\alpha},\bar{\beta} + \angle\bar{\beta},\bar{\mu} = \angle\bar{\alpha},\bar{\mu}$

2) If $\angle\bar{\alpha},\bar{\beta} \sim \angle\bar{\gamma},\bar{\delta}$ and $\angle\bar{\lambda},\bar{\mu} \sim \angle\bar{\rho},\bar{\sigma}$ then

$$\angle\bar{\alpha},\bar{\beta} + \angle\bar{\lambda},\bar{\mu} \sim \angle\bar{\gamma},\bar{\delta} + \angle\bar{\rho},\bar{\sigma} .$$

Theorem 9.2.2 For any angles

$$\angle\bar{\alpha},\bar{\beta} + \angle\bar{\lambda},\bar{\mu} \sim \angle\bar{\lambda},\bar{\mu} + \angle\bar{\alpha},\bar{\beta} .$$

THEOREM 9.2.4 For any angle $\angle\bar{\alpha},\bar{\beta}$

$$\angle\bar{\alpha},\bar{\beta} + \angle\bar{\beta},\bar{\alpha} = \angle\bar{\alpha},\bar{\alpha}$$

...we call the angle $\angle\bar{\alpha},\bar{\alpha}$ a zero angle. Each two such are clearly equivalent. We also denote $\angle\bar{\beta},\bar{\alpha}$ by $-\angle\bar{\alpha},\bar{\beta}$.

THEOREM 9.2.5 For any angles $\angle\bar{\alpha},\bar{\beta}$, $\angle\bar{\lambda},\bar{\mu}$, $\angle\bar{\rho},\bar{\sigma}$,

$$(\angle\bar{\alpha},\bar{\beta} + \angle\bar{\lambda},\bar{\mu}) + \angle\bar{\rho},\bar{\sigma} = \angle\bar{\alpha},\bar{\beta} + (\angle\bar{\lambda},\bar{\mu} + \angle\bar{\rho},\bar{\sigma}),$$

that is, addition of angles is associative.

<u>DEFINITION (10.1)</u>

Let μ be a positive real number. Given the points A and B in Π, we define $d_\mu(A,B)$ by the equation

$$d_\mu(A,B) = \mu \, \| B - A \| \quad .$$

Then $d_\mu(A,B)$ is defined for all $A,B \in \Pi$. We call d_μ a <u>distance function</u>.

<u>THEOREM 10.2.1</u> For each $\mu > 0$ and all $A,B,C \in \Pi$, we have the following:

(i) $d_\mu(A,B) = d_\mu(B,A)$;

(ii) $d_\mu(A,B) \geq 0$, with equality if and only if $A = B$.

(iii) $d_\mu(A,C) \leq d_\mu(A,B) + d_\mu(B,C)$, with equality if and only if B [A,C]. In particular if ABC is a triangle,

$$d_\mu(A,C) < d_\mu(A,B) + d_\mu(B,C) \ .$$

<u>DEFINITION</u> Given any distinct points A,B in Π, there is a unique value of μ such that $d_\mu(A,B) = 1$.

For then $\|B-A\| > 0$, and $d_\mu(A,B) = \mu \, \| B-A \| = 1$, if and only if $\mu = \dfrac{1}{\|B-A\|}$ ... our <u>unit of distance</u>.

<u>THEOREM 10.2.2</u> The Theorem of Pythagoras

Let A,B,C form a triangle and the lines AB and BC be perpendicular. Then

$$d(A,B)^2 + d(B,C)^2 = d(A,C)^2 \ .$$

<u>THEOREM 10.2.6</u> Let A,B,C be a triangle and D the mid-point of B and C. Then lines AD and BC are perpendicular if and only if $d(A,B) = d(A,C)$.

<u>THEOREM 11.1.1</u> distance is invariant under each translation i.e. $d(X,Y) = d(t_k(X), t_k(Y))$, for all $X, Y \in \Pi$.

<u>THEOREM 11.1.2</u> Distance is invariant under a linear transformation ℓ if and only if ℓ is either a rotation with center 0 or axial symmetry in a line λ thru 0.

<u>THEOREM 11.2.1</u> Let ν be the bisector of the angle $\angle\overline{\lambda},\overline{\mu}$. For any point $X \in \nu$, let Y be the foot of the perpendicular from X on λ and Z be the foot of the perpendicular from X on μ . Then $d(X,Y) = d(X,Z)$.

<u>Definition (12.1)</u> Given any vector $A = (a_1, a_2)$ in $\mathbb{R}^2$ we define $A^* = (-a_2, a_1)$.

