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ROBERT ROSS FOSSUM for the DOCTOR OF PHILOSOPHY
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Donald Guthrie, Jr.

The purpose of this thesis is to construct several stochastic process models for combined statistical dynamic prediction of the 500-millibar pressure surface for the northern hemisphere. To achieve this, a random forcing function is added to the spectral form of the nondivergent vorticity equation. Three models, one linear and two nonlinear are developed based upon the theory of stochastic ordinary differential equations. Mean value and covariance solutions to these equations are then found. Each model is converted to a difference equation and statistical estimation techniques suggested.

The techniques are sufficiently general to be applied to simple multi-level models. Suggestions for extended range models and other areas of generalization are given.

Models for Statistical Dynamic Prediction of the 500-Millibar Surface

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Robert Ross Fossum

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[redacted]

Chairman of Department of Statistics

[redacted]

Dean of Graduate School

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I. INTRODUCTION

Prediction of meteorological fields is accomplished by techniques belonging to two broad areas--synoptic forecasting and objective forecasting. Synoptic forecasting is an intricate process of analysis and predictions carried out by the forecaster himself with a great deal of subjective decision making. Objective forecasting is carried out by means of digital computers such that the major decisions are made by the computer on the basis of the objective criteria used. There are two basic methods of objective forecasting--statistical and dynamic or numerical. In this chapter, a background in the major objective forecast techniques is presented.

The advent of the digital computer in the late 1940's motivated a renewed interest in research on objective forecasting. During the 1950's extensive work was undertaken to achieve both dynamic (numerical) forecasts and statistical forecasts. Occasionally, attempts at combined statistical and dynamic forecasts were made, but, by and large, these were not highly successful. Nor were the purely statistical forecast methods very successful. On the other hand, dynamic forecasting has improved continually until today the routine forecasts are almost all based upon dynamic prognostications using a complicated six-level model (Shuman and Hovermale, 1968).

A basic problem in medium-range forecasting of the weather is the prediction of the pressure field, both at the surface and aloft. This is true because the pressure field determines the circulation, and the circulation, in turn, determines the movement of the air masses and their boundaries, the fronts. The movement of air masses under the influence of the pressure gradients is instrumental in determining the changes in temperature and humidity; the relative vertical motion of the different masses to some extent determines the precipitation. In addition, the space variations of the pressure field give rise to accelerations which cause divergence and consequent vertical motion. If the air mass is moist and the vertical motion is favorable, considerable precipitation may result. Thus, a fundamental problem is the prediction of the pressure field.

The objective of this thesis is to introduce new dynamic statistical models for pressure surface prediction based on random processes and random differential equations. In this manner, it is hoped that a start toward effective combined dynamic statistical forecasting can be made.

Specifically, three new random differential equation models for forecasting the 500 mb surface are derived. These models depart from prior models for prediction in several ways:

(1) The models differ from the usual deterministic models by

treating the pressure surfaces as random fields.

(2) The models differ from prior statistical forecast methods by the use of the hydrodynamic equations to remove much of the non linear prediction problems, reserving the application of linear statistical methods for the forecasting of a "residual field."

(3) The models differ from prior dynamic-statistical forecast methods and models by introducing the randomness through the use of a random forcing function in the hydrodynamic equations. Prior methods introduce randomness only through the initial conditions and assume, like the deterministic methods, that the model equations completely describe the atmospheric processes.

(4) The new models are more general than prior models. In the absence of the forcing functions they reduce to the deterministic or random initial conditions models.

(5) In addition to determining the evolution of the height field (mean height field) a variance field is predicted which gives the forecaster some measure of uncertainties in the predicted height.

(6) The form of the equations modeling the evolution of the mean field and variance fields are quite general. Because of this, application of the techniques to more general

meteorological problems, e.g., prediction with baroclinic models, should be possible.

In the next chapter, background material in stochastic processes, hydrodynamics and objective forecasting is given. The following chapter contains a simple linear dynamic model which is used to diagnose some of the problems encountered in purely dynamic or purely statistical forecasting. This simple model is further used to suggest the proper type combined dynamic-statistical barotropic model. Following these hueristics and diagnostics, three barotropic combined models are presented based upon the spectral form of the vorticity equation. Next, the discrete form of the models is presented and statistical techniques discussed. The final chapter presents conclusions and research plans for future work.

II. BACKGROUND MATERIAL IN OBJECTIVE FORECAST METHODS

The objective forecast methods in use today are based upon linear statistical models and on the hydrodynamic equations for atmospheric motion. This study combines the two basic methods through the use of stochastic differential equations, based on the hydrodynamics, with a random forcing function estimated by linear statistical methods. This chapter presents the background material necessary to understand the prior forecast methods and the tools needed for constructing the combined statistical-dynamic models.

Random Vectors, Vector Processes and Fields

The important concepts of random processes which are of interest in this study may be found in the book of Yaglom (1962), or in the more advanced books of Loève (1963), Cramér and Leadbetter (1967), and Rosanov (1963). The reader is assumed to be familiar with the definitions and properties of sample function, spectral representation theorems, concepts of almost sure continuity and differentiability, mean square continuity and differentiability, etc. The main emphasis in this section is a review of a particular orthogonal expansion which will be useful in development of a method of solution of a system of ordinary differential equations and a review of expansions used in statistical objective forecast techniques.

Statisticians have long made use of the concept of principal components for multivariate random variables whose variances exist. In particular, if $\vec{X}$ is a random vector with mean $\vec{\mu}$ and covariance matrix Σ, then there exists an orthogonal matrix A, such that

$$\Sigma = ADA'$$

where D is a diagonal matrix having the eigenvalues λ_i of Σ on its main diagonal in descending order; i. e.,

$$\lambda_1 \geq \lambda_2 \cdots$$

The new vector $\vec{Y}$ defined by

$$\vec{Y} = A'(\vec{X} - \vec{\mu})$$

has components taking values on the "principal axes" of the ellipsoids of concentration of the original distribution. Further,

$$\text{Var}\,(Y_i) = \lambda_1$$

$$\text{Cov}\,(Y_i Y_j) = \vec{\alpha}_i{}' \Sigma \vec{\alpha}_j \qquad i = j$$

$$= 0 \qquad i \neq j$$

where $\vec{\alpha}_i$ is the eigen vector corresponding to λ_i with length 1. The eigen vectors $\vec{\alpha}_i$ make up the columns of A. Principle components analysis is used extensively in statistical forecasting.

The random processes considered in this study will be assumed to have finite second moments, i. e., second order random functions. A very brief review of some topics of interest in the theory of second order random functions follows. Of particular interest will be topics related to the existence of solutions to random differential equations and certain types of orthogonal expansions useful in determining solutions to these equations.

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and consider the space $L_2 = L_2(\Omega, \mathcal{B}, \mu)$ of all complex valued random variables on Ω with finite second moments. Random variables which differ on a set of μ measure zero will be considered identical. For an index set T (possibly multidimensional), a family of random variables $\{Z(t) = Z(t, \omega), t \in T\}$ is called a stochastic process (random process or random function). For such processes, it is convenient to consider the space of values $Z(t, \omega)$ as a Hilbert space, $H = L_2(\Omega, \mathcal{B}, \mu)$ with inner product defined as the second mixed central moment or covariance. The norm in H is defined by the inner product. Mean square convergence is then strong convergence in H. A random process will have strong continuity, strong differentiability, integrability, etc., according to the properties of the covariance functions. A concise summary of such properties and relations is given in Syski (1965).

Turning now to second order random vector processes, let

$\vec{Z}(t) = (Z_1(t), Z_2(t), \ldots)$ be a vector valued random variable, for each t in an interval $T = [a, b]$ and where $Z_i(t)$ is complex valued for each $i = 1, 2, 3 \ldots$. The expectation of $\vec{Z}(t)$ is the vector of expectation of its components. The covariance matrix of the vector process is defined as the matrix of elements

$$R_{ij}^{Z}(s, t) = E(Z_i(s)\overline{Z_j(t)}),$$

which are the cross covariances of lag (s-t) of the individual component processes.

Most of the work in this study is concerned with practical applications of the theory of stochastic processes to hydrodynamics. Consequently, the analytic properties of sample functions $\vec{Z}(t, \omega)$ are of great importance. A random function is said to be sample-continuous, sample-measurable, or sample integrable at a point $\omega \epsilon \Omega$ if the corresponding property is true for sample functions. Sample properties hold almost everywhere if they hold apart from a set of probability zero. In general, the analytic properties of sample functions are stronger than the corresponding properties relative to the space H. For example, almost everywhere sample continuity of $\vec{Z}(t, \omega)$ implies almost everywhere continuity of the stochastic process $\vec{Z}(t)$ but the converse is not true.

Suppose that the second-order random vector process $\vec{U}(t)$ is defined on the closed interval $[a, b]$ and possess continuous covariance functions $R^U_{ij}(t,s)$. It can be shown that the random process $\vec{U}(t)$ is mean-square continuous, hence measurable, and is also sample integrable and sample square-integrable. Furthermore, the continuity of R^U implies that the indefinite mean-square integral

$$\vec{Z}(t) = \int_a^t \vec{U}(s)ds \qquad a \le t \le b$$

exists, and is a random variable (up to an equivalence). It can further be shown that its mean-square derivative is

$$\frac{d\vec{Z}(t)}{dt} = \vec{U}(t).$$

The following important result on the relations between random processes and their sample functions holds in the present case. If the random function is mean-square continuous, then its stochastic integral $\vec{Z}(t)$, and its sample integral defined as a random variable $\vec{Z}(t,\omega)$ with values (for almost all individual sample functions)

$$\vec{Z}(t,\omega) = \int_a^t \vec{U}(s,\omega)ds \qquad a \le t \le b$$

coincide.

For second order vector random process there are various orthogonal decompositions, that is, decompositions into sums of orthogonal random variables or decomposition as a stochastic integral with respect to a process with orthogonal increments. For univariate or one-dimensional processes, Loeve (1963, Section 34.5) gives three of particular interest: the proper orthogonal decomposition theorem, and the harmonic orthogonal decomposition theorem, and the harmonic orthogonal decomposition theorem for stationary processes. As a corollary, the harmonic orthogonal decomposition theorem for stationary vector processes is also given. Rozanov (1963) gives the same results which are due in the univariate case to Cramér (1942).

The extension of the proper orthogonal decomposition theorem will be useful in this study as an aid in finding the mean and covariance solutions of certain ordinary stochastic differential equations. Kelly and Root (1960) have shown that if $\vec{Z}(t)$ is defined over a finite interval $T = [a, b]$, $E(\vec{Z})(t)) \equiv 0$ and

(1) $R_{ij}^{Z}(s, t)$ exists for all i, j and are continuous on $[a, b] \times [a, b]$ and is uniformly bounded

(2) $$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_a^b \int_a^b |R_{ij}^{Z}(s, t)|^2 ds dt < \infty$$

(3) There exists constants c_j, $j = 1, 2, \ldots,$ such that

$$\int_a^b |R_{ij}^{Z}(s, t)|^2 \, dt < c_j^2 \quad \text{for all} \quad i = 1, 2, \ldots, \quad a \leq s \leq b,$$

and such that $\sum_{j=1}^{\infty} c_j^2 = c,$

then

$$Z_h(t) = \sum_{\nu=1}^{\infty} V_\nu \varphi_{\nu h}(t) \qquad \text{(m. s.)}$$

where

$$V_\nu = \sum_{i=1}^{\infty} \int_a^b Z_i(t)\varphi_{i\nu}(t)dt \qquad \text{(m. s.)}$$

and

$\varphi_{\nu k}$, v = 1, 2, . . ., are the eigenfunctions,,

$$\sum_{j=1}^{\infty} \int_a^b R_{ij}(t, s)\varphi_{jk}(s)ds = \lambda_k \varphi_{ik}(t) \qquad a \leq t \leq b.$$

Further, V_ν are orthogonal, i. e. ,

$$E(V_\nu \overline{V}_\mu) = \delta_{\nu\mu}\lambda_\nu, \quad \lambda_\nu \geq 0.$$

where $\delta_{\nu\mu}$ is the Kronecker delta function.

The mean square covergence in the basic expansion is uniform in t. If the vector process is finite dimensional then if (1) holds then (2) and (3) are automatically satisfied. This will be the case in this study.

To this point, the processes considered have been vector valued with the parameter time. Now consider a univariate or scalar process over time. Corresponding to stationary univariate processes is the spectral representation of Cramer. This represents the process as an integral or sum of orthogonal random variables. A corresponding representation exists for scalar random processes where now the parameter is no longer time, but is a parameter P varying in space. Throughout this study, P wil be a point on the sphere, i.e., $P = (\lambda, \theta)$ where λ is longitude and θ is latitude. Of particular interest is the representation for isotropic random fields varying over the unit sphere.

Following Jones (1963a), let $\xi(P)$ be a collection of real valued random variables indexed by P, the points on the surface of a unit sphere S in R_3. Let $\xi(P, \omega) = \xi(\theta, \lambda, \omega)$ by the sample functions of $\xi(P)$ and suppose that

$$\int_S \xi^2(\lambda, \theta, \omega) \cos \theta d\theta d\lambda < \infty$$

for all $\omega \in \Omega$. The random process $\xi(P)$ is called a second order scalar random field over the unit sphere.

The set of functions which are complete and orthogonal over the unit sphere are the spherical harmonics Y_n^m of degree n and order m. Obukbov (1947), has shown that

$$\xi(P) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Z_{nm} Y_n^m(P)$$

where the characteristics of the random variables Z_{nm} depend upon the spatial variation characteristics of the field, $Y_n^m = e^{im\lambda} P_n^m (\cos \theta')$, θ' being colatitude, and P_n^m is the associated Legendre function of order m and degree n.

It is often desirable to assume the meteorological scalar fields are isotropic, that is, the covariance between two points P and Q depends only on the spherical distance between the two points and $E(\xi(P))$ is constant, where E is the expectation operator. Without loss of generality, it will be assumed that $E(\xi(P)) = 0$. This implies $E(Z_{mn}) = 0$ for all m, n. As mentioned above, the concept of isotropic scalar random fields is analogous to the concept of stationary second order random processes. A somewhat less restrictive assumption on the spatial covariance function is made in the next section.

For normalized spherical harmonics, i. e.,

$$\int_S [Y_n^m(\theta,\lambda)]^2 \cos \theta d\theta d\lambda = 1$$

the necessary and sufficient conditions for isotropy are

$$E(Z_{nm} Z_{ij}) = \delta_{ni} \delta_{mj} f_n \geq 0$$

δ being the Kronecker delta and f_n a spatial spectrum.

By use of the following relations, the complex representation above may be converted to a real representation (Kaula, 1967):

$$\xi(P) = \mathrm{Re} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Z_{nm} Y_n^m(P)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} (Z'_{nm} C_{nm} + Z''_{nm} S_{nm})$$

where

$$Y_n^m = \frac{(-1)^m}{(2-\delta_{0m})^{1/2}} (C_{nm} + iS_{nm}) \qquad m \geq 0$$

$$Y_n^m = \frac{1}{(2-\delta_{0m})^{1/2}} (C_{nm} - iS_{n|m|}) \qquad m \leq 0$$

$$Z_{nm} = \frac{(-1)^m}{(2-\delta_{0m})^{1/2}} (Z'_{nm} - iZ''_{nm}) \qquad m \geq 0$$

$$Z_{nm} = \frac{1}{(2-\delta_{0m})^{1/2}} (Z'_{n|m|} + iZ''_{n|m|}) \qquad m \leq 0$$

Thus

$$C_{nm} = \cos m\lambda P_n^m(\cos\theta'), \quad S_{nm} = \sin m\lambda P_n^m(\cos\theta').$$

The real representation is therefore

$$\xi(P) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \{Z'_{nm} \cos m\lambda + Z''_{nm} \sin m\lambda\} P_n^m (\cos \theta') \tag{2.1}$$

Let P and Q be two points on the unit sphere and let Δ be the central angle between P and Q. Then the covariance function $\gamma(P, Q)$ for $\xi(P)$ is

$$E[\xi(P)\xi(Q)] = \gamma(P, Q)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=-n}^{n} \sum_{j=-m}^{m} E(Z_{mi} Z_{nj}) Y_n^i(P) Y_m^j(Q).$$

Because of isotrophy, this reduces to

$$r(\Delta) = \sum_{n=0}^{\infty} \left(\frac{2n+1}{4\pi}\right) f_n P_n (\cos \Delta) \tag{2.2}$$

where P_n is the Legendre polynomial of degree n and f_n was defined before.

The above spectral representation, Equations (2. 1) and (2. 2), for the isotropic process and its covariance function defined over the sphere corresponds to the Cramér and Wiener-Khintchine representations for stochastic processes with a time parameter.

