Stochastic Collocation Error Analysis Within a Kinematic Magnetohydrodynamics Framework

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Abstract. We explore one numerical method for dealing with uncertainty quantification, stochastic collocation. We adapt this method for the uncertain kinematic magnetohydrodynamic system. We then demonstrate well-posedness of the uncertain forward problem. We also describe the method in detail, and perform an error analysis of the method, describing the necessary assumptions and regularity properties required of the parameters for convergence of the method.

1 Introduction

Magnetohydrodynamics (MHD) is the study of an electrically-conductive medium flowing through a magnetic field [13]. It is a multi-physics problem, governing the behavior of fluid flow, electric fields and currents, magnetic fields, and their interactions. MHD has applications in many different areas of study, such as astrophysics [15], medicine [9] and power generation [13].

These different applications operate under separate conditions, and thus varying assumptions are made on the system. We focus on the power-generation capabilities of MHD, which involves applying a large magnetic field to an electrically conductive plasma, artificially creating an electric field [13]. Here, we prescribe the fluid-flow, and focus on the electromagnetic behavior of the MHD system, also called the kinematic MHD equations. This greatly simplifies the necessary computational load, and provides an appropriate representation of an MHD generator, as MHD flows have been studied extensively [1] [8].

MHD power generation has been studied previously. However, they have recently become a viable source of power, due to improvements on the production of stronger magnets. This has led to renewed interest in MHD generators for many applications. Optimization of these generators is therefore paramount, especially in applications where size limitations are critical, such as in rockets or naval vessels. As with all generators, to implement some form of optimal design, the numerical model must first be validated. The numerical validation of the deterministic model is

done via implemention in COMSOL [2] and can be seen in [12]. We thus consider the feasibility of real-time optimization of an operating MHD generator. By prescribing the parameters, we have introduced aleatoric uncertainty into the fluid-flow, conductivity, electron mobility, and ion-mobility. Furthermore, we hope to estimate these parameters in real-time, and thus will recover an estimated set of values for given measurements. Therefore, the optimal design of the generator must consider uncertainty, with each random parameter being governed by some probability space and estimated distribution. We establish well-posedness of the forward problem, building on work seen in [12], and then propose a non-intrusive approach to quantifying the uncertainty in the forward problem. This will be incorporated into to an optimization under uncertainty problem in future work.

There are many different approaches for dealing with uncertainty in a system in general. Although simplistic for implementation, other methods, such as Monte Carlo, come with a high computational burden [7]. The method we utilize lessens the computational burden both through the choice of points in which we sample the domain, and the way in which we choose to average the solutions. In the following work, we implement an approximation method called *stochastic collocation* (SC) [14]. Utilizing a sparse grid with SC can result in a diminishing of the 'curse of dimensionality' that often adversely affects uncertainty quantification, while maintaining a high degree of accuracy [5], which will allow the utilization of the model in the inverse problem under uncertainty.

We begin by stating the well-posedness of the system under uncertainty. Following, we will go into detail regarding the SC method and the random grid. Finally, we conclude with an error analysis of the SC method, showing that the random error diminishes sub-exponentially. Numerical demonstration of the method can be seen in [11].

2 Well Posedness

First, note a notational choice made for this paper. Any mathematical object, function, parameter, etc., is a vector in $\mathbb{R}^3$ or a mapping to $\mathbb{R}^3$ if it is **bolded**. All other mappings will be defined as the paper proceeds.

Let the spatial domain for our system be given by $D \subset \mathbb{R}^3$, open with compact closure and denote the boundary as ∂D. We denote the random space $(\Omega, \mathcal{H}, p)$, where Ω is the set of outcomes, $\mathcal{H}$ is a given sigma algebra of events, and p is some continuous probability measure. As noted in the introduction, we work on the kinematic MHD system, with prescribed fluid-flow, $\mathbf{u}$. We assume that the induced magnetic field is negligible compared to the applied magnetic field [4], $\mathbf{B}$, which we also assign. Under the generator model, we then prescribe conductivity, σ, electron-mobility, μ_e, and ion-mobility, μ_i. Note that physically, we have that $mu_e, \mu_i > 0$ on their domains. We use the standard definition of the hall parameter and ion-slip parameter [13], e.g. for $\mathbf{x} \in D,\ \omega \in \Omega$

$$\beta_e(\mathbf{x}, \omega) = \mu_e(\mathbf{x}, \omega)||\mathbf{B}(\mathbf{x})||_{l^2}, \text{ and } \beta_i(\mathbf{x}, \omega) = \mu_e(\mathbf{x}, \omega)\mu_i(\mathbf{x}, \omega)||\mathbf{B}(\mathbf{x})||_{l^2}^2.$$

Finally, we use the conductivity tensor, as defined in [12] but adjusted to include the random domain, given by

$$\overline{\sigma}(\mathbf{x}, \omega) = \sigma(\mathbf{x}, \omega)\left(\mathcal{I} - \frac{\beta_e(\mathbf{x}, \omega)}{||\mathbf{B}||_{l^2}}[\mathbf{B}]_\times - \frac{\beta_i(\mathbf{x}, \omega)}{||\mathcal{B}||_{l^2}^2}[\mathbf{B}]_\times^2\right)^{-1},$$

where $\mathcal{I}$ denotes the identity matrix in $\mathbb{R}^{3\times 3}$ and $[\mathbf{B}]_\times$ is the matrix form of the cross-product. Invertibility of this matrix is guaranteed by the physical restriction $\mu_e, \mu_i > 0$. Under these definitions, the random MHD kinematic system is given by: find the induced current density, $\mathbf{J}_i : D \times \Gamma \to \mathbb{R}^3$, and electric potential, $\mathcal{V} : D \times \Omega \to \mathbb{R}$ that satisfy

$$\overline{\underline{\sigma}}^{-1}\mathbf{J}_i(\mathbf{x},\omega) - \nabla\mathcal{V}(\mathbf{x},\omega) = 0 \ \forall \mathbf{x} \in D, \ p.a.e. \ \omega \in \Omega, \tag{1a}$$

$$-\nabla \cdot \mathbf{J}_i(\mathbf{x},\omega) = \nabla \cdot \big(\overline{\underline{\sigma}}(\mathbf{x},\sigma)(\mathbf{u}(\mathbf{x},\sigma) \times \mathbf{B}(\mathbf{x}))\big) \ \forall \mathbf{x} \in D, \ p.a.e. \ \omega \in \Omega, \tag{1b}$$

with boundary conditions

$$\mathbf{J}_i \cdot \mathbf{n} = -\big(\overline{\underline{\sigma}}(\mathbf{u} \times \mathbf{B})\big) \cdot \mathbf{n} \text{ and } \mathrm{tr}(\mathcal{V}) = 0 \text{ on } \partial D, \ p.a.e. \ \omega \in \Omega,$$

where n is a unit-vector normal to the boundary of D. We denote the system (1) the *random strong form* of the kinematic MHD governing equations. We establish well-posedness by following a similar approach to what was seen in [12], converting (1) into a weak form, then subsequently an operator form, and finally applying the Babuska-Brezzi-Kovalevskaya (BBK) theorem [6] for existence and uniqueness of solutions. Continuous dependence on the parameters follows immediately from continuity of integration, the dependence shown in [12], and the continuity of the composure of two continuous functions.

