SIAM J. NUMER. ANAL.
Vol. 12, No. 3, June 1975

A PRIORI ERROR ESTIMATES FOR APPROXIMATION OF PARABOLIC BOUNDARY VALUE PROBLEMS*

R. E. SHOWALTER†

Abstract. The L^2-error estimates are established for the continuous time Faedo–Galerkin approximation to solutions of a linear parabolic initial boundary value problem that has elliptic part of order $2m$. Properties of analytic semigroups are used to obtain these estimates directly from the L^2-estimates for the corresponding steady state elliptic problem under hypotheses only on the data in the problem (initial condition, elliptic operator).

1. Introduction. We obtain estimates for the error resulting from a continuous time Faedo–Galerkin approximation of the linear parabolic boundary value problem

$$u'(t) + Au(t) = 0, \qquad t > 0, \qquad u(0) = u_0, \tag{1.1}$$

where A is a realization in $L^2(G)$ of an elliptic partial differential operator of order $2m$. These estimates are best possible: the rate of convergence is the same as that for the Galerkin approximation of the corresponding (variational) elliptic steady state problem whose exact solution is the initial condition, u_0.

These estimates for the rate of convergence are well known; our contribution here is that they are obtained from hypotheses on the *data* in the problem—the regularity properties of the elliptic operator, an approximation assumption on the rate of convergence in the corresponding elliptic problem and the initial condition—and without the usual ad-hoc assumptions on the solution $u(\cdot)$ of the problem. The proofs depend on the existence-regularity theory for the evolution problem. See [1], [2], [12] for related results.

An exposition of the well-known results for the steady state problem is given in §2, where we briefly discuss the approximation of solutions and the interpolation of various estimates associated with these regular elliptic boundary value problems. Section 3 begins with a description of the regularity properties of the solution of the abstract Cauchy problem (1.1) where $-A$ is the generator of an analytic semigroup of contractions. We use the notions of interpolation theory and fractional powers of the operator A to relate the growth of $u(t)$ in various norms as $t \to 0^+$ to the (regularity of the) initial condition, u_0. After these preliminaries, the error estimates are proved and stated as our Theorem.

2. Elliptic operators: Interpolation and approximation. Let V and H be Hilbert spaces for which V is a dense subspace of H and the injection is continuous. Denote the inner product and norm on H by $(\cdot, \cdot)$ and $|\cdot|$, respectively, and the norm on V by $\|\cdot\|$. Let $a(\cdot, \cdot)$ be a continuous bilinear form on V which is *coercive*: there is a

* Received by the editors March 29, 1974, and in revised form August 5, 1974.

† Department of Mathematics, University of Texas, Austin, Texas 78712. This work was supported in part by the National Science Foundation under Grant GP-34261.

number $c > 0$ such that

$$a(v, v) \geqq c\|v\|^2, \qquad v \in V. \tag{2.1}$$

The triple $\{a(\cdot, \cdot), V, H\}$ defines an unbounded operator A on H with domain $D(A)$ by

$$a(u, v) = (Au, v), \qquad u \in D(A), \quad v \in V,$$

where $D(A)$ is the set of all $u \in V$ for which the linear form $v \mapsto a(u, v)$ is continuous on V with the (weaker) norm of H. From (2.1) it follows that A is an injection of $D(A)$ onto H, and $D(A)$ with the norm

$$\|v\|_A = |Av|, \qquad v \in D(A),$$

is a Hilbert space (with the obvious inner product). Further, $D(A)$ is dense in H, and the identity map from $D(A)$ into H is continuous: there is a number $c_1 > 0$ such that

$$\|v\|_A \geqq c_1|v|, \qquad v \in D(A). \tag{2.2}$$

Fractional powers of operators such as A have been constructed and discussed by many authors. (See the bibliographies of [3], [6], [10].) For each α, $0 < \alpha < 1$, there is an operator A^α which is a closed linear injection with dense domain $D(A^\alpha)$ and range H. $D(A^\alpha)$ is a Hilbert space with norm

$$\|v\|_{A^\alpha} = |A^\alpha v|, \qquad v \in D(A^\alpha).$$

and A^α has properties appropriate for the αth root of A. Under hypotheses more general than those above on A, T. Kato proved the following [5].