Fundamental to the study of various meteorological fields are two types of transformations. The first type is derived from the deterministic hydrodynamic equations of motion of the atmosphere, usually nonlinear. These are studied in future sections. The second type, reviewed in this section, are linear operators which commute with those operations describing the homogeniety of the process. This type of linear operator is called a linear space invariant filter. Most common filters of interest are differential and integral operators. According to Hannan (1965), the only differential operator which is a linear filter of this type for fields on the sphere is the Laplacian operator

$$\nabla^2 = \frac{\partial}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{(\sin\theta)^2}\frac{\partial^2}{\partial\lambda}$$

or polynomials in this operator.

As in the case of processes over time, the effect of such a linear filter is completely described by its response function. For the differential operator above the response function is

$$h(m, n) = n(n+1).$$

Time varying linear filters often occur in time parameter processes. The corresponding concept in random scalar fields is that of a space varying linear filter. A basic field measured in weather observations is the constant pressure height field. A typical example of such a field for 500 millibars is shown in Figure 1. Such fields are very important in forecasting for reasons outlined in the introduction. The derivation of the dynamics of such a field depends on its relation to the vorticity field which is directly a function of the equations of motion and the velocity field. In particular, it can be shown that, if $\vec{V}$ is the geostrophic wind field (a reasonable approximation to the actual velocity field), then the vertical component of relative geostrophic vorticity ξ is

$$\xi = \frac{g}{f} \nabla^2 h$$

where h is the height, f is the Coriolis parameter and g is the gravitational constant. This widely used relation in meteorology is an example of a linear filter which varies over space, since the Coriolis parameter varies with latitude.

A generalization of the random scalar field is the concept of random vector field. Random vector fields are studied extensively in the theory of turbulence. The only example of interest in this study is the velocity or wind field. In this case, at each point in space, the random quantity is a vector, rather than a scalar. Because of the Helmholtz theorem of fluid dynamics, study of this vector field may be reduced to the sum of two vector fields, a solenoidal (nondivergent) field and an irrotational field. In meteorological applications it is often assumed that the irrotational part of the wind field is zero. The

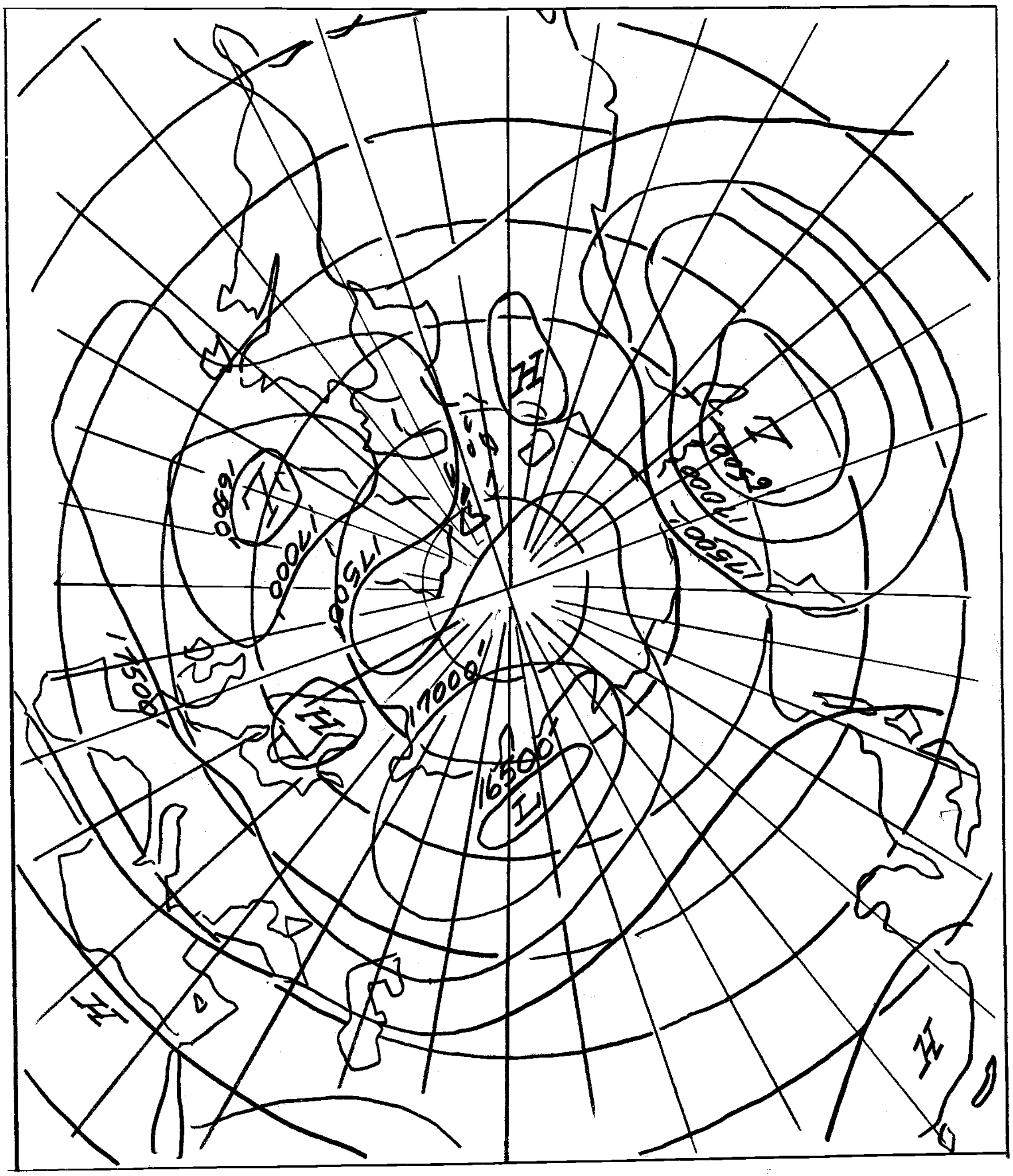

Figure 1. Typical 500 mb height field (heights in feet).

remaining nondivergent part of the field can then be expressed in terms of a scalar stream function, thus reducing the study to scalar fields. A justification of this for large scale horizontal motions is found in Eliassen and Kleinschmidt (1957, p. 21). Thus random vector fields are of only occasional interest in this study.

Statistical Techniques in Forecasting

The use of statistics in weather forecasting has had a long and not too successful history. The techniques may be classified to include simple graphical techniques, correlation studies and spectrum searches for periodicity, the application of multivariate analysis, in particular multiple discriminant analysis, and the use of orthogonal functions of various types. Serious attempts at statistical objective short-term forecasting were initiated by Byrant and Wadsworth (1948) in the late nineteen forties. This work made use of the then developing theory of prediction. Following this, Lorenz continued statistical forecasting research, concentrating on the use of principal component techniques (or "empirical orthogonal functions"). According to Lorenz (1959) this research had somewhat limited success, probably due to the linearity of the techniques used. Lorenz did attempt some diagnostic research into the reasons for limited success and the advantages of nonlinear techniques. Despite the limited success, these techniques are still used by some agencies for extended range forecasting (Thomas, 1963) apparently with somewhat greater success. The idea of using principal components has also been discussed by Obukhov (1960).

To achieve a forecast using empirical orthogonal function, the meteorological field (usually pressure or temperature) is represented in terms of the empirical functions. This representation is found by the usual methods of principle components; i.e., determining the orthogonal transformation matrix by Gram Schmidt orthogonalization. As mentioned before, the columns of this matrix are the eigenvectors or empirical orthogonal functions. Once the representation of field in terms of these is determined, statistical regression techniques are used for prediction.

A second branch of statistical forecasting has been developed by Miller (1962). The techniques are primarily variants of discrimination (classification) analysis and ordinary regression. A very useful technique for screening groups of variables and selecting good predictors has been developed by Miller.

The most significant recent work in statistical forecasting is due to Jones (1963b). In this work can be found the beginning of a realization by statisticians that meteorological time series analysis should be more closely related to the underlying physical processes. The basic result which makes the problem tractable is a spectral representation of a time varying scalar field on a sphere in the following separation of variables form:

$$\xi(P,t) = \sum_{\nu} Z_{\nu}(t)\varphi_{\nu}(P)$$

where ξ is the field to be represented, $\{Z_{\nu}(t)\}$ are the random processes, and $\{\varphi_{\nu}(P)\}$ is a complete orthonormal system. The set of functions $\{\varphi_{\nu}\}$ orthogonal over a sphere are surface spherical harmonics as developed above. In the case discussed before, isotropic fields were assumed. If axial symmetry of the field is assumed, a natural physical assumption in this problem, the representation becomes

$$\xi(P,t) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} (Z'_{\nu m}(t) \cos m\lambda + Z''_{\nu m}(t) \sin m\lambda) P_{\nu}^{m} (\cos \theta')$$

where as before $P = (\theta, \lambda)$, λ is the longitude, θ' is the colatitude, and P_{ν}^{m} are associated Legendre functions. For fixed t, the $\{Z_{\nu m}\}$ processes have the relationships

$$E(Z'_{\nu m} Z''_{\mu n}) = E(Z''_{\nu m} Z'_{\mu n}) = \delta_{mn} f_{\nu\mu m}$$

$$E(Z''_{\nu m} Z'_{\mu n}) = E(Z'_{\nu m} Z''_{\mu n}) = -\delta_{mn} f'_{\nu\mu n}$$

The quantities $f_{\nu\mu m}$ and $f'_{\nu\mu m}$ form a spatial spectrum.

In order to predict the field $\xi(P,t)$ at some time in the future, a time series analysis is done on the $\{Z(t)\}$ processes and some sort of prediction scheme applied.

Suppose $\xi(P, t, \omega)$ is a realization of the process. Then

$$\xi(P, t, \omega) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} (Z'_{\nu m}(t, \omega) \cos m\lambda + Z''_{\nu m}(t, \omega) \sin m\lambda) P_{\nu}^{m}(\cos \theta')$$

where

$$Z'_{\nu m}(t, \omega) = \int_S \xi(\lambda, \theta', \omega) \cos m\lambda P_{\nu}^{m} (\cos \theta') \sin \theta' d\theta' d\lambda$$

$$Z''_{\nu m}(t, \omega) = \int_S \xi(\lambda, \theta', \omega) \sin m\lambda P_{\nu}^{m} (\cos \theta') \sin \theta' d\theta' d\lambda.$$

Thus using a sequence of maps, $\{\xi(P, t, \omega)\}$, the sequences $\{Z'_{\nu m}(t, \omega)\}$ and $\{Z''_{\nu m}(t, \omega)\}$ can be found. Then, using some prediction scheme, these time series can be estimated for future values and the predicted field constructed.

Jones achieved good predictions for periods as long as five days using these techniques. As will be demonstrated, this is probably due to use of the spherical harmonic representation which is closely related to a simple, but meaningful, dynamic forecase model.

Dynamics of Random Fields

Dynamic forecasting depends on the basic hydrodynamic and thermodynamic equations describing atmospheric processes. These equations are assumed to exactly model the processes without error or residual. The simplest such model is the non-divergent barotropic model. This will be developed in this section both for illustrative purposes and because it will form the basis of the dynamic-statistical models of Chapter IV.

There are two types of derivatives used in hydrodynamics. The usual partial derivative $\partial/\partial t$ denotes the rate of change at a fixed point with respect to some frame of reference. The Eulerian or individual time derivative D/Dt, on the other hand, denotes the rate of change at some point embedded in and moving with the fluid. Thus the Eulerian operator is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V}' \cdot \text{grad}$$

where $\vec{V}'$ is the fluid velocity, grad is the gradient operator and $\cdot$ is the inner product operation. This may be applied to vector or scalar fields (Eliassen and Kleinschmidt, 1957).

Using the individual time derivative, the equations of motion of a particle of air (unit mass) with respect to a reference system fixed in the earth and rotating with it are

$$\frac{D\vec{V}'}{Dt} = \vec{g} - 2\vec{\Omega} \times \vec{V}' - \frac{1}{\rho}\,\text{grad}\, p + \frac{1}{\rho}\,\text{div}\, \mathcal{F} .$$

This equation says that the velocity of the particle is changed by the force of gravity $\vec{g}$, the earths rotation effects $2\vec{\Omega} \times \vec{V}'$ where $\vec{\Omega}$ is earths constant angular velocity, the pressure gradient effect $\frac{1}{\rho}$ grad p, where p is pressure and ρ is density and by friction effect, where $\mathcal{F}$ is frictional stress. This equation is derived from Newton's second law.

In studying large scale motions in the atmosphere, i.e., those with horizontal dimensions of the order of 300 miles or more, it is customary to make several approximations. These approximations and their justification are discussed by Eliassen and Kleinschmidt (1957, p. 21-22). First, the vertical acceleration is ignored in the equations of motion. Second, the horizontal component of $\vec{\Omega}$ is ignored in the expression for the Coriolis force. In addition to these major approximations, two small mathematical approximations are made. The resulting equations, called the quasi static equations, are derived using the following individual time derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \text{grad}_h + w \frac{\partial}{\partial z}$$

where $\vec{V}$ is the horizontal velocity, the subscript h refers to horizontal components, z is the height or vertical direction, and w is vertical velocity. Without the friction term the equations of motion are

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\,\mathrm{grad}_h\,p - f\,\vec{k} \times \vec{V}$$

for horizontal motion. $\vec{K}$ is the vertical unit vector.

Because of certain simplifications, meteorologists have traditionally converted the vertical coordinate from simple height to pressure, which of course varies monotonically with height.

In the pressure coordinate system the vertical "velocity" component is formally

$$\omega = \frac{Dp}{Dt}.$$

ω is positive downward. The horizontal velocity components are

$$u = r \cos\theta \frac{D\lambda}{Dt}$$

$$v = r \frac{D\theta}{Dt}$$

positive toward the east and pole, respectively. The constant r is the earth's radius. In terms of these velocity components the relation between the two kinds of time derivative for a spherical earth is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{1}{r\cos\theta}\frac{\partial}{\partial\lambda} + v\frac{1}{r}\frac{\partial}{\partial\theta} + \omega\frac{\partial}{\partial p}$$

The terms involving space derivatives and velocity components are referred to as the "advection terms". They are also known as convective terms.

Using the vector notation once more, the Eulerian operator becomes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \text{grad}_p + \omega \frac{\partial}{\partial p}$$

The suffix p in the vector operators (and usually omitted in partial differentiations) indicates that p is to be held constant in the appropriate differentiations. If h is the height of an isobaric surface, then let

$$B = gh$$

where g is the gravitational constant. B is called the geopotential.

The equations of motion are

$$\frac{D\vec{V}'}{Dt} = -\vec{g} - 2\vec{\Omega} \times \vec{V}' - \frac{1}{\rho}\,\text{grad}_p\, p$$

where $\vec{g}$ is the effect of gravity; $2\vec{\Omega} \times \vec{V}'$ is the earth's rotations effect, $\vec{\Omega}$ being the constant earth's angular velocity; ρ is density and p is pressure so that $\frac{1}{\rho}\,\text{grad}\, p$ is the pressure gradient force.

The horizontal components of this motion are described by

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\,\text{grad}_p\, B - f\,\vec{k} \times \vec{V} \qquad (2.3)$$

where $\vec{k}$ is the vertical unit vector. The vertical component is

$$\frac{\partial B}{\partial p} = -\frac{1}{\rho}$$

Following a similar derivation, in pressure coordinates the equation of continuity is

$$\text{div}_p \vec{V} = -\frac{\partial \omega}{\partial p}$$

The fundamental physical quantity used in dynamic forecasting is the vertical component of vorticity

$$\xi_p = \text{curl}_p \vec{V} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad (2.4)$$

The equation describing vertical vorticity is found, as indicated in the equation above, by applying the curl operator to the horizontal equations of motion. When this is done, the following vorticity equation is obtained

$$\frac{D}{Dt}(\xi_p+f) = -(\xi_p+f)\,\text{div}_p \vec{V} + \vec{k} \times \frac{\partial \vec{V}}{\partial p} \cdot \text{grad}_p \omega \qquad (2.5)$$

The above equation states that total or absolute vertical vorticity of a particle changes due to the factors on the right hand side, which include divergence (the first term) and a "vortex tube" term (the second term). Both of these terms on occasion contribute significantly to the change in vorticity.

In addition to the vorticity equation, a second fundamental equation is the divergence equation, which is obtained by applying the horizontal divergence operator to the equations of motion. This yields

$$\frac{D}{Dt}\,\mathrm{div}_p\,\vec{V} + \left[\left(\frac{\partial u}{\partial x}\right)^2 + 2\,\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial y}\right)^2\right] + \mathrm{grad}_p\,\omega \cdot \frac{\partial \vec{V}}{\partial P} + \frac{\partial f}{\partial y}u$$

$$= f(\xi_p - \frac{1}{f}\nabla_p^2 B) \qquad (2.6)$$

where ∇_p^2 is the Laplacian operator in pressure coordinates.