We now define a random function space. We only seek solutions that are square-integrable with respect to the random space. Notationally, this space is given by

$$L_{2,p}(\Omega) := \Big\{ f : \Omega \to \mathbb{R} \ \Big| \ \int_\Omega f^2(\omega)p(\omega) \ d\omega < \infty \Big\}$$

with norm

$$||f||_{L_{2,p}} := \left(\int_\Omega f^2(\omega)p(\omega) \ d\omega \right)^{1/2}$$

We also define the deterministic solution spaces for $\mathbf{J}_i$ and $\mathcal{V}$ respectively, i.e.

$$V(D) := \Big\{ \mathbf{f} \in (L^2(D))^3 : \mathbf{f}\cdot n = -\big(\overline{\underline{\sigma}}(\mathbf{u}\times\mathbf{B})\big)\cdot\mathbf{n} \Big\}, \quad W(D) := W_0^{1,2}(D) = \Big\{ f \in H^1(D) : T(f) = 0 \Big\}.$$

where $\mathbf{n}$ is a vector normal to the surface of D, and $T(f)$ is the trace of f on D.

Taking the tensor product between these two, we define the *random solution space* for $\mathbf{J}_i$ as

$$\overline{V} := V(D) \times L_{2,p}(\Omega) = \{ f : D \times \Omega \to \mathbb{R}^3 \ \Big| \ f(\cdot, y) \in \big(L_{2,p}(\Omega)\big)^3, \ f(\mathbf{x}, \cdot) \in V \}.$$

Similarly, define $\overline{W} := W(D) \times L_{2,p}(\Omega)$ as the random solution space for $\mathcal{V}$. We define the norm on $\overline{V}$ as the averaging norm, i.e.

$$||\mathbf{F}||_{\overline{V}} = \Big(\mathbb{E}\Big[||\mathbf{F}||_V^2 \Big] \Big)^{1/2}$$

and similarly define the norm on $\overline{W}$ to be the averaging norm using $||\cdot||_W$. Given that V, W are both Hilbert spaces, as showin in [12], it follows that $\overline{V}, \overline{W}$ are also Hilbert spaces. We also define the expected value of a real-valued function to be

$$\mathbb{E}[f] := \int_D f(\omega)p(\omega) \ d\omega.$$

Multiplying (1) by appropriate test functions and integrating, the spatially-weak form of the system is given by: find $\mathbf{J}_i \in \overline{V}, \mathcal{V} \in \overline{W}$ that satisfy

$$\mathbb{E}\left[\int_D \underline{\overline{\sigma}}^{-1}\mathbf{J}\cdot\phi\right] - \mathbb{E}\left[\int_D \nabla\mathcal{V}\cdot\phi\right] = 0 \ \forall\phi \in \widetilde{V}, \tag{2a}$$

$$-\mathbb{E}\left[\int_D \mathbf{J}_i\cdot\nabla\psi\right] = \mathbb{E}\left[\int_D \big(\underline{\overline{\sigma}}(\mathbf{u}\times\mathbf{B})\big)\cdot\nabla\psi\right] \ \forall\psi\in\widetilde{W}. \tag{2b}$$

We denote (2) the *random weak form*. Defining the bilinear operator $\overline{\mathcal{A}} : \overline{V}\times\overline{V}\to\mathbb{R}$ as

$$\overline{\mathcal{A}}(\mathbf{F})(\mathbf{G}) = \mathbb{E}\left[\int_D \underline{\overline{\sigma}}^{-1}\mathbf{F}\cdot\mathbf{G}\right].$$

Similarly, we define $\overline{\mathcal{B}} : \overline{V}\times\overline{W}\to\mathbb{R}$ as

$$\overline{\mathcal{B}}(\mathbf{F})(g) = -\mathbb{E}\left[\int_D \mathbf{F}\cdot\nabla g\right].$$

Finally, define

$$\overline{G}(\psi) = \mathbb{E}\left[\int_D \big(\underline{\overline{\sigma}}(\mathbf{u}\times\mathbf{B})\big)\cdot\nabla\psi\right].$$

Using these, the *random operator form* is given: find $\mathbf{J}_i \in \overline{V}, \mathcal{V}\in\overline{W}$ that satisfy

$$\overline{\mathcal{A}}(\mathbf{J}_i) + \overline{\mathcal{B}}'(\mathcal{V}) = 0 \ \in \overline{V}', \tag{3a}$$

$$\overline{\mathcal{B}}(\mathbf{J}_i) = \overline{G} \ \in \overline{W}'. \tag{3b}$$

With this operator form defined, we now present a theorem stating that the system (3) is well-posed.

Theorem: 1. *Given $\boldsymbol{u}, \boldsymbol{B} \in L_{2,p}(\Omega)\times(L^2(D))^3$, bounded, and $\sigma, \beta_e, \beta_i \in L_{2,p}(\Omega)\times L^\infty_+(D)$, i.e. for each $\omega\in\Omega$, assume σ, β_e, and β_i are positive and bounded. Then there exist unique solutions, $\boldsymbol{J}_i, \mathcal{V}$, to*

$$\overline{\mathcal{A}}(\boldsymbol{J}_i) + \overline{\mathcal{B}}'(\mathcal{V}) = F_1 \in \overline{V}', \tag{4a}$$

$$\overline{\mathcal{B}}(\boldsymbol{J}_i) = F_2 \in \overline{W}', \tag{4b}$$

which obey the following a priori estimates:

$$||\boldsymbol{J}_i||_{\overline{V}} \le ||F_1||_{\overline{V}'} + \frac{1}{b}\Big(||\overline{\mathcal{A}}||_{\mathcal{L}(\overline{V},\overline{V}')} + 1\Big)||F_2||_{\overline{W}'}, \tag{5}$$

$$||\mathcal{V}||_{\overline{W}} \le \frac{1}{b}\Big(||F_1||_{\overline{V}'} + ||\overline{\mathcal{A}}||_{\mathcal{L}(\overline{V},\overline{V}')}||F_2||_{\overline{V}'}\Big), \tag{6}$$

where b is the bounding constant for $\overline{\mathcal{B}}$, i.e., $b = (1 + C_{p.f.})^{1/2}$.

The proof of this theorem follows immediately from [12], with appropriate extensions to the arguments regarding the coercivity and bounding constants of the operators, and the fact that, as V, W are Hilbert spaces [12], $\overline{V}, \overline{W}$ must be as well. Note now that letting $F_1 = 0,\ F_2 = G$ implies well-posedness of our system. The bounds on the solutions will prove vital in the error analysis of our numerical method.

