AN ABSTRACT OF THE DISSERTATION OF

Stephen A. Krughoff for the degree of Doctor of Philosophy in Mathematics presented on May 30, 2019.

Title: On the Topologies of Moduli Spaces of Polygons: A Taxonomic Approach

Abstract approved: ______________________________________

Ren Guo

Consider a polygon lying in the Euclidean plane with labeled edge lengths. The moduli space of polygons is the space of all polygons with the same labeled edge lengths, modulo orientation preserving isometries. It is well known that this space is generically a smooth manifold. For certain combinations of edge lengths, however, non-smooth points can arise. We show these points to be isolated, each with a neighborhood homeomorphic to a cone over a product of spheres. We proceed to explicitly compute the homeomorphism types that arise from non-smooth moduli spaces of pentagons. We then turn our attention to the number of different (up to diffeomorphism) smooth manifolds that can arise as moduli spaces of polygons. It is known that, for a fixed number of edges, this number is finite. The exact number is only known up to the case of pentagons, however. We provide a new structure that summarizes the possible manifold topologies of a moduli space of polygons in a directed graph. We then use this structure to provide bounds on the number of diffeomorphism types that can arise as moduli spaces of polygons with no more than eight edges.

On the Topologies of Moduli Spaces of Polygons: A Taxonomic Approach

by
Stephen A. Krughoff

A DISSERTATION

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Presented May 30, 2019
Commencement June 2019

Doctor of Philosophy dissertation of Stephen A. Krughoff presented on May 30, 2019.

APPROVED:

Major Professor, representing Mathematics

Head of the Department of Mathematics

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Stephen A. Krughoff, Author

ACKNOWLEDGMENTS

This dissertation would have been impossible without the guidance, collaboration, and comfort provided by dozens of friends, colleagues, and mentors. This work is dedicated

- to my advisor Dr. Ren Guo for providing me with the intuition to visualize the objects in seemingly every subfield of topology and geometry, and for the freedom to pursue my own path through my graduate career.
- to the geometry and topology students in my cohort, Charles Camacho and Dong Zheting. You were always available to hash out problems on a chalkboard, and I wouldn't have made it through this program without your intellectual support.
- to the faculty members who introduced me to the beauty of geometry and topology. To Dr. Christine Escher for teaching me the interaction between algebraic topology and geometry. To Dr. Bill Bogley for breathing life into group theory. To Dr. Tevian Dray and Dr. Juha Pohjanpelto for showing me the interplay between the infinitesimal and the macroscopic embodied in Lie theory. My view of mathematics, and of the world as a whole, would not be the same without all of you.
- to the incredible friends that I met in Corvallis, Jake and Megan Pratt, Cory Langhoff, and Lisa Tedder. I have met few people as warm, thoughtful, and downright fun as you four. More than friends of convenience, you make me hope to be an Oregonian forever.
- finally, and most importantly, to my wife Lucy. I can't imagine where I would be without you, but I know my life would be poorer. You make me a better person than I have any right to be, and this work is as much yours as it is mine.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

LIST OF FIGURES

Nomenclature

$[n]$ The set of natural numbers $\{1, \ldots, n\}$

$\operatorname{aff} A$ The affine hull of the set $A \subseteq \mathbb{R}^n$

$\bar{P}_n$ The space of all planar n-gons with marked vertices, modulo orientation preserving isometries of the plane

$\operatorname{conv} A$ The convex hull of the set $A \subseteq \mathbb{R}^n$

$\mathbb{R}$ The set of real numbers

$\mathbb{R}^n$ The n-fold product of $\mathbb{R}$

$\mathcal{M}_r$ The moduli space of polygons with edge-length vector r

$\operatorname{relint} A$ The relative interior of $A \subseteq \mathbb{R}^n$.

Σ_n The singular set of D_n

$\tilde{D}_n$ The fundamental complex in D_n

$\varnothing$ The empty set

D_n The edge-length polytope of n-gons

$F^k(P)$ The set of all k-faces of a polytope P

$H_p(f)$ The Hessian of $f : M \to \mathbb{R}$ at $p \in M$

P_n The subspace of $\bar{P}_n$ of unit perimeter n-gons

S_n The symmetric group on n-elements

1. Introduction

This dissertation principally concerns the study of the so-called *moduli space of polygons.* The idea is as follows. Consider a polygon lying in the Euclidean plane. Label the edges of this polygon by their edge-lengths, $(r_1, \ldots, r_n) = r \in \mathbb{R}^n_{\geq 0}$. By varying the interior angles of the polygon, one can generally form many different polygons with the same edge-length vector. For example, a rhombus shares the same edge-lengths as a square of equal perimeter. We allow our polygons to have self-intersections, and it will be useful at times to allow one or more of the edge-lengths to shrink to 0. By considering all polygons with a given edge-length vector r, and identifying those polygons that differ by an orientation preserving isometry of the plane, we obtain the moduli space of polygons $\mathcal{M}_r$.

The study of configuration spaces like the moduli space of polygons has a long history. Early applications involved the problem of transforming rotational motion into linear motion in mechanical linkages. For example, James Watt described what is essentially the moduli space of a certain quadrilateral in his 1784 patent for a steam engine (Figure 1.1). Watt's linkage can still be found in automobile suspensions to this day! Modern applications include motion planning in robotics [27], and the obvious generalization to three dimensions has applications in chemistry [26].

The primary inspiration for this dissertation comes from a 1995 paper by Michael Kappovich and John Millson [18]. Portions of their results are also found in [10, 13, 14, 24, 30, 17]. Kappovich and Millson's main insight was that the diffeomorphism types of $\mathcal{M}_r$ could be parametrized by a certain polytope subdivided by hyperplanes. They were then able use Morse theory and a wall crossing argument to

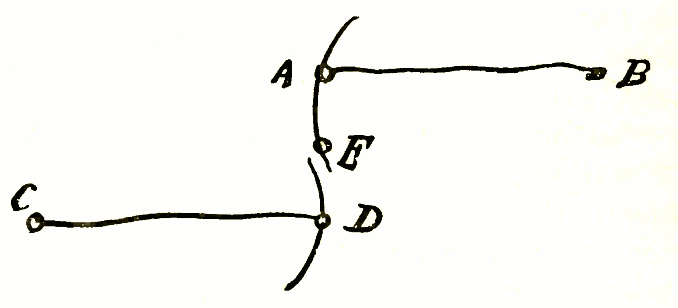

Figure 1.1.: James Watt's hand drawn diagram of his linkage, from a letter to his son. In this diagram, points B and C are fixed, and point E is the midpoint of a segment connecting point A to point D. Thus, this linkage has four fixed lengths CD, DA, AB, and BC forming a (self-intersecting) quadrilateral. The space of all possible configurations of this linkage, modulo rotations and translations, forms a moduli space of quadrilaterals. (James Watt. *The Kinematics of Machinery Fig. 1.* 1808. <wikipedia.org>. Accessed May 9, 2019.)

identify all of the smooth manifolds that arise as moduli spaces of pentagons, and they gave a finite list of diffeomorphism types for the moduli spaces of hexagons, though this list was not tight in the sense that some of the diffeomorphism types are not actually realized in a moduli space of hexagons.

Since Kappovich and Millson's 1995 paper, a body of literature has developed around moduli spaces of polygons. Kappovich and Millson themselves extended their results to 3-dimensional Euclidean space, 3-dimensional hyperbolic space, and 2-dimensional spherical space [19, 21, 20]. In this process, they found an additional symplectic structure on the moduli space of spatial polygons. This symplectic structure proved very fruitful, and a great deal of attention in this field has been focused on the 3-dimensional case. For example, Hausmann and Knutson [9], and later Mandini using different methods [25], computed the cohomology rings of the moduli spaces of spatial polygons. The symplectic volume of the moduli spaces of spatial polygons was computed first by Kamiyama and Tezuka [16] and later by Khoi [22].

As it turns out, the moduli spaces of polygons are generically smooth manifolds. The sense that this is generic will be made clear in Section 2.3. However, for certain combinations of edge-lengths, the moduli space $\mathcal{M}_r$ can contain nonmanifolds points. Relatively little attention has been paid to this non-smooth case. It was known to Walker [30] that the non-manifold moduli spaces of quadrilaterals are homeomorphic to either (i) a wedge of two circles, (ii) two circles joined at two points, or (iii) three circles, any pair of which share a unique point of intersection. In this dissertation, we turn our attention to the non-manifold spaces that arise as moduli spaces of pentagons. It turns out that all non-manifold points in these spaces are locally modeled on a cone $C(S^0 \times S^1)$. Collectively, we will refer to spaces of this type as *crimped manifolds* (Definiton 1.1.2). In Chapter 3 we compute, up to homeomorphism, all of the crimped manifolds that arise as moduli spaces of pentagons. We then turn our attention to bounding the number of diffeomorphism types of moduli spaces of polygons in Chapter 4. Kappovich and Millson were able to compute the manifolds that arise as moduli spaces of pentagons, and they produced a list of manifolds which gave an upper bound on the number of distinct diffeomorphism types of moduli spaces of hexagons. We improve the bound in the hexagon case, and provide bounds on the number of diffeomorphism types for the moduli spaces of 7- and 8-gons.

Remark 1.0.1. After writing this dissertation, we became aware that the results of Chapter 4 were discovered by Hausmann and Rodriquez [12] and later improved upon by Hausmann [11]. Our methods are substantially similar to theirs, though they did not produce the graphs of the networks found in Figures 4.1 and 4.2.

1.1. Main results

Here we summarize the main results of this dissertation. The chief result of Chapter 3 is providing a list of spaces illustrating the homeomorphism types of the moduli spaces of pentagons. Before we state the result, we require some notation.

Definition 1.1.1. We will denote the set of natural numbers no larger than n by

$$[n] = \{1, \ldots, n\}.$$

Definition 1.1.2. Let M be a topological space, and $\Pi = \{p_1, \ldots, p_n\} \in M$ be a finite set of points. Assume that $M \backslash \Pi$ is a smooth manifold of dimension n, and about each $p_i \in \Pi$ there is a neighborhood $N_i \subseteq \Pi$ homeomorphic to a cone over a product of spheres $p_i \in N_i \cong C\left((S^j \times S^k\right)$, with $j + k = n$. Then we will call M a *crimped manifold* and the $p_i \in \Pi$ will be called *crimped points.*

Definition 1.1.3. Let Σ_g denote an orientable surface of genus g. For distinct points $p_1, \ldots, p_n, q_1, \ldots, q_n \in \Sigma_g$, define the equivalence relation $p_i \sim q_i$ for $i \in [n]$. Denote by $\Sigma_{g,n}$ the quotient space

$$\Sigma_{g,n} = \Sigma_g / \sim .$$

We call $\Sigma_{g,n}$ a *surface of genus g with $2n$-points pairwise identified.*

Definition 1.1.4. Let M and N be topological spaces. For distinct points

$$m_1, \ldots, m_q \in M,\ n_1, \ldots, n_q \in N,$$

define the equivalence relation $m_i \sim n_i$ for $i \in [q]$. We denote the disjoint union of M and N by $M \bigsqcup N$. Then the *q-fold wedge sum $M \vee_q N$* is the quotient

$$M \vee_q N = \left(M \bigsqcup N\right) / \sim .$$

Remark 1.1.5. Notice that both $\Sigma_{g,n}$ and $\Sigma_{g'} \vee_q \Sigma_{g'}$, $g, g', q, n \in \mathbb{N} \cup \{0\}$, are examples of crimped manifolds, where each of the crimped points has a neighborhood homeomorphic to a cone $C(S^1 \times S^0)$.

It turns out the non-manifold moduli spaces of pentagons are homeomorphic to either a surface with $2n$ points pairwise identified or q-fold wedge sum of two tori. We list the specific crimped manifolds in the tables below. The notations of the first columns will become apparent in the introduction to Chapter 3.

Theorem 1.1.6. *The following tables list all homeomorphism types of non-manifold moduli spaces of hexagons. The d-cells of D_5, $0 \leq d \leq 3$, are listed in the left column, and the homeomorphism types of $\mathcal{M}_r$ for r in the relative interior of each d-cell is listed in the right column.*

3-*cells in* D_5	*Topology of* $\mathcal{M}_r$
$\text{conv}\{h_{i,j}, h_{i,k}, h_{i,\ell}, t_i\}$	$\Sigma_{0,1}$
$\text{conv}\{h_{i,j}, h_{i,k}, q_\ell, t_i\}$, $\ell \notin \{i,j,k\}$	$\Sigma_{1,1}$
$\text{conv}\{h_{i,j}, h_{i,k}, q_\ell, q_m\}$, $\ell, m \notin \{i,j,k\}$	$\Sigma_1 \vee \Sigma_1$
$\text{conv}\{h_{i,j}, q_k, q_\ell, t_i\}$, $k, \ell, \notin \{i,j\}$	$\Sigma_{2,1}$
$\text{conv}\{q_i, q_j, q_k, t_\ell, t_m\}$, $\ell, m \notin \{i,j,k\}$	$\Sigma_{3,1}$

2-*cells in* D_5	*Topology of* $\mathcal{M}_r$
$\text{conv}\{h_{i,j}, h_{i,k}, t_i\}$	$\Sigma_{0,2}$
$\text{conv}\{h_{i,j}, q_k, t_i\}$, $k \notin \{i,j\}$	$\Sigma_{1,2}$
$\text{conv}\{h_{i,j}, q_k, q_\ell\}$, $k, \ell \notin \{i,j\}$	$\Sigma_1 \vee_2 \Sigma_1$
$\text{conv}\{q_i, q_j, t_k\}$	$\Sigma_{2,2}$

1-*cells in* D_5	*Topology of* $\mathcal{M}_r$
$\text{conv}\{h_{i,j}, t_i\}$	$\Sigma_{0,3}$
$\text{conv}\{q_i, t_j\}$	$\Sigma_{1,3}$
$\text{conv}\{q_i, q_j\}$	$\Sigma_1 \vee_3 \Sigma_1$

0-*cells in* D_5	*Topology of* $\mathcal{M}_r$
$\{t_i\}$	$\Sigma_{0,4}$

In the course of proving Theorem 1.1.6 we realized that the Kappovich-Millson polytope D_n contained a large amount of redundant data. In order to push our census of the topology of the moduli spaces of polygons to higher dimensions, it became necessary to find a way to simplify that data. We pursue that process in Chapter 4. As a result, we were able to provide a bound on the number of non-diffeomorphic smooth manifolds that arise as moduli spaces of n-gons for $n \leq 8$. Denote by t_n the number of non-diffeomorphic, maximal dimensional smooth manifolds that arise as a moduli space of n-gons. It is shown in [18] that $t_3 = 1$, $t_4 = 2$, and $t_5 = 6$. We extend those results with the following theorem.

Theorem 1.1.7. *Let t_n denote the number of distinct (up to diffeomorphism) maximal-dimensional, smooth manifolds that arise as a moduli space of n-gons. Then we have the following bounds:*

$$t_6 \leq 20,\ t_7 \leq 134,\ t_8 \leq 2469.$$

Remark 1.1.8. After completing the work of this dissertation, we were made aware that these bounds were previously known to Hausmann and Rodriguez [12]. Their approach to providing these bounds was more combinatorial, while ours is more geometric. We both attempt to count the number of connected components of a certain subdivision of $\tilde{D}_n \subseteq \mathbb{R}^n$. For a definition of $\tilde{D}_n$, see Section 4.1. Hausmann and Rodriguez's approach is to induce a certain partial order on the components of $\tilde{D}_n$. They then give each chamber of $\tilde{D}_n$ a label they call the *genetic code*. They develop a set of rules that produce a finite set of *virtual* genetic codes, and then they use polyhedral computations to check if their virtual genetic codes are, indeed, realized as the genetic code of a chamber of $\tilde{D}_n$.

Our approach is much more geometric. We consider $\tilde{D}_n$ to be the fundamental region of a larger polytopal complex modulo the action of a certain reflection group. We then identify one component of $\tilde{D}_n$ for arbitrary n. Finally, we identify the remaining components of $\tilde{D}_n$ by inductively searching for the chambers adjacent to all of the known chambers of $\tilde{D}_n$.

2. Background

Before we proceed with the proofs of our main results, we require some background material on the tools of convex geometry and Morse theory that form the basis of our arguments. We provide those backgrounds in Sections 2.1 and 3.2. We then give an overview of the main results of Kappovich and Millson in Section 2.3.

2.1. Polytopes

Here we collect some basic facts about one of the fundamental objects in combinatorial geometry: the *polytope*. The main source of this material is Grünbaum's *Coinvex Polytopes* [8]. A polytope allows one to generalize the combinatorial properties (vertices, edges, faces, etc.) of polygons and polyhedra to higher dimensions. Before describing the manner of this generalization, we require language allowing us to discuss subsets of $\mathbb{R}^n$.

Definition 2.1.1. A *hyperplane* $H \subseteq \mathbb{R}^n$ is a solution to a linear equation

$$H_{h,b} = \{x \in \mathbb{R}^n \mid h \cdot x = b\}$$

for some fixed $h \in \mathbb{R}^n \backslash \{0\}$, $b \in \mathbb{R}$. In the above formula, $\cdot$ represents the usual Euclidean inner product. The vector h is called a *normal* to the hyperplane $H_{h,b}$. Every hyperplane splits $\mathbb{R}^n$ into two (closed) *half-spaces* $H^{\pm}_{h,b}$ defined by

$$H^{+}_{h,b} = \{x \in \mathbb{R}^n \mid h \cdot x \geq b\}$$

and

$$H^-_{h,b} = \{x \in \mathbb{R}^n \mid h \cdot x \leq b\}.$$

Definition 2.1.2. Let $p_1, \ldots, p_m \in \mathbb{R}^n$. A *convex combination* of $p_1, \ldots, p_m$ is a sum

$$\sum_{i=1}^{m} \lambda_i p_i$$

for some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_{\geq 0}$ satisfying $\sum_{i=1}^{m} \lambda_i = 1$. On the other hand, if the $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ are possibly negative, and $\sum_{i=1}^{m} \alpha_i = 1$, then we call

$$\sum_{i=1}^{m} \alpha_i p_i$$

an *affine combination* of $p_1, \ldots, p_m$. Now consider any $A \subseteq \mathbb{R}^n$. The *convex (affine) hull* of A is

$$\operatorname{conv} A \,(\operatorname{aff} A) = \{x \in \mathbb{R}^n \mid x \text{ is a convex (affine) combination of points in } A\}.$$

We call set $D \subseteq \mathbb{R}^n$ *convex* provided $D = \operatorname{conv} D$.

Example 2.1.3. Consider a half-space $H^+_{h,b} \subseteq \mathbb{R}^n$. Let

$$y = \sum_{i=1}^{m} \lambda_i x_i$$

be a convex combination of points $x_1, \ldots, x_m \in H^+_{h,b}$. Then

$$h \cdot y = \sum_{i=1}^{m} \lambda_i (h \cdot x_i) \geq \sum_{i=1}^{m} \lambda_i b = b,$$

so $H^+_{h,b}$ is convex. Notice the importance that each $\lambda_i \geq 0$ to conclude $\lambda_i(h \cdot x_i) \geq \lambda_i b$. Indeed, no half-space is closed under affine combinations. To see this, notice that if $z_1 = \frac{b}{|h|^2} h$ and $z_2 = z_1 + \frac{h}{|h|^2}$, then

$$h \cdot z_1 = b,$$

and

$$h \cdot z_2 = b + 1 > b.$$

We conclude $z_1, z_2 \in H^+_{h,b}$. We now have $z = 2z_1 - z_2 \in \operatorname{aff} H^+_{h,b}$, but $z \notin H^+_{h,b}$ because

$$h \cdot z = 2b - (b + 1) < b.$$

Definition 2.1.4. A subset $A \subseteq \mathbb{R}^n$ is called an *affine subspace* if, for fixed $a \in A$, the set

$$A - a = \{x - a \in \mathbb{R}^n \mid x \in A\}$$

is a linear subspace of $\mathbb{R}^n$

Proposition 2.1.5. *Let $S \subseteq \mathbb{R}^n$ be a nonempty subset, and $A = \operatorname{aff} S$. Then A is an affine subspace of $\mathbb{R}^n$. Conversely, suppose that $A \subseteq \mathbb{R}^n$ is an affine subspace. Then* $\operatorname{aff} A = A$.

Proof. Let $S \subseteq \mathbb{R}^n$ be nonempty, set $A = \operatorname{aff} S$, and choose $a \in A$. Now, as $A - a \supseteq S - a \neq \varnothing$ is nonempty, we proceed to check closure under scalar multiplication. Let $\lambda \in \mathbb{R}$ and $x - a \in A - a$. Write $x = \sum_{i=1}^m \alpha_i s_i$ as an affine combination of $s_1, \dots, s_m \in S$. Then

$$\lambda(x - a) = \left(\sum_{i=1}^m (\lambda \alpha_i) s_i + (1 - \lambda) a \right) - a \in A - a$$

since $\sum_{i=1}^m \lambda \alpha_i + (1 - \lambda) = 1$ and $s_1, \dots, s_m, a \in S$. To see closure under vector addition, let $y - a \in A - a$, and write $y = \sum_{j=1}^k \beta_j t_j$ as an affine combination of $t_1, \dots, t_k \in S$. Then

$$(x - a) + (y - a) = 2 \left(\sum_{i=1}^m \frac{\alpha_i}{2} s_i + \sum_{j=1}^k \frac{\beta_j}{2} t_j - a \right) \in A - a,$$

since $\sum_{i=1}^m \alpha_i/2 + \sum_{j=1}^k \beta_j/2 = 1$. Thus $A - a$ is closed under vector addition, and therefore a linear subspace of $\mathbb{R}^n$.

For the converse, assume that $A \subseteq \mathbb{R}^n$ is an affine subspace. It is immediate from the definition that $A \subseteq \operatorname{aff} A$, so it suffices to show the reverse containment. Assume $x \in \operatorname{aff} A$, and write $x = \sum_{i=1}^m \alpha_i a_i$ as an affine combination of $a_1, \dots, a_m \in A$. Then, since $\sum_{i=1}^m \alpha_i = 1$,

$$x - a_1 = x - \sum_{i=1}^m \alpha_i a_1 = \sum_{i=1}^m \alpha_i (a_i - a_1)$$

is a linear combination of elements in the linear space $A - a_1$, hence $x - a_1 \in A - a_1$. Therefore $x \in A$. □

Definition 2.1.6. Let $A \subseteq \mathbb{R}^n$. The *relative interior* of A is denoted $\operatorname{relint} A$ and represents the interior of A in the subspace topology of $\operatorname{aff} A \subseteq \mathbb{R}^n$, i.e.,

$$\operatorname{relint} A = A^\circ \subseteq \operatorname{aff} A \subseteq \mathbb{R}^n.$$

We are now equipped with the language to define a polytope. As motivation, consider two ways to view a polyhedron. In one respect, a polyhedron can be viewed as being carved out of $\mathbb{R}^3$ by making a series of straight cuts. This view gives rise to the h-representation (h stands for "half-space") of a polytope. On the other hand, a polyhedron can be viewed as taking a finite set of points in $\mathbb{R}^n$, and tautly stretching a membrane across them. This gives rise to the v-representation (v for "vertex") of a polytope. We give both definitions here.

Definition 2.1.7. Let $H_1, \dots, H_m \subset \mathbb{R}^n$ be a collection of closed half-spaces. A *polytope* $P \subseteq \mathbb{R}^n$ is the intersection

$$P = \bigcap_{i=1}^m H_i.$$

Such a description of P as an intersection of finitely many half-spaces is called an *h-representation* of P, and the set $\mathcal{H} = \{H_1, \dots, H_m\}$ is called an *h-generating set* for P.