<u>THEOREM 12.1.1</u> For all vectors A, we have:

(i) A^* is orthogonal to A

(ii) $\|A^*\| = \|A\|$

(iii) $(A^*)^* = -A$

(iv) $(kA)^* = kA^*$

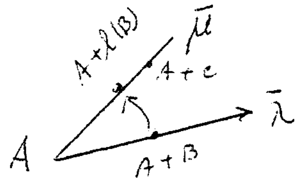
(v) $(A+B)^* = A^* + B^*$

(vi) If is a rotation with center 0, then $\ell(A^*) = \{\ell(A)\}^*$

(vii) If $A \neq 0$, then A and A^* are not linearly dependent.

<u>Definition 12.2</u> Given an angle $\angle\overline{\lambda},\overline{\mu}$ we define $\cos \angle\overline{\lambda},\overline{\mu}$, $\sin \angle\overline{\lambda},\overline{\mu}$ as follows:
Let $\overline{\lambda} = \{A + kB : k \geq 0\}$, $\overline{\mu} = \{A + kC : k \geq 0\}$ there is a unique rotation ℓ such that $\ell\{kB : k \geq 0\} = \{kC : k \geq 0\}$ ... there exist unique numbers m_1 and m_2 such that $\ell(B) = m_1 B + m_2 B^*$. We define

$$\cos \angle\overline{\lambda},\overline{\mu} = m_1, \quad \sin \angle\overline{\lambda},\overline{\mu} = m_2 .$$



Special Cosines and Sines Defined (12.2)

If $\angle\overline{\lambda},\overline{\lambda} = 0$ angle $\cos\angle\overline{\lambda},\overline{\lambda} = 1$, $\sin\angle\overline{\lambda},\overline{\lambda} = 0$.

If $\overline{\mu}$ is the complementary half-line to $\overline{\lambda}$

$\cos\angle\overline{\lambda},\overline{\mu} = -1$, $\sin\angle\overline{\lambda},\overline{\mu} = 0$.

If $\overline{\mu} = \{A + kB^*;\ k\geq 0\}$ so that $\lambda\perp\mu$

$\cos\angle\overline{\lambda},\overline{\mu} = 0$, $\sin\angle\overline{\lambda},\overline{\mu} = 1$.

If $\overline{\mu} = \{ A - kB^*;\ k\geq 0 \}$

$\cos\angle\overline{\lambda},\overline{\mu} = 0$, $\sin\angle\overline{\lambda},\overline{\mu} = -1$.

THEOREM 12.2.1 If $\angle\overline{\lambda},\overline{\mu} \sim \angle\overline{\rho},\overline{\sigma}$, then

$\cos\angle\overline{\lambda},\overline{\mu} = \cos\angle\overline{\rho},\overline{\sigma}$ and $\sin\angle\overline{\lambda},\overline{\mu} = \sin\angle\overline{\rho},\overline{\sigma}$.

THEOREM 12.2.2 For all angles $\angle\overline{\lambda},\overline{\mu}$

$\cos\angle\overline{\mu},\overline{\lambda} = \cos\angle\overline{\lambda},\overline{\mu}$ and $\sin\angle\overline{\mu},\overline{\lambda} = -\sin\angle\overline{\lambda},\overline{\mu}$.

THEOREM 12.2.3 For any angle $\angle\overline{\lambda},\overline{\mu}$

$\cos^2\angle\overline{\lambda},\overline{\mu} + \sin^2\angle\overline{\lambda},\overline{\mu} = 1$ and the *corollary

$-1 \leq \cos\angle\overline{\lambda},\overline{\mu} \leq 1$, $-1 \leq \sin\angle\overline{\lambda},\overline{\mu} \leq 1$.

THEOREM 12.2.4 For any angles $\angle\overline{\alpha},\overline{\beta}$ and $\angle\overline{\lambda},\overline{\mu}$

$\cos\{\angle\overline{\alpha},\overline{\beta} + \angle\overline{\lambda},\overline{\mu}\} = \cos\angle\overline{\alpha},\overline{\beta}\cos\angle\overline{\lambda},\overline{\mu} - \sin\angle\overline{\alpha},\overline{\beta}\sin\angle\overline{\lambda},\overline{\mu}$,

$\sin\{\angle\overline{\alpha},\overline{\beta} + \angle\overline{\lambda},\overline{\mu}\} = \cos\angle\overline{\alpha},\overline{\beta}\sin\angle\overline{\lambda},\overline{\mu} + \sin\angle\overline{\alpha},\overline{\beta}\cos\angle\overline{\lambda},\overline{\mu}$.