3 Stochastic Collocation

We now turn to a practical approach of solving (2). We seek a numerical approximation to $\mathbf{J}_i, \mathcal{V}$ in some finite-dimensional approximation of the solution spaces. We do so by implementing stochastic collocation, a numerical method used to approximate solutions to random PDEs.

As a method, stochastic collocation is similar to Monte Carlo. Both are non-intrusive and involve sampling the random space. Each uses these sampled values of random parameters to solve the deterministic form of (2), and approximate the desired moments by averaging the sampled solutions appropriately [14]. However, the choice of sampling of the random inputs is not done arbitrarily for stochastic collocation. Rather, we specifically choose to sample at the zeros of orthogonal polynomials of the random-space, and thus greatly reduce the number of samples for the same accuracy comparatively to Monte Carlo [7]. As well, stochastic collocation uses a weighted average over a numerical average, with weights corresponding from the choice of approximation to the random space, and subsequently the basis chosen to represent it. Under appropriate assumptions on the regularity of the system, this will result in sub-exponential convergence in the random direction, as seen in the error analysis of the method. With this in mind, we now describe in detail the collocation method, but first make a necessary assumption to apply the method.

3.1 Finite-Dimensional Noise

To use the stochastic collocation method, one must show that there are a finite number of random variables describing the noise [3]. One such way is to truncate a KL expansion of each random parameter. To this end, we assume that the only random variables in the system are real-valued random parameters describing the electron mobility, μ_e, ion-mobility, μ_i, conductivity, σ, and fluid-flow, $\mathbf{u}$, and furthermore, that each are described by a finite number of independent random variables. For notational simplicity, let the set of random variables be denoted $\{m_k,\ k = 1, \dots, M\}$. With a change of variables, one can transform all of the random variables into uniform random variables, and therefore we make the final assumption that $\forall k,\ m_k \sim U(0,1)$.

Now, let $\Gamma_k := m_k(\Omega)$, or the image of the events under the real-valued random variable, and define $\Gamma := \Pi_{k=1}^{M} \Gamma_k$, the tensor product of each Γ_k. Let ρ be the joint independent probability distribution for the random variables $[m_1, \dots, m_M]$, $\rho : \Gamma \to \mathbb{R}^+,\ \rho \in L^\infty(\Gamma)$. Thus, by the Doob-Dynkin's Lemma, [10], we have that the solutions $\mathbf{J}_i, \mathcal{V}$ can be described by a finite number of random variables as well, e.g.

$$\mathbf{J}_i(\mathbf{x}, \omega) = \mathbf{J}_i(\mathbf{x}, m_1(\omega), \dots, m_M(\omega))$$

and similarly for $\mathcal{V}$. Applying this idea to the above problem, we attempt to find $\mathbf{J}_i \in V \times L_{2,\rho}(\Gamma),\ \mathcal{V} \in W \times L_{2,\rho}(\Gamma)$ that satisfy

$$\int_D \overline{\sigma}^{-1} \mathbf{J}_i(\mathbf{x}, y) \cdot \phi(\mathbf{x}, y)\ d\mathbf{x}\ - \int_D \nabla \mathcal{V}(\mathbf{x}, y) \cdot \phi(\mathbf{x}, y)\ d\mathbf{x}\ = 0 \quad \forall \phi \in V, \text{ for } \rho.a.e.\ y \in \Gamma \tag{7a}$$

$$-\int_D \mathbf{J}_i(\mathbf{x}, y) \cdot \nabla \psi(\mathbf{x}, y)\ d\mathbf{x}\ = \int_D \overline{\sigma}(\mathbf{x}, y)\big(\mathbf{u}(\mathbf{x}, y) \times \mathbf{B}(\mathbf{x})\big) \cdot \nabla \psi(\mathbf{x}, y)\ d\mathbf{x} \quad \forall \psi \in W, \text{ for } \rho.a.e.\ y \in \Gamma \tag{7b}$$

For notational convenience, we now define $\widetilde{V} := V \times L_{2,\rho}(\Gamma)$ and $\widetilde{W} := W \times L_{2,\rho}(\Gamma)$ as the *new* random solution spaces for which we seek a numerical approximation. Note now that this is equivalent to (2), only with the alternative probability space (Γ, H, ρ).

We can now define the finite-dimensional (FD) random solution subspaces in which we search for our approximate solutions. We begin with the spatial dimension. Define $V_h \subset V$ to be the standard finite element approximation to $\mathcal{V}$, with quadratic polynomials, on some Delauney triangular prism mesh t_h with max side length h. Similarly define $W_h \subset W$ on the same mesh t_h. The meshes used for this problem have been described extensively in [12], and thus are not discussed further here. We now turn to the FD approximation to the random function space, $L^2_\rho(\Gamma)$. We do so iteratively, for each subspace Γ_k. For $k = 1, \ldots, M$, define $P_{N_k}(\Gamma_k) \subset L_{2,\rho}(\Gamma_k)$ as the span of all polynomials on Γ_k of degree up to N_k, for $N_k \in \mathbb{N}$. In each direction Γ_k, we choose a basis of orthogonal Legendre polynomials $\{r_k^j\}_{j=0}^{N_k-1}$ [14] that satisfy

$$\int_\Gamma r_k^j r_k^l \rho(y)\ dy = \delta_{jl}$$

where δ_{jl} is the Dirac delta function. Note that Legendre polynomials have well-documented zeroes and allows for the nesting of the zeroes at subsequent levels [5]. Define $N = [N_1, \ldots, N_M]$, as a set of orders for each dimension of Γ. Then we approximate our random $L_2(\Gamma)$ space with a tensor product of these polynomial spaces, i.e.

$$P_N(\Gamma) = \prod_{k=1}^{M} P_{N_k}(\Gamma_k).$$

Note that the dimension of P_N is $\prod_{k=1}^{M} N_k < \infty$. The FD approximation to the random solution spaces are thus given by:

$$\widetilde{V}_{h,N} := V_h \otimes P_\rho(\Gamma) \text{ and } \widetilde{W}_{h,N} := W_h \otimes P_N(\Gamma).$$

With these spaces defined, we can apply the stochastic collocation method. It is worth noting that the FD noise assumption is a crucial step in turning the stochastic system of equations into a deterministic one, and subsequently allowing the use of finite-element and finite-difference techniques in approximating the solutions [3]. It follows that this system is well-posed, by the continuity of the measure ρ, and the well-posedness of the equivalent form, as seen in Section 2.