Definition 2.1.8. Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be a collection of points. A *polytope* $P \subset \mathbb{R}^n$ is the convex hull

$$P = \operatorname{conv}\{v_1, \ldots, v_m\}.$$

Such a description of P as a convex hull of finitely many points is called a *v-representation* of P, and the set $V = \{v_1, \ldots, v_m\}$ is called a *v-generating set* for P.

A subset $B \subseteq \mathbb{R}^n$ is *bounded* provided $B \subseteq B_r(0)$, $0 \in \mathbb{R}^n$, $r < \infty$, is contained in a finite radius ball. It should be noted that some authors require a polytope to be bounded. For our purposes it will be useful to consider the unbounded case, so we do not impose that restriction. Regardless, a fundamental theorem of convex geometry is the Minkowski-Weyl Theorem which states that, so long as a polytope *is* bounded, it has both a v-representation and an h-representation.

Theorem 2.1.9 (Minkowski-Weyl)**.** *Assume that $P \subseteq \mathbb{R}^n$ is compact. Then the following are equivalent.*

1. *There exist half-spaces $H_1, \ldots, H_s \subseteq \mathbb{R}^n$ such that*

$$P = \bigcap_{i=1}^{s} H_i.$$

2. *There exist a finite list of points $v_1, \ldots, v_t \in \mathbb{R}^n$ such that*

$$P = \operatorname{conv}\{p_1, \ldots, p_t\}.$$

A proof of the Minkowski-Weyl theorem can be found in, e.g., [31] where it appears as Ziegler's "Main Theorem" of Chapter 1. For an algorithm to switch between v- and h-representations of a polytope see [1]. In addition to the utility of being able to describe a polytope in two different ways, the equivalence of of the

v- and h-representations of a polytope has practical significance. For example, the goal of linear programming is to maximize a linear function, called the *objective function*, subject to a finite set of linear constraints. The constraints then define a polytope, and, so long as this polytope is non-empty and bounded, the objective is maximized at some (not necessarily unique) vertex. Thus, linear programming is equivalent to switching from the h-representation to the v-representation of a polytope.

We now make precise the combinatorial properties of a polytope.

Definition 2.1.10. Let $P \subseteq \mathbb{R}^n$ be a polytope. The *dimension* of P is

$$\dim P = \dim (\operatorname{aff} P) .$$

We say P is *full dimensional* if $\dim P = n$. A hyperplane $H \subset \mathbb{R}^n$ is called a *supporting hyperplane* of P provided $P \cap H \neq \varnothing$, and P lies entirely in one of the closed half-spaces defined by H. Alternatively, the hyperplane H *cuts* P provided $H \cap \operatorname{relint} P \neq \varnothing$.

Remark 2.1.11. Notice that if $P \subseteq \mathbb{R}^n$ is full dimensional, and H is a supporting hyperplane of a polytope P, then $H \cap P \subseteq \partial P \subseteq \mathbb{R}^n$ lies in the boundary of P as a subset of $\mathbb{R}^n$.

Definition 2.1.12. A *face* of P is the intersection of P with a closed half-space whose bounding hyperplane does not cut P. It follows immediately from the h-representation that a nonempty face $F \subseteq P$ of a polytope P is, itself, a polytope with dimension $\dim F \leq \dim P$. If $\dim F = k$, we will call F a k-face. If $\dim P = n$, we call the 0-faces *vertices*, the 1-faces *edges*, and the $(n-1)$-faces *facets*. We denote the set of all k-faces of P

$$F^k(P) = \{F \subseteq P \mid F \text{ is a } k\text{-face of } P\} .$$

Remark 2.1.13. As a consequence of the definition, the polytope P is a face of itself, as is the empty set $\varnothing$. By convention, we declare $\dim \varnothing = -1$. We call a k-face *proper* provided $-1 < k < \dim P$.

Notice that if $H \subset \mathbb{R}^n$ is a supporting hyperplane of a polytope $P \subseteq \mathbb{R}^n$, then either $H \cap P = H^+ \cap P$ or $H \cap P = H^- \cap P$. Therefore the intersection of a supporting hyperplane with P is a face of P. Moreover, every proper face of P is realized by the intersection of P with a supporting hyperplane.

Definition 2.1.14. A supporting hyperplane $H \subset \mathbb{R}^n$ of a polytope $P \subseteq \mathbb{R}^n$ is an *essential supporting hyperplane* provided

$$\dim (H \cap P) = (\dim P) - 1.$$

Definition 2.1.15. A v- (h-) description of a polytope P is *minimal* provided the v- (h-) generating set S is minimal amongst all v- (h-) generating sets.

The next goal is to verify the fact that minimal v- and h-generating sets essentially correspond with the vertices and facets of a polytope. To prove this, though, a lemma is required.

Lemma 2.1.16. *Assume $A, B \subseteq \mathbb{R}^n$ are compact, convex subsets with $A \cap B = \varnothing$. Then there exists a hyperplane $H \subseteq \mathbb{R}^n$ such that $A \subseteq H^+$, $B \subseteq H^-$, and $A \cap H = \varnothing = B \cap H$.*

Proof. Consider the Euclidean distance function $\delta : A \times B \to \mathbb{R}$ defined by $\delta(a, b) = |a - b|$. Since $A \times B$ is compact, δ attains a minimum. Let $(a, b) \in A \times B$ be such a minimum, and set $d = a - b$. Since $A \cap B = \varnothing$, we have $d \neq 0 \in \mathbb{R}^n$. Now let $p = (a + b)/2$, and set $c = d \cdot p$. I claim that

$$H_{d,c} = \{x \in \mathbb{R}^n \mid d \cdot x = c\}$$

is the desired hyperplane. Calculating

$$\begin{aligned} d \cdot a &= \frac{d \cdot a + d \cdot b}{2} + \frac{1}{2}(d \cdot a - d \cdot b) \\ &= d \cdot p + \frac{1}{2}\left(|a|^2 - 2a \cdot b + |b|^2\right) \\ &= d \cdot p + \frac{1}{2}|d|^2 \\ &> d \cdot p, \end{aligned}$$

we see $a \in \left(H^+_{d,c}\right)^\circ$ is in the interior of one of the half-spaces bounded by H. We show that, in fact, $A \subset (H^+)^\circ$. For a contradiction, assume that $a' \in H^- \cap A$. Then

$$(a' - a) \cdot d = a' \cdot d - a \cdot d < d \cdot p - d \cdot p = 0,$$

so the angle between the vectors $a' - a$ and d is obtuse. But d is the exterior normal at the point a for the sphere of radius $|d|$ centered at b. Therefore, there is a section of the line segment from a to a' that lies within the ball of radius $|d|$ centered at b. Since A is convex, this contradicts the minimality of the point (a, b) for the function δ. We conclude that $A \subset (H^+)^\circ$, and a symmetric argument shows $B \subset (H^-)^\circ$. □

Proposition 2.1.17. *Let $P \subseteq \mathbb{R}^n$ be a polytope.*

1. *A v-description of P with v-generating set V is minimal if and only if $V = F^0(P)$.*

2. *Assume P is full dimensional. An h-description of P with h-generating set $\mathcal{H}$ is minimal if and only if every half-space $H \in \mathcal{H}$ has the property that ∂H is an essential supporting hyperplane of P.*

Proof of (1). We first show that

Claim. If V is any v-generating set of P, then $F^0(P) \subseteq V$.

For a contradiction, assume that V is a v-generating set of a polytope P, and

$v \in F^0(P)\backslash V$. Then we may write

$$v = \sum_{i=1}^{m} \lambda_i v_i$$

as a convex combination of $v_1, \ldots, v_m \in V$. Since v is a vertex, there is a half-space $H \subseteq \mathbb{R}^n$ such that $H \cap P = \{v\}$. Since $v_1, \ldots, v_m \in V \subseteq P$, we conclude $v_1, \ldots, v_m \in H^c$ are in the complement of H. But H^c is convex, and v is a convex combination of points in H^c. Therefore $v \in H^c$, a contradiction. We conclude $v \in V$, and the Claim is proved.

To prove (1), it now suffices to show that if V is a minimal v-generating set of P, then $V \subseteq F^0(P)$. Let $V = \{v_1, \ldots, v_j\}$ be such a minimal v-generating set, select $k \in \{1, \ldots, j\}$, and denote $Q = \text{conv}\{v_i \in V \mid i \neq k\}$. Then $v_k \notin Q$ by the minimality of V as a v-generating set. Thus Q and $\{v_k\}$ are compact, convex, disjoint subsets of $\mathbb{R}^n$, and we may apply Lemma 2.1.16 to find a separating hyperplane $H \subseteq \mathbb{R}^n$ satisfying $Q \subseteq (H^+)^\circ$ and $v_k \in (H^-)^\circ$. Assume H is defined by the equation $d \cdot x = a$. Let H' be the translate of H defined by the equation $d \cdot x = d \cdot v_k$. Consider any convex combination $\sum_{i=1}^{j} \lambda_i v_i$. Then

$$d \cdot \sum_{i=1}^{j} \lambda_i v_i \geq \lambda_k d \cdot v_k + a \sum_{i \neq k} \lambda_i \geq \lambda_k d \cdot v_k + (d \cdot v_k) \sum_{i \neq k} \lambda_i = d \cdot v_k,$$

with equality holding if and only if $\lambda_k = 1$ and $\lambda_i = 0$ for $i \neq k$. Since any point in P can be expressed as such a convex combination, we conclude that $(H')^- \cap P = \{v_k\}$, and $v_k \in F^0(P)$. $\square$

Proof of (2). Denote the set of essential supporting hyperplanes of $P \subseteq \mathbb{R}^n$ by $E(P)$. Assume that P is full dimensional. We first claim

Claim. If $\mathcal{H}$ is any h-generating set of P, then $E(P) \subseteq \mathcal{H}$.

Assume that $\Pi \in E(P)$. First notice that

$$\partial P = \partial \bigcap_{H \in \mathcal{H}} H \subseteq \bigcup_{H \in \mathcal{H}} \partial H.$$

Second, since P is full-dimensional and Π is a supporting hyperplane of P, $\Pi \cap P \subseteq \partial P$. Combining these two observations yields

$$\Pi \cap P \subseteq \Pi \cap \bigcup_{H \in \mathcal{H}} \partial H = \bigcup_{H \in \mathcal{H}} (\Pi \cap \partial H).$$

Since Π is an essential supporting hyperplane of the full dimensional polytope P, we have

$$\dim(\Pi \cap P) = n - 1 \leq \max_{H \in \mathcal{H}} \dim(\Pi \cap \partial H) \leq n - 1.$$

We conclude $\dim \Pi \cap H = n - 1$ for some $H \in \mathcal{H}$, whence $\Pi = H \in \mathcal{H}$.

It now suffice to show that if $\mathcal{H}$ is a minimal h-generating set, then $\partial H \in E(P)$ for each $H \in \mathcal{H}$. Assuming $\mathcal{H}$ is such a minimal h-generating set, fix $K \in \mathcal{H}$, and set $\mathcal{H}' = \mathcal{H} \backslash \{K\}$. Now define

$$P' = \bigcap_{H \in \mathcal{H}'} H,$$

and $L = \partial K \cap P'$. Toward a contradiction, assume that $\dim L < n - 1$. Since $P \subseteq P'$ is full-dimensional, so, too, is P'. Therefore, since $\dim L < n - 1$, ∂K does not cut P'. We conclude P' lies entirely in one of the half-spaces defined by ∂K. But $P \subseteq K$ and $\varnothing \subsetneq P \subseteq P'$, so we must have $P' \subseteq K$. Therefore $P = P' \cap K = P'$, contradicting the minimality of $\mathcal{H}$. We conclude that, in fact, $\dim L = n - 1$. Now

$$L = \partial K \cap P' = (\partial K \cap K) \cap P' = \partial K \cap (K \cap P') = \partial K \cap P,$$

so ∂K must be an essential supporting hyperplane of P. $\square$

To gain familiarity with the setting, let us consider the v- and h-representations

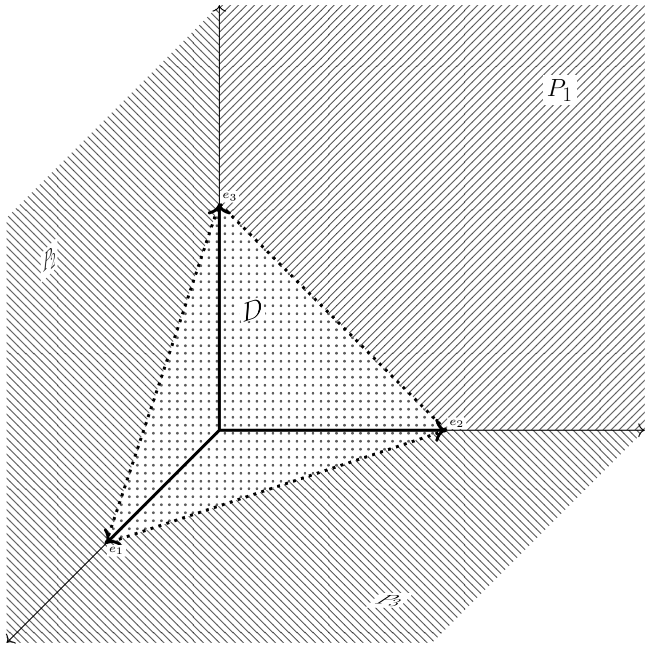

Figure 2.1.: The standard 2-simplex Δ^2.

of a few important families of polytopes.

Example 2.1.18. Let $e_i \in \mathbb{R}^{n+1}$ denote the standard i-th basis vector. The *standard n-simplex* $\Delta^n \subseteq \mathbb{R}^{n+1}$ (Figure 2.1) has a v-representation

$$\Delta^n = \operatorname{conv}\{e_1, \ldots, e_{n+1}\}.$$

Alternatively, let $P_j \subseteq \mathbb{R}^{n+1}$ denote the j-th coordinate hyperplane, i.e.,

$$P_{e_j,0} = \left\{x \in \mathbb{R}^{n+1} \mid e_j \cdot x = 0\right\}.$$

Define $d = \sum_{i=1}^{n+1} e_i$ and the hyperplane

$$D_{d,1} = \left\{x \in \mathbb{R}^{n+1} \mid d \cdot x = 1\right\}.$$

Then an h-representation of Δ^n is the following intersection of closed half-spaces

$$\Delta^n = D^+_{d,1} \cap D^-_{d,1} \cap \bigcap_{j=1}^{n+1} P^+_{e_j,0}.$$

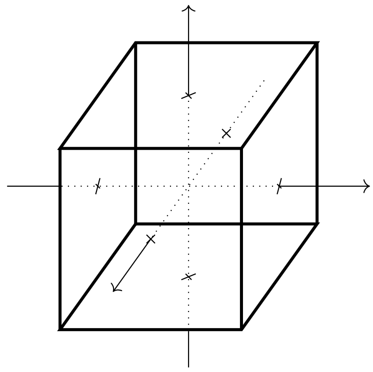

Figure 2.2.: The 3-cube.

Example 2.1.19. Let

$$V = \{(\delta_1, \ldots, \delta_n) \in \mathbb{R}^n \mid \delta_i \in \{\pm 1\}, 1 \leq i \leq n\}.$$

The *n-cube* $\square^n \subseteq \mathbb{R}^n$ (Figure 2.2) has *v*-representation

$$\square^n = \operatorname{conv} V.$$

On the other hand, define the hyperplane

$$P_{j^\pm} = \{x \in \mathbb{R}^n \mid e_j \cdot x = \pm 1\},$$

where, as before, e_j denotes the j-th standard basis vector in $\mathbb{R}^n$. Then $\square^n$ has *h*-representation

$$\square^n = \bigcap_{i=1}^{n} P_{i^+}^- \cap \bigcap_{j=1}^{n} P_{j^-}^+.$$

Example 2.1.20. Again, let $e_i \in \mathbb{R}^n$ denote the i-th standard basis vector. Let

$$V = \{\pm e_1, \ldots, \pm e_n\}.$$

The *n-cross-polytope* $\diamondsuit^n \subseteq \mathbb{R}^n$ (Figure 2.3) has a *v*-representation

$$\diamondsuit^n = \operatorname{conv} V.$$

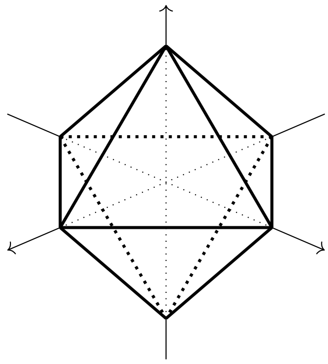

Figure 2.3.: The 3-cross-polytope $\diamondsuit^3$ is also known as the octahedron.

Toward an h-representation of $\diamondsuit^n$, let

$$D = \{(\delta_1, \ldots, \delta_n) \in \mathbb{R}^n \mid \delta_i = \pm 1, 1 \le i \le n\}.$$

Let for each $d \in D$, consider the hyperplane $H_{d,1} = \{x \in \mathbb{R}^n \mid d \cdot x = 1\}$. Then

$$\mathcal{H} = \left\{H_{d,1}^- \mid d \in D\right\}$$

is an h-generating set of $\diamondsuit^n$, i.e.,

$$\diamondsuit^n = \bigcap_{d \in D} H_{d,1}^-.$$

For example, the 3-cross-polytope in Figure 2.3 is defined by the eight inequalities

$$\begin{aligned} (1,1,1) \cdot x &\le 1 & \qquad (1,-1,-1) \cdot x &\le 1 \\ (1,1,-1) \cdot x &\le 1 & \qquad (-1,1,-1) \cdot x &\le 1 \\ (1,-1,1) \cdot x &\le 1 & \qquad (-1,-1,1) \cdot x &\le 1 \\ (-1,1,1) \cdot x &\le 1 & \qquad (-1,-1,-1) \cdot x &\le 1. \end{aligned}$$

These examples show that the size of minimal h- and v-generating sets can be meaningfully different. While the n-simplex has a minimal v-generating set of size

$n+1$ and a minimal h-generating set of size $n+3$, the n-cube has a minimal v-generating set of size 2^n, but a minimal h-generating set of only size only $2n$. The n-cross-polytope is just the reverse; it has a minimal v-generating set of size $2n$, but a minimal h-generating set of size 2^n. When dealing with high dimensional polytopes, then, there can be significant computational savings by selecting one description over the other.

The concept of a polytope generalizes naturally, and saliently for the sequel, to a *polytopal complex*.

Definition 2.1.21. Let $\mathcal{C}$ be a collection of polytopes in $\mathbb{R}^n$. We call $\mathcal{C}$ a *polytopal complex* provided the following two axioms are satisfied.

1. If $P \in \mathcal{C}$, then every d-face of P is contained in $\mathcal{C}$, i.e., $F^d(P) \subseteq \mathcal{C}$.
2. If $P, P' \in \mathcal{C}$, then $P \cap P' \in \mathcal{C}$.

Definition 2.1.22. A *d-cell in $\mathcal{C}$* is a polytope $P \in \mathcal{C}$ satisfying $\dim P = d$. As an abuse of notation, we will refer to the d-cells in $\mathcal{C}$ as

$$F^d(\mathcal{C}) = \{P \in \mathcal{C} \mid \dim P = d\}.$$

In the sequel, we will study a certain polytopal complex that arises as a subdivision of a given polytope by hyperplanes. We describe that process in general here. Assume $P \subseteq \mathbb{R}^n$ is a polytope with h-generating set $\mathcal{B} = \{B_1, \ldots, B_i\}$ where each $B_k \subseteq \mathbb{R}^n$ is a half-space. Now let $\mathcal{W} = \{W_1, \ldots, W_j\}$ be a collection of hyperplanes $W_k \subseteq \mathbb{R}^n$, $1 \leq k \leq j$. Let $\mathcal{Q}$ be the collection of all polytopes of the form

$$Q = P \cap \bigcap_{k=1}^{j} W_k^*,$$

where each W_k^* is a choice of half-space W_k^+ or W_k^-. Finally define

$$\mathcal{C} = \bigcup_{Q \in \mathcal{Q}} \bigcup_{d=0}^{n} F^{(d)}(Q).$$

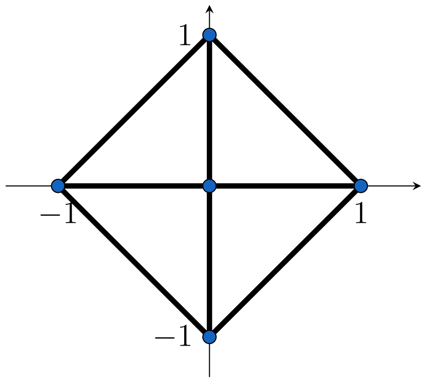

Figure 2.4.: A subdivision of a polytope by hyperplanes.

We call the the polytopal complex $\mathcal{C}$ a *subdivision of P by the collection of hyperplanes $\mathcal{W}$.*

Example 2.1.23. Let

$$P = \operatorname{conv}\{(\pm 1, 0), (0, \pm 1)\} \subseteq \mathbb{R}^2,$$

$$W_x = \{(x, 0) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2,$$

$$W_y = \{(0, y) \mid y \in \mathbb{R}\} \subseteq \mathbb{R}^2,$$

and $\mathcal{W} = \{W_x, W_y\}$. The complex $\mathcal{C}$ that results from the subdivision of P by $\mathcal{W}$ is shown in Figure 2.4. The complex $\mathcal{C}$ contains four 2-faces, eight 1-faces, and five 0-faces.

2.2. Morse theory

In this section we provide an overview of Morse theory. The Morse theory in this section follows Milnor [28]. For a reference on the wider differential topology discussed here, see Bröcker and Jänich [2].

Just as a CW-complex provides a systematic way to resolve a topological space into a collection of fundamental pieces, Morse theory provides a setting for modifying the topological structure of a smooth manifold. There are various flavors of Morse theory, but the essential ingredient is that the presence of a sufficiently well-behaved function on a smooth manifold in turn provides a well-behaved set of coordinates with which to study that manifold. This section will focus on the classical Morse theory.

Definition 2.2.1. Let M be a smooth manifold of dimension n, and let $f : M \to \mathbb{R}$ be a smooth function. A *critical point* of f is a point $p \in M$ where the differential $df_p : T_pM \to T_{f(p)}\mathbb{R}$ vanishes. Equivalently, if $(x^1, \ldots, x^n)$ are coordinates near p, then $(\partial f/\partial x^i)(p) = 0$ for all i.

Definition 2.2.2. Let M be a smooth manifold of dimension n, let $f : M \to \mathbb{R}$ be a smooth function, and let $p \in M$ be a critical point of f. The *Hessian* of f at p is a bilinear form $H_p(f) : T_pM \times T_pM \to \mathbb{R}$ defined as follows. For $v, w \in T_pM$, let $V, W \in \Gamma(TM)$ be vector fields that extend v and w. Then

$$H_p(f)(v, w) = V_p(Wf).$$

Proposition 2.2.3. *The Hessian is a well-defined, symmetric bilinear form. If $(x^1, \ldots, x^n)$ are coordinates for M near p, then the matrix of $H_p(f)$ in the basis $\{\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p\}$ is*

$$H_p(f) = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)(p).$$

Proof. We follow Milnor [28]. We first show $H_p(f)$ is symmetric. Let $v, w \in T_pM$. Then

$$H_p(f)(v, w) - H_p(f)(w, v) = V_p(Wf) - W_p(Vf) = [V, W]_p f = 0,$$

where $[V, W]$ is the Lie bracket of V and W. That $[V, W]_p(f) = 0$ follows from the fact that $[V, W]_p \in T_pM$, and p is a critical point of f.

Next, $H_p(f)(v, w) = V_p(Wf) = v(Wf)$ is independent of the choice of extension

V of v. On the other hand, $H_p(f)(v,w) = H_p(f)(w,v) = W_p(Vf) = w(Vf)$ is independent of the choice of extension W of w. Thus $H_p(f)$ is well-defined.