Let A_1 be an unbounded operator on a Hilbert space H_1 with properties like A on H. If $T \in L(H, H_1)$ and $T \in L(D(A), D(A_1))$, where $L(X, Y)$ denotes the Banach space of continuous linear maps of X into Y, then for each θ, $0 < \theta < 1$, we have $T \in L(D(A^\theta), D(A_1^\theta))$ with the estimate

$$\|T\|_{L(D(A^\theta),D(A_1^\theta))} \leqq c_2\|T\|_{L(H,H_1)}^{1-\theta}\|T\|_{L(D(A),D(A_1))}^{\theta}. \tag{2.3}$$

This is an *interpolation theorem* which we can apply with $T = A_1 = I$ on $H_1 = D(A_1) = H$ to obtain from (2.2) the estimate

$$|A^\theta v| \geqq c_1^\theta|v|, \qquad 0 \leqq \theta \leqq 1, \quad v \in D(A^\theta). \tag{2.4}$$

There are various equivalent methods of constructing from a pair of Hilbert spaces H_0 and H_1, with H_0 dense and continuously embedded in H_1, a family of intermediate spaces $[H_0, H_1]_\theta$, $0 < \theta < 1$, and we refer to [10] for references. Such spaces have an interpolation property like (2.3) for bounded operators, and we have (e.g., [9])

$$[D(A), H]_\theta = D(A^{1-\theta}), \qquad 0 \leqq \theta \leqq 1,$$

with equivalent norms, for the operator A on H constructed above.

In order to construct an elliptic differential operator, we let G be a bounded open set in $\mathbb{R}^n$ with smooth boundary ∂G locally on one side of G, and define for integer $m \geqq 0$ the Hilbert space $H^m(G)$ of (equivalence classes of) functions $v \in L^2(G)$

$\equiv H^0(G)$ such that each (distribution) derivative $D^p v$ of order $|p| \leqq m$ belongs to $L^2(G)$. The inner product on $H^m(G)$ is

$$(u, v)_m = \sum\left\{\int_G D^p u(x) D^p v(x)\, dx : |p| \leqq m\right\}$$

and the norm is given by $\|v\|_m = (v, v)_m^{1/2}$([8]). Let V be a closed subspace of $H^m(G)$ containing the space $C_0^\infty(G)$ of infinitely differentiable functions with compact support in G. Take $H = L^2(G)$, so V is dense in H, and prescribe a bilinear form on V by

$$a(u, v) = \sum\left\{\int_G a_{pq}(x) D^q u(x) D^p v(x)\, dx : |p|, |q| \leqq m\right\},$$

where the coefficients a_{pq} are smooth functions on $\bar{G}$ and the coercive estimate (2.1) holds. The unbounded operator A determined by the triple $\{a, V, H\}$ is given by

$$Au = \sum\{(-1)^{|p|} D^p(a_{pq} D^q u) : |p|, |q| \leqq m\}, \qquad u \in D(A). \tag{2.5}$$

To determine $D(A)$, we define

$$V = \{v \in H^m(G) : B_j v = 0 \text{ on } \partial G, 0 \leqq j \leqq p - 1\},$$

where $0 \leqq p \leqq m$ and each B_j is a *normal* boundary differential operator of (normal) order $m_j < m$ [4], [10]. We choose additional boundary operators to augment this set and then obtain by Green's formula a collection $\{B_j : p \leqq j \leqq m - 1\}$ of normal boundary operators with $m \leqq m_j < 2m$, $p \leqq j \leqq m - 1$. (See, e.g., [10, II. 9] for details.) Assume that the partial differential operator (2.5) and the boundary operators $\{B_j : 0 \leqq j \leqq m - 1\}$ constitute a *regular* boundary value problem on G. Then $D(A) = \{v \in H^{2m}(G) : B_j v = 0 \text{ on } \partial G,\ 0 \leqq j \leqq m - 1\}$ and we have the estimate

$$|Au| \geqq c_3 \|u\|_{2m}, \qquad u \in D(A), \tag{2.6}$$

where $c_3 > 0$.