3.2 Method

The SC method solves the deterministic system numerous times, whose solutions are then used to build an interpolate approximation to the solutions of (7). The deterministic system is given by: for fixed $y' \in \Gamma$, find $\mathbf{J}_i^h \in V_h, \mathcal{V}^h \in W_h$ such that

$$\int_D \underline{\overline{\sigma}}^{-1}(\mathbf{x}, y')\mathbf{J}_i^h(\mathbf{x}, y') \cdot \phi(\mathbf{x}, y^*)\ d\mathbf{x} - \int_D \nabla \mathcal{V}^h(\mathbf{x}, y') \cdot \phi(\mathbf{x}, y')\ d\mathbf{x} = 0 \quad \forall \phi \in V_h, \tag{8a}$$

$$-\int_D \mathbf{J}_i^h(\mathbf{x}, y') \cdot \nabla \psi(\mathbf{x}, y')\ d\mathbf{x} = \int_D \overline{\sigma}(\mathbf{x}, y')\big(\mathbf{u}(\mathbf{x}, y') \times \mathbf{B}(\mathbf{x})\big) \cdot \nabla \psi(\mathbf{x}, y')\ d\mathbf{x} \quad \forall \psi \in W_h. \tag{8b}$$

We now perform the *collocation*, i.e. the collecting of solutions sampled at the zeros $\{y_{k,l}^m\}$, $l = 1, \ldots, m$ of each polynomial r_k^m in each direction Γ_k and building a polynomial chaos interpolant. By using the Legendre polynomials, we are able to use the Clenshaw-Curtis (CC) or Fejér method of numerical quadrature, which guarantees **nesting** of the zeros in each random direction [14]. Furthermore, we reduce the number of points by constructing a Smolyak sparse grid [5]. Although there is not a closed-form method of giving the number of nodes required in each domain [7], we let $\widetilde{N}_k$ denote the total number of points in the Γ_k direction. A more detailed discussion of the construction of such grids can be seen in [5] or [7]. A representation of the difference between full and sparse CC or Fejér grids can be seen in [11].

With this in mind, we let $y_k^{m_k}$ for $m_k = 1, \ldots, \tilde{N}_k$, $k = 1, \ldots, N$ be the m^{th} unique zero in the direction Γ_k. To ease the notation, define $m = [m_1, \ldots, m_M]$ as an array of indices, and define $y_m = [y_1^{m_1}, \ldots, y_M^{m_M}]$ as a collection of zeroes in each random direction. Lastly, define the product of the polynomials of a given order in each direction as

$$r_m(y) = \prod_{j=1}^{M} r_j^{m_j}(y_j).$$

Thus, the polynomial chaos expansion of $\mathbf{J}_i$ is given by

$$\mathbf{J}_i^{h,N}(\mathbf{x}, y) = \sum_{m_1=1}^{\widetilde{N}_1} \cdots \sum_{m_M=1}^{\widetilde{N}_M} \mathbf{J}_i(\mathbf{x}, y_m) r_m(y) \tag{9}$$

and for $\mathcal{V}$,

$$\mathcal{V}^{h,N}(\mathbf{x}, y) = \sum_{m_1=1}^{\widetilde{N}_1} \cdots \sum_{m_M=1}^{\widetilde{N}_M} \mathcal{V}(\mathbf{x}, y_m) r_m(y) \tag{10}$$

Let $C^0(\Gamma)$ denote the set of continuous functions on Γ and we can define an interpolation operator, $\mathcal{I}_V^N : C^0(\Gamma) \times V(D) \to P_N(\Gamma) \times V(D)$ as, for $f \in C^0(\Gamma) \times V(D)$

$$\mathcal{I}_V^N(\mathbf{f}) := \sum_{m_1=1}^{\widetilde{N}_1} \cdots \sum_{m_M=1}^{\widetilde{N}_M} \mathbf{f}(\mathbf{x}, y_k) r_m(y). \tag{11}$$

This implies immediately that $\mathbf{J}_i^{h,N} = \mathcal{I}_V^N(\mathbf{J}_i)$, under appropriate assumptions. We similarly define $\mathcal{I}_W^N(g) : C^0(\Gamma) \times W(D) \to P_N(\Gamma) \times W(D)$ as, for $g \in C^0(\Gamma) \times W(D)$

$$\mathcal{I}_W^N(g) := \sum_{m_1=1}^{\widetilde{N}_1} \cdots \sum_{m_M=1}^{\widetilde{N}_M} g(\mathbf{x}, y_k) r_m(y). \tag{12}$$

and we have that $\mathcal{V}^{h,N} = \mathcal{I}_W^N(\mathcal{V})$, under appropriate assumptions. For both $\mathbf{J}_i$ and $\mathcal{V}$, these assumptions are discussed in the Section 3.3. Using the interpolation, we arrive at a deterministic form of estimating the expected values of the true solutions. Using Gaussian quadrature to approximate the integral yields

$$\mathbb{E}\left[\mathbf{J}_i^{h,N}\right] = \sum_{m_1=1}^{\widetilde{N}_1} \cdots \sum_{m_M=1}^{\widetilde{N}_M} w_m \mathbf{J}_i(\mathbf{x}, y_k), \quad \text{and} \quad \mathbb{E}\left[\mathcal{V}^{h,N}\right] = \sum_{m_1=1}^{\widetilde{N}_1} \cdots \sum_{m_M=1}^{\widetilde{N}_M} w_m \mathcal{V}(\mathbf{x}, y_k)$$

where $w_m := \prod_{j=1}^{M} w_j^{m_j}$ and $w_j^{m_j} := \int_{\Gamma_j} \left(r_j^{m_j}(y_j)\right)^2 \rho(y_j^{m_j})\, dy_j$, e.g. the weights of the polynomial in each direction.

3.3 Regularity Assumptions

Before going through the error analysis of the stochastic collocation method, we must first establish some regularity properties of the solutions. We do so through assumptions about the random parameters, and then show the implications on $\mathbf{J}_i, \mathcal{V}$. These results will prove necessary to guarantee convergence of the collocation method. For simplicity of discussing these assumptions, we define

$$\mathbf{g}(\mathbf{x}, y) := \overline{\underline{\sigma}}(\mathbf{x}, y)\big(\mathbf{u}(\mathbf{x}, y) \times \mathbf{B}(\mathbf{x})\big).$$

We begin with a lemma that will be used to show that the continuity of the random parameters with respect to y can be directly transferred to $\mathbf{J}_i, \mathcal{V}$, under suitable conditions.

Lemma 1. *Under the assumption that* $\mathbf{g} \in C^0(\Gamma) \times W$, $\mathbf{J}_i \in C^0(\Gamma) \times V$ *and* $\mathcal{V} \in C^0(\Gamma) \times W$.

The proof of this lemma follows immediately from the bounds given in Theorem 1. We thus turn to the more complex task of bounding the derivatives of $\mathbf{J}_i, \mathcal{V}$ in each random direction. For notational simplicity, let

$$\partial_n^k := \frac{\partial^k}{\partial y_n^k}$$

We now make another major assumption regarding the implementation of the SC method. We require that for the remainder of this paper, Γ_k is bounded for each k. [1] For the power-generation application, this assumption is rooted in the physical limitations of what the expected parameters values can take, and how regular their distributions are.