Finally, in coordinates write $v = \sum_i v^i \left.\frac{\partial}{\partial x^i}\right|_p$, $w = \sum_j w^j \left.\frac{\partial}{\partial x^j}\right|_p$, and use the local extension $W = \sum_j w^j \frac{\partial}{\partial x^j}$ near p. Then

$$\begin{aligned} H_p(f) &= \left(\sum_i v^i \left.\frac{\partial}{\partial x^i}\right|_p \right) \sum_j w^j \frac{\partial f}{\partial x^j} \\ &= \sum_{i,j} v^i w^j \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right)(p). \end{aligned}$$

Therefore, the matrix of $H_p(f)$ in the basis $\{\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p\}$ is $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)(p)$, as claimed. □

Definition 2.2.4. Let M be a smooth n-dimensional manifold, $f : M \to \mathbb{R}$ be a smooth function, and $p \in M$ be a critical point of f. We say that p is a *non-degenerate critical point* provided $H_p(f)$ is non-singular, i.e., $H_p(f)(v,w) = 0$ if and only if either $v = 0$ or $w = 0$. If p is a non-degenerate critical point of f, let λ_- and λ_+ be the maximal dimensions of the subspaces on which $H_p(f)$ is negative and positive definite, respectively. The *index of f at p* is the pair (λ_-, λ_+). If every critical point of f is non-degenerate, then we say f is a *Morse function.*

Remark 2.2.5. Since $H_p(f)$ is nonsingular at p, we have $\lambda_- + \lambda_+ = n$. For this reason many author's simply specify λ_- as the index of f.

Remark 2.2.6. When working in coordinates, the eigenvalues of the matrix $(\partial^2 f/\partial x^i \partial x^j)(p)$ depend on the choice of coordinates. Due to Sylvester's Law of Inertia, though, the index of f at p is coordinate independent.

We are now ready to state the key lemma of Morse theory. For a proof, see [28].

Lemma 2.2.7 (Morse Lemma)**.** *Let M be an n-dimensional smooth manifold, and let $p \in M$ be a non-degenerate critical point of $f : M \to \mathbb{R}$. Denote the index of f at p by (λ_-, λ_+). Then there are coordinates $(x^1, \ldots, x^{\lambda_-}, y^1, \ldots, y^{\lambda_+})$ near p such* *that f has the form*

$$f = f(p) - \left(x^1\right)^2 - \cdots - \left(x^{\lambda_-}\right)^2 + \left(y^1\right)^2 + \cdots + \left(y^{\lambda_+}\right)^2.$$

Corollary 2.2.8. *If $f : M \to \mathbb{R}$ is a Morse function, then the critical points of f are isolated.*

The final goal of this section is to use Lemma 2.2.7 to study how the topology of the fibers of a Morse function change when passing through a critical value. It should be noted that if $f : M \to \mathbb{R}$ is a Morse function on a smooth manifold M, and $p \in \mathbb{R}$ is a regular value of f (i.e., $f^{-1}(p)$ contains no critical points of f), then the fiber $f^{-1}(p)$ is a smooth submanifold of M. This is due to the so-called Regular Level Set theorem [23].

Theorem 2.2.9 (Regular Level Set Theorem)**.** *Let $f : M \to N$ be a smooth map between smooth manifolds, and let $p \in N$ be a regular value of f. Then $f^{-1}(p)$ is a properly embedded submanifold whose codimension is equal to the dimension of the codomain.*

The presence of a Morse function gives a great deal of topological information. Our interest is mainly in understanding how the topology of the level sets of a Morse function changes when the levels cross a critical value. To facilitate that discussion, we develop the language of a *surgery.*

Definition 2.2.10. Let M be a smooth n-dimensional manifold. Denote by $D^\lambda \subseteq \mathbb{R}^\lambda$ the closed unit disk, and $S^{n-\lambda} \subseteq \mathbb{R}^{n-\lambda}$ the unit sphere. Assume that $\phi : D^\lambda \times S^{n-\lambda} \to M$ is a smooth embedding. Then we a have a diffeomorphism

$$\partial(\operatorname{im}\phi) \cong S^{\lambda-1} \times S^{n-\lambda} = \partial\left(S^{\lambda-1} \times D^{n-\lambda+1}\right).$$

Denote the interior of the image of ϕ by $(\operatorname{im}\phi)^\circ$. Then $M \backslash (\operatorname{im}\phi)^\circ$ is a smooth manifold with boundary, and the restriction $\phi|_{S^{\lambda-1}\times S^{n-\lambda}}$ provides a diffeomorphism between the boundary of $M \backslash (\operatorname{im}\phi)^\circ$ and

$$S^{\lambda-1} \times S^{n-\lambda} = \partial \left(S^{\lambda-1} \times D^{n-\lambda+1} \right).$$

Now, produce a new manifold M' by identifying the common boundary of $M \backslash (\operatorname{im}\phi)^\circ$ and $S^{\lambda-1} \times D^{n-\lambda+1}$ via the diffeomorphism $\phi|_{S^{\lambda-1} \times S^{n-\lambda}}$:

$$M' = (M \backslash (\operatorname{im}\phi)^\circ) \cup_{\phi|_{S^{\lambda-1} \times S^{n-\lambda}}} (S^{\lambda-1} \times D^{n-\lambda+1}).$$

We say that M' is obtained from M *by a surgery of index* λ.

Remark 2.2.11. Note that in the literature, our definition of surgery is generally called a surgery of index $(n-\lambda)$. For our purpose, though, it will be more natural to refer to the index of the surgery as the dimension of the embedded disc factor.

Remark 2.2.12. Strictly speaking, we did not define a smooth structure on M'. That can be remedied, though. A *collar* of the boundary component $\partial(\operatorname{im}\phi) \subseteq M \backslash (\operatorname{im}\phi)^\circ$ is a set $\kappa_0 \supseteq \partial(\operatorname{im}\phi)$ that is diffeomorphic to the product $\kappa_0 \cong \partial(\operatorname{im}\phi) \times [0,1)$. After selecting such a collar κ_0, choose a collar κ_1 of the boundary of $S^{\lambda-1} \times D^{n-\lambda+1}$ satisfying $S^{\lambda-1} \times S^{n-\lambda} \subseteq \kappa_1 \subseteq S^{\lambda-1} \times D^{n-\lambda+1}$ and $\kappa_1 \cong \left(S^{\lambda-1} \times S^{n-\lambda} \right) \times [0,1)$ (diffeomorphism). We now have normal vector fields $\partial/\partial t_0$ and $\partial/\partial t_1$ to the boundary $S^{\lambda-1} \times S^{n-\lambda}$ of κ_0 and κ_1, respectively. The smooth structure on M' is now defined by identifying the boundary of κ_0 and κ_1 so that the normal fields $\partial/\partial t_0$ and $\partial/\partial t_1$ align into a smooth vector field in a neighborhood of $\partial(\operatorname{im}\phi) \subseteq M'$. Moreover, this smooth structure turns out to be independent of the choice of κ_0 and κ_1. For the details of identifying the boundaries of collared manifolds, see [2].

We now consider the following situation. Assume $f : M \to \mathbb{R}$ is a Morse function on a smooth n-dimensional manifold M. Assume that $c \in \mathbb{R}$ is a critical value of f with a unique critical point $p \in f^{-1}(c)$. In this setting we can choose an $\epsilon > 0$ small enough so that $f^{-1}[c-\epsilon, c+\epsilon]$ contains no other critical point than

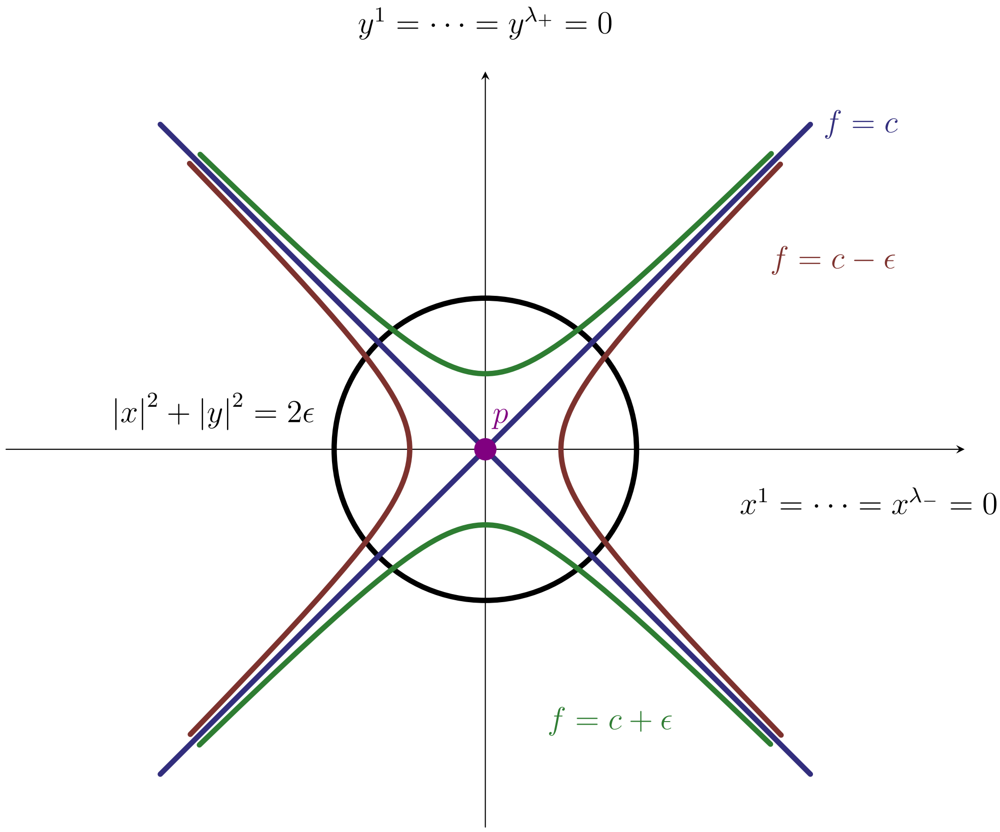

Figure 2.5.: The neighborhood of a critical point of a Morse function f of index (λ_-, λ_+).

p. Assume further that $f^{-1}[c-\epsilon, c+\epsilon]$ is compact. If the index of the critical point p is (λ_-, λ_+), we intend to show that $M^- = f^{-1}(c-\epsilon)$ is obtained from $M^+ = f^{-1}(c+\epsilon)$ by a surgery of index λ_-.

We first consider the local behavior of the level sets of f near p. By choosing ϵ small enough, we may apply Lemma 2.2.7 to find a neighborhood $U \subseteq M$ containing p, and coordinates $(x,y) : U \to \mathbb{R}^{\lambda_-} \times \mathbb{R}^{\lambda_+}$ so that $(x,y)(p) = (0,0)$, and

$$f = c - \left(x^1\right)^2 - \cdots - \left(x^{\lambda_-}\right)^2 + \left(y^1\right)^2 + \cdots + \left(y^{\lambda_+}\right)^2$$

whenever and $|x|^2 + |y|^2 \leq 2\epsilon$. We denote by $B_{\sqrt{2\epsilon}} \subseteq M$ the set

$$B_{\sqrt{2\epsilon}} = \left\{p \in U \mid |x(p)|^2 + |y(p)|^2 \leq 2\epsilon\right\}.$$

An illustration of the situation is presented in Figure 2.5. Now it is apparent that

$$\{f = c - \epsilon\} \cap B_{\sqrt{2\epsilon}} = \left\{|x|^2 = |y|^2 + \epsilon \mid |y|^2 \leq \epsilon/2\right\} \cong S^{\lambda_- - 1} \times D^{\lambda_+},$$

and

$$\{f(x, y) = c + \epsilon\} \cap B_{\sqrt{2\epsilon}} = \left\{|y|^2 = |x|^2 + \epsilon \mid |x|^2 \leq \epsilon/2\right\} \cong D^{\lambda_-} \times S^{\lambda_+ - 1}.$$

So, local to p, at least, the level set M^- is produced from M^+ by a surgery of index λ_-.

The final step is to show that M^- and M^+ are diffeomorphic on the closure of the complement of $B_{\sqrt{2\epsilon}}$. The full proof is provided by Milnor in [28], but we outline the key steps here. The required hypotheses are that $f : M \to \mathbb{R}$ is a Morse function, $f(p) = c$ is a critical point / critical value pair, and $f^{-1}[c - \epsilon, c + \epsilon]$ and is compact with no other critical value of f other than p. Then Milnor is able to produce a function $F : M \to \mathbb{R}$ such that $F^{-1}[c - \epsilon, c + \epsilon] \subseteq f^{-1}[c - \epsilon, c + \epsilon]$, and the level sets of F coincide with those of f outside of $B_{\sqrt{2\epsilon}}$, yet F has no critical points in $F^{-1}[c - \epsilon, c + \epsilon]$. Thus F is a submersion on the compact set $F^{-1}[c - \epsilon, c + \epsilon]$. Using Ehresmann's Lemma, we then find that that the levels of of F over $[c - \epsilon, c + \epsilon]$ are diffeomorphic. For a proof, see e.g. Bröcker and Jänich [2].

Theorem 2.2.13 (Ehresmann's Lemma)**.** *Let $f : E \to M$ be a proper submersion of differentiable manifolds. Then f is a locally trivial fibration, that is, if $p \in M$ and $F = f^{-1}(p)$ the fiber of p, then there exists a neighborhood U of p in M and a diffeomorphism $\phi : U \times F \to f^{-1}(U)$, so that the following diagram is commutative:*

$$\begin{array}{ccc} U \times F & \xrightarrow{\ \phi\ } & f^{-1}(U) \\ & {\scriptstyle pr_1} \searrow \quad \swarrow {\scriptstyle f|f^{-1}(U)} & \\ & U & \end{array}$$

Since the level sets of F coincide with those of f outside of $B_{\sqrt{2\epsilon}}$, we conclude that M^- and M^+ are diffeomorphic on the closure of the complement of $B_{\sqrt{2\epsilon}}$.

Finally, what of the level $f^{-1}(c)$? Inside $B_{\sqrt{2\epsilon}}$, we have homeomorphisms

$$\begin{aligned} f^{-1}(c) \cap B_{\sqrt{2\epsilon}} &= \left\{|x|^2 = |y|^2\right\} \cap \left\{|x|^2 + |y|^2 \leq 2\epsilon\right\} \\ &\cong \left\{(x,y) \in \mathbb{R}^{\lambda_-} \times \mathbb{R}^{\lambda_+} \mid |x|^2, |y|^2 = r, 0 \leq r \leq \epsilon\right\} \\ &\cong C\left(S^{\lambda_- -1} \times S^{\lambda_+ -1}\right). \end{aligned}$$

Thus, near $p \in M$, $f^{-1}(c)$ is homeomorphic to a cone $C(S^{\lambda_- -1} \times S^{\lambda_+ -1})$. We summarize these results as a theorem.

Theorem 2.2.14. *Assume $f : M \to \mathbb{R}$ is a Morse function. Assume that $c \in \mathbb{R}$ is a critical value of f. Finally, assume that $f^{-1}[c-\epsilon, c+\epsilon]$ is compact and contains no other critical point than $p \in f^{-1}(c)$, whose index we denote by (λ_-, λ_+). Then*

1. *The level sets $f^{-1}(c-\epsilon)$ and $f^{-1}(c+\epsilon)$ are compact, smoothly embedded codimension 1 submanifolds of M.*
2. *The level set $f^{-1}(c-\epsilon)$ is obtained from $f^{-1}(c+\epsilon)$ by a surgery of index λ_-.*
3. *The level set $f^{-1}(c)\backslash\{p\}$ is a smooth manifold, and there is a neighborhood $U \subseteq f^{-1}(c)$ of p homeomorphic to a cone $U \cong C(S^{\lambda_- -1} \times S^{\lambda_+ -1})$.*
4. *The level $f^{-1}(c)$ is obtained from $f^{-1}(c+\epsilon)$ by deleting a region diffeomorphic to the interior of $D^{\lambda_-} \times S^{\lambda_+ -1}$ and coning over the resulting boundary.*

2.3. The moduli space of polygons in the Euclidean plane

The goal of this section is to summarize the main results of Kappovich and Millson's "On the moduli space of polygons in the Euclidean plane" [18], upon which the results of Chapters 3 and 4 are built. Unless otherwise stated, all results in this section are due to Kappovich and Millson.

Denote by $\overline{P}_n$ the space of all planar n-gons with marked vertices, modulo orientation preserving isometries of of the Euclidean plane. We allow all degeneracies of polygons, except for degeneration to point. For a class $[P] \in \overline{P}_n$ we may identify the representative P by its vertex vector $P = (v_1, \ldots, v_n) \in \mathbb{C}^n$. By a translation and rotation, each $[P] \in \overline{P}_n$ has a unique representative with its initial vertex lying at the origin, and its final edge lying along the positive x-axis. This allows us to identify

$$\overline{P}_n = \{(0, v_1, \ldots, v_n) \in \mathbb{C}^n \mid v_n \in \mathbb{R}_{\geq 0}\} \backslash \{(0, \ldots 0)\} = \mathbb{C}^{n-2} \times \mathbb{R}_{\geq 0} \backslash \{(0, \ldots, 0)\}.$$

For $[P] = [(v_1, \ldots, v_n)] \in \overline{P}_n$, define the edge length vector of $[P]$ to be

$$r = (r_1, \ldots, r_n) = (|v_2 - v_1|, \ldots, |v_1 - v_n|) \in \mathbb{R}^n.$$

Since this is independent of the representative, we obtain a projection $\bar{\pi} : \overline{P}_n \to \mathbb{R}^n$ by $\bar{\pi}([P]) = (r_1, \ldots, r_n)$. The moduli space of polygons with side-length vector r is then the fiber of this projection: $\mathcal{M}_r = \bar{\pi}^{-1}(r)$.

For $\lambda \in \mathbb{R}_{>0}$, the scaling $r \mapsto \lambda r$ induces a diffeomorphism $\mathcal{M}_r \cong M_{\lambda r}$. In order to study the topology of $\mathcal{M}_r$, we may thus assume our polygons to have perimeter 1. We assume henceforth that $\sum r_i = 1$. Let $P_n \subset \overline{P}_n$ denote the subspace of unit perimeter pentagons, and $\pi = \bar{\pi}|_{P_n}$. We define the Kapovich-Millson edge-length polytope $D_n = \pi(P_n)$. By definition, $D_n \subseteq \Delta^{n-1}$, the standard $(n-1)$-simplex. Moreover, for each $j \in [n]$, $\sum_{i \neq j} r_i \geq r_j$, so we must also have $r_j \leq 1/2$. Now it is apparent that the moduli space of unit perimeter triangles is nonempty if and only if $r_j \leq 1/2$ for $j \in [3]$. An inductive argument now yields the

Lemma 2.3.1. *The edge-length polytope $D_n \subset \mathbb{R}^n$ is an $(n-1)$-polytope with h-generating set*

$$\sum r_i \leq 1$$
$$\sum r_i \geq 1$$
$$0 \leq r_i \leq 1/2,\ i \in [n].$$

We will find the structure of D_n to provide important insights into the topology of $\mathcal{M}_r$. To facilitate a discussion of that structure, we require additional definitions.

Definition 2.3.2. We call a polygon $[P] \in \mathcal{M}_r$ *degenerate* provided $[P]$ is contained in a line.

Definition 2.3.3. The intersection of the interior $(D_n)^\circ$ with a hyperplane $\sum_i \epsilon_i r_i = 0$, $\epsilon_i \in \{-1, 1\}$, $i \in [n]$, is called a *wall* of D_n. The vector $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ will be called a *signature* of the wall. We call the union of all the walls of D_n the *critical locus* of D_n, denoted by Σ_n.

Remark 2.3.4. The definitions of D_n, the walls of D_n, and the critical locus of D_n appear in [18], while the verbage of the signature of a wall is original to this dissertation.

Collectively, the intersection of all the walls with D_n forms a polytopal complex. Hereafter, by D_n, we will refer to this entire polytopal complex. For brevity, we will refer to the $(n-1)$-cells of D_n as *chambers.* These $(n-1)$-cells arise as the components of $D_n \backslash \Sigma_n$.

It is immediately obvious that $\mathcal{M}_r$ admits degenerate polygons if and only if r lies in a wall of D_n. Now we provide the fundamental lemma justifying the language we have developed.

Lemma 2.3.5. *If $[P] \in \overline{P}_n$ is not degenerate, then π is a smooth submersion at $[P]$.*

Proof. First recall that

$$\overline{P}_n = \mathbb{C}^{n-2} \times \mathbb{R}_{\geq 0} \backslash \{(0, \ldots, 0)\},$$

so we are working in the smooth category. Consider the space of all marked polygons in the plane $Q_n = \mathbb{C}^n$, where the points of Q_n are identified with the vertices of a polygon. We have the quotient map $q : Q_n \to \overline{P}_n$. Let $p : Q_n \to \mathbb{R}^n$ by

$$p(v_1, \ldots, v_n) = (|v_2 - v_1|, \ldots, |v_1 - v_n|)\,.$$

Then p factors through the quotient by $p = \pi \circ q$. Writing $v_j = x^j + iy^j \in \mathbb{C}$, we have

$$\partial p^j / \partial x^j = \frac{x^j - x^{j+1}}{|v_{j+1} - v_j|},$$

$$\partial p^j / \partial x^{j+1} = \frac{x^{j+1} - x^j}{|v_{j+1} - v_j|}$$

$$\partial p^j / \partial y^j = \frac{y^j - y^{j+1}}{|v_{j+1} - v_j|}$$

$$\partial p^j / \partial y^{j+1} = \frac{y^{j+1} - y^j}{|v_{j+1} - v_j|},$$

and all other derivatives are zero. Ordering our coordinates $(x^1, y^1, \ldots, x^n, y^n)$, and writing $e_j = v_{j+1} - v_j \in \mathbb{C}$, we find

$$dp|_{(v_1,\ldots,v_n)} = \begin{pmatrix} \frac{x^1-x^2}{|e_1|} & \frac{y^1-y^2}{|e_1|} & \frac{x^2-x^1}{|e_1|} & \frac{y^2-y^1}{|e_1|} & 0 & \cdots & 0 & 0 \\ 0 & \frac{x^2-x^3}{|e_2|} & \frac{y^2-y^3}{|e_2|} & \frac{x^3-x^2}{|e_2|} & \frac{y^3-y^2}{|e_2|} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{x^1-x^n}{|e_n|} & \frac{y^1-y^n}{|e_n|} & 0 & 0 & 0 & \cdots & \frac{x^n-x^1}{|e_n|} & \frac{y^n-y^1}{|e_n|} \end{pmatrix}.$$

Assume $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^{2n}$ lies in the kernel of $dp|_{(v_1,\ldots,v_n)}$. Then either $\xi = (z, \ldots, z) \in \mathbb{R}^{2n}$ for $z \in \mathbb{R}^2$, or $(\xi_{i+1} - \xi_i) \cdot e_i = 0$ for each i, where we are using the usual inner product on $\mathbb{R}^2$. Writing $\hat{\xi}_i = \xi_{i+1} - \xi_i$ we see that

$$\begin{aligned} \ker dp|_{(v_1,\ldots,v_n)} = &\left\{ (\hat{\xi}_1, \ldots, \hat{\xi}_n) \in \mathbb{R}^{2n} \mid \hat{\xi}_i \cdot e_i = 0, \sum \hat{\xi}_i = 0 \right\} \\ &\cup \left\{ (z, \ldots, z) \in \mathbb{R}^{2n} \mid z \in \mathbb{R}^2 \right\}. \end{aligned}$$

Assume that $(v_1, \ldots, v_n)$ are not contained in a line. Then $\{e_1, \ldots, e_n\}$ span $\mathbb{R}^2$. Therefore the equation $\sum \hat{\xi}_i = 0$ is a nondegenerate system of two equations, and

$\dim\ker dp|_{(v_1,\ldots,v_n)} = n-2$. But then

$$\dim\ker dp|_{(v_1,\ldots,v_n)} = (n-2)+2 = n = \dim\mathbb{R}^n = \dim\operatorname{im} p.$$

We conclude that p is a submersion, and therefore so, too, is π. □

The following corollary is implicit in Section 11 of [18].