We apply the interpolation theorem to our regular boundary value problem. Considering the identity map of $H \to H$ and $D(A) \to H^{2m}(G)$, we obtain from the estimate (2.6) and the identity $[H^j(G), H^k(G)]_\alpha = H^l(G)$, $l = (1 - \alpha)j + \alpha k$, the estimates

$$|A^{k/2m} u| \geqq c_4 \|u\|_k, \qquad u \in D(A), \quad 0 \leqq k \leqq 2m. \tag{2.7}$$

If the boundary value problem is k-regular, $k \geqq 1$, that is, $Au \in H^r(G)$ implies $u \in H^{2m+r}(G)$ with

$$\|Au\|_r \geqq c^{(k)} \|u\|_{2m+r}, \qquad 0 \leqq r \leqq k,$$

then we obtain the estimate

$$|A^{1+k/2m} u| \geqq c_1^{(k)} \|u\|_{2m+k}, \tag{2.8}$$

where powers of A larger than one are defined by composition or, e.g., as in [3, II. 14]. Finally, we recall that Grisvard [4] characterized the domains of the fractional powers by $D(A^\theta) = \{v \in H^{2m\theta}(G): B_j v = 0 \text{ on } \partial G \text{ if } m_j < 2m\theta - \frac{1}{2}\}$ whenever $0 \leqq \theta \leqq 1$ and $2m\theta - \frac{1}{2}$ was not an integer. (We shall be concerned only with the case in which $2m\theta$ is an integer.)

Let $f \in H$ and consider the problem of approximating the $u \in D(A)$ for which $Au = f$. The solution is determined by the variational equality

$$u \in V, \quad a(u, v) = (f, v), \quad v \in V, \tag{2.9}$$

since V is dense in H. The Galerkin approximation of u determined by a given finite-dimensional subspace M of V is the solution w of the problem

$$w \in M, \quad a(w, v) = (f, v), \quad v \in M. \tag{2.10}$$

Then w is the projection of u onto M with respect to the bilinear form $a(\cdot, \cdot)$, and (2.10) is an algebraic problem for the coefficients of the expansion of w by a basis for M. We shall assume that M belongs to a family $\mathscr{M} = \{M_h: 0 < h < 1\}$ of finite-dimensional subspaces of V which satisfy the following *approximation assumption*: there is an integer $k \geqq m$ depending on $\mathscr{M}$ and a constant $c > 0$ depending on $\mathscr{M}$ and the regular boundary value problem such that the Galerkin approximation $w \in M$ of the solution $u \in H^k(G)$ of (2.9) satisfies the error estimate

$$\begin{aligned} |u - w| &\leqq ch^{2(k-m)}\|u\|_k \quad \text{if } m \leqq k \leqq 2m, \\ |u - w| &\leqq ch^k\|u\|_k \quad \text{if } 2m \leqq k. \end{aligned} \tag{2.11}$$

Such L^2-estimates are typical, e.g., for finite element spaces of degree $k - 1$ with mesh parameter $h > 0$ [11, Thm. 3.7].

3. Parabolic boundary value problems. Let A be the unbounded operator on H constructed from a continuous coercive bilinear form as in §2. Then $-A$ generates an analytic semigroup $\{S(t): t \geqq 0\}$ of contractions on H, and for each $u_0 \in H$ the function $u(t) \equiv S(t)u_0$ is the unique solution of (1.1). In particular, $u \in C([0, \infty), H)$ and at each $t > 0$, u is analytic with $u(t) \in D(A^p)$, all $p \geqq 0$, and satisfies $u^{(k)}(t) = (-A)^k u(t)$ for integer $k \geqq 0$. For $\beta \geqq 0$ and $t > 0$ we have the estimates

$$\|A^\beta S(t)\|_{L(H)} \leqq M_\beta / t^\beta.$$

(See [3, II. 14] or [7, IX. 1.6] for details.)

Suppose, hereafter, that $u_0 \in D(A^\alpha)$ for some $\alpha \geqq 0$. Since each fractional power of A commutes with the semigroup, the estimate above gives

$$|A^\beta u(t)| \leqq (M_{\beta-\alpha}/t^{\beta-\alpha})|A^\alpha u_0|, \qquad \beta \geqq \alpha, \quad t > 0. \tag{3.1}$$

Similarly, since each $S(t)$ is a contraction on H we obtain

$$|A^\alpha u(t)| \leqq |A^\alpha u_0|, \qquad t \geqq 0, \tag{3.2}$$

and $u \in C([0, \infty), D(A^\alpha))$.