To bound the derivatives of the solutions, we will make use of the bounds in Theorem 1, and begin by taking the derivative of (8) with respect to one of the random directions. We also consider the deterministic form of (2), i.e. for fixed y, and let $\mathcal{A}, \mathcal{B}$ represent the deterministic equivalent operators of $\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}$ respectively. Through an iterative application of the product rule and solving for the desired operators on the left-hand side of the system, the differentiated solutions to (7) satisfy

$$\mathcal{A}(\partial_n^k \mathbf{J}_i) + \mathcal{B}'(\partial_n^k \mathcal{V}) = F^k \ \in V', \tag{13a}$$

$$\mathcal{B}(\partial_n^k \mathbf{J}_i) = G^k \ \in W'. \tag{13b}$$

Here,

$$F^k(\phi) := -\int_D \Big(\sum_{j=0}^{k-1} \binom{k}{j} \partial_n^{k-j}(\overline{\underline{\sigma}}^{-1})\ \partial_n^j(\mathbf{J}_i)\Big) \cdot \phi, \text{ and } G^k(\psi) := \int_D \partial_n^k \mathbf{g} \cdot \nabla\psi. \tag{14}$$

Note that it is obvious by their definition that $F^k \in V'$ and $G^k \in W'$ for all $k \in \mathbb{N}$, by construction. Through an application of Theorem 1, we have that the solutions $\partial_n^k \mathbf{J}_i, \partial_n^k \mathcal{V}$ exist, are unique, and obey the following bounds:

$$||\partial_n^k \mathbf{J}_i||_V \leq ||F^k||_{V'} + C||G^k||_{W'}, \tag{15}$$

[1] Should this be stated more obviously? Should we make the assumptions a bit more obvious?

$$||\partial_n^k \mathcal{V}||_W \leq \frac{1}{b}\Big(||F^k||_{V'} + ||A||_{\mathcal{L}(V,V')}||G^k||_{V'}\Big), \tag{16}$$

where $C = \frac{1}{ab}\Big(||\mathcal{A}||_{\mathcal{L}(V,V')} + a\Big)$. Using this system, we now place bounds on the derivatives of $\mathbf{J}_i, \mathcal{V}$ in the random direction that will guarantee the convergence of the SC method. We will need restrictive assumptions regarding the decay of the derivatives of $\underline{\sigma}^{-1}, \mathbf{g}$. These restrictions may differ in each random direction by the values of the constants only. Thus, for the purposes of the rest of this section, fix $n \in \{1, \ldots, M\}$, and thus fix our direction Γ_n. Dependence of a parameter on the direction is denoted through the subscript n. We now state the following regularity assumptions and the resulting regularity properties of $\mathbf{J}_i, \mathcal{V}$.

Lemma 2. *Assume that*

$$||\partial_n^k \underline{\sigma}^{-1}||_{V'} \leq \alpha_{1,k}\frac{k!}{\gamma_n^k},\ ||\partial_n^k \mathbf{g}||_W \leq \alpha_{2,k}\frac{k!}{\gamma_n^k} \quad \forall k \in \mathbb{N}$$

where $\gamma_n > 0$, and the other constants satisfy the inequality

$$\frac{\alpha_{2,k}}{||g||_W} + \Big(\alpha_{1,k} + \sum_{j=1}^{k-1} \alpha_{1,k-j}\Big) \leq 1.$$

Then

$$||\partial_n^k \mathbf{J}_i||_V \leq C_{1,k}\frac{k!}{\gamma_n^k},\ \textit{and}\ ||\partial_n^k \mathcal{V}||_V \leq C_{2,k}\frac{k!}{\gamma_n^k}$$

where

$$C_{1,k} \leq C||\mathbf{g}||_W := C_0,\ C_{2,k} \leq \frac{1}{b}||\mathbf{g}||_W\ \forall k \in \mathbb{N}$$

Proof. (By induction for $\mathbf{J}_i$) First, let $k = 1$. Then by (15), we have

$$||\partial \mathbf{J}_i||_V \leq ||F^1||_{V'} + C||G^1||_{V'}.$$

By definition of F^1, G^1, we have

$$||\partial_n \mathbf{J}_i||_V \leq ||\partial_n \underline{\sigma}^{-1}||_{V'}||\mathbf{J}_i||_V + C||\partial_n \mathbf{g}||_V.$$

By the bounds given in Theorem 1, we have

$$\begin{aligned} ||\partial_n \mathbf{J}_i||_V &\leq ||\partial_n \underline{\sigma}^{-1}||_{V'} C||\mathbf{g}||_W + C||\partial_n \mathbf{g}||_V \\ &\leq \alpha_{1,1} C||\mathbf{g}||_W + C\alpha_{2,1}. \end{aligned}$$

Letting $C_{1,k} = \alpha_{1,1}C||g||_W + C\alpha_{2,1}$ yields the first desired inequality. To see that $C_{1,k} \leq C_0$, consider that as $j = 1$, the sum in (15) is 0. Then we have

$$1 \geq \alpha_{1,1} + \frac{\alpha_{2,1}}{||\mathbf{g}||_W} \qquad \text{by assumption.}$$

Multiplying each side by C_0 gives

$$C_0 = C||g||_W \geq C\alpha_{1,1}||\mathbf{g}||_W + C\alpha_{2,1} = C_{1,k}$$

and the second desired inequality, the bounding constant inequality, is satisfied.

Now, let $k \in \mathbb{N}$, and assume that

$$||\partial^j \mathbf{J}_i||_V \leq C_{1,j} \frac{j!}{\alpha^j}$$

holds for some $C_{1,j} \leq C_0,\ j = 1, \ldots, k-1$. Again, by (15), we have

$$\begin{aligned} ||\partial_n^k \mathbf{J}_i||_V &\leq C||G^k||_{W'} + ||F^k||_{V'} \\ &\leq C||\partial_n^k \mathbf{g}||_W + \sum_{j=0}^{k-1} \binom{k}{j} ||\partial_n^{k-j} \overline{\sigma}^{-1}||_{V'} ||\partial^j \mathbf{J}_i||_V \end{aligned}$$

By the induction assumption, we have

$$||\partial_n^k \mathbf{J}_i||_V \leq C\alpha_{2,k} \frac{k!}{\gamma_n^k} + \left(\alpha_{1,k} \frac{k!}{\gamma_n^k} C_0 + \sum_{j=1}^{k-1} \binom{k}{j} \alpha_{1,k-j} \frac{(k-j)!}{\gamma_n^{k-j}} C_{1,j} \frac{j!}{\gamma_n^j} \right)$$

Here, we separated the $j = 0$ case from the rest, as these bounds stem from the deterministic inequality, and not the lemma assumptions. As well, note that $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. Thus, we have

$$\begin{aligned} ||\partial_n^k \mathbf{J}_i||_V &\leq C\alpha_{2,k} \frac{k!}{\gamma_n^k} + \left(\alpha_{1,k} \frac{k!}{\gamma_n^k} C_0 + \sum_{j=1}^{k-1} \frac{k!}{j!(k-j)!} \alpha_{1,k-j} \frac{(k-j)!}{\gamma_n^{k-j}} C_{1,j} \frac{j!}{\gamma_n^j} \right) \\ &= \frac{k!}{\gamma_n^k} \underbrace{\left(C\alpha_{2,k} + \left(\alpha_{1,k} C_0 + \sum_{j=0}^{k-1} \alpha_{1,k-j} C_j \right) \right)}_{C_{1,k}} \end{aligned}$$

which yields the first inequality. To see that $C_{1,k} < C_0$, consider that by assumption

$$1 \geq \frac{\alpha_{2,k}}{||\mathbf{g}||_W} + \Big(\alpha_{1,k} + \sum_{j=1}^{k-1} \alpha_{1,k-j} \Big).$$