Corollary 2.3.6. *If $r \in D_n$ does not lie on a wall, then $\mathcal{M}_r$ is a compact, smooth, orientable manifold of dimension $n-3$.*

Proof. By Lemma 2.3.5, r is a regular value of π, so we may apply the Regular Level Set Theorem (see, e.g., [23]) to conclude that $\mathcal{M}_r = \pi^{-1}(r)$ is a smooth submanifold of P_n of codimension $(n-1)$. Since

$$\overline{P}_n = \mathbb{C}^{n-2} \times \mathbb{R}_{\geq 0} \backslash \{(0,\ldots,0)\},$$

we see that $\dim\overline{P}_n = 2n-3$. Adding the condition that all polygons have perimeter 1 implies $\dim P_n = 2n-4$. Thus the dimension of $\mathcal{M}_r$ is $2n-4-(n-1) = n-3$.

For orientability, consider the hyperplane

$$H = \left\{(e_1,\ldots,e_n) \in \mathbb{C}^n \mid \sum e_i = 0\right\}$$

of edge vectors of n-gons. Then P_n is diffeomorphic to the projective space $P(H) \cong \mathbb{C}P^{n-2}$. By Ehresmann's lemma, π is a locally trivial fibration near r. Thus there is a neighborhood $U \subseteq D_n$ such that $\pi^{-1}(U) \cong U \times \mathcal{M}_r \cong \mathbb{R}^{n-1} \times \mathcal{M}_r$. Were $\mathcal{M}_r$ non-orientable, we would have found a non-orientable open subset of the orientable manifold $\mathbb{C}P^{n-2}$, a contradiction. This viewpoint also allows us to conclude that $\mathcal{M}_r$ is compact, as $\mathcal{M}_r = \pi^{-1}(r) \subseteq P_n \cong \mathbb{C}P^{n-2}$ is a closed subset of a compact manifold. □

Corollary 2.3.7. *Let $C \subseteq D_n \backslash \Sigma_n$ be a chamber of D_n, and assume $r, r' \in C$. Then there is a diffeomorphism $\mathcal{M}_r \cong \mathcal{M}_{r'}$.*

Proof. Let $C \subseteq D_n \backslash \Sigma_n$ be a chamber. Fix $r \in C$, and consider any other $r' \in C$. By convexity of C, there is a line segment $L \subseteq C$ connecting r and r'. Since $\pi : P_n \to D_n$ is a submersion on $\pi^{-1}(C)$, the map π is transverse to L. Indeed, $d\pi_p$ is surjective for every $p \in \pi^{-1}(C)$, hence π is transverse to any submanifold $M \subseteq C$. Consequently, $K = \pi^{-1}(L) \subseteq P_n$ is an embedded submanifold of P_n. By compactness, $\pi|_K$ is a proper submersion. Ehresmann's lemma now implies that $\pi|_K$ is a fibration, so $\pi^{-1}(r)$ and $\pi^{-1}(r')$ are diffeomorphic. □

As a first step to studying the topology of $\mathcal{M}_r$ we note that disconnected space can arise. The following is Theorem 1 of [18].

Proposition 2.3.8. *The moduli space $\mathcal{M}_r$, $r \in D_n$, is disconnected if and only if there are three edge lengths r_i, r_j, r_k such that each of*

$$r_i + r_j > 1/2, \;\; r_i + r_j > 1/2, \;\; r_j + r_k > 1/2.$$

If this is the case, then $\mathcal{M}_r$ is homeomorphic to a disjoint union of two $(n-3)$-dimensional tori.

The proof of Proposition 2.3.8 is too involved to be reproduced here, but we will provide the geometric intuition behind it. From basic trigonometry, the moduli space of triangles is disconnected. Now consider building an n-gon from a triangle by expanding the vertices of the triangle into a number of edges. If the conditions of Proposition 2.3.8 are satisfied, then consider growing an n-gon from the triangle with edge-lengths r_i, r_j, and r_k. The resulting n-gon will be too "close" to the original triangle, and it is impossible to find a continuous deformation from the polygon whose edges $e_1, \ldots, e_n$ are arranged in a clockwise ring, to that polygon whose edges are arranged in a counter-clockwise ring.

The final piece is to study how the topology of $\mathcal{M}_r$ changes as r crosses a wall in D_n. For $\hat{r} = (r_1, \ldots, r_{n-1}) \in \mathbb{R}^{n-1}$, define $F_{\hat{r}}$ to be the space of "free-linkages" modulo orientation preserving isometries of the Euclidean plane (Fig-

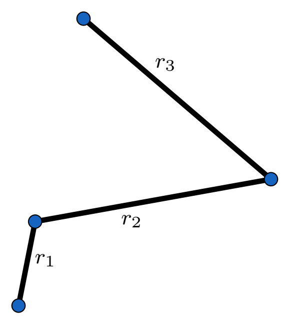

Figure 2.6.: A free linkage of 3 rods with length vector $\hat{r} = (r_1, r_2, r_3)$.

ure 2.6). A free-linkage is a sequence of marked line segments place end-to-end. Let $(v_1, \ldots, v_n) \in \mathbb{C}^n$ be the vertices of a free-linkage Λ of $(n-1)$-many rods. Let θ_i be the argument of the edge vector $e_i = v_{i+1} - v_i \in \mathbb{C}$, $1 \leq i \leq n-1$. We identify Λ with its vector of edges

$$\Lambda = \left(r_1 e^{i\theta_1}, \ldots, r_{n-1} e^{i\theta_{n-1}}\right).$$

Let $S^1_\rho \subseteq \mathbb{C}$ be the circle of radius ρ centered at the origin. Then

$$F_{\hat{r}} = S^1_{r_1} \times \cdots \times S^1_{r_{n-1}} / SO(2),$$

where $SO(2)$ acts diagonally. Given a linkage $[\ell] = \left[r_1 e^{i\theta_1}, \ldots, r_{n-1} e^{i\theta_{n-1}}\right] \in F_{\hat{r}}$, we define the function $r_n : F_{\hat{r}} \to \mathbb{R}$ by

$$r_n[\ell] = \left|\sum r_j e^{i\theta_j}\right|^2.$$

Notice that r_n is independent of the representative ℓ, and r_n gives the distance between the first and last vertices in ℓ. For technical reasons, we wish to restrict ourselves to the space $E_{\hat{r}} = F_{\hat{r}} \backslash \{r_n = 0\}$. Now for each linkage in $E_{\hat{r}}$, there is a canonical representative whose final vertex lies on the positive x-axis. As usual, we define a linkage to be degenerate if it is contained in a line. At a degenerate canonical linkage Λ, let$f(\Lambda)$ denote the the number of edge vectors pointing to the right, and $b(\Lambda)$ denote the number of edge vectors pointing to the left. Then we have the following lemma, which is Lemma 11 in [18].

Lemma 2.3.9. *The function r_n is a Morse function on $E_{\hat{r}}$ whose critical points are those linkages contained in a line. If Λ is such a critical point, then the Morse index of Λ is $(f(\Lambda)-1, b(\Lambda))$.*

Now consider a line segment $\gamma : (-\delta, \delta) \to D_5$ with the following properties:

1. $\gamma' > 0$, hence γ is a diffeomorphism onto its range.
2. $\gamma(-\delta, 0)$ lies on the interior of a chamber C.
3. $\gamma(0, \delta)$ lies on the interior of an adjacent chamber C'.
4. $\gamma(0)$ lies in a unique wall $W \supseteq C \cap C'$.
5. The first $(n-1)$-coordinates $(r_1, \ldots, r_{n-1})$ are fixed along γ.

Then $\Gamma = \pi^{-1}(\operatorname{im}\gamma) \subseteq E_{\hat{r}}$ is an open submanifold, and r_n is a Morse function on Γ with a unique critical point Λ in the fiber over $\gamma(0)$. Moreover, if W has the equation $\sum \epsilon_i r_i = 0$ with $\epsilon_n = 1$, $C \subseteq W^-$ is in the lower half-space, and $C' \subseteq W^+$ is in the upper half-space, then $f(\Lambda)$ is the number of positive entries in the signature ϵ of W, and $b(\Lambda) - 1$ is the number of negative entries in the signature ϵ of W (recall that the final edge is not a member of Λ, viewed as a linkage). Taken with Lemma 2.3.9 we have the most important result of [18] for our purposes.

Theorem 2.3.10. *Assume W is a wall in D_n with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_{n-1}, 1)$. Let λ_- be one fewer than the number of 0-entries in ϵ, and let λ_+ be one fewer than the number of 1-entries in ϵ. Let C and C' be chambers of D_n adjacent so that $C \cap C' \subseteq W$. Assume that C is contained in the lower half-space $\sum(-1)^{\epsilon_i} r_i \leq 0$, and C' is contained in the upper half-space $\sum(-1)^{\epsilon_i} r_i \geq 0$. Let $r \in \operatorname{relint} C$, and $r' \in \operatorname{relint} C'$. Then $\mathcal{M}_{r'}$ is obtained from $\mathcal{M}_r$ by a λ_- surgery. Explicitly, to* *obtain $\mathcal{M}_{r'}$, (i) delete the interior of an embedded $D^{\lambda_-} \times S^{\lambda_+ - 1} \subseteq \mathcal{M}_r$, then (ii) sew in a $S^{\lambda_- - 1} \times D^{\lambda_+}$ along the resulting common boundary.*

In [18], Kappovich and Millson used Theorem 2.3.10 to compute the smooth manifolds that arise as moduli spaces of pentagons.

Theorem 2.3.11. *The smooth manifolds that arise as moduli spaces of pentagons are the surfaces of genus no greater than 4, and a disjoint union of two tori.*

They also found a list that contained all possible smooth manifolds that arise as a moduli space of hexagons.

Theorem 2.3.12. *If the moduli space of hexagons is nonsingular and connected, then it is either diffeomorphic to a connected sum of k copies of $S^2 \times S^1$ and of the product $\Sigma_g \times S^1$, or it is diffeomorphic to a connected sum of $T^3 \# T^3$ and t copies of $S^2 \times S^1$. Here $k \leq 4$, the genus g of the surface Σ_g is not greater than 4, and $t \leq 2$. If it is nonsingular and disconnected, then it is diffeomorphic to the disjoint union of two tori $T^3 \bigsqcup T^3$.*

Notice that if all values of t, k, and g in Theorem 2.3.12 are achieved as moduli spaces of hexagons, then one will find $(5)(5) + 3 + 1 = 29$ non-diffeomorphic smooth manifolds that arise as moduli spaces of hexagons. However, we will show in Chapter 4 that, there can be no more than 20 diffeomorphism types of smooth moduli spaces of hexagons.

3. Singularities in the moduli spaces of pentagons

The topology of non-smooth moduli spaces of polygons has so far received little attention. Since the moduli spaces of quadrilaterals are one dimensional, analysis of connected components was enough for Kapovich and Millson to deduce the three singular topologies that arise as moduli spaces of quadrilaterals [18]: they are (i) a wedge of two circles, (ii) two circles connected at two points, and (iii) a union of three circles, each pair sharing a unique point of intersection. To this point, the topology of a non-smooth moduli space of n-gons, $n > 4$, has not been explicitly computed, however. It should be noted that the Betti-numbers of $\mathcal{M}_r$ have been computed as a function of r, first by Kamiyama and Tezuka [17], who computed the Betti-numbers in the equilateral case, and later by Farber and Schütz [4] for all edge-lengths vectors. In three dimensions, Kamiyama computed the rational homology of singular polygon spaces [15].

The goal of this chapter is to analyze the topology of non-smooth moduli spaces of pentagons. We will see that these spaces are all crimped manifolds (Definition 1.1.2), and we will produce a list containing a member of each homeomorphism class of non-smooth moduli spaces of pentagons (Theorem 1.1.6).

We summarize our results here. The first step of the process is to analyze the combinatorial structure of the polytopal complex D_5. In Section 3.1 we implement a computer search to find all the vertices in D_5, and the v-description of all the chambers of D_5. The result is the following lemma.

Lemma 3.0.1. *Denote by e_i the i-th standard basis vector in $\mathbb{R}^5$. The vertices in*

D_5 are

$$h_{i,j} = \frac{1}{2}(e_i + e_j), \quad q_k = \frac{1}{4}\left(-e_k + \sum_{i=1}^{5} e_i\right), \quad t_\ell = \frac{1}{6}\left(e_\ell + \sum_{i=1}^{5} e_i\right)$$

for $i \neq j, k, \ell \in [5]$.

The chambers in D_5 are

$$C_4 = \operatorname{conv}\{q_1, \ldots, q_5, t_1, \ldots t_5\}$$

$$C_3^{i,j} = \operatorname{conv}\left(\{t_i, t_j, h_{i,j}\} \cup \{q_k\}_{k \notin \{i,j\}}\right)$$

$$C_2^{i,j,k} = \operatorname{conv}\left(\{t_i, h_{i,j}, h_{i,k}\} \cup \{q_\ell\}_{\ell \notin \{i,j,k\}}\right)$$

$$C_{1,1}^{i,j} = \operatorname{conv}\left(\{q_i, q_j\} \cup \{h_{k,\ell}\}_{k \neq \ell \notin \{i,j\}}\right)$$

$$C_1^{i,j} = \operatorname{conv}\left(\{t_i, q_j\} \cup \{h_{i,k}\}_{k \notin \{i,j\}}\right)$$

$$C_0^{i} = \operatorname{conv}\left(\{t_i\} \cup \{h_{i,j}\}_{j \neq i}\right)$$

for $i \neq j \neq k \neq i \in [5]$.

Having computed the v-description of the d-cells in D_5, their pairwise intersections easily produce the v-descriptions of the $(d-1)$-cells, and so Lemma 3.0.1 provides all of the information necessary to compute every cell in D_5.

Next, we extend Corollary 2.3.7 to the lower dimensional cells of D_5.

Theorem 3.0.2. *Let C be a d-cell, $0 \leq d \leq 4$, lying in the relative interior of D_5, and let r and r' lie in the relative interior of C. Then $\mathcal{M}_r$ and $\mathcal{M}_{r'}$ are homeomorphic. Moreover, the $\mathcal{M}_r$ is homeomorphic to a crimped manifold with $(4-d)$-many crimped points.*

The proof of Theorem 3.0.2 is contained in Section 3.2. In light of Theorem 3.0.2, we have the following definition.

Definition 3.0.3. Let $C \subseteq D_5$ be a d-cell, $0 \leq d \leq 4$. The *topological type* of C is the homeomorphism class of $\mathcal{M}_r$ for $r \in \operatorname{relint} C$.

We conclude this chapter with Section 3.3, in which we compute the topological type of each d-cell, $0 \leq d \leq 3$, in D_5, yielding the proof of Theorem 1.1.6.

3.1. The edge-length polytope D_5

We begin this section by studying the edge-length polytope

$$D_5 = \left\{ (r_1, \ldots, r_5) \in \mathbb{R}^5_{\geq 0} \mid \sum_{j=1}^{5} r_j = 1,\ r_k \leq 1/2 \text{ for all } k \right\}.$$

Remark 3.1.1. One immediately observes that D_5 is a 4-polytope lying in the affine space

$$A = \left\{ (r_1, \ldots, r_5) \in \mathbb{R}^5 \mid \sum_{j=1}^{5} r_j = 1 \right\}$$

with essential supporting hyperplanes

$$H_j^0 = \{(r_1, \ldots, r_5) \in A \mid r_j = 0\}$$

and

$$H_k^{1/2} = \{(r_1, \ldots, r_5) \in A \mid r_k = 1/2\}.$$

Remark 3.1.2. If $r \in H_j^0$ for some j, then $\mathcal{M}_r$ is diffeomorphic to the moduli space of quadrilaterals $\mathcal{M}_{\hat{r}}$ where $\hat{r}$ is obtained from r by deleting the j-th coordinate. If $r \in H_k^{1/2}$ for some k, then $\mathcal{M}_r$ is a single point corresponding to a line segment of length $1/2$. Thus we may restrict our attention to the relative interior of D_5 in order to study the topologies of the moduli space of pentagons.

Now $\mathcal{M}_r$ contains degenerate pentagons if and only if r is contained in a wall. The intersection of D_5 with all the half-spaces arising from all the walls turns D_5 into a polytopal complex. Hereafter, any reference to D_5 will be to this polytopal complex.

We study the combinatorics of D_5. The walls of D_5 are of two types:

$$B_i = \left\{ (r_1, \ldots, r_5) \in D_5 \mid r_i = \sum_{j \in [5] \setminus \{i\}} r_j \right\}$$

$$I_{j,k} = \left\{ (r_1, \ldots, r_5) \in D_5 \mid r_j + r_k = \sum_{\ell \in [5] \setminus \{j,k\}} r_\ell \right\}.$$

That every point in D_5 satisfies $\sum_{i=1}^{5} r_i = 1$ implies $B_i = H_i^{1/2}$ lies in a supporting hyperplane of D_5. We will refer to the walls $I_{j,k}$ as *interior walls*, and to the B_i as *boundary walls*. As a first step, we identify the 0-cells and 4-cells of D_5.

Proof of Lemma 3.0.1. For each interior wall $I_{j,k}$, denote by $I_{j,k}^{+}$ ($I_{j,k}^{-}$) the half-spaces defined by

$$r_j + r_k \leq (\geq) \sum_{\ell \in [5] \setminus \{j,k\}} r_\ell.$$

Each cell in D_5 is determined by a choice of half-spaces of the $\binom{5}{2} = 10$ interior walls. Additionally, all points in D_5 satisfy the inequalities $0 \leq r_i \leq 1/2$, $i \in [5]$, $\sum r_j \leq 1$, and $\sum r_k \geq 1$. The chambers of D_5 are then solution sets to a system of inequalities of the form $Ar \geq b$, where A is 22×5 matrix, b is a 22×1 column vector, and the inequality is to be valid in each coordinate. This system now forms an h-description for a (possibly empty) bounded polytope lying in $\mathbb{R}^5$. Algorithms exist, e.g. [1], to compute a minimal v-description for these polytopes. See Appendix A for a SageMath 8.4 implementation of a script that computes the v-description of each of the 76 chambers in D_5, and collects a list of all of the vertices that appear in at least one of the chambers. The output of that script are exactly the vertices and chambers listed above. □

Corollary 3.1.3. *The vertex description of all d-cells, $0 \leq d \leq 4$, lying in the relative interior of D_5 are algorithmically computable.*

Proof. Inductively having found all d-cells, $0 < d \leq 4$, the $(d-1)$-cells of D_5

appear as an intersection of two d-cells. Moreover, if V_1 and V_2 are the vertex sets of two d-cells C_1 and C_2, respectively, then $V_1 \cap V_2$ is the vertex set of $C_1 \cap C_2$. Thus, by finding the pairwise intersection of all vertex sets of d-cells, we compute the minimal v-description of all $(d-1)$-cells. The initial step of the induction is provided by Lemma 3.0.1. □

To avoid a proliferation of notation, we do not carry out the process described in Corollary 3.1.3 here. Instead, we refer the reader to the full statement of Theorem 1.1.6 which contains a complete list of all interior d-cells in D_5, $0 \leq d \leq 4$.

3.2. The Morse theory above walls

Let us first give an alternate description of $\overline{P}_5$. Let $e_i = v_{i+1} - v_i \in \mathbb{C}$ (indices modulo 5) denote the i-th edge vector of $[P] \in \overline{P}_5$. In polar coordinates, write $e_j = r_j e^{i\theta_j}$. Then

$$\overline{P}_5 = \frac{\left\{(\theta_1, \ldots, \theta_5, r_1, \ldots, r_5) \in (S^1)^5 \times \mathbb{R}^5_{\geq 0} \mid \sum_{i=1}^5 e_i = 0 \in \mathbb{C}, (r_1, \ldots, r_5) \neq 0 \in \mathbb{R}^5\right\}}{SO(2)},$$

where $SO(2)$ acts diagonally on the first five factors. Since there is a unique representative of each $[P] \in \overline{P}_5$ whose terminal edge lies on the positive x-axis, we obtain coordinates for $\overline{P}_5$

$$\overline{P}_5 = \left\{(\theta_1, \ldots, \theta_4, r_1, \ldots, r_4) \in (S^1)^4 \times \mathbb{R}^4_{\geq 0} \mid \sum_{j=1}^4 r_j \sin\theta_j = 0, \sum_{j=1}^4 r_j \cos\theta_j > 0\right\}.$$

Such a pentagon will be considered in *standard position.* For a pentagon

$$[P] = (\theta_1, \ldots, \theta_4, r_1, \ldots, r_4) \in \overline{P}_5,$$

we call the vector $(\theta_1, \ldots, \theta_4, \pi)$ the *angle vector* of $[P]$. Let $\pi : \overline{P}_5 \to \mathbb{R}^5$ be the projection to the edge-length vector, and define $F : \overline{P}_5 \to \mathbb{R}$ to be the length of the final edge. In coordinates

$$F(\theta_1, \ldots, \theta_4, r_1, \ldots, r_4) = \sum_{j=1}^{4} r_j \cos\theta_j.$$

Towards the proof of Theorem 3.0.2, we have our first key Lemma.