*corollary 1)

$\cos\{\angle\overline{\alpha},\overline{\beta} - \angle\overline{\lambda},\overline{\mu}\} = \cos\angle\overline{\alpha},\overline{\beta}\cos\angle\overline{\lambda},\overline{\mu} + \sin\angle\overline{\alpha},\overline{\beta}\sin\angle\overline{\lambda},\overline{\mu}$,

$\sin\{\angle\overline{\alpha},\overline{\beta} - \angle\overline{\lambda},\overline{\mu}\} = \sin\angle\overline{\alpha},\overline{\beta}\cos\angle\overline{\lambda},\overline{\mu} - \cos\angle\overline{\alpha},\overline{\beta}\sin\angle\overline{\lambda},\overline{\mu}$.

*corollary 2) If $\angle\overline{\alpha},\overline{\beta} \sim \angle\overline{\lambda},\overline{\mu}$

$\cos\{\angle\overline{\alpha},\overline{\beta} + \angle\overline{\lambda},\overline{\mu}\} = \cos 2\angle\overline{\alpha},\overline{\beta} = \cos^2\angle\overline{\alpha},\overline{\beta} - \sin^2\angle\overline{\alpha},\overline{\beta}$,

$\sin\{\angle\overline{\alpha},\overline{\beta} + \angle\overline{\lambda},\overline{\mu}\} = \sin 2\angle\overline{\alpha},\overline{\beta} = 2\sin\angle\overline{\alpha},\overline{\beta}\cos\angle\overline{\alpha},\overline{\beta}$.

DEFINITION Given a half-line $\overline{\lambda} = \{A + kB : k\_0\}$, we call $\angle IOB$ the angle of inclination of $\overline{\lambda}$, where $0 = (0,0)$ and $I = (1,0)$ and denote it by $I(\overline{\lambda})$.

THEOREM 12.2.4

*corollary 3) For any angle $\angle \overline{\alpha},\overline{\beta}$

$$\cos 2 \angle \overline{\alpha},\overline{\beta} = 2 \cos^2 \angle \overline{\alpha},\overline{\beta} - 1 , \qquad \text{and}$$

$$\cos 2 \angle \overline{\alpha},\overline{\beta} = 1 - 2 \sin^2 \angle \overline{\alpha},\overline{\beta} .$$

*corollary 4) For any angle $\angle \overline{\alpha},\overline{\beta}$

$$\cos^2 \tfrac{1}{2} \angle \overline{\alpha},\overline{\beta} = \frac{1 + \cos \angle \overline{\alpha},\overline{\beta}}{2}$$

$$\sin^2 \tfrac{1}{2} \angle \overline{\alpha},\overline{\beta} = \frac{1 - \cos \angle \overline{\alpha},\overline{\beta}}{2} .$$

THEOREM 12.2.5 If $\angle \overline{\alpha},\overline{\beta} + \angle \overline{\lambda},\overline{\mu}$ is a straight angle, then $\cos \angle \overline{\lambda},\overline{\mu} = -\cos \angle \overline{\alpha}\ \overline{\beta}$, and $\sin \angle \overline{\lambda},\overline{\mu} = \sin \angle \overline{\alpha},\overline{\beta}$.

THEOREM 12.2.6 Let $\overline{\alpha} = \{A + kB ; k \geq 0\}$, $\overline{\gamma} = \{A + kB^*; k \geq 0\}$ $\overline{\delta} = \{A - kB^*; k \geq 0\}$. If $\angle \overline{\alpha},\overline{\beta} + \angle \overline{\lambda},\overline{\mu} = \angle \overline{\alpha},\overline{\gamma}$, then $\cos \angle \overline{\lambda},\overline{\mu} = \sin \angle \overline{\alpha},\overline{\beta}$, and $\sin \angle \overline{\lambda},\overline{\mu} = \cos \angle \overline{\alpha},\overline{\beta}$.
If $\angle \overline{\alpha},\overline{\beta} + \angle \overline{\lambda},\overline{\mu} = \angle \overline{\alpha},\overline{\delta}$, then $\cos \angle \overline{\lambda},\overline{\mu} = -\sin \angle \overline{\alpha},\overline{\beta}$ and $\sin \angle \overline{\lambda},\overline{\mu} = -\cos \angle \overline{\alpha},\overline{\beta}$.

THEOREM 12.3.1 Given ppints B and C, distinct from each other and from 0

$$\cos \angle B0C = \frac{B \cdot C}{\|B\| \|C\|} , \quad \sin \angle B0C = \frac{B^* \cdot C}{\|B\| \|C\|} .$$

THEOREM 12.3.2 Let A,B,C be a triangle such that AB $\perp$ BC. Then

$$\cos \angle BAC = \frac{d(A,B)}{d(A,C)} , \quad \left| \sin \angle BAC \right| = \frac{d(B,C)}{d(A,C)} .$$