Multiplying each side by C_0 yields

$$C_0 \geq C\alpha_{2,k} + \Big(C_0 \alpha_{1,k} + \sum_{j=1}^{k-1} C_0 \alpha_{1,k-j} \Big).$$

Using the induction assumption that $C_{1,j} \leq C_0$ yields

$$C_0 \geq C\alpha_{2,k} + \left(\alpha_{1,k} C_0 + \sum_{j=0}^{k-1} \alpha_{1,k-j} C_{1,j} \right) = C_{1,k}$$

and the second inequality follows. A similar proof will achieve the same result for $\mathcal{V}$ and the constants $C_{2,k}$. $\square$

Note that the constants $\alpha_{1,k}$ and $\alpha_{2,k}$ may also depend on n, but that dependence is neglected for notational sake. With these regularity assumptions, we can define an appropriate analytic extension to each function of $\mathbf{J}_i$ and $\mathcal{V}$.

Lemma 3. *Under the assumptions of Lemma 2,* $\mathbf{J}_i$ *and* $\mathcal{V}$ *admit analytic extensions in the region of the complex plane* $\Sigma(\Gamma_n, \gamma_n) := \{z = y_n + iy_2 \in \Gamma_n \times \mathbb{C} : y_n \in \Gamma_n \text{ and } |y_2| \leq a\}$.

Proof. We show this for $\mathcal{V}$ only, as a similar approach will show the same in any spatial direction x_j for $\mathbf{J}_{i,x_j}$. [2] Here, we define $y_n^* = \{y_k\}_{k \neq n}$ as a set of values in Γ_k for $k \neq N$. Define the extension of $\mathcal{V}$ on $\Sigma(\Gamma_n, \gamma_n)$ with an analytic power-series,

$$\mathcal{V}(z, y_n^*, x) = \sum_{k=0}^{\infty} \frac{(z - y_n)^k}{k!} \left|\left| \partial_n^k \mathcal{V}(y_n, y_n^*, \mathbf{x}) \right|\right|_W . \tag{17}$$

We now seek to show that this series converges. To this end, consider

$$\begin{aligned}
\mathcal{V}(z, y_n^*, x) &= \sum_{k=0}^{\infty} \frac{(z - y_n)^k}{k!} \left|\left| \partial_n^k \mathcal{V}(y_n, y_n^*, \mathbf{x}) \right|\right|_W \\
&\leq \sum_{k=0}^{\infty} \frac{(z - y_n)^k}{k!} C_{2,k} \frac{k!}{\gamma_n^k} && \text{by Lemma 2} \\
&\leq \frac{1}{b} ||\mathbf{g}||_W \sum_{k=0}^{\infty} \frac{(z - y_n)^k}{\gamma_n^k} && \text{as } C_{2,k} < \frac{1}{b} ||\mathbf{g}||_W \ \forall k.
\end{aligned}$$

This converges by a geometric series argument for $(z - y_n) < \gamma_n$, implying that the region of convergence for our series is given by the subspace $\Sigma(\Gamma_n, \gamma_n)$, yielding the desired result. □

Note that this implies for both $\mathbf{J}_i, \mathcal{V}$, the domain in which there exists an analytic extension is directly related to the decay of the derivatives of $\mathbf{g}, \overline{\sigma}^{-1}$ in the direction Γ_n, as this domain is given by, for each Γ_n, $\Sigma(\Gamma_n, \gamma_n)$. A faster decay of the derivatives results in a larger area in which the analytic extension exists. We now move onto the error analysis section.

4 Error Analysis

Prior to proving the convergence of the SC method, we present two crucial lemmas necessary for bounding the random approximation error. These are as given in Babuska [3], and pertain to bounding 1-D random polynomial interpolation. They will be then adapted to bound the interpolation error on the entire random-space Γ. To this end, similar to the notation seen in Section 3.3, fix $n \in \{1, \ldots, M\}$, which in turn fixes our direction, Γ_n, and all other constants dependent upon the direction Γ_n, such as the Γ_n interpolation order, N_n. Then we have a general interpolation lemma on a 1-D random space.

[2] Recall here that $\mathbf{J}_i = (\mathbf{J}_{i,x_1}, \mathbf{J}_{i,x_2}, \mathbf{J}_{i,x_3})$

Lemma 4. *For any Banach function space, H, define the tensor product norm of H with $L^2_\rho(\Gamma_n)$ as*

$$||f||_{L^2_\rho(\Gamma_n)\times H} := \int_{\Gamma_n} ||f||^2_H \ \rho_n(y_n) \ dy_n$$

Then the 1-D random interpolation $\mathcal{I}^{N_n}_H : C^0(\Gamma_n)\times H \to L^2_\rho(\Gamma_n)\times H$, defined for $f \in C^0(\Gamma_n)\times H$,

$$\mathcal{I}^{N_n}_H(f) := \sum_{m_n=1}^{\widetilde{N}_n} f(y_n, \boldsymbol{x}) \ r_n^{m_k}(y_n)$$

is continuous under this norm. We define the tensor product norm of H with $C^0(\Gamma_n)$, as

$$||f||_{C^0(\Gamma_n)\times H} := \max_{y_n\in\Gamma_n} ||f(y_n, \boldsymbol{x})||_H$$

Then the interpolation error satisfies

$$||f - \mathcal{I}^{N_n}_H(f)||_{L^2_\rho(\Gamma_n)\times H} \leq K_H \inf_{w\in P_{N_n}(\Gamma_n)\times H} ||f(y_n) - w(y_k)||_{C^0(\Gamma_n)\times H} \tag{18}$$

with constant K_H independent of the choice of N_n.

We also wish to bound the *best-approximation* error. Thus, we consider this next lemma.

Lemma 5. *For some Banach function space, H, let $v \in C^0(\Gamma_n)\times H$. Assume v also admits an analytic extension in the region of the complex plane $\Sigma(\Gamma_n;\tau)$ for some $\tau > 0$. Then it holds:*

$$\min_{w\in P_{N_n}(\Gamma_n)\times H} ||v(y_n) - w(y_n)||_{C^0(\Gamma)\times H} \leq \frac{2}{\Psi_n - 1} \exp\big(-N_n \log(\Psi_n)\big) \max_{z\in\Sigma(\Gamma_n;\tau)} ||v(z)||_H$$

where $1 < \Psi_n = \frac{2\tau}{|\Gamma_n|} + \sqrt{1 + \frac{4\tau^2}{|\Gamma_n|^2}}$.

