AN ABSTRACT OF THE THESIS OF

Dionysus Birnbaum for the degree of Doctor of Philosophy in Mathematics presented on August 28, 2020.

Title: On the Logical Independence of DR and CLA

Abstract approved: ______________________________

William A. Bogley

In 1941, J.H.C. Whitehead posed the question of whether asphericity is a hereditary property for 2-dimensional CW complexes. This question remains unanswered, but has led to the development of several algebraic and topological properties that are sufficient (but not necessary) for the asphericity of presentation 2-complexes. While many of the logical relationships between these flavors of asphericity are known, there remain a few to be answered. In particular, it has long been known that Cohen-Lyndon aspherical (CLA) complexes are not necessarily diagrammatically reducible (DR), but the existence of a (DR) complex which is not (CLA) remains open. We resolve this by showing that the presentation 2-complex associated to the presentation

$$\langle x, y, t : x = txyx^{-1}y^{-1}t^{-1}, y \rangle$$

is (DR) but not (CLA). In fact, we prove that if a nontrivial group G occurs as the fundamental group of a (DR) 2-complex, then there is a (DR) 2-complex with fundamental group G that is not (CLA). This completes the study of flavors of asphericity that was begun in [8] and also recovers the main result of [4].

On the Logical Independence of DR and CLA

by

Dionysus Birnbaum

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Presented August 28, 2020
Commencement June 2021

Doctor of Philosophy thesis of Dionysus Birnbaum presented on August 28, 2020

APPROVED:

Major Professor, representing Mathematics

Head of the Department of Mathematics

Dean of the Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Dionysus Birnbaum, Author

ACKNOWLEDGEMENTS

Academic

I am indebted to to my advisor, Bill Bogley, without whose support and guidance I would not have made it to this point. I would also like to thank my committee, Ren Guo and Christine Escher, who have always been available and supportive, and Tom Schmidt and Deb Pence, who were willing to step in on short notice as I struggled with the logistics of wrangling academics.

Personal

I could not possibly list all the friends and family I want to thank in this space; you already know who you are and I love you all.

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ON THE LOGICAL INDEPENDENCE OF DR AND CLA

1. INTRODUCTION

1.1. History of Asphericity

In his 1936 article on homotopy groups, W. Hurewicz [25] called an arcwise connected space X *aspherical* if all its higher homotopy groups vanish. That is, X is aspherical if for all $k \geq 2$, each map of the k-sphere $S^k \to X$ factors through a map of the $(k+1)$-ball into X. Furthermore, Hurewicz discovered that, for a finite, aspherical, simplicial complex X, its homotopy type is determined entirely by its fundamental group $\pi_1(X)$. In general, aspherical CW complexes with fundamental group G are called Eilenberg-Maclane or $K(G, 1)$ spaces. As these spaces are uniquely determined up to homotopy type by their fundamental group G, the topological invariants of a $K(G, 1)$ may also be seen as invariants of the group G. For example, Heinz Hopf's integral homology formula [20] (coinciding with Schur's formula for the Schur multiplier) is an invariant of the group G and, in fact, can be seen in the homology of the universal cover $\widetilde{K(G, 1)}$, see [40, Sec. 4] for group homology from the perspective of projective resolutions, provided by the cellular chain complex of $\widetilde{K(G, 1)}$.

In 1941, J.H.C. Whitehead [41] posed the question of whether asphericity is a hereditary property for 2-dimensional CW-complexes. That is, given a connected, aspherical 2-complex K, is every connected subcomplex $L \subseteq K$ also aspherical? A likely motivation to Whitehead's question is the fact that the complement of any tame knot in the 3-sphere has the homotopy type of a 2-dimensional complex that can be embedded in a finite, contractible (hence aspherical) 2-complex, see [5, Sec.2] for more on the context of Whitehead's question. If the answer to Whitehead's question is "yes" tame knots in the 3-sphere are aspherical. The asphericity of tame knot complements has since been proved by Papakyriakopoulos' sphere theorem[34]. Whitehead's question, in general, however, remains unanswered, despite considerable effort, but the assumption that the answer is "yes" is known as the *Whitehead Conjecture*. None of the currently available framings of asphericity shed any light on the heredity of asphericity for 2-complexes.

The Lyndon Identity Theorem [29] for one-relator presentations gave the first major result in asphericity in general (and is a major jumping off point for the study of group (co)homology). A consequence of the Lyndon Identity Theorem is that, given a one-relator presentation whose relator is not a proper power, the associated presentation 2-complex is aspherical. Heredity of asphericity in the one-relator case is no question: the only proper subcomplexes are 1-complexes, which are aspherical (because the universal cover of a 1-complex is a tree). Allowing for a proper power relator led to the formulation of combinatorial asphericity (CA), see [14]. Asking whether subcomplexes of (CA) complexes are (CA) turns out to be equivalent to the Whitehead Conjecture [27].

Given a group presentation $\mathcal{P} = \langle X : R \rangle$ (see Section 2.2.), we have the free group $F = F(X)$, the normal closure $N = \ll R \gg \leq F$, and the group presented $G = G(\mathcal{P}) = F/N$. The Lyndon Identity Theorem is a result describing the structure of the relation module $N/[N, N]$ when R consists of a single relator $\{r\}$. Cohen-Lyndon [10] then lifts this structure to a description of N, in terms of conjugates of the single relator of R. Generalizing this led to the notion of a Cohen-Lyndon aspherical (CLA) presentation.

Diagrammatic methods generalizing Dehn's algorithm were featured in [30] and [31]. Subtleties around various diagram-based forms of asphericity led to the formulation of various hereditary forms of asphericity, including diagrammatic asphericity (DA) and diagrammatic reducibility (DR).

Known implications between these flavors of asphericity are shown in Figure 1.1.

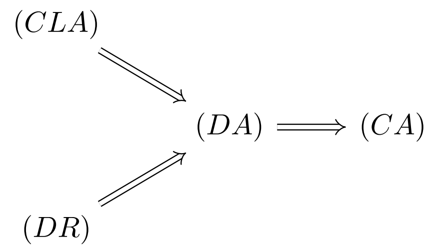

FIGURE 1.1: Implications between Flavors of Asphericity

The irreversibility of $(DA) \Rightarrow (CA)$ was shown by Sieradski [36] and Chiswell [8]. The irreversability of $(CLA) \Rightarrow (DA)$ is a result of Biskup [4]. The irreversibility of $(DR) \Rightarrow (DA)$ is a well-known piece of folklore, with counterexample provided by the dunce cap, the topological model of the presentation $\langle x : xxx^{-1} \rangle$. This same piece of folklore tells us that (CLA) does *not* imply (DR). Somewhat interesting is the fact that the dunce cap, as well as the counterexamples of [8], [36], and [4] all arise from balanced presentations of the trivial group. Altogether, these results leave only the question of whether (DR) implies (CLA).

1.2. Free Kernels and Distingushing (DR) and (CLA)

The *Whitehead setting* consists of an aspherical complex K and a subcomplex $L \subseteq K$. The inclusion $L \subseteq K$ naturally induces a group homomorphism $\pi_1(L) \to \pi_1(K)$. A number of positive results on the Whitehead conjecture have been proven under group-theoretic conditions on the kernel of $\pi_1(L) \to \pi_1(K)$. For example, if the kernel is locally finite [2], locally non-perfect [21], or locally indicable [22], then L ias aspherical.

Our primary technical results pertain to the π_1-kernel for either (DR) or (CLA) 2-complexes:

Lemma 3.1.0.1. *Let (K, L) be a Whitehead setting where K is (DR). Then, the associated*

π_1-kernel is a locally free group.

Corollary 3.2.0.2. *Let $\mathcal{P} = \langle X : R \rangle$ be a (CLA) presentation of the group $G(\mathcal{P})$. Let $\mathcal{Q} = \langle Y : S \rangle$ be any subpresentation and presenting a group $H = H(\mathcal{Q})$. Then, $M = \ker\{H \to G\}$ is free*[1]*.*

(This is a slight generalization of Theorem 3.2.0.1. Theorem 3.2.0.1 is sufficient to prove Theorem 4.5.0.1, but may be extended to the more general setting of Corollary 3.2.0.2.)

We produce a class of (DR) complexes where a certain subcomplex does not have free kernel, thereby violating Corollary 3.2.0.2, and giving:

Theorem 4.5.0.1. *Every non-trivial group G with (DR) presentation admits a (DR) presentation which is not (CLA).*

With theorem 4.5.0.1 established, the status of all implications in Figure 1.1 are resolved: the implications are exactly and only as pictured.

[1] While these two results are phrased, respectively, in terms of 2-complexes and group presentations, there is a suitable equivalence between these two notions, as discussed in chapter 2, that would allow rephrasing either in terms of 2-complexes or group presentations.

2. METHODS AND CONSTRUCTIONS

Virtually all of Chapter 2 consists of background material. Our main goals in this chapter are to establish notation and terminology for two-dimensional CW complexes (2-complexes, for short), including complexes modeled on group presentations, and to formulate properties related to asphericity in topological, algebraic, and combinatorial terms. Some standard references for the material on 2-complexes in this chapter are Brown [7], Hatcher [19], and Sieradski [40]; results given without proof are drawn from these texts or various places in the literature and cited as such. Results in this chapter are not novel, generally being either implicit in the standard references or using fairly typical arguments.

2.1. Asphericity of 2-Complexes

Definition 2.1.0.1. *A path-connected space K is said to be* ***aspherical*** *if the homotopy groups $\pi_n(K, x)$ are all trivial for $n \geq 2$.*

Let K be a connected (hence path-connected) 2-complex and let $p : \widetilde{K} \to K$ be the universal covering projection. As described in [39, Ch. 15.3], characteristic maps for the cells of K lift through the projection p to provide characteristic maps for a cell decomposition of $\widetilde{K}$. In this way, the total space $\widetilde{K}$ inherits a two-dimensional CW structure with the property that the projection p carries each open cell of $\widetilde{K}$ homeomorphically onto an open cell of K. In particular, the projection p is a cellular map satisfying $p(\widetilde{K}^i) = K^i$ on the i-skeleta for $i = 0, 1, 2$. We wish to describe the asphericity of K in terms of the cellular homology of the total space $\widetilde{K}$.

Lemma 2.1.0.1. *A connected 2-complex K is aspherical if and only if the homological boundary map $\partial : H_2(\widetilde{K}, \widetilde{K}^1) \to H_1(\widetilde{K}^1)$ from the long exact homology sequence for the* *skeleton pair* $(\widetilde{K}, \widetilde{K}^1)$ *is an isomorphism.*

Proof. Choose a basepoint $x \in K$ and a fiber point $\widetilde{x} \in p^{-1}(x) \subseteq \widetilde{K}$. Covering space theory [19, Prop. 4.1] provides that the covering projection induces an isomorphism $p_* : \pi_n(\widetilde{K}, \widetilde{x}) \to \pi_n(K, x)$ for all $n \geq 2$. Thus, K is aspherical if and only if the universal covering complex $\widetilde{K}$ is aspherical.

Since $\widetilde{K}$ is two-dimensional, the augmented cellular chain complex

$$\cdots \to C_3(\widetilde{K}) \to C_2(\widetilde{K}) \to C_1(\widetilde{K}) \to C_0(\widetilde{K}) \to \mathbb{Z} \to 0$$

vanishes in dimension three and above, that is, $C_i(\widetilde{K}) = 0$ for $i \geq 3$. Thus, the cellular homology vanishes in dimension three and above, that is, $H_i(\widetilde{K}) = 0$ for $i \geq 3$. Next, we show that $\widetilde{K}$ is aspherical if and only if $H_2(\widetilde{K}) = 0$.

Since $\widetilde{K}$ is simply-connected, the Hurewicz theorem [19, Thm. 4.32] tells us that we have an isomorphism $h : \pi_2(\widetilde{K}, \widetilde{*}) \xrightarrow{\cong} H_2(\widetilde{K})$. Thus, if K is aspherical, then $H_2(\widetilde{K}) = 0$. Conversely, if $H_2(\widetilde{K}) = 0$, then $\pi_2(\widetilde{K}, \widetilde{x}) = 0$ and so the Hurewicz isomorphisms [19, Thm. 4.32] imply that $h : \pi_3(\widetilde{K}, \widetilde{x}) \to H_3(\widetilde{K}) = 0$ is an isomorphism. Proceeding inductively, the vanishing of the higher homology groups of $\widetilde{K}$ allows us to conclude that $\pi_n(\widetilde{K}) = 0$ for all $n \geq 2$. Thus, we have proved that K is aspherical if and only if $H_2(\widetilde{K}) = 0$.

Since $C_3(\widetilde{K})$ is trivial, $H_2(\widetilde{K})$ is isomorphic to the kernel of the cellular boundary homomorphism $C_2(\widetilde{K}) \to C_1(\widetilde{K})$, which factors as

$$C_2(\widetilde{K}) = H_2(\widetilde{K}, \widetilde{K}^1) \xrightarrow{\partial} H_1(\widetilde{K}^1) \xrightarrow{j} H_1(\widetilde{K}^1, \widetilde{K}^0) = C_1(\widetilde{K})$$

where the map $j : H_1(\widetilde{K}^1) \to H_1(\widetilde{K}^1, \widetilde{K}^0)$ is taken from the long exact homology sequence for the pair $(\widetilde{K}^1, \widetilde{K}^0)$ [19, Lem. 2.34]. Since $H_1(\widetilde{K}^0) = 0$, this long exact sequence shows that $j : H_1(\widetilde{K}^1) \to H_1(\widetilde{K}^1, \widetilde{K}^0)$ is injective. Thus, K is aspherical if and only if $\partial : H_2(\widetilde{K}.\widetilde{K}^1) \to H_1(\widetilde{K}^1)$ is injective. That fact that ∂ is surjective follows from the long exact sequence of the pair $(\widetilde{K}, \widetilde{K}^1)$ and the fact that $\widetilde{K}$ is is simply connected, so that $H_1(\widetilde{K}) = 0$. Thus, K is aspherical if and only if ∂ is an isomorphism, as desired. □

In Section 2.4., we give an explicit (and well-known) algebraic interpretation of ∂ in quite a general setting. First, it is worth noting that the foregoing criterion for asphericity provides little insight into Whitehead's conjecture about the heredity of asphericity. The reason for this is as follows: let L be a connected subcomplex of a connected, aspherical 2-complex K, and let $\overline{L}$ be a connected component of the pre-image $p^{-1}(L)$ of L under the universal covering projection $p : \widetilde{K} \to K$. Then, p restricts to a covering projection $\overline{p} : \overline{L} \to L$. By Lemma 2.1.0.1, asphericity of K implies $H_2(\widetilde{K}) = 0$. Since $\overline{L}$ is a subcomplex of the 2-complex $\widetilde{K}$, it follows that $H_2(\overline{L}) = 0$. However, we cannot use the Hurewicz theorem to conlude that $\pi_2(\overline{L})$ is trivial (and thereby argue that L is aspherical), because $\overline{L}$ is not simply-connected in general. Indeed, the fact that L is a subcomplex of the aspherical 2-complex K gives little practical information about a simply connected covering $\widetilde{L}$ of L or its homology group $H_2(\widetilde{L})$. One general observation is the following, which is well known.

Lemma 2.1.0.2. *Let $i : L \to K$ denote the inclusion of a nonempty, connected subcomplex L in a connected 2-complex K. Let $\overline{L}$ be a connected component of the pre-image $p^{-1}(L)$ under the universal covering projection $p : \widetilde{K} \to K$ and let $\overline{p} = p|_{\overline{L}} : \overline{L} \to L$ be the restriction. Choose a basepoint $x \in L$ and a fiber point $\overline{x} \in p^{-1}(x) \cap \overline{L}$. Then, $\overline{p}$ induces an isomorphism from $\pi_1(\overline{L}, \overline{x})$ onto the kernel of the inclusion-induced homomorphism $i_* : \pi_1(L, x) \to \pi_1(K, x)$:*

$$\overline{p}_* : \pi_1(\overline{L}, \overline{x}) \overset{\cong}{\to} \ker\left(\pi_1(L, x) \overset{i_*}{\to} \pi_1(K, x)\right)$$

Proof. The homomorphism $\overline{p} : \pi_1(\overline{L}, \overline{x}) \to \pi_1(L, x)$ is injective because $\overline{p}$ is a covering projection [19, Prop 1.31]. That the image $\overline{p}_*(\pi_1(\overline{L}, \overline{x}))$ is contained in the kernel of i_* follows from the fact that the composite $i \circ \overline{p} : \overline{L} \to L \to K$ factors through the simply connected total space $\widetilde{K}$.

For the reverse containment, choose a basepoint $y \in S^1$ and let $\alpha : (S^1, y) \to (L, x)$ be a loop in L based at x such that the path homotopy class $[\alpha]$ is in the kernel of

$i_* : \pi_1(L, x) \to \pi_1(K, x)$. Then, α extends over the disc B^2 to a based map of pairs $F : (B^2, S^1, y) \to (K, L, x)$ where $F|_{S^1} = \alpha$. Since the disc B^2 is simply connected, the map F lifts through p to a map $\widetilde{F} : (B^2, y) \to (\widetilde{K}, \overline{x})$. The connected image $\widetilde{F}(S^1)$ contains the fiber point $\overline{x}$ and lies in the pre-image $p^{-1}(L)$, so $\widetilde{F}(S^1) \subseteq \widetilde{L}$. Thus, $[\widetilde{F}|_{S^1}] \in \pi_1(\widetilde{L}, \widetilde{x})$ and $\overline{p}_*([\widetilde{F}|_{S^1}]) = [\overline{p} \circ \widetilde{F}|_{S^1}] = [F|_{S^1}] = [\alpha]$ so $\widetilde{p}_*(\pi_1(\widetilde{L}, \widetilde{x})) = \ker i_*$. □

2.2. Presentation 2-Complexes

A **group presentation** is a pair $\mathcal{P} = \langle X : R \rangle$, where X is a set and R is a set of words in the alphabet $X \cup X^{-1}$ consisting of symbols x, x^{-1} where $x \in X$. The elements of X are called the **generators** and the elements of R are called the **relators**. Each relator $r \in R$ uniquely determines an element of the free group $F = F(X)$ with basis X. The group defined by $\mathcal{P}$ is the quotient

$$G = G(\mathcal{P}) = F(X)/ \ll R \gg = F/N$$

where $N = \ll R \gg$ is the smallest normal subgroup of the free group F that contains each of the relators, ie, the **normal closure of** R. The normal subgroup N is the set of **consequences** of the relators. By the Nielsen-Schreier theorem [39, Thm 15.3.6], N is a free group. Furthermore, N is generated as a group by conjugates of the form wrw^{-1}, where $w \in F$ and $r \in R$. An important issue that will arise later is whether or not the free group N has a free basis consisting of conjugates of the relators. See Section 2.5. below. Given a word $w \in F$, the element $wN \in F/N = G$ will be denoted by $\overline{w} \in G$.

Associated to the presentation $\mathcal{P} = \langle X : R \rangle$ for the group $G = F/N$ is a 2-complex $K = K(\mathcal{P})$, constructed as follows. Starting with a single 0-cell c^0, we attach a 1-cell for each generator $x \in X$ to obtain a one-point union of circles $K^1 = \vee_{x \in X} S^1_x$. As in [40, Lem. 2.1], there is an isomorphism

$$\tau : F \xrightarrow{\cong} \pi_1(K^1, c^0)$$

that carries each basis element $x \in X$ to the path homotopy class $\tau(x) = [\eta_x] \in \pi_1(K^1, c^0)$ where $\eta_x : (S^1, y) \to (K^1, c^0)$ is a based loop that conducts a degree one wrap around the circle $S^1_x \subseteq K^1$. The complement $S^1_x - \{c^0\}$ is an open 1-cell denoted by c^1_x. The choice of the elements $\tau(x) = [\eta_x]$ for $x \in X$ amounts to an orientation for the 1-cells of K^1. For each relator $r \in R$, we attach a 2-cell c^2_r with based attaching map $\dot{\phi}_r : (S^1, y) \to (K^1, c^0)$ that realizes $[\dot{\phi}_r] = \tau(r) \in \pi_1(K^1, c^0)$. In particular, for $r = x_1^{\epsilon_1} \dots x_n^{\epsilon_n} \in R$ where $x_i \in X$ and $\epsilon_i \in \{1, -1\}$, we cellulate S^1 as an n-gon, fix one of the vertices y of the n-gon as our global basepoint, and set $\dot{\phi}_r : (S^1, y) \to (K^1, c^0)$ to be the map which sends each 0-cell of S^1 to c^0 and the ith 1-cell, when read from the global basepoint, homeomorphically to the open 1-cell $c^1_{x_i}$ with the same orientation when $\epsilon_i = 1$ and the reverse orientation when $\epsilon_i = -1$. (Useful for later purposes is that under this particular choice of attaching map, the presentation 2-complex is a *combinatorial* complex, see Section 2.6..)

A well-known application of the Seifert-van Kampen Theorem provides that the inclusion-induced homomorphism $i_* : \pi_1(K^1, c^0) \to \pi_1(K, c^0)$ is surjective and has kernel equal to $\tau(N)$, see [40, Thm. 2.3]. Thus, we have the following commutative diagram, where the vertical maps are isomorphisms and $q : F \to F/N$ is the natural quotient given by $q(w) = wN = \overline{w}$ for all $w \in F$:

$$\begin{array}{ccc} F & \xrightarrow{\quad q \quad} & F/N \\ \big\downarrow{\scriptstyle \tau} & & \big\downarrow \\ \pi_1(K^1, c^0) & \xrightarrow{\ i_*\ } & \pi_1(K, c^0) \end{array} .$$

The 2-complex $K = K(\mathcal{P})$ is the **presentation 2-complex** associated to $\mathcal{P}$. The topology of $K(\mathcal{P})$ depends on the choice of 2-cell attaching maps $\dot{\phi}^2_r$ in the path homotopy class $\tau(r) \in \pi_1(K, c^0)$, but the homotopy type of $K(\mathcal{P})$ is uniquely determined by $\mathcal{P}$ due to the following fact:

Theorem 2.2.0.1. *[40, Thm 1.6] Skeletal pairs (K^2, K^1) and (M^2, M^1) having the same 1-skeleton $K^1 = M^1$ and 2-cells $\{c^2_\alpha\}$ that are attached via freely homotopic maps $\{\dot{\phi}_\alpha \sim \dot{\lambda}_\alpha : S^1 \to K^1 = M^1\}$ are homotopy equivalent.*

By [40, Thm. 1.9], the homotopy type of K is uniquely determined by $\mathcal{P}$ and every connected 2-complex has the homotopy type of a presentation 2-complex.

2.3. Cayley Complexes

Following [19, Ch. 1.3], we now develop notation to describing the natural CW structure and fundamental group action on the universal covering complex $\widetilde{K}$ for the presentation complex $K = K(\mathcal{P})$ of the presentation $\mathcal{P} = \langle X : R \rangle$. By the construction in Section 2.2., we have:

$$K = \bigvee_{x \in X} S^1_x \cup \bigcup_{r \in R} c^2_r = \{c^0\} \cup \bigcup_{x \in X} c^1_x \cup \bigcup_{r \in R} c^2_r.$$

Let $p : \widetilde{K} \to K$ be the universal covering projection. The pre-image $p^{-1}(c^0)$ is a discrete subset of the total space $\widetilde{K}$ and will serve as the 0-skeleton. We choose a **preferred 0-cell** $\widetilde{c}^0 \in p^{-1}(c^0) = \widetilde{K}^0$.

The **group of deck transformations** for p is the group $Aut(p)$ of self-homeomorphisms $\alpha : \widetilde{K} \to \widetilde{K}$ that are p-equivariant in the sense that $p \circ \alpha = p$. As in [19, Prop. 1.39], the choice of preferred 0-cell $\widetilde{c}^0$ determines an isomorphism $\Phi_{\widetilde{c}^0} : \pi_1(K, c^0) \to Aut(p)$. Given any based loop in K, represented by a path $\gamma : (I, 0, 1) \to (K, c^0, c^0)$, the isomorphism $\Phi_{\widetilde{c}^0}$ satisfies and is uniquely determined by path lifting in the sense that

$$\Phi_{\widetilde{c}^0}([\gamma])(\widetilde{c}^0) = \widetilde{\gamma}_{\widetilde{c}^0}(1)$$

where $\widetilde{\gamma}_{\widetilde{c}^0} : (I, 0) \to (\widetilde{K}, \widetilde{c}^0)$ is the unique lift of the path γ through the covering projection p that begins at $\widetilde{c}^0 \in p^{-1}(\gamma(0)) = p^{-1}(c^0) = \widetilde{K}^0$. Thus, the group $G = G(\mathcal{P}) \cong \pi_1(K, c^0) \cong Aut(p)$ acts on the total space $\widetilde{K}$ on the left via

$$g \cdot \widetilde{x} = \Phi_{\widetilde{c}^0}(g)(\widetilde{x})$$

for all $g \in G$ and $\widetilde{x} \in \widetilde{K}$. In particular, the fact that $\Phi_{\widetilde{c}^0}$ is a homomorphism implies that $g \cdot (h \cdot \widetilde{x}) = (gh) \cdot \widetilde{x}$ for all $g, h \in G$ and $\widetilde{x} \in \widetilde{K}$. This G-action is free, by the uniqueness of lifts, and transitive on each fiber of p, since deck transformations are p-equivariant and $\widetilde{K}$ is path-connected. In particular, there is a one-to-one correspondence $G \equiv p^{-1}(c^0) = \widetilde{K}^0$ between the fundamental group and the 0-skeleton, given by identifying each $g \in G$ with its action on the preferred 0-cell, $g \cdot \widetilde{c}^0$. Under this identification, the G-action on $p^{-1}(c^0) = \widetilde{K}^0$ coincides with the action of G on itself by left multiplication: $g \cdot (h \cdot \widetilde{c}^0) = (gh) \cdot \widetilde{c}^0$.

Now, for each generator $x \in X$, a characterisitc map for the 1-cell $c^1_x \subseteq K$ lifts at the preferred 0-cell to a characteristic map for a **preferred lift**, which is a 1-cell $\widetilde{c}^1_x$ in $\widetilde{K}$. Letting x range through the generating set X and g range through the automorphism group G, we obtain lifted 1-cells $g \cdot \widetilde{c}^1_x$ that are permuted freely under the action of G. The lifted 1-cell $g \cdot \widetilde{c}^1_x$ originates at $g \equiv g \cdot \widetilde{c}^0 \in \widetilde{K}^0$ and terminates at $g\overline{x} \equiv g\overline{x} \cdot \widetilde{c}^0 \in \widetilde{K}^0$.

In the same fashion, for each relator $r \in R$, a based characteristic map for the 2-cell $c^2_r \subseteq K$ lifts at the preferred 0-cell to a characteristic map for a **preferred lift** $\widetilde{c}^2_r$, which is a 2-cell in $\widetilde{K}$. Letting r range through the relators R and g range through the automorphism group G, we obtain lifted 2-cells $g \cdot \widetilde{c}^2_r$ that are permuted freely under the action of G. We thereby obtain the following CW structure on the universal covering complex:

$$\widetilde{K} = \bigcup_{g \in G} g \cdot \widetilde{c}^0 \cup \bigcup_{x \in X, g \in G} g \cdot \widetilde{c}^1_x \cup \bigcup_{r \in R, g \in G} g \cdot \widetilde{c}^2_r.$$

As in [19, P. 77], we refer to the 2-complex $\widetilde{K}$ as the **Cayley complex** for the group presentation $\mathcal{P} = \langle X : R \rangle$. In summary, the Cayley complex for $\mathcal{P}$ is equipped with a left $G(\mathcal{P})$-action that is free and transitive on the fibers of the universal covering $p : \widetilde{K} \to K$ and also freely permutes cells of $\widetilde{K}$ with orbits in one-to-one correspondence with the cells of K. In particular, the covering projection $p : \widetilde{K} \to K$ can be identified with the natural orbit map $\widetilde{K} \to G \backslash \widetilde{K}$.

It will also be useful to have explicit notation to refer to path homotopy classes in the groupoid $\pi_1(\widetilde{K}^1, \widetilde{K}^0)$ (that is, the groupoid of homotopy classes of paths in the 1-skeleton

$\widetilde{K}^1$, with initial and terminal vertices in the 0-skeleton $\widetilde{K}^0$, and groupoid operation given by path concatenation). Specifically, given a word $w \in F$, choose a based loop $\lambda : (I, 0, 1) \to (K, c^0, c^0)$ such that $\tau(w) = [\lambda]$ under the isomorphism $\tau : F \to \pi_1(K^1, c^0)$. Then, given $g \in G \equiv p^{-1}(c^0) = \widetilde{K}^0$, we have the lifted path $\widetilde{\lambda}_g$ beginning at $g \equiv g \cdot \widetilde{c}^0$. Then, $[g, w]$ is the path homotopy class

$$[g, w] = [\widetilde{\lambda}_g] \in \pi_1(\widetilde{K}^1, \widetilde{K}^0).$$

With a slight abuse of terminology, we say that the path homotopy class $[g, w]$ **lies over** the word w and **begins at** the 0-cell g.

We will make use of the following combinatorial modifications on path homotopy classes in $\pi_1(\widetilde{K}^1, \widetilde{K}^0)$ in order to pass between equivalent representatives of a given class.

Lemma 2.3.0.1. *Let $g, h \in G$ and let u, v be words in the alphabet $X \cup X^{-1}$. Then:*

1. *If u and v are freely equal in $F = F(X)$, then $[g, u] = [g, v]$.*
2. *In terms of the groupoid operation in $\pi_1(\widetilde{K}^1, \widetilde{K}^0)$, we have $[g, uv] = [g, u][g\overline{u}, v]$.*
3. *In terms of the groupoid operation in $\pi_1(\widetilde{K}^1, \widetilde{K}^0)$, we have $[g, u]^{-1} = [g\overline{u}, u^{-1}]$.*
4. *If $n \in N$, then $[g, unu^{-1}] = [g, u][g\overline{u}, n][g, u]^{-1}$.*

Proof. Represent the words u, v as $u = [\lambda]$ and $v = [\mu]$ in $F \cong \pi_1(K^1, c^0)$, where λ, μ are based loops in K^1.

1. When u and v are freely equal, the loops λ, μ are path homotopic in K^1, so the result follows by path homotopy lifting.
2. The lifted paths $\widetilde{\lambda}_g$ ends at the 0-cell $g\overline{u} \in G \equiv \widetilde{K}^0$, and the lifted path $\widetilde{\mu}_{g\overline{u}}$ begins at $g\overline{u}$, so the concatenated path $\widetilde{\lambda}_g * \widetilde{\mu}_{g\overline{u}}$ is defined and lies over the word uv. Thus, $[g, uv] = [\widetilde{\lambda}_g * \widetilde{\mu}_g\overline{u}] = [\widetilde{\lambda}_g][\widetilde{\mu}_{g\overline{u}}] = [g, u][g\overline{u}, v]$.

3. This follows from (2) by setting $v = u^{-1}$.

4. Using (2) and (3) as appropriate and the fact that $\overline{n} = 1 \in G$, we have

$$[g, unu^{-1}] = [g,u][g\overline{u}, v][g\overline{un}, u^{-1}] = [g,u][g\overline{u}, n][g\overline{u}, u^{-1}] = [g,u][g\overline{u}, n][g,u]^{-1}.$$

□

2.4. The Relation Module

In this section, we spell out a well-known combinatorial algebraic description of the boundary map $\partial : H_2(\widetilde{K}^2, \widetilde{K}^1) \to H_1(\widetilde{K}^1)$ in the case where $K = K(\mathcal{P})$ is the presentation complex asociated to the group presentation $\mathcal{P} = \langle X : R \rangle$. We will discuss how the homology group $H_1(\widetilde{K^1})$ is identified with the relation module of the presentation $\mathcal{P}$ (see [7, P. 43]), and how the Lyndon Identity Theorem [29] describes the structure of that module in the case where the presentation $\mathcal{P}$ has a single relation. This motivates the definition of combinatorial asphericity for group presentations as a natural generalization of asphericity for 2-complexes.

Recall from Lemma 2.1.0.1 that ∂ is an isomorphism if and only if K is aspherical. Recall that $N = \ll R \gg \leq F$ is the free normal subgroup of consequences of the relators $r \in R$ in the free group $F = F(X)$. The abelianized group $N^{ab} = N/[N,N]$ is called the **relation module** for $\mathcal{P}$; it is a left $\mathbb{Z}G$-module with action given by $\overline{w} \cdot n[N,N] = wnw^{-1}[N,N]$ for all $w \in F$ and $n \in N$. As seen in the proof of [7, Thm. I.5.3], there is the relationship between the topology of $\widetilde{K}^1$ and the relation module:

Lemma 2.4.0.1. *[7, P. 42] There is a surjective group homomorphism* $d : N \to H_1(\widetilde{K}^1)$ *given by*

$$d(n) = h([1, n])$$

for all $n \in N$ *where* $h : \pi_1(\widetilde{K}^1, \widetilde{c}^0) \to H_1(\widetilde{K}^1)$ *is the Hurewicz homomorphism. The homomorphism* d *has these properties:*

1. *The kernel of* d *is the commutator subgroup* $\ker d = [N, N]$.

2. *For all* $w \in F$ *and* $n \in N$, $d(wnw^{-1}) = \overline{w} \cdot d(n)$.

Proof. Lemma 2.1.0.2 implies that we have a pair of isomorphisms $\tau : N \overset{\cong}{\to} \tau(N)$ and $\overline{p}_* : \pi_1(\widetilde{K}^1, \widetilde{c}^0) \overset{\cong}{\to} \tau(N)$ where $\overline{p}_*^{-1}(\tau(n)) = [1, n] \in \pi_1(\widetilde{K}^1, \widetilde{c}^0)$. Composing with h, the homomorphism d is given by $d(n) = h([1, n]) \in H_1(\widetilde{K}^1)$. Since $\widetilde{K}^1$ is connected, the Hurewicz homomorphism $h : \pi_1(\widetilde{K}^1, \widetilde{c}^0) \to H_1(\widetilde{K}^1)$ is surjective with kernel equal to the commutator subgroup of $\pi_1(\widetilde{K}^1, \widetilde{c}^0)$ [40, Thm. 3.1]. Thus, the homomorphism d given as the composite

$$N \overset{\tau}{\to} \tau(N) \overset{\overline{p}_*^{-1}}{\to} \pi_1(\widetilde{K}^1, \widetilde{c}^0) \overset{h}{\to} H_1(\widetilde{K}^1)$$

is a surjective homomorphism with kernel equal to the commutator subgroup of N, as in statement 1. For the second statement, using Lemma 2.3.0.1 and the fact that h is additive and respects the G-action on $\widetilde{K}$ [40, Sec. 3.1], we have:

$$\begin{aligned} d(wnw^{-1}) &= h([1, wnw^{-1}]) \\ &= h([1, w][\overline{w}, n][1, w]^{-1}) \\ &= h([1, w]) + h([\overline{w}, n]) - h([1, w]) \\ &= h([\overline{w}, n]) \\ &= \overline{w} \cdot h([1, n]) \\ &= \overline{w} \cdot d(n) \end{aligned}.$$

Here we use the fact that the cellular action of G on $\widetilde{K}$ determines an action on the group $\pi_1(\widetilde{K}^1, \widetilde{K}^0)$ for the skeleton pair. □

The account of the preceding calculation given in [7, P. 42] is that if we represent w, n by based loops ω, ν in K^1 and form the concatenated loop $\omega * \nu * \omega^{-1}$, then the lift of this loop through p beginning at $\tilde{c}^0$ is the concatenated loop $\widetilde{\omega}_{\tilde{c}^0} * \tilde{\nu}_{\bar{w}\cdot\tilde{c}^0} * (\widetilde{\omega}_{\tilde{c}^0})^{-1}$. It follows that the associated homology class in $H_1(\widetilde{K}^1)$ is supported on $\tilde{\nu}_{\bar{w}\cdot\tilde{c}^0}$, which is the translate of $\tilde{\nu}_{\tilde{c}^0}$ under the action of $\bar{w} \in G$ where $[\tilde{\nu}_{\tilde{c}^0}] = [1, n] \in \pi_1(\widetilde{K}^1, \widetilde{K}^0)$.

Here is a well-known interpretation of the boundary homomorphism ∂ from Lemma 2.1.0.1 in the case of a presentation complex $K = K(\mathcal{P})$. This interpretation is detailed in the proof of [7, Thm. I.5.3] and in [14, Cor. 2.2] for the case of one-relator presentations. The following formulation is tailored to our needs.

Theorem 2.4.0.1. *Let $K = K(\mathcal{P})$ be the model of a group presentation $\mathcal{P} = \langle X : R\rangle$ for a group $G = G(\mathcal{P})$. There is a commutative square of $\mathbb{Z}G$-homomorphisms*

$$\begin{array}{ccc} \oplus_{r\in R}\mathbb{Z}G & \xrightarrow{\Delta} & N^{ab} \\ \downarrow\cong & & \cong\downarrow d \\ H_2(\widetilde{K}, \widetilde{K}^1) & \xrightarrow{\partial} & H_1(\widetilde{K}^1) \end{array}$$

where a $\mathbb{Z}G$-basis $\{\widetilde{c}^2_r : r \in R\}$ for $C_2(\widetilde{K}) = H_2(\widetilde{K}, \widetilde{K}^1)$ corresponding to preferred lifts of 2-cells satisfies $\partial(\widetilde{c}^2_r) = d(r[N, N])$ for all $r \in R$. Viewing these preferred lifts as a $\mathbb{Z}G$-basis for the free module $\oplus_{r\in R}\mathbb{Z}G$, we can therefore identify the boundary homomorphism ∂ with the homomorphism Δ that is given by $\mathbb{Z}G$-linearity and the basis assignments

$$\Delta(\widetilde{c}^2_r) = r[N, N]$$

for all $r \in R$

Proof. Lemma 2.4.0.1 provides an isomorphism $N^{ab} \to H_1(\widetilde{K}^1)$ of $\mathbb{Z}G$ modules, which we also denote by d, that is given by $\mathbb{Z}G$-linearity and

$$d(r[N, N]) = \partial(\widetilde{c}^2_r)$$

for all $r \in R$. That the second cellular chain group $C_2(\widetilde{K}) = H_2(\widetilde{K}, \widetilde{K}^1)$ is a free $\mathbb{Z}G$-module with $\mathbb{Z}G$-basis in one-to-one correspondence with preferred lifts $\widetilde{c}^2_r, r \in R$, for the 2-cells of K is a standard element of cellular homology for the universal cover $\widetilde{K}$, see [40, Sec. 3.2]. To be explicit, if we choose a characteristic map $\phi_r : B^2 \to K$ with corresponding based attaching map $\dot{\phi}_r : S^1 \to K^1$ satisfying $[\dot{\phi}_r] = \tau(r) \in \pi_1(K^1, c^0) \cong F$, then the preferred lift $\widetilde{c}_r^2$ has characteristic and attaching maps obtained by lifting at the preferred 0-cell $\widetilde{c}^0 \in p^{-1}(c^0) = \widetilde{K}^0$. The lifted characteristic map $\widetilde{\phi}_r : (B^2, S^1, *) \to (\widetilde{K}, \widetilde{K}^1, \widetilde{c}^0)$ gives rise to the homotopy class $[\widetilde{\phi}_r] \in \pi_2(\widetilde{K}, \widetilde{K}^1, \widetilde{c}^0)$. The preferred lift $\widetilde{c}_r^2$ is viewed as a homology class in $C_2(\widetilde{K}) = H_2(\widetilde{K}^2, \widetilde{K}^1)$ via

$$\widetilde{c}_r^2 = h([\widetilde{\phi}_r]) \in H_2(\widetilde{K}, \widetilde{K}^1)$$

where $h : \pi_2(\widetilde{K}, \widetilde{K}^1, \widetilde{c}^0) \to H_2(\widetilde{K}, \widetilde{K}^1)$ is the Hurewicz homomorphism.

Under the homotopy boundary $\partial : \pi_2(\widetilde{K}, \widetilde{K}^1, \widetilde{c}^0) \to \pi_1(\widetilde{K}^1, \widetilde{c}^0)$ we have

$$\partial([\widetilde{\phi}_r]) = [1, r] \in \pi_1(\widetilde{K}^1, \widetilde{c}^0) \subseteq \pi_1(\widetilde{K}^1, \widetilde{K}^0).$$

Naturality of the Hurewicz homomorphisms yields a commutative square the relates the homotopy and homology boundary maps.

$$\begin{array}{ccc} \pi_2(\widetilde{K}, \widetilde{K}^1, \widetilde{c}^0) & \xrightarrow{\partial} & \pi_1(\widetilde{K}^1, \widetilde{c}^0) \\ \downarrow h & & \downarrow h \\ H_2(\widetilde{K}, \widetilde{K}^1) & \xrightarrow{\partial} & H_1(\widetilde{K}^1 \end{array}$$

Under the homology boundary $\partial : H_2(\widetilde{K}, \widetilde{K}^1) \to H_1(\widetilde{K}^1)$, we therefore have

$$\partial(\widetilde{c}_r^2) = \partial(h([\widetilde{\phi}_r])) = h(\partial([\widetilde{\phi}_r])) = h([1, r]) = d(r[N, N]),$$

completing the proof. □

2.5. (CA) and (CLA)

In this section, we introduce combinatorial algebraic concepts for group presentations that are closely related to asphericity of 2-complexes. These concepts are motivated by results on one-relator groups, due to Lyndon [29] and Cohen-Lyndon [10]. The definitive treatement for these and other asphericity concepts for group presentations is in [8].

A presentation $\mathcal{P} = \langle X : r \rangle$ with a single defining relation $R = \{r\}$ is called a **one-relator** presentation. A special case of the Lyndon Identity Theorem [29] provides that if $\mathcal{P} = \langle X : r \rangle$ is a one-relator presentation for which the relator r is not expressible as a proper power in $F = F(X)$, then the boundary homomorphism $\Delta : \mathbb{Z}G \to N^{ab}$ is an isomorphism. It follows from Theorem 2.4.0.1 and Lemma 2.1.0.1 that the presentation complex $K = K\langle X : r \rangle$ is aspherical. This generalizes the fact that closed surfaces other than S^2 and $\mathbb{R}P^2$ are aspherical, which was a well-known and motivating fact at the time of Lyndon's work. In fact, Lyndon proved something even more general.

For any word $r \in F = F(X)$, there is a positive integer $e = e(r)$ and a reduced word $\mathring{r}$ for which r is freely equivalent to $\mathring{r}^e$ and e is maximal. The word $\mathring{r}$ is the **root** of r and e is the **exponent**. We set $C_r = \langle \mathring{r} \rangle \leq F = F(X)$. Note that C_r centralizes r in F: $\mathring{r} r \mathring{r}^{-1} = r$. It follows that if $\mathcal{P} = \langle X : R \rangle$ is any group presentation for a group G and having relation module N^{ab}, then for each $r \in R$, the map $\Delta : \oplus_{r \in R} \mathbb{Z}G \to N^{ab}$, we have

$$\Delta((\overline{\mathring{r}} - 1) \cdot \tilde{c}_r^2) = 0.$$

In his 1950 paper [29], Lyndon proved that as long as the relator r for a one-relator presentation $\langle X : r \rangle$ for $G = F/N = F/ \ll r \gg$ is freely nontrivial, the kernel of Δ is $\mathbb{Z}G$-generated by the element $(\overline{\mathring{r}} - 1) \cdot \tilde{c}_r^2$. It follows that the relation module N^{ab} is isomorphic to the permutation $\mathbb{Z}G$-module $\mathbb{Z}[G/\overline{C}_r] \cong \mathbb{Z}[F/NC_r]$ where $\overline{C}_r$ is the image of C_r under the natural projection $F \to F/N = G$.

Lyndon's theorem led to the formulation of generalized asphericity properties that accommodate the presence of proper power relators. Following [8], a presentation $\mathcal{P} = \langle X : R \rangle$ for a group $G = F/N$ is **combinatorially aspherical** if the kernel of the homomorphism $\Delta : \oplus_{r \in R} \mathbb{Z}G \to N^{ab}$ is $\mathbb{Z}G$ generated by elements of the form $(\overline{\mathring{r}} - 1) \cdot \tilde{c}_r^2$

for $r \in R$.[2] As in [8, Prop. 1.3], this is equivalent to saying that the relation module N^{ab} is the direct sum of permutation modules

$$N^{ab} \cong \bigoplus_{r \in R} \mathbb{Z}[F/NC_r]$$

where the coset $1NC_r \in \mathbb{Z}[F/NC_r]$ corresponds to the element $r[N,N] \in N/[N,N] = N^{ab}$ for each $r \in R$. As in [8, Prop. 1.3], Theorem 2.4.0.1 and Lemma 2.1.0.1 imply that for a presentation $\mathcal{P}$ with no proper power relators, combinatorial asphericity of $\mathcal{P}$ is logically equivalent to asphericity of the presentation complex $K(\mathcal{P})$.

Note: in general, (CA) presentations do not yield aspherical presentation 2-complexes. Consider, for example, the presentation $\mathcal{P} = \langle x : x^n \rangle$ of the cyclic group of order n, C_n, for $n \geq 2$. Being a one-relator presentation, this is (CA) by the Lyndon Identity Theorem [29]. The root of the relator x^n is x, which has non-trivial image in $G(\mathcal{P}) \cong C_n$ (in fact $\overline{x}$ is a generator for the cyclic group). It follows that $(\overline{x} - 1) \cdot \widetilde{c}^2_{x^n}$ is a non-trivial element of the kernel of Δ, and so by Theorem 2.4.0.1 and Lemma 2.1.0.1, Δ is not an isomorphism and the presentation 2-complex $K\langle x : x^n \rangle$ for $n \geq 2$ is *not* aspherical. For $n = 2$, we have the presentation 2-complex $K\langle x : x^2 \rangle$, which is well-known to be homotopy equivalent to $\mathbb{R}P^2$, which is not aspherical. Significantly, however, the generalization from asphericity to heredity of (CA) is equivalent to Whitehead's original question.

Theorem 2.5.0.1. *[27, Thm. 4] Every subpresentation of a (CA) presentation is (CA) if and only if every connected subcomplex of a connected, aspherical 2-complex is aspherical.*

Suppose now that $\mathcal{P}$ is (CA). Recall that the normal group $N \leq F$ of consequences is a free group, so that in turn the relation module N^{ab} is free abelian. The fact that $\mathcal{P}$ is (CA) implies that if, for each relator $r \in R$, a transversal $U(r)$ for NC_r in $F = F(X)$

[2] In [8], a slightly more general formulation of (CA) allows for the preliminary removal of relators that are **redundant** in the sense that they are freely trivial are else freely equal to conjguates of other relators or their inverses. We will neither need nor consider this more general form of the concept.

is given, then the relation module N^{ab} has a $\mathbb{Z}$-basis of the form $\{wrw^{-1}[N,N] : r \in R, w \in U(r)\}$. In particular, if $\mathcal{P}$ is (CA) and has no proper powers, $NC_r = N$ and so N^{ab} has $\mathbb{Z}G$-basis of the form $\{r[N,N] : r \in R\}$, that is, every element of N^{ab} is an integral linear combination of the $r[N,N]$ for $r \in R$, along with the G-action. By forgetting the G-action and just taking N^{ab} to be a free abelian group (ie, a $\mathbb{Z}$-module), we have $\mathbb{Z}$-basis of the form $\{g \cdot r[N,N] : r \in R, g \in G\}$. For each $r \in R$, a transversal $U(r)$ for $NC_r = N$ in $F(X)$ is then in one-to-one correspondence with G and so for $w \in U(r)$ we have $\overline{w} \cdot r[N,N] = wrw^{-1}[N,N]$.

In [10], Cohen and Lyndon showed that it is possible to lift the $\mathbb{Z}$-basis for the relation module of a one-relator presentation to a *free* basis for the group N of consequences. Lyndon's original Identity Theorem is an immediate consequence [10, Cor. 4.5].

Theorem 2.5.0.2. *(Cohen-Lyndon thm. 4.1] [10]) Let $\mathcal{P} = \langle X : r \rangle$ be a one-relator presentation for a group $G = F/N$ where $1 \neq r \in F$. There exists a transversal $U = U(r)$ for NC_r in $F = F(X)$ such that N is free with basis $\{wrw^{-1} : w \in U\}$.*

The Cohen-Lyndon theorem motivates the definition of a related asphericity concept.

Definition 2.5.0.1. *[8, P. 9] The presentation $\mathcal{P} = \langle X : R \rangle$ for the group $G = F/N$ is* ***Cohen-Lyndon aspherical (CLA)*** *if for each relator $r \in R$, there is a transveral $U(r)$ for NC_r in $F = F(X)$ such that the group N of consequences of the relators has a free basis of the form*

$$\mathcal{B} = \{wrw^{-1} : r \in R, w \in U(r)\}.$$

Such a basis is called a ***Cohen-Lyndon basis****.*

As in [8, Prop. 1.7], we have $(CLA) \Rightarrow (CA)$, but it is known [8, Pp. 8-9], [36] that this implication is not reversible. Additionally, and significantly in the study of Whitehead's conjecture, [8, Prop. 2.4] shows that (CLA) is hereditary: any subpresentation of a (CLA) presentation is also (CLA).

Now, it is important to note, given a (CLA) presentation $\mathcal{P} = \langle X : R \rangle$, the transversals do not necessarily coincide, that is, in general the transversals $U(r) \neq U(s)$ for $r \neq s \in R$.

Example 2.5.0.1. Consider the presentation $\mathcal{P} = \langle x, y : r, s \rangle$ where $r = xyx^{-1}$ and $s = yxy^{-1}$. It is obvious that the normal closure $N = \ll xyx^{-1}, yxy^{-1} \gg$ is the whole group $F = F(x, y)$; $G(\mathcal{P})$ is the trivial group. The relators are not proper powers in F, hence $NC_r = N$. Given $t \in \{r, s\}$, a transversal $U(t)$ is simply a choice of word $w \in F$. Thus, we choose the transversals $U(r) = \{x^{-1}\}$ and $U(s) = \{y^{-1}\}$. Then, $N = F$ clearly has a Cohen-Lyndon basis, as

$$\{x^{-1} \cdot r \cdot x, y^{-1} \cdot s \cdot y\} = \{y, x\}$$

is a basis of conjugates of the relators, with conjugating elements drawn from a transversal for each relator. Thus, $\mathcal{P}$ is (CLA).

The transversals $U(r)$ and $U(s)$ do not coincide, morever, there is *no* choice of transversals that coincide and gives a Cohen-Lyndon basis. To see this fact, observe first that the subgroup of F generated by the relators $\{xyx^{-1}, yxy^{-1}\}$ is a *proper* subgroup (in particular, it plainly omits the elements x and y; subgroup containment is algorithmically decidable in a free group, see [31, Prop. 2.21]). Any choice of transversals which coincide, $U(xyx^{-1}) = U(yxy^{-1}) = \{w\}$ for $w \in F$, gives the subgroup $\langle w \cdot xyx^{-1} \cdot w^{-1}, w \cdot yxy^{-1} \cdot w^{-1} \rangle$, which is conjugate to $\langle xyx^{-1}, yxy^{-1} \rangle$, hence a proper subgroup as well. Thus, $\{w \cdot xyx^{-1} \cdot w^{-1}, w \cdot yxy^{-1} \cdot w^{-1}\}$ could not be a basis and the transversals for any Cohen-Lyndon basis for the presentation $\langle x, y : xyx^{-1}, yxy^{-1} \rangle$ cannot coincide.

An additional subtlety of (CLA) is the fact that, given a presentation $\mathcal{P} = \langle X : R \rangle$, it is possible that N *does* have a basis of conjugates of the relators R, but which is *not* a *Cohen-Lyndon* basis, that is, the conjugating words for some relator $r \in R$ may not be drawn from a transversal for NC_r in $F = F(X)$, as the following example shows.

Example 2.5.0.2. Consider the presentation $\mathcal{P} = \langle x, y : r, s\rangle$ where $r = x^2$ and $s = xyx^{-1}y^{-1}$ and associated presentation 2-complex $K = K(\mathcal{P})$. Then, $\mathcal{P}$ defines the abelian group

$$G(\mathcal{P}) = \langle \overline{x}\rangle \oplus \langle \overline{y}\rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} = \{(i, j) : i \in \mathbb{Z}/2\mathbb{Z}, j \in \mathbb{Z}\}$$

where $\overline{x} = (1, 0)$ has order two and $\overline{y} = (0, 1)$ has infinite order. Here we will show that the normal free group $N = \ll r, s \gg \leq F$ has a basis consisting of conjugates of r and s, but that N does not possess a Cohen-Lyndon basis.

By Lemma 2.1.0.2, N is isomorphic to the fundamental group of the 1-skeleton of $\widetilde{K}$, which is depicted in Figure 2.1.

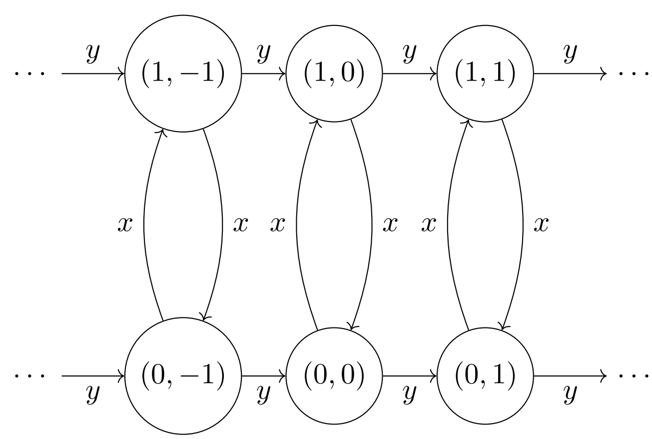

FIGURE 2.1: The 1-skeleton of the Cayley Complex of $\langle x, y : x^2, xyx^{-1}y^{-1}\rangle$

Under the equivalence $G \equiv \widetilde{K}^0$, we take our global basepoint to be the identity, labeled in Figure 2.1 as $(0, 0)$; the other vertices are labeled according to the isomorphism with $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. We take our preferred lifts of c_x^1 and c_y^1 to have initial vertex $(0, 0)$. The various lifts of the 1-cells c_x^1 and c_y^1 are labeled x and y, respectively.

A basis for $\pi_1(\widetilde{K}^1, (0, 0))$ and hence for N is obtained by selecting a maximal tree in $\widetilde{K}^1$, for which we take

$$\mathcal{T} = \widetilde{K}^0 \cup \bigcup_{i\in\mathbb{Z}} \{(0,i)\cdot \widetilde{c}^1_y\} \cup \bigcup_{i\in\mathbb{Z}} \{(0,i)\cdot \widetilde{c}^1_x\}.$$

A basis for N arises by considering the edges of $\widetilde{K}^1$ that lie outside $\mathcal{T}$ and considering the associated loops based at the preferred 0-cell $\widetilde{c}^0$. Projecting to K^1 and considering the corresponding elements of $N \leq F \cong \pi_1(K^1, c^0)$, these loops produce a basis

$$\mathcal{B} = \{y^j x^2 y^{-j} : j \in \mathbb{Z}\} \cup \{y^j xyx^{-1}y^{-1}y^{-j} : j \in \mathbb{Z}\} = \{y^j r y^{-j} : j \in \mathbb{Z}\} \cup \{y^j s j^{-j} : j \in \mathbb{Z}\}$$

for N that consists entirely of conjugates of the relators r and s.

However, this is not a Cohen-Lyndon basis for N. Indeed, no such basis exists because $\mathcal{P}$ is not (CA) [8, Prop. 1.7]. To see this, consider the 2-chain

$$\eta = (1-\overline{y})\widetilde{c}^2_r - (1+\overline{x})\widetilde{c}^2_s$$

inside the free cellular chain module $C_2(\widetilde{K}) \cong H_2(\widetilde{K}^2, \widetilde{K}^1) \cong \mathbb{Z}G\cdot\widetilde{c}^2_r \oplus \mathbb{Z}G\cdot\widetilde{c}^2_s$ with $\mathbb{Z}G$ basis $\widetilde{c}^2_r, \widetilde{c}^2_s$ (see Theorem 2.4.0.1). We verify that $\eta \in \ker\Delta$:

$$\begin{aligned} \Delta(\eta) &= r\cdot yr^{-1}y^{-1}\cdot s^{-1}\cdot xs^{-1}x^{-1}[N,N] \\ &= x^2\cdot yx^{-2}y^{-1}\cdot yxy^{-1}x^{-1}\cdot xyxy^{-1}x^{-2}[N,N] \\ &= x^2\cdot yx^{-2}y^{-1}yx^2y^{-1}x^{-2}[N,N] \\ &= 1N \end{aligned}.$$

Since the relator $s = xyx^{-1}y^{-1}$ is not a proper power in F and $1+\overline{x} \neq 0 \in \mathbb{Z}G$, the 2-cycle η does not lie in the $\mathbb{Z}G$-submodule of $\mathbb{Z}G\cdot\widetilde{c}^2_r \oplus \mathbb{Z}G\cdot\widetilde{c}^2_s$ generated by $(\overline{\mathring{r}}-1)\widetilde{c}^2_r$ and $(\overline{\mathring{s}}-1)\widetilde{c}^2_s = 0$. Thus, the presentation $\mathcal{P}$ is not (C)A and hence is not (CLA).

Regardless of this example, it is not entirely inappropriate to think of Cohen-Lyndon asphericity in terms of N having a basis of conjugates. Indeed, given a (CA) presentation $\mathcal{P} = \langle X : R\rangle$, if $N = \ll R \gg_{F(X)}$, the normal closure of the relators in $F(X)$, has *some* basis of conjugates of the relators, then it has a Cohen-Lyndon basis [8, Lem. 1.8].

2.6. (DR)

In this section, we review a topological flavor of asphericity, which was first described by Sieradski [38] and named and further examined by Gersten [15]. We will give the basic definition of a diagrammatically reducible 2-complex, as well as a characterization of diagrammatic reducibility in terms of the universal cover, originally conjectured by Brick [6] and proved by Corson-Trace [11].

Following the definitions as given by Gersten [15], a cellular map $f : X \to Y$ of CW complexes is **combinatorial** if the restriction of f to each open cell of X is a homeomorphism onto its image. A 2-complex K is **combinatorial** if for each 2-cell α of K, the attaching map $f_\alpha : S^1 \to K^1$ is combinatorial for a suitable subdivision of the 1-sphere S^1. Presentation complexes can be chosen to be combinatorial as long as all relators are freely nontrivial. (But they are only defined up to homotopy type.)

A **diagram** is a combinatorial map $f : C \to K$, where C is a cell structure on the 2-sphere S^2. We say that two distinct faces (ie, closed 2-cells) F and F' of C are **opposite** with respect to f if F and F' have an edge e in common, and if there is an orientation reversing homeomorphism $g : F' \to F$ fixing $F \cap F'$ pointwise with $f \circ g|_{F'} = f|_{F'}$. The diagram $f : C \to X$ is **reducible** if there is an opposite pair of faces and **reduced** otherwise.

Definition 2.6.0.1. *A combinatorial 2-complex K is **diagrammatically reducible** (DR) if every diagram is reducible.*

It is easy to see that (DR) is a hereditary property: if K is a (DR) 2-complex and $L \subseteq K$ some subcomplex, then any diagram $f : C \to L$ is naturally a diagram over K, hence is reducible. Additionally, (DR) 2-complexes are aspherical: transversality techniques allow one to approximate any map from the sphere with a combinatorial map (see [40, Thm. 1.8]). There is then a well-known procedure to replace a reducible diagram

$f : C \to K$ with a homotopy equivalent diagram $f' : C \to K$ with two fewer faces: given the diagram $f : C \to K$ with faces F, F', edge e, and homeomorphism g between the faces F, F', one deletes $F \cup F'$ and identifies the boundary points via g. An inductive argument then shows that any spherical diagram is the zero element in $\pi_2(K, c^0)$. To avoid some of the subtleties in this heuristic argument, we will give a more complete proof following the Corson-Trace characterization. To that end, we need to define collapsability. The following definition is taken from Cohen [13], albeit modified to fit our purposes.

Definition 2.6.0.2. *An* ***elementary collapse*** *in dimension $n \geq 1$ is a pair (K, L) of CW complexes where $K = L \cup e^{n-1} \cup e^n$ is obtained from L by attaching an n-cell and an $(n-1)$-cell and with additional proeprties that imply the existence of a strong deformation retraction $r : K \to L$. In particular, the inclusion $i : L \to K$ is a homotopy equivalence. See Cohen [13] for the precise definitions. We say that K* ***collapses*** *to the subcomplex L if there are subcomplexes $K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_i \supseteq K_{i+1} \supseteq \cdots \supseteq K_N = L$ such that $(K_i.K_{i+1})$ is an elementary collapse (possibly in various dimensions) for $i = 0, \ldots, N-1$. In this case, we write $K \searrow L$ and note that the inclusion $i : L \to K$ is a homotopy equivalence.*

Theorem 2.6.0.1. *[11, Thm. 2.4] A complex K is (DR) if and only if every finite subcomplex of the universal cover $\widetilde{K}$ collapses to a 1-complex.*

It follows quickly from Theorem 2.6.0.1 that any finite, simply-connected, (DR) 2-complex must have a very simple structure:

Lemma 2.6.0.1. *Suppose K is simply connected, compact and (DR). Then, K collapses to a point.*

Note: the result of Lemma 2.6.0.1 was originally an unpublished result, first proven by Gersten [16].

Proof. Since K is (DR), it has universal cover wherein every finite subcomplex collapses to a 1-complex. Since K is simply connected, K is its own universal cover, hence every finite subcomplex must collapse to a 1-complex. Since K is compact, it has finitely many cells, and therefore K itself collapses to a 1-complex. K is simply connected, though, and collapses are a homotopy equivalence, thus it must collapse to a finite, simply connected 1-complex, ie, a tree. Any finite tree in turn collapses to a point, hence K is collapsible. □

The Corson-Trace characterization also gives a straight-forward proof that (DR) complexes are aspherical, with no need for transversality techniques:

Corollary 2.6.0.1. *Let K be a (DR) 2-complex. Then, K is aspherical.*

Proof. Let K be a (DR) 2-complex. By Theorem 2.6.0.1, every finite subcomplex of the universal cover $\widetilde{K}$ collapses to a 1-complex.

Let $f : S^2 \to K$ be any spherical map. Since S^2 is simply connected, this lifts to a map $\widetilde{f} : S^2 \to \widetilde{K}$. Since S^2 is compact, $\widetilde{f}(S^2)$ is a finite subcomplex of $\widetilde{K}$. In turn, $\widetilde{f}(S^2)$ collapses to a 1-complex $L \subseteq \widetilde{K}^1$. Thus: composing $\widetilde{f}$ with the collapse (that is, a strong deformation retract on the 2-cells) homotopes $\widetilde{f}$ to a map $\widetilde{g}$, contained entirely in the 1-skeleton. Pushing down along the covering projection gives a homotopy $f \sim g$, where g maps to the 1-skeleton of K. Thus, any map $f : S^2 \to K$ can be homotoped to a map $g : S^2 \to K^1$.

The map g (and hence f) is therefore trivial, as any 1-complex K^1 is aspherical (by, say, observing that cellular homology obviously vanishes in dimension 2). □

2.7. Some Miscellaneous Algebra

We now conclude this section with certain miscellaneous algebraic facts we will later use. In particular, we review the notion of a locally free group and state the Freiheitssatz, a result on one-relator groups that we will make brief use of.

First, recall that a **locally free** group is a group all of whose finitely generated subgroups are free. For example, the additive group of rational numbers is locally free. For our purposes, we need the following result:

Lemma 2.7.0.1. *Suppose that X is a connected CW complex with the property that every finite subcomplex of X has free fundamental group. Then, for any choice of basepoint $x \in X$, the fundamental group $\pi_1(X, x)$ is locally free. That is, every finitely generated subgroup of $\pi_1(X, x)$ is a free group.*

Proof. Let G be a finitely generated subgroup of $\pi_1(X, x)$. We can choose a positive integer N and a set of based loops $\lambda_m : (S^1, y) \to (X, x)$ for $m = 1, \ldots, N$ such that $H = \langle [\lambda_1], \ldots, [\lambda_N] \rangle$ where N is as small as possible. We show that H is free with basis $\{[\lambda_1], \ldots, [\lambda_N]\}$. By compact supports, there is a finite subcomplex Q of X such that each of the loops λ_m factors as $k \circ \mu_m$ where $\mu_m : (S^1, y) \to (Q, x)$ and $k : Q \to X$ is the inclusion. By hypothesis, the subgroup $H = \langle [\mu_1], \ldots, [\mu_n] \rangle$ is a free group and its rank is at most N. Since $k_*(H) = G$ and N is the minimal size of a generating set for G, the rank of H is at least N and so $\{[\mu_1], \ldots, [\mu_N]\}$ is a basis of H [31, Prop. I.3.5]. It suffices to prove that $k_* : \pi_1(Q, x) \to \pi_1(X, x)$ maps H injectively onto G. Suppose $h \in H$ and $k_*(h) = 1$. By compact supports, there is a finite subcomplex P of X containing Q such that $i_*(h) = 1$ where $i : Q \to P$ is the inclusion. Letting $j : P \to X$ be the inclusion, we have $k = j \circ i$ and so $G = k_*(H) = j_*(i_*(H))$. By hypothesis, the subgroup $i_*(H) = \langle i_*([\mu_1]), \ldots, u_*([\mu_N]) \rangle$ of $\pi_1(P, x)$ is a free group and its rank is at most N. Since $j_*(i_*(H)) = G$ and N is the minimal size of a generating set for G, it follows that the rank of the free subgroup $i_*(H)$ of $\pi_1(P, x)$ is equal to N. Thus, i_* determines a surjective homomorphism $H \to i_*(H)$ between free groups of rank N. It follows that $i_* : H \to i_*(H)$ is an isomorphism [31, Prop. I.3.5] and so $h = 1$ as desired. □

We now state the Freiheitssatz of Magnus.

Theorem 2.7.0.1. *[31, Prop. 5.1] Let F be a free group with basis X and r a cyclically reduced element of F that contains a certain generator x from X. Then, every non-trivial element of the normal clsoure of r in F also contains x.*

For our purposes, we will use the equivalent formulation, stated in terms of a one-relator presentation: given a one-relator presentation $\langle X : r \rangle$ with r cyclically reduced, freely non-trivial, and containing an element x from X, then (the image of) $X - \{x\}$ is a basis for a free subgroup of G.

3. FREE KERNEL THEOREM

Our goal in this chapter is to prove a technical result, the Free Kernel Theorem, (Corollary 3.2.0.1 below) which will enable us to distinguish (CLA) from (DR). Recally that our overall goal is to produce an example of a 2-complex that is (DR) but not (CLA). We work in the **Whitehead Setting**, consisting of a 2-complex K and a connected subcomplex L, where K is either (DR) or (CLA). Both of these properties are inherited by L, as appropriate, but we will see the distinction between (DR) and (CLA) by considering the kernel of the inclusion-induced homomorphism $i_* : \pi_1(L, c^0) \to \pi_1(K, c^0)$ on fundamental groups. To ease exposition, we will refer to this as the π_1-kernel associated to the Whitehead setting (K, L). We will interested only in the isomorphism class of the π_1-kernel in the case where both K and L are (path) connected, so there is no ambiguity in choice of basepoint, which is generally taken to be a 0-cell. We often work in the setting of presentation 2-complexes, in which case there is a single 0-cell c^0 to serve as the basepoint.

Characterizations of (DR) and (CLA) have been developed in [11] and [37]. Using Lemma 2.1.0.2, these characterizations yield information about the π_1-kernels in both cases. For a Whitehead setting (K, L) where K is (DR), the Corson-Trace [11] result of Theorem 2.6.0.1 implies that the π_1-kernel is a locally free group. In the case where K is (CLA), Sieradski's analysis [37] in the non-proper case implies that the π_1-kernel is a free group. This was originally observed by W. Bogley (unpublished). We prove this result in full generality for an arbitrary (CLA) complex K and produce an explicit description of a basis for the π_1-kernel.

3.1. (DR) Presentations and Locally Free Kernels

In this section, we show that (DR) presentations have locally free π_1-kernel. The argument, using the Corson-Trace characterization, is brief and is primarily given in contrast to the free kernel theorem for (CLA) presentations.

In this section, we show that the π_1-kernel in the (DR) setting is locally free. The argument relies on the Corson-Trace characterization of (DR), is brief, and is given primarily in contract with the Free Kernel Theorem for the (CLA) setting.

Lemma 3.1.0.1. *Let (K, L) be a Whitehead setting where K is (DR). Then, the associated π_1-kernel is a locally free group.*

Proof. Ler $p : \widetilde{K} \to K$ be the universal covering projection. Let $c^0 \in L$ be the basepoint and let $\overline{L}$ be the connected component of the pre-image $p^{-1}(L)$ that contains the preferred 0-cell $\widetilde{c}^0 \in p^{-1}(c^0) \subseteq \widetilde{K}^0$. By Lemma 2.1.0.2, the π_1-kernel is isomorphic to $\pi_1(\overline{L}, \widetilde{c}^0)$. By [11, Thm. 2.4] (see Theorem 2.6.0.1 above), every finite subcomplex of $\overline{L}$ has the homotopy type of a 1-complex and so has free fundamental group. By Lemma 2.7.0.1, it follows that π_1-kernel free. □

3.2. Free Kernel Theorem

In this section, we prove the Free Kernel Theorem for (CLA) presentations (Cor. 3.2.0.1). Given a Whitehead setting (K, L) where K is (CLA), we use Lemma 2.3.0.1 to modify attaching maps for 2-cells in the universal cover $\widetilde{K}$ within their free homotopy classes to give an explicit description for the associated π_1-kernel.

As before, for a presentation $\mathcal{P} = \langle X : R \rangle$, we have the free group $F = F(X)$, the normal closure $N = \ll R \gg$ in F, and the group $G = G(\mathcal{P}) = F/N$. We also have presentation 2-complex $K = K(\mathcal{P})$ with universal cover $p : \widetilde{K} \to K$ as described in Section

2.2. and Section 2.3..

Given $r \in R$, suppose $r = \mathring{r}^{e(r)}$ with $e(r)$ maximal. For $r \in R$, let C_r be the centralizer in $F(X)$ (so that $C_r = \langle \mathring{r} \rangle$). Let $\overline{C}_r = \langle \overline{\mathring{r}} \rangle \leq G$.

For each $r \in R$, let $U(r)$ be an arbitrary chosen transversal for NC_r in F. Then, if $U(r)$ is a transversal for NC_r in $F(X)$, then $\overline{U}(r) = \{\overline{u} : u \in U(r)\}$ is a transversal for $\overline{C}_r$ in $G = F/N$. To see this, first let $g \in G$, which can be represented as $g = wN \in F/N = G$ for some $w \in F$. Sunice $U(r)$ is a transversal for NC_r in F, there exists $u \in U(r)$ such that $w \in uNC_r$, and so $g = \overline{w} \in \overline{u}NC_r/N = \overline{u}\overline{C}_r$. Next, if $\overline{u}, \overline{v} \in \overline{U}(r)$ and $\overline{u}\overline{C}_r \cap \overline{v}\overline{C}_r \neq \emptyset$, then $\overline{u} \in \overline{v}\overline{C}_r$ so $uN = v\overline{\mathring{r}^m}N$ for some $m \in \mathbb{Z}$ and hence $u \in vNC_r$. Since u, v lie in the transversal for NC_r in F, this implies $u = v$ and hence $\overline{u} = \overline{v}$. Thus, G is the disjoint union of cosets $\overline{u}\overline{C_r}$ where $u \in U(r)$ as desired.

Thus, we have

$$G = \bigcup_{u \in U(r)} \overline{u}\overline{C}_r.$$

Under this representation, each $g \in G$ can be uniquely written as some $\overline{uc}$ where $u \in U(r)$ and $\overline{\mathring{r}^j} = \bar{c} \in \overline{C}_r$.

Recall from Section 2.3. that the Cayley complex has cell structure

$$\widetilde{K} = \widetilde{K}^1 \cup \bigcup_{g \in G, r \in R} g \cdot \widetilde{c}_r^2$$

where the 2-cell $g \cdot \widetilde{c}_r^2$ is attached by a based loop with path homotopy class $[g, r] \in \pi_1(\widetilde{K}^1, \widetilde{K}^0)$ that begins at $g \in G \equiv \widetilde{K}^0$ and lies over $\tau(r) \in \pi_1(K^1, c^0)$. Up to homotopy equivalence, we therefore write

$$\widetilde{K} \simeq \widetilde{K}^1 \cup \bigcup_{g \in G, r \in R} c_{[g,r]}^2$$

to indicate the path homotopy classes of the attaching maps for the 2-cells of the Cayley complex. We will modify these attaching maps within their *free* homotopy classes. These modifications preserve homotopy type by Theorem 2.2.0.1.

We now consider a Whitehead setting (K, L) where L is the subcomplex of $K = K(\mathcal{P})$ modeled on a subpresentation $\langle X : S\rangle$ of $\mathcal{P} = \langle X : R\rangle$ where $S \subseteq R$. In this case, the preimage $p^{-1}(L) = \overline{L}$ is connected (since $L^1 = K^1$). By Lemma 2.1.0.2, the π_1-kernel is isomorphic to $\pi_1(\overline{L}, \widetilde{c}^0)$.

Lemma 3.2.0.1. *With $S \subseteq R$ as above, the pre-image $\overline{L} = p^{-1}(L)$ is homotopy equivalent to the space*

$$\overline{L} \simeq \widetilde{K}^1 \cup \bigcup_{s\in S, u\in U(S), c\in C_s} c^2_{[1,ucsc^{-1}u^{-1}]}$$

where the subscripts indicate path homotopy classes for 2-cell attaching maps that are loops based at the preferred 0-cell $\widetilde{c}^0 \in \widetilde{K}^0$. Moreover, the π_1-kernel associated to the Whitehead setting (K, L) has this description:

$$\pi_1(\overline{L}, \widetilde{c}^0) \cong \pi_1(\widetilde{K}^1 \cup \bigcup_{s\in S, u\in U(S)} c^2_{[1,usu^{-1}]}).$$

Proof. Recall that we have

$$\widetilde{K} = \widetilde{K}^1 \cup \bigcup_{r\in R, g\in G} g \cdot \widetilde{c}^2_r.$$

Rewriting the 2-cell $g\cdot\widetilde{c}^2_r$ according to our notation for path homotopy classes in $\pi_1(\widetilde{K}^1, \widetilde{K}^0)$, this cell is $c^2_{[g,r]}$. Rewriting $G = \cup_{u\in U(r)} \overline{u}\overline{C}_r$ and for each $g \in G$ writing $g = \overline{uc}$, we have

$$\widetilde{K} = \widetilde{K}^1 \cup \bigcup_{r\in R, u\in U(r), c\in C_r} c^2_{[\overline{uc},r]}.$$

By Lemma 2.3.0.1, the path homotopy class $[1, uru^{-1}]$ satisfies

$$[1, uru^{-1}] = [1, uc \cdot r \cdot c^{-1}u^{-1}] = [1, uc][\overline{uc}, r][1, uc]^{-1}.$$

On the other hand, the path homotopy class $[\overline{uc}, r]$ is freely homotopic to $[1, uc][\overline{uc}, r][1, uc]^{-1}$ and hence to $[1, uru^{-1}]$ (by dragging the initial vertex of the loop $[\overline{uc}, r]$ to the point $1 \in G \equiv \widetilde{K}^0$, forming a "tail"). Replacing a 2-cell by a 2-cell with homotopic attaching map gives a homotopy equivalence [40, Thm. 1.6], and so we have that the universal cover $\widetilde{K}$ is homotopy equivalent to the space

$$\widetilde{K}^1 \cup \bigcup_{r\in R, u\in U(r), \overline{c}\in\overline{C}_r} c^2_{[1,uru^{-1}]}.$$

For the subcomplex $L \subseteq K$, the previous calculation yields the following sequence of equality and homotopy equivalences:

$$\begin{aligned}
p^{-1}(L) &= \widetilde{K}^1 \cup \textstyle\bigcup_{g\in G, s\in S} g\cdot \tilde{c}^2_s \\
&= \widetilde{K}^1 \cup \textstyle\bigcup_{g\in G, s\in S} c^2_{[g,s]} \\
&= \widetilde{K}^1 \cup \textstyle\bigcup_{s\in S, u\in U(s), c\in C_s} c^2_{[\bar{u}c,s]} \\
&\simeq \widetilde{K}^1 \cup \textstyle\bigcup_{s\in S, u\in U(s), c\in C_s} c^2_{[\bar{u},csc^{-1}]} \\
&\simeq \widetilde{K}^1 \cup \textstyle\bigcup_{s\in S, u\in U(s), c\in C_s} c^2_{[\bar{u},s]} \\
&\simeq \widetilde{K}^1 \cup \textstyle\bigcup_{s\in S, u\in U(s), c\in C_s} c^2_{[1,usu^{-1}]}
\end{aligned}$$

Now, for each proper power relator $s \in S$, we have multiple 2-cells with same attaching loop, one for each centralizing element $c \in C_s$. Thus the fundamental group is unaffected by the removal of these 2-cells and so:

$$\pi_1(p^{-1}(L), 1) \cong \pi_1\left(\widetilde{K}^1 \cup \bigcup_{s\in S, u\in U(s)} c^2_{(1,usu^{-1})}\right).$$

□

With the groundwork of Lemma 3.2.0.1 lain, the free kernel theorem follows readily:

Theorem 3.2.0.1. *Let K be the model of a CLA presentation $\langle X : R\rangle$ for a group G. Let S be a subset of R and let L be the model of a subpresesentation $\langle X : S\rangle$ so that L is a subcomplex of K that contains the one-skeleton K^1. Let $\mathcal{B} = \bigcup_{r\in R}\mathcal{B}_r$ where $\mathcal{B}_r = \{uru^{-1} : U(r)\}$ be a Cohen-Lyndon basis basis for the normal closure $N = \ll R \gg$in the free group $F = F(X)$. Then the kernel of the inclusion induced homomorphism $\pi_1(L) \to \pi_1(K)$ is the free group with basis $\bigcup_{r\in R-S}\mathcal{B}_r$.*

Proof. Let $K = K(\mathcal{P})$ and $L = L(\mathcal{Q})$ be the associated presentation 2-complexes. Let $p : \widetilde{K} \to K$ be the universal cover. Then, M is isomorphic to the kernel of the inclusion induced map $\pi_1(L, c^0) \to \pi_1(K, c^0)$ and $\ker\{\pi_1(L, c^0) \to \pi_1(K, c^0)\}$ is isomorphic to $\pi_1(p^{-1}(L), 1)$ by Lemma 2.1.0.2.

Since $\mathcal{P}$ and $\mathcal{Q}$ had the same generators, $K^1 = L^1$, and $p^{-1}(L^1) = \widetilde{K}^1$. Now, the generators of $\pi_1(p^{-1}(L), 1)$ are exactly the generators of $\pi_1(\widetilde{K}^1, 1) \cong N = \ker\{F(X) \to G\}$ and the relators from the lifted 2-cells for each $s \in S$. Since we have the full one-skeleton, we may modify all the 2-cells of $\widetilde{K}$, including those of $p^{-1}(L)$, as in Lemma 3.2.0.1. Thus: $p^{-1}(L)$ is homotopy equivalent to

$$\widetilde{K}^1 \cup \bigcup_{s \in S, u \in U(s), \bar{c} \in \overline{C}_s} c^2_{(1, usu^{-1})}.$$

Since $\mathcal{P}$ is (CLA), the free normal subgroup N of F has a Cohen-Lyndon basis that is constructed in terms of special transversals $U(r), r \in R$. In therms of these transversals, this basis is

$$\mathcal{B} = \cup_{r \in R}\{uru^{-1} : u \in U(r)\}$$

where $U(r)$ is a transversal for NC_r in $F(X)$. This gives our generators for $\pi_1(p^{-1}(L), 1)$ under the isomorphism of Lemma 2.1.0.2 and since every 2-cell of $p^{-1}(L)$ is attached by a word in these generators, based at the global basepoint, we have the following presentation for $\pi_1(p^{-1}(L), 1)$:

$$\langle uru^{-1}; r \in R, u \in U(r) : usu^{-1}; s \in S, u \in U(s), \bar{c} \in \overline{C}_s\rangle.$$

Deleting the redundant relators from $\bar{c} \in \overline{C}_s$, we have more simply:

$$\langle uru^{-1}; r \in R, u \in U(r) : usu^{-1}; s \in S, u \in U(s)\rangle.$$

This is clearly a free group (the relations are simply a subset of the generators), completing the proof. $\square$

Corollary 3.2.0.1. *Let* $\mathcal{P} = \langle X : R \rangle$ *be a (CLA) presentation of the group* $G(\mathcal{P})$*, such that no relator is a proper power nor conjugate to any other relator or its inverse. Let* $\mathcal{Q} = \langle X : S \rangle$ *be a subpresentation, sharing all the generators, and presenting a group* $H = H(\mathcal{Q})$*. Then,* $M = \ker\{H \to G\}$ *is free and naturally isomorphic to* $F(G \times (R - S))$*.*

Proof. Since there are no proper powers, $NC_r = N$ and so the transversal $U(r)$ is in one-to-one correspondence with G. The result then follows directly from the construction in Theorem 3.2.0.1. □

Corollary 3.2.0.2. *Let* $\mathcal{P} = \langle X : R \rangle$ *be a (CLA) presentation of the group* $G(\mathcal{P})$*. Let* $\mathcal{Q} = \langle Y : S \rangle$ *be any subpresentation and presenting a group* $H = H(\mathcal{Q})$*. Then,* $M = \ker\{H \to G\}$ *is free.*

Proof. Consider the group H' presented by $\langle X : S \rangle$. Then, the inclusion $H \to G$ factors through H' and $\ker\{H \to G\}$ factors through $\ker\{H' \to G\}$. H' satisfies the conditions of Theorem 3.2.0.1 hence $\ker\{H' \to G\}$ is free. Thus, $\ker\{H \to G\}$ is the subgroup of a free group, hence is free by the Nielsen-Schreier theorem. □

4. A (DR) PRESENTATION THAT IS NOT (CLA)

With the free kernel theorem established, we can fulfill our mission to find a presentation which is (DR) but not (CLA). This completes the map of logical implications between flavors of asphericity shown in Figure 1.1, that is, there are no implications other than exactly what is shown. Furthermore, examples of (DR) presentations which are not (CLA) are bountiful, as we can produce such a presentation for any non-trivial group which admits a (DR) presentation.

4.1. Where to Look for a Counterexample

Among the various counterexamples to implications between flavors of asphericity, a common theme emerges: accessible counterexamples tend to be finite, balanced presentations of the trivial group (where a **balanced** presentation $\mathcal{P} = \langle X : R \rangle$ is a finite presentation such that $|X| = |R|$, ie, a presentation which has the same number of generators as relators). The properties (DA) and (CA) were distinguished by Sieradski [36, Sec. 6] and Chiswell [8, Sec. 1], with examples of (CA) presentations of the trivial group which are not (DA). (CLA) and (DA) were distinguished by Biskup [4, Thm. 4.1], again using a balanced presentation of the trivial group. Likewise, the folklore example of a (CLA), but not (DR) (additionally: (DA) but not (DR)), is given by the dunce cap, a balanced presentation of the trivial group. For posterity, we repeat this example. The dunce cap, the topological model of $\langle x : xxx^{-1} \rangle$, is shown in Figure 4.1, as a triangle with identified sides.

The spherical map to the dunce cap with no identified sides is shown in Figure 4.2. Here, the sphere's north and south poles are cellulated as cones, each cone mapping to the dunce cap according to the labeling on the cells. Observe that the global basepoint for

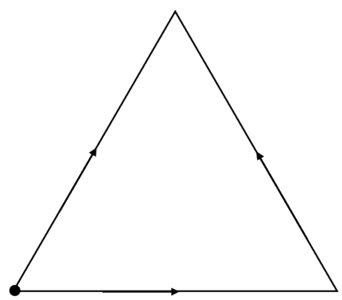

FIGURE 4.1: The Dunce Cap

the two cells are *rotated* away from each other, hence the two cells are *not* opposite, and the dunce cap is not (DR). The presentation $\langle x : xxx^{-1} \rangle$ of the trivial group obviously has Cohen-Lyndon basis. In particular, $\langle x : xxx^{-1} \rangle$ is a presentation of the trivial group, and so the kernel of the natural map $F(x) \to 1$ is $F(x)$ itself. The relator xxx^{-1} is freely equal to x and so the kernel has basis of conjugates of $\{xxx^{-1}\}$, namely $\{x\}$ itself.

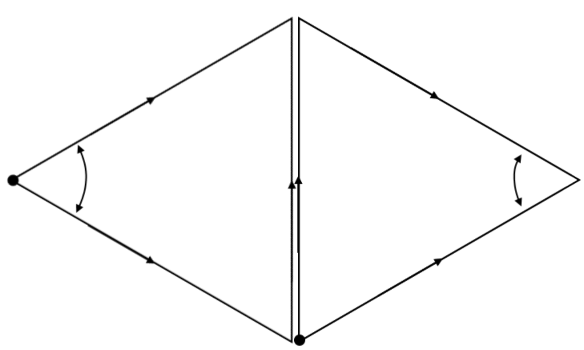

FIGURE 4.2: As Spherical Map to the Dunce Cap

It is worth noting that the presentation $\langle x : xxx^{-1} \rangle$ is not cyclically reduced; the failure to be cyclically reduced is not why this presentation is not (DR). Observe, the presentation $\langle a, b : aab, ab\rangle$ also fails to be (DR); in fact it is a subdivision of the dunce cap, as shown in Figure 4.3, and admits the same spherical map as in the dunce cap counterexample.

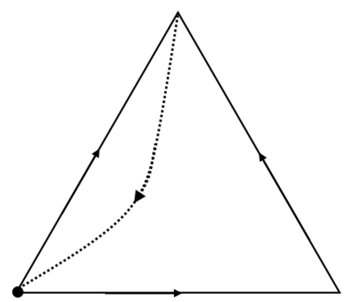

FIGURE 4.3: A Subdivision of the Dunce Cap

Now, given an arbitrary, balanced presentation of the trivial group, it is comparatively straight-forward to check whether or not it is (CLA). A balanced presentation of the trivial group is (CLA) if and only if there is a basis of conjugates of the relators. There are many known balanced presentations of the trivial group and one might hope that one of these might be DR but not CLA. However this is not possible, as we now demonstrate.

A routine calculation shows:

Lemma 4.1.0.1. *The cellular model of a finite presentation of the trivial group is aspherical if and only if it is balanced.*

Proof. Let K be the model of a finite presentation of the trivial group. Since K is simply connected, K is aspherical if and only if $H_2(K) = 0$. Taking the cellular chain complex for K, we have

$$\oplus_{r\in R}\mathbb{Z} \to \oplus_{x\in X}\mathbb{Z} \to \mathbb{Z}.$$

Since K is simply connected, $H_1(K) = 0$ by the Hurewicz theorem. The map $\oplus_{x\in X}\mathbb{Z} \to \mathbb{Z}$

is the 0-map, hence $\oplus_{r\in R}\mathbb{Z} \to \oplus_{x\in X}\mathbb{Z}$ is surjective. Since K is 2-dimensional, $H_2(K)$ is the kernel of the map $\oplus_{r\in R}\mathbb{Z} \to \oplus_{x\in X}\mathbb{Z}$, which is 0 if and only if $|R| = |X|$, by the rank-nullity theorem. □

Combining this with the Corson-Trace characterization of (DR), we have the following:

Lemma 4.1.0.2. *Suppose $\langle X : R\rangle$ is finite presentation of the trivial group and $K\langle X : R\rangle$ is (DR). Then, $\langle X : R\rangle$ is (CLA).*

Proof. To show a presentation of the trivial group is (CLA), it suffices to give a basis for the group in terms of conjugates of the relators.

K is (DR) and therefore aspherical by Corollary 2.6.0.1. By Lemma 4.1.0.1, $\langle X : R\rangle$ is therefore a balanced presentation with generators $\{x_1, \dots, x_n\}$ and relators $\{r_1, \dots, r_n\}$. Moreover, by Lemma 2.6.0.1, K collapses to a point (again, see Cohen [13] for the precise definition). Identifying the generators and relators with their respective 1- and 2-cells, we order the generators and relators by the order in which collapses occur $\{x_1 < \cdots < x_n\}$ and $\{r_1 < \cdots < r_n\}$. That is, the 2-cell corresponding to r_n collapses across the 1-cell corresponding to x_n, then the $n-1$st pair, and so on. Collapsing a 2-cell across a free edge corresponds to the deletion of the generator t and the relator $t \cdot w$ for the presentation $\langle X : R\rangle \to \langle X \cup t : R \cup t \cdot w\rangle$, where t is some generator and w is a word in $X \cup X^{-1}$.

Since r_n collapses across x_n, x_n must occur exactly once in the relator r_n and not at all in any relator $r_i < r_n$. In this case, up to cyclic permutation (but not free deletion) and inversion, we have $r_n = x_n w(x_1, \dots, x_{n-1})$, where $w(x_1, \dots, x_{n-1})$ is some word in $x_1, \dots, x_{n-1}$.

We now induct on the number of generators or relators to show that the topological model of a (DR) presentation $\langle x_1, \dots, x_n : r_1, \dots, r_n\rangle$ collapses to the topological model of a (DR) presentation with one fewer generator or relator. The base case of no generators or relators holds trivially. Now suppose $r_1, \dots, r_{n-1}$ is a basis for $F(x_1, \dots, x_{n-1})$ of the form

$\{x_1, x_2w(x_1), \ldots, x_{n-1}w(x_1, \ldots, x_{n-2})\}$. Then, the word $w(x_1, \ldots, x_{n-1})$ is contained in the group generated by $r_1, \ldots, r_{n-1}$ and $x_n = r_n w(x_1, \ldots, x_{n-1})^{-1}$ is contained in the group generated by $r_1, \ldots, r_n$, thus $F(x_1, \ldots, x_n)$ is generated by the r_i in the fashion required. □

The proof of Lemma 4.1.0.2 shows that any balanced, (DR) presentation of the trivial group is of a very explicit form, namely, up to cyclic permutation and inversion, but not free deletion, it must be of the form $\langle x_1, \ldots, x_n : x_1, x_2w(x_1), \ldots, x_nw(x_1, \ldots, x_{n-1})\rangle$. Note: it is not enough that the r_i form a basis for $F(x_1, \ldots, x_n)$ for a balanced presentation of the trivial group to yield a collapsible model, as the presenation $\langle a, b : aab, ab\rangle$ shows. The relators $\{aab, ab\}$ are a basis for $F(a, b)$, but this presentation does not yield a (DR) complex, as already discussed.

To get a better understanding of what sort of presentation might be (DR) but not (CLA), it was useful to consider in what ways the proof of Lemma 4.1.0.2 might be extended. Naturally, any counterexample must arise from a non-trivial, infinite (because aspherical presentations are torsion-free) group. Passing to the universal cover necessarily gives an infinite complex, hence one which does not collapse to a point, but it may be the ascending union of collapsible complexes and perhaps one could argue after the same fashion as Lemma 4.1.0.2.

Consider, for example, the presentation $\mathcal{P} = \langle x, y, t : x, xtyt^{-1}\rangle$ of $\mathbb{Z}$, generated by t, with x and y trivial. The kernel of $F(x, y, t) \to \mathbb{Z}$ arising from this presentation is the normal closure of $\{x, y\}$ in $F(x, y, t)$. This has a basis of the form $\{t^ixt^{-i}, t^iyt^{-i}\}$. Replacing each basis element t^iyt^{-i} with

$$t^{i-1}xt^{-(i-1)} \cdot t^iyt^{-i} = t^{i-1}xtyt^{-i} = t^{i-1}(xtyt^{-1})t^{-(i-1)}$$

gives a new basis, in fact a Cohen-Lyndon basis.

Setting $K = K_{\mathcal{P}}$ and passing to the universal cover $\widetilde{K}$, we have a complex with 1-skeleton depicted in Figure 4.4.

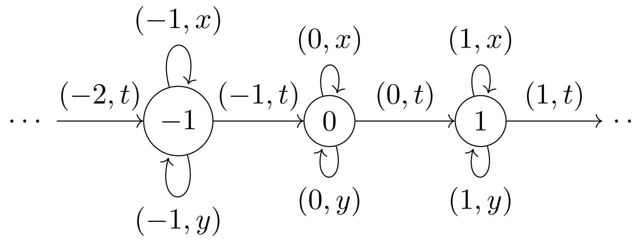

FIGURE 4.4: 1-skeleton of $\widetilde{K}$

Here, $\mathcal{P}$ is a presentation of the integers and so the 0-cells are labeled according to the equivalence $\mathbb{Z} \cong \pi_1(K, c^0) \equiv \widetilde{K}^0$. We take 0 to be the preferred lift of the unique 0-cell c^0 of K. The 1-cells are labeled by (j, z) for $j \in J$, $z \in \{x, y, t\}$, with (j, z) being the lift of c^1_z based at the 0-cell j.

Working modulo the unique spanning tree, there is a fairly natural presentation for $\pi_1(\widetilde{K}, 1)$:

$$\langle x_i, y_i : x_i, x_i y_{i+1} \rangle_{i \in \mathbb{Z}}.$$

Here we are simply taking $x_i := t^i x t^{-i}$ and $y_i := t^i y t^{-i}$. It is not difficult to see that this is the ascending union of collapsible subcomplexes, hence (DR), but further the $\{x_i, x_i y_i\}$ are a basis for $F(x_i, y_i)$ exactly as in Lemma 4.1.0.2.

Altogether, this gives a delicate balance to maintain: we want to produce (1) a contractible 2-complex, so that it is simply-connected and aspherical, where (2) every finite subcomplex collapses to a 1-complex, so that it is (DR), and (3) the collapses to a 1-complex are poorly enough behaved that we do not have the hereditary homotopy type of a wedge of disks. Comparing the information we have about the fundamental groups of subcomplexes of the universal cover of (DR) and (CLA) complexes, namely Corollary 3.1.0.1 which states that, in the Whitehead setting, the inclusion of a subcomplex in the (DR) case must be locally free, and Corollary 3.2.0.2 which states the inclusion of a subcomplex in (CLA) case must be free, we will try and force our complex to collapse edges "off to infinity" so that, although the finite portions of the complex have free fundamental group, the complex as a whole does not. Indeed, this approach does end up working, in a charmingly classical fashion. It arises from a $\mathbb{Z}$-cover of a one-relator group. In fact, the non-example shown in Figure 4.4 is extremely near to the actual counterexample we will produce

4.2. Introduction to the Counterexample

Fix the presentation $\mathcal{P} = \langle x, y, t : r, s\rangle$, where $r = x^{-1}txyx^{-1}y^{-1}t^{-1}$ and $s = y$. It is plain to see that $\mathcal{P}$ presents the integers, generated by t, with both x and y trivial. Let $K = K(\mathcal{P})$ and $p : \widetilde{K} \to K$ be the universal cover. The universal cover $\widetilde{K}$ has 1-skeleton depicted in Figure 4.5

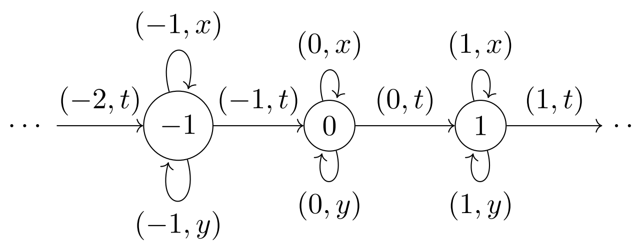

FIGURE 4.5: 1-skeleton of $\widetilde{K}$

As in chapter 3, we are denoting a path based at g and spelling w by (g, w). Here, since $\mathcal{P}$ presents $\mathbb{Z}$, g is simply an integer, and our labeled paths w are a word in the generators x, y, t and their inversesm corresponding to oriented lifts of the 1-cells of K. The lift $g\cdot\tilde{c}_r^2$ has boundary path $(g, x^{-1}txyx^{-1}y^{-1}t^{-1})$, which meets, in order (and disregarding orientation), the 1-cells labeled

$$(g, x), (g, t), (g + 1, x), (g + 1, y), (g + 1, x), (g + 1, y), (g, t).$$

We are writing $(g, x^{-1}txyx^{-1}y^{-1}t^{-1})$ to denote the combinatorial path with initial vertex g and edges traversed, in order, to spell the word $x^{-1}txyx^{-1}y^{-1}t^{-1}$. This is a *specific* path and not just the path homotopy class, per the particular combinatorial attaching map for a presentation 2-complex, as described in Section 2.2.. We have, in particular, $g \cdot \tilde{c}_r^2$

meets (g, x) exactly once and $(g, t), (g+1, x)$, and $(g+1, y)$ exactly twice. Likewise, the lift $g \cdot \tilde{c}_s^2$ has boundary path (g, y) and meets only the 1-cell (g, y). (Geometrically, $\widetilde{K}$ consists of $\mathbb{Z}$-many toruses, each with an open disk deleted, then a cylinder connecting the loop x in the gth torus to the boundary of deleted disk in the $g+1$st torus, then finally a disk glued along each loop y. See Figure 4.7 for a sketch the lifted cell $\tilde{c}_r^2$. Similarly, K consists of a torus, with an open disk deleted, then a cylinder connecting the loop x to the boundary of the deleted disk, and finally a disk glued along the loop y. The loop which traverses from the basepoint meeting x, along the cylinder to the boundary of the disk, then back to the basepoint, is the generating loop t.) See Figure 4.6.

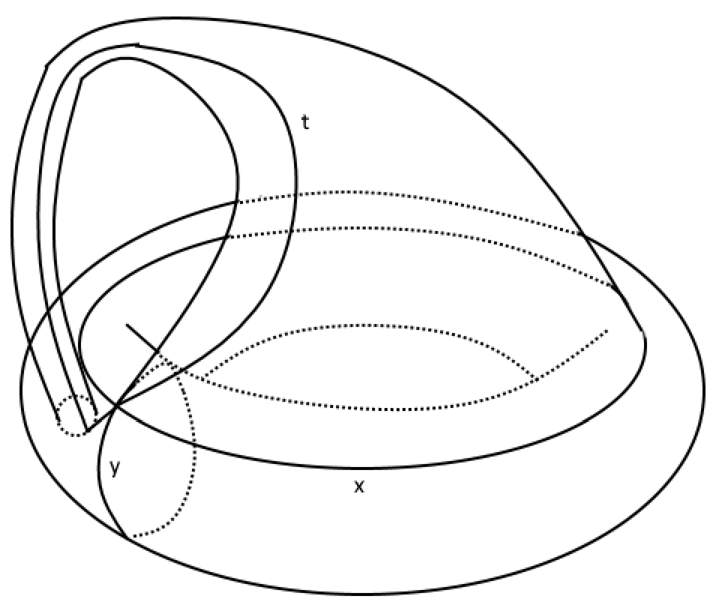

FIGURE 4.6: The Complex K

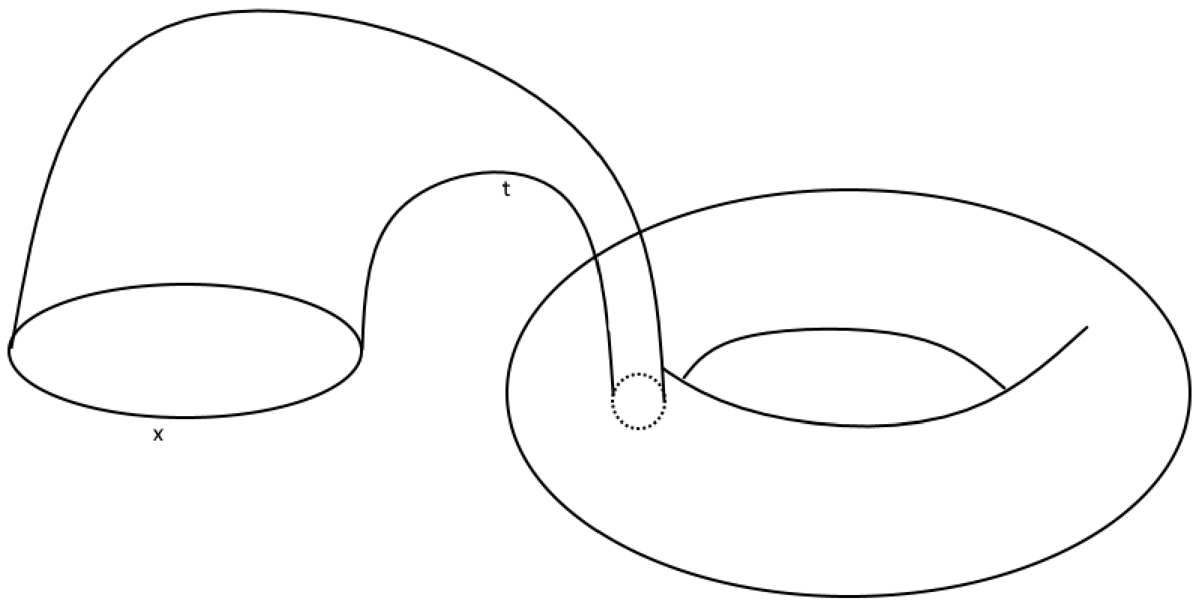

FIGURE 4.7: The Cell $\tilde{c}_r^2$

4.3. $K_{\mathcal{P}}$ Is (DR)

We now show that $K_{\mathcal{P}}$ is (DR).

Lemma 4.3.0.1. *Let* $\mathcal{P} = \langle x, y, t : r, s\rangle$ *where* $r = x^{-1}txyx^{-1}y^{-1}t^{-1}$ *and* $s = y$. *Let* $K = K(\mathcal{P})$ *and* $\widetilde{K}$ *be the universal cover of* K. *Then, every finite subcomplex of* $\widetilde{K}$ *collapses to a 1-complex.*

Proof. Recall that $\mathcal{P}$ presents the integers and $K = K_{\mathcal{P}}$ has universal cover $\widetilde{K}$ with 1-skeleton as in Figure 4.5.

Given a finite 2-complex which collapses to a 1-complex, any subcomplex also collapses to a 1-complex. For integers $a \leq b$, we denote the interval of integers $\{a, a+1, \ldots, b-1, b\}$ by $[a, b]$. Any finite subcomplex of $\widetilde{K}$ is contained in the full subcomplex containing the 0-cells in $[a, b]$ for some integers $a, b \in \mathbb{Z}$. Thus, it suffices to consider the full subcomplex containing the 0-cells in $[a, b]$ to show that every finite subcomplex collapses to a 1-complex. Without loss of generality, we show the result holds for the subcomplex $\widetilde{K}_n$, the full subcomplex on the 0-cells $[0, n]$.

$\widetilde{K}_n$ has 0-cells $g \in [0, n]$, but slightly more care is needed to list the 1- and 2-cells. $\widetilde{K}_n$ has the lifts $g \cdot \tilde{c}_x^1$ and $g \cdot \tilde{c}_y^1$ for $g \in [0, n]$, however, the 1-cell $n \cdot \tilde{c}_t^1$ has terminal vertex $n+1$. Thus, $\widetilde{K}_n$ only has the lifts $g \cdot \tilde{c}_t^1$ for $g \in [0, n-1]$. Likewise, $\widetilde{K}_n$ has all the lifts $g \cdot \tilde{c}_s^2$ for $g \in [0, n]$, but only the lifts $g \cdot \tilde{c}_r^2$ for $g \in [0, n-1]$, for the same reason (the boundary loop of $n \cdot \tilde{c}_r^2$ would meet the 0-cell $n+1$).

For the full subcomplex on $[a, n]$, we induct on $n - a$, that is, we show that if the full subcomplex on $[a, n]$ collapses to a 1-complex, then the full subcomplex on $[a-1, n]$ collapses to a 1-complex. For $n = 0$, the complex $\widetilde{K}_0$ is a wedge of a disk and a circle, ie, it has 0-cell 0, two 1-cells $0 \cdot \tilde{c}_x^1$ and $0 \cdot \tilde{c}_y^1$, and one 2-cell $0 \cdot \tilde{c}_s^2$. The 2-cell $0 \cdot \tilde{c}_x^2$ has a free edge, the 1-cell $0 \cdot \tilde{c}_y^1$, and so collapsing leaves a 1-complex.

Now, consider $\widetilde{K}_{n-1}$, the full subcomplex on the 0-cells $[1, n]$, and suppose it collapses to a 1-complex. We show $\widetilde{K}_n$ collapses to $\widetilde{K}_{n-1}$, completing the inductive step. The 2-cell $0 \cdot \tilde{c}_r^2$ has boundary path $[0, x^{-1}txyx^{-1}y^{-1}t^{-1}] = [g, r]$, which meets, in order (and disregarding orientation), the 1-cells labeled

$$(0, x), (0, t), (1, x), (1, y), (1, x), (1, y), (0, t)$$

as in the discussion of Figure 4.5. Likewise, the 0-cell $0 \cdot \tilde{c}_s^2$ has boundary path $(0, y)$, meeting only the 1-cell labeled $(0, y)$. These 2-cells have free faces $(0, x)$ and $(0, y)$. Collapsing across these faces deletes the 2-cells $0 \cdot \tilde{c}_r^2$ and $0 \cdot \tilde{c}_s^2$, as well as the 1-cells $0 \cdot \tilde{c}_x^1$ and $0 \cdot \tilde{c}_y^1$. This now leaves the 1-cell $0 \cdot \tilde{c}_t^1$, with free face the vertex 0, so this two may be collapsed away. This now leaves the 0-cells $[1, n]$ and all connected 1- and 2-cells, ie, the subcomplex $\widetilde{K}_{n-1}$, as desired. □

Corollary 4.3.0.1. *Let $\mathcal{P} = \langle x, y, t : r, s\rangle$ where $r = x^{-1}txyx^{-1}y^{-1}t^{-1}$ and $s = y$. Then, $K = K(\mathcal{P})$ is (DR).*

Proof. Every finite subcomplex of the universal cover of K collapses to a 1-complex, by Lemma 4.3.0.1. This characterizes (DR), by [11, Thm 2.4](here: 2.6.0.1), hence K is (DR). □

4.4. A Subpresentation With Non-Free Kernel

Let $\mathcal{Q} = \langle x, y, t : r\rangle = \langle x, y, t : x^{-1}txyx^{-1}y^{-1}t^{-1}\rangle$ be the subpresentation of some one relator group $H = H(\mathcal{Q})$. Let $L = L(\mathcal{Q})$ be the associated subcomplex $L \subseteq K$. Ultimately, our goal is to show that $M = \ker\{H \to \mathbb{Z}\}$ induced by the inclusion $\mathcal{Q} \to \mathcal{P}$ is not a free group. To do so, we will produce a presentation for $\pi_1(p^{-1}(L)) \cong \ker\{\pi_1(L) \to \pi_1(K)\} \cong M$ as in Lemma 2.1.0.2. Note: $p^{-1}(L)$ is connected: $\mathcal{Q}$ has the same set of generators as $\mathcal{P}$, hence L has the same 1-skeleton as K, and so $p^{-1}(L)^1 = \widetilde{K}^1$.

L has the same 1-skeleton as K, however, it omits the 2-cell c^2_s, corresponding to the relator s. As such, $p^{-1}(L)$ is the subcomplex of $\widetilde{K}$ which omits every lift $g \cdot \tilde{c}^2_s$. Contracting the unique spanning tree (each lift $g \cdot \tilde{c}^1_t$) and reading off a presentation by the boundary labels of the 2-cells, in terms of the remaining edges is routine, and yields the presentation:

Lemma 4.4.0.1. *The kernel $M = \ker\{H \to \mathbb{Z}\}$ induced by the inclusion $\mathcal{Q} \to \mathcal{P}$ has presentation*

$$\langle x_i, y_i : x_i = x_{i+1}y_{i+1}x_{i+1}^{-1}y_{i+1}^{-1}\rangle_{i\in\mathbb{Z}}.$$

Here, the generators x_i and y_i correspond, respectively, to the edges $i \cdot \tilde{c}^1_x$ and $i \cdot \tilde{c}^1_y$ in $p^{-1}(L)$. In terms of the presentations $\mathcal{Q} \to \mathcal{P}$, the generators x_i and y_i are the words

$t^i \cdot x \cdot t^{-i}$ and $t^i \cdot y \cdot t^{-i}$. The kernel M is therefore the subgroup of H generated by $\{t^i x t^{-i}, t^i y t^{-i}\}_{i \in \mathbb{Z}}$.

We claim that M is not a free group. To prove this, we will show that M is not residually nilpotent. Recall a group G is **nilpotent** if the lower central series, inductively defined as $G_{(0)} = G$ and $G_{(n)} = [G_{(n-1)}, G]$, stablizes at the identity for some integer n. A group is **residually nilpotent** if every nontrivial element has nontrivial image in some nilpotent quotient. Equivalently, a group is residually nilpotent if

$$\bigcap_{i=0}^{\infty} G_{(i)} = \{1\}.$$

By [31, Prop. I.10.2], free groups are residually nilpotent.

Lemma 4.4.0.2. *Let M be the group presented by $\langle x_i, y_i : x_i = x_{i+1} y_{i+1} x_{i+1}^{-1} y_{i+1}^{-1} \rangle_{i \in \mathbb{Z}}$. Then,*

$$x_i \in \bigcap_{j=0}^{\infty} M_{(j)}$$

for all $i \in \mathbb{Z}$.

Proof. We proceed by induction on the kth term $M_{(j)}$ of the lower central series of M, showing at each step that $x_j \in M_{(k)}$ for all $j \in \mathbb{Z}$. The base case is immediate: $x_j \in M_{(0)} = M$ is trivial.

For the inductive step, suppose $x_j \in M_{(k)}$ for all $j \in \mathbb{Z}$ and $k \geq 0$. Then, it follows from the relations $x_i = x_{i+1} y_{i+1} x_{i+1}^{-1} y_{i+1}^{-1}$ for all $i \in \mathbb{Z}$ that $x_j = [x_{j+1}, y_{j+1}]$ is in $M_{(k+1)} = [M_{(k)}, M]$, since we have $x_j \in M_{(k)}$ by the inductive hypothesis. □

Theorem 4.4.0.1. *The presentation $\mathcal{P} = \langle x, y, t : r, s \rangle$, where $r = x^{-1} t x y x^{-1} y^{-1} t^{-1}$ and $s = y$ is not (CLA).*

Proof. By Lemma 4.4.0.1, the kernel M of the inclusion induced homomorphism $H \to \mathbb{Z}$ has presentation $\langle x_i, y_i : x_i = x_{i+1} y_{i+1} x_{i+1}^{-1} y_{i+1}^{-1} \rangle_{i \in \mathbb{Z}}$. By Lemma 4.4.0.2, $x_0 \in \bigcap_{j=0}^{\infty} N^{(j)}$.

Viewing the presentation $\langle x_i, y_i : x_i = x_{i+1}y_{i+1}x_{i+1}^{-1}y_{i+1}^{-1}\rangle_{i\in\mathbb{Z}}$ as a subgroup of the one-relator group $\langle x, y, t : x = txyx^{-1}y^{-1}t^{-1}\rangle$ by $x_i = t^ixt^{-i}, y_i = t^iyt^{-i}$, we conclude by the Freiheitssatz, $x_0 \neq 1 \in N$. By [31, Prop. 10.2], free groups are residually nilpotent, and so N is not free. Finally, by the Free Kernel Theorem 3.2.0.1, $\mathcal{P}$ is not (CLA). □

Thus, combining Theorem 4.4.0.1 and Corollary 4.3.0.1, we have:

Theorem 4.4.0.2. *The presentation $\langle x, y, t : x = txyx^{-1}y^{-1}t^{-1}, y\rangle$ is (DR) but not (CLA).*

Additionally, since (DR) implies (DA) [15], we recover, as a corollary, a result of Biskup [4]:

Corollary 4.4.0.1. *There exists a presentation which is (DA) but not (CLA).*

4.5. A Counterexample for Every Non-Trivial Group with (DR) Presentation

The counterexample $\mathcal{P}$ can be extended, in fact, to any non-trivial group which admits a (DR) presentation. Fundamentally, the distinction between (DR) and (CLA) is not a distinction between *group* properties, but *presentation* properties.

Lemma 4.5.0.1. *Let $\mathcal{P} = \langle X : R\rangle$ be a presentation of the group $G = G(\mathcal{P})$, with some fixed $t \in X$. Then, $\mathcal{P}' = \langle X \cup \{x, y\} : R \cup \{x = txyx^{-1}y^{-1}t^{-1}, y\}\rangle$ presents a group G', isomorphic to G.*

Proof. The generators x, y are trivial in G'. The identity on the remaining generators therefore gives an isomorphism between G and G'. □

Lemma 4.5.0.2. *Let $\mathcal{P} = \langle X : R\rangle$ be a (DR) presentation of a non-trivial group G. Then, there is a generator $t \in X$ such that $\mathcal{P}' = \langle X \cup \{x, y\} : R \cup \{x = txyx^{-1}y^{-1}t^{-1}, y\}\rangle$ is (DR).*

Proof. Since G is non-trivial, some generator $t \in X$ must be non-trivial. Furthermore, since (DR) presentations are aspherical, they are torsion-free, thus t has infinite order and the subgroup $\langle t \rangle$ is isomorphic to the integers.

Forming $K = K(\mathcal{P}')$ and passing to the universal cover $\widetilde{K}$, every $\langle t \rangle$ coset gives a connected subcomplex of $\widetilde{K}$ homeomorphic to the space in Figure 4.5. That is, at every vertex $g \in \widetilde{K}$, we have a pair of loops for the trivial generators x, y as well as an edge t connecting g to its t-translate, as well as the associated lifted 2-cells. Because t has infinite order, we do indeed have the same "line" of t-edges as in Figure 4.5.

Now, fixing some finite subcomplex of $\widetilde{K}$, the edges of the lifted cells for x and y collapse as in Lemma 4.3.0.1, leaving only 1- and 2-cells drawn from X and R, respectively. This is exactly a finite subcomplex of the universal cover of $L = L(\mathcal{P})$. By Theorem 2.6.0.1, this collapses to a 1-complex as $\mathcal{P}$ is (DR). Thus: any finite subcomplex of $\widetilde{K}$ collapses to a 1-complex, hence it is (DR). □

Theorem 4.5.0.1. *Every non-trivial group G with (DR) presentation admits a (DR) presentation which is not (CLA).*

Proof. By Lemmas 4.5.0.1 and4.5.0.2, we may build a presentation $\mathcal{P}'$ of G which is (DR) and has a subpresentation $\langle x, y, t : x = txyx^{-1}y^{-1}t^{-1}, y \rangle$.

The presentation $\langle x, y, t : x = txyx^{-1}y^{-1}t^{-1}, y \rangle$ is not (CLA) by Theorem 4.4.0.1 and so by [8, Prop 1.4], then, $\mathcal{P}'$ is not (CLA), as it has a subpresentation which is not (CLA). □

5. CONCLUSIONS

An additional combinatorial geometric form of asphericity was introduced by G. Huck and S. Rosebrock [26]. A spherical diagram $f : S^2 \to K$ which admits a pair of faces c, d (closed 2-cells) which share a vertex v, and an orientation reversing homeomorphism $\phi : S^2 \to S^2$ fixing v with $f \circ \phi|_c = f|_d$ is said to be **vertex reducible**. A map which is not vertex reducible is **vertex reduced**. A 2-complex K is **vertex aspherical** (VA), if there is no nonempty spherical diagram $f : S^2 \to K$ which is vertex reduced. (VA) gives a slight generalization of (DR), where a pair of faces are folded together, not necessarily across a shared edge, but at least across shared vertex. Trivially, we have the implication $(DR) \Rightarrow (VA)$ and [26, Lem. 2.1] gives $(VA) \Rightarrow (DA)$.

It is a very simple matter to place (VA) in among the other flavors of asphericity. First, it is shown in [26, Ex. 2.3] that the dunce cap is (VA), which we already know is not (DR). Thus, the implication $(DR) \Rightarrow (VA)$ cannot be reversed. The presentation $\langle x, y, t : x = txyx^{-1}y^{-1}t^{-1}, y\rangle$ is a (DR), hence (VA) presentation which is not (CLA), as shown in Theorem 4.4.0.2.

Finally, (CLA) and hence (DA) do not imply (VA) as the following example demonstrates. Figure 5.1 shows a spherical diagram for the 2-complex K modeled on the presentation $\langle x : x^2 \cdot x \cdot x^{-2}\rangle$. Just as for the dunce cap, $\{x\} = \{x^2 \cdot x \cdot x^{-2}\}$ gives a Cohen-Lyndon basis for $F(x)$, so this presentation is (CLA). One checks, however, that the spherical diagram in Figure 5.1 is vertex reduced, so this presentation is not (VA).

Figure 5.1 gives a sphere by identifying the unbounded region of the plane to a point. It is then mapped to K according to the labeled identifications. The pentagonal faces of this sphere are mapped to the 2-cell corresponding to the relator $x^2 \cdot x \cdot x^{-2}$, read either counter-clockwise from the global basepoint (for the left pentagon) or clockwise (for the right). Notice that, in order, no vertex lies in the same component of the plane, ie,

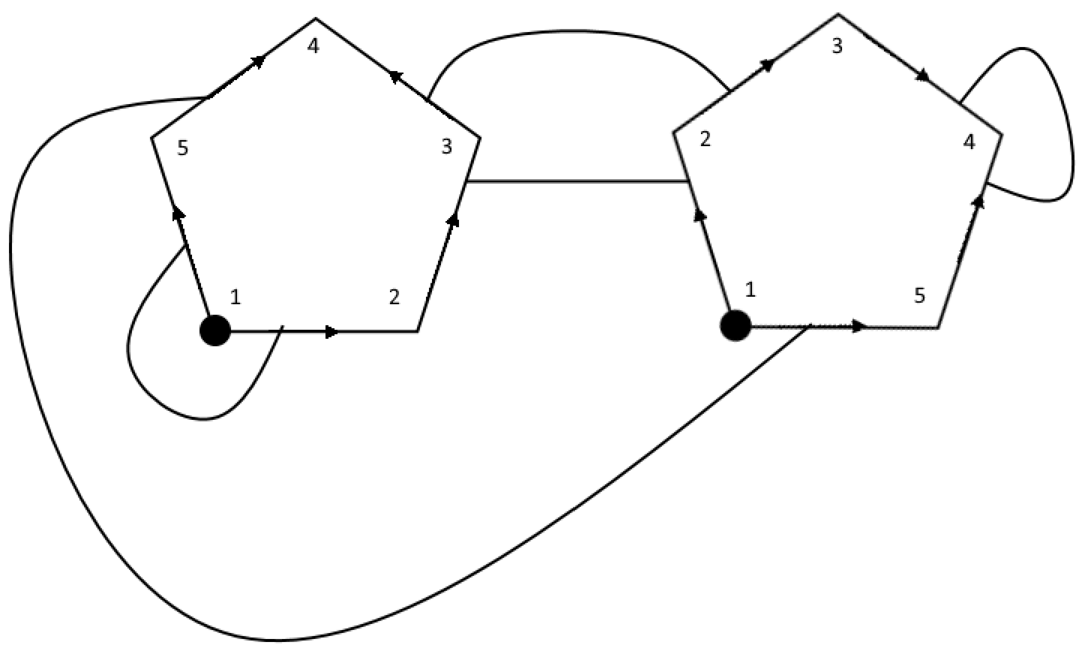

FIGURE 5.1: A Spherical Map to K

there is no shared vertex across which these two pantagons are reflected. Thus, much as before, including (VA) in the flavors of asphericity introduces no implications save for the essentially immediate implications.

The presentation $\mathcal{P} = \langle x, y, t : x = txyx^{-1}y^{-1}t^{-1}, y \rangle$ of Theorem 4.4.0.2 is itself a subpresentation of a balanced presentation of the trivial group, $\langle x, y, t : x = txyx^{-1}y^{-1}t^{-1}, y, t \rangle$. Moreover, the topological model associated with this presentation 3-deforms to a point. Thus, by [23, Thm 4.2], the model of $\mathcal{P}$ has the homotopy type of a ribbon disc complement 2-spine, which is the complement in the 4-ball of a properly embedded 2-disc, see [24] for more on ribbon disc complements. Ribbon disc complements and LOT presentations (presentations modeled on labeled oriented trees, which combinatorially parametrize the homotopy types of ribbon disc complements) constitute a significant class of test cases for the Whitehead conjecture. There are various recent results in [18], [17], and [26] showing the asphericity of LOTs using (VA). A fairly straightforward question is:

Question 5.0.0.1. *Are LOT presentations (CLA)?*

The presentation $\mathcal{P}$ of Theorem 4.4.0.2 presents an interesting potential counterexample. It is unclear, however, if there is an effective method to 3-deform the model of $\mathcal{P}$ while still preserving the subpresentation essential to the counterexample.

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