From this equation, the balance equation and the geostrophic approximation equation may be obtained.

From the vorticity and divergence equations, model forecast equations are derived using various simplifying assumptions which allow integration of the equations by numerical techniques. The simplest dynamic forecasting model, called the "barotropic nondivergent model" simply treats large scale motion as horizontal (or isobaric, which is about the same) and nondivergent. With these assumptions the vorticity equation becomes, neglecting vortex tube terms

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \mathrm{grad}_p\right)(\xi_p + f) = 0 \qquad (2.7)$$

and the continuity equation becomes

$$\mathrm{div}_p \vec{V} = 0.$$

Applied to the 500 mb level, where nondivergence is a reasonable physical assumption, this forms the basis of simple numerical forecasting techniques.

It is important to look at the left side of Equation (2. 7). The equation has the time derivative at a point and the advection terms, mentioned before. Thus, the equation states that vorticity at a point can change only through advection, a space-velocity process.

Practically, use of the vorticity equation requires a knowledge of the horizontal velocity field, $\vec{V}$. There are many possible wind laws (Ellsasser, 1968). Most of these laws can be derived by means of an approximation to the divergence equation known as the balance equation. In the divergence equation (2. 6) set the Eulerian derivative term and the term involving ω equal to zero. Then the divergence equation simplifies to

$$f\xi_p - [(\frac{\partial u}{\partial x})^2 + 2\frac{\partial u}{\partial g}\frac{\partial v}{\partial x} + (\frac{\partial v}{\partial y})^2] - \beta u = \nabla_p^2 B$$

where $\beta = \frac{\partial f}{\partial y}$ and $B = gh$, as defined before.

Now decompose the actual velocity field $\vec{V}$ into a nondivergent part and an irrotational part (Helmholtz Theorem)

$$\vec{V} = \vec{V}_\psi + \vec{V}_\chi$$

where

$$\vec{V}_\psi = \vec{k} \times \mathbf{grad}_p \psi$$

$$\vec{V}_\chi = \mathbf{grad}_p \chi$$

and χ is the velocity potential function and ψ is the stream function. Assume now that $\vec{V}_\chi = 0$, that is, the velocity field is approximated by its nondivergent part. Then,

$$u = -\frac{\partial\psi}{\partial y} \tag{2.8}$$

$$v = \frac{\partial\psi}{\partial x} \tag{2.9}$$

and

$$\xi_p = \nabla_p^2 \psi. \tag{2.10}$$

In this case the balance equation becomes

$$f\nabla_p^2\psi + 2\left[\frac{\partial^2\psi}{\partial x^2}\,\frac{\partial\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right] + \beta\,\frac{\partial\psi}{\partial y} = \nabla_p^2 B. \tag{2.11}$$

This form of the balance equation relates the geopotential B (and, thus, height) to the velocity field determined by ψ. It is unfortunately difficult to handle even numerically (Ellsaesser, 1968).

Certain further simplifications result in a useful equation. If the nonlinear terms and the term involving β are neglected, the relative vorticity may be related simply to height

$$\xi_p = \frac{g}{f} \nabla_p^2 h \qquad (2.12)$$

(which was the example used in the discussion of the spatial filtering of random scalar fields). The velocity field corresponding to this approximation is the geostrophic velocity field.

$$\vec{V}_g = \vec{k} \times \frac{g}{f} \mathrm{grad}_p\, h .$$

The extent of the approximate nature of this relation can be seen by returning to the divergence equation. Because of this, the balance equation is usually used rather than the last approximation.

The nondivergent barotropic model can be used with any number of wind laws. However, the most common, are the geostrophic approximation law above and the wind field determined by the nonlinear balance equation. In the first case, the forecast equation will contain the geopotential directly, while in the latter, it will contain the stream functions and, as a second equation in a system of two forecast equations, the balance equation.

In a study of various wind laws for hemispheric forecasting using the barotropic model, Ellsasser (1968) found a linearized form of the balance equations to be "optimum." In general, the nonlinear balance equation uses an order of magnitude more of computer time and yields, according to Ellsasser, almost no improvement in mean height and standard error of height forecasts. The geostrophic approximation is only slightly inferior to the linearized balanced equation in these two measures. In subsequent sections of this study, the geostrophic approximation will therefore be used exclusively.

Using the geostrophic wind, the geostrophic vorticity is defined as

$$\xi_g = \mathrm{curl}_p \vec{V}_g .$$

A geostrophic vorticity equation at the nondivergent level can then be derived as

$$\frac{\partial \xi_g}{\partial t} + \vec{V}_g \cdot \mathrm{grad}_p (\xi_g + f) = 0. \qquad (2.14)$$

Here the variation of f with latitude has been neglected in the last term.

Using the relation between the vorticity field and the height field, the basic prognostic equation of the nondivergent barotropic model for the nondivergent level is

$$\frac{\partial}{\partial t} \nabla_p^2 h + \vec{V}_g \cdot \mathrm{grad}_p (\nabla_p^2 h) + \beta \frac{\partial h}{\partial x} = 0 \qquad (2.15)$$

The nondivergent barotropic model above applies to an atmosphere which is very idealized. It therefore is primarily useful conceptually. The basic form of the prognostic equations is retained if a more realistic equivalent barotropic atmosphere is considered.

In the equivalent barotropic model the wind vectors at all altitudes are assumed to have the same direction, but the magnitude of the wind vector may vary. In equation form, this assumption may be written as

$$\vec{V}_g(x, y, p, t) = A(p)\,\overline{\vec{V}_g(x, y, t)}$$

The "bar" is an averaging or mean value operation taken over the height variable p which is usually taken to vary from $p = 0$ (in space) up to $p_o = 1000$ mb (at the lowest level considered). The actual form of the function $A(p)$ is usually found from climatological data and normalized so that $\overline{A} = 1$.

Applying the mean value or averaging operator to the geostrophic vorticity equation gives

$$\overline{\frac{\partial \xi_g}{\partial t}} + \overline{\vec{V}_g \cdot \operatorname{grad}_p (\xi_g + f)} = \overline{-f \operatorname{div}_p \vec{V}}$$

Frequently used boundary conditions are,

$$\omega = 0 \quad \text{at} \quad p = 0 \quad \text{and at} \quad p = p_o$$

although the latter condition is not strictly correct. Application of the averaging operator to the equation of continuity gives

$$\overline{\mathrm{div}_p \vec{V}} = -\frac{\omega_o}{p_o}$$

where ω_o and p_o are surface values; from the above boundary conditions, $\overline{\mathrm{div}_p \vec{V}} = 0$; hence, the geostrophic vorticity equation becomes

$$\overline{\frac{\partial \xi_g}{\partial t}} + \overline{\vec{V}_g \cdot \mathrm{grad}_p (\xi_g + f)} = 0;$$

thus

$$\frac{\partial \overline{\xi}_g}{\partial t} \doteq -\overline{\vec{V}} \cdot \mathrm{grad}_p f - \overline{A^2}\, \overline{\vec{V}}_g \cdot \mathrm{grad}_p \overline{\xi}_g$$

Moreover,

$$\frac{\partial \xi_g}{\partial t} + \vec{V}_g \cdot \mathrm{grad}_p \left(\frac{\overline{A^2}}{A(p)} \xi_g + f\right) = 0$$

which is valid at any isobaric level. In particular, the above equation reduces to the vorticity equation for the geostrophic form of the barotropic nondivergent model at the level $p*$ defined by

$$A(p*) = \overline{A^2}$$

which is therefore called the nondivergent level of this model. Therefore, at $p*$, usually assumed to be 500 mb, the vorticity equation is

$$\frac{\partial \xi_g}{\partial t} + \vec{V} \cdot \mathrm{grad}_p (\xi_g + f) = 0$$

From this equation it is clear that the nondivergent barotropic model and the equivalent barotropic model are essentially equivalent. In the meteorological literature, the equivalent barotropic model is most often used.

The steps necessary to obtain a numerical production using this model are:

(1) Analyze the height field of the 500 mb surface and interpolate the height at each point in a mathematically convenient grid.

(2) Calculate the Laplacian of the height using finite differences.

(3) Obtain the absolute vorticity by multiplying $\nabla^2 h$ by g/f and adding f to the result at each grid point.

(4) Advect the absolute vorticity values with the initial geostrophic wind for whatever small Δt has been chosen as the basic time increment.

(5) Subtract the old vorticity field from this new one to get $\Delta(f+\xi)$ (same as $\Delta\xi$).

(6) Multiply by f/g to convert to changes in $\nabla^2 h$ (same as $\nabla^2[\Delta h]$).

(7) Integrate $\nabla^2[\Delta h]$ by relaxation or other methods, assuming $\Delta h = 0$ on the boundary. This gives Δh everywhere.

(8) Add Δh to the initial h field to get a new h field at

time Δt later.

(9) Repeat the whole process until a forecast for whatever desired future time is reached. Discard the results just inside the boundary.

The various fields and equations of dynamic forecasting are simply related. However, the relationships are not always clear to the nonmeteorologist. As an aid in progressing through these equations, the diagram of Figure 2 is presented.

There are four basic fields: the height field, the stream function, the velocity field, and the vorticity field. Observations on the height of the 500 mb surface are taken every 12 hours. These observations are taken by means of upper air soundings at the various weather stations throughout the world (at a common time). These are then interpolated to a grid covering the earth and a continuous height field drawn in terms of contour lines. Thus the height surface is the basic quantity "observed."

The three other fields are not observed directly. By means of the balance equation, the height field is related to the stream function field. The stream function field is related to the velocity field and vorticity field. To make these relations valid, it is necessary to assume that the solenoidal part of the actual wind field dominates, an assumption which is valid for the large scale motion being studied. These last three fields are of prime importance because their

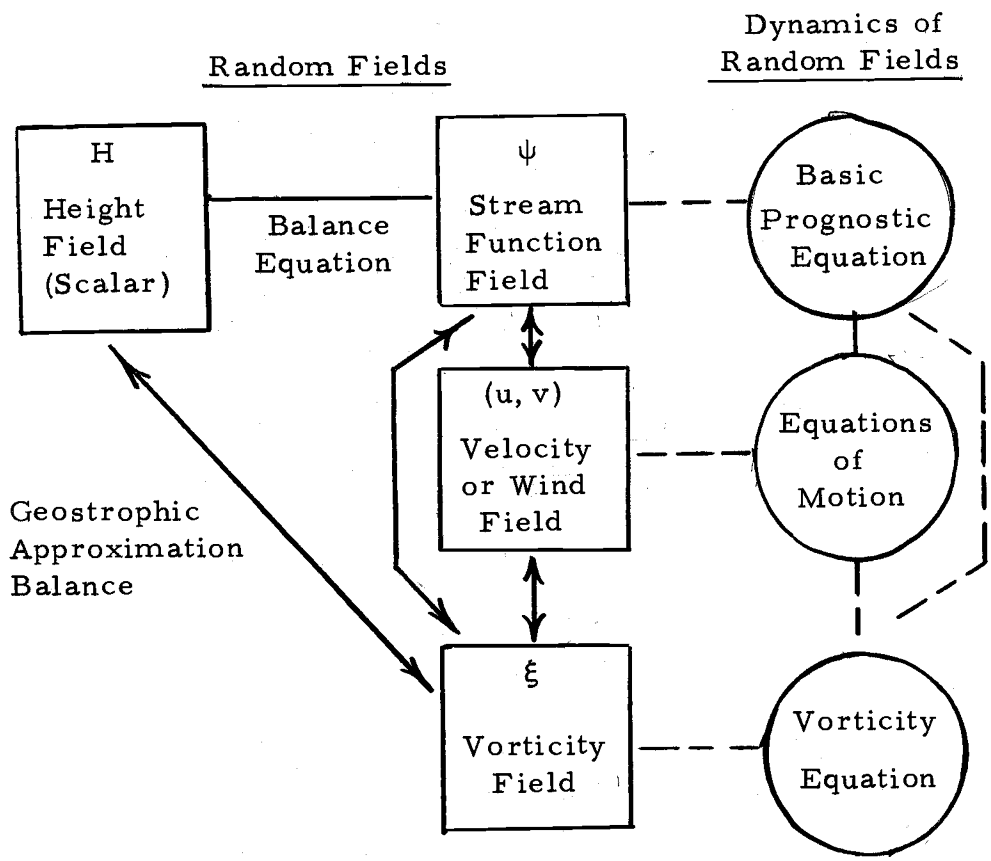

Figure 2. Fields and dynamics of fields with equation numbers referring to text.

dynamic equations can be found, beginning with Newton's second law as given. The equations relating the fields and their corresponding dynamics are given in parentheses in Figure 2.

If the geostrophic wind assumption is made the height field can be directly related to the vorticity field. The dynamics of the vorticity field then determine the dynamics of the height field which, under the geostrophic assumption, is almost the same (except for a constant multiplier) as the stream function field. This situation will be assumed in this study. Thus, the four "peripheral" fields of Figure 2 and their equations will be used in the following chapters.

III. HUERISTICS AND DIAGNOSTICS

In the last chapter, the dynamic or numerical methods based on the vorticity equation for the nondivergent level were discussed. It was pointed out that these may be used to achieve reasonably good short term forecasts.. Also discussed was the statistical method based on the prediction of spectral coefficients in a spherical harmonic expansion. In this chapter, a simplified dynamic and a simplified statistical model are studied. The objectives are to discover the advantages of each and the proper means of constructing a combined statistical-dynamic model.

A Simplified Dynamic Model

The equation of the equivalent barotropic model states that geostrophic vorticity at a point P can change only through advection by the geostrophic wind field. In effect this says that we can find the vorticity of the particle of air at point P at time $t + \Delta t$ by moving back along the geostrophic stream lines (or equivalently the contour of the height field) a distance which a particle would travel in time Δt and observing the vorticity at that point at time t. Thus for the advection operation **A**,

$$\xi(P, t+1) = A\xi(P, t) = \xi(P+\Delta P, t)$$

where ΔP is determined by the numerical process, applied N times, i.e., $N\Delta t$ is the forecast period.

Recall that the vorticity field over the sphere has the following natural representation

$$\xi(P, t) = \sum_{\nu} Z_{\nu}(t)\varphi_{\nu}(P).$$

When the advection operation is applied to the vorticity field, expanded in terms of spherical harmonics, the predicted field is

$$\begin{aligned}\xi(P, t+1) &= A[\xi(P, t)] \\ &= A[\sum_{\nu} Z_{\nu}(t)\varphi_{\nu}(P)] \\ &= \sum_{\nu} Z_{\nu}(t)\varphi_{\nu}(P+\Delta P_{\nu}).\end{aligned}$$

Thus application of the dynamics may be heuristically thought of as operating on the second factor of each term of the expansion of the field.

Recalling the relation between the height and vorticity field (using the geostrophic wind law) an expression for the predicted height field is

$$h(P, t+1) = \sum_{\nu} K(\nu) Z_{\nu}(t) \varphi_{\nu}(P + \Delta P_{\nu})$$

where $K(\nu)$ is the response function of the linear space filter (the Laplacian operator) relating height and vorticity.

To examine this explicitly for a dynamic prediction model, consider the model of Charney and Eliassen (1949). In this model, the motion at the equivalent barotropic level (500 mb) is considered as consisting of small perturbations superimposed on a constant zonal current U. The height perturbation is considered independent of the latitude, but depends on longitude. Under these assumptions, the vorticity equation (2. 14), written in terms of height, becomes

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \frac{\partial^2 h}{\partial x^2} + \beta \frac{\partial h}{\partial x} = 0$$

which reduced by means of a single integration (Thompson, 1953, p. 60) to an equation for deviations from an average height:

$$\frac{\partial^2 h}{\partial t \partial x} + U \frac{\partial^2 h}{\partial x^2} + \beta h = 0 \qquad (3.1)$$

where β has been defined before as

$$\beta = \frac{\partial f}{\partial y} .$$

A fundamental set of wave solutions to this equation is given by (Thompson, 1953)

$$h(x, t) = a \cos \alpha\ (x - ct + \zeta)$$

where $c = U - \beta\alpha^{-2}$, and ζ is a phase angle. Since both the amplitude function a and the phase angle δ are arbitrary, functions of this type form a complete set of solutions.