The solution of (1.1) satisfies the variational equation

$$(u'(t), v) + a(u(t), v) = 0, \qquad v \in V, \quad t > 0. \tag{3.3}$$

If M is a finite-dimensional subspace of V, we define a corresponding Faedo–Galerkin approximation of $u(\cdot)$ as the solution $U \in C^1([0, \infty), M)$ of

$$(3.4) \qquad (U'(t), v) + a(U(t), v) = 0, \qquad v \in M, \quad t > 0,$$

with $U(0)$ to be prescribed below. (Since M has finite dimension, (3.4) is equivalent to a system of ordinary differential equations.)

Assume hereafter that $2m\alpha$ is an integer and $\alpha \geqq \frac{1}{2}$. Since $D(A^{1/2}) = V$ ([9], [10]) we have $u \in C([0, \infty), V)$ and we can define $W \in C([0, \infty), M)$ as the pointwise elliptic Galerkin projection onto u:

$$(3.5) \qquad a(W(t), v) = a(u(t), v), \qquad v \in M, \quad t \geqq 0.$$

It follows then that we have

$$(3.6) \qquad a(W'(t), v) = a(u'(t), v), \qquad v \in M, \quad t > 0.$$

Following [11], we have from (3.3), (3.4) and (3.5),

$$(u'(t) - W'(t), v) = (U'(t) - W'(t), v) + a(U(t) - W(t), v), \qquad v \in M, \quad t > 0,$$

and setting $v = U(t) - W(t)$ gives

$$(\tfrac{1}{2})D_t(|U(t) - W(t)|^2) \leqq (u'(t) - W'(t), U(t) - W(t)).$$

Note that if f is a continuously differentiable H-valued function, then it is locally Lipschitz and so then is $|f(t)|$. Hence, $|f(t)|$ is differentiable almost everywhere. Applying this remark to the above estimate gives

$$|U(t) - W(t)|D_t(|U(t) - W(t)|) \leqq |U(t) - W(t)| \cdot |u'(t) - W'(t)|$$

for almost every $t > 0$. Consider the set $Z \equiv \{t > 0 : |U(t) - W(t)| = 0\}$. If t is not in this set, then we have

$$(3.7) \qquad D_t|U(t) - W(t)| \leqq |u'(t) - W'(t)|.$$

If t is an accumulation point of Z then the indicated derivative is zero and (3.7) holds. But there are a countable number of isolated points of Z, so (3.7) holds almost everywhere, and we can integrate it to obtain

$$|U(t) - W(t)| \leqq |U(0) - W(0)| + \int_0^t |u' - W'|, \qquad t \geqq 0.$$

From the triangle inequality we obtain the fundamental error estimate

$$(3.8) \qquad |u(t) - U(t)| \leqq |u(t) - W(t)| + |U(0) - W(0)| + \int_0^t |u' - W'|, \qquad t \geqq 0.$$

We can easily estimate the error contributed by the first term in (3.8). If $\frac{1}{2} \leqq \alpha \leqq 1$, we obtain from (2.11), (2.7) and (3.2),

$$|u(t) - W(t)| \leqq (\text{const.})|A^\alpha u_0| \cdot \begin{cases} h^{2m(2\alpha - 1)}, & 2m\alpha \leqq k, \\ h^{2(k-m)}, & m \leqq k \leqq 2m\alpha. \end{cases}$$

Similarly, if $\alpha \geqq 1$ we obtain the estimates

$$|u(t) - W(t)| \leqq (\text{const.})|A^\alpha u_0| \cdot \begin{cases} h^{2m\alpha}, & 2m\alpha \leqq k, \\ h^k, & 2m \leqq k \leqq 2m\alpha, \\ h^{2(k-m)}, & m \leqq k \leqq 2m. \end{cases}$$

We intend to obtain from (3.6) estimates of the same order in h for the third term in (3.8) on the interval $0 \leqq t \leqq T$, and we consider the preceding five cases separately.