Lemma 3.2.1. *Let $I = \cap_{k=1}^{d} I_{i_k,j_k}$ be an intersection of exactly d-many walls in D_5. There exist disjoint open sets $N_1, \ldots, N_d \subseteq \pi^{-1}(I)$ such that $\pi^{-1}(I) \setminus \cup N_i$ is a smooth, compact, manifold with boundary. Moreover, for each r in the relative interior of I and $i \in [d]$, $\mathcal{M}_r \cap N_i$ is homeomorphic to a cone $C(S^1 \times S^0)$.*

Proof. We construct $N_1 \subseteq \pi^{-1}(I)$. Assume $(i_1, j_1) = (3, 5)$ are the indices of one of the member walls of I, and the other cases are handled similarly. Given $r \in I$, there is a unique degenerate pentagon $P_r^1 \in \mathcal{M}_r$ We study the Morse theory of F, defined above, in a neighborhood of P_r^1. Now F is not a classical Morse function since P_r^1 is not an isolated critical point of F. Instead, we apply the following Morse lemma *with parameters.*

Lemma 3.2.2. *(pg. 97 in [3]) Let $F : \mathbb{R}^n \times \mathbb{R}^r, 0 \to \mathbb{R}$ be a smooth map, where we use coordinates x_i in $\mathbb{R}^n$ and u_j in $\mathbb{R}^r$ and 0 here stands for all x_i and u_j equal to 0. Suppose that $\partial F/\partial x_i(0) = 0$ for all i and that the matrix $(\partial^2 F/\partial x_i \partial x_j(0))$ is nonsingular. (Thus F_0, defined by $F_0(x) = F(x, 0)$, is a Morse function.) Then there is a map $\psi : \mathbb{R}^n \times \mathbb{R}^r, 0 \to \mathbb{R}^n, 0$ with*

1. *the matrix $(\partial\psi_i/\partial x_j(0))$ nonsingular, so that $x \mapsto \psi(x, u)$ is a local diffeomorphism for $u = 0$ and therefore for all u close to 0;*
2. *$F(\psi(x, u), u) = F(0, 0) + \sum_{i=1}^{n} \epsilon_i x_i^2 + h(u)$, where each ϵ_i is ± 1 and $h : \mathbb{R}^r, 0 \to \mathbb{R}, 0$ is a smooth map.*

Since $\cos\theta_4 > 0$ in a neighborhood $V_{P_r^1}$ of P_r^1, we may invert the constraint

$\sum r_i \sin\theta_i = 0$ to find

$$F(\theta_1,\ldots,\theta_3,r_1,\ldots,r_4) = \sum_{i=1}^{3} r_i \cos\theta_i + \sqrt{r_4^2 - \left(\sum_{i=1}^{3} r_i \sin\theta_i\right)^2}.$$

Moreover, by intersecting with the open set

$$\{(\theta_1,\ldots,\theta_4,r_1,\ldots,r_4) \mid -\pi < \theta_1,\theta_2,\theta_4 < \pi, 0 < \theta_3 < 2\pi\},$$

we may assume the only degenerate pentagons contained in $V_{P_r^1}$ have angle vector $(0,0,\pi,0,\pi)$. Assume P_r^1 has edge-length vector $(R_1,\ldots,R_5)$. Setting $x_1 = \theta_1$, $x_2 = \theta_2$, $x_3 = \theta_3 - \pi$, and $u_i = r_i - R_i$, $i \in [4]$, we find

$$F(x_1,x_2,x_3,u_1,u_2,u_3,u_4) \quad =$$

$$\sum_{i=1}^{3}(-1)^{\delta_{i,3}}(u_i+R_i)\cos x_i + \sqrt{(u_1+R_4)^2 - \left(\sum_{i=1}^{3}(-1)^{\delta_{i,3}}(u_i+R_i)\sin x_i\right)^2},$$

where $\delta_{i,j}$ is the Kronecker delta. We now check the hypotheses of the Lemma. One immediately sees that $\partial F/\partial x_i(0) = 0$ for all i, and a machine calculation gives

$$(\partial^2 F/\partial x_i \partial x_j(0)) = -R_4^{-1} \begin{pmatrix} R_1R_4 + R_1^2 & R_1R_2 & -R_1R_3 \\ R_1R_2 & R_2R_4 + R_2^2 & -R_2R_3 \\ -R_1R_3 & -R_2R_3 & -R_3R_4 + R_3^2 \end{pmatrix}$$

with determinant

$$\det(\partial^2 F/\partial x_i \partial x_j(0)) = \frac{R_1R_2R_3(R_1+R_2-R_3+R_4)}{R_4}.$$

Notice that $R_1 + R_2 - R_3 + R_4 = F(P_r^1) > 0$ since $(R_1,\ldots,R_5)$ lies in the relative interior of I.

Having verified the hypotheses of Lemma 3.2.2, we conclude there is a neighborhood $U_{P_r^1} \subseteq V_{P_r^1}$ of P_r^1 on which F has the coordinate representation

$$F(\psi(x,u),u) = F(0,0) - x_1^2 - x_2^2 + x_3^2 + h(u).$$

To see these are the correct values of the ϵ_i, notice that $(\partial^2 F/\partial x_i \partial x_j(0))$ is not positive definite. Indeed,

$$\begin{aligned} \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} (\partial^2 F/\partial x_i \partial x_j(0)) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} &= -R_4^{-1}(R_1+R_2)(R_1+R_2+R_3+R_4) \\ &\quad -R_4^{-1}(R_1+R_2+R_3-R_4) \\ &< 0. \end{aligned}$$

Since $\det(\partial^2 F/\partial x_i \partial x_j(0)) > 0$, we conclude $(\partial^2 F/\partial x_i \partial x_j(0))$ must have two negative eigenvalues (including multiplicity). Now on $U_{P_r^1}$, $F_u = F(\psi(\cdot,u),u)$ is a classical Morse function whose critical points are where the x_i simultaneously vanish. The vanishing of the x_i coincides with those pentagons that are contained in a line. For each u, the level set of F_u passing through a critical point is locally homeomorphic to an open cone $C(S^1 \times S^0)$ by the classical Morse lemma [28]. Now by holding u, constant, the edge-length vector is constant, so we find that $U_{P_r^1} \cap \mathcal{M}_s \cong C(S^1 \times S^0)$ for each $s \in \pi(U_{P_r^1})$. Complete the construction by setting $N_1 = \cup_{r \in \pi^{-1}(I)} U_{P_r^1}$.

Inductively construct N_k. For each $r \in I$ there is a unique pentagon P_r^k with angle vector containing π in the i_k and j_k entries, and all other entries 0. By construction $P_r^k \notin \cup_{i<k} N_i$. Follow the process above to construct $N_k \subseteq \pi^{-1}(I) \backslash \cup_{i<k} N_i$ such that $\mathcal{M}_s \cap N_k \cong C(S^1 \times S^0)$ for each $s \in \pi(N_k)$. Finally, note that every degenerate pentagon lies in one of the N_k, so Lemma 2.3.5 along with the implicit function theorem imply $\pi^{-1}(I) \backslash \cup N_i$ is a smooth manifold with boundary. For compactness, notice that $\pi^{-1}(I) \subseteq P_5 \subseteq \overline{P}_5$ is a closed subspace of the compact

space P_5 of unit perimeter pentagons. □

Armed with Lemmas 3.2.1, Theorem 3.0.2 is almost a corollary.

Proof of Theorem 3.0.2. Let C be a d-cell lying in the relative interior of D_5. Then C is contained in the intersection $I = \cap_{k=1}^{4-d} I_{i_k,j_k}$ of exactly $(4-d)$-many interior walls. Let $N_1, \ldots, N_{4-d}$ be the open subsets of $\pi^{-1}(I)$ guaranteed by Lemma 3.2.1. Let r and r' lie in the relative interior of C. Since $d\pi$ is nondegenerate on the compact manifold with boundary $I' = \pi^{-1}(I) \setminus \cup_{k=1}^{4-d} N_k$, Ehressman's lemma implies $\mathcal{M}_r \cap I' \cong \mathcal{M}_{r'} \cap I'$ (diffeomorphism). Moreover, $\mathcal{M}_r \cap N_k \cong C(S^1 \times S^0) \cong \mathcal{M}_{r'} \cap N_k$ (homeomorphism) for each k, yielding homeomorphisms

$$\mathcal{M}_r \cong (\mathcal{M}_r \cap I') \cup_{k=1}^{4-d} C(S^1 \times S^0) \cong \mathcal{M}_{r'},$$

where the cones are sewn into the boundary of $\mathcal{M}_r \cap I'$ along their boundaries $\partial C(S^1 \times S^0) = S^1 \times S^0$. □

It remains to show how the topology on the smooth portions of the $\mathcal{M}_r$ changes as r moves between cells in D_5. Consider a linear path γ in D_5 not fixing r_5. Then γ can be parametrized by $r_5 \in (a, b)$ in some small interval, so we may write

$$\gamma(r_5) = (L_1(r_5), \ldots, L_4(r_5), r_5)$$

for some linear functions L_i.

Now consider the space of chains of four bars in $\mathbb{E}^2$. Allow the lengths of the bars to vary according to the first four coordinates of γ for $r_5 > 0$. Let X denote this space modulo orientation preserving isometries. If $L_i \equiv r_i$ are constant for all $i \in [4]$, then we may identify X with $(S^1)^4/SO(2)$. If, on the other hand, one of the L_i is not constant, we may identify X with $(S^1)^4 \times \mathbb{R}_{>0}/SO(2)$, where $SO(2)$ acts diagonally on the first four coordinates. To see this, let r_i be the length of the i-th bar of a chain in X, and $e_j = v_{j+1} - v_j = r_j e^{i\theta_j} \in \mathbb{C}$ be the j-th edge vector. Then $X \equiv (S^1)^4 \times \mathbb{R}_{>0}/SO(2)$ by

$$[L_1(r_5)e^{i\theta_1}, \ldots, L_4(r_5)e^{i\theta_4}] = [\theta_1, \ldots, \theta_4, r_5].$$

Remark 3.2.3. Notice X is a smooth manifold since the $SO(2)$-action is free.

For each chain in X, there is a unique representative whose initial point lies at the origin, and whose terminal vertex lies on the positive x-axis. This yields coordinates

$$X = \left\{ (\theta_1, \ldots, \theta_4, r_5) \in (S^1)^4 \times \mathbb{R}_{>0} \mid \sum_{i=1}^{4} L_i(r_5)\sin\theta_i = 0, \sum_{j=1}^{4} L_j(r_5)\cos\theta_j > 0 \right\}.$$

Obtain a projection $p : X \to D_5$ by

$$p(\theta_1, \ldots, \theta_4, r_5) = \left(L_1(r_5), \ldots, L_4(r_5), \sum_{j=1}^{4} L_j(r_5)\cos\theta_j \right)$$

with fibers $p^{-1}(r) = \mathcal{M}_r$. We study the Morse theory of F on $\Gamma = p^{-1}(\mathrm{Im}\gamma)$ as γ crosses a wall. Unfortunately, Γ is not guaranteed to be smooth.

Lemma 3.2.4. *If the L_i described above are constant for $i \in [4]$, then Γ is a smooth manifold. If γ is contained in a wall $I_{j,k}$, then Γ is not smooth. However, if $P \in \Gamma$ is not a degenerate pentagon, then P is a regular point of p.*

Proof. If the L_i are constant for $i \in [4]$, then Γ is an open subset of the smooth manifold $(S^1)^4/SO(2)$. Now suppose γ is contained in a wall $I_{j,k}$. Without loss, assume $(j,k) = (3,5)$. Define $f : X \to \mathbb{R}$ by

$$f(\theta_1, \ldots, \theta_4, r_5) = \sum_{i=1}^{4} (-1)^{\delta_{i,3}} L_i(r_5) - \sum_{i=1}^{4} L_i(r_5)\cos\theta_i.$$

Then $\Gamma \subseteq f^{-1}(0)$. Since $\partial f/\partial\theta_j = L_i(r_5)\sin\theta_j$, and $L_i(r_5) \neq 0$ for all i, the critical points of f must be pentagons which are contained in a line. $\square$

Lemma 3.2.5. *Assume γ is contained in a wall. Then there are disjoint open* *sets* $N_1, \ldots, N_d \subseteq \Gamma$ *such that* $\Gamma \backslash \cup N_i$ *is a smooth manifold with boundary, and* $\mathcal{M}_r \cap N_i \cong C(S^1 \times S^0)$ *for each* $i \in [d]$ *and* $r \in Im\gamma$.

Proof. Let $I = \cap_{i=1}^d I_{j_i,k_i}$ be the intersection of all walls containing γ. Regarding $\Gamma \subseteq P_5$, Lemma 3.2.1 guarantees neighborhoods $N_1, \ldots, N_d$ so that $\mathcal{M}_r \cap N_i \cong C(S^1 \times S^0)$ for each $i \in [d]$ and $r \in \mathrm{Im}\gamma$. Denote $N = \cup_{i=1}^d N_i$. It remains to be shown that if $P \in \Gamma \backslash N$ is a degenerate pentagon, then P is not a singular point of Γ. Assume $P \in \Gamma \backslash N$ is a degenerate pentagon. If $(\theta_1, \ldots, \theta_5)$ is the angle vector of P, then $\delta = (\cos\theta_1, \ldots, \cos\theta_5)$ is the signature of a wall which intersects γ only in the point $\pi(P)$. Assume $I_{3,5}$ is a wall containing γ, and the other cases are handled similarly. Then, defining f as in the proof of Lemma 3.2.4, we see $\Gamma \subseteq f^{-1}(0)$. It now suffices to show that

$$\partial f / \partial r_5 = L_1' + L_2' - L_3' + L_4' - \sum_{i=1}^{4} L_i' \cos\theta_i$$

is nonzero at P. Let $v = (L_1', \ldots, L_4', 1)$ be the velocity of γ. Since γ lies in $I_{3,5}$, we have

$$v \cdot (1, 1, -1, 1, -1) = 0.$$

We reduce

$$\partial f / \partial r_5(P) = 1 - \sum_{j=1}^{4} \delta_j L_j'.$$

Since γ does not lie in the wall of signature δ, $v \cdot \delta \neq 0$, and hence $\partial f / \partial r_5(P) \neq 0$. □

Possessing an understanding of the singular set $\Sigma \subseteq \Gamma$, we analyze the Morse theory of $F|_{\Gamma \backslash \Sigma}$.

Lemma 3.2.6. *The critical points of* $F|_{\Gamma \backslash \Sigma}$ *are degenerate pentagons.*

Proof. The proof is by Lagrange multipliers. We solve the more complicated system when γ is contained in a wall. The case when γ is not contained in a wall is handled in the proof of Lemma 11 of [18]. Without loss, γ is contained in $I_{3,5}$.

The critical points of $F|_{\Gamma\setminus\Sigma}$ coincide with those of $\tilde{F}:\mathbb{R}^4 \to \mathbb{R}$, $\tilde{F}(\theta_1,\ldots,\theta_4,r_4) = \sum_{i=1}^4 r_i \cos\theta_i$ subject to the constraints $g(\theta_1,\ldots\theta_4,r_5) = \sum_{i=1}^4 L(r_5)\sin\theta_i = 0 = f(\theta_1,\ldots,\theta_4,r_5)$, where f is as in the proof of Lemma 3.2.4. Applying Lagrange multipliers,

$$d\tilde{F} = \lambda_1 dg + \lambda_2 df = \lambda_1 dg + \lambda_2(dr_5 - d\tilde{F}).$$

If $\lambda_2 = -1$, then

$$dr_5 = \lambda_1 dg = \lambda_1 \left(\sum_{i=1}^4 L_i' \sin\theta_i\right) dr_5 + \lambda_1 \sum_{i=1}^4 L_i(r_5)\cos\theta_i d\theta_i,$$

implying $\cos\theta_i = 0$ for all i. But then $F(\theta_1,\ldots,\theta_4,r_5) = 0$, an impossibility since γ lies in an interior wall. We conclude that $\lambda_2 \neq -1$, and

$$d\tilde{F} = \frac{\lambda_1}{1+\lambda_2} dg = \lambda dg.$$

We now have the reduced system

$$\begin{aligned} &\left(\sum_{i=1}^4 L_i' \cos\theta_i\right) dr_5 - \sum_{i=1}^4 L_i(r_5)\sin\theta_i d\theta_i \\ = \; &\lambda \left(\sum_{i=1}^4 L_i' \sin\theta_i\right) dr_5 + \lambda \sum_{i=1}^4 L_i(r_5)\cos\theta_i d\theta_i \end{aligned}$$

Equating the $d\theta_i$ yields

$$\tan\theta_i = -\lambda,$$

for all i. Therefore, for all $i,j \in [4]$, either $\theta_i = \theta_j$ or $\theta_i - \theta_j = \pm\pi$, and a critical point of $F|_{\Gamma\setminus\Sigma}$ is contained in a line. □

We are ready to prove our key lemma that will allow us to determine the homeomorphism types of the non-smooth moduli spaces of pentagons.

Lemma 3.2.7. *Let C_- and C_+ be two d-cells, $1 \leq d \leq 4$, lying in the common intersection of $(4-d)-$many walls satisfying $T = C_- \cap C_+$ is a $(4-d-1)$-cell lying the in the wall of signature ϵ. Let $\gamma(r_5^0 - \varepsilon, r_5^0 + \varepsilon) \to D_5$ as defined above be a path* *with* $\gamma(r_5^0-\varepsilon, r_5^0) \subseteq relint C_-$, $\gamma(r_5^0, r_5^0+\varepsilon) \subseteq relint C_+$, *and* $\gamma(r_5^0) \in relint T$. *Let* $N_1, \ldots, N_d \subseteq \Gamma$ *be given so that* $\Gamma \backslash N$, *where* $N = \cup_i N_i$, *is smooth. Then* $F|_{\Gamma\backslash N}$ *has a nondegenerate critical point* $P \in \mathcal{M}_{\gamma(r_5^0)}$. *Moreover, if* f *is the number of positive entries in* ϵ, *and* $v = \gamma'$, *then the Morse index of* $F|_{\Gamma\backslash N}$ *at* P *is*

$$4 - f \text{ if } v\cdot\epsilon > 0$$

$$f - 1 \text{ if } v\cdot\epsilon < 0.$$

Proof. There is a unique degenerate pentagon $P \in \mathcal{M}_{\gamma(r_5^0)}$ with angle vector $\theta = (\theta_1, \ldots, \theta_5)$ satisfying $\epsilon = (\cos\theta_1, \ldots, \cos\theta_5)$. We show this critical point is nondegenerate. Near P we have in coordinates

$$F|_\Gamma = \sum_{i=1}^{3} L_i(r_5)\cos\theta_i + \epsilon_4\sqrt{L_4(r_5)^2 - \left(\sum_{i=1}^{3} L_i(r_5)\sin\theta_i\right)^2}.$$

But $r_5 = F|_\Gamma$, so we implicitly differentiate to find

$$\begin{aligned}\partial^2 F|_\Gamma/\partial\theta_i\partial\theta_j(P) &= \sum_{k=1}^{4} L_k'\epsilon_k\partial^2 F|_\Gamma/\partial\theta_i\partial\theta_j(P) - \delta_{i,j}\epsilon_i L_i(r_5^0)\\ &- \epsilon_4\frac{\epsilon_j L_j(r_5^0)\epsilon_i L_i(r_5^0)}{L_4(r_5^0)}.\end{aligned}$$

Since γ does not lie in the wall of signature ϵ, the velocity $v = (L_1', \ldots, L_4', 1)$ of γ satisfies

$$\sum_{i=1}^{4}\epsilon_i L_i' - 1 = v\cdot\epsilon \neq 0.$$

If $(R_1, \ldots, R_5) = \gamma(r_5^0)$, we find the Jacobian matrix

$$\partial^2 F|_\Gamma/\partial\theta_i\partial\theta_j(P) = \frac{\epsilon_4}{R_4(v\cdot\epsilon)}\begin{pmatrix}\epsilon_1\epsilon_4 R_1R_4 + R_1^2 & \epsilon_1\epsilon_2R_1R_2 & \epsilon_1\epsilon_3R_1R_3\\ \epsilon_1\epsilon_2R_1R_2 & \epsilon_2\epsilon_4R_2R_4 + R_2^2 & \epsilon_2\epsilon_3R_2R_3\\ \epsilon_1\epsilon_3R_1R_3 & \epsilon_2\epsilon_3R_2R_3 & \epsilon_3\epsilon_4R_3R_4 + R_3^2\end{pmatrix},$$

with determinant

$$\det \partial^2 F|_\Gamma/\partial\theta_i\partial\theta_j(P) = \frac{\epsilon_1\epsilon_2\epsilon_3\epsilon_4 R_1R_2R_3\sum_{i=1}^4 \epsilon_i R_i}{R_4(v\cdot\epsilon)^3}$$
$$= -\frac{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5 R_1R_2R_3R_5}{R_4(v\cdot\epsilon)^3}.$$

Let f be the number of positive entries in the signature ϵ. We wish to show that $(v\cdot\epsilon)\partial^2 F|_\Gamma/\partial\theta_i\partial\theta_j(P)$ has $f-1$ positive eigenvalues. Write

$$A = (v\cdot\epsilon)\partial^2 F|_\Gamma/\partial\theta_i\partial\theta_j(P).$$

Then we may decompose A into

$$A = \frac{\epsilon_4}{R_4}\begin{pmatrix}\epsilon_1R_1\\ \epsilon_2R_2\\ \epsilon_3R_3\end{pmatrix}\begin{pmatrix}\epsilon_1R_1 & \epsilon_2R_2 & \epsilon_3R_3\end{pmatrix} + \mathrm{diag}\begin{pmatrix}\epsilon_1R_1 & \epsilon_2R_2 & \epsilon_3R_3\end{pmatrix} = H + P$$

where H and P are both Hermitian matrices. The eigenvalues of H are

$$\frac{\epsilon_4}{R_4}\left(R_1^2+R_2^2+R_3^2\right),\ 0,\ 0,$$

and the eigenvalues of P are $\epsilon_i R_i$ for $i \in [3]$. Denote by $h_1 \geq h_2 \geq h_3$ the spectrum of H, and by $p_1 \geq p_2 \geq p_3$ the spectrum of P. We analyze the spectrum $a_1 \geq a_2 \geq a_3$ of A by cases. The key ingredient are Weyl's inequalities which state that if

$$j+k-3 \geq i \geq r+s-1,$$

then

$$h_j + p_k \leq a_i \leq h_r + p_s.$$

For a discussion of Weyl's inequalities, including proof, see [5].

Case 1. Suppose $f = 1$. Then $h_i, p_i \geq 0$ for all i, so Weyl's inequalities imply

A has only positive eigenvalues.

Case 2. Suppose $f = 2$. Then by relabelling we may assume $h_3 < 0$ and $p_i > 0$ for all i. Since P is positive definite, Weyl's inequalities imply $a_1, a_2 > 0$. To see that $a_3 < 0$ note the determinant of A

$$\det A = -\frac{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5 R_1 R_2 R_3 R_5}{R_4} < 0.$$

Case 3. Suppose $f = 3$. By relabelling, assume $h_3, p_3 < 0$. Then Weyl's inequalities gives

$$a_1 \geq h_2 + p_2 > 0.$$

To see that $a_2, a_3 < 0$, note the determinant of A

$$\det A = -\frac{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5 R_1 R_2 R_3 R_5}{R_4} > 0.$$

Case 4. Suppose $f = 4$. By relabelling, assume $p_i < 0$ for all i. Since P is negative definite, Weyl's inequalities imply $a_2, a_3 < 0$. Finally, to see that $a_1 < 0$, we appeal to the determinant of A

$$\det A = -\frac{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5 R_1 R_2 R_3 R_5}{R_4} < 0.$$

□

3.3. The homeomorphism types of non-smooth moduli spaces of pentagons

Before we begin our computations, let us first make a key simplifying observation. Let $r = (r_1, \ldots, r_n) \in D_n$. Then we may identify $\mathcal{M}_r$ with

$$\mathcal{M}_r = \left\{ \left(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}\right) \in \mathbb{C}^n \mid \sum r_j e^{i\theta_j} = 0 \right\} / SO(2),$$

where $SO(2)$ is acting diagonally. In light of the commutativity of addition in $\mathbb{C}$, we see that if $\sigma \in S_n$ is a permutation, and $\hat{r} = (r_{\sigma(1)}, \dots, r_{\sigma(n)})$, then we have a homeomorphism $\mathcal{M}_r \cong \mathcal{M}_{\hat{r}}$. Thus, we have proved

Lemma 3.3.1. *Let the symmetric group S_n act on D_n by coordinate permutation. Then $\mathcal{M}_r \cong \mathcal{M}_{\hat{r}}$ (homeomorphism) for r and $\hat{r}$ in a common orbit of this action.*

Definition 3.3.2. If C and C' are d-cells in D_n that lie in a common orbit of the coordinate permutation action, then we say C and C' have the *same combinatorial type.*

In light of Lemma 3.3.1, to classify the homeomorphism types of non-smooth moduli spaces of polygons, we need only identify the topological type of one cell from each combinatorial type.

As a first step to understanding the topology of the non-smooth moduli spaces of pentagons, we must first understand how the smooth manifolds found by Kappovich and Millson in Theorem 2.3.11 relate to the 4-cells in D_5. The calculations below are substantially those found in [18], but translated into our notation for the chambers of D_5.

C_0^i: These chambers are adjacent to the boundary wall $r_i = \sum_{j \neq i} r_j$. On this wall, the moduli space is a point corresponding to the pentagon that is a straight line segment of length 1/2. This point is the location of a local maximum of the Morse function r_5, so C_0^i has the topological type of a sphere S^2.