As stated previously, proofs of both Lemma 4 and Lemma 5 can be found in [3]. With these established, we now turn to bounding our complete Γ interpolation error. We do so iteratively. We first break apart our function space, separating as

$$L^2_\rho(\Gamma)\times V \equiv L^2_\rho(\Gamma_k) \times \Big(L^2_\rho(\Gamma_{k^*})\times V\Big)$$

where Γ_{k^*} are all random directions **except** Γ_k. Similarly define the product of the values of any parameter in every direction but the k^{th} as $\circ_{k^*} := \prod_{j\neq k} \circ_j$, with this being a real product, tensor product, or set of values where appropriate. [3] We define the norm on the latter space as the averaging norm, i.e. for $f \in L^2_\rho(\Gamma_{k^*})\times V$,

$$||f||_{L^2_\rho(\Gamma_{k^*})\times V} := \int_{\Gamma_{k^*}} ||f(y_{k^*}, \mathbf{x})||^2_V \rho_{k^*}(y_{k^*}) \ dy_{k^*},$$

[3] Wording is hard here.

which corresponds to the Hilbert tensor product construction. Of course, this also implies that it is a Banach space, and thus Lemmas 4 and 5 both apply. Denote the product space of the analytic extension region as

$$\Sigma(\Gamma, \gamma) := \prod_{j=1}^{M} \Sigma(\Gamma_j, \gamma_j)$$

where γ_j is as defined in Lemma 3. Then we have the following interpolation bounds for our specific operators.

Lemma 6. *The interpolation operators, $\mathcal{I}_V^N, \mathcal{I}_W^N$ are continuous, and obey the following interpolation bounds. For $\boldsymbol{f} \in C^0(\Gamma) \times V$ and $g \in C^0(\Gamma) \times W$, with analytic extensions in the region $\Sigma(\Gamma, \gamma)$, we have*

$$||\boldsymbol{f} - \mathcal{I}_V^N(\boldsymbol{f})||_{L^2_\rho(\Gamma)\times V} \leq K_1 \max_{z\in\Sigma(\Gamma,\tau)} ||\boldsymbol{f}(z)||_V \sum_{j=1}^{M} \frac{2}{\Psi_j - 1} \exp\big(-N_j \log(\Psi_j)\big) \tag{19a}$$

$$||g - \mathcal{I}_W^N(g)||_{L^2_\rho(\Gamma)\times W} \leq K_2 \max_{z\in\Sigma(\Gamma,\tau)} ||g(z)||_W \sum_{j=1}^{M} \frac{2}{\Psi_j - 1} \exp\big(-N_j \log(\Psi_j)\big) \tag{19b}$$

where $\Psi_j := \frac{2\gamma_j}{|\Gamma_j|} + \sqrt{1 + \frac{4\gamma_j^2}{|\Gamma_j|^2}} > 1$, and K_1, K_2 are two constants independence of the choice of N.

Proof. We begin with continuity, and must prove this iteratively. As it will follow with similar logic, we show these properties for the interpolation operator $\mathcal{I}_V^N$ only. For notational convenience, define

$$V_{k^*} := L^2_\rho(\Gamma_{k^*}) \times V$$

We define the k^{th} interpolation operator $\mathcal{I}_{V_{K^*}}^{N_k} : C^0(\Gamma_k) \times V_{k^*} \to L^2_\rho(\Gamma_k) \times V_{k^*}$ as

$$\mathcal{I}_{V_{k^*}}^{N_k}(\mathbf{f}) := \sum_{m_k=1}^{\widetilde{N}_k} \mathbf{f}(y_k^{m_k}, \mathbf{x})\ r_k^{m_k}(y_k) \tag{20}$$

It follows that the interpolation operator satisfies

$$\mathcal{I}_V^N = \mathcal{I}_{V_{1^*}}^{N_1} \circ \ldots \circ \mathcal{I}_{V_{M^*}}^{N_M}$$

Note now that V_{k^*} is a Hilbert space for all $k = 1, \ldots, M$, as it is the finite tensor product of Hilbert spaces. Therefore, Lemma 4 applies, and as the composure of continuous functions are continuous, continuity holds. To see that the bounds hold, consider that

$$\begin{aligned}
||\mathbf{f} - \mathcal{I}_V^N(\mathbf{f})||_{L^2_\rho(\Gamma)\times V} &\leq \Big|\Big|\mathbf{f} - \big(\mathcal{I}_{V_{1^*}}^{N_1} \circ \ldots \circ \mathcal{I}_{V_{M^*}}^{N_M}\big)(\mathbf{f})\Big|\Big|_{L^2_\rho(\Gamma)\times V} \\
&\leq \Big|\Big|\Big(\mathbf{f} - \mathcal{I}_{V_{1^*}}^{N_1}\mathbf{f}\Big) + \Big(\mathcal{I}_{V_{1^*}}^{N_1}\mathbf{f} - \mathcal{I}_{V_{1^*}}^{N_1} \circ \mathcal{I}_{V_{2^*}}^{N_2}\mathbf{f}\Big) + \ldots\Big|\Big|_{L^2_\rho(\Gamma)\times V} \\
&\leq \Big|\Big|\mathbf{f} - \mathcal{I}_{V_{1^*}}^{N_1}\mathbf{f}\Big|\Big|_{L^2_\rho(\Gamma)\times V} + \Big|\Big|\mathcal{I}_{V_{1^*}}^{N_1}\mathbf{f} - \mathcal{I}_{V_{1^*}}^{N_1} \circ \mathcal{I}_{V_{2^*}}^{N_2}\mathbf{f}\Big|\Big|_{L^2_\rho(\Gamma)\times V} + \ldots
\end{aligned}$$

We now note that $L^2_\rho(\Gamma) \times V = L^2_\rho(\Gamma_k) \times V_{k^*}$, and thus $||\cdot||_{L^2_\rho(\Gamma)\times V} = ||\cdot||_{L^2_\rho(\Gamma_k)\times V_{k^*}}$. It is also immediate that $\mathbf{f} \in C^0(\Gamma)$ implies that $\mathcal{I}^{N_k}_{V_k}(\mathbf{f}) \in C^0(\Gamma_k)$. Finally, for notational convenience in the purposes of this proof, we define $P_k := P_{N_k}(\Gamma_k) \times V_{k^*}$ WIth this in mind, we apply the bounds from Lemma 4, and have

$$||\mathbf{f} - \mathcal{I}^N_V(\mathbf{f})||_{L^2_\rho(\Gamma)\times V} \leq K_{V_{1^*}} \inf_{w_1 \in P_1} ||\mathbf{f} - w_1||_{C^0(\Gamma_1)\times V_{1^*}} + K_{V_{2^*}} \inf_{w_2 \in P_2} ||\mathcal{I}^{N_1}_{V_{1^*}}(\mathbf{f}) - w_2||_{C^0(\Gamma_2)\times V_{2^*}} + \dots$$