For a fixed latitude θ, solutions of interest are those whose wave lengths are submultiples of

$$L = 2\pi r \cos \theta,$$

the length of a latitude circle at latitude θ. Thus a complete solution may be written as

$$h(x, t) = \sum_{\nu=1}^{\infty} a_{\nu} \cos \frac{\pi\nu}{L} (x - c_{\nu} t - \zeta_{\nu}) \tag{3.2}$$

$$= \sum_{\nu=1}^{\infty} b_{\nu} \cos \frac{\pi\nu}{L} (x - c_{\nu} t) + \sum_{\nu=1}^{\infty} b'_{\nu} \sin \frac{\pi\nu}{L} (x - c_{\nu} t). \tag{3.3}$$

$$= \sum_{\nu=1}^{\infty} d_{\nu}(t) \cos \frac{\pi\nu}{L} x + \sum_{\nu=1}^{\infty} d'_{\nu}(t) \sin \frac{\pi\nu}{L} x \tag{3.4}$$

where

$$d_\nu(t) = b_\nu \cos \frac{\pi \nu c_\nu t}{L} - b'_\nu \sin \frac{\pi \nu c_\nu t}{L}$$

$$d'_\nu(t) = b'_\nu \cos \frac{\pi \nu c_\nu t}{L} + b'_\nu \sin \frac{\pi \nu c_\nu t}{L}$$

The phase speeds c_ν are uniquely determined as

$$c_\nu = U - \beta \frac{L^2}{\pi^2 \nu^2}.$$

If the constant b_ν and b'_ν are determined using the initial conditions, forecasts for a given latitude can be made.

Using the phase speeds, the effect of the advection operation is

$$\Delta P_\nu = c_\nu \Delta t.$$

A Simplified Statistical Model

Recall that a successful statistical model (Jones, 1963b) made use of the same orthogonal expansion as used in the previous section,

$$h(P, t) = \sum_\nu K(\nu) Z_\nu(t) \varphi_\nu(P).$$

In this technique, the coefficients $Z_\nu(t) K(\nu)$ were statistically predicted. Thus the predicted field was

$$h(P, t+1) = \sum_{\nu} K(\nu) Z_{\nu}(t+1) \varphi_{\nu}(P).$$

Reducing this model to the fixed latitude case as was done for the simplified dynamic model, the expansion[1] becomes

$$h(x, t) = \sum_{\nu=1}^{\infty} d_{\nu}(t) \cos \frac{\nu \pi x}{L} + \sum_{\nu=1}^{\infty} d'_{\nu}(t) \sin \frac{\pi \nu x}{L} \qquad (3.5)$$

and the prediction is

$$\hat{h}(x, t+1) = \sum_{\nu=1}^{\infty} \hat{d}_{\nu}(t+1) \cos \frac{\nu \pi}{L} + \sum_{\nu=1}^{\infty} \hat{d}'_{\nu}(t+1) \sin \frac{\nu \pi}{L} \qquad (3.6)$$

Thus, the statistical prediction does not involve spatial considerations.

A comparison of the simple statistical forecast model with the dynamic model of the last section is revealing in two ways. First, the use of spectral expansions (3.5) in the forecast technique is natural in light of the solutions (3.4) to the dynamic model. Second, Jones (1963a) remarks that when using simple straight line extrapolations as a prediction scheme for the spectral coefficients, his results are significantly better when using the phase-amplitude representation

---

[1] According to Sommerfeld (1964, p. 294), for a fixed t, expansion of $d_{\nu}(t)$ and $d'_{\nu}(t)$ in terms of the argument $\cos \theta$, results in an overall expansion of $h(P, t)$ in terms of the spherical harmonic expansion used by Jones (1963b).

$$h(x, t) = \sum_{\nu=1}^{\infty} c_\nu(t) \cos \frac{\nu\pi}{L} (x+\delta_\nu(t)) \tag{3.7}$$

rather than the form (3. 5). A comparison with the dynamic model (3. 2) shows that phase is linear, i. e.,

$$\delta_\nu(t) = c_\nu t + \zeta_\nu.$$

Thus a partial answer to better results with an equation of the form (3. 7) is the linearity of the phase suggested by the dynamic model. Conversely, using (3. 6), results in straight line extrapolation of oscillatory functions, which would be good only over short intervals.

To see how actual data for 500 mb height looks, the empirical study of Eliasen (1958) is valuable. Eliasen has done a Fourier analysis of the height for various fixed latitudes. Examples of his results are shown in Figures 3, 4, 5, and 6 for latitude 50°N. Using the form (3. 7) the evolution of $c_\nu(t)$ and $\delta_\nu(t)$ may be examined for certain wave numbers.

From the data, two ideas emerge. First, if a straight line is fit to the phase data, the resulting phase speeds differ significantly from those prdicted by the dynamic model. In fact, for low wave numbers, the waves appear stationary, contrary to the dynamic model. More will be said about this later. Second, both phase and

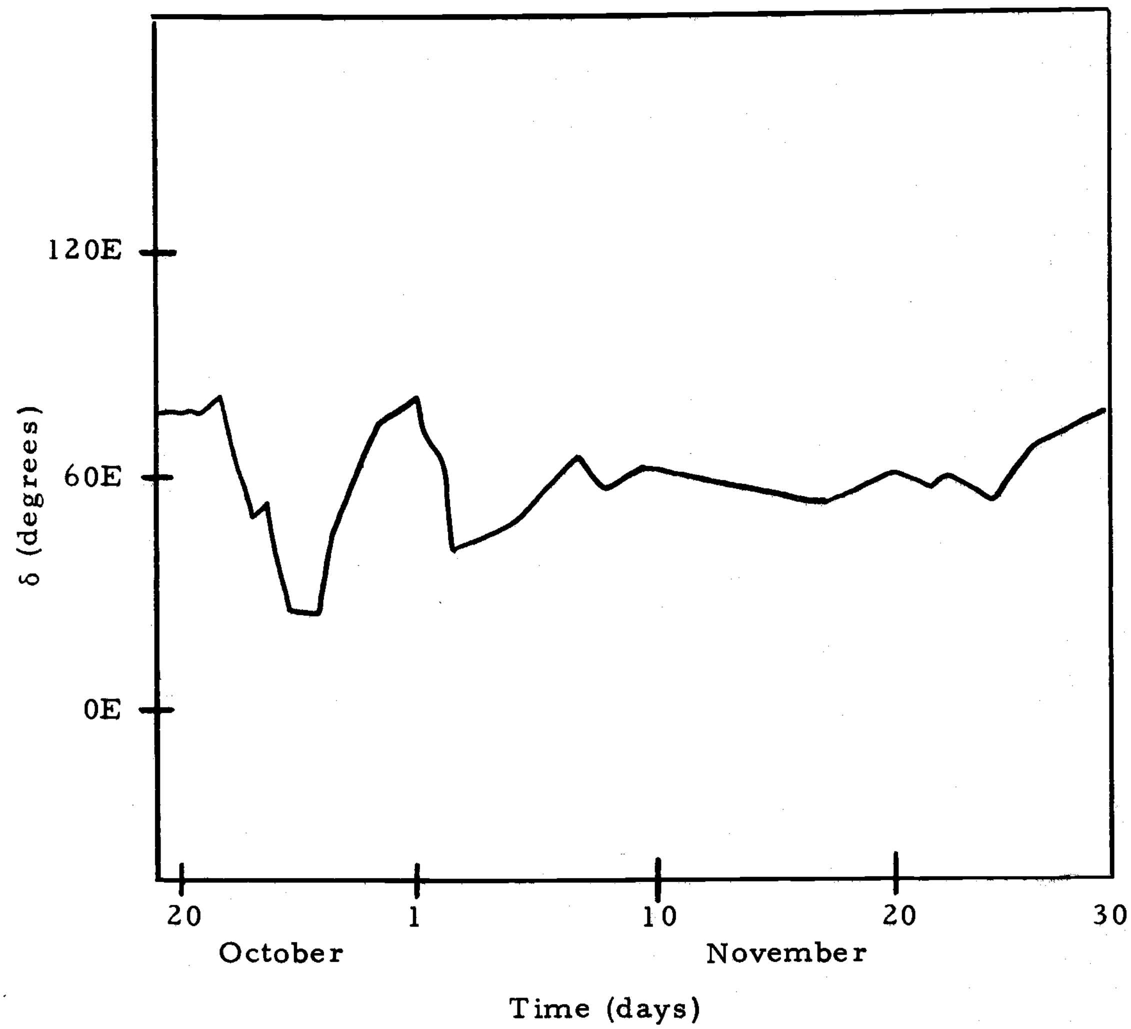

Figure 3. Phase angle for wave number 3 for 500 mb height contour on 50° N latitude (after Eliasen, 1958).

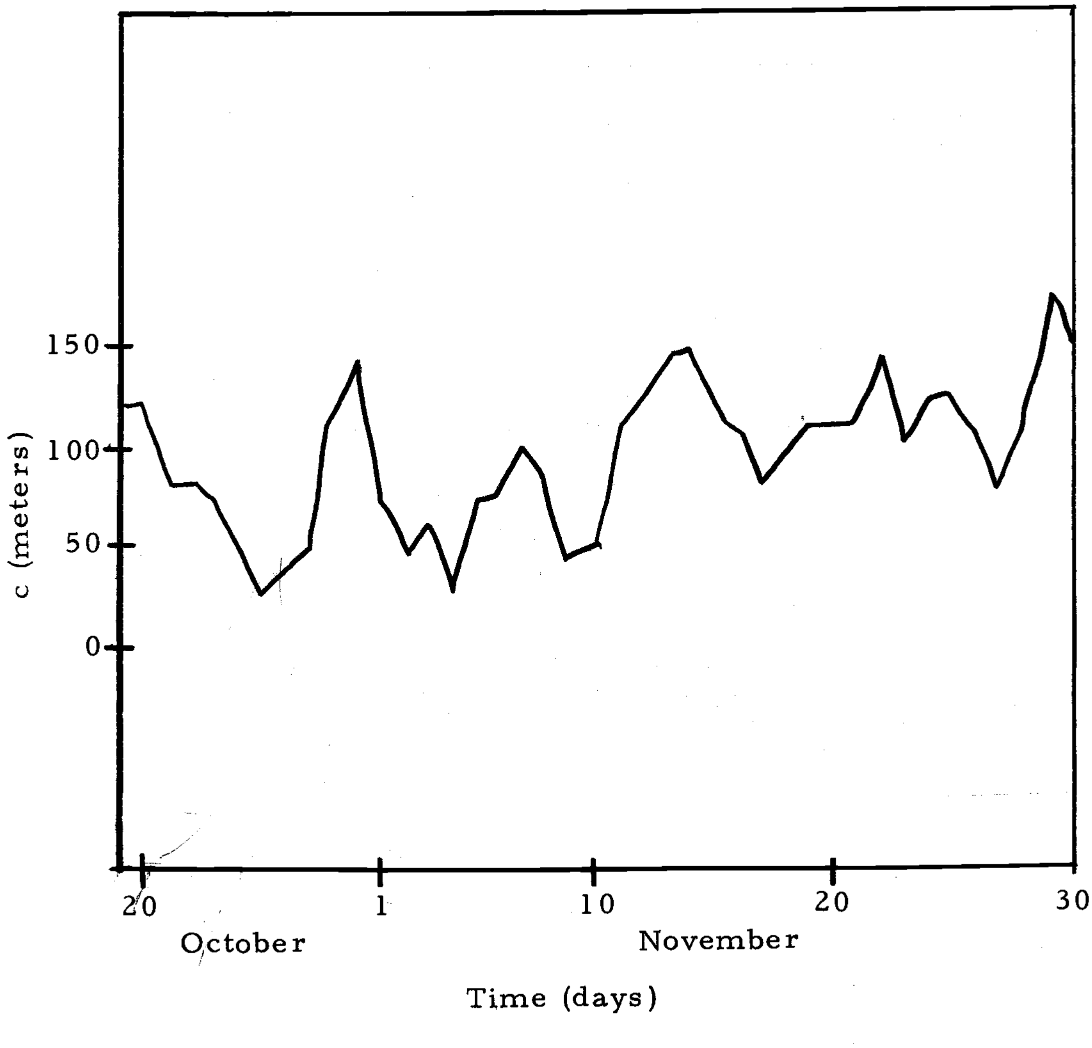

Figure 4. Amplitude coefficient for wave number 3 for 500 mb height contour on 50° N latitude (after, Eliasen, 1958).

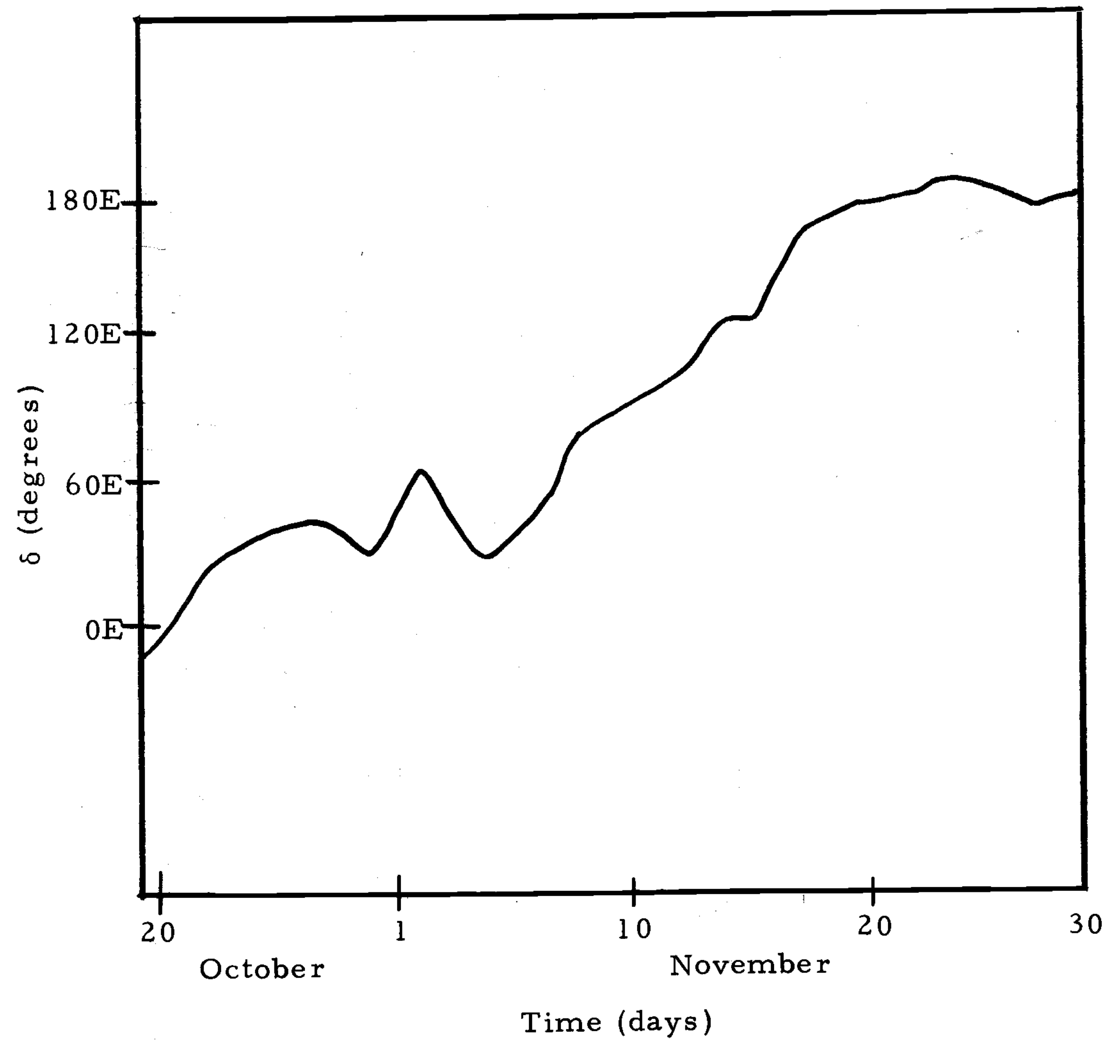

Figure 5. Phase angle δ for wave number 5 for 500 mb height contour on 50° N latitude (after Eliasen, 1958).

