AN ABSTRACT OF THE DISSERTATION OF

Bogdana A. Georgieva for the degree of Doctor of Philosophy in Mathematics presented on June 5, 2001.

Title: Noether-type Theorems for the Generalized Variational Principle of Herglotz

Abstract approved: Ronald B. Guenther

Ronald B. Guenther

The generalized variational principle of Herglotz defines the functional whose extrema are sought by a differential equation rather than an integral. It reduces to the classical variational principle under classical conditions. The Noether theorems are not applicable to functionals defined by differential equations.

For a system of differential equations derivable from the generalized variational principle of Herglotz, a first Noether-type theorem is proven, which gives explicit conserved quantities corresponding to the symmetries of the functional defined by the generalized variational principle of Herglotz. This theorem reduces to the classical first Noether theorem in the case when the generalized variational principle of Herglotz reduces to the classical variational principle.

Applications of the first Noether-type theorem are shown and specific examples are provided.

A second Noether-type theorem is proven, providing a non-trivial identity corresponding to each infinite-dimensional symmetry group of the functional defined by the generalized variational principle of Herglotz. This theorem reduces to the classical second Noether theorem when the generalized variational principle of Herglotz reduces to the classical variational principle.

A new variational principle with several independent variables is defined. It reduces to Herglotz's generalized variational principle in the case of one independent variable, time. It also reduces to the classical variational principle with several

independent variables, when only the spatial independent variables are present. Thus, it generalizes both. This new variational principle can give a variational description of processes involving physical fields. One valuable characteristic is that, unlike the classical variational principle with several independent variables, this variational principle gives a variational description of nonconservative processes *even when the Lagrangian function is independent of time.* This is not possible with the classical variational principle.

The equations providing the extrema of the functional defined by this generalized variational principle are derived. They reduce to the classical Euler-Lagrange equations (in the case of several independent variables), when this new variational principle reduces to the classical variational principle with several independent variables.

A first Noether-type theorem is proven for the generalized variational principle with several independent variables. One of its corollaries provides an explicit procedure for finding the conserved quantities corresponding to symmetries of the functional defined by this variational principle. This theorem reduces to the classical first Noether theorem in the case when the generalized variational principle with several independent variables reduces to the classical variational principle with several independent variables. It reduces to the first Noether-type theorem for Herglotz generalized variational principle when this generalized variational principle reduces to Herglotz's variational principle.

A criterion for a transformation to be a symmetry of the functional defined by the generalized variational principle with several independent variables is proven.

Applications of the first Noether-type theorem in the several independent variables case are shown and specific examples are provided.

Noether-type Theorems for the Generalized Variational Principle of Herglotz

by

Bogdana A. Georgieva

A DISSERTATION

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of
Doctor of Philosophy

Completed June 5, 2001
Commencement June 2002

Doctor of Philosophy dissertation of Bogdana A. Georgieva presented on June 5, 2001

APPROVED:

Major Professor, representing Mathematics

Chairman of Department of Mathematics

Dean of Graduate School

I understand that my dissertation will become a part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Bogdana A. Georgieva, Author

ACKNOWLEDGMENTS

I would like to express my deep gratitude to my advisor, Professor Ronald B. Guenther, for his teaching, help and guidance in writing this dissertation. In addition, I am thankful to him for his unconditional friendship and for giving me a hand every time I needed it.

I also like to acknowledge the contribution of my uncle and coworker, Theodore G. Bodurov, to this work. I am grateful to him for his love, support and direct help with the physics and the mathematics.

I like to thank my mother Maria G. Bodurova for teaching me not to be afraid of difficulties, and for loving me and encouraging me during this long period of time.

Finally, I am thankful to my grandfather, George G. Bodurov, for rasing me from babyhood and for first showing to me the pleasure of learning.

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NOETHER–TYPE THEOREMS FOR THE GENERALIZED VARIATIONAL PRINCIPLE OF HERGLOTZ

INTRODUCTION

The generalized variational principle, proposed by G. Herglotz, defines the functional whose extrema are sought by a differential equation, rather than an integral. For such functionals the classical Noether theorems are not applicable. In this thesis Noether-type theorems are formulated and proved which do apply to the Generalized Variational Principle, and contain the first and second Noether theorems as special cases. The first Noether-type theorem gives explicit conserved quantities for non-conservative (and conservative) systems described by the Generalized Variational Principle, corresponding to symmetries of the functional under an one-parameter symmetry group. The second Noether-type theorem gives an identity, which reduces to the identity provided by the classical second Noether theorem in the case when the functional is defined by an integral.

In the middle of the nineteenth century Sophus Lie made the far-reaching discovery that the seemingly unrelated techniques for solving differential equations like separable, homogeneous or exact, were in fact all special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of transformations. This observation unified and significantly extended the available integration techniques. Lie devoted the remainder of his mathematical career to the development and applications of his monumental theory of continuous groups. These groups, now known as Lie groups, have had a profound impact on all areas of mathematics, both pure and applied, as well as physics, engineering and other mathematically based sciences. The applications of Lie groups include such fields as algebraic topology, differential geometry, bifurcation theory, invariant theory, numerical analysis, special functions, control theory, classical mechanics, quantum mechanics, relativity, continuum mechanics, etc.

After the pioneering work of Lie and Noether, the works of Caratheodory, Herglotz, George Birkhoff, Ovsiannikov, Lefschetz, Kähler and numerous others followed. George Birkhoff called attention to the unexploited applications of Lie groups to the differential equations of fluid mechanics. Subsequently Ovsiannikov and his school began a systematic program of successfully applying these methods to a wide range of physically important problems. During the last two decades there has been an explosion of research activity in this field, both in the applications to concrete physical systems, as well as extensions of the scope and depth of the theory itself. Nevertheless, many questions remain unsolved, and the full range of applicability of Lie group methods to differential equations is yet to be found.

Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. In the classical framework of Lie, these groups consist of geometric transformations on the space of independent and dependent variables for the system, and act on solutions by transforming their graphs. He called them *contact transformations.* In mechanics a special case called canonical transformations were used. Typical examples are the groups of rotations and translations, and the scaling symmetries, but these do not exhaust the range of possibilities. The great advantage of looking at continuous symmetry groups, as opposed to discrete symmetries such as reflections, is that they can all be found using computational methods. This is not to say that discrete groups are not important in the study of differential equations (see, for example, Hejhal,[1]), but rather that one must employ quite different methods to find or utilize them.

Lie's fundamental discovery was that the complicated *nonlinear* conditions of invariance of the system under the group transformations could, in the case of a continuous group, be replaced by equivalent, but far simpler, *linear* conditions reflecting a form of "infinitesimal" invariance of the system under the generators of the group. In most physically important systems of differential equations, these infinitesimal symmetry conditions – the so-called defining equations of the symmetry group of the system – can be explicitely solved in closed form and thus the most general continuous symmetry group of the system can be explicitly determined.

Once the symmetry group of a system of differential equations has been determined, a number of applications become available. To begin with, one can directly use the defining property of such a group and construct new solutions from known ones. The symmetry group thus provides a means of classifying different symmetry classes of solutions, where two solutions are considered equivalent if one can be transformed into the other by some group element. Alternatively, one can use the symmetry groups to effect a classification of families of differential equations depending on arbitrary parameters or functions; often there are good physical or mathematical reasons for preferring those equations with as high a degree of symmetry as possible. Another approach is to determine which types of differential equations admit a prescribed group of symmetries; this problem is also answered by infinitesimal methods using the theory of differential invariants.

In the case of ordinary differential equations, invariance under a one-parameter symmetry group implies that the order of the equation can be reduced by one. The solutions to the original equations can be recovered from those of the reduced equation by a single quadrature. For a single first order equation, this method provides an explicit formula for the general solution. Multi-parameter symmetry groups lead to further reductions in order, but, unless the group satisfies an additional "solvability" requirement, the solutions of the original equation may not be recoverable from those of the reduced equation by quadratures alone. If the system of ordinary differential equations is derivable from the classical variational principle, either as the Euler-Lagrange equations of some functional defined by an integral, or as a Hamiltonian system, then the power of the symmetry group reduction method is doubled. A one-parameter group of variational symmetries allows one to reduce the order of the system by two; the case of multi-parameter variational symmetry groups is more delicate.

For systems of partial differential equations one can use general symmetry groups to determine explicitly special types of solutions which are themselves invariant under some subgroup of the full symmetry group of the system. These group-invariant solutions are found by solving a reduced system of differential equations involving fewer independent variables than the original system, which presumably makes it easier to solve. For example, the solutions to a partial differential equation in two independent variables which are invariant under a given one-parameter symmetry group are all found by solving a system of ordinary differential equations. For many nonlinear systems these are the only explicit exact solutions which are available, and, as such, play an important role in both the mathematical analysis and the physical applications of the system.

Symmetries and their properties were investegated by Herglotz, [3], Klein [1] and Kneser, [1]. Emmy Noether was inspired by their work when she carried out her oun fundamental investigations. In 1918, she proved two remarkable theorems relating symmetry groups of a variational integral to properties of its associated Euler-Lagrange equations, see Noether, [1], [2]. For modern derivations and discussions of these theorems see Logan, [1], Olver, [5], Bluman and Kumei, [1]. In the first of these theorems, Noether shows how each one-parameter variational symmetry group gives rise to a conservation law of the Euler-Lagrange equations. Conservation of energy, for example, comes from invariance of the problem under time translations, while conservation of linear and angular momenta reflect invariance of the system under spatial translations and rotations. This first theorem gives a one-to-one correspondence between symmetry groups and conservation laws. Each one-parameter group of symmetries of the classical variational problem gives rise to a conservation law, and , conversely, every conservation law arises in this manner.

Noether's second theorem involves an infinite-dimensional symmetry group depending on an arbitrary function. Another reason for the importance of the infinite-dimensional symmetry groups is that a class of systems in general relativity arrises as those systems whose variational integral admits an infinite-dimensional symmetry group. Noether's second theorem shows that there is a non-trivial relation between the Euler–Lagrange expressions and provides an identity which holds on solutions of the Euler–Lagrange equations.

Conservation laws have many important applications, both physical and mathematical. These applications include existence results, shock waves, scattering theory, stability, relativity, fluid mechanics, elasticity, etc. Conservation laws have an old origin, although the idea of conservation of energy was not conceptualized until the work of Mayer and Helmholtz, in the late 1830's, and Joule in the 1840's. See Elkana, [1], for an interesting study of the historical development of this idea.

Lax, [1], uses conservation laws (called entropy-flux pairs in his context) to prove global existence theorems and determine realistic conditions for shock wave solutions to hyperbolic systems. This is further developed in DiPerna, [1],[2], where extra conservation laws are applied to the decay of shock waves and further existence theorems are proved. Conservation laws have been applied to stability theory starting with the work of Poincare and Liapunov. Also, Benjamin, [1], and Holm, Marsden, Ratiu and Weinstein, [1] have applied conservation laws to problems of stability. Morawetz, [1] and Strauss, [1], use them in scattering theory. In elasticity, conservation laws are of key importance in the study of cracks and dislocations; see the papers in Bilby, Miller and Wills, [1]. Knops and Stuart, [1], have used them to prove uniqueness theorems for elastic equilibria. This is only a small sample of all the applications which have appeared. Trivial conservation laws were known for a long time by researchers in general relativity. Those of the second kind go under the name of "strong conservation laws" since they hold regardless of the underlying field equations; see the review papers of J. G. Fletcher, [1], and Goldberg, [1]. The characteristic form of a conservation law appears in Steudel, [1], but the connection between trivial characteristics and trivial conservation laws is due to Alonso, [1]. See Vinogradov, [1], and Olver, [3], for related results.

The concept of a variational symmetry, including the basic infinitesimal criterion, is due to Lie, [1]. The first people to notice a connection between symmetries and conservation laws were Jacobi, [1], and later, Schütz, [1]. Engel, [1], developed the correspondence between the conservation of linear and angular momenta and linear motion of the center of mass with invariance under translational, rotational and Galilean symmetries in the context of classical mechanics.

Klein and Hilbert's investigations into Einstein's theory of general relativity inspired Noether to her remarkable paper, [1], in which both the concept of a variational symmetry group and the connection with conservation laws were set in complete generality. The extension of Noether's methods to include divergence symmetries is due to Bessel-Hagen, [1]. The next significant reference to Noether's paper is in a review article by Hill, [1]. During the following twenty years many papers appeared which rederived Noether's original result or special cases of it. The lack of immediate appreciation of Noether's theorem has had some interesting consequences. Eshelby's energy-momentum tensor, which has much importance in the study of cracks and dislocations in the study of elastic media, was originally found using ad hoc techniques, Eshelby, [1]. It was not related to symmetry properties of the media until the work of Günther, [1], and Knowels and Sternberg, [1].

Frobenius found all the crystallographic groups for the elastic equation in three dimensions. An extension to the equations of linear elastodynamics was made by D. C. Fletcher, [1]. Subsequently, Olver, [1],[2],[4], found some new conservation laws. Similarly, the important identities of Morawetz, [1], used in scattering theory for the wave equation were initially derived from scratch. Subsequently Strauss, [1], showed how these were related to the conformal invariance of the equation. A similar development holds for the work of Baker and Tavel, [1], on conservation laws in optics.

A version of the theorem showing the use of variational symmetry groups to reduce the order of ordinary differential equations which are the Euler-Lagrange equations of some variational integral, for Lagrangians depending on only first order derivatives of the dependent variables, is given in Whittaker, [1; p.55], Olver's book [5] gives the full version of the theorem.

Noether's theorems are applicable only to the classical variational principle, in which the functional is defined by an integral. They do not apply to functionals defined by differential equations. The Generalized Variational Principle, proposed by Gustav Herglotz in 1930, see Herglotz [2], generalizes the classical variational principle by defining the functional, whose extrema are sought, by a differential equation. It reduces to the classical variational integral under classical conditions. Herglotz's original idea was published in 1979 in his collected works, see Herglotz, [1]; this publication was supervised by Schwerdtfeger. Immediately thereafter, Scherdtfeger and R.B. Guenther published Herglotz's Vorlesungen über die Mechanik der Kontinua which appeared in the series Teubner-Archive zur Mathematik, B. G. Teubner Verlagsgesellschaft, Leipzig, 1985. Herglotz reached the idea of the Generalized Variational Principle through his work on contact transformations and their connections with Hamiltonian systems and Poisson brackets. His work was motivated by ideas from S. Lie, C. Caratheodory and other researchers. For historical remarks through 1935, see C. Caratheodory's Variationsrechnung und Partielle Differentialgleichungen Erster Ordnung, Teubner Verlagsgesellschaft, Leipzig 1956. An important reference on the Generalized Variational Principle is The Herglotz Lectures on Contact Transformations and Hamiltonian Systems published in 1996 by R.B. Guenther, J.A. Gottsch and C. M. Guenther, [1].