Case 1. $\frac{1}{2} < \alpha \leqq 1$, $2m\alpha < k$. Let $0 < \delta < T$. Then (2.11), (1.1), (2.7) and (3.1) give

$$\int_0^\delta |u' - W'| \leqq \text{const.}\, |A^\alpha u_0| \int_0^\delta t^{\alpha - 3/2}\, dt = \text{const.}\, |A^\alpha u_0| \delta^{\alpha - 1/2},$$

and, similarly, we obtain (with $k \leqq 2m$)

$$\int_\delta^T |u' - W'| \leqq \int_\delta^T \|Au\|_k \cdot h^{2(k-m)} \leqq \text{const.}\, |A^\alpha u_0| \int_\delta^T t^{\alpha - 1 - k/2m}\, dt \leqq \text{const.}\, |A^\alpha u_0| \delta^{\alpha - k/2m} \cdot h^{2(k-m)}.$$

Adding these with $\delta = h^{4m}$ gives

$$\int_0^T |u' - W'| \leqq \text{const.}\, |A^\alpha u_0| \cdot h^{2m(2\alpha - 1)}. \tag{3.9}$$

Case 2. $\frac{1}{2} < \alpha \leqq 1$, $m \leqq k \leqq 2m\alpha$. If $k = 2m\alpha$, then for each $\varepsilon > 0$ we can apply the proof of Case 1 with α replaced by $\alpha - \varepsilon$ to obtain

$$\int_0^T |u' - W'| \leqq \text{const.}\, |A^{\alpha - \varepsilon} u_0| \cdot h^{2m(2\alpha - 1 - 2\varepsilon)}.$$

But (2.4) shows $|A^{\alpha-\varepsilon} u_0| \leqq c_1^{-\varepsilon} |A^\alpha u_0|$, so

$$\int_0^T |u' - W'| \leqq \text{const.}\, |A^\alpha u_0| \cdot h^{2m(2\alpha - 1 - 2\varepsilon)},$$

where the constant is independent of ε, and we obtain (3.9). If $k < 2m\alpha$ we obtain the estimate

$$\int_0^T |u' - W'| \leqq C \int_0^T \|Au\|_k \cdot h^{2(k-m)} \leqq \text{const.}\, |A^\alpha u_0| \int_0^T t^{\alpha - 1 - k/2m}\, dt \cdot h^{2(k-m)} = \text{const.}\, |A^\alpha u_0| \cdot h^{2(k-m)}$$

from (2.11), (2.7) and (3.1).

Case 3. $1 < \alpha$, $2m\alpha \leqq k$. Let $0 < \delta < T$ and assume $k = 2m\alpha$. For each ε, $0 < \varepsilon < 1$, we have from (2.11), (2.8), (3.1) and (2.4),

$$\int_\delta^T |u' - W'| \leqq C \int_\delta^T \|Au\|_k \cdot h^k \leqq \text{const.} \int_\delta^T |A^{1+\alpha}u| \cdot h^k$$

$$\leqq \text{const.}\, |A^{\alpha-\varepsilon}u_0| \int_\delta^T t^{-1-\varepsilon}\, dt \cdot h^k$$

$$\leqq \text{const.}\, |A^\alpha u_0| \cdot \delta^{-\varepsilon} \cdot h^k,$$

where the constant is independent of ε and δ. Letting $\varepsilon \to 0^+$, then $\delta \to 0^+$, we obtain

$$\int_0^T |u' - W'| \leqq \text{const.}\, |A^\alpha u_0| \cdot h^{2m\alpha}.$$

Case 4. $1 < \alpha$, $2m \leqq k < 2m\alpha$. Choose β so that $2m\beta = 1 + k$. Then $\beta \leqq \alpha$, so

$$\int_0^T |u' - W'| \leqq \text{const.} \int_0^T |A^{1+k/2m}u| \cdot h^k$$

$$\leqq \text{const.}\, |A^\beta u_0| \int_0^T t^{-1+1/2m}\, dt \cdot h^k$$

$$\leqq \text{const.}\, |A^\alpha u_0| \cdot h^k.$$

Case 5. $1 < \alpha$, $m \leqq k \leqq 2m$. The preceding procedure gives the estimate

$$\int_0^T |u' - W'| \leqq \text{const.}\, |A^\alpha u_0| \cdot h^{2(k-m)}.$$

Finally, we note that if $U(0)$ is chosen by any of the usual methods, i.e., interpolation, Galerkin projection or L^2-projection, the resulting estimates are (at least) as good as the above. The preceding discussion is summarized in the following.