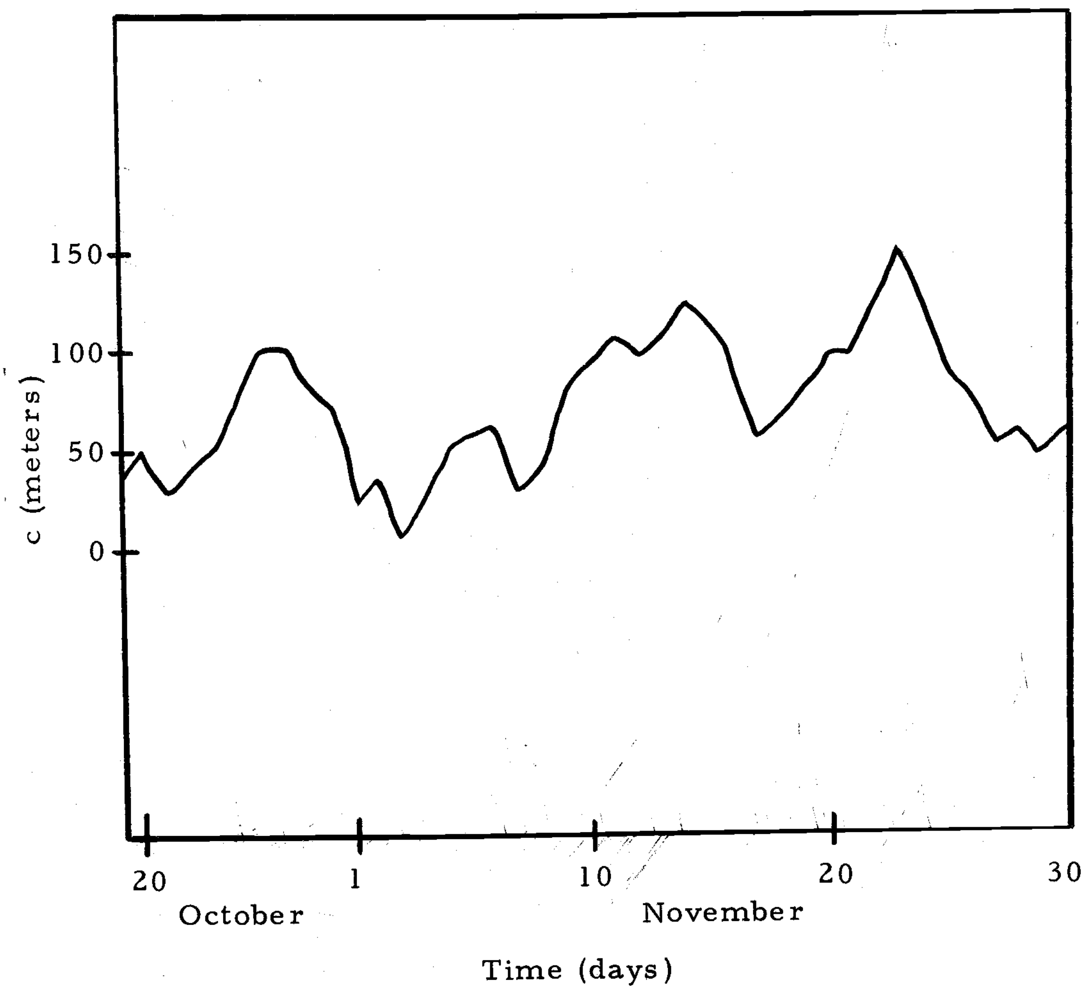

Figure 6. Amplitude coefficient for wave number 5 for 500 mb height contour of 50° N latitude (after Eliasen, 1958).

amplitude appear to have random variations in addition to the systematic trends predicted dynamically. Because of this, the concept of introducing a random term into the hydrodynamic equations is quite reasonable.

Random Forcing Functions

Randomness can enter the differential equations for the dynamics of meterological fields in three ways (Syski, 1965):

(1) Random initial conditions

(2) Random forcing functions

(3) Random differential operators (e. g., random coefficients in linear differential equations).

To construct an adequate differential equation model, it is worthwhile considering the natural way in which randomness should enter the barotropic model. The simple linear models of the last two sections yield some insight into this problem.

An appealing and simple approach to a dynamic statistical model is to combine both of the above operations as follows. First apply the dynamic advection operator to arrive at the dynamically predicted field at time $t + 1$

$$\xi^{(d)}(P, t+1) = \sum_{\nu} Z_{\nu}(t)\varphi_{\nu}(P+\Delta P_{\nu}).$$

This field can itself now be expanded as

$$\xi^{(d)}(P, t+1) = \sum_{\nu} Z_{\nu}^{(d)}(t+1)\varphi_{\nu}(P).$$

Now apply a statistical correction to the $Z^{(d)}(t+1)$ to arrive at the estimated $Z_{\nu}(t+1)$, i. e., now

$$\xi(P, t+1) = \sum_{\nu} Z_{\nu}(t+1)\varphi_{\nu}(P),$$

where

$$Z_{\nu}(t+1) = Z^{(d)}(t+1) + \epsilon_{\nu}(t+1) \quad \text{and} \quad \epsilon(t+1)$$

is the statistical correction.

To arrive at the estimates $\epsilon(t+1)$, a time series analysis is done on the process

$$Z_{\nu}^{(d)}(t) - Z(t) = \epsilon(t),$$

i. e., the difference between the dynamically predicted field and the observed field. Estimates are then constructed for several time periods and applied to the dynamically predicted field for each period.

In the purely statistical prodiction scheme, both dynamic and random variations were absorbed in the process $\{Z(t)\}$. A considerable amount of the variation is removed by the dynamic operator in the proposed scheme. Hopefully, variance is significantly reduced, the process is stationary, and the forecasts can be extended correspondingly.

To determine the form of the operator $S_Z(t, t+1)$, recall that coefficients of the spectral expansion are

$$Z_\nu(t+1) = \int \xi(P, t+1)\varphi_\nu(P)dP,$$

where dP refers to the element of integration on surface.

Now using the representation for $\xi(P, t+1)$, write

$$Z_\nu(t+1) = \int \sum_{i=1}^{n} \varphi_i(P)Z_i(t+1) \quad \varphi_\nu(P)dP$$

$$= \int \sum_{i=1}^{n} [Z_i^d(t)+\epsilon_i(t)]\varphi_i(P) \quad \varphi_\nu(P)dP,$$

$$= \int \sum_{i=1}^{n} Z_i(t)\varphi_i(P+\Delta P_i) \quad \varphi_\nu(P)dP + \int \sum_{i=1}^{n} \epsilon_i(t)\varphi_i(P)\varphi_\nu(P)dP$$

$$= \sum_{i=1}^{n} Z_i(t) \int \varphi_i(P+\Delta P)\varphi_\nu(P_i)dP + \sum_{i=1}^{n} \epsilon_i(t) \int \varphi_\nu(P)\varphi_i(P)dP$$

$$= \sum_{i=1}^{n} s_{i\nu}Z_i(t) + \epsilon_\nu(t),$$

where

$$s_{i\nu} = \int \varphi_i(P+\Delta P_i)\varphi_\nu(P)dP.$$

The evolution of the random field is completely determined by the evolution of the random vector $\vec{Z}(t) = [Z_1(t), Z_2(t), \ldots]$. Actually, the field can be adequately represented by a finite number of components. This finite vector representation will be used. Thus, calling these the state variables of the process, it is convenient to consider their evolution as follows:

$$\vec{Z}(t+1) = S_Z(t, t+1)\vec{Z}(t) + \vec{\epsilon}(t)$$

where

$$\vec{\epsilon}(t) = [\epsilon_1(t), \epsilon_2(t) \ldots \epsilon_n(t)]' \quad \text{and} \quad S_Z(t, t+1)$$

is a transition operator representing the effect of the dynamics. The matrix form of the operator S_Z is

$$S_Z(t, t+1) = [s_{ij}(t)]$$

Continuing formally, suppose that the error field is sufficiently smooth that the vector $\vec{\epsilon}(t)$ may be written as an integral of an integrable process $U(s)$ as follows

$$\vec{\epsilon}(t+\Delta t) = \int_t^{t+\Delta t} \vec{U}(s)ds$$

for any time increment Δt. Then

$$\vec{Z}(t+\Delta t) = S_Z(t, t+\Delta t)\vec{Z}(t) + \int_t^{t+\Delta t} \vec{U}(s)ds.$$

Subtracting $\vec{Z}(t)$ from both sides

$$\vec{Z}(t+\Delta t) - \vec{Z}(t) = S_Z(t, t+\Delta t)\vec{Z}(t) - \vec{Z}(t) + \int_t^{t+\Delta t} \vec{U}(s)dx.$$

Assuming $S_Z(t, t+(t)$ is differentiable, which is reasonable for the hydrodynamic processes of interest in meteorology, expand $S_Z(t, t+\Delta t)$ in a Taylor series about t

$$S_Z(t, t+\Delta t) \cong S_Z(t, t) + \dot{S}_Z(t, t)\Delta t$$

where the remainder term, which goes to zero as Δt goes to zero, has been neglected.

Substituting this into the last expression and dividing by Δt

$$\frac{\vec{Z}(t+\Delta t)-\vec{Z}(t)}{\Delta t} \cong \frac{[S_Z(t, t)+\dot{S}_Z(t, t)\Delta t]\vec{Z}(t)}{\Delta t} - \frac{\vec{Z}(t)}{\Delta t} + \int_t^{t+\Delta t} \frac{\vec{U}(s)ds}{\Delta t}$$

Recall that $s_{ij} = \int \varphi_i(P+\Delta P_i)\varphi_j(P)dP$. Now $\Delta P_i \to 0$ as $\Delta t \to 0$ and thus $s_{ij} \to \delta_{ij}$ as $\Delta t \to 0$ since $\{\varphi_\nu\}$ or orthonormal functions. Hence $S_Z(t, t) = I$, the identity matrix. Also, formally using the mean value theorem on the integral term

$$\int_{t}^{t+\Delta t} \vec{U}(s)ds = \vec{U}(s*) \int_{t}^{t+\Delta t} ds$$

where $s* \rightarrow t$ as $\Delta t \rightarrow 0$. With these things in mind,

$$\frac{\vec{Z}(t+\Delta t)-\vec{Z}(t)}{\Delta t} \cong \dot{S}_Z(t, t)\vec{Z}(t) + \vec{U}(s*)$$

Taking the mean square limit on both sides

$$\frac{d}{dt}\vec{Z}(t) = \dot{S}_Z(t, t)\vec{Z}(t) + \vec{U}(t)$$

or

$$\frac{d}{dt}\vec{Z}(t) - \dot{S}_Z(t, t)\vec{Z}(t) = \vec{U}(t)$$

where $\vec{U}(t)$ is the random forcing function.

Thus the form of a reasonable statistical-dynamic prediction scheme suggests that the randomness should be introduced into the underlying stochastic model by means of a random forcing function.

Aside from the above formal argument, there are several other more pragmatic arguments for using random forcing functions. The simple dynamic model above has been used to make predictions of the 500 mb surface. As mentioned before Elliasen (1958) observed that the low wave numbers ($\nu = 1, 2, 3, 4$) are stationary while the waves for numbers $\nu \geq 5$ are traveling. On the other hand, the simple dynamic model predicts traveling waves for low wave numbers. This discrepancy results in large errors. This phenomena is not restricted to the linear model but also occurred in the case of the nonlinear barotropic model.

To correct this problem, considerable meteorological research took place from 1956 through 1958 on the nonlinear barotropic model. Williams (1958) was the first to attempt a correction in a purely empirical manner. By computing the difference between the forecast and observed field of 500 mb height he constructed an error field. He then averaged the error field over several days. This average 24 hour error field was then subtracted after each 24 hour period of forecast before integration of the barotropic equations for the next forecast. Thus in a sense, Williams did the first "time series analysis" combined with dynamic forecasting in the manner suggested above. While his prediction of the error field was simply a numerical average, it can be thought of as a particular type of autoregression prediction. His results were excellent. The success demonstrated suggests the use of a random forcing function in the combined statistical dynamic prediction will be successful.

Wolff (1958) correctly diagnosed the trouble in forecasts by observing that the barotropic model yielded waves which progressed rapidly westward in 24 hours for low wave numbers contrary to the behavior observed for the actual waves as noted above in the linearized model. To correct this, he devised and applied an empirical correction which stabilized waves number 1, 2 and 3. The effect of this correction was quite good.

Cressman (1958), following Rossby's suggestion, that a "divergence term" to be added to the barotropic prognostic equation, derived the following equation:

$$\frac{\partial}{\partial t} \nabla_p^2 h + \vec{V} \cdot \text{grad}_p \nabla_p^2 h + \beta \frac{\partial h}{\partial x} = g(h)$$

where

$$g(h) = k \frac{\partial h}{\partial t}$$

and k is a constant empirically determined. Thus Cressman physically derived a forcing function dependent on the state variable h. Again the results were highly significant (Shuman, 1966) and resulted in much better forecasts. However, subsequent research has shown k not to be constant but to take on many values (Deland, 1967) which suggests it could be a random function of time. Thus, in all three cases of correction of the nonlinear barotropic model, forcing functions were used and in all three cases, the improvement was significant.

A final reason for using random forcing function is the form of the vorticity equation (2.5). To arrive at the barotropic nondivergent model, all the terms on the right side of the equation were assumed negligible. In actual atmospheric processes it is quite probable that these terms do contribute from time to time. These contributions are naturally accounted for by a random forcing function.

Summary

In summary, the hueristics and diagnostics of this chapter have motivated

(1) Use of the spectral expansion as naturally related to the dynamics of the 500 mb surface

(2) Use of the random forcing function technique as a natural introduction of randomness into the models.

IV. DYNAMIC STATISTICAL MODELS

The basic notion that a combined dynamic and statistical forecast model would be very useful is not new. Meteorologists have made attempts to motivate such a model (Gleeson, 1968) and some actual predictions have been made using the barotropic model combined with empirical orthogonal functions (Sellars, 1957). In these approaches, randomness has entered only through initial conditions or only through use of principle components of sample covariance functions. Suggestions on combining the dynamics and statistics have been given by Frieburger and Grenander (1965) in an excellent general paper.

Based upon the motivation of the last chapter, this study departs from the prior approaches and uses the concept of a random forcing function in the dynamic equations. Statistical analysis discussed in Chapter V enters through estimation problems associated with the forcing functions, the form of which is not assumed known a priori. The various possible forms of the dynamic equations and the forcing functions define the three models considered.

This chapter has the following outline. First, based on the spherical harmonic expansion of the random field, the random spectral vorticity equation is derived. Sufficient conditions for the existence of a solution to this equation are then given. Next, mean value and covariance solutions are found for various forms of the dynamics (linear and nonlinear) and the forcing function (dependent on the time parameter and dependent on the time and state variables) in the spectral vorticity equation.

Spectral Form of the Vorticity Equation

The vorticity equation has the following form

$$\frac{\partial}{\partial t} \xi_p + \vec{V} \cdot \text{grad}_p(\xi_p + f) = -(f+\xi_p)\,\text{div}_p\,\vec{V} + \vec{k} \times \frac{\partial \vec{V}}{\partial p} \cdot \text{grad}_p\,\omega \qquad (4.1)$$

where the symbols are the same as in Equation (2. 5) of Chapter II. Replacing the terms on the right side with a random forcing function n(t), the equation takes the form

$$\frac{\partial}{\partial t} \xi_p + \vec{V} \cdot \text{grad}_p(\xi_p + f) = n(t). \qquad (4.2)$$

In this form, the vorticity equation is a random partial differential equation. Since $\vec{V} = \vec{V}_g$, $\xi_p = \xi_g$ in this model.

The forcing function process n(t) accounts for terms neglected in the vorticity equation, for physical effects unaccounted for in the fluid dynamic model (because of assumptions made to simplify the mathematical derivations), and randomness in the atmospheric process. Hence, the function is not known analytically and its effects must therefore be statistically estimated (Chapter V).

The theory of partial differential equations with random forcing functions is not well developed. More is known about ordinary differential equations with random forcing functions. For this reason, the vorticity equation will be transformed into the spectral form before seeking a solution in terms of the mean and variance.

Recall that the 500 mb level was chosen as a field for prediction because it is a level of nondivergence. This allowed the introduction of stream functions (and consequently height through the balance equation). A similar assumption is made here about the forcing function, that is, there exists a stream function η such that

$$n(t) = \nabla_p^2 \eta(t).$$

Physically, this means that the small part of the velocity field arising from the velocity forcing function corresponding to the vorticity forcing function is also nondivergent.

The following derivation of the random spectral vorticity equation is patterned after Silberman's (1954) derivation of the deterministic spectral vorticity equations. The deterministic form of the vorticity equation is well known and has been rederived by other authors (Platzman, 1960; Merilees, 1968).