$C_1^{i,j}$: By the symmetry of D_5 we may assume that $j = 5$. Then $C_1^{i,j}$ is adjacent to C_0^i across the wall $-r_i - r_5 + \sum_{j \neq i,5} r_j = 0$. Moreover, C_0^i lies in the lower half-space $r_i + r_5 > \sum_{j \neq i,5} r_j$, so the topological type of $C_1^{i,j}$ is that of a sphere S^2 with a handle $S^2 \times D^1$ attached. Therefore $C_1^{i,j}$ has the topological type of a torus Σ_1.

$C_{1,1}^{i,j}$: The barycenter of $C_{1,1}^{i,j}$ is $r_i = r_j = 1/20$ and $r_k = 3/10$ for $k \neq i, j$. Then the r_k satisfy the "three long edges" condition of Proposition 2.3.8, so the topological type of $C_{1,1}^{i,j}$ is that of a disjoint union of tori $\Sigma_1 \bigsqcup \Sigma_1$.

$C_2^{i,j,k}$: Without loss, assume $i, j, k \neq 5$. Then $C_2^{i,j,k}$ is adjacent to $C_1^{i,\ell}$, $\ell \notin \{i, j, k, 5\}$, across the wall $-r_i - r_5 + \sum_{j \neq i,5} r_j = 0$. Again, $C_1^{i,\ell}$ lies in the lower half-space $r_i + r_5 > \sum_{j \neq i,5} r_j$, so the topological type of $C_2^{i,j,k}$ is that of a torus Σ_1 with a handle $S^2 \times D^1$ attached. Therefore $C_2^{i,j,k}$ has the topological type of a genus 2 surface Σ_2.

$C_3^{i,j}$: Without loss, assume $i, j \neq 5$. Then $C_3^{i,j}$ is adjacent to $C_2^{i,j,5}$ across the wall $-r_i - r_5 + \sum_{j \neq i,5} r_j = 0$. Once again, $C_2^{i,j,5}$ lies in the lower half-space $r_i + r_5 > \sum_{j \neq i,5} r_j$, so the topological type of $C_3^{i,j}$ is that of the surface Σ_2 with a handle $S^2 \times D^1$ attached. Therefore $C_3^{i,j}$ has the topological type of a genus 3 surface Σ_3.

C_4: Notice that C_4 is adjacent to $C_3^{1,5}$ across the wall $-r_1 + r_2 + r_3 + r_4 - r_5 = 0$. Since $C_3^{1,5}$ lies in the lower half-space $r_1 + r_5 > r_2 + r_3 + r_4$, we find the topological type of C_4 to be that of a surface Σ_3 with a handle $S^2 \times D^1$ attached. We conclude that C_4 has the topological type of a genus 4 surface Σ_4.

The remainder of this chapter is dedicated to the proof of Theorem 1.1.6. We organize our work by the dimension of the cells.

3.3.1. 3-cells

Every interior 3-cell lies in the intersection of two adjacent 4-cells. Thus the topological type of the 3-cells is given by the intermediate step of a Morse surgery as detailed in Theorem 2.2.14. There are five different combinatorial types of 3-cells. We deal with each individually.

$\operatorname{conv}\{h_{i,j}, h_{i,k}, h_{i,\ell}, t_i\}$: This cell lies in the intersection $C_0^i \cap C_1^{i,m}$ where $m \notin \{i, j, k, \ell\}$. Thus the topological type of this cell is that of sphere with two discs removed, and a cone formed over the resulting boundary. Equivalently, the topological type of this cell is a sphere with two points identified $\Sigma_{0,1}$.

conv $\{h_{i,j}, h_{i,k}, q_\ell, t_i\}$, $\ell \notin \{i,j,k\}$: This cell lies in the intersection $C_1^{i,\ell} \cap C_2^{i,j,k}$. Thus the topological type of this cell is that of a torus Σ_1 with two points identified $\Sigma_{1,1}$.

conv $\{h_{i,j}, h_{i,k}, q_\ell, q_m\}$, $\ell, m \notin \{i,j,k\}$: This cell lies in the intersection $C_{1,1}^{\ell,m} \cap C_2^{i,j,k}$. Without loss, assume that $j = 5$. Then $C_{1,1}^{\ell,m}$ and $C_2^{i,j,k}$ are separated by the wall $-r_k - r_5 + \sum_{i \neq k,5} r_i = 0$. Moreover $C_{1,1}^{\ell,m}$ lies in the lower half-space $r_k + r_5 > \sum_{i \neq k,5}$. Thus the topological type of the 3-cell presently considered is that of a disjoint union of two tori with a pair of points identified. Moreover, the space must be connected since the topological type of $C_2^{i,j,k}$ is connected. Therefore the topological type of the intermediate 3-cell is $\Sigma_1 \vee \Sigma_1$.

conv $\{h_{i,j}, q_k, q_\ell, t_i\}$, $k, \ell, \notin \{i,j\}$: This cell lies in the intersection $C_2^{i,j,m} \cap C_3^{i,j}$ where $m \notin \{i,j,k,\ell\}$. Thus the topological type of this cell is that of Σ_2 with two points identified $\Sigma_{2,1}$.

conv $\{q_i, q_j, q_k, t_\ell, t_m\}$, $\ell, m \notin \{i,j,k\}$: This cell lies in the intersection $C_3^{\ell,m} \cap C_4$. Thus the topological type of this cell is that of Σ_3 with two points identified $\Sigma_{3,1}$.

3.3.2. 2-cells

Note that every d-cell, $d \leq 3$ is contained in some wall $I_{j,k}$. Since all internal walls of D_5 are of the same combinatorial type, by Lemma 3.3.1 it suffices to classify the topologies of the cells that lie in a specific wall. Consider the wall $I_{3,5}$ displayed in an "exploded" view in Figure 3.1. The wall contains three tetrahedra in a ring around the perimeter. The top and bottom sections are volcano shaped, composed of of six tetrahedra in a ring around a base tetrahedron. The very center of the

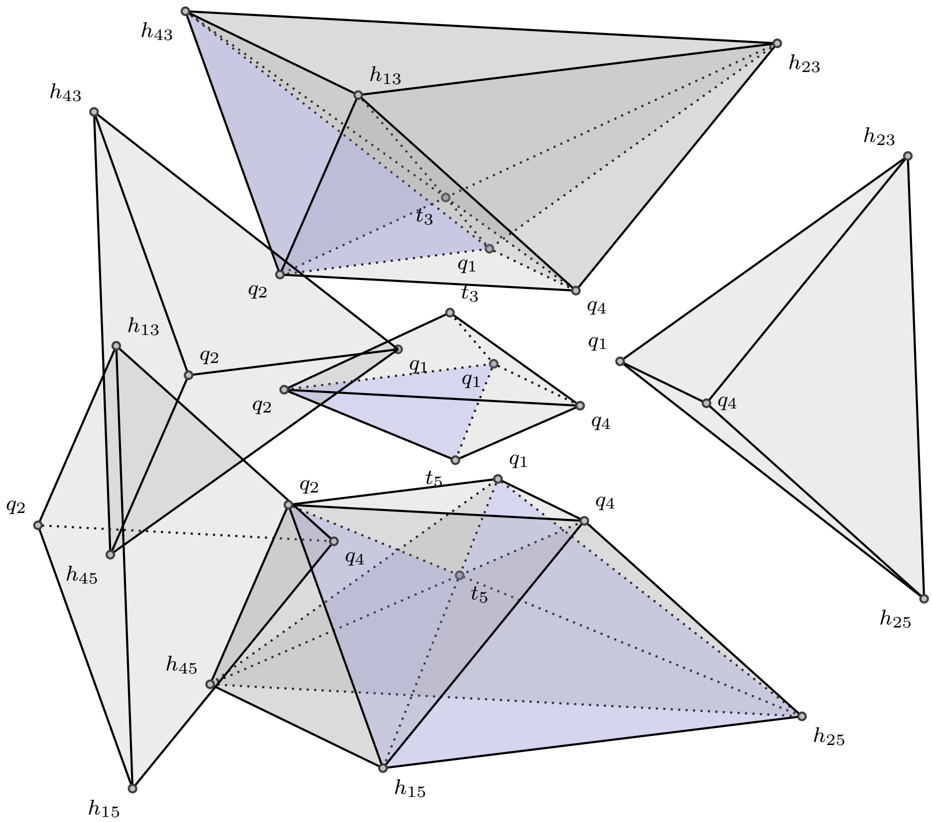

Figure 3.1.: The wall $I_{3,5}$. The blue plane is the intersection $I_{4,5} \cap I_{3,5}$.

wall is a single 3-cell composed of two tetrahedra glued together across a common face. Moreover, very 2-cell in the interior of $I_{3,5}$ has the combinatorial type of a 2-cell in $I_{4,5} \cap I_{3,5}$. We compute the combinatorial type of such 2-cells.

conv $\{h_{1,5}, h_{2,5}, t_5\}$**:** This cell lies in the intersection of $C = \operatorname{conv}\{h_{1,5}, h_{2,5}, t_5, q_4\}$ and $C' = \operatorname{conv}\{h_{1,5}, h_{2,5}, t_5, h_{4,5}\}$. The barycenter of C is $c = (11, 11, 5, 2, 19)/48$, and the barycenter of C' is $c' = (8, 8, 2, 8, 22)/48$. Let γ be the straight line path starting at c and ending at c' with $\gamma' = (-1, -1, -1, 2, 1)$. The wall $I_{4,5}$ between C and C' has signature $\epsilon = (1, 1, 1, -1, -1)$, and

$$\gamma' \cdot \epsilon = -6 < 0,$$

so by Lemma 3.2.7 the topological type of C is obtained from C' by an index 2 surgery. Therefore the topological type of the 2-cell in question is obtained from $\Sigma_{0,1}$ by identifying a pair of points to form $\Sigma_{0,2}$.

conv$\{h_{2,5}, q_1, t_5\}$: This cell lies in the intersection of $C = \text{conv}\{h_{2,5}, q_1, t_5, q_4\}$ and $C' = \text{conv}\{h_{2,5}, q_1, t_5, h_{4,5}\}$. The barycenter of C is $c = (5, 14, 8, 5, 16)/48$, and the barycenter of C' is $c' = (2, 11, 5, 11, 19)/48$. Let γ be the straight line path starting at c and ending at c' with $\gamma' = (-1, -1, -1, 2, 1)$. Again, the wall $I_{4,5}$ between C and C' has signature $\epsilon = (1, 1, 1, -1, -1)$, so $\gamma' \cdot \epsilon < 0$. Thus the topological type of C is obtained from C' by an index 2 surgery. Since the topological type of C' is $\Sigma_{1,1}$ we find the topological type of our 2-cell to be $\Sigma_{1,2}$.

conv$\{h_{3,4}, q_1, q_2\}$: This cell lies in the intersection of $C = \text{conv}\{h_{3,4}, q_1, q_2, t_3\}$ and $C' = \text{conv}\{h_{3,4}, q_1, q_2, h_{4,5}\}$. The barycenter of C is $c = (5, 5, 16, 14, 8)/48$, and the barycenter of C' is $c' = (3, 3, 12, 18, 12)/48$. Let γ be the straight line path starting at c and ending at c' with $\gamma' = (-1, -1, -2, 2, 2)$. The wall between C and C' agin has signature $\epsilon = (1, 1, 1, -1, -1)$, so $\gamma' \cdot \epsilon < 0$. We find the topological type of C to be obtained from C' by an index 2 surgery. Now the topological type of C' is $\Sigma_1 \vee \Sigma_1$, while the topological type of C is $\Sigma_{2,1}$. Therefore the surgery operation must be to delete a disc from each factor of Σ_1 in the wedge sum that forms C', and sew in a handle $S^1 \times D^2$. Our 2-cell of interest must therefore have the topological type $\Sigma_1 \vee_2 \Sigma_1$.

conv$\{q_1, q_2, t_5\}$: This cell lies in the intersection of $C = \text{conv}\{q_1, q_2, t_5, q_4, t_3\}$ and $C' = \text{conv}\{q_1, q_2, t_5, h_{4,5}\}$. The barycenter of C is $c = (8, 8, 12, 8, 12)/48$ and the barycenter of C' is $c' = (5, 5, 8, 14, 16)/48$. Let γ be the straight line path starting at c and ending at c' with $\gamma' = (-3, -3, -4, 6, 4)$. Letting ϵ be the signature of $I_{4,5}$ we find that $\gamma' \cdot \epsilon < 0$. We find the topological type of C to be obtained from C' by an index 2 surgery, whence the topological type of our 2-cell of interest is $\Sigma_{2,2}$.

3.3.3. 1-cells

All interior 1-cells of D_5 have the combinatorial type of a 1-cell lying in the intersection $I_{3,5} \cap I_{4,5}$. Two of those 1-cells lie in the intersection $I_{2,5} \cap I_{3,5} \cap I_{4,5}$. We first consider those cells.

$\mathrm{conv}\{h_{1,5}, t_5\}$: This 1-cell lies in the intersection of the 2-cells $C = \mathrm{conv}\{h_{1,5}, t_5, q_2\}$ and $C' = \mathrm{conv}\{h_{1,5}, t_5, h_{2,5}\}$. The barycenter of C is $c = (11, 2, 5, 5, 13)/36$, and the barycenter of C' is $c' = (8, 8, 2, 2, 16)/36$. Let γ be the line segment from c to c' with velocity $\gamma' = (-3, 5, -3, -3, 3)$. The signature of the wall separating C from C' is $\epsilon = (1, -1, 1, 1, -1)$, and

$$\gamma' \cdot \epsilon = -17 < 0,$$

so by Lemma 3.2.7, the topological type of C is obtained from C' by an index 2 surgery. Thus the topological type of our 1-cell of interest is formed from $\Sigma_{0,2}$ by deleting 2 discs and coning over the boundary, resulting in the topology $\Sigma_{0,2}$.

$\mathrm{conv}\{q_1, t_5\}$: This 1-cell lies in the intersection of the 2-cells $C = \mathrm{conv}\{q_1, t_5, q_2\}$ and $C' = \mathrm{conv}\{q_1, t_5, h_{2,5}\}$. The barycenter of C is $c = (5, 5, 8, 8, 10)/36$, and the barycenter of C' is $c' = (2, 11, 5, 5, 13)/36$. Let γ be the line segment from c to c' with velocity $\gamma' = (-3, 6, -3, -3, 3)$. Again, we have $\gamma' \cdot \epsilon < 0$, where ϵ is the signature of the wall $I_{2,5}$. So the topological type of C is obtained from C' by an index 2 surgery. Thus the topological type of our 1-cell of interest is formed from $\Sigma_{1,2}$ by deleting two discs and coning over the boundary, resulting in the topology $\Sigma_{1,3}$.

The final combinatorial type of a 1-cell has a representative lying in the intersection of walls $I_{3,4} \cap I_{3,5} \cap I_{4,5}$.

$\mathrm{conv}\{q_1, q_2\}$: This 1-cell lies in the intersection of the 2-cells $C = \mathrm{conv}\{q_1, q_2, h_{3,4}\}$ and $C' = \mathrm{conv}\{q_1, q_2, t_5\}$. The barycenter of C is $c = (3, 3, 12, 12, 6)/36$, and the barycenter of C' is $c' = (5, 5, 8, 8, 10)/36$. Let γ be the line segment from c to c' with velocity $\gamma' = (1, 1, -2, -2, 2)$. The signature of the wall $I_{3,4}$ separating C an C' is $\epsilon = (-1, -1, 1, 1, -1)$, so $\gamma' \cdot \epsilon < 0$. Again we have the topological type of C is obtained from C' by an index 1 surgery. Now the topological type of C is $\Sigma_1 \vee_2 \Sigma_1$, and the topological type of C' is $\Sigma_{2,2}$. Thus the surgery operation must be to first delete the neighborhood of an essential loop in $\Sigma_{2,2}$ that is homologically trivial, to form $(\Sigma_1 \backslash D^1) \vee_2 (\Sigma_1 \backslash D^1)$. Coning over the resulting boundary gives the topological type of our 1-cell of interest to be $\Sigma_1 \vee_3 \Sigma_1$.

3.3.4. 0-cell

Up to combinatorial type, there is a unique 0-cell in the interior of D_5. A representative of this combinatorial type is t_5. Now t_5 lies in the intersection $I_{1,5} \cap I_{2,5} \cap I_{3,5} \cap I_{4,5}$. Let $C = \text{conv}\{q_1, t_5\}$ and $C' = \text{conv}\{h_{1,5}, t_5\}$. Let γ be the line segment from C to C' with velocity $\gamma' = (1, -2, -2, -2, 2)$. At t_5, γ crosses the wall $I_{1,5}$ of signature $\epsilon = (-1, 1, 1, 1, -1)$. Therefore $\gamma' \cdot \epsilon < 0$, and the topological type of C is obtained from C' by a surgery of index 2. Since the topological type of C' is $\Sigma_{0,2}$, we conclude that the topological type of t_5 is $\Sigma_{0,3}$. This concludes the proof of Theorem 1.1.6, and this chapter.

4. Bounding the number of diffeomorphism types of higher dimensional, smooth moduli spaces of polygons

In light of Chapter 3, it is now apparent that the polytopal complex structure of D_n contains far more $(n-1)$-cells than are necessary to describe the diffeomorphism types of moduli spaces of n-gons. For example, the computation in Lemma 3.0.1 shows there are 76 4-cells in D_5, yet there are only six different diffeomorphism types possible for a smooth moduli space of pentagons. The main idea of this chapter is to mod out by the action of the symmetric group on D_n to find a smaller complex that contains exactly one top dimensional cell for each combinatorial type. We will then be able to prove Theorem 1.1.7.

Theorem 1.1.7. *Let t_n denote the number of distinct (up to diffeomorphism) maximal-dimensional, smooth manifolds that arise as a moduli space of n-gons. Then we have the following bounds:*

$$t_6 \leq 20,\ t_7 \leq 134,\ t_8 \leq 2469.$$

Remark 4.0.1. If $r = (r_1, \ldots, r_n)$ has $r_i = 0$ for some i, then let

$$\hat{r}_i = (r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n) \in \mathbb{R}^{n-1}$$

denote the vector r with its i-th coordinate deleted. Then $\mathcal{M}_r \cong \mathcal{M}_{\hat{r}_i}$ can be viewed as a moduli space of $(n-1)$-gons. Thus by insisting that t_n only count the maximal dimensional manifolds, we insist that t_n only counts the new diffeomorphism types that arise when moving from a moduli space of $(n-1)$-gons to

n-gons.

Now recall that multiplication $r \mapsto \lambda r$, $r \in \mathbb{R}^n$, $\lambda \in \mathbb{R}_{>0}$, induces a diffeomorphism $\mathcal{M}_r \cong \mathcal{M}_{\lambda r}$. Thus we may remove the equation $r_1 + \cdots + r_n = 1$ from the computation of D_n at the price that D_n is no longer a compact polytopal complex. Throughout the rest of this chapter we make that simplification. Combinatorially, this simplification simply moves every cell in D_n up a dimension, and creates a single vertex at the origin. Such an object is referred to in the literature as a *polyhedral fan [6].*

Let us now concretely state the action of S_n on D_n. Throughout this chapter, the S_n *action on* D_n will refer to the action

$$(r_1, \ldots, r_n) \mapsto (r_{\sigma(1)}, \ldots, r_{\sigma(n)})$$

for $r = (r_1, \ldots, r_n) \in D_n$, $\sigma \in S_n$. It is immediately apparent that this satisfies the axioms of a left group action on a set, namely (i) $1 \cdot r = r$ for the identity $1 \in S_n$ and any $r \in D_n$, and (ii)

$$(\sigma_1\sigma_2) \cdot r = (r_{\sigma_1\sigma_2(1)}, \ldots, r_{\sigma_1\sigma_2(n)}) = (r_{\sigma_1(\sigma_2(1))}, \ldots, r_{\sigma_1(\sigma_2(n))}) = \sigma_1 \cdot (\sigma_2 \cdot r)$$

for $\sigma_1, \sigma_2 \in S_n$, $r \in D_n$. Moreover, the S_n action on D_n is *cellular*, i.e., if $C \subseteq D_n$ is a d-cell, then $\sigma C \subseteq D_n$ is a d-cell for all $\sigma \in S_n$.

4.1. Constructing a fundamental region in D_n

It is well known that the symmetric group is generated by transpositions. When acting on $\mathbb{R}^n$ by coordinate permutation, a transposition of the i-th and j-th coordinate acts as a reflection of $\mathbb{R}^n$ through the hyperplane $x_i = x_j$. Thus we can study the structure of D_n/S_n using the language of finite reflection groups [7].

Definition 4.1.1. Let X be a subset of a finite dimensional vector space V invariant under a finite group G acting on V by reflections. We call a set $F \subseteq X$ a *fundamental region* provided

1. F is relatively open in X.

2. $F \cap TF = \varnothing$ if $1 \neq T \in G$.

3. $X = \cup \left\{ \overline{TF} \cap X \mid T \in G \right\}$.

Lemma 4.1.2. *The inequalities*

$$0 < r_1 < \cdots < r_n < \sum_{i=1}^{n-1} r_i$$

define a fundamental region $F \subseteq D_n$ for the S_n-action on D_n.

Proof. First we recall the defining equations of the boundary of D_n. For $r \in \mathbb{R}^n$, we have $r \in D_n$ if and only if both

$$r_i \geq 0, \text{ for all } i$$
$$r_j \leq \frac{1}{2} \sum_{i=1}^{n} r_i, \text{ for all } j.$$

It is apparent that F satisfies the first condition, and adding r_n to the final inequality defining F yields $2r_n < \sum_{i=1}^{n} r_i$, so $F \subseteq D_n$ as claimed. Since all inequalities are not strict, F is open in D_n. Consider a cycle $s = (s_1 \cdots s_j) \in S_n$, $s_1, \ldots, s_j \in [n]$. We can cyclically permute s to ensure that $s_1 < s_k$ for all $k \in \{2, \ldots, j\}$. Let $r \in F$. Then writing

$$s \cdot r = r' \in D_n$$

we see that

$$r'_{s_1} = r_{s_2} > r_{s_1} = r'_{s_j}.$$

Since $s_1 < s_j$ we have $r' \notin F$. As any element of S_n is a product of disjoint cycles, we have proved that $F \cap TF = \varnothing$, provided T is not the identity of S_n. Finally, consider any $\sigma \in S_n$. Then $\overline{\sigma F}$ is defined by the inequalities

$$0 \leq r_{\sigma(1)} \leq \cdots \leq r_{\sigma(n)} \leq \sum_{i \neq \sigma(n)} r_i.$$

Since, for any $r \in D_n$, we my find a permutation coordinate permutation $\psi \in S_n$ so that the components of $\psi \cdot r$ are non-decreasing, we conclude that $r \in \overline{\psi^{-1} F}$. This verifies the third axiom of the definition of a fundamental region, and the proof is completed. □

We now define $\tilde{D}_n = \bar{F}$ to be the *fundamental complex in* D_n. The polytopal complex structure of $\tilde{D}_n$ is induced by cutting the polytope $\bar{F}$ with the singular set Σ of D_n.

Now it turns out F satisfies a stronger condition than that the union of the closures of its images cover X. Notice that the boundary $\partial(\sigma F) \subseteq \Sigma \subseteq D_n$ is contained in the singular set of D_n. But Σ is formed from an intersection of a union of hyperplanes with D_n, and, since D_n is n-dimensional, Σ is nowhere dense in D_n. Therefore, the orbit of F under the S_n action is in fact *dense* in D_n.