THEOREM. *With the notation given in* §2, *let V be a closed subspace of $H^m(G)$ determined by a (possibly empty) collection of normal homogeneous boundary operators of order $<m$, $a(\cdot,\cdot)$ be the given continuous and coercive bilinear form on V, and let A be the elliptic partial differential operator* (2.5) *of order $2m$ determined on $L^2(G)$ by a choice of additional boundary operators and Green's theorem on G. Let $u_0 \in L^2(G)$ and denote by $u(\cdot)$ the unique solution of* (1.1). *Let M be a finite-dimensional subspace of V and denote by $U(\cdot)$ the unique solution of* (3.4) *for which, e.g., $U(0)$ is the $L^2(G)$-projection of u_0 onto M.*

Assume the following:

(i) *$\{A, B_j : 0 \leqq j \leqq m - 1\}$ is a k-regular elliptic boundary value problem.*

(ii) *M is taken from a collection of subspaces of V which satisfy the approximation assumption* (2.11), *and*

(iii) *$u_0 \in D(A^\alpha)$, where $2m\alpha$ is an integer $\geqq m$.*

Then we have the estimate

$$\|u(t) - U(t)\|_{L^2(G)} \leqq \text{const.}\, \|A^\alpha u_0\|_{L^2(G)} \cdot h^p, \tag{3.10}$$

where $p = 2(k - m)$ if $m \leqq k \leqq 2m \cdot \min\{\alpha, 1\}$ and $p = k$ if $2m \leqq k \leqq 2m\alpha$.

Remarks.

1. The hypothesis (i) is standard and one can consult, e.g., [10] for conditions on the coefficients and region G which imply it. Similarly, (ii) is standard [9], and (iii) can be obtained from regularity of u_0 and boundary conditions on $A^j u_0$ for j integer and $\leqq$ the integer part of α [4].

2. In the case $k \geqq 2m$ and $\alpha = 1$ we have convergence of order $2m$ in h. Trace theory [8, VII] implies that, in general, $u' \in L^2((0, T), D(A^{1/2})) = L^2((0, T), V)$, so we cannot hope to obtain (3.10) by estimating u' with the norm of $L^2((0, T), D(A))$ or (equivalently) $L^2((0, T), H^{2m}(G))$.

3. The assumption that M be a subspace of V restricts the method to natural boundary conditions or very special domains. The construction of subspaces satisfying essential boundary conditions is almost always impossible.

REFERENCES

[1] T. DUPONT, *Some L^2 error estimates for parabolic Galerkin methods*, Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equation, Academic Press, New York, 1972.

[2] J. BRAMBLE AND V. THOMEE, *Discrete time Galerkin methods for a parabolic boundary value problem*, Math. Research Center Rep. 1240, Univ. of Wisconsin, Madison, 1972.

[3] A. FRIEDMAN, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.

[4] P. GRISVARD, *Caracterisation de quelques espaces d'interpolation*, Arch. Rational Mech. Anal., 25 (1967), pp. 40–63.

[5] T. KATO, *A generalization of the Heinz inequality*, Proc. Japan Acad., 6 (1961), pp. 305–308.

[6] ———, *Fractional powers of dissipative operators*, J. Math. Soc. Japan, 13 (1961), pp. 246–274; 14 (1962), pp. 242–248.

[7] ———, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966.

[8] J. LIONS, *Equations differentielles operationnelles et problemes aux limite*, Springer, Berlin, 1961.

[9] ———, *Espaces d'interpolation et domaines de puissances fractionnaires d'operateurs*, J. Math. Soc. Japan, 14 (1962), pp. 233–241.

[10] J. LIONS AND E. MAGENES, *Problems aux limites non homogenes et applications*, vol. I, Dunod, Paris, 1968.

[11] G. STRANG AND G. FIX, *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, 1973.

[12] V. THOMEE, *Some convergence results for parabolic boundary value problems*, Conference on Finite Element Methods, Math. Research Center, Univ. of Wisconsin, Madison, 1974.