Consider an isotropic random field, $\xi_p(P)$. As mentioned before the vorticity field $\xi_p(P)$ has the representation

$$\xi_p(P) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Z_{mn} Y_n^m(P)$$

where

$$Y_n^m(P) = P_n^m(\sin\theta')e^{im\lambda}$$

and P_n^m is the associated (normalized) Legendre function, θ' is the colatitude and λ is longitude. Introducing time parameter

$$\xi_p(P,t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Z_{mn}(t) Y_n^m(P).$$

Proceeding now to derive a spectral vorticity equation, write (4. 2)

$$\frac{\partial \xi_p}{\partial t} = \frac{1}{r}\left(v \frac{\partial}{\partial \theta'} + \frac{u}{\sin\theta'} \frac{\partial}{\partial \lambda}\right)(\xi_p + 2|\vec{\Omega}| \cos\theta') + n \tag{4. 3}$$

Now, setting $\xi_p = \nabla_p^2 \psi$, $n = \nabla_p^2 \eta$, the above tendency equation becomes

$$\nabla_p^2 \frac{\partial \psi}{\partial t} = \frac{1}{r^2 \sin\theta'}\left(\frac{\partial \psi}{\partial \lambda}\frac{\partial}{\partial \theta'} - \frac{\partial \psi}{\partial \theta'}\frac{\partial}{\partial \lambda}\right)(\nabla_p^2 \psi + 2|\vec{\Omega}| \cos\theta') + \nabla_p^2 \eta . \tag{4. 4}$$

Now expand the stream functions in terms of spherical harmonics:

$$\psi(t) = r^2|\vec{\Omega}| \sum K_n^m(t) Y_n^m$$

$$\eta(t) = r^2 |\vec{\Omega}| \sum Z_n^m(t) Y_n^m$$

At this point it will be convenient to truncate the expansion at a finite number of terms (possibly different for the separate expansions). Eliasen and Machenhauer (1965) have done an extensive analysis of the spectral distribution of kinetic energy for the large planetary flow patterns represented by the stream function. As would be expected, their data (Eliasen and Machenhaur, 1965, p. 225) clearly indicates almost no energy is contained in the larger wave number part of the spectrum. Because of these energy considerations, it is physically reasonable to truncate the spectral expansions. Thus, rewrite the expression as

$$\psi = r^2 |\vec{\Omega}| \sum_{m=-m'}^{m'} \sum_{n=|m'|}^{n'} K_n^m Y_n^m$$

$$\eta = r^2 |\vec{\Omega}| \sum_{m=-m'''}^{m'''} \sum_{n=|m'''|}^{n'''} K_n^m Y_n^m$$

To derive the ordinary differential equations from the partial differential equation, it is necessary to do the following: First, derive each of the terms in the equation using the spherical harmonic expansion. For example, the first term is derived by differentiating as

$$\frac{\partial\psi}{\partial t} = r^2|\vec{\Omega}| \sum_{m=-m''}^{m''} \sum_{n=|m|}^{n''} (\frac{d}{dt}K_n^m(t))Y_n^m(P).$$

Observing that Y_n^m are eigenfunctions of the LaPlacian operator ∇_p^2, i.e.,

$$\nabla_p^2 Y_n^m = -n(n+1)Y_n^m$$

it is easy to see that

$$\nabla_p^2 Y_n^m = r^2|\vec{\Omega}| \sum_{m=-m''}^{m''} \sum_{n=|m|}^{n''} (\frac{d}{dt}K_n^m(t))(-n[n+1])Y_n^m(P) .$$

Second, having derived all of the needed expressions in terms to their spherical harmonic expansion, substitute them into the expression (4.4). To extract from the resulting expression the equations for the evolution of the coefficients, multiply the resulting expressions by Y_n^{-m} sin θ' and integrate θ' from 0 to π and λ from 0 to 2π. By making use of the orthogonality of the spherical harmonics the coefficient equations are found to be the following form for each m and n:

$$\frac{d}{dt}K_n^m(t) = \frac{2i|\vec{\Omega}|}{n(n+1)} K_n^m + 2i|\vec{\Omega}|Z_n^m$$

$$+\frac{i|\vec{\Omega}|}{2} \sum_{r=m'}^{m'} \sum_{s=|r|}^{n'} \sum_{j=m}^{m'} \sum_{k=|j|}^{n'} K_k^j K_s^r H_{kns}^{jmr} \qquad (4.5)$$

where H^{jmr}_{kns} is zero unless $j + r = m$, in which case

$$H^{jmr}_{kns} = \frac{s(s+1)-h(k+1)}{n(n+1)} \int_0^{\pi} P^m_n \times (jP^j_k \frac{d}{d\theta} P^r_s - r \frac{d}{d\theta} P^j_k P^r_s) d\theta.$$

The quantities H^{jmr}_{kns} are called the interaction coefficients. Rules for evaluating these interaction coefficients are found in numerous papers (Silberman, 1954; Ellsaesser, 1966; Platzman, 1960; Bear and Platzman, 1961; Merilees, 1968).

As a simplification notationally, this may be written as

$$\frac{d}{dt} K^m_n(t) = F(K^m_n(t), Z^m_n(t), t).$$

Written in vector form this system of equation is

$$\frac{d\vec{K}}{dt} = F(\vec{K}, \vec{Z}, t). \qquad (4.6)$$

By letting the subscript γ correspond to m, n (Platzman, 1962) the expressions may be notationally simplified. A further simplification results by converting the equations to real differential equations

$$\frac{d}{dt} C_\gamma = k_1 S_\gamma + k_3 \sum_\alpha \sum_\beta (C_\alpha S_\beta + S_\alpha C_\beta) H_{\alpha\beta r} + k_2 S^*_\gamma$$

$$\frac{d}{dt} S_\gamma = k_1 C_\gamma + k_3 \sum_\alpha \sum_\beta (C_\alpha C_\beta + S_\alpha S_\beta) H_{\alpha\beta r} + k_2 C^*_\gamma \qquad (4.7)$$

Where C_γ and S_γ are the real spectral coefficients and S^*_γ and C^*_γ are corresponding random forcing function real coefficients and k_1, k_2, and k_3 are constants. In this form, the problem has been transformed into a system of ordinary quadratic differential equations with random forcing functions.

A final simplification of notation is desirable. The above system of equations will be rewritten as

$$\frac{dA_r}{dt} = f_r(t, A_1, A_2, \ldots, A_h, R_1, R_2, \ldots, R_h) \quad (4.8)$$

where r corresponds to (m, n), A corresponds to the spectral coefficients and R corresponds to the random forcing functions.

For the remainder of this study the general form of the vorticity equation represented by equation (4.8) will be used. Merilees (1968) has shown that a quite general set of dynamic equations can be transformed explicitly into spectral form and given tables of functions to accomplish this transformation. This general set includes not only the vorticity equation but the divergence equation, continuity equation, the adiabatic thermodynamic equation and the hydrostatic equation. With this set of equations, more general baroclinic models may be placed in spectral form. In each case, the resulting ordinary differential equations for the spectral coefficients of the various fields have the same general form as the general form of the vorticity equation.

Therefore, statistical dynamic models based on the general form (4.8) of the spectral vorticity equation will have application to more general spectral baroclinic forecast systems as they are developed.

Existence

The existence and uniqueness theorem for random differential equations is of essentially the same form as the existence theorem (involving Lipshitz conditions) for ordinary differential equations. An explicit statement is given in Syski (1965). Writing Equation (4.8) as

$$\frac{d\vec{A}}{dt} = f(t, \vec{A}) + \vec{R}$$

where $\vec{R}$ is the random forcing function, sufficient conditions on f and $\vec{r}$ may be summarized as

(1) f is continuous and satisfies the conditions, for M a constant,

$$\| f(t, \vec{\zeta}_1) - f(t, \vec{\zeta}_2) \| \leq M \| \vec{\zeta}_1 - \vec{\zeta}_2 \|$$

uniformly for $t \in T = [a, b]$, the interval of interest and for all $\vec{\zeta}_1, \vec{\zeta}_2 \in H$, the space $H = L_2(\Omega, \mathcal{B}, \mu)$.

(2) $\| f(t, \vec{\zeta}) \| \leq F(1 + \| \zeta \|^2)^{1/2}$ for F a constant and $\vec{\zeta} \in H$.

(3) $\vec{R}$ is m. s. differentiable.

For the spectral vorticity equation, the function f is quadratically nonlinear but is continuous and differentiable (for all orders of derivatives). It will therefore satisfy the conditions above and over a finite interval $T = [a, b]$.

In the derivation of the spectral vorticity equation, it was tacitly assumed that the random differential equation could be found from the corresponding equation for the sample functions. This procedure is followed in most physical derivations of random differential equations. It turns out that the space $H = L_2(\Omega, \mathcal{B}, \mu)$ has properties necessary to insure that this is valid (Syski, 1965, p. 441). In succeeding sections, the same procedure will be used without specific reference to the fact.

Finally, in the following sections, it will be necessary to interchange the expectation operator with an integration over time. Again, according to Syski (1965, p. 442) this interchange is valid for the processes being studied.

Random Forcing Function Models

The solution of a random differential equation is itself a random function. The description of the random function is usually in terms of its distribution function structure. In the case that the solution is a Markov process the Fokker-Planck and Kolmogorov equations describe the evolution of the distribution function. When the random differential equation is linear and the random forcing function is a second order process, the evolution of the distribution function can be found through equations for the evolution of the cumulant generating function (Syski, 1965, p. 389). Unfortunately, neither of these cases are general enough for the nonlinear random differential equations describing the 500 mb surface dynamics. It is therefore necessary to select some function which is representative of the distribution functions. The ones chosen are the mean and covariance functions.

To find differential equations for an estimate of the mean value function and covariance matrix of the vector random process, various techniques are available; linearization of the spectral vorticity equation itself by means of meteorological principles, use of the truncated Taylor Series expansion, use of the generalized Karhunen-Loeve expansion.

Each of these methods will be used in subsequent sections to determine the "solution" of the random spectral vorticity equations. Each method constitutes a different stochastic model.

Model 1, Linearization Based upon Meteorological Principles

A principle often used by meteorologists to linearize equations involving hemispheric flow patterns assumes the stream function ψ may be written as

$$\psi = \overline{\psi} + \psi'$$

where

$$\overline{\psi} = r^2|\vec{\Omega}| \sum_{n=0}^{n'} K_n^o P_n^o$$

is the stream function for the zonal flow and ψ' is a small perturbation on the zonal flow, represented as

$$\psi' = r^2|\vec{\Omega}| \sum_{n=|m|}^{n'} \sum_{m=-m'}^{m'} K_n^m Y_n^m \quad . \quad m \neq 0$$

The values of K_n^o are associated with $\overline{\psi}$, the constant zonal flow and, thus, do not vary with time, This representation is similar to the one used in Chapter III to derive the simple dynamic model.

The values of $|K_n^m|$ for $m \neq 0$ are now very small in comparison with the values of K_n^o. Thus, every term on the right side of the spectral vorticity equation (4. 5) which does not contain the coefficient of a zonal harmonic as one of its factors may be neglected. Thus, the equation becomes

$$\frac{d}{dt} K_n^m = i|\vec{\Omega}| \sum_{s=|m|}^{n'} K_s^m G_{ns}^m + 2i|\vec{\Omega}| Z_n^m \tag{4. 8}$$

where

$$G_{ns}^m = \frac{2m\delta_{ns}}{n(n+1)} + \sum_{k=1}^{n'} K_k^o H_{kns}^{omm}$$

and H_{kns}^{omn} has been defined before. The system of equations resulting is a linear system with random forcing function represented by the vector of Z's.

As before, this may be converted to real valued differential equations,

$$\frac{d}{dt}\vec{A} = T_A\vec{A} + \vec{R}.$$

After changing this system of equations to its real form the assumption that the random forcing function is an "ideal white" noise, allows application of Jones' techniques of using Kalman (1960) techniques for achieving predictions (Jones, 1965). Thus, this particular model when combined with Jones' work is a first step in the combining of Kalman techniques with meteorological forecasting.

According to the theory of linear differential equations, there corresponds to T_A a transition matrix $S_A(t_o, t_s)$ such that the solution of the equation is

$$\vec{A}(t) = S_A(t, t_o)\vec{A}(t_o) + \int_{t_0}^{t} S_A(t, s)\vec{R}(s)ds \qquad (4.9)$$

where

$$S_A(t_o, t_o) = I$$

$$\frac{d}{dt} S_A(t, t_o) = T_A S_A(t, t_o)$$

To determine the mean value solution, it will be necessary to apply the expectation operator to Equation (4. 9). As remarked in the last section, the interchange of the expectation operation with the integral is valid in the processes being studied. When the expectation operation is applied (4. 9) the equations for the mean values become

$$\vec{m}^A(t) = S_A(t, t_o)\vec{m}^A(t_o) + \int_{t_0}^{t} S_A(t, s)\vec{m}^R(s)ds$$

or

$$\frac{d\vec{m}^A}{dt} = T_A \vec{m}^A + \vec{m}^R$$

where $\vec{m}^A$ and $\vec{m}^R$ are the expected values of $\vec{A}$ and $\vec{R}$. This solution is well defined in terms of the deterministic mean value functions, provided the value $\vec{m}^R(s)$ is known.

If the initial conditions are known exactly, i. e., $\vec{A}(t_o)$ is observed, then $\vec{m}^A(t_o) = A(t_o)$ in the above equation. However, in actual observations, the initial conditions are random variables also, due to the uncertainty in observation of the height of the 500 mb surface. This uncertainty will therefore contribute to the general covariance structure derived in the following paragraphs.

To determine the solution in terms of the covariance function, it is convenient to make use of the random variables $\vec{A}^o$ and $\vec{R}^o$

which are centered at the expectations of $\vec{A}$ and $\vec{R}$, found from the above mean solution. $\vec{A}^o$ and $\vec{A}$ have the same covariance structure which is also true for $\vec{R}^o$ and $\vec{R}$.

By definition

$$\begin{aligned} \text{Cov}\,(\vec{A}(t)) &= E(\vec{A}(t)-\vec{m}^A(t))(\vec{A}(t)-\vec{m}^A(t))') \\ &= E(\vec{A}^o(t)\vec{A}^{o'}(t)) \end{aligned}$$

where $\vec{A}^o$ is a random vector centered at its mean value $\vec{m}^A$, that is

$$\vec{A}^o(t) = \vec{A}(t) - \vec{m}^A(t).$$

Consider the equations for the evolution of the centered random variables $\vec{A}^o$:

$$\begin{aligned} \vec{A}^o(t) &= S_A(t, t_o)\vec{A}(t_o) + \int_{t_0}^{t} S_A(t, s)\vec{R}(s)ds \\ &\qquad - S_A(t, t_o)\vec{m}^A(t_o) - \int_{t_0}^{t} S_A(t, s)\vec{m}^R(s)ds \\ &= S_A(t, t_o)\vec{A}^o(t_o) + \int_{t_0}^{t} S_A(t, s)\vec{R}^o(s)ds \end{aligned}$$

Thus, in this case, the centered random variables have the same transition matrices as the noncentered random variables. Substituting the last equation into the definition of the covariance functions,

$$\mathrm{Cov}\,(\vec{A}^{o}) = E(\vec{A}^{o}\,\vec{A}^{o\prime})$$

$$= E[S_A(t,t_o)\vec{A}_o(t_o) + \int_{t_0}^{t} S_A(t,s)\vec{R}^{o}(s)ds]$$

$$\times\ [\vec{A}^{o\prime}(t_o)S'_A(t,t_o) + \int_{t_0}^{t} \vec{R}^{o\prime}(s)S'_A(t,s)ds]$$

$$= E[S_A(t,t_o)\vec{A}^{o}(t_o)\vec{A}^{o\prime}(t_o)S'_A(t,t_o)]$$

$$+ E[\int_{t_0}^{t} S_A(t,t_o)\vec{A}(t_o)\vec{R}^{o\prime}(s)S'_A(t,s)ds]$$

$$+ E[\int_{t_0}^{t} S_A(t,s)\vec{R}^{o}(s)\vec{A}^{o\prime}(t_o)S'_A(t,t_o)ds]$$

$$+ E[\int_{t_0}^{t} S_A(t,s)\vec{R}^{o}(s)ds \int_{t_0}^{t} \vec{R}^{o\prime}(r)S'_A(t,r)dr]\ .$$

Now, by moving the expected value operation inside the integral the last expression for the covariance matrix of solutions becomes

$$\mathrm{Cov}\,[\vec{A}(t)] = S_A(t,t_o)E[\vec{A}^{o}(t_o)\vec{A}^{o\prime}(t_o)]S'_A(t,t_o)$$

$$+ \int_{t_0}^{t} S_A(t,t_o)E[\vec{A}^{o}(t_o)\vec{R}^{o\prime}(s)]S'_A(t,s)ds$$

$$+ \int_{t_0}^{t} S_A(t,s)E[\vec{R}^{o}(s)\vec{A}^{o\prime}(t_o)]S'_A(t,t_o)ds$$

$$+ \int_{t_0}^{t}\int_{t_0}^{t} S_A(t,s)E[\vec{R}^{o}(s)\vec{R}^{o\prime}(r)]S'_A(t,r)drds.$$

Thus, the evolution of the covariance matrix involves the covariances and cross covariances of $\vec{A}^o$ and $\vec{R}^o$. In particular the following are needed:

(1) The covariance of $\vec{A}(t_o)$, the initial state of the spectral coefficients.

(2) The covariance of the initial state $\vec{A}(t_o)$ and the random forcing functions $\vec{R}(t)$ for values of $t \geq t_o$.

(3) Covariance function of the random forcing function $R(s)$ for values $t_o \leq s \leq t$.

Observe that the initial conditions occur in the first three terms of the covariance equation. If it is assumed that the initial field is observed with no uncertainty, the first, second and third terms drop out since $\vec{A}^o(t_o) = \vec{0}$ in this case. The total uncertainty then is due to the random forcing function alone.