Choosing $K_1 := \max_{j \in \{1,\dots,M\}} K_{V_{j^*}}$ yields

$$||\mathbf{f} - \mathcal{I}^N_V(\mathbf{f})||_{L^2_\rho(\Gamma)\times V} \leq K_1 \Big(\inf_{w_1 \in P_1} ||\mathbf{f} - w_1||_{C^0(\Gamma_k)\times V_{1^*}} + \inf_{w_2 \in P_2} ||\mathcal{I}^{N_1}_{V_{1^*}}(\mathbf{f}) - w_2||_{C^0(\Gamma_2)\times V_{2^*}} + \dots \Big)$$

Now, given that all of V_{k^*} are Hilbert spaces, it follows that they are Banach spaces, and we can apply Lemma 5, which yields

$$||\mathbf{f} - \mathcal{I}^N_V(\mathbf{f})||_{L^2_\rho(\Gamma)\times V} \leq K_1 \max_{z \in \Sigma(\Gamma,\tau)} ||\mathbf{f}(z)||_V \sum_{j=1}^{M} \frac{2}{\Psi_j - 1} \exp\big(- N_j \log(\Psi_j)\big)$$

with Ψ_j defined as in Lemma 5. The independence of K_1 on N follows from the independence of each constant $K_{V_{j^*}}$ on N. This is the exact approximation bound we claimed. With similar logic, the continuity and approximation error bounds for $\mathcal{I}^N_W$ also follow. □

We have now established that the random polynomial chaos interpolation has error bounded by the max value of the true solutions. However, we used the complete deterministic solution space, V and W, only out of notational convenience. Both V and it's finite-dimensional approximation, V_h, and subsequently W, W_h, are assumed to have the same random-space regularity, and thus the Lemma 6 also applies when considering the interpolation operators acting on V_h, W_h. Combining this lemma with the regularity assumptions in Section 3.3 and the finite-dimensionality of Γ (see Section 3.1), we now bound the approximation error in the SC method.

Theorem: 2. *Under the assumptions of Lemma 1, Lemma 2, the finite-dimensional noise assumption, and that the Γ is bounded, we have the following error bounds on the approximate solutions resulting from the SC method. They satisfy*

$$||\boldsymbol{J}_i - \boldsymbol{J}^{h,N}_i||_{\widetilde{V}} \leq C||\mathbf{g} - \mathbf{g}^h||_{\widetilde{W}} + K_1 \max_{z \in \Sigma(\Gamma,\tau)} ||\boldsymbol{J}_i(z)||_V \sum_{j=1}^{M} \frac{2}{\Psi_j - 1} \exp\big(- N_j \log(\Psi_j)\big) \tag{21a}$$

$$||\mathcal{V} - \mathcal{V}^{h,N}||_{\widetilde{W}} \leq \frac{1}{b}||\mathbf{g} - \mathbf{g}^h||_{\widetilde{W}} + K_2 \max_{z \in \Sigma(\Gamma,\tau)} ||\mathcal{V}(z)||_W \sum_{j=1}^{M} \frac{2}{\Psi_j - 1} \exp\big(- N_j \log(\Psi_j)\big) \tag{21b}$$

with $\Psi_j < 1$ defined as in Lemma 5, and gf^h is the polynomial interpolation of $\mathbf{g}$ on t_h, of the same degree as $\boldsymbol{J}^h_i$ and $\mathcal{V}^h$.

Proof. We will show that the bounds hold for $\mathbf{J}_i$ only, as similar logic will follow for $\mathcal{V}$. We first expand the approximations into the spatial discretization and random polynomial interpolations, e.g.

$$||\mathbf{J}_i - \mathbf{J}^{h,N}_i||_{\widetilde{V}} \leq ||\mathbf{J}_i - \mathbf{J}^h_i||_{\widetilde{V}} + ||\mathbf{J}^h_i - \mathbf{J}^{h,N}_i||_{\widetilde{V}}$$

The first of the two terms stem from standard finite-element theory and polynomial interpolation of $\mathbf{J}_i$, and corresponds to the first term in the bound above. It is clear that the spatial approximation of $\mathbf{J}_i^h$ satisfies the system

$$\widetilde{\mathcal{A}}(\mathbf{J}_i^h) + \widetilde{\mathcal{B}}'(\mathcal{V}^h) = 0 \in \widetilde{V}',$$
$$\widetilde{\mathcal{B}}(\mathbf{J}_i^h) = G_h \ \in \widetilde{W}'.$$

where $G_h(\psi) := \mathbb{E}\left[\int_D \mathbf{g}^h \cdot \nabla\psi\right]$, and $\mathbf{g}^h$ is as stated in the theorem. Subtracting this system from (2) and applying the bounds from Theorem 1, we have

$$||\mathbf{J}_i - \mathbf{J}_i^h||_{\widetilde{V}} \leq C||\mathbf{g} - \mathbf{g}^h||_{\widetilde{W}}$$

We now turn to the second of the two terms, which stems from the random space polynomial chaos expansion and subsequent approximation. It is clear from Lemma 6 that the interpolation operator corresponding to V, when applied to $\mathbf{J}_i^h$ satisfies

$$||\mathbf{J}_i^h - \mathcal{I}_V^N(\mathbf{J}_i^h)||_{L^2_\rho(\Gamma)\times V} \leq K_1 \max_{z\in\Sigma(\Gamma,\tau)} ||\mathbf{J}_i^h(z)||_V \sum_{j=1}^{M} \frac{2}{\Psi_j - 1} \exp\big(-N_j \log(\Psi_j)\big) \tag{23}$$

with the previously defined ψ_j, and K_1. This yields the desired result for $||\mathbf{J}_i - \mathbf{J}_i^{h,N}||_{\widetilde{V}}$ and a similar argument argument holds for the bounds on the error of approximating $\mathcal{V}$. □

Note now that it is sufficient to bound the spatial approximation error by $||\mathbf{g} - \mathbf{g}^h||_{\widetilde{W}}$, as this converges as $h \to \infty$ by standard polynomial interpolation arguments [6], and is thus not discussed further here. We also can see that the random error, as desired, achieves the sub-exponential convergence, as $\psi_j > 1$. Thus, the SC method will converge, as desired.

5 Conclusion

In this paper, we have explored the random-kinematic MHD equations. The system has wide-ranging applications with vastly different assumptions regarding the system itself. To this end, we focused on one in particular: power-generation. As with any generator, optimization of the operation is desired. However, without being able to precisely assign the system parameters, the optimization of the MHD generator must be done with uncertainty in mind. As well, optimization of an MHD generator in real-time requires both estimation of the current state of the system, as well as optimization for the given state. Recovery of the parameters are greatly affected by noise [12], and thus both the recovery and subsequent optimization must consider uncertainty. To deal with this, we proposed the stochastic collocation method. This method was shown to not only be numerically efficient when combined with the anisotropic sparse grid, but also had sub-exponential convergence. Numerical demonstrations of this method can be seen in [11]. The efficiency of the SC method will allow for the inverse problem of optimization, as well as parameter estimation under uncertainty, to be feasible. Future work will investigate this.

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