Corollary 4.1.3. *Let $C \subseteq D_n$ be an n-cell. Then there is a $\sigma \in S_n$ and exactly one n-cell $\tilde{C} \subseteq \tilde{D}_n$ such that*

$$\operatorname{relint} \tilde{C} = \operatorname{relint} \left((\sigma C) \cap \tilde{D}_n \right).$$

Proof. Since the orbit of F in D_n is dense, there is a $\sigma \in S_n$ such that $(\sigma C) \cap \tilde{D}_n \neq \varnothing$. Moreover, since the S_n action is cellular, we have $\sigma C = C'$ for some n-cell in D_n. Defining the n-cell $\tilde{C} = C' \cap \tilde{D}_n$, we have

$$(\sigma C) \cap \tilde{D}_n = \tilde{C}.$$

Now suppose that $\psi C = C''$. Then C' and C'' lie in the same orbit under the S_n action, so there is an $\eta \in S_n$ so that $\eta C'' = C'$. We conclude that if $C'' \neq C'$, then relint $\left(C'' \cap \tilde{D}_n\right) = \varnothing$. To see this, notice that $C'' \neq C'$ implies η is not the identity. By the second axiom of fundamental regions,

$$\varnothing = F \cap \left(\eta^{-1}(\operatorname{relint} C')\right) = F \cap C'' = \operatorname{relint}\left(C'' \cap \tilde{D}_n\right).$$

□

The consequence of Corollary 4.1.3 is that the n-cells in the fundamental complex $\tilde{D}_n$ are in one-to-one correspondence with the orbits of the n-cells in D_n.

Remark 4.1.4. The same cannot be said for lower dimensional cells. The boundary $\partial\tilde{D}_n$ contains cells with essential supporting hyperplanes of the form $r_i = r_j$. These hyperplanes are not members of the singular set Σ of D_n, hence the cells in $\tilde{D}_n$ they support may or may not correspond to a cell in D_n. More careful analysis is required to obtain information about the lower dimensional cells of D_n from those of $\tilde{D}_n$.

4.2. Computing the chambers of $\tilde{D}_n$

Realizing that the n-cells of $\tilde{D}_n$ are in one-to-one correspondence with the combinatorial types of the n-cells in D_n, we now attempt to enumerate the n-cells of $\tilde{D}_n$. The cell enumeration is done algorithmically. The first step of the algorithm is to identify an n-cell in $\tilde{D}_n$.

Lemma 4.2.1. *For $n \geq 3$, let*

$$\mathcal{I} = \{I \subseteq [n-1] \mid \varnothing \neq I \neq [n-1]\}.$$

For $I \in \mathcal{I}$, define $W_I \subseteq \mathbb{R}^n$ to be the half-space defined by the equation

$$x_n + \sum_{i \in I} x_i \geq \sum_{j \in [n-1] \setminus I} x_j.$$

Then

$$C = \tilde{D}_n \cap \bigcap_{I \in \mathcal{I}} W_I$$

is an n-cell in $\tilde{D}_n$.

Proof. It suffices to find a point $r \in \mathbb{R}^n$ that solves each of the defining inequalities of $\tilde{D}_n$ and W_I for all $I \in \mathcal{I}$ without achieving equality. Such a point will have a small neighborhood ball $B_\epsilon(r) \subseteq \mathring{C}$ contained in the interior of C, proving the claim. Define

$$r = \left(1, 2, \ldots, n-1, \frac{n(n-1)-1}{2}\right) \in \mathbb{R}^n.$$

Then $r \in \mathring{\tilde{D}}_n$ since $r_i < r_j$ for all $i, j \in [n]$ with $i < j$, and

$$r_n = \frac{n(n-1)-1}{2} < \frac{n(n-1)}{2} = \sum_{i=1}^{n-1} i = \sum_{i=1}^{n-1} r_i.$$

Let $I \in \mathcal{I}$. It now suffices to show that r solves the defining equality of W_I without achieving equality. To that end, we compute

$$r_5 + \sum_{i \in I} r_i > r_5 = \frac{n(n-1)-1}{2} > \sum_{i=2}^{n-1} r_i \geq \sum_{i \in [n-1] \setminus I} r_i,$$

where the final inequality follows since r_1 is the minimum of $\{r_i \in [n-1]\}$. □

We proceed to outline the steps of the algorithm here, and provide an implementation of the algorithm in SageMath 8.4 in Appendix B.

1. First we build the outer boundary of $\tilde{D}_n$. This is done by solving the matrix

inequality $Ax \geq 0$, where $x \in \mathbb{R}^n$, and

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -1 \end{pmatrix}.$$

Call this boundary polytope B_n.

2. Next we find the interior walls of D_n which cut the polytope found above. Let Φ denote the set of these walls, collectively called the *interior walls of* $\tilde{D}_n$. Together these walls form the fundamental complex $\tilde{D}_n$.

3. Let ϵ be the signature of a wall in Φ. There are two choices of ϵ. For each $W \in \Phi$, we specify the signature of W to be the one with $\epsilon_n = 1$. Let W^+ denote the upper half-space with this choice of signature. I.e., W^+ is defined by the inequality $\sum_{i=1}^n \epsilon_i r_i \geq 0$.

4. By Lemma 4.2.1,

$$\bigcap_{W \in \Phi} W^+ \cap B_n$$

is an n-cell in $\tilde{D}_n$.

5. A choice of half-spaces with boundary walls in Φ determines a cell in $\tilde{D}_n$. Let C be an n-cell in $\tilde{D}_n$. Compute the minimal h-representation of C. The essential supporting hyperplanes of this representation are either in Φ, or they are boundary walls of $\tilde{D}_n$. A neighbor of C across an $(n-1)$-cell is found by switching the half-space of exactly one essential supporting hyperplane of C that lies in Φ.

6. Having found one n-cell $C \subseteq \tilde{D}_n$, we recursively compute all neighbors across

an $(n-1)$-cell of C. We then compute the neighbors of the neighbors, and so on.

7. This algorithm terminates due to the finiteness of the set Φ.

The output of the function "chambers(n)" defined in Appendix B is a pair $[C, H]$, where C is a list of all n-cells in $\tilde{D}_n$, and H is a list of the half-spaces from Φ that define each of these n-cells. By counting the size of C, we have proved Theorem 1.1.7.

Remark 4.2.2. I ran the algorithm in Appendix B on a 2018 model MacBook Pro. Computing "chambers(7)" required about 15 seconds. The runtime of "chambers(8)" was under 3 hours. I allowed "chambers(9)" to run for several days, but that wasn't enough time for it to terminate.

4.3. The network of $\tilde{D}_n$

We conclude this chapter by noting that $\tilde{D}_n$ has more information than simply the number of cells. Suppose C and C' are adjacent across a wall W of signature ϵ where $\epsilon_n = 1$, with $C \subseteq W^+$ and $C' \subseteq W^-$. Then on a path from C to C' with $r_1, \ldots, r_{n-1}$ fixed that crosses $C \cap C'$ in the relative interior, r_n decreases. Thus by Theorem 2.3.10, the topological type of C' is obtained from C by a surgery. Let f be the number of -1 entries in ϵ, and b be the number of $+1$ entries in ϵ. Then the Morse index of the surgery is given by $(f-1, b-1)$. We proceed to build the adjacency graph of the n-cells of $\tilde{D}_n$. We add a directed edge pointing from C to C' labeled with the number of $f-1$. We call this graph the *network of* $\tilde{D}_n$. In a sense the network of $\tilde{D}_n$ contains all of the information of the Morse theory of the moduli space of n-gons. See Figures 4.1 and 4.2.

Now, the initial cell found in Lemma 4.2.1 is bounded by the hyperplane $r_n = \frac{1}{2}\sum_{i\in[n]} r_i$. The for r contained in this hyperplane, $\mathcal{M}_r$ is a single point corresponding to a rigid line segment of length $\frac{1}{2}\sum_{i\in[n]} r_i$. Therefore, for r in the

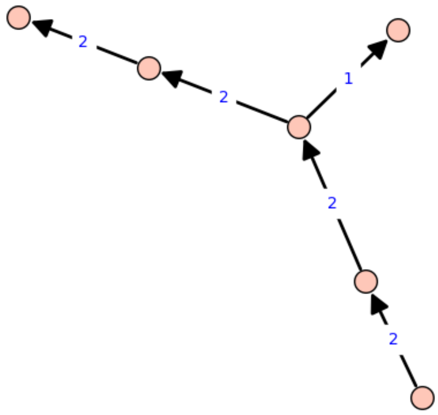

Figure 4.1.: The network of $\tilde{D}_5$.

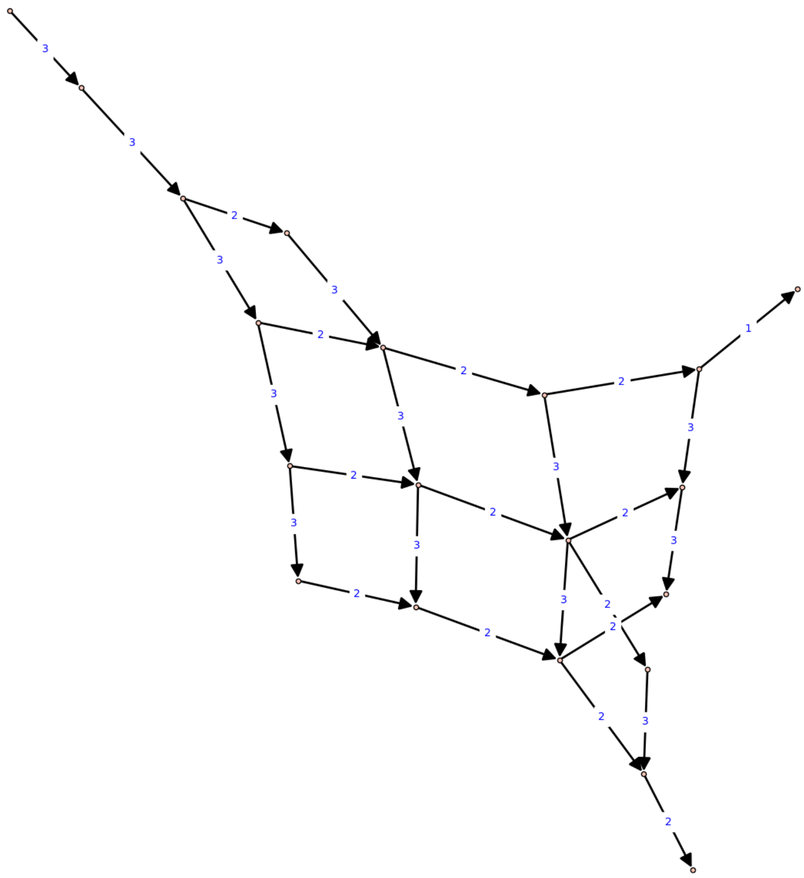

Figure 4.2.: The network of $\tilde{D}_6$.

initial cell of the network of $\tilde{D}_n$, $\mathcal{M}_r \cong S^{n-3}$ is a sphere.

Next, consider an edge of $\tilde{D}_n$ labeled by $n-3$. Then the topological type of the head vertex of that edge is obtained from that of the tail by deleting a region diffeomorphic to $D^{n-3} \times S^0$ and sewing in a "handle" $S^{n-4} \times D^1$. Consider the effect of such a surgery on a sphere S^{n-3}. The first step of this surgery applied to S^{n-3} results in a handle $S^{n-4} \times D^1$. We then double this handle across it's boundary. Assuming the resulting manifold is orientable, we obtain $S^{n-4} \times S^1$. Since we can always move this surgery into a region diffeomorphic to a disk, the topological type of the head vertex of an edge labeled by $n-3$ is obtained from the tail by a connected sum with $S^{n-4} \times S^1$.

Applying these results to the network of $\tilde{D}_6$ in Figure 4.2, we obtain the following theorem.

Theorem 4.3.1. *There exist moduli spaces diffeomorphic to S^3 and $(S^2 \times S^1)^{\#k}$ for $1 \leq k \leq 5$.*

Remark 4.3.2. After I completed the work of Section 4.3 I found that Hausmann, through similar methods, had already obtained a handle description of all smooth moduli spaces of hexagons. For completion I include his results here.

Theorem 4.3.3. *[11] Let $H = S^2 \times S^1$. The following table gives a complete list of the diffeomorphism types realized as smooth moduli spaces of hexagons.*

$S^3 \# H^{\#k}$	$0 \leq k \leq 5$	$\Sigma_4 \times S^1$	
$T^3 \# H^{\#k}$	$0 \leq k \leq 3$	$T^3 \bigsqcup T^3$	
$(\Sigma_2 \times S^1) \# H^{\#k}$	$0 \leq k \leq 2$	$(T^3 \# T^3) \# H^{\#k}$	$0 \leq k \leq 2$
$(\Sigma_3 \times S^1) \# H^{\#k}$	$0 \leq k \leq 1$		

5. Future work

Having completed a census of the crimped manifolds that arise as moduli spaces of pentagons, it seems logical to consider the topology of non-smooth moduli spaces of hexagons. The key problem is that there are degenerate hexagons at which the Morse index of the final edge length function is $(2,2)$.When performing a Morse surgery of such index, the singularity that arises is a cone over a torus $C(T^2)$. Unlike the surgeries performed on the moduli spaces of pentagons, the topology of the singular space depends on how the removed solid torus was embedded in the initial manifold. Relevant questions are: is the torus knotted? Is the midline of the torus contractible in the ambient manifold? Though Hausmann [11] computed all of the smooth topologies for the moduli spaces of hexagons, he did not apply any index $(2,2)$ surgeries. For example, comparing his list in Theorem 4.3.3 to the network of $\tilde{D}_6$ in Figure 4.2, apparently there is a moduli space $\mathcal{M}_r \cong (S^2 \times S^1)\#(S^2 \times S^1)$ and another $\mathcal{M}_{r'} \cong T^3$ such that $\mathcal{M}_{r'}$ is obtained from $\mathcal{M}_r$ by removing a solid torus, and gluing another solid torus in its place. How can this torus be identified?

A promising line of reasoning is indicated by Panina [29]. She developed a CW-structure on $\mathcal{M}_r$ which we describe here. Let $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. Let $s \subseteq [n]$. We call s *short* provided $\sum_{i \in s} r_i < \frac{1}{2} \sum_{j \in [n]} r_j$. A *label* is a cyclic ordering $(s_1, \ldots, s_k)$ such that $\cup s_i = [n]$ and $s_i \cap s_j = \varnothing$ if $i \neq j$. We consider all cyclic permutations of a label to be the same. Also the ordering of the objects with each of the s_i is not considered.

Example 5.0.1. Let $n = 4$. Then, as labels,

$$(\{1\},\{2\},\{3,4\}) = (\{2\},\{3,4\},\{1\}) = (\{2\},\{4,3\},\{1\}) \neq (\{4,3\},\{2\},\{1\}).$$

For a label $\ell = (s_1, \ldots, s_k)$, we call one the s_i a *part* of ℓ. We call a label *admissible* if all of its parts are short. We call a label ℓ' a *refinement* of ℓ if ℓ' can be obtained by subdividing one or more of the parts of ℓ.

Example 5.0.2. Consider the label of $\ell = (\{1\},\{2,3\},\{4,5\})$. Then

$$(\{1\},\{2\},\{3\},\{5\},\{4\})$$

is a refinement of ℓ, but $(\{2\},\{1\},\{3\},\{4,5\})$ is not a refinement of ℓ.

In this language we have the following CW-structure on $\mathcal{M}_r$.

Theorem 5.0.3. *[29] Let $r \in D_n$ be generic, i.e., r does not lie on a wall. Then there is a CW-structure $K(r)$ on $\mathcal{M}_r$ produced as follows.*

1. *The d-cells of $K(r)$ are admissible labels containing $(d+3)$-many nonempty parts.*
2. *A closed cell C' lies in the boundary of C if and only if C is a refinement of C'.*

To simplify notation, let us remove the set braces from the label, and insert bars between the disparate sets. E.g., we will write

$$(\{1,2\},\{3\},\{4\}) = (1,2 \mid 3 \mid 4).$$

Example 5.0.4. Consider $\mathcal{M}_r$ for $r = (1,2,3,5)$. The following are the only

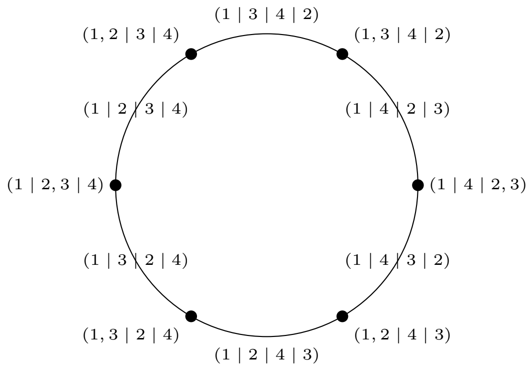

Figure 5.1.: The CW-structure for $\mathcal{M}_r$, $r = (1, 2, 3, 5)$.

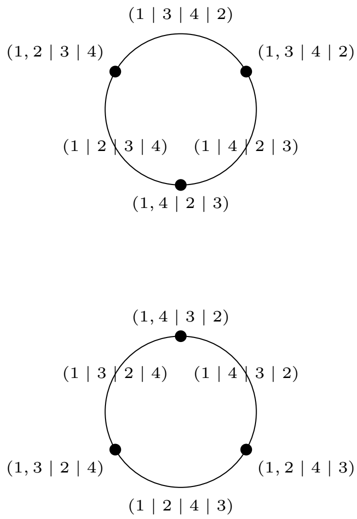

Figure 5.2.: The CW-structure for $\mathcal{M}_r$, $r = (1, 2, 3, 3)$.

admissible labels of $K(r)$ containing 3 parts:

$$\begin{array}{ll} (1,2 \mid 3 \mid 4) & (1,2 \mid 4 \mid 3) \\ (1,3 \mid 2 \mid 4) & (1,3 \mid 4 \mid 2) \\ (1 \mid 2,3 \mid 4) & (1 \mid 4 \mid 2,3). \end{array}$$

We have drawn a representation of the complex $K(r)$ in Figure 5.1.

Example 5.0.5. Consider $\mathcal{M}_r$ for $r = (1, 2, 3, 3)$. The following are the only admissible labels of $K(r)$ containing 3 parts:

$$\begin{array}{ll} (1,2 \mid 3 \mid 4) & (1,2 \mid 4 \mid 3) \\ (1,3 \mid 2 \mid 4) & (1,3 \mid 4 \mid 2) \\ (1,4 \mid 2 \mid 3) & (1,4 \mid 3 \mid 2). \end{array}$$

We have drawn a representation of the complex $K(r)$ in Figure 5.2.

Notice the complexes in Examples 5.0.4 and 5.0.5 only differ in one set of vertex labels. Moreover, the topology of the complex in Example 5.0.5 is obtained from that of Example 5.0.4 by removing a small neighborhood of the non-common label, and inserting a new label. Is there an analogous procedure that can be performed on moduli spaces of hexagons that differ by a $(2,2)$-surgery? Can this process be automated from information contained in the network of $\tilde{D}_6$?

Bibliography

[1] David Avis and Komei Fukuda. A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra. *Discrete Comput. Geom.*, 8(3):295–313, 1992. ACM Symposium on Computational Geometry (North Conway, NH, 1991).

[2] Theodor Bröcker and Klaus Jänich. *Introduction to differential topology.* Cambridge University Press, Cambridge-New York, 1982. Translated from the German by C. B. Thomas and M. J. Thomas.

[3] J. W. Bruce and P. J. Giblin. *Curves and singularities.* Cambridge University Press, Cambridge, second edition, 1992. A geometrical introduction to singularity theory.

[4] M. Farber and D. Schütz. Homology of planar polygon spaces. *Geom. Dedicata*, 125:75–92, 2007.

[5] Joel N. Franklin. *Matrix theory.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1968.

[6] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[7] L. C. Grove and C. T. Benson. *Finite reflection groups*, volume 99 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1985.

[8] Branko Grünbaum. *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

[9] J.-C. Hausmann and A. Knutson. The cohomology ring of polygon spaces. *Ann. Inst. Fourier (Grenoble)*, 48(1):281–321, 1998.

[10] Jean-Claude Hausmann. Sur la topologie des bras articulés. In *Algebraic topology Poznań 1989*, volume 1474 of *Lecture Notes in Math.*, pages 146–159. Springer, Berlin, 1991.

[11] Jean-Claude Hausmann. Geometric descriptions of polygon and chain spaces. In *Topology and robotics*, volume 438 of *Contemp. Math.*, pages 47–57. Amer. Math. Soc., Providence, RI, 2007.

[12] Jean-Claude Hausmann and Eugenio Rodriguez. The space of clouds in Euclidean space. *Experiment. Math.*, 13(1):31–47, 2004.

[13] Timothy F. Havel. Some examples of the use of distances as coordinates for Euclidean geometry. *J. Symbolic Comput.*, 11(5-6):579–593, 1991. Invariant-theoretic algorithms in geometry (Minneapolis, MN, 1987).

[14] B. Jaggi. Configuration spaces of point sets with distance constraints. PhD thesis Univ. of Bern, 1992.

[15] Yasuhiko Kamiyama. The homology of singular polygon spaces. *Canad. J. Math.*, 50(3):581–594, 1998.

[16] Yasuhiko Kamiyama and Michishige Tezuka. Symplectic volume of the moduli space of spatial polygons. *J. Math. Kyoto Univ.*, 39(3):557–575, 1999.

[17] Yasuhiko Kamiyama and Michishige Tezuka. Topology and geometry of equilateral polygon linkages in the Euclidean plane. *Quart. J. Math. Oxford Ser. (2)*, 50(200):463–470, 1999.

[18] Michael Kapovich and John Millson. On the moduli space of polygons in the Euclidean plane. *J. Differential Geom.*, 42(2):430–464, 1995.

[19] Michael Kapovich and John J. Millson. The symplectic geometry of polygons in Euclidean space. *J. Differential Geom.*, 44(3):479–513, 1996.

[20] Michael Kapovich and John J. Millson. On the moduli space of a spherical polygonal linkage. *Canad. Math. Bull.*, 42(3):307–320, 1999.

[21] Michael Kapovich, John J. Millson, and Thomas Treloar. The symplectic geometry of polygons in hyperbolic 3-space. *Asian J. Math.*, 4(1):123–164, 2000. Kodaira's issue.

[22] Vu The Khoi. On the symplectic volume of the moduli space of spherical and Euclidean polygons. *Kodai Math. J.*, 28(1):199–208, 2005.

[23] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[24] W. J. Lenhart and S. H. Whitesides. Reconfiguring closed polygonal chains in euclidean d-space. *Discrete & Computational Geometry*, 13(1):123–140, Jan 1995.

[25] Alessia Mandini. The Duistermaat-Heckman formula and the cohomology of moduli spaces of polygons. *J. Symplectic Geom.*, 12(1):171–213, 2014.

[26] Shawn Martin, Aidan Thompson, Evangelos Coutsias, and Jean-Paul Watson. Topology of cyclooctane energy landscape. *The Journal of chemical physics*, 132:234115, 06 2010.

[27] R. James Milgram and J. C. Trinkle. The geometry of configuration spaces for closed chains in two and three dimensions. *Homology Homotopy Appl.*, 6(1):237–267, 2004.

[28] J. Milnor. *Morse theory*. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[29] Gaiane Panina. Moduli space of a planar polygonal linkage: a combinatorial description. *Arnold Math. J.*, 3(3):351–364, 2017.

[30] K. Walker. Configuration spaces of linkages. Undergraduate thesis Princeton University, 1985.

[31] Günter M. Ziegler. *Lectures on Polytopes*. Number 152 in Graduate Texts in Mathematics. Springer, New York, 1995.

APPENDIX

A. Computing 4-cells and vertices of D_5

The following code was run in SageMath 8.4 to find all 4-cells and vertices in the polytopal complex D_5.

```
index = cartesian_product([
GF(2),GF(2),GF(2),GF(2),GF(2),
GF(2),GF(2),GF(2),GF(2),GF(2)
])
I = matrix.identity(5)
b = matrix([0,0,0,0,0,0,0,0,0,0,-1,1,1,1,1,1,1,0,0,0,0,0])
b = b.transpose()
A = matrix([
[1,1,1,-1,-1],
[1,1,-1,1,-1],
[1,-1,1,1,-1],
[-1,1,1,1,-1],
[1,1,-1,-1,1],
[1,-1,1,-1,1],
[-1,1,1,-1,1],
[1,-1,-1,1,1],
[-1,1,-1,1,1],
[-1,-1,1,1,1],
[1,1,1,1,1],
[-1,-1,-1,-1,-1]
])
```

```
A = A.stack(-2*I)
A = A.stack(I)
Cells = []
Verts = []

for x in index:
        C = copy(A)
        for i in range(len(x)):
                C[i] = (-1)^x[i]*C[i]
        C = b.augment(C)
        P = Polyhedron(ieqs = C)
        if P.dim() == 4:
                Cells.append(P)
                Q = P.vertices_list()
                for i in range(len(Q)):
                        if not Q[i] in Verts:
                                Verts.append(Q[i])

for i in range(len(Cells)):
        print 'Cell', i
        print (Cells[i].vertices_matrix()).transpose()
        print'\n'

print 'List of vertices'
print matrix(Verts)
```

The result of running this code follows.

```
Cell 0
[1/6 1/6 1/6 1/6 1/3]
[  0 1/4 1/4 1/4 1/4]
[1/4   0 1/4 1/4 1/4]
[1/4 1/4   0 1/4 1/4]
[1/4 1/4 1/4   0 1/4]
[1/3 1/6 1/6 1/6 1/6]
[1/6 1/3 1/6 1/6 1/6]
[1/4 1/4 1/4 1/4   0]
[1/6 1/6 1/3 1/6 1/6]
[1/6 1/6 1/6 1/3 1/6]

Cell 1
[1/2 1/2   0   0   0]
[1/4 1/4   0 1/4 1/4]
[1/4 1/4 1/4   0 1/4]
[1/6 1/3 1/6 1/6 1/6]
[1/3 1/6 1/6 1/6 1/6]
[1/4 1/4 1/4 1/4   0]

Cell 2
[1/6 1/6 1/3 1/6 1/6]
[1/4   0 1/4 1/4 1/4]
[1/2   0 1/2   0   0]
[1/4 1/4 1/4   0 1/4]
[1/4 1/4 1/4 1/4   0]
[1/3 1/6 1/6 1/6 1/6]

Cell 3
[1/2 1/2   0   0   0]
[1/2   0 1/2   0   0]
[1/4 1/4 1/4   0 1/4]
[1/4 1/4 1/4 1/4   0]
[1/3 1/6 1/6 1/6 1/6]

Cell 4
[1/6 1/6 1/3 1/6 1/6]
[  0 1/4 1/4 1/4 1/4]
[  0 1/2 1/2   0   0]
[1/4 1/4 1/4   0 1/4]
[1/4 1/4 1/4 1/4   0]
[1/6 1/3 1/6 1/6 1/6]

Cell 5
[1/2 1/2   0   0   0]
[  0 1/2 1/2   0   0]
[1/4 1/4 1/4   0 1/4]
[1/4 1/4 1/4 1/4   0]
[1/6 1/3 1/6 1/6 1/6]

Cell 6
[1/6 1/6 1/3 1/6 1/6]
[  0 1/2 1/2   0   0]
[1/4 1/4 1/4   0 1/4]
[1/2   0 1/2   0   0]
[1/4 1/4 1/4 1/4   0]
```

Cell 7

[1/4 1/4 1/4 1/4 0]
[0 1/2 1/2 0 0]
[1/2 0 1/2 0 0]
[1/4 1/4 1/4 0 1/4]
[1/2 1/2 0 0 0]

Cell 8

[1/6 1/6 1/6 1/3 1/6]
[1/4 0 1/4 1/4 1/4]
[1/2 0 0 1/2 0]
[1/4 1/4 0 1/4 1/4]
[1/4 1/4 1/4 1/4 0]
[1/3 1/6 1/6 1/6 1/6]

Cell 9

[1/2 1/2 0 0 0]
[1/2 0 0 1/2 0]
[1/4 1/4 0 1/4 1/4]
[1/4 1/4 1/4 1/4 0]
[1/3 1/6 1/6 1/6 1/6]

Cell 10

[1/3 1/6 1/6 1/6 1/6]
[1/4 0 1/4 1/4 1/4]
[1/2 0 0 1/2 0]
[1/4 1/4 1/4 1/4 0]
[1/2 0 1/2 0 0]

Cell 11

[1/3 1/6 1/6 1/6 1/6]
[1/2 0 0 1/2 0]
[1/2 0 1/2 0 0]
[1/4 1/4 1/4 1/4 0]
[1/2 1/2 0 0 0]

Cell 12

[1/6 1/6 1/6 1/3 1/6]
[1/4 1/4 0 1/4 1/4]
[0 1/4 1/4 1/4 1/4]
[0 1/2 0 1/2 0]
[1/4 1/4 1/4 1/4 0]
[1/6 1/3 1/6 1/6 1/6]

Cell 13

[1/4 1/4 1/4 1/4 0]
[0 1/2 0 1/2 0]
[1/4 1/4 0 1/4 1/4]
[1/2 1/2 0 0 0]
[1/6 1/3 1/6 1/6 1/6]

Cell 14

[1/6 1/3 1/6 1/6 1/6]
[0 1/4 1/4 1/4 1/4]
[0 1/2 0 1/2 0]
[0 1/2 1/2 0 0]
[1/4 1/4 1/4 1/4 0]

Cell 15

[1/6 1/3 1/6 1/6 1/6]
[0 1/2 0 1/2 0]
[0 1/2 1/2 0 0]
[1/4 1/4 1/4 1/4 0]
[1/2 1/2 0 0 0]

Cell 16

[1/6 1/6 1/6 1/3 1/6]
[1/4 1/4 0 1/4 1/4]
[0 1/2 0 1/2 0]
[1/2 0 0 1/2 0]
[1/4 1/4 1/4 1/4 0]

Cell 17

[1/4 1/4 1/4 1/4 0]
[0 1/2 0 1/2 0]
[1/2 0 0 1/2 0]
[1/4 1/4 0 1/4 1/4]
[1/2 1/2 0 0 0]

Cell 18

[1/6 1/6 1/6 1/3 1/6]
[0 1/4 1/4 1/4 1/4]
[1/4 0 1/4 1/4 1/4]
[0 0 1/2 1/2 0]
[1/6 1/6 1/3 1/6 1/6]
[1/4 1/4 1/4 1/4 0]

Cell 19

[1/6 1/6 1/3 1/6 1/6]
[0 0 1/2 1/2 0]
[1/4 0 1/4 1/4 1/4]
[1/2 0 1/2 0 0]
[1/4 1/4 1/4 1/4 0]

Cell 20

[0 1/4 1/4 1/4 1/4]
[1/6 1/6 1/3 1/6 1/6]
[0 0 1/2 1/2 0]
[0 1/2 1/2 0 0]
[1/4 1/4 1/4 1/4 0]

Cell 21

[1/6 1/6 1/3 1/6 1/6]
[0 0 1/2 1/2 0]
[1/2 0 1/2 0 0]
[0 1/2 1/2 0 0]
[1/4 1/4 1/4 1/4 0]

Cell 22

[1/6 1/6 1/6 1/3 1/6]
[0 0 1/2 1/2 0]
[1/4 0 1/4 1/4 1/4]
[1/2 0 0 1/2 0]
[1/4 1/4 1/4 1/4 0]

Cell 23

[1/4 1/4 1/4 1/4 0]
[0 0 1/2 1/2 0]
[1/4 0 1/4 1/4 1/4]
[1/2 0 0 1/2 0]
[1/2 0 1/2 0 0]

Cell 24

[0 1/4 1/4 1/4 1/4]
[1/6 1/6 1/6 1/3 1/6]
[0 0 1/2 1/2 0]
[0 1/2 0 1/2 0]
[1/4 1/4 1/4 1/4 0]

Cell 25

[0 1/4 1/4 1/4 1/4]
[1/4 1/4 1/4 1/4 0]
[0 0 1/2 1/2 0]
[0 1/2 0 1/2 0]
[0 1/2 1/2 0 0]

Cell 26

[1/6 1/6 1/6 1/3 1/6]
[1/2 0 0 1/2 0]
[0 0 1/2 1/2 0]
[0 1/2 0 1/2 0]
[1/4 1/4 1/4 1/4 0]

Cell 27

[1/6 1/6 1/6 1/6 1/3]
[1/4 1/4 0 1/4 1/4]
[1/4 0 1/4 1/4 1/4]
[1/2 0 0 0 1/2]
[1/4 1/4 1/4 0 1/4]
[1/3 1/6 1/6 1/6 1/6]

Cell 28

[1/4 1/4 1/4 0 1/4]
[1/2 0 0 0 1/2]
[1/4 1/4 0 1/4 1/4]
[1/2 1/2 0 0 0]
[1/3 1/6 1/6 1/6 1/6]

Cell 29

[1/3 1/6 1/6 1/6 1/6]
[1/4 0 1/4 1/4 1/4]
[1/2 0 0 0 1/2]
[1/4 1/4 1/4 0 1/4]
[1/2 0 1/2 0 0]

Cell 30

[1/3 1/6 1/6 1/6 1/6]
[1/2 0 0 0 1/2]
[1/2 0 1/2 0 0]
[1/4 1/4 1/4 0 1/4]
[1/2 1/2 0 0 0]

Cell 31

[1/3 1/6 1/6 1/6 1/6]
[1/2 0 0 1/2 0]
[1/4 0 1/4 1/4 1/4]
[1/2 0 0 0 1/2]
[1/4 1/4 0 1/4 1/4]

Cell 32

[1/3 1/6 1/6 1/6 1/6]
[1/2 0 0 0 1/2]
[1/2 0 0 1/2 0]
[1/4 1/4 0 1/4 1/4]
[1/2 1/2 0 0 0]

Cell 33

[1/3 1/6 1/6 1/6 1/6]
[1/4 0 1/4 1/4 1/4]
[1/2 0 0 0 1/2]
[1/2 0 0 1/2 0]
[1/2 0 1/2 0 0]

Cell 34

[1/3 1/6 1/6 1/6 1/6]
[1/2 0 0 0 1/2]
[1/2 0 0 1/2 0]
[1/2 0 1/2 0 0]
[1/2 1/2 0 0 0]

Cell 35

[1/6 1/6 1/6 1/6 1/3]
[0 1/2 0 0 1/2]
[1/4 1/4 0 1/4 1/4]
[0 1/4 1/4 1/4 1/4]
[1/4 1/4 1/4 0 1/4]
[1/6 1/3 1/6 1/6 1/6]

Cell 36

[1/4 1/4 1/4 0 1/4]
[0 1/2 0 0 1/2]
[1/4 1/4 0 1/4 1/4]
[1/2 1/2 0 0 0]
[1/6 1/3 1/6 1/6 1/6]

Cell 37

[1/6 1/3 1/6 1/6 1/6]
[0 1/2 0 0 1/2]
[0 1/4 1/4 1/4 1/4]
[0 1/2 1/2 0 0]
[1/4 1/4 1/4 0 1/4]

Cell 38

[1/6 1/3 1/6 1/6 1/6]
[0 1/2 0 0 1/2]
[0 1/2 1/2 0 0]
[1/4 1/4 1/4 0 1/4]
[1/2 1/2 0 0 0]

Cell 39

[1/6 1/3 1/6 1/6 1/6]
[0 1/2 0 1/2 0]
[0 1/4 1/4 1/4 1/4]
[0 1/2 0 0 1/2]
[1/4 1/4 0 1/4 1/4]

Cell 40

[1/6 1/3 1/6 1/6 1/6]
[0 1/2 0 0 1/2]
[0 1/2 0 1/2 0]
[1/4 1/4 0 1/4 1/4]
[1/2 1/2 0 0 0]

Cell 41

[0 1/2 1/2 0 0]
[0 1/4 1/4 1/4 1/4]
[0 1/2 0 0 1/2]
[0 1/2 0 1/2 0]
[1/6 1/3 1/6 1/6 1/6]

Cell 42

[1/6 1/3 1/6 1/6 1/6]
[0 1/2 0 0 1/2]
[0 1/2 0 1/2 0]
[0 1/2 1/2 0 0]
[1/2 1/2 0 0 0]

Cell 43

[1/6 1/6 1/6 1/6 1/3]
[1/4 1/4 0 1/4 1/4]
[0 1/2 0 0 1/2]
[1/2 0 0 0 1/2]
[1/4 1/4 1/4 0 1/4]

Cell 44

[1/4 1/4 1/4 0 1/4]
[0 1/2 0 0 1/2]
[1/2 0 0 0 1/2]
[1/4 1/4 0 1/4 1/4]
[1/2 1/2 0 0 0]

Cell 45

[1/6 1/6 1/6 1/6 1/3]
[0 1/4 1/4 1/4 1/4]
[0 0 1/2 0 1/2]
[1/4 0 1/4 1/4 1/4]
[1/6 1/6 1/3 1/6 1/6]
[1/4 1/4 1/4 0 1/4]

Cell 46

[1/6 1/6 1/3 1/6 1/6]
[0 0 1/2 0 1/2]
[1/4 0 1/4 1/4 1/4]
[1/2 0 1/2 0 0]
[1/4 1/4 1/4 0 1/4]

Cell 47

[0 0 1/2 0 1/2]
[1/6 1/6 1/3 1/6 1/6]
[0 1/4 1/4 1/4 1/4]
[0 1/2 1/2 0 0]
[1/4 1/4 1/4 0 1/4]

Cell 48

[1/6 1/6 1/3 1/6 1/6]
[0 0 1/2 0 1/2]
[0 1/2 1/2 0 0]
[1/2 0 1/2 0 0]
[1/4 1/4 1/4 0 1/4]

Cell 49

[1/6 1/6 1/3 1/6 1/6]
[0 1/4 1/4 1/4 1/4]
[1/4 0 1/4 1/4 1/4]
[0 0 1/2 0 1/2]
[0 0 1/2 1/2 0]

Cell 50

[1/6 1/6 1/3 1/6 1/6]
[0 0 1/2 0 1/2]
[0 0 1/2 1/2 0]
[1/4 0 1/4 1/4 1/4]
[1/2 0 1/2 0 0]

Cell 51

```
[1/6 1/6 1/3 1/6 1/6]
[  0   0 1/2 1/2   0]
[  0 1/4 1/4 1/4 1/4]
[  0   0 1/2   0 1/2]
[  0 1/2 1/2   0   0]
```

Cell 52

```
[1/6 1/6 1/3 1/6 1/6]
[  0   0 1/2   0 1/2]
[  0   0 1/2 1/2   0]
[  0 1/2 1/2   0   0]
[1/2   0 1/2   0   0]
```

Cell 53

```
[1/6 1/6 1/6 1/6 1/3]
[  0   0 1/2   0 1/2]
[1/4   0 1/4 1/4 1/4]
[1/2   0   0   0 1/2]
[1/4 1/4 1/4   0 1/4]
```

Cell 54

```
[1/4 1/4 1/4   0 1/4]
[  0   0 1/2   0 1/2]
[1/4   0 1/4 1/4 1/4]
[1/2   0   0   0 1/2]
[1/2   0 1/2   0   0]
```

Cell 55

```
[1/6 1/6 1/6 1/6 1/3]
[  0 1/4 1/4 1/4 1/4]
[  0   0 1/2   0 1/2]
[  0 1/2   0   0 1/2]
[1/4 1/4 1/4   0 1/4]
```

Cell 56

```
[1/4 1/4 1/4   0 1/4]
[  0 1/4 1/4 1/4 1/4]
[  0   0 1/2   0 1/2]
[  0 1/2   0   0 1/2]
[  0 1/2 1/2   0   0]
```

Cell 57

```
[1/6 1/6 1/6 1/6 1/3]
[1/2   0   0   0 1/2]
[  0   0 1/2   0 1/2]
[  0 1/2   0   0 1/2]
[1/4 1/4 1/4   0 1/4]
```

Cell 58

```
[1/6 1/6 1/6 1/6 1/3]
[  0 1/4 1/4 1/4 1/4]
[  0   0   0 1/2 1/2]
[1/4   0 1/4 1/4 1/4]
[1/6 1/6 1/6 1/3 1/6]
[1/4 1/4   0 1/4 1/4]
```

Cell 59

```
[1/6 1/6 1/6 1/3 1/6]
[  0   0   0 1/2 1/2]
[1/4   0 1/4 1/4 1/4]
[1/2   0   0 1/2   0]
[1/4 1/4   0 1/4 1/4]
```

```
Cell 60
[1/6 1/6 1/6 1/3 1/6]
[  0   0   0 1/2 1/2]
[  0 1/4 1/4 1/4 1/4]
[  0 1/2   0 1/2   0]
[1/4 1/4   0 1/4 1/4]

Cell 61
[1/6 1/6 1/6 1/3 1/6]
[  0   0   0 1/2 1/2]
[  0 1/2   0 1/2   0]
[1/2   0   0 1/2   0]
[1/4 1/4   0 1/4 1/4]

Cell 62
[1/6 1/6 1/6 1/3 1/6]
[  0   0   0 1/2 1/2]
[  0 1/4 1/4 1/4 1/4]
[1/4   0 1/4 1/4 1/4]
[  0   0 1/2 1/2   0]

Cell 63
[1/6 1/6 1/6 1/3 1/6]
[  0   0   0 1/2 1/2]
[  0   0 1/2 1/2   0]
[1/4   0 1/4 1/4 1/4]
[1/2   0   0 1/2   0]

Cell 64
[  0   0 1/2 1/2   0]
[1/6 1/6 1/6 1/3 1/6]
[  0 1/4 1/4 1/4 1/4]
[  0   0   0 1/2 1/2]
[  0 1/2   0 1/2   0]

Cell 65
[1/6 1/6 1/6 1/3 1/6]
[  0   0   0 1/2 1/2]
[  0   0 1/2 1/2   0]
[  0 1/2   0 1/2   0]
[1/2   0   0 1/2   0]

Cell 66
[1/6 1/6 1/6 1/6 1/3]
[  0   0   0 1/2 1/2]
[1/4   0 1/4 1/4 1/4]
[1/2   0   0   0 1/2]
[1/4 1/4   0 1/4 1/4]

Cell 67
[1/2   0   0 1/2   0]
[  0   0   0 1/2 1/2]
[1/4   0 1/4 1/4 1/4]
[1/2   0   0   0 1/2]
[1/4 1/4   0 1/4 1/4]

Cell 68
[1/6 1/6 1/6 1/6 1/3]
[  0   0   0 1/2 1/2]
[  0 1/4 1/4 1/4 1/4]
[  0 1/2   0   0 1/2]
[1/4 1/4   0 1/4 1/4]
```

Cell 69

[0 1/4 1/4 1/4 1/4]

[1/4 1/4 0 1/4 1/4]

[0 0 0 1/2 1/2]

[0 1/2 0 0 1/2]

[0 1/2 0 1/2 0]

Cell 70

[1/6 1/6 1/6 1/6 1/3]

[0 0 0 1/2 1/2]

[1/2 0 0 0 1/2]

[0 1/2 0 0 1/2]

[1/4 1/4 0 1/4 1/4]

Cell 71

[1/6 1/6 1/6 1/6 1/3]

[0 1/4 1/4 1/4 1/4]

[0 0 0 1/2 1/2]

[0 0 1/2 0 1/2]

[1/4 0 1/4 1/4 1/4]

Cell 72

[0 1/4 1/4 1/4 1/4]

[1/4 0 1/4 1/4 1/4]

[0 0 1/2 0 1/2]

[0 0 0 1/2 1/2]

[0 0 1/2 1/2 0]

Cell 73

[1/6 1/6 1/6 1/6 1/3]

[0 0 1/2 0 1/2]

[0 0 0 1/2 1/2]

[1/4 0 1/4 1/4 1/4]

[1/2 0 0 0 1/2]

Cell 74

[0 0 1/2 0 1/2]

[1/6 1/6 1/6 1/6 1/3]

[0 1/4 1/4 1/4 1/4]

[0 0 0 1/2 1/2]

[0 1/2 0 0 1/2]

Cell 75

[1/6 1/6 1/6 1/6 1/3]

[0 0 0 1/2 1/2]

[0 0 1/2 0 1/2]

[0 1/2 0 0 1/2]

[1/2 0 0 0 1/2]

```
List of vertices
[1/6 1/6 1/6 1/6 1/3]
[  0 1/4 1/4 1/4 1/4]
[1/4   0 1/4 1/4 1/4]
[1/4 1/4   0 1/4 1/4]
[1/4 1/4 1/4   0 1/4]
[1/3 1/6 1/6 1/6 1/6]
[1/6 1/3 1/6 1/6 1/6]
[1/4 1/4 1/4 1/4   0]
[1/6 1/6 1/3 1/6 1/6]
[1/6 1/6 1/6 1/3 1/6]
[1/2 1/2   0   0   0]
[1/2   0 1/2   0   0]
[  0 1/2 1/2   0   0]
[1/2   0   0 1/2   0]
[  0 1/2   0 1/2   0]
[  0   0 1/2 1/2   0]
[1/2   0   0   0 1/2]
[  0 1/2   0   0 1/2]
[  0   0 1/2   0 1/2]
[  0   0   0 1/2 1/2]
```

B. Enumerating the n-cells in $\tilde{D}_n$

```
def build_D(n):
        D = []
        L = [0 for i in range(n+1)]
        L[1] = 1
        D.append(L)
        for i in range(n-1):
                L = [0 for j in range(n+1)]
                L[i+1] = -1
                L[i+2] = 1
                D.append(tuple(L))
        L = [1 for i in range(n+1)]
        L[0] = 0
        L[n] = -1
        D.append(tuple(L))
        return tuple(D)


def all_walls(n):
        walls=[]
        for i in range(n//2-kronecker_delta(0,n%2)):
                S=Subsets(range(n),i+1)
                for j in S:
                        w = [1 for k in range(n)]
                        for l in j:
```

```
                    w[l]=-w[l]
                s = list([0])
                s.extend(w)
                if s[n] < 0:
                    v = vector(s)
                    v = -v
                    s = list(v)
                walls.append(s)
    if 2.divides(n):
        S=Subsets(range(n-1),n/2-1)
        for j in S:
            w = [1 for k in range(n-1)]
            for l in j:
                w[l] = -w[l]
            w.append(-1)
            s = list([0])
            s.extend(w)
            if s[n] < 0:
                v = vector(s)
                v = -v
                s = list(v)
            walls.append(s)
    return walls

def necessary_walls(P,L):
    walls=[]
    for w in L:
        Q = Polyhedron(ieqs = [w])
        R = Polyhedron(ieqs = [-vector(w)])
```

```
            if dim(P.intersection(Q)) == dim(P.intersection
                (R)):
                    walls.append(vector(w))
        return tuple(walls)

def chambers(n):
        cells = []
        halfspaces = []
        boundary_list = build_D(n)
        D = Polyhedron(ieqs = boundary_list)
        interior_list = necessary_walls(D, all_walls(n))
        C = D.intersection(Polyhedron(ieqs = interior_list))
        cells.append(C)
        halfspaces=[list(interior_list)]
        i = 0
        while i < len(cells):
                minimal_walls = cells[i].inequalities_list()
                for j in range(len(halfspaces[i])):
                        if list(halfspaces[i][j]) in
                            minimal_walls:
                                new_walls = copy(halfspaces[i])
                                new_walls[j] = -new_walls[j]
                                C = D.intersection(Polyhedron(
                                    ieqs = new_walls))
                                if dim(C) == n and not C in
                                    cells:
                                        cells.append(C)
                                        halfspaces.append(
                                            new_walls)
                i = i+1
```

```
return [cells,halfspaces]
```