The equations above describe the evolution of continuous time of the mean spectral coefficients and the covariances. Knowing these, the evolution of the mean field over the sphere may be determined as well as an "uncertainty" field or variance field. The solution above is not really complete because:

(1) For the mean vector, it is necessary to know the evolution of the mean of the random forming functions. This is not known a priori from the hydrodynamic model and must be statistically estimated.

(2) A similar situation holds in the case of the covariance functions, statistical estimation being necessary also.

(3) To construct a prediction algorithm it is necessary to rewrite the model in terms of a discrete time model since observations of the 500 mb surface occurs only at discrete times. Thus the statistical estimation should be in terms of the discrete time. This is the subject of Chapter V.

Model 2, Use of Taylor Series Expansions

Model 1 made use of linearized differential equations. The resulting combined statistical dynamic model was linear. Model 2 departs from linearity for the dynamic prediction of the mean field and makes use of nonlinear differential equations. Linear evolution of the covariance field is assumed, however.

Recall the basic spectral vorticity equation (4. 8) is

$$\frac{dA_r}{dt} = f_r(t, A_1 \ldots A_h, R_1 \ldots R_h), \quad r = 1, 2, \ldots, h$$

Suppose now that $R_s = R_s(t)$, i. e., that the random forcing functions are functions of time only and independent of the variables A. Following Rao (1965, p. 321) expand f_r in a Taylor series about the expectations of $\vec{A}$ and $\vec{R}$,

$$f_r(t, A_1 \ldots A_h, R_1 \ldots R_h) = f_r(t, m_1^A \ldots m_h^A, m_1^R \ldots m_h^R)$$

$$+ \sum_{p=1}^{h} \frac{\partial f_r}{\partial m_p^A} A_p^o + \sum_{p=1}^{h} \frac{\partial f_r}{\partial m_p^R} R_p^o + B_r$$

where $A_p^o = (A_p - m_p^A)$ and $R_p^o = (R_p - m_p^R)$ are random variables centered at their expectations, and

$$\frac{\partial f_r}{\partial m_p^A} = \left.\frac{\partial f_r}{\partial A_p}\right|_{A_p = m_p^A}$$

$$\frac{\partial f_r}{\partial m_p^R} = \left.\frac{\partial f_r}{\partial R_p}\right|_{R_p = m_p^R}$$

The remainder B is

$$B_r = \delta_r \left[\sum_{p=1}^{h} (A_p - m_p^A)^2 + \sum_{p=1}^{h} (R_p - m_p^R)^2 \right]^{1/2}$$

and

$$\delta_r \to 0 \quad \text{as} \quad \vec{A} \to \vec{m}^A, \quad \vec{R} \to \vec{m}^R.$$

Substituting the above expansion in the original equations and neglecting the remainder B_r, the following results:

$$\frac{d}{dt}(m_r^A + A_r^o) \cong f_r(t, m_1^A \ldots m_h^A, m_1^R \ldots m_h^R) + \sum_{p=1}^{h} \frac{\partial f_r}{\partial m_p^A} A_p^o + \sum_{p=1}^{h} \frac{\partial f_r}{\partial m_p^R} R_p^o$$

To find approximate equations for the expectations of the random variable A_r, apply the expectation operator to the above approximate differential equations. Thes results in

$$\frac{dm_r^A}{dt} \cong f_r(t, m_1^A \ldots m_h^A, m_1^R \ldots m_h^R), \quad r = 1, 2, \ldots, h,$$

since the expectation of the centered random variables is zero. Thus, the approximate equations for mean values retain the nonlinearity of the original numerical equation.

To determine the variance field, consider the linear approximate differential equations for the centered random functions A^o in terms of R^o:

$$\frac{dA_r^o}{dt} = \sum_{p=1}^{h} \frac{\partial f_r}{\partial m_p^A} A_p^o + \sum_{p=1}^{h} \frac{\partial f_r}{\partial m_p^R} R_p^o$$

Setting

$$a_{rp}(t) = \left. \frac{\partial f_r}{\partial A_p} \right|_{A_p = m_p^A}$$

$$b_{rp}(t) = \left. \frac{\partial f_r}{\partial R_p} \right|_{R_p = m_p^R}$$

These become

$$\frac{dA_r^o}{dt} = \sum_{p=1}^{h} a_{rp}(t) A_p^o + \sum_{p=1}^{h} b_{rp}(t) R_p^o .$$

Written in matrix form, the equations become

$$\frac{d\vec{A}^o}{dt} = T_A(t)\vec{A}^o + T_B(t)\vec{R}_p^o$$

where

$$T_A(t) = [a_{ij}(t)] \quad \text{and} \quad T_B(t) = [b_{ij}(t)] .$$

Since $\vec{R}^o$ is not known in any functional form and must eventually be statistically estimated, it is convenient to replace the random variable with

$$\vec{R}^{o'} = T_B(t)\vec{R}^o.$$

The equation for the centered random vectors becomes identical to that for the last model except for the variation of T_A with time in a known manner dependent on the mean solution

$$\frac{d\vec{A}^o}{dt} = T_A(t)\vec{A}^o + \vec{R}^{o'}.$$

To summarize, the mean values evolve according to the approximate nonlinear differential equations. If the mean of the random forcing function is known, solutions to these equations can be evaluated. Once these are known the partial derivatives $a_{rp}(t)$ and $b_{rp}(t)$ may be determined. Hence the transformation matrix $T_A(t)$ is determined. This time varying transformation determines a transition matrix for the centered random variables. Thus the methods applied to Model 1, which were not restricted to time invariant transition matrices, can be applied to find the covariance field for the centered random variables. This covariance field is the same as the covariance field of the noncentered random variables.

Model 3, Solution using the Karhunen-Loève Expansion

The solutions for mean values and covariances in the last section were based upon the assumption that the forcing functions were dependent on the time variable but independent of the state vector $\vec{A}$. Recalling the vorticity equation, it is evident that a different but realistic assumption is that the forcing function $\vec{R}$ is dependent not only time but also on the state variables themselves. In this section, this assumption is made and the mean value solutions found.

Again, write the basic vorticity equation as

$$\frac{dA_k}{dk} = f_k(t, A_1 A_2 \ldots A_h, R_1 \ldots R_h)$$

$$k = 1, 2, \ldots, h.$$

Now assume

$$R_s = R_s(t, A_1 \ldots A_h),$$

that is, the forcing function depends on the state vector $\vec{A}$. The generalized Karhunen-Loève Theorem allows an expansion of the forcing function

$$R_s = m_s^R + \sum_\nu V_\nu \Phi_{\nu s}$$

where

$$m_s^R = m_s^R(t, A_1 \ldots A_h)$$

and

$$\Phi_{\nu s} = \Phi_{\nu s}(t, A_1 \ldots A_h).$$

The functions $\Phi_{\nu s}$ will be referred to as coordinate functions. Recall that the random variables V_ν are uncorrelated and have zero means. Substituting the expansion form of R_s in the vorticity equation makes the right side depend on $t, A_1 \ldots A_h$ and the random variables V. Integration of the differential equations, now results in expressions for $A_1, \ldots, A_h$ in terms of t and the random parameters $V_1, \ldots, V_h$. Denote the results of such an integration as

$$A_k = \Lambda_k(t, V_1 V_2 \ldots)$$

which represents the state variables as functions of the V_ν's and t.

Now expand the functions Λ_k in a Taylor series as was done in the last section and disregard the remainder term. In this case, as before, the expansion is made around the mean of the uncorrelated random variables V_ν. A critical assumption here is that the approximation made by disregarding the remainder term is valid; that is, that the deviations of the random variables V_ν are sufficiently small that the linear approximation to Λ_k is adequate in the neighborhood of the means.

Since

$$E(V_\nu) = 0 \quad \text{for all } \nu \text{ and } t$$

$$E(A_k) \cong \Lambda_k(m_k^V, \ldots, m_h^V),$$

an approximate Karhunen-Loève expansion for A_k, based on the Taylor expansion, is

$$A_k \cong \Lambda_k(m_1^V \ldots m_h^V) + \sum_\nu \left.\frac{\partial \Lambda_k}{\partial V_\nu}\right|_{V_\nu = 0}$$

$$A_k(t) \cong E[A_k(t)] + \sum_\nu V_\nu a_{\nu k}(t)$$

$$A_k(t) \cong m_k^A(t) + \sum_\nu V_\nu a_{\nu k}$$

In terms of the centered random variables,

$$A_k^o(t) = A_k(t) - m_k^A(t) \cong \sum_{\nu=1} V_\nu a_{\nu k}(t).$$

The last equation says that once the mean value functions are evaluated, the centered random variables evolve according to a time varying linear system. Thus to determine the evolution of the variance-covariance functions it will be necessary to determine the evolution of the functions $a_{\nu k}(t)$ as well as the mean value function.

To determine the equations for the evolution of the means, substitute the Karhunen-Loève expansions in the basic spectral vorticity equation

$$A_k(t) \cong m_k^A(t) + \sum_\nu V_\nu a_{\nu k}(t)$$

$$R_k(t) = m_k^R(t) + \sum_\nu V_\nu \Phi_{\nu s}(t)$$

$$\frac{d(m_k^A(t) + \sum_\nu V_\nu a_{k\nu}(t))}{dt} \cong f_k(t, m_1^A(t) + \sum_\nu V_\nu a_{k\nu}(t), \ldots, m_n^A(t) + \sum_\nu V_\nu a_{h\nu},$$

$$m_1^R(t) + \sum_\nu V_\nu \Phi_{\nu 1}(t), \ldots, m_h^R(t) + \sum_\nu V_\nu a_{h\nu}) .$$

This holds uniformly for all values of the expansion parameters V_ν, in particular for $V_\nu = 0$ for all ν. Therefore, the equations reduce in this case to

$$\frac{dm_k^A(t)}{dt} = f_k[t, m_1^A(t) \ldots m_h^A(t), m_1^R(t), \ldots, m_h^R(t)]$$

$$m^R(t) = m^R(t, m_1^Y, \ldots, m_h^Y) .$$

Based upon the values of the mean of the state variables, the mean of the forcing function is determined by statistical regression techniques and the above tendencies evaluated.

The determination of the variance field is considerably more complicated and has not yet been done.

Summary

In the last section, three models for the evolution of the 500 mb constant pressure surface were given. Mean value solutions were desired for all three and the covariance derived for the first two. In each case, it was assumed that the forcing functions and covariance functions were explicitly known. This is not the case. Further, the fields are observed only periodically. To make use of the models it will be necessary to convert them to discrete time models and to apply statistical estimation techniques. This will be done in Chapter V.

V. STATISTICAL AND NUMERICAL TECHNIQUES

Introduction

In the last chapter probabilistic models for the evolution of the 500 mb surface were derived. These models assumed complete knowledge of the random forcing function process. Of course such knowledge is not available from physical considerations. It is therefore necessary to statistically estimate the effect of the forcing functions. In addition, the height field is observed only periodically. Hence, the continuous models must be converted to discrete time models. These factors are the considerations of this chapter.

Random Sequence Models

Model 1

Recall the linear model, based upon linearization of the meteorological equations had the general form

$$\frac{d\vec{A}}{dt} = T_A \vec{A} + \vec{R}$$

where T_A is a time invariant matrix. The entries on the matrix T_A are determined by the zonal wind profile at the initial time. It is quite likely that the zonal winds, and hence T_A, can be considered invariant for short periods, say 24 hours. However, for longer periods, this assumption may not be good. Consequently, it will be desirable to allow for T_A to change from time to time. The period over which it may be considered constant can be determined by numerical experimentation. In the methods of this chapter, T_A will be allowed to vary thus making the form of the equation

$$\frac{d\vec{A}}{dt} = T_A(t)\vec{A} + \vec{R} \tag{5.1}$$

where for Model 1, $T_A(t)$ is assumed piecewise constant.

Corresponding to the matrix $T_A(t)$ is the transition matrix $S_A(t, t_o)$ which allows the equivalent form of the above random differential equation

$$\vec{A}(t) = S_A(t, t_o)\vec{A}(t_o) + \int_{t_0}^{t} S_A(t, s)\vec{R}(s)ds. \tag{5. 2}$$

The transition matrix satisfies the differential equation

$$\frac{dS_A(t, t_o)}{dt} = T_A S_A(t, t_o)$$

$$S_A(t, t) = I.$$

Since $S_A(t, t)$ will have to be evaluated numerically it is worthwhile to examine briefly some characteristics of the transition matrix $S_A(t, t_o)$. If $S_A(t, t_o)$ satisfies the last conditions, then a solution of the form

$$\vec{A}(t) = S_A(t, t_o)\vec{A}(t_o) \tag{5.3}$$

satisfies the homogeneous differential equation

$$\frac{d\vec{A}}{dt} = T_A(t)\vec{A}.$$

Using (5. 3) it is easy to see that

$$d\vec{A}(t) = S_A(t, t_o)dA(t_o) \tag{5.4}$$

In differential form, $dA_i(t)$ may be written as

$$dA_i(t) = \sum_{j=1}^{h} \frac{\partial A_j(t)}{\partial A_i(t_o)} dA_j(t_o) \qquad i = 1, \ldots, h \tag{5.5}$$

Comparing these last two equations, it is clear that

$$S_{ij}(t, t_o) = \frac{\partial A_j(t)}{\partial A_i(t_o)} \qquad i, j = 1, 2, \ldots, h$$

In approximate form

$$S_{ij}(t, t_o) \cong \frac{\Delta A_j(t)}{\Delta A_i(t_o)} .$$

which is useful numerically.

The 500 mb observations occur at 12 hour intervals. Consider the sequence of equally spaced observation times as $\ldots t_{-1}, t_o, t_1 \ldots$. Then from (5. 2) it can be seen that

$$\vec{A}(t_{k+1}) = S_A(t_{k+1}, t_k)\vec{A}(t_k) + \int_{t_k}^{t_{k+1}} S_A(t_{k+1}, s)\vec{R}(s)ds.$$

Define

$$\vec{U}(k+1) = \int_{t_k}^{t_{k+1}} S_A(t_{k+1}, s)\vec{R}(s)ds.$$

Now make a simple change in notation

$$S_A(k) = S_A(t_{k+1}, t_k)$$

$$A(t_k) = A(k) \qquad k = 0, \pm 1, \pm 2, \ldots$$

The following difference equation then results

$$\vec{A}(k+1) = S_A(k)\vec{A}(k) + \vec{U}(k+1) \tag{5. 6}$$

which is the basic discrete equation.

If the forcing sequence $\{\vec{U}(k)\}$ is known, then Equation (5. 6) determines the spectral coefficients uniquely. Since this is not the case, statistical estimation must be used. It is known that the least squares estimate of $\vec{U}(k)$, $k > 0$ based on the realizations $\vec{U}(k)$, $k \leq 0$ is given by the conditional expectation of $\vec{U}(k)$, $k > 0$, conditioned on the realizations. Applying this conditional expectation operator to Equation (5. 6) results in

$$E(\vec{A}(k+1)\,|\,\vec{U}(s),\ s \le 0) = S_A(k)E(\vec{A}(k)\,|\,\vec{U}(s),\ s \le 0)$$

$$+\ E(U(k)\,|\,U(s),\ s \le 0)$$

or, changing to a more convenient notation

$$\vec{m}^A(k+1) = S_A(k)\vec{m}^A(k) + \vec{m}^U(k+1)$$

which describes the evolution of the conditional mean of the process in discrete form. Notationally, $k \le 0$ are observation times.

Using the discrete form above, if estimates $\hat{\vec{m}}^U(k)$ can be found, then

$$\hat{\vec{m}}^A(k+1) = S_A(k)\hat{\vec{m}}^A(h) + \hat{\vec{m}}^U(k+1) \tag{5.7}$$

describes the evolution of the system. An excellent mehtod of estimating $\vec{m}^U$ has been developed by Jones (1964) for vector random process in terms of an autoregressive model. Autoregression models are natural in problems of estimating conditional expectations.