The generalized variational principle is important for a number of reasons:

1. The solutions of the equations, which give the extrema of the functional defined by the generalized variational principle, when written in terms of x_i and $p_i = \partial L/\partial \dot{x}_i$, determine a family of *contact transformations*. This family is a one-parameter group in a certain case. See Guenther et al, [1]. The significance of contact transformations in mathematics and mathematical physics is well recognized. See Caratheodory [1] and Eisenhart [1].

2. The generalized variational principle gives a *variational* description of nonconservative processes. Unlike the classical variational principle, the generalized one provides such a description *even when the Lagrangian, denoted by L, is independent of time.*

3. For a process, conservative or nonconservative, which can be described with the generalized variational principle, one can *systematically derive conserved quantities*, as shown in this thesis, by application of the first Noether-type theorem.

4. For any process described by the generalized variational principle the second Noether-type theorem, proved in this thesis, can be applied to produce a non-trivial identity which holds on solutions of the equations which provide the extrema of the functional defined by the generalized variational principle.

5. There are two major methods in control and optimal control theories. The first is based on the Laplace transformation. The second, and more recent one, is the so called "state variable" method, see Furta [1]. In this method the controlled physical system is described by a normal system of ODE's. Each of these ODE's has the same form as the defining equation of the generalized variational principle. Thus, there is a link between the mathematical structure of control / optimal control theories and the generalized variational principle.

6. The contact transformations, which can always be derived from the generalized variational principle, have found applications in thermodynamics. Mrugala [1] shows that the processes in equilibrium thermodynamics can be described by successions of contact transformations acting in a suitably defined *thermodynamic phase space*. The latter is endowed with a *contact structure*, which is closely related to the *symplectic structure* (occuring in mechanics).

7. In physical applications the dimensionality of the Lagrangian L in the generalized variational principle is energy. From this and the defining equation of this variational principle follows that, when a process is described by it, the dimensionality of the functional is [action].

The remaining part of this introduction gives a brief description of the contributions of this thesis.

The classical Noether theorems apply only to functionals defined by integrals. For a system of differential equations derivable from the generalized variational principle of Herglotz, a first Noether-type theorem is proven, which provides explicit conserved quantities corresponding to the symmetries of the functional defined by the generalized variational principle of Herglotz. This theorem reduces to the classical first Noether theorem in the case when the generalized variational principle of Herglotz reduces to the classical variational principle.

A criterion for a transformation to be a symmetry of the functional defined by the generalized variational principle of Herglotz is proved.

Three corollaries of the first Noether-type theorem are stated and proved, giving the conserved quantities, which correspond to time translations, space translations and rotations. The correspondence between these new results and the classical results of conservation of energy, linear and angular momentum is observed. Applications of the first Noether-type theorem are shown and specific examples are provided.

A second Noether-type theorem is proven, providing a non-trivial identity corresponding to each infinite-dimensional symmetry group of the functional defined by the generalized variational principle of Herglotz. This theorem reduces to the classical second Noether theorem when the generalized variational principle of Herglotz reduces to the classical variational principle.

A new variational principle with several independent variables is defined. It can give a variational description of processes involving several independent variables, with one independent variable being the time and the rest of the independent variables being the spatial variables. It reduces to Herglotz's generalized variational principle, when only one independent variable, the time–variable, is present. It also reduces to the classical variational principle with several independent variables, when only spatial variables are involved. Thus, it generalizes both Herglotz's variational principle and the classical variational principle with several independent variables. I call this new variational principle *generalized variational principle with several independent variables.*

One valuable characteristic of the generalized variational principle with several independent variables is that, unlike the classical variational principle with several independent variables, it can give a variational description of nonconservative processes where the *Lagrangian function does not explicitly depend time.* This is not possible with the classical variational principle. Some of the applications of this new variational principle involve giving a time-independent variational description of non-conservative processes involving physical fields.

The equations providing the extrema of the functional defined by the generalized variational principle with several independent variables are derived. I call them *generalized Euler-Lagrange equations with several independent variables.* They reduce to the classical Euler-Lagrange equations with several independent variables, when the new variational principle reduces to the classical variational principle with several independent variables. The generalized Euler-Lagrange equations for the new variational principle reduce to the generalized Euler-Lagrange equations of Herglotz's variational principle in the case when only one independent variable, the time–variable, is present.

A first Noether-type theorem is proved for the generalized variational principle with several independent variables. One of its corollaries provides an explicit procedure for finding the conserved quantities corresponding to symmetries of the functional defined by this variational principle. This theorem reduces to the classical first Noether theorem in the case when the generalized variational principle with several independent variables reduces to the classical variational principle with several independent variables. It reduces to the first Noether-type theorem for Herglotz's generalized variational principle when this generalized variational principle reduces to Herglotz's variational principle.

A criterion for a transformation to be a symmetry of the functional defined by the generalized variational principle with several independent variables is proved.

Applications of the first Noether-type theorem in the several independent variables case are shown and specific examples are provided.

CHAPTER 1

FIRST AND SECOND NOETHER–TYPE THEOREMS FOR THE GENERALIZED VARIATIONAL PRINCIPLE OF HERGLOTZ

1. INTRODUCTION

The generalized variational principle, proposed by G. Herglotz, defines the functional whose extrema are sought by a differential equation, rather than an integral. For such functionals, the classical Noether theorems are not applicable. Here, Noether-type theorems are formulated and proved which do apply to the generalized variational principle of Herglotz, and contain the classical Noether theorems as special cases. The first of these theorems gives explicit conserved quantities for non-conservative (and conservative) systems described by the generalized variational principle of Herglotz, corresponding to symmetries of the functional. The conserved quantities corresponding to translations in time, translations in space and rotations in space are derived in the case of non-conservative systems. The relationship with the fundamental conservation laws of physics is discussed and examples for applications are given.

The second Noether-type theorem provides an identity which holds on solutions of the generalized Euler–Lagrange equations, which give the extrema of the functional. This theorem reduces to the classical second Noether theorem when the variational principle of Herglotz reduces to the classical variational principle.

2. FIRST NOETHER–TYPE THEOREM FOR THE GENERALIZED VARIATIONAL PRINCIPLE OF HERGLOTZ

The generalized variational principle of Herglotz defines the functional z, whose extrema are sought, by the differential equation

$$\frac{dz}{dt} = L\Big(t, x(t), \frac{dx(t)}{dt}, z\Big) \ , \tag{2.1}$$

where t is the independent variable, $x \equiv (x^1, \dots, x^n)$ stands for the dependent variables and $dx(t)/dt$ denotes the derivatives of the dependent variables. In order for the equation (2.1) to define a functional z of $x(t)$ we must solve equation (2.1) with the same fixed initial condition $z(0)$ for all argument functions $x(t)$, and evaluate the solution $z(t)$ at the same fixed final time $t = T$ for all argument functions $x(t)$.

The equations which produce the extrema of the functional z defined by the generalized variational principle of Herglotz are

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_i} = 0 \,, \qquad i = 1, \ldots, n \,.$$

Herglotz called them *generalized Euler-Lagrange equations*. See Guenther, [1].

Consider the one-parameter group of invertible transformations

$$\begin{aligned} \bar{t} &= \phi(t, x, \varepsilon), \\ \bar{x}_k &= \psi_k(t, x, \varepsilon), \qquad k = 1, \ldots, n \end{aligned} \tag{2.2}$$

where ε is the parameter, $\phi(t, x, 0) = t$, and $\psi_k(t, x, 0) = x_k$. Let the generators of the corresponding infinitesimal transformation be

$$\tau(t, x) = \frac{d\phi}{d\varepsilon}(t, x, 0) \qquad \text{and} \qquad \xi_k(t, x) = \frac{d\psi_k}{d\varepsilon}(t, x, 0) \,. \tag{2.3}$$

Denote by $\zeta = \zeta(t)$ the total variation of the functional $z = z[x; t]$ produced by the family of transformations (2.2), i.e.

$$\zeta(t) = \frac{d}{d\varepsilon} z[x; t, \varepsilon]\Big|_{\varepsilon=0} \,.$$

We make the following

Remark: $\zeta(0) = 0$. Indeed, as explained earlier, in order to have a well-defined functional z as a functional of $x(t)$, we must evaluate the solution $z(t)$ of the equation (2.1) with the same fixed initial condition $z(0)$, independently of the function $x(t)$. Then $z(0)$ is independent of ε. Hence, the variation of z evaluated at $t = 0$ is

$$\zeta(0) = \frac{d}{d\varepsilon} z[x; 0, \varepsilon]\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} z(0)\Big|_{\varepsilon=0} = 0 \,.$$

Throughout the chapter we assume that the summation convention on repeated indices holds.

Theorem 2.1. *If the functional* $z = z[x(t);t]$ *defined by the differential equation* $\dot{z} = L(t,x,\dot{x},z)$ *is invariant under the one-parameter group of transformations (2.2) then the quantities*

$$\exp\left(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\right)\left(\left(L - \dot{x}_k\frac{\partial L}{\partial \dot{x}_k}\right)\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k\right) \tag{2.4}$$

are conserved along the solutions of the generalized Euler-Lagrange equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_i} = 0\ , \qquad i = 1,\ldots,n\ . \tag{2.5}$$

Proof: By Taylor's theorem we have

$$\begin{aligned}\bar{t} &= t + \tau(t,x)\,\varepsilon + O(\varepsilon^2)\\ \bar{x}_k &= x_k + \xi_k(t,x)\,\varepsilon + O(\varepsilon^2)\ .\end{aligned} \tag{2.6}$$

Let us apply the transformation (2.6) to the differential equation (2.1). Observing that $d\bar{z}/d\bar{t} = (d\bar{z}/dt)\,(dt/d\bar{t}\,)$ we get

$$\begin{aligned}\frac{d\bar{z}}{d\bar{t}} &= L\left(\bar{t},\bar{x}(\bar{t}),\frac{d\bar{x}(\bar{t})}{d\bar{t}},\bar{z}\right)\\ \frac{d\bar{z}}{dt} &= L\left(\bar{t},\bar{x}(\bar{t}),\frac{d\bar{x}(\bar{t})}{d\bar{t}},\bar{z}\right)\frac{d\bar{t}}{dt}\ .\end{aligned} \tag{2.7}$$

Differentiate (2.7) with respect to ε and set $\varepsilon = 0$ to obtain

$$\frac{d}{d\varepsilon}\left(\frac{d\bar{z}}{dt}\right)\bigg|_{\varepsilon=0} = \frac{d}{dt}\left(\frac{d\bar{z}}{d\varepsilon}\right)\bigg|_{\varepsilon=0} = \frac{dL}{d\varepsilon}\bigg|_{\varepsilon=0}\frac{d\bar{t}}{dt}\bigg|_{\varepsilon=0} + L\frac{d}{d\varepsilon}\left(\frac{d\bar{t}}{dt}\right)\bigg|_{\varepsilon=0}\ . \tag{2.8}$$

Now observe that

$$\frac{d\bar{t}}{dt}\bigg|_{\varepsilon=0} = 1\ ,$$

since by Taylor's theorem

$$\frac{d\bar{t}}{dt} = \frac{d\varphi(t,x,\varepsilon)}{dt} = \frac{d}{dt}\left(\varphi(t,x,0) + \frac{d\varphi}{d\varepsilon}(t,x,0)\,\varepsilon + O(\varepsilon^2)\right)$$

and $\varphi(t,x,0) = t$. Also we observe that

$$\frac{d}{d\varepsilon}\left(\frac{d\bar{t}}{dt}\right)\bigg|_{\varepsilon=0} = \frac{d}{dt}\left(\frac{d}{d\varepsilon}\varphi(t,x,\varepsilon)\right)\bigg|_{\varepsilon=0} = \frac{d}{dt}\left(\frac{d\varphi}{d\varepsilon}(t,x,0) + o(\varepsilon)\right)\bigg|_{\varepsilon=0} = \frac{d}{dt}\tau(t,x)\ .$$

Thus, equation (2.8) becomes

$$\frac{d\zeta}{dt} = \frac{dL}{d\varepsilon}\Big|_{\varepsilon=0} + L\frac{d\tau}{dt} \ .$$

Expanding the derivative $dL/d\varepsilon$ and setting $\varepsilon = 0$, we obtain

$$\begin{aligned}\frac{d\zeta}{dt} &= \frac{\partial L}{\partial t}\frac{d\bar{t}}{d\varepsilon}\Big|_{\varepsilon=0} + \frac{\partial L}{\partial x_k}\frac{d\bar{x}_k}{d\varepsilon}\Big|_{\varepsilon=0} + \frac{\partial L}{\partial \dot{x}_k}\frac{d}{d\varepsilon}\Big(\frac{d\bar{x}_k}{d\bar{t}}\Big)_{\varepsilon=0} + \frac{\partial L}{\partial z}\frac{d\bar{z}}{d\varepsilon}\Big|_{\varepsilon=0} + L\frac{d\tau}{dt}\\ \frac{d\zeta}{dt} &= \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\frac{d}{d\varepsilon}\Big(\frac{d\bar{x}_k}{d\bar{t}}\Big)_{\varepsilon=0} + \frac{\partial L}{\partial z}\zeta + L\frac{d\tau}{dt} \ . \end{aligned} \tag{2.9}$$

We still need to calculate and insert in equation (2.9) the expression

$$\frac{d}{d\varepsilon}\Big(\frac{d\bar{x}_k}{d\bar{t}}\Big)\Big|_{\varepsilon=0} \ .$$

For this we proceed as follows:

$$\frac{d\bar{x}_k}{dt} \equiv \frac{\partial \bar{x}_k}{\partial t} + \frac{\partial \bar{x}_k}{\partial x_h}\dot{x}_h = \frac{d\bar{x}_k}{d\bar{t}}\frac{d\bar{t}}{dt} \equiv \frac{d\bar{x}_k}{d\bar{t}}\Big(\frac{\partial \bar{t}}{\partial t} + \frac{\partial \bar{t}}{\partial x_h}\dot{x}_h\Big) \ . \tag{2.10}$$

Set $\varepsilon = 0$ to obtain

$$\frac{d\bar{x}_k}{d\bar{t}}\Big|_{\varepsilon=0} = \delta^k_h \dot{x}_h = \dot{x}_k \ . \tag{2.11}$$

Differentiate (2.10) with respect to ε

$$\frac{d}{d\varepsilon}\Big(\frac{\partial \bar{x}_k}{\partial t} + \frac{\partial \bar{x}_k}{\partial x_h}\dot{x}_h\Big) = \frac{d}{d\varepsilon}\Big(\frac{d\bar{x}_k}{d\bar{t}}\Big(\frac{\partial \bar{t}}{\partial t} + \frac{\partial \bar{t}}{\partial x_h}\dot{x}_h\Big)\Big)$$

and expand both sides to get

$$\begin{aligned}&\frac{d}{d\varepsilon}\Big(\frac{\partial \bar{x}_k}{\partial t}\Big) + \frac{d}{d\varepsilon}\Big(\frac{\partial \bar{x}_k}{\partial x_h}\Big)\dot{x}_h\\ &= \frac{d\bar{x}_k}{d\bar{t}}\Big(\frac{d}{d\varepsilon}\Big(\frac{\partial \bar{t}}{\partial t}\Big) + \frac{d}{d\varepsilon}\Big(\frac{\partial \bar{t}}{\partial x_h}\Big)\dot{x}_h\Big) + \frac{d}{d\varepsilon}\Big(\frac{d\bar{x}_k}{d\bar{t}}\Big)\Big(\frac{\partial \bar{t}}{\partial t} + \frac{\partial \bar{t}}{\partial x_h}\dot{x}_h\Big) \ . \end{aligned} \tag{2.12}$$

We set $\varepsilon = 0$ in this equation, substitute in it (2.11) and use the following relations:

$$\frac{d}{d\varepsilon}\Big(\frac{\partial \bar{x}_k}{\partial t}\Big)\Big|_{\varepsilon=0} = \frac{\partial \xi_k}{\partial t} \ , \qquad \frac{d}{d\varepsilon}\Big(\frac{\partial \bar{t}}{\partial t}\Big)\Big|_{\varepsilon=0} = \frac{\partial \tau}{\partial t} \ , \qquad \frac{d}{d\varepsilon}\Big(\frac{\partial \bar{t}}{\partial x_h}\Big)\Big|_{\varepsilon=0} = \frac{\partial \tau}{\partial x_h} \ ,$$

$$\frac{\partial \bar{t}}{\partial t}\Big|_{\varepsilon=0} = 1 \ , \qquad \frac{\partial \bar{t}}{\partial x_h}\Big|_{\varepsilon=0} = 0 \ .$$

Then equation (2.12) becomes

$$\frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial x_h}\dot{x}_h = \dot{x}_k\left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x_h}\dot{x}_h\right) + \frac{d}{d\varepsilon}\left(\frac{d\bar{x}_k}{d\bar{t}}\right)\Big|_{\varepsilon=0} .$$

Now observe that the total derivatives of ξ_k and τ appear in the last equation. Solving for the last term in it, we obtain

$$\frac{d}{d\varepsilon}\left(\frac{d\bar{x}_k}{d\bar{t}}\right)\Big|_{\varepsilon=0} = \frac{d\xi_k}{dt} - \dot{x}_k\frac{d\tau}{dt} .$$

We insert the last result in equation (2.9) to obtain the differential equation for the variation ζ of the functional z. This equation is

$$\frac{d\zeta}{dt} = \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\left(\frac{d\xi_k}{dt} - \dot{x}_k\frac{d\tau}{dt}\right) + \frac{\partial L}{\partial z}\zeta + L\frac{d\tau}{dt} . \tag{2.13}$$

Its solution ζ is given by

$$\begin{aligned}
&\exp\left(-\int_0^t \frac{\partial L}{\partial z}\, d\theta\right)\zeta - \zeta^0 \\
&= \int_0^t \exp\left(-\int_0^s \frac{\partial L}{\partial z}\, d\theta\right)\left(\frac{\partial L}{\partial s}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\left(\frac{d\xi_k}{ds} - \dot{x}_k\frac{d\tau}{ds}\right) + L\frac{d\tau}{ds}\right)ds .
\end{aligned}$$

By the remark preceeding the Theorem, $\zeta^0 = \zeta(0) = 0$. Also, since by hypothesis the one–parameter family of transformations (2.2) leaves the functional $z = z[x(t);t]$ stationary, we have $\zeta(t) = 0$. Thus, one obtains

$$\int_0^t \exp\left(-\int_0^s \frac{\partial L}{\partial z}\, d\theta\right)\left(\frac{\partial L}{\partial s}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\left(\frac{d\xi_k}{ds} - \dot{x}_k\frac{d\tau}{ds}\right) + L\frac{d\tau}{ds}\right)ds = 0 . \tag{2.14}$$

An integration by parts of the last equation produces

$$\begin{aligned}
&\exp\left(-\int_0^s \frac{\partial L}{\partial z} d\theta\right)\left(L\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k - \frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\tau\right)\Bigg|_{s=0}^{s=t} \\
&+ \int_0^t \exp\left(-\int_0^s \frac{\partial L}{\partial z} d\theta\right)\left(\frac{\partial L}{\partial s}\tau - \dot{L}\tau + L\frac{\partial L}{\partial z}\tau + \frac{\partial L}{\partial x_k}\xi_k - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}_k}\right)\xi_k + \frac{\partial L}{\partial \dot{x}_k}\frac{\partial L}{\partial z}\xi_k \right. \\
&\left. - \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\tau + \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}_k}\right)\dot{x}_k\tau + \frac{\partial L}{\partial \dot{x}_k}\ddot{x}_k\tau\right)ds = 0 ,
\end{aligned}$$

which on solutions of the generalized Euler-Lagrange equations becomes

$$\begin{aligned}
&\exp\left(-\int_0^s \frac{\partial L}{\partial z} d\theta\right)\left(L\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k - \frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\tau\right)\Bigg|_{s=0}^{s=t} \\
&+ \int_0^t \exp\left(-\int_0^s \frac{\partial L}{\partial z} d\theta\right)\left(-\frac{\partial L}{\partial z}\dot{z} + L\frac{\partial L}{\partial z} - \dot{x}_k\left(\frac{\partial L}{\partial x_k} - \frac{d}{ds}\frac{\partial L}{\partial \dot{x}_k} + \frac{dL}{dz}\frac{\partial L}{\partial \dot{x}_k}\right)\right)\tau\, ds = 0.
\end{aligned}$$

Taking into consideration the fact that $\dot{z} = L$, we obtain that along the solutions of the generalized Euler-Lagrange equations (2.5)

$$\exp\left(-\int_0^s \frac{\partial L}{\partial z}\,d\theta\right)\left(L\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k - \frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\tau\right)\Bigg|_{s=0}^{s=t} = 0\ .$$

Since the last equation holds for all t, it follows that

$$\exp\left(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\right)\left(\left(L - \dot{x}_k\frac{\partial L}{\partial \dot{x}_k}\right)\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k\right) = \text{constant}$$

along solutions of the generalized Euler-Lagrange equations, as claimed.

It should be observed that the exponential factor

$$\exp\left(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\right) = \frac{1}{\rho} \tag{2.15}$$

which is present in the conserved quantities (2.4) is the reciprocal of the *multiplier function* ρ which appears in the definition

$$P_i\,dX_i - dZ = \rho\,(p_i\,dx_i - dz)$$

of contact transformations (since $\partial L/\partial z = -\partial H/\partial z$). See Guenther, [1].

The conserved quantities (2.4) have a remarkable form — they are products of ρ^{-1} with the expressions

$$\left(L - \dot{x}_k\frac{\partial L}{\partial \dot{x}_k}\right)\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k \tag{2.16}$$

whose form is exactly the same as that of the conserved quantities obtained from the classical first Noether theorem. Consequently, in the special case $\partial L/\partial z = 0$, when the functional z is defined by the integral

$$z = \int_0^t L(t, x, \dot{x})\,d\theta$$

we have $\rho = 1$. Hence, in this case Theorem 2.1 reduces to the classical first Noether theorem. Three modern references on the classical first Noether theorem are Logan [1], Olver [5] and Bluman, Kumei [1]. For applications to physics see Goldstein [1] and Roman [1].

3. CONSERVED QUANTITIES IN GENERATIVE AND DISSIPATIVE SYSTEMS

Physical systems described by the generalized Euler-Lagrange equations (2.5) or by the canonical equations (see system (0.16) in the Appendix) are not conservative in general. Since the Lagrangian functional of such systems cannot be expressed as an integral, the first Noether theorem cannot be used for finding conserved quantities. Below we show how the first Noether-type theorem can be used to find conserved quantities in non-conservative systems. For this we need to describe the physical system with the generalized Euler-Lagrange equations or the canonical equations and then find symmetries of the functional $z = z[x(t); t]$ defined by the differential equation $\dot{z} = L(t, x, \dot{x}, z)$, that is, transformations of both dependent and independent variables which leave $z[x(t); t]$ invariant.

To test whether a transformation is a symmetry of the functional $z[x(t); t]$ we use the following

Proposition 3.1. *The transformation*

$$\begin{aligned} \bar{t} &= \varphi(t, x, \varepsilon) \\ \bar{x}_k &= \psi_k(t, x, \varepsilon) \end{aligned} \tag{3.1}$$

leaves the functional z, *defined by the differential equation* $\dot{z} = L(t, x, \dot{x}, z)$ *invariant if and only if*

$$L\Big(\bar{t},\, \bar{x},\, \frac{d\bar{x}}{dt}\Big(\frac{d\bar{t}}{dt}\Big)^{-1}, z\Big)\frac{d\bar{t}}{dt} = L\Big(t,\, x(t),\, \frac{dx(t)}{dt},\, z\Big) \tag{3.2}$$

holds for all t, x, z *in the domain of consideration.*

Proof: Apply the transformation (3.1) to the differential equation defining z to obtain

$$\frac{d\bar{z}}{d\bar{t}} = L\Big(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}(\bar{t})}{d\bar{t}}, \bar{z}\Big)\ .$$

By the chain rule we get

$$\frac{d\bar{z}}{dt} = L\Big(\bar{t}, \bar{x}, \frac{d\bar{x}}{dt}\Big(\frac{d\bar{t}}{dt}\Big)^{-1}, \bar{z}\Big)\frac{d\bar{t}}{dt}\ . \tag{3.3}$$

If condition (3.2) is satisfied then the differential equation defining $\bar{z}$ is the same as the differential equation defining z. Thus, $z = \bar{z}$, i.e. the transformation leaves z invariant. Conversely, if (3.1) leaves z invariant, then $d\bar{z}/dt = dz/dt$. Comparing (3.3) with the equation $\dot{z} = L(t, x, \dot{x}, z)$ defining the non-transformed z we obtain condition (3.2).

We are now ready to apply the first Noether-type theorem to find specific conserved quantities corresponding to several basic symmetries. Because of their generality and physical significance we state the results as corollaries to the first Noether-type theorem.

Corollary 3.2. *Let the functional z defined by the differential equation $\dot{z} = L(t, x, \dot{x}, z)$ be invariant with respect to translation in time, $\bar{t} = t + \varepsilon$, $\bar{x} = x$. Then the quantity*

$$E = \exp\left(-\int_0^t \frac{\partial L}{\partial z}\, d\theta\right)\left(L(x, \dot{x}, z) - \frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\right) \tag{3.4}$$

is conserved on solutions of the generalized Euler-Lagrange equations.

Proof: By Proposition 3.1 we see that $\partial L/\partial t = 0$. The infinitesimal generator of the group translation in time is $\partial/\partial t$. To obtain the conclusion of the corollary, apply the first Noether-type theorem with

$$\tau = \left.\frac{d\bar{t}}{d\varepsilon}\right|_{\varepsilon=0} = 1\ , \qquad \xi_k = \left.\frac{d\bar{x}_k}{d\varepsilon}\right|_{\varepsilon=0} = 0\ .$$

Noticing that the Hamiltonian is $H = p_k\dot{x}_k - L$,we see that on solutions of the generalized Euler-Lagrange equations

$$E = \frac{1}{\rho}H = \text{constant}\ , \tag{3.5}$$

where ρ is the multiplier function (2.15). We observe a correspondence with the classical law of conservation of energy: If a physical system is described by a time-independent Lagrangian, which does not depend on z, then the Hamiltonian H is conserved and is identified with the energy of the system. If we continue to interpret H as the energy of the system when L does depend on z, then we see from formula (3.5) that H varies proportionally to ρ since E is constant.

Let us now verify by a direct computation that the quantity E in (3.4) is conserved. Indeed,

$$\begin{aligned}
\frac{dE}{dt} &= \Big(L - \frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\Big)\frac{d}{dt}\exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\Big) + \exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\Big)\frac{d}{dt}\Big(L - \frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\Big) \\
&= -\exp\Big(-\int_0^t \frac{\partial L}{\partial z}d\theta\Big)\frac{\partial L}{\partial z}\Big(L - \frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\Big) \\
&\quad + \exp\Big(-\int_0^t \frac{\partial L}{\partial z}d\theta\Big)\Big(\frac{\partial L}{\partial x_k}\dot{x}_k + \frac{\partial L}{\partial \dot{x}_k}\ddot{x}_k + \frac{\partial L}{\partial z}L - \frac{\partial L}{\partial \dot{x}_k}\ddot{x}_k - \dot{x}_k\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k}\Big) \\
&= \exp\Big(-\int_0^t \frac{\partial L}{\partial z}d\theta\Big)\Big(\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_k}\Big)\dot{x}_k = 0
\end{aligned}$$

on solutions of the generalized Euler-Lagrange equations.

Corollary 3.3. *Let the functional z defined by the differential equation $\dot{z} = L(t, x, \dot{x}, z)$ be invariant with respect to translation in space direction x_k, i.e, $\bar{t} = t$, $\bar{x}_k = x_k + \varepsilon$, $\bar{x}_i = x_i$ for $i = 1, ..., k-1, k+1, ..., n$. Then the quantity*

$$M_k = \exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\Big)\frac{\partial L}{\partial \dot{x}_k} \tag{3.6}$$

is conserved on solutions of the generalized Euler-Lagrange equations.

Proof: By proposition 3.1 we know that $\partial L/\partial x_k = 0$. The infinitesimal generator of the group of translations in direction x_k is $\partial/\partial x_k$. To get the conserved quantity M_k apply the first Noether-type theorem with

$$\tau = \frac{d\bar{t}}{d\varepsilon}\Big|_{\varepsilon=0} = 0\,, \quad \xi_k = \frac{d\bar{x}_k}{d\varepsilon}\Big|_{\varepsilon=0} = 1\,, \quad \xi_i = \frac{d\bar{x}_i}{d\varepsilon}\Big|_{\varepsilon=0} = 0\,, \quad \text{for} \quad i \neq k\,.$$

In terms of the multiplier function ρ the expression (3.6) takes the form

$$M_k = \frac{1}{\rho}\frac{\partial L}{\partial \dot{x}_k} = \text{constant.} \tag{3.7}$$

Again, we observe a correspondence with the classical law of conservation of linear momentum. If we retain the definition of linear momentum $\partial L/\partial \dot{x}_k$ then the result (3.7) says that the linear momentum is not conserved, but changes proportionally to ρ.

Because of the importance of this result, we shall verify it directly:

$$\frac{d}{dt}M_k = \frac{d}{dt}\left(\exp\left(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\right)\frac{\partial L}{\partial \dot{x}_k}\right) = \exp\left(-\int_0^t \frac{\partial L}{\partial z}d\theta\right)\left(-\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_k} + \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k}\right)$$

$$= -\exp\left(-\int_0^t \frac{\partial L}{\partial z}d\theta\right)\left(\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_k}\right) = 0$$

which holds on solutions of the generalized Euler-Lagrange equations. Notice the use of $\partial L/\partial x_k = 0$.

Corollary 3.4. *Let the functional z defined by the equation $\dot{z} = L(t, x, \dot{x}, z)$ be invariant with respect to rotations in the $x_i x_j$-plane. Then the quantity*

$$A_{ij} = \exp\left(-\int_0^t \frac{\partial L}{\partial z}d\theta\right)\left(\frac{\partial L}{\partial \dot{x}_i}x_j - \frac{\partial L}{\partial \dot{x}_j}x_i\right) \tag{3.8}$$

is conserved along solutions of the generalized Euler-Lagrange equations.

Proof: By Proposition (3.1) we know that the Lagrangian has the form

$$L = L\left(t, {x_i}^2 + {x_j}^2, x_r, \dot{x}_i^2 + \dot{x}_j^2, \dot{x}_r, z\right)$$

where x_r stands for all coordinates distinct from x_i or x_j. Indeed, $d\bar{t}/dt = 1$ and the invariance of z under rotations in the $x_i x_j$-plane implies the specific form of L.

The infinitesimal generator of the group of rotations is

$$x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}\ .$$

Thus, to obtain the conserved quantity A_{ij}, we apply the first Noether-type theorem with $\tau = 0$, $\xi_i = x_j$, $\xi_j = -x_i$, $\xi_r = 0$ for $r \neq i, j$.

Once again, in the case of z-independent L, we have a correspondence with the classical law of conservation of angular momentum

$$x_j \frac{\partial L}{\partial x_i} - x_i \frac{\partial L}{\partial x_j}\ .$$

The importance of (3.8) demands a verification by a direct calculation:

$$\begin{aligned}
\frac{d}{dt}A_{ij} =& \exp\Big(-\int_0^t \frac{\partial L}{\partial z}, d\theta\Big) \\
&\Big(-\frac{\partial L}{\partial z}\Big(\frac{\partial L}{\partial \dot x_i}x_j - \frac{\partial L}{\partial \dot x_j}x_i\Big) + \frac{d}{dt}\frac{\partial L}{\partial \dot x_i}x_j + \frac{\partial L}{\partial \dot x_i}\dot x_j - \frac{d}{dt}\frac{\partial L}{\partial \dot x_j}x_i - \frac{\partial L}{\partial \dot x_j}\dot x_i\Big) \\
=& \exp\Big(-\int_0^t \frac{\partial L}{\partial z}d\theta\Big)\Big(-\frac{\partial L}{\partial z}\Big(\frac{\partial L}{\partial \dot x_i}x_j - \frac{\partial L}{\partial \dot x_j}x_i\Big) + \frac{d}{dt}\frac{\partial L}{\partial \dot x_i}x_j - \frac{d}{dt}\frac{\partial L}{\partial \dot x_j}x_i\Big) \\
=& \exp\Big(-\int_0^t \frac{\partial L}{\partial z}d\theta\Big) \\
&\Big(x_i\Big(\frac{\partial L}{\partial x_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot x_j} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot x_j}\Big) - x_j\Big(\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot x_i} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot x_i}\Big)\Big) = 0
\end{aligned}$$

on solutions of the generalized Euler-Lagrange equations. Notice the use of *both*

$$\frac{\partial L}{\partial \dot x_i}\dot x_j = \frac{\partial L}{\partial \dot x_j}\dot x_i \qquad \text{and} \qquad \frac{\partial L}{\partial x_i}x_j = \frac{\partial L}{\partial x_j}x_i$$

which hold since $L = L(t, {x_i}^2 + {x_j}^2, \dot x_i^2 + \dot x_j^2, z)$.

4. ADDITIONAL APPLICATIONS OF THE FIRST NOETHER–TYPE THEOREM

It is known that dissipation effects in physical processes can often be accounted for in the equations describing these processes by terms which are proportional to the first time derivatives $\dot x_i(t) = dx_i/dt$ of the dependent variables. (See Goldstein [7]). For example, the viscous frictional forces acting on an object which is moving in a resistive medium, such as a gas or a liquid, are proportional to the object's velocity. Similarly, the dissipative effects (due to the ohmic resistance) in electrical circuits can often be modeled by including terms which are proportional to the first time-derivative of the corresponding dependent variables, such as the electric charge.

All such dissipative processes can be given a unified description by the generalized variational principle.

For example, let us consider the motion of a small object with mass m (point mass) under the action of some potential $U = U(t, x)$ with $x = (x_1, x_2, x_3)$ in a resistive medium. We assume that the velocity of the object is not extremely high so that the resistive forces are proportional to the velocity. Thus, the equations of motion of such an object, according to Newton's Second law, are

$$m\,\ddot{x}_i = -\frac{\partial U}{\partial x_i} - k\,\dot{x}_i\,, \qquad i = 1,2,3 \tag{4.1}$$

where $k > 0$ is a constant. All equations of this form can be obtained from the generalized variational principle by choosing for the Lagrangian function L the expression

$$L = \frac{m}{2}(\dot{x}_1^2 + \cdots + \dot{x}_n^2) - U(t, x_1, \ldots, x_n) - \alpha\, z \tag{4.2}$$

where $U = U(t, x_1, \ldots, x_n)$ is the potential energy of the system and $\alpha > 0$ is a constant. From (4.2) we obtain the generalized Euler-Lagrange equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) + \frac{\partial L}{\partial z}\,\frac{\partial L}{\partial \dot{x}_i} = -\frac{\partial U}{\partial x_i} - \frac{d}{dt}(m\,\dot{x}_i) - m\alpha\,\dot{x}_i = 0$$

which are the same as (4.1) for $n = 3$ and $k = m\alpha$.

Depending on the choice of the function U, equations (4.1) can describe a variety of systems. For instance:

1. When $U = kr^2 = c(x_1^2 + \cdots + x_n^2)$, with $c > 0$ constant, (4.1) describe one-dimensional or multi-dimensional *isotropic damped harmonic oscillators.*

2. When $U = -c/r = -k/\sqrt{x_1^2 + x_2^2 + x_3^2}$, equations (4.1) describe the motion of a point mass m under Coulomb (electrostatic) or gravitational forces in a resistive medium characterized by the constant α.

3. The equations describing the currents, voltages and charges in single or coupled electrical circuits have the same form as equations (4.1) with $i = 1, \ldots, n$, where n is the number of *state variables* (currents, voltages and charges) and U is an appropriately chosen function (see Goldstein, [1], p. 52). Hence, the processes in electrical circuits can also be derived from a Lagrangian function of the form (4.2) via the generalized variational principle.

As an illustration of the preceeding discussion consider a system whose Lagrangian is of the form (4.2) and assume that the potential U is time-independent. Then, $\partial L/\partial t = 0$ and it follows from the Noether-type Theorem that the quantity

$$\exp\left(-\int_0^t \frac{\partial L}{\partial z}\,d\vartheta\right)\left(\dot{x}_i\,\frac{\partial L}{\partial \dot{x}_i} - L\right) = e^{\alpha t}\left(\frac{m}{2}(\dot{x}_1^2 + \cdots + \dot{x}_n^2) + U(x) + \alpha\, z\right)$$

is conserved. Recognizing that $H = \dot{x}_i\,(\partial L/\partial \dot{x}_i) - L$ is the Hamiltonian of the system, we conclude that the value of the Hamiltonian decreases exponentially in time, i.e.

$$H = e^{-\alpha t}\left(\frac{m}{2}\left(\dot{x}_1^2 + \cdots + \dot{x}_n^2\right) + U(x)\right)\Big|_{t=0} = e^{-\alpha t} H_0 \tag{4.3}$$

where H_0 is the initial value of the Hamiltonian, that is, the initial total energy of the system.

5. SECOND NOETHER–TYPE THEOREM FOR THE GENERALIZED VARIATIONAL PRINCIPLE OF HERGLOTZ

The classical second Noether theorem does not apply to functionals defined by differential equations. In this section we prove a theorem which extends the second Noether theorem to the generalized variational principle of Herglotz, which defines the functional, whose exterma are sought, by a differential equation. This theorem provides an identity, involving the generalized Euler-Lagrange expressions, which corresponds to an infinite–dimensional Lie group. This new theorem reduces to the classical second Noether theorem when the generalized variational principle reduces to the classical variational principle.

Consider the differential equation

$$\frac{dz}{dt} = L\Big(t, x(t), \frac{dx(t)}{dt}, z\Big) \tag{5.1}$$

where t is the independent variable, $x \equiv (x^1, \ldots, x^n)$ stands for the dependent variables and $dx(t)/dt$ stands for the derivatives of the dependent variables. Equation (5.1) defines the functional z whose extrema are provided by the generalized Euler-Lagrange equations

$$\mathrm{Q}_i \equiv \frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_i} = 0\ , \qquad i = 1, \ldots, n\ . \tag{5.2}$$

We call the left hand sides Q_i of equations (5.2) the generalized Euler-Lagrange expressions.

Theorem 5.1 *If the infinite continuous group of transformations*

$$\begin{aligned} \bar{t} &= \phi(t, x, p(t), p^{(1)}(t), \ldots, p^{(r)}(t)) \\ \bar{x}^k &= \psi^k(t, x, p(t), p^{(1)}(t), \ldots, p^{(r)}(t))\ , \qquad k = 1, \ldots, n\ , \end{aligned} \tag{5.3}$$

which depends on the arbitrary function $p(t) \in \mathrm{C}^{r+2}$ *, with* $\bar{t} = t$ *and* $\bar{x}^k = x^k$ *when* $p(t) \equiv p^{(1)}(t) \equiv \ldots \equiv p^{(r)}(t) \equiv 0$ *, is a symmetry group of the functional* z *defined by the differential equation (5.1) then the identity*

$$\tilde{X}^k(\mathrm{E}\,\mathrm{Q}_k) - \tilde{U}(\mathrm{E}\,\mathrm{Q}_k\,\dot{x}^k) = 0 \tag{5.4}$$

holds. Here U *and* X^k *are the linear differential operators*

$$\begin{aligned} U &= \frac{\partial\phi}{\partial p} + \frac{\partial\phi}{\partial p^{(1)}}\frac{d}{dt} + \ldots + \frac{\partial\phi}{\partial p^{(r)}}\frac{d^r}{dt^r} \\ X^k &= \frac{\partial\psi^k}{\partial p} + \frac{\partial\psi^k}{\partial p^{(1)}}\frac{d}{dt} + \ldots + \frac{\partial\psi^k}{\partial p^{(r)}}\frac{d^r}{dt^r}\ , \qquad k = 1, \ldots, n\ , \end{aligned} \tag{5.5}$$

evaluated at $p(t) \equiv p^{(1)}(t) \equiv \ldots \equiv p^{(r)}(t) \equiv 0$ *,* $\tilde{U}$ *,* $\tilde{X}^k$ *are the adjoints of* U *and* X^k *respectively, the quantities* Q_k *are defined in (5.2), and*

$$\mathrm{E} = \exp\left(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\right)\ . \tag{5.6}$$

Proof: By Lie theory we know that close to the identity transformation the action of the group (5.3) is the same as the action of the infinitesimal group corresponding to (5.3). Thus, we must find the infinitesimal transformation which corresponds to (5.3). For this reason we replace p in (5.3) with $\varepsilon\,p$ to obtain

$$\begin{aligned} \bar{t} &= \phi(t, x, \varepsilon\,p(t), \varepsilon\,p^{(1)}(t), \ldots, \varepsilon\,p^{(r)}(t)) \\ \bar{x}^k &= \psi^k(t, x, \varepsilon\,p(t), \varepsilon\,p^{(1)}(t), \ldots, \varepsilon\,p^{(r)}(t))\ , \qquad k = 1, \ldots, n\ . \end{aligned} \tag{5.7}$$

Expand (5.7) in Taylor series with respect to ε and retain the zero and first order terms only:

$$\begin{aligned} \bar{t} &= t + \varepsilon\frac{d\phi}{d\varepsilon}\Big|_{\varepsilon=0} = t + \varepsilon\Big(\frac{\partial\phi}{\partial p}p + \frac{\partial\phi}{\partial p^{(1)}}p^{(1)} + \ldots + \frac{\partial\phi}{\partial p^{(r)}}p^{(r)}\Big)\Big|_{\varepsilon=0} \\ \bar{x}^k &= x^k + \varepsilon\frac{d\psi^k}{d\varepsilon}\Big|_{\varepsilon=0} = x^k + \varepsilon\Big(\frac{\partial\psi^k}{\partial p}p + \frac{\partial\psi^k}{\partial p^{(1)}}p^{(1)} + \ldots + \frac{\partial\psi^k}{\partial p^{(r)}}p^{(r)}\Big)\Big|_{\varepsilon=0}\ . \end{aligned}$$

Then the infinitesimal transformation corresponding to (5.3) is

$$\begin{aligned} \bar{t} &= t + \varepsilon\,U\,p \\ \bar{x}^k &= x^k + \varepsilon\,X^k\,p\ , \qquad k = 1, \ldots, n\ , \end{aligned} \tag{5.8}$$

where U and X^k are defined by (5.5). Apply (5.8) to the differential equation (5.1) to obtain

$$\frac{d\bar{z}}{d\bar{t}} = L\left(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}(\bar{t})}{d\bar{t}}, \bar{z}\right) .$$

By the chain rule

$$\frac{d\bar{z}}{dt} = \frac{d\bar{t}}{dt} L\left(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}(\bar{t})}{d\bar{t}}, \bar{z}\right) ,$$

or written more explicitly

$$\frac{d\bar{z}}{dt} = \left(1 + \varepsilon \frac{d(Up)}{dt}\right) L\left(t + \varepsilon\, Up, x + \varepsilon Xp, \frac{d\bar{x}}{d\bar{t}}, \bar{z}\right) . \tag{5.9}$$

Here $x + \varepsilon Xp = (x^1 + \varepsilon X^1\, p, \ldots, x^n + \varepsilon X^n\, p)$. Differentiate (5.9) with respect to ε, set $\varepsilon = 0$ and recall that

$$\zeta(t) = \frac{d}{d\varepsilon}\left(\bar{z}[\bar{x}; \bar{t}]\right)\Big|_{\varepsilon=0} . \tag{5.10}$$

Then from (5.9) we obtain the differential equation for the total variation ζ produced in the functional z by (5.3), namely

$$\frac{d\zeta}{dt} = \frac{d(Up)}{dt} L + \frac{\partial L}{\partial t} Up + \frac{\partial L}{\partial x^k} X^k\, p + \frac{d}{d\varepsilon}\frac{d\bar{x}^k}{d\bar{t}}\Big|_{\varepsilon=0} \frac{\partial L}{\partial \dot{x}^k} + \frac{\partial L}{\partial z}\zeta \tag{5.11}$$

We now need to calculate the term

$$\frac{d}{d\varepsilon}\frac{d\bar{x}^k}{d\bar{t}}\Big|_{\varepsilon=0} .$$

For this we observe that

$$\frac{d\bar{x}^k}{dt} = \frac{d\bar{x}^k}{d\bar{t}}\frac{d\bar{t}}{dt}$$

or more explicitely

$$\varepsilon \frac{d(X^k p)}{dt} = \frac{d\bar{x}^k}{d\bar{t}}\left(1 + \varepsilon \frac{d(Up)}{dt}\right) . \tag{5.12}$$

Set $\varepsilon = 0$ in (5.12) to obtain $d\bar{x}^k / d\bar{t}\big|_{\varepsilon=0} = 0$. Differentiate (5.12) with respect to ε to get

$$\frac{d(X^k p)}{dt} = \frac{d\bar{x}^k}{d\bar{t}}\frac{d(Up)}{dt} + \left(1 + \varepsilon \frac{d(Up)}{dt}\right)\frac{d}{d\varepsilon}\frac{d\bar{x}^k}{d\bar{t}} .$$

Set $\varepsilon = 0$ in the last equation to find

$$\frac{d(X^k p)}{dt} = \frac{d}{d\varepsilon}\frac{d\bar{x}^k}{d\bar{t}}\Big|_{\varepsilon=0} .$$

Thus, equation (5.11) becomes

$$\frac{d\zeta}{dt} = \frac{d(Up)}{dt}L + \frac{\partial L}{\partial t}Up + \frac{\partial L}{\partial x^k}X^k\,p + \frac{d(X^kp)}{dt}\,\frac{\partial L}{\partial \dot{x}^k} + \frac{\partial L}{\partial z}\zeta\,. \tag{5.13}$$

The solution ζ of equation (5.13) is given by

$$\begin{aligned}&\exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,dt\Big)\zeta - \zeta^0 \qquad (5.14)\\ &= \int_0^t \exp\Big(-\int_0^s \frac{\partial L}{\partial z}\,d\theta\Big)\Big(L\frac{d(Up)}{ds} + \frac{\partial L}{\partial s}Up + \frac{\partial L}{\partial x^k}X^k\,p + \frac{\partial L}{\partial \dot{x}^k}\frac{d(X^kp)}{ds}\Big)ds\,.\end{aligned}$$

Since (5.3) is a symmetry group of the functional z, the total variation produced by it is zero, i.e., $\zeta = 0$ in (5.14). Also, as explained earlier, in order to have a well–defined functional z as a functional of the function $x(t)$ we must evaluate the solution $z(t)$ of the equation (5.1) with the same fixed initial condition $z(0)$ independently of the function $x(t)$. Then $z^0 \equiv z(0;\varepsilon)$ is independent of ε. Hence, the total variation of z evaluated at $t = 0$ is

$$\zeta^0 = \frac{d}{d\varepsilon}z[x;0,\varepsilon]\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}z(0)\Big|_{\varepsilon=0} = 0\,.$$

We integrate the terms involving total derivatives of Up and X^kp by parts. Then (5.14) becomes

$$\begin{aligned}&\int_0^t \exp\Big(-\int_0^s \frac{\partial L}{\partial z}\,d\theta\Big)\Big(-\frac{dL}{ds}Up + \frac{\partial L}{\partial z}LUp + \frac{\partial L}{\partial s}Up + \frac{\partial L}{\partial x^k}X^kp \qquad (5.15)\\ &+ \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}^k}X^kp - \frac{d}{ds}\frac{\partial L}{\partial \dot{x}^k}X^kp\Big)ds + \exp\Big(-\int_0^s \frac{\partial L}{\partial z}d\theta\Big)\Big(LUp + \frac{\partial L}{\partial \dot{x}^k}X^kp\Big)\Big|_{s=0}^{t}.\end{aligned}$$

Since $p(s)$ is arbitrary, we may choose $p(s)$ such that $p(0) = p^{(1)}(0) = \ldots$ $= p^{(r)}(0) = 0$ and $p(t) = p^{(1)}(t) = \ldots = p^{(r)}(t) = 0$. Then, we get from (5.15)

$$\begin{aligned}&\int_0^t \exp\Big(-\int_0^s \frac{\partial L}{\partial z}d\theta\Big)\Big(\Big(-\frac{\partial L}{\partial s} - \frac{\partial L}{\partial z}L - \frac{\partial L}{\partial x^k}\dot{x}^k - \frac{\partial L}{\partial \dot{x}^k}\ddot{x}^k + \frac{\partial L}{\partial z}L + \frac{\partial L}{\partial s}\Big)Up\\ &\qquad + \Big(\frac{\partial L}{\partial x^k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}^k} - \frac{d}{ds}\frac{\partial L}{\partial \dot{x}^k}\Big)X^kp\Big)ds = 0\,.\end{aligned}$$

One more integration by parts yields

$$\begin{aligned}&\int_0^t \exp\Big(-\int_0^s \frac{\partial L}{\partial z}d\theta\Big)\Big(\Big(-\frac{\partial L}{\partial x^k} - \frac{\partial L}{\partial \dot{x}^k}\frac{\partial L}{\partial z} + \frac{d}{ds}\frac{\partial L}{\partial \dot{x}^k}\Big)\dot{x}^k\,Up\\ &\qquad + \Big(\frac{\partial L}{\partial x^k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}^k} - \frac{d}{ds}\frac{\partial L}{\partial \dot{x}^k}\Big)X^kp\Big)ds = 0\,. \qquad (5.16)\end{aligned}$$

Let $\tilde{U}$ be the adjoint of U and $\tilde{X}^k$ be the adjoint of X^k. This allows us to put (5.16) into the form

$$\int_0^t \Big(\tilde{X}^k(\mathrm{E}\,\mathrm{Q}_k) - \tilde{U}(\mathrm{E}\,\mathrm{Q}_k\,\dot{x}^k)\Big)p(s)ds + [\,\cdot\,]_{s=0}^t = 0\,, \tag{5.17}$$

where $\mathrm{Q_k}$ and E are defined in (5.2) and (5.6). Equation (5.17) is obtained by repeated integration by parts in (5.16). We may force the boundary terms $[\,\cdot\,]_{s=0}^t$ to zero using the arbitrariness of p. Since t is arbitrary, applying the fundamental lemma of the calculus of variations we obtain the identity (5.4).

We observe that the identity (5.4) reduces to the identity provided by the second Noether theorem, namely,

$$\tilde{X}^k\left(\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k}\right) - \tilde{U}\left(\left(\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k}\right)\dot{x}^k\right) = 0$$

when the generalized variational principle of Herglotz reduces to the classical variational principle, i.e., when L does not depend on z.

Theorem 5.1 reduces to the classical second Noether's theorem when the generalized variational principle of Herglotz reduces to the classical variational principle.

CAPTER 2

GENERALIZED VARIATIONAL PRINCIPLE WITH SEVERAL INDEPENDENT VARIABLES. FIRST NOETHER-TYPE THEOREM. CONSERVED QUANTITIES IN DISSIPATIVE AND GENERATIVE SYSTEMS

1. THE GENERALIZED VARIATIONAL PRINCIPLE WITH SEVERAL INDEPENDENT VARIABLES

1. *Extention of the Herglotz variational principle to the case of several independent variables*

We would like to extend the generalized variational principle of Herglotz, i.e.,

$$\frac{dz}{dt} = L(t, x, \dot{x}, z)$$

with generalized Euler-Lagrange equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_i} = 0 \ , \qquad i = 1, \dots, n$$

to one with several independent variables.

As in physics applications, the t variable will again stand for time and the rest of the independent variables $x \equiv (x^1, \dots, x^n)$ will stand for spatial variables. The argument function of the functional z defined by the new variational principle will be $u = u(t, x)$.

When there are no spatial variables involved, this new generalized variational principle should reduce to the generalized variational principle of Herglotz. In addition, the new variational principle should contain the classical variational principle with several independent variables as a special case.

The summation convention is assumed to hold throughout the chapter.

The integro-differential equation

$$\frac{dz}{dt} = \int_\Omega \mathcal{L}(t, x, u(t,x), u_t, u_x, z)\, dx \ , \tag{1.1}$$

where $dx \equiv dx^1 \dots dx^n$ and $u_x \equiv (u_{x^1}, \dots, u_{x^n})$, defines the functional z, so that together with the generalized Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}} + \frac{\partial \mathcal{L}}{\partial u_t}\int_\Omega \frac{\partial \mathcal{L}}{\partial z}\, dx = 0 \tag{1.2}$$

derived in section 3 below, it satisfies the above requirements for a variational principle. We will call the variational principle defined by (1.1) and (1.2) the *Generalized Variational Principle with Several Independent Variables.*

In order for the equation (1.1) to define a functional z of $u(t,x)$ we must solve equation (1.1) with the same fixed initial condition $z(0)$ and evaluate the solution $z(t)$ at the same fixed final time $t = T$ for all argument functions $u(t,x)$.

2. *Physical significance of the Generalized Variational Principle with Several Independent Variables*

The generalized variational principle with several independent variables gives a variational description of processes involving physical fields. This is done by the generalized Euler-Lagrange equations (1.2), which are derived in section 3 below. The dependent variables $u = u(t,x)$ can describe an electromagnetic or a gravitational field, a temperature distribution of a body, a flow of a gas or liquid, etc. Most of these fields and the processes which involve them have well known variational descriptions with the classical variational principle, with several independent variables. What is new here? If a non-conservative process is describable by the classical variational principle then the Lagrangian function $\mathcal{L}$ necessarily depends on time. This dependence on time of the Lagrangian function cannot be avoided if a non-conservative process is described by the classical variational principle with several independent variables. However, if a non-conservative process is described by the generalized variational principle with several independent variables then the process does have a variational description, yet the Lagrangian function does not have to depend on time.

2. INFINITESIMAL CRITERION FOR INVARIANCE

We are interested in finding an infinitesimal criterion for the invariance of the functional z defined by the generalized variational principle with several independent variables. Let us consider a functional z defined by the integro-differential equation (1.1) where $x = (x^1, \ldots, x^n)$, u_t stands for the partial derivative of u with respect to the independent variable t and u_x stands for all first partial derivatives of u with respect to the spatial variables $x^1, \ldots, x^n$. Given a one–parameter group of transformations of the independent and dependent variables $(t, x^1, \ldots, x^n)$, we would like to know how this family of transformations affects the functional z defined by (1.1), when the transformation is applied to (1.1). So, consider the one–parameter group of transformations

$$\begin{aligned} \bar{t} &= \phi(t, x, u; \varepsilon) \\ \bar{x}^k &= \varphi^k(t, x, u; \varepsilon), \qquad k = 1, \ldots n \\ \bar{u} &= \psi(t, x, u; \varepsilon) \ . \end{aligned} \tag{2.1}$$

We need to recall how (2.1) transforms the function $u = u(t, x)$. The process is as follows: For a *given* function $u = u(t, x)$, the system

$$\begin{aligned} \bar{t} &= \phi(t, x, u(t, x); \varepsilon) \\ \bar{x}^k &= \varphi^k(t, x, u(t, x); \varepsilon), \qquad k = 1, \ldots n, \end{aligned} \tag{2.2}$$

is a system of $n+1$ equations with $n+1$ unknowns $t, x^1, \ldots, x^n$ and a parameter ε. We invert this system to get t and $x^1, \ldots, x^n$ as functions

$$\begin{aligned} t &= \Theta(\bar{t}, \bar{x}; \varepsilon) \\ x^k &= \Upsilon^k(\bar{t}, \bar{x}; \varepsilon), \qquad k = 1, \ldots n, \end{aligned} \tag{2.3}$$

of $\bar{t}$ and $\bar{x}^1, \ldots, \bar{x}^n$. Then we substitute (2.3) into the last equation of (2.1) to get $\bar{u}$ as a function of $\bar{t}$ and $\bar{x}^1, \ldots, \bar{x}^n$ and ε, which we denote by $\bar{u} = \bar{u}(\bar{t}, \bar{x}; \varepsilon)$. This is how (2.1) transforms the function $u = u(t, x)$.

We now define what is meant by the "transformed functional" of a functional z defined by (1.1).

Definition 2.1. The transformed functional $\bar{z}$, of a functional z defined by (1.1), is a solution of the transformed differential equation

$$\frac{d\bar{z}}{d\bar{t}} = \int_{\bar{\Omega}} \mathcal{L}(\bar{t}, \bar{x}, \bar{u}(\bar{t}, \bar{x}), \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{z}) \, d\bar{x} \ , \tag{2.4}$$

where $\bar{\Omega}$ is the transformed domain of the domain Ω.

There are two important observations to be made here:

1. The transformed differential equation (2.4) differs from the non-transformed (1.1) only in the argument function and in the domain of integration, i.e. it contains $\bar{u}$ in place of u and $\bar{\Omega}$ in place of Ω. In other words, we can write the transformed equation as

$$\frac{d\bar{z}}{dt} = \int_{\bar{\Omega}} \mathcal{L}(t, x, \bar{u}(t,x), \bar{u}_t, \bar{u}_x, \bar{z})\, dx$$

where t and x are the new independent variables. We choose to adhere to the notation of (2.4) in which the transformed variables have the names $\bar{t}$ and $\bar{x}$.

2. If the one-parameter family of invertible transformations (2.1) is used to transform the equation (1.1), then the result is meaningful only if ϕ is independent of x and u. We state and prove this fact as

Lemma 2.1. *The most general form of a one-parameter family of invertible transformations of the independent and dependent variables which transforms equation (1.1) in a meaningful way is*

$$\begin{aligned} \bar{t} &= \phi(t;\varepsilon) \\ \bar{x}^k &= \varphi^k(t, x, u(t,x);\varepsilon), \qquad k = 1, \dots n, \\ \bar{u} &= \psi(t, x, u(t,x);\varepsilon) \; . \end{aligned} \tag{2.5}$$

Proof: First observe that for a known fixed $u = u(x,t)$ every solution z of the non-transformed equation (1.1) is a function of t only. That is, z as a solution function of (1.1) does not depend on x. This is because x is integrated out in the right hand side of (1.1), so $dz/dt = f(t)$, and hence $z = F(t)$ is a function of t only.

Assume that we transform (1.1) with (2.1) where ϕ *does* depend on either x or $u(t,x)$ or both. The result is

$$\frac{d\bar{z}}{d\bar{t}} = \int_{\bar{D}} \mathcal{L}(\bar{t}, \bar{x}, \bar{u}(\bar{t},\bar{x}), \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{z})\, d\bar{x} \tag{2.6}$$

and since $\bar{x}$ is integrated out, $d\bar{z}/d\bar{t} = f(\bar{t};\varepsilon)$. Hence $\bar{z} = F(\bar{t};\varepsilon)$ is a function of $\bar{t}$ and ε only. Now we use the fact that: if we apply an invertible transformation to an equation, solve the transformed equation and then apply the inverse transformation to a solution of the transformed equation, we obtain a solution of the non-transformed equation. Let's apply the transformation $\bar{t} = \phi(t, x, u; \varepsilon)$ to $\bar{z} = F(\bar{t}; \varepsilon)$. We get $\bar{z} = F(\phi(t, x, u; \varepsilon); \varepsilon)$. The above fact now asserts that $\bar{z} = F(\phi(t, x, u; \varepsilon); \varepsilon)$ is a solution of the non-transformed equation (1.1) which depends explicitely on x. This is a contradiction with the fact that no solution functions of (1.1) depend on x.

Now we need to define what it means for the functional z to remain invariant under the transformation (2.1). Loosely speaking, it means that the transformed functional $\bar{z}$ is identically equal to the non-transformed functional z. The precise definition is

Definition 2.2. Let Φ, Ω and Ψ be the spaces where t, x and $u(t, x)$ vary. A local group of transformations G acting on the independent and dependent variables $\Phi \times \Omega \times \Psi$ is a *symmetry group* of the functional z defined by the integro-differential equation (1.1) if whenever D is a subdomain with closure $D^{cl} \subset \Omega$ and $u = f(t, x)$ is a function defined over $\Phi \times D$ whose graph lies in $\Phi \times \Omega \times \Psi$ with continuous second partial derivatives, and $g \in G$ is such that

$$\bar{u} = \bar{f}(\bar{t}, \bar{x}) = g \circ f(t, x)$$

is a single valued function defined over $\bar{\Phi} \times \bar{D} \subset \Phi \times \Omega$ then the functional defined by the transformed integro-differential equation

$$\frac{d\bar{z}}{d\bar{t}} = \int_{\bar{D}} \mathcal{L}(\bar{t}, \bar{x}, \bar{u}(\bar{t}, \bar{x}), \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{z})\, d\bar{x}$$

is equal to the functional defined by the non-transformed integro-differential equation

$$\frac{dz}{dt} = \int_{D} \mathcal{L}(t, x, u(t, x), u_t, u_x, z)\, dx$$

for all t. Here $\bar{D}$ and $\bar{\Omega}$ denote the transformed D and Ω under G.

If a group G is a symmetry group of a functional z as above then we say that the group leaves the functional invariant.

We can now address the question of finding an infinitesimal criterion for the invariance of the functional z defined by (1.1) under a one-parameter group of transformations (2.1). This condition will be necessary and sufficient for a connected group of transformations to be a symmetry group of the functional.

Proposition 2.3. *The one–parameter group of transformations* G

$$\begin{aligned} \bar{t} &= \phi(t;\varepsilon) \\ \bar{x}^k &= \varphi^k(t,x,u;\varepsilon), \qquad k=1,\dots n \\ \bar{u} &= \psi(t,x,u;\varepsilon) \end{aligned} \tag{2.7}$$

is a symmetry group of the functional defined by the integro-differential equation

$$\frac{dz}{dt} = \int_\Omega \mathcal{L}(t,x,u(t,x),u_t,u_x,z)\,dx \tag{2.8}$$

if and only if

$$\frac{d\tau}{dt}\mathcal{L} + \frac{d\bar{t}}{dt}\bigg|_{\varepsilon=0}\Big(\mathrm{pr}^{(1)}v(\mathcal{L}) + \mathcal{L}\mathrm{Div}\,\xi\Big) = 0 \tag{2.9}$$

for all t,x,u,u_t *and* u_x *in the domain of definition, where*

$$v = \tau(t)\frac{\partial}{\partial t} + \xi^i(t,x,u)\frac{\partial}{\partial x^i} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.10}$$

is the infinitesimal generator of the group G *and* $\mathrm{Div}\,\xi$ *denotes the total divergence of the* n *-tuple* $\xi \equiv (\xi^1,\dots,\xi^n)$.

For the definition of $\mathrm{pr}^{(1)}v$ see the Appendix. Equation (2.9) can be written without the prolongation notation as

$$\frac{d\tau}{dt}\mathcal{L} + \frac{d\bar{t}}{dt}\bigg|_{\varepsilon=0}\Big(\frac{d}{d\varepsilon}\mathcal{L}\Big|_{\varepsilon=0} + \mathcal{L}\mathrm{Div}\,\xi\Big) = 0\,. \tag{2.11}$$

Proof: The functions τ, ξ^k and η in the formula for v are

$$\tau \equiv \frac{d\phi}{d\varepsilon}\bigg|_{\varepsilon=0}, \qquad \xi^k \equiv \frac{d\varphi^k}{d\varepsilon}\bigg|_{\varepsilon=0}, \quad k=1,\dots,n\,, \qquad \eta \equiv \frac{d\psi}{d\varepsilon}\bigg|_{\varepsilon=0}.$$

For each $g \in G$ the group transformation

$$(\bar{t},\bar{x},\bar{u}) = g\circ(t,x,u) = (\phi_g(t),\varphi_g(t,x,u),\psi_g(t,x,u))$$

can be regarded as a change of variables, so we can rewrite the transformed equation

$$\frac{d\bar{z}}{d\bar{t}} = \int_{\bar{D}} \mathcal{L}(\bar{t},\bar{x},\bar{u}(\bar{t},\bar{x}),\bar{u}_{\bar{t}},\bar{u}_{\bar{x}},\bar{z})\,d\bar{x}$$

as

$$\frac{d\bar{z}}{dt} = \frac{d\bar{t}}{dt} \int_D \mathcal{L}(\bar{t}, \bar{x}, \mathrm{pr}^{(1)}(g \circ f)(\bar{t}, \bar{x}), \bar{z}) \det J_g(t, x, \mathrm{pr}^{(1)} f(t, x))\, dx \tag{2.12}$$

where the Jacobi matrix has the entries

$$J_g^{ij}(t, x, u^{(1)}) = \frac{d}{dx^i} \varphi_g^j(t, x, u^{(1)}) \, .$$

For the definition of $u^{(1)}$ see the Appendix. If G is a symmetry group of the functional z defined by (2.8), then the functional z defined by (2.8) is identical with the functional $\bar{z}$ defined by (2.12), for all subdomains D of Ω and all functions $u = f(t, x)$. Hence

$$\begin{aligned} &\frac{d\bar{t}}{dt} \int_D \mathcal{L}(\bar{t}, \bar{x}, \mathrm{pr}^{(1)}(g \circ f)(\bar{t}, \bar{x}), \bar{z}) \det J_g(t, x, \mathrm{pr}^{(1)} f(t, x))\, dx \\ &= \int_D \mathcal{L}(t, x, u(t, x), u_t, u_x, z)\, dx \end{aligned}$$

holds for all subdomains D of Ω, all functions $u = f(t, x)$ and all t in its domain of definition. Since $\bar{t} = \phi(t; \varepsilon)$ does not depend on $x^1, \ldots, x^n$, the arbitrariness of D as a subdomain of Ω now implies that

$$\begin{aligned} &\frac{d\bar{t}}{dt} \mathcal{L}\big(\bar{t}, \bar{x}, \mathrm{pr}^{(1)}(g \circ f)(\bar{t}, \bar{x}), \bar{z}\big) \det J_g\big(t, x, \mathrm{pr}^{(1)} f(t, x)\big) \\ &= \mathcal{L}\big(t, x, u(t, x), u_t, u_x, z\big) \end{aligned} \tag{2.13}$$

holds for all t, x, u, u_t, u_x in the domain of definition. To obtain the infinitesimal version of (2.13) we set $g = g_\varepsilon = \exp(\varepsilon v)$ and differentiate with respect to ε. We need the formula

$$\frac{d}{d\varepsilon}(\det J_{g_\varepsilon}(t, x, u^{(1)})) = \mathrm{Div}\, \xi\, (\mathrm{pr}^{(1)} g_\varepsilon \circ (t, x, u^{(1)})) \det J_{g_\varepsilon}(t, x, u^{(1)}) \tag{2.14}$$

expressing the fact that the divergence of a vector field measures the rate of change of volume under the induced flow. The derivative of (2.13) with respect to ε when $g = g_\varepsilon = \exp(\varepsilon v)$ is

$$\frac{d\tau}{dt} \mathcal{L} \det J_{g_\varepsilon} + \frac{d\bar{t}}{dt} \big(\mathrm{pr}^{(1)} v\, (\mathcal{L}) + \mathcal{L}\, \mathrm{Div}\, \xi \,\big) \det J_{g_\varepsilon} = 0$$

or

$$\left(\frac{d\tau}{dt} \mathcal{L} \; + \; \frac{d\bar{t}}{dt} \left(\mathrm{pr}^{(1)} v \left(\mathcal{L} \right) \; + \; \mathcal{L} \, \mathrm{Div} \, \xi \right) \right) \; \det \, J_{g_\varepsilon} \; = \; 0 \; , \tag{2.15}$$

the expression in parentheses being evaluated at $(\bar{t}, \bar{x}, \bar{u}_\varepsilon^{(1)}) = \mathrm{pr}^{(1)} g_\varepsilon \circ (t, x, u^{(1)})$. In particular, when $\varepsilon = 0$, g_ε is the identity map and we obtain that

$$\frac{d\tau}{dt} \mathcal{L} + \left. \frac{d\bar{t}}{dt} \right|_{\varepsilon=0} \left(\mathrm{pr}^{(1)} v(\mathcal{L}) + \mathcal{L} \, \mathrm{Div} \, \xi \right) = 0$$

for all $(t, x, u^{(1)})$ in the domain of definition. The last identity can be written without the prolongation notation as

$$\frac{d\tau}{dt} \mathcal{L} + \left. \frac{d\bar{t}}{dt} \right|_{\varepsilon=0} \left(\frac{d}{d\varepsilon} \mathcal{L} \Big|_{\varepsilon=0} + \mathcal{L} \, \mathrm{Div} \, \xi \right) = 0 \; .$$

Conversely, if

$$\frac{d\tau}{dt} \mathcal{L} + \left. \frac{d\bar{t}}{dt} \right|_{\varepsilon=0} \left(\mathrm{pr}^{(1)} v(\mathcal{L}) + \mathcal{L} \, \mathrm{Div} \, \xi \right) = 0$$

for all (t, x, u, u_t, u_x) in the domain of definition, then (2.15) holds for ε sufficiently small. The left hand side of (2.15) is just the derivative of the left hand side of (2.13) (for $g = g_\varepsilon$) with respect to ε. Thus, integrating from 0 to ε we prove (2.13) for g sufficiently near the identity. The usual connectivity arguments complete the proof of (2.13) for all $g \in G$. The assertion of the proposition now follows.

We observe that the infinitesimal criterion which the above proposition provides reduces to the infinitesimal criterion for the classical variational integral under a group of transformations, when the generalized variational principle with several independent variables reduces to the classical variational principle.

3. GENERALIZED EULER–LAGRANGE EQUATIONS FOR THE GENERALIZED VARIATIONAL PRINCIPLE WITH SEVERAL INDEPENDENT VARIABLES

In this section we derive the equations which provide the extrema of the functional defined by the generalized variational principle with several independent variables. Due to the obvious correspondence, we call these equations *generalized Euler-Lagrange equations with several independent variables.*

Let us consider the defining integro-differential equation for the functional z

$$\frac{dz}{dt} = \int_\Omega \mathcal{L}(t, x, u(t,x), u_t, u_x, z)\, dx \tag{3.1}$$

where $x = (x^1, \dots, x^n)$, $u_x = (u_{x^1}, \dots, u_{x^n})$, $dx = dx^1 \dots dx^n$. As explained in Section 1, equation (3.1) defines z as a functional of $u = u(t,x)$. We write this as $z = z[u;t]$. Our goal is to derive the equations whose solutions $u = u(t,x)$ will make the functional z stationary.

Theorem 3.1 *If a function $u = u(t,x)$ produces an extremum of the functional $z = z[u;t]$ defined by the integro-differential equation (3.1) then u is a solution of the equations*

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}} + \frac{\partial \mathcal{L}}{\partial u_t}\int_\Omega \frac{\partial \mathcal{L}}{\partial z}\, dx = 0 \ . \tag{3.2}$$

Proof: We will need the total variation ζ produced in the functional z by the variation in the argument function u. The precise definition of ζ is

$$\zeta(t) \equiv \frac{d}{d\varepsilon} z[u + \varepsilon\eta; t]\Big|_{\varepsilon=0} \ . \tag{3.3}$$

Give a variation to the argument function u of the functional z, namely $u + \varepsilon\eta$. Here η is an arbitrary function of t and x with continuous first partial derivatives. We assume that η is zero on the boundary $\partial\Omega$ of Ω, $\eta(0,x) = 0$ and $\eta(T,x) = 0$, where $[0,\ T]$ is the interval in which the time variable t varies. We now consider the varied integro-differential equation

$$\frac{dz[u + \varepsilon\eta; t]}{dt} = \int_\Omega \mathcal{L}(t, x, u + \varepsilon\eta, u_t + \varepsilon\eta_t, u_x + \varepsilon\eta_x, z)\, dx \ . \tag{3.4}$$

We differentiate equation (3.4) with respect to ε

$$\frac{d}{d\varepsilon}\frac{dz[u + \varepsilon\eta; t]}{dt} = \frac{d}{d\varepsilon}\int_\Omega \mathcal{L}(t, x, u + \varepsilon\eta, u_t + \varepsilon\eta_t, u_x + \varepsilon\eta_x, z)\, dx \ .$$

Since ε, t and $x^1, \dots, x^n$ are independent of each other, we may interchange the order of differentiation in the left hand side of the last equation and the order of integration and differentiation in the right hand side to obtain

$$\frac{d\zeta}{dt} = \int_\Omega \Big(\frac{\partial \mathcal{L}}{\partial u}\eta + \frac{\partial \mathcal{L}}{\partial u_t}\eta_t + \frac{\partial \mathcal{L}}{\partial u_{x^k}}\eta_{x^k} + \frac{\partial \mathcal{L}}{\partial z}\zeta\Big)\, dx$$

Since ζ does not depend on $x^1, \ldots, x^n$, the last equation can be written as

$$\frac{d\zeta}{dt} = \int_\Omega \Big(\frac{\partial \mathcal{L}}{\partial u}\eta + \frac{\partial \mathcal{L}}{\partial u_t}\eta_t + \frac{\partial \mathcal{L}}{\partial u_{x^k}}\eta_{x^k} \Big) dx + \zeta \int_\Omega \frac{\partial \mathcal{L}}{\partial z}\, dx \; . \tag{3.5}$$

Equation (3.5) is the equation for the total variation ζ of the functional z produced by the variation in the argument function $u(t,x)$. For convenience let's denote by $A(t)$ and $B(t)$ the quantities

$$A(t) = \int_\Omega \Big(\frac{\partial \mathcal{L}}{\partial u}\eta + \frac{\partial \mathcal{L}}{\partial u_t}\eta_t + \frac{\partial \mathcal{L}}{\partial u_{x^k}}\eta_{x^k} \Big) dx \; , \qquad B(t) = \int_\Omega \frac{\partial \mathcal{L}}{\partial z}\, dx \; .$$

With this notation equation (3.5) becomes

$$\frac{d\zeta(t)}{dt} = A(t) + B(t)\,\zeta(t) \; .$$

Its solution $\zeta(t)$ is given by

$$\exp\Big(-\int_0^t B(\theta)\,d\theta\Big)\,\zeta(t) - \zeta(0) = \int_0^t \exp\Big(-\int_0^s B(\theta)\,d\theta\Big)\,A(s)\,ds \; . \tag{3.6}$$

We note that $\zeta(0) = 0$. Indeed, as explained earlier, in order to have a well-defined functional z of the function $u(t,x)$ we must evaluate the solution $z(t)$ of the equation (3.1) with the same fixed initial condition $z(0)$ independently of the function $u(t,x)$. Then $z(0)$ is independent of ε. Hence, the variation of z evaluated at $t=0$ is

$$\zeta(0) = \frac{d}{d\varepsilon} z[u;0,\varepsilon]\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} z(0)\Big|_{\varepsilon=0} = 0 \; .$$

We are interested in those functions u which leave the functional z stationary, i.e. those for which the total variation $\zeta \equiv 0$ identically. With these observations (3.6) becomes

$$\int_0^t \exp\Big(-\int_0^s B(\theta)\,d\theta\Big)\,A(s)\,ds = 0 \; . \tag{3.7}$$

Let's denote the exponent function by

$$E(t) \equiv \exp\Big(-\int_0^t B(\theta)\,d\theta\Big) \; .$$

Then equation (3.7) becomes

$$\int_0^t E(s) \int_\Omega \Big(\frac{\partial \mathcal{L}}{\partial u}\eta + \frac{\partial \mathcal{L}}{\partial u_s}\eta_s + \frac{\partial \mathcal{L}}{\partial u_{x^k}}\eta_{x^k} \Big) dx\, ds = 0$$

or

$$\int_0^t E(s) \int_\Omega \Big(\frac{\partial \mathcal{L}}{\partial u}\eta + \frac{\partial \mathcal{L}}{\partial u_s}\eta_s + \frac{d}{dx^k}\Big(\frac{\partial \mathcal{L}}{\partial u_{x^k}}\eta\Big) - \eta \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}} \Big) dx\, ds = 0 \ . \tag{3.8}$$

Application of Gauss theorem produces

$$\int_0^t E(s) \int_{\partial\Omega} \eta \frac{\partial \mathcal{L}}{\partial u_x} \cdot N\, dA\, ds + \int_0^t E(s) \int_\Omega \Big(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}} \Big) \eta\, dx\, ds$$
$$+ \int_\Omega \int_0^t E(s) \frac{\partial \mathcal{L}}{\partial u_s} \eta_s\, dx\, ds = 0$$

where

$$\frac{\partial \mathcal{L}}{\partial u_x} = \Big(\frac{\partial \mathcal{L}}{\partial u_{x^1}}, \ldots, \frac{\partial \mathcal{L}}{\partial u_{x^n}} \Big) \ ,$$

N is the normal unit vector to the boundary $\partial\Omega$ and dA is an element of $\partial\Omega$. Taking into consideration the fact that η vanishes on $\partial\Omega$, and integrating by parts the term involving η_s, we obtain

$$\int_0^t E(s) \int_\Omega \Big(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}} \Big) \eta\, dx\, ds - \int_\Omega \int_0^t \eta \frac{d}{ds}\Big(E(s) \frac{\partial \mathcal{L}}{\partial u_s} \Big) ds\, dx = 0$$

since $\eta(0,x) = 0$ and $\eta(s,x) = 0$. Expanding the second term in the last equation we get

$$\int_0^t \int_\Omega E(s) \Big(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}} - \frac{d}{ds}\frac{\partial \mathcal{L}}{\partial u_s} + \frac{\partial \mathcal{L}}{\partial u_s} \int_\Omega \frac{\partial \mathcal{L}}{\partial z}\, dx \Big) \eta\, dx\, ds \ = 0 \ .$$

The arbitrariness of η together with $E(t) > 0$, for all t, imply that the generalized Euler-Lagrange equations, which produce the extrema of the functional z, are

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} + \frac{\partial \mathcal{L}}{\partial u_t} \int_\Omega \frac{\partial \mathcal{L}}{\partial z}\, dx = 0 \ ,$$

which concludes the proof of the theorem.

It is important to observe that the generalized Euler-Lagrange equations (3.2) reduce to the classical Euler-Lagrange equations when $\mathcal{L}$ does not depend on z, i.e., when the generalized variational principle with several independent variables reduces to the classical variational integral with several independent variables. Also, equations (3.2) reduce to the generalized Euler-Lagrange equations (2.5) in Chapter 1, for the generalized variational principle of Herglotz.

4. FIRST NOETHER–TYPE THEOREM FOR THE GENERALIZED VARIATIONAL PRINCIPLE WITH SEVERAL INDEPENDENT VARIABLES

In this section we prove a theorem which provides an identity corresponding to each symmetry of the functional z defined by the generalized variational principle with several independent variables. We call it a first Noether-type theorem for the generalized variational principle with several independent variables because this theorem reduces to the classical first Noether theorem for a variational integral with several independent variables when the generalized variational principle with several independent variables reduces to the classical variational principle with several independent variables.

As corollaries to this theorem we show that there is a correspondence between the symmetries of the functional z defined by the generalized variational principle with several independent variables and the conserved quantities of the system described by it. One of the corollaries to the theorem provides a systematic procedure for finding the conserved quantities.

Let us consider again the generalized variational principle with several independent variables in which the functional z is defined by

$$\frac{dz}{dt} = \int_\Omega \mathcal{L}(t, x, u(t,x), u_t, u_x, z)\, dx \ , \tag{4.1}$$

where $x = (x^1, \ldots, x^n)$, u_t stands for the partial derivative of u with respect to the independent variable t and u_x stands for the gradient of u in the spatial variables $x^1, \ldots, x^n$. The equations for the extrema of z are

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}} + \frac{\partial \mathcal{L}}{\partial u_t}\int_\Omega \frac{\partial \mathcal{L}}{\partial z}\, dx = 0 \ . \tag{4.2}$$

Let also a one-parameter symmetry group

$$\begin{aligned} \bar{t} &= \phi(t,\varepsilon) \\ \bar{x}^k &= \varphi^k(t,x,u,\varepsilon), \qquad k = 1,\dots n \\ \bar{u} &= \psi(t,x,u,\varepsilon)\ , \end{aligned} \tag{4.3}$$

of the functional z defined by (4.1) be given. The form of the transformation (4.3) is the most general form of a transformation which meaningfully transforms the functional z defined by (4.1). See Lemma 2.1 in section 2 of this chapter.

By Taylor's theorem we can write this transformation as

$$\begin{aligned} \bar{t} &= t + \tau(t)\,\varepsilon + O(\varepsilon^2) \\ \bar{x}^k &= x^k + \xi^k(t,x,u)\,\varepsilon + O(\varepsilon^2) \qquad k = 1,\dots n \\ \bar{u} &= u + \eta(t,x,u)\,\varepsilon + O(\varepsilon^2) \end{aligned}$$

where τ , ξ^k and η are

$$\tau \equiv \left.\frac{d\phi}{d\varepsilon}\right|_{\varepsilon=0}, \qquad \xi^k \equiv \left.\frac{d\varphi^k}{d\varepsilon}\right|_{\varepsilon=0}, \qquad \eta \equiv \left.\frac{d\psi}{d\varepsilon}\right|_{\varepsilon=0}.$$

The infinitesimal generator of the group (4.3) is

$$v = \tau(t)\frac{\partial}{\partial t} + \xi^i(t,x,u)\frac{\partial}{\partial x^i} + \eta(t,x,u)\frac{\partial}{\partial u}\ ,$$

where here and throughout the rest of the chapter we assume the summation convention on repeated indices $i = 1,\dots,n$ to hold.

We are interested in the total variation ζ produced in the functional z by the action of the symmetry group (4.3). We know from Lie theory that near the identity transformation the action of the *nonlinear* group (4.3) is the same as the action of the infinitesimal *linear* group

$$\begin{aligned} \bar{t} &= t + \tau(t)\,\varepsilon \\ \bar{x}^k &= x^k + \xi^k(t,x,u)\,\varepsilon \qquad k = 1,\dots n \\ \bar{u} &= u + \eta(t,x,u)\,\varepsilon\ . \end{aligned} \tag{4.4}$$

The following theorem provides the identity of the physical or mathematical system described by the generalized Euler-Lagrange equations (4.2), which corresponds to the symmetry group (4.3).

Theorem 4.1 *Let (4.3) be a given symmetry group of the functional* z *, defined by (4.1), with infinitesimal generator*

$$v = \tau(t)\frac{\partial}{\partial t} + \xi^i(t,x,u)\frac{\partial}{\partial x^i} + \eta(t,x,u)\frac{\partial}{\partial u} \ . \tag{4.5}$$

Then the following identity

$$\int_D \left(\frac{d}{dt}\left(E\left(\left(\frac{\partial \mathcal{L}}{\partial u_t} u_t - \mathcal{L} \right) \tau + \frac{\partial \mathcal{L}}{\partial u_t}\left(\xi^j u_{x^j} - \eta \right) \right) \right) \right.$$

$$\left. + \frac{d}{dx^k}\left(E\left(\frac{\partial \mathcal{L}}{\partial u_{x^k}} u_t \tau - \mathcal{L}\xi^k + \frac{\partial \mathcal{L}}{\partial u_{x^k}}\left(\xi^j u_{x^j} - \eta \right) \right) \right) \right) dx = 0 \tag{4.6}$$

holds on solutions of the generalized Euler-Lagrange equations (4.2) for the functional z *. Here* D *is any subdomain of* Ω *, including* Ω *itself, whose closure* $D^{cl} \subset \Omega^{cl}$ *and* E *is*

$$E \equiv \exp\left(-\int_0^t \int_D \frac{\partial \mathcal{L}}{\partial z} \, dx \, d\theta \right) \ . \tag{4.7}$$

Remarks: When the transformation (4.3) or equivalently (4.4) is applied to the functional $z[u;t]$, the limits where the functional is evaluated (after the equation (4.1) is solved) are also transformed by the transformation. Thus, we must account for this transformation of the limits and compare the nontransformed functional $z[u;t]$ having nontransformed limits of evaluation with the transformed functional $\bar{z}[\bar{u};\bar{t}]$ having transformed limits of evaluation. The two tricks which make possible this procedure are:

1. Apply the transformation (4.4) to the defining equation (4.1).

2. In the resulting integro-differential equation perform a change of the independent variables $\bar{t}$ and $\bar{x}^k$ to go back to t and x^k , $k = 1, \ldots, n$.

Proof: Apply the transformation (4.4) to

$$\frac{dz}{dt} = \int_D \mathcal{L}(t, x, u(t,x), u_t, u_x, z) \, dx \ , \tag{4.8}$$

to obtain

$$\frac{d\bar{z}}{d\bar{t}} = \int_{\bar{D}} \mathcal{L}(\bar{t}, \bar{x}, \bar{u}(\bar{t}, \bar{x}), \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{z})\, d\bar{x} \;, \tag{4.9}$$

where D is *any* subdomain of Ω, including Ω itself, with closure $D^{cl} \subset \Omega^{cl}$. Here $\bar{D}$ denotes the result of transforming D with (4.4). Note that $\bar{D} = \bar{D}(\bar{t}, \bar{u}, \varepsilon)$ depends on $\bar{t}$, $\bar{u}$ and ε. Now we perform a change of the independent variables in (4.9) to go back to the original independent variables t and x^k, $k = 1, \ldots, n$. The resulting equation is

$$\frac{d\bar{z}}{dt} = \frac{d\bar{t}}{dt} \int_D \mathcal{L}(\bar{t}, \bar{x}, \bar{u}(\bar{t}, \bar{x}), \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{z}) \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) dx^1 \ldots dx^n \;, \tag{4.10}$$

where the determinant of the Jacobi matrix arrises from the change of the spatial variables x^k, $k = 1, \ldots, n$. Differentiate equation (4.10) with respect to ε and set $\varepsilon = 0$ to obtain

$$\begin{aligned} \frac{d}{dt}\frac{d\bar{z}}{d\varepsilon}\bigg|_{\varepsilon=0} &= \frac{d\bar{t}}{dt}\bigg|_{\varepsilon=0} \left(\int_D \frac{d\mathcal{L}}{d\varepsilon}\bigg|_{\varepsilon=0} \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right)\bigg|_{\varepsilon=0} dx + \int_D \mathcal{L} \frac{d}{d\varepsilon} \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right)\bigg|_{\varepsilon=0} dx \right) \\ &\quad + \frac{d}{d\varepsilon}\frac{d\bar{t}}{dt}\bigg|_{\varepsilon=0} \int_D \mathcal{L}\, dx \;. \end{aligned} \tag{4.11}$$

To get the above equation we have also interchanged the order of differentiation with respect to ε and t and the order of integration with respect to the spatial variables and differentiation with respect to ε. Observe that

$$\frac{d\bar{t}}{dt}\bigg|_{\varepsilon=0} = 1 \;,$$

since

$$\frac{d\bar{t}}{dt}\bigg|_{\varepsilon=0} = \frac{d\,\phi(t,\varepsilon)}{dt}\bigg|_{\varepsilon=0} = \frac{d}{dt}\left(\phi(t,0) + \frac{d}{d\varepsilon}\,\phi(t,0)\varepsilon + O(\varepsilon^2)\right)\bigg|_{\varepsilon=0} = 1 \;.$$

A similar calculation yields

$$\det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right)\bigg|_{\varepsilon=0} = 1 \;.$$

Also,

$$\frac{d}{d\varepsilon}\frac{d\bar{t}}{dt}\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left(1 + \frac{d\tau}{dt}\varepsilon + O(\varepsilon^2)\right)\bigg|_{\varepsilon=0} = \frac{d\tau}{dt} \;.$$

Recall that the precise definition of the total variation ζ produced by the group of transformations (4.3) in the functional z is

$$\zeta(t) \equiv \frac{d}{d\varepsilon} z[u+\varepsilon\eta; t]\bigg|_{\varepsilon=0} .$$

Thus, equation (4.11) becomes

$$\frac{d\zeta}{dt} = \int_D \frac{d\mathcal{L}}{d\varepsilon}\bigg|_{\varepsilon=0} dx + \int_D \mathcal{L} \frac{d}{d\varepsilon} \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right)\bigg|_{\varepsilon=0} dx + \frac{d\tau}{dt}\int_D \mathcal{L}\, dx . \tag{4.12}$$

Now we must calculate

$$\frac{d}{d\varepsilon} \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right)\bigg|_{\varepsilon=0} . \tag{4.13}$$

If $a = \det({a_j}^i)$, where ${a_j}^i$ are functions of the same parameter ε, then

$$\frac{da}{d\varepsilon} = \frac{d{a_j}^i}{d\varepsilon} {A_i}^j ,$$

where summation is performed on i and j and ${A_i}^j$ is the cofactor of ${a_j}^i$ in the determinant. Apply this formula to $\det\left(\partial \bar{x}^i/\partial x^j\right)$ to obtain

$$\frac{d}{d\varepsilon} \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) = {A_i}^j \frac{d}{d\varepsilon}\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \tag{4.14}$$

where ${A_i}^j$ is the cofactor of $\partial \bar{x}^i/\partial x^j$. Next,

$$\frac{d}{d\varepsilon}\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\frac{d\bar{x}^i}{d\varepsilon} = \frac{\partial^2 \bar{x}^i}{\partial x^j\,\partial\varepsilon} + \frac{\partial^2 \bar{x}^i}{\partial u\,\partial\varepsilon}\frac{\partial u}{\partial x^j} ,$$

because $\bar{x}^i = \bar{x}^i(t, x, u(t,x); \varepsilon)$, $\quad i = 1, \ldots, n$. Hence (4.14) becomes

$$\frac{d}{d\varepsilon} \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) = {A_i}^j\left(\frac{\partial^2 \bar{x}^i}{\partial x^j\,\partial\varepsilon} + \frac{\partial^2 \bar{x}^i}{\partial u\,\partial\varepsilon}\frac{\partial u}{\partial x^j}\right) .$$

Observing that ${A_i}^j\Big|_{\varepsilon=0} = {\delta_i}^j$ is a cofactor of the identity matrix, we get

$$\frac{d}{d\varepsilon} \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right)\bigg|_{\varepsilon=0} = \left(\frac{\partial \xi^i}{\partial x^j} + \frac{\partial \xi^i}{\partial u}\frac{\partial u}{\partial x^j}\right){\delta_i}^j = \frac{d\xi^i}{dx^j}{\delta_i}^j = \frac{d\xi^i}{dx^i} .$$

Thus, equation (4.12) becomes

$$\frac{d\zeta}{dt} = \int_D \frac{d\mathcal{L}}{d\varepsilon}\bigg|_{\varepsilon=0} dx + \int_D \mathcal{L}\frac{d\xi^i}{dx^i}\, dx + \frac{d\tau}{dt}\int_D \mathcal{L}\, dx . \tag{4.15}$$

The integrand of the first integral in (4.15) is

$$\left.\frac{d\mathcal{L}}{d\varepsilon}\right|_{\varepsilon=0}=\left.\left(\frac{\partial\mathcal{L}}{\partial\bar{t}}\frac{d\phi}{d\varepsilon}+\frac{\partial\mathcal{L}}{\partial\bar{x}^k}\frac{d\varphi^k}{d\varepsilon}+\frac{\partial\mathcal{L}}{\partial\bar{u}}\frac{d\psi}{d\varepsilon}+\frac{\partial\mathcal{L}}{\partial\bar{u}_{\bar{t}}}\frac{d}{d\varepsilon}\frac{\partial\bar{u}}{\partial\bar{t}}+\frac{\partial\mathcal{L}}{\partial\bar{u}_{\bar{x}^k}}\frac{d}{d\varepsilon}\frac{\partial\bar{u}}{\partial\bar{x}^k}+\frac{\partial\mathcal{L}}{\partial\bar{z}}\frac{d\bar{z}}{d\varepsilon}\right)\right|_{\varepsilon=0}$$

which when written with ζ and the infinitesimal generators of the group becomes

$$\left.\frac{d\mathcal{L}}{d\varepsilon}\right|_{\varepsilon=0}=\frac{\partial\mathcal{L}}{\partial t}\tau+\frac{\partial\mathcal{L}}{\partial x^k}\xi^k+\frac{\partial\mathcal{L}}{\partial u}\eta+\frac{\partial\mathcal{L}}{\partial u_t}\frac{d}{d\varepsilon}\left.\left(\frac{\partial\bar{u}}{\partial\bar{t}}\right)\right|_{\varepsilon=0}+\frac{\partial\mathcal{L}}{\partial u_{x^k}}\frac{d}{d\varepsilon}\left.\left(\frac{\partial\bar{u}}{\partial\bar{x}^k}\right)\right|_{\varepsilon=0}+\frac{\partial\mathcal{L}}{\partial z}\zeta\,. \tag{4.16}$$

To calculate

$$\frac{d}{d\varepsilon}\left.\left(\frac{\partial\bar{u}}{\partial\bar{t}}\right)\right|_{\varepsilon=0},$$

which appears in the above expression, proceed as follows: Differentiate the equation $\bar{u}=\bar{u}(\bar{t},\bar{x};\varepsilon)=\psi(t,x,u;\varepsilon)$ in two different ways with respect to t. The results are equal, so

$$\frac{\partial\bar{u}}{\partial t}+\frac{\partial\bar{u}}{\partial u}u_t=\frac{\partial\bar{u}}{\partial\bar{t}}\frac{d\bar{t}}{dt}+\frac{\partial\bar{u}}{\partial\bar{x}^k}\frac{d\bar{x}^k}{dt}\equiv\frac{\partial\bar{u}}{\partial\bar{t}}\frac{d\bar{t}}{dt}+\frac{\partial\bar{u}}{\partial\bar{x}^k}\left(\frac{\partial\bar{x}^k}{\partial t}+\frac{\partial\bar{x}^k}{\partial u}u_t\right). \tag{4.17}$$

Set $\varepsilon=0$ and take into consideration the identities:

$$\left.\frac{\partial\bar{u}}{\partial t}\right|_{\varepsilon=0}=0\,,\quad\left.\frac{\partial\bar{u}}{\partial u}\right|_{\varepsilon=0}=1\,,\quad\left.\frac{\partial\bar{t}}{\partial t}\right|_{\varepsilon=0}=1\,,\quad\left.\frac{\partial\bar{x}^k}{\partial t}\right|_{\varepsilon=0}=0\,,\quad\left.\frac{\partial\bar{x}^k}{\partial u}\right|_{\varepsilon=0}=0\,.$$

Substitute these in (4.17) and solve the resulting equation for $\left.\bar{u}_{\bar{t}}\right|_{\varepsilon=0}$ to find

$$\left.\bar{u}_{\bar{t}}\right|_{\varepsilon=0}=u_t\,. \tag{4.18}$$

Differentiate the equations $\bar{u}=\bar{u}(\bar{t},\bar{x};\varepsilon)$ and $\bar{u}=\psi(t,x,u;\varepsilon)$ with respect to x^j. The results are equal, namely,

$$\frac{\partial\bar{u}}{\partial x^j}+\frac{\partial\bar{u}}{\partial u}u_{x^j}=\frac{\partial\bar{u}}{\partial\bar{x}^k}\frac{d\bar{x}^k}{dx^j}\equiv\frac{\partial\bar{u}}{\partial\bar{x}^k}\left(\frac{\partial\bar{x}^k}{\partial x^j}+\frac{\partial\bar{x}^k}{\partial u}u_{x^j}\right). \tag{4.19}$$

Set $\varepsilon=0$ and then substitute the identities:

$$\left.\frac{\partial\bar{u}}{\partial x^j}\right|_{\varepsilon=0}=0\,,\quad\left.\frac{\partial\bar{u}}{\partial u}\right|_{\varepsilon=0}=1\,,\quad\left.\frac{\partial\bar{x}^k}{\partial x^j}\right|_{\varepsilon=0}=\delta_j{}^k\,,\quad\left.\frac{\partial\bar{x}^k}{\partial u}\right|_{\varepsilon=0}=0$$

in (4.19). Then solve the resulting equation for $\left.\bar{u}_{\bar{x}^j}\right|_{\varepsilon=0}$ to obtain

$$\left.\bar{u}_{\bar{x}^j}\right|_{\varepsilon=0}=u_{x^j}\,. \tag{4.20}$$

Differentiate (4.17) with respect to ε to get

$$\frac{d}{d\varepsilon}\Big(\frac{\partial \bar{u}}{\partial t}+\frac{\partial \bar{u}}{\partial u}u_t\Big)=\bar{u}_{\bar{t}}\,\frac{d}{d\varepsilon}\frac{d\bar{t}}{dt}+\frac{d\bar{t}}{dt}\,\frac{d}{d\varepsilon}\bar{u}_{\bar{t}}+\bar{u}_{\bar{x}^k}\Big(\frac{d}{d\varepsilon}\frac{\partial \bar{x}^k}{\partial t}+\frac{d}{d\varepsilon}\Big(\frac{\partial \bar{x}^k}{\partial u}\Big)u_t\Big)$$
$$+\Big(\frac{\partial \bar{x}^k}{\partial t}+\frac{\partial \bar{x}^k}{\partial u}u_t\Big)\frac{d}{d\varepsilon}\bar{u}_{\bar{x}^k}\ . \tag{4.21}$$

Set $\varepsilon=0$ in (4.21) and then substitute (4.18) and (4.20) into (4.21). Take into consideration the fact that

$$\frac{d}{d\varepsilon}\frac{\partial \bar{u}}{\partial t}\Big|_{\varepsilon=0}=\frac{\partial \eta}{\partial t}\ ,\quad \frac{d}{d\varepsilon}\frac{\partial \bar{u}}{\partial u}\Big|_{\varepsilon=0}=\frac{\partial \eta}{\partial u}\ ,\quad \frac{d}{d\varepsilon}\frac{\partial \bar{t}}{\partial t}\Big|_{\varepsilon=0}=\frac{\partial \tau}{\partial t}\ ,\quad \frac{\partial \bar{t}}{\partial t}\Big|_{\varepsilon=0}=1\ ,$$

$$\frac{d}{d\varepsilon}\frac{\partial \bar{x}^k}{\partial t}\Big|_{\varepsilon=0}=\frac{\partial \xi^k}{\partial t}\ ,\quad \frac{d}{d\varepsilon}\frac{\partial \bar{x}^k}{\partial u}\Big|_{\varepsilon=0}=\frac{\partial \xi^k}{\partial u}\ ,\quad \frac{\partial \bar{x}^k}{\partial t}\Big|_{\varepsilon=0}=0\ ,\quad \frac{\partial \bar{x}^k}{\partial u}\Big|_{\varepsilon=0}=0\ .$$

Then equation (4.21) becomes

$$\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial u}u_t=u_t\frac{\partial \tau}{\partial t}+\frac{d}{d\varepsilon}\bar{u}_{\bar{t}}\Big|_{\varepsilon=0}+u_{x^k}\Big(\frac{\partial \xi^k}{\partial t}+\frac{\partial \xi^k}{\partial u}u_t\Big)\ . \tag{4.22}$$

Observe that the total derivatives of η, τ and ξ appear in the last equation. Hence, (4.22) becomes

$$\frac{d\eta}{dt}=u_t\frac{d\tau}{dt}+\frac{d}{d\varepsilon}\bar{u}_{\bar{t}}\Big|_{\varepsilon=0}+u_{x^k}\frac{d\xi^k}{dt}\ .$$

Solving this last equation for

$$\frac{d}{d\varepsilon}\bar{u}_{\bar{t}}\Big|_{\varepsilon=0}\ ,$$

we obtain

$$\frac{d}{d\varepsilon}\bar{u}_{\bar{t}}\Big|_{\varepsilon=0}=\frac{d\eta}{dt}-u_t\frac{d\tau}{dt}-u_{x^k}\frac{d\xi^k}{dt}\ . \tag{4.23}$$

We must now calculate

$$\frac{d}{d\varepsilon}\Big(\frac{\partial \bar{u}}{\partial \bar{x}^k}\Big)\Big|_{\varepsilon=0}$$

which appears in (4.16). For this purpose differentiate (4.19) with respect to ε.

$$\frac{d}{d\varepsilon}\frac{\partial \bar{u}}{\partial x^j}+\frac{d}{d\varepsilon}\frac{\partial \bar{u}}{\partial u}u_{x^j}=\frac{d}{d\varepsilon}\Big(\bar{u}_{\bar{x}^k}\Big)\Big(\frac{\partial \bar{x}^k}{\partial x^j}+\frac{\partial \bar{x}^k}{\partial u}u_{x^j}\Big)+\bar{u}_{\bar{x}^k}\Big(\frac{d}{d\varepsilon}\frac{\partial \bar{x}^k}{\partial x^j}+u_{x^j}\frac{d}{d\varepsilon}\frac{\partial \bar{x}^k}{\partial u}\Big)\ . \tag{4.24}$$

Set $\varepsilon = 0$ in (4.24), substitute (4.18) and (4.20) into (4.24) and observe that

$$\frac{d}{d\varepsilon}\frac{\partial \bar{u}}{\partial x^j}\bigg|_{\varepsilon=0} = \frac{\partial \eta}{\partial x^j}\ , \quad \frac{d}{d\varepsilon}\frac{\partial \bar{u}}{\partial u}\bigg|_{\varepsilon=0} = \frac{\partial \eta}{\partial u}\ , \quad \frac{\partial \bar{x}^k}{\partial x^j}\bigg|_{\varepsilon=0} = \delta_j^{\ k}\ ,$$

$$\frac{\partial \bar{x}^k}{\partial u}\bigg|_{\varepsilon=0} = 0\ , \quad \frac{d}{d\varepsilon}\frac{\partial \bar{x}^k}{\partial x^j}\bigg|_{\varepsilon=0} = \frac{\partial \xi^k}{\partial x^j}\ , \quad \frac{d}{d\varepsilon}\frac{\partial \bar{x}^k}{\partial u}\bigg|_{\varepsilon=0} = \frac{\partial \xi^k}{\partial u}\ .$$

Then