Given the evolution of the spectral coefficients $\vec{A}$ described by the above equation and statistical estimates of the conditional mean of the random forcing vectors, the estimates of the covariance matrix may be found. Again considering the centered random variables $\vec{A}^o(t)$, the equation for their evolution can be found:

$$\vec{A}^o(k+1) = S_A(k)\vec{A}^o(k) + \vec{U}^o(k+1) \tag{5.8}$$

where $\vec{U}^o(k)$ has an analogous relation to $\vec{U}(k)$ in terms of the centered random variables $\vec{R}^o(t)$. Now

$$\text{Cov}\,[\vec{A}(k+1)] = E[\vec{A}(k+1)-\vec{m}_A(k+1)][\vec{A}(k+1)-\vec{m}_A(k+1)]'$$
$$= E[\vec{A}^o(k+1)\vec{A}^{o\,\prime}(k+1)]$$

Using Equation (5. 8)

$$\text{Cov}\,[\vec{A}(k+1)] = E[S_A(k)\vec{A}^o(k) + \vec{U}^o(k+1)][S_A(k)\vec{A}^o(k) + \vec{U}(k+1)]'$$
$$= E[S_A(k)\vec{A}^o(k) + \vec{U}^o(k+1)][\vec{A}^{o\,\prime}(k)S_A'(k) + \vec{U}'(k+1)]$$
$$= S_A(k)E[\vec{A}^o(k)\vec{A}^{o\,\prime}(k)]S_A'(k) + S_A(k)E[\vec{A}^o(k)\vec{U}^{o\,\prime}(k+1)]$$
$$+ E[\vec{U}^o(k+1)\vec{A}^{o\,\prime}(k)]S_A'(k) + E[\vec{U}^o(k+1)\vec{U}^{o\,\prime}(k+1)]. \tag{5. 9}$$

In Chapter IV, it was assumed that the random forcing function was not dependent on the state variable at any particular time point. For this reason, the center terms in the above expansion are zero. Replacing the expectations by covariances,

$$\text{Cov}\,[\vec{A}(k+1)] = S_A(k)\,\text{Cov}\,[\vec{A}(k)]S_A'(k) + \text{Cov}\,[\vec{U}(k+1)] \tag{5. 10}$$

which describes the evolution of the covariance field for this model.

The actual covariance matrices are not known and must be estimated. In the process of fitting the autoregressive scheme to obtain conditional means for Equation (5. 10) using Jones (1964), estimates of the k-step prediction covariance for $\{\vec{U}(k)\}$ are obtained.

When these are substituted into the last equation, the covariance estimates become

$$\hat{\text{Cov}}\,[\vec{A}(k+1)] = S_A(k)\,\hat{\text{Cov}}\,[\vec{A}(k)]S'_A(k) + \hat{\text{Cov}}\,[\vec{U}(k+1)]$$

A discussion of the steps necessary to obtain a forecast using these expressions is given in the Forecasting section of this chapter.

Model 2

In Model 2, the equation for the evolution of the mean values is nonlinear and thus a somewhat more complicated scheme is necessary to evaluate the mean value field. Recalling the equation has the form

$$\frac{dm_r^A}{dt} \cong f_r(t, m_1^A \ldots m_h^A, m_1^R \ldots m_h^R)$$

where f_r is a nonlinear expression. Assuming that the forcing function means were known, a reasonable numerical integration scheme for the nonlinear equations is

$$m_r^A(t+\Delta t) \cong m_r^A(t) + \Delta t \frac{dm_r^A(t)}{dt}$$

where the tendencies $\frac{dm_r^A}{dt}$ are found from the basic differential equations, so that

$$m_r^A(t+\Delta t) \cong m_r^A(t) + \Delta t\, f_r(m_1^A(t) \ldots m_h^A(t), m_1^R(t) \ldots m_h^R(t))$$

where Δt is the integration increment, which is chosen small enough to insure numerical stability.

Recall that the forcing function $\vec{R}$ is additive and the nonlinearity is confined to the quadratic terms of the tendency equation involving the spectral coefficients $\vec{A}$. Thus the last equation can be written

$$m_r^A(t+\Delta t) \cong m_r^A(t) + \Delta t\, g_r(m_1^A \ldots m_h^A) + m_r^R(t)\Delta t \qquad (5.11)$$

Because of numerical integration problems, Δt must be not more than three hours. On the other hand, observations are taken at 12 hour intervals. Because observations are taken at the longer intervals, estimates of the forcing function are also given for 12 hour intervals. Therefore, the above equation will be approximated by

$$\vec{m}^A(k+1) \cong N_A(\vec{m}^A(k)) + \vec{m}^U(k+1)$$

where N_A is the nonlinear operator corresponding to the first two terms and

$$m_r^U(k+1) = \int_{t_k}^{t_{k+1}} m_r^R(t)dt$$

is the effect of the random forcing function. If $\vec{m}^U$ is estimated by the autoregressive model, the estimation equation becomes

$$\hat{\vec{m}}^A(k+1) = N_A(\hat{\vec{m}}^A(k)) + \hat{\vec{m}}^U(k+1)$$

which is the same as the estimation equation for the linear model except for replacement of the linear transition matrix with a nonlinear operator.

To determine the evolution of the variance field, it is necessary to know the evolution of the mean field. Once this is known, the transformation matrix $T_A(t)$ is known. Through solution of the equation

$$\frac{d}{dt}S_A(t_o, t) = T_A(t)S_A(t_o, t)$$

$$S_A(t_o, t) = I,$$

by numerical means, the transition matrix S_A becomes known for the centered random variables of Model 2. The discrete equation for evolution of the centered random vector is then

$$\vec{A}^o(k+1) \cong S_A(k)\vec{A}^o(k) + \vec{U}^o(k+1)$$

which is identical to that of Model 1 variance field evolution apply.

The equation above is approximate because the Taylor series approximations (which makes linearity approximate) and because T_A

is known only from estimates of partial derivatives at times $k > 0$.

Following the arguments in Model 1 for the covariance function, the covariance estimates are

$$\text{Cov}\,(\vec{A}^{o}(k)) = \text{Cov}\,(\vec{A}(k))$$

$$\text{Cov}\,(\vec{U}^{o}(k)) = \text{Cov}\,(\vec{U}(k))$$

and

$$\widehat{\text{Cov}}\,(\vec{A}(k+1)) = S_A(k)\,\widehat{\text{Cov}}\,(\vec{A}(k))S'_A(k) + \widehat{\text{Cov}}\,(\vec{U}(k+1))$$

The covariance estimate of $\vec{U}(k)$ is the k-step prediction covariance determined in the process of fiting the autoregressive scheme. As with Model 1, the cross covariance of $\vec{U}(k+1)$ and $\vec{A}(k)$ have been assumed zero for each k.

Model 3

The method of solution for the conditional mean is identical to that for Model 2. Since the equations for the underlying model of the variance field were not derived, no discrete equation is given.

The estimation of $\vec{U}^{o}$ in this model is considerably simpler than in Models 1 and 2. In this model, the forcing function is dependent on the state variable $\vec{A}$ but not on the past values. But both $\vec{A}$ and $\vec{U}$ are spectral coefficient vectors and hence the only non-zero correlation is with like components. Thus $U_j = f_j(A_j)$ is the regression equation, and simple regression estimation is used.

Forecasting

To achieve a forecast using the random forcing function models, the processing is divided into two phases. The first phase is the preprocessing necessary to set up the statistical autoregression model for the effect of the forcing function. The second phase processing makes the actual forecast. The steps necessary are outlined below.

Preprocessing

The preprocessing begins by selection of a period, say 30 days, prior to forecast time. The 60 field observations during this time are each converted to spectral form. The tendency equation for the model being used is set up at each observation time and a pure dynamic prediction made for the next 12 hours. A comparison is then made between the dynamically predicted coefficients and the observed coefficients. The difference between the predicted and observed coefficients, called the residual, is then stored and the process repeated at each observation time.

When using Model 1 or Model 2, the autoregressive scheme is then fitted to the residual coefficients. Jones' technique successively fits higher order regressions until it is found that the coefficient matrix for an additional term is effectively zero. In addition, the technique generates unbiased estimates of the one step prediction covariance matrix. Based on these, unbiased estimates of the k-step prediction covariance matrices are generated.

When using Model 3, simple regressions are set up to predict the residual based on the values of the corresponding spectral coefficient.

Prediction

For the first 12 hour forecast, a dynamic prediction of the spectral coefficients is made using Model 1 or Model 2 based on observed data at the time the forecast is made. Using the autoregression statistical model, a one step prediction of the residual is made. The one step statistical prediction is then added to the dynamic prediction as a statistical correction. These coefficients are then used to generate a 12 hour predicted 500 mb height chart. The one step prediction covariance matrix is used to generate a variance or standard deviation chart. In this case, the first step, it is assumed that the height field at forecast time is observed with no error. Having produced the 12 hour forecast, this forecast is used as initial conditions for the next step. The above process is repeated. A dynamic forecase and a two step statistical prediction of the residual are made and these are combined. The initial conditions for this second stage forecast are the first stage combined forecast and thus are random. Hence the covariance of the second stage forecast, as noted, in each of the covariance prediction equations is the sum of the two stage autoregression prediction covariance and the term accounting for the random output of the first stage prediction. A variance field can be constructed and a chart made up. The process is then repeated for as many stages as needed or until the uncertainty reflected in the variance field makes the forecast of doubtful use.

The prediction of the mean field for Model 3 is the same as for Model 2 except that residuals are predicted by regression at individual times.

VI. CONCLUSIONS AND FURTHER RESEARCH

Summary and Conclusions

In the preceding two chapters, the development of statistical dynamic models for preodiction of the 500 mb pressure surface over the northern hemisphere was carried out. The primary objective was to demonstrate a reasonable approach to combined statistical-dynamic techniques for forecasting such a pressure surface. An examination of the literature has shown that such techniques, when properly implemented on a digital computer, will significantly improve forecasts based on the nondivergent barotropic dynamic model. These methods are adequately general to handle prediction of other fields as well. Further, they suggest techniques for extended range forecasting which may be a significant improvement over present methods.

Further Research

The material presented in Chapters III, IV and V has laid the foundation and suggests avenues of further work. These may be outlined in two areas--numerical experimentation and theoretical work on further models.

Theoretical Research

(1) The derivations of the nonlinear combined model are not complete for Model 3. In this case, no equations for the evolution of the variance field were derived. This should be completed and the corresponding discrete time model derived.

(2) In the models developed, each observation was assumed perfect. While this is done in current meteorological forecasting, it is unrealistic. Kalman models have been developed to account for observation error but are still to restrictive to be directly applied to the problems of this study. Research in Kalman models continues and should be closely followed for possible application to forecasting problems.

(3) The techniques developed in this study were based on general equations in order to allow application to baroclinic or multilevel dynamic models. Probably the first research in this direction should be on two parameter or two level models. There are many to choose from. The reasons for selecting one of these are simplicity of the models and, second, the controversy associated with them. A great deal of effort has been expended on perfecting two level models (Shuman and Hovermale, 1968) but without much success. The barotropic models do better. The reasons for the lack of success of these simple baroclinic models has

never been adequately explained. To improve the two level prediction by a combined statistical-dynamic model presents a definite challenge.

(4) There are several indications that the model based on linearization of the meteorological equations (Model 1) has greater utility than heretofore thought. These indications are:

(a) Haurwitz (1953) has derived a slightly more general model than that of Chapter III by linearization of the equations of motion and vorticity equation over the sphere. This corresponds closely to the dynamics of Model 1 above. For the same reasons recounted in Chapter III, combined statistical dynamic models based on this equation are natural especially in light of Jones' success in spherical harmonic prediction by statistical methods.

(b) Blinova (1943) used such a model for short term prediction by spherical harmonics with quite good results, thus indicating its short term utility.

(c) More important, perhaps, is comparison of the model with the latter data on 500 mb height analyzed by Eliasen and Machenhaur (1963). An example of the evolution of several spectral coefficients is shown in Figure 7 (Deland and Lin (1965) confirm their data). Of primary interest is the empirical evidence that the coefficients evolve in an almost

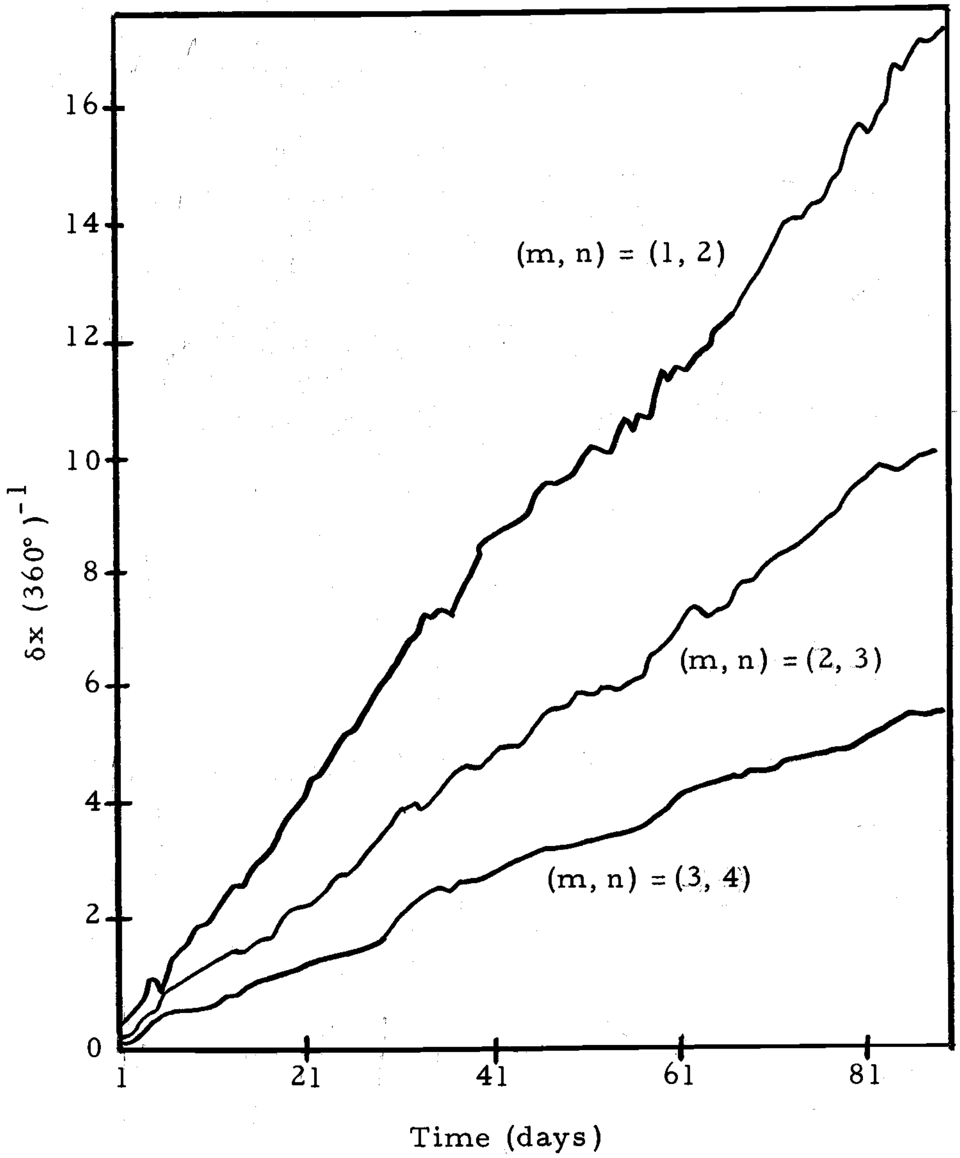

Figure 7. Values of phase angle for tendency fields for 500 mb surface; three harmonic components (after Eliasen and Machenhaur, 1963).

linear manner for remarkably long times. This opens the possibility of prediction of weekly means of 500 mb height 30 days in advance. Combined with prediction of thickness and temperature by an advective model, the possibility of extended range precipitation predictions emerges. Such prediction could be made by regression on the prediction mean fields (Cohen and Jones, 1967).

(5) An area of strong contemporary research is in diffusion models. Consider the stochastic differential equation

$$dx_n(t) = m[x_n(t), t]dt + \sigma[x_n(t), t]dy_n(t)$$

where this equation means

$$x_n(t) = x_n(t_o) + \int_{t_0}^{t} m[x_n(s), s]dx + \int_{t_0}^{t} \sigma[x_n(s), s]dy_n(s)$$

Under certain conditions $x_n(t)$ may be approximated in mean square by a diffusion process $x(t)$ (continuous Marhov process) with $x(t)$ defined by the equation

$$dx(t) = m[x(t), t]dt + \frac{1}{2}\sigma[x(t), t]\frac{\partial\sigma[x(t), t]}{\partial x(t)}dt + \sigma(x(t), t)dy(t)$$

where $y(t)$ is a Brownian motion process (Wong and Zakai, 1965). A similar result holds for vector process (Clark, 1967).

It is easy to see the utility of such a representation. First, by similar techniques to those used in Chapter IV, the conditional mean value could be found. It seems evident that the form of the estimation problem for the mean value would be the same as in the above models. The advantage of the diffusion process representation is the existence of the Fokker-Plank and Kolomogorov equations which describe the evolution of the distribution function.

Numerical Research

(1) Each of the models suggested above should be programmed and their prediction skill evaluated. The skill should be compared with the corresponding model without statistical correction.

(2) Extended range forecasting with the linear model should be checked. In addition, attempts to predict mean total precipitation for one week, 30 days in advance, should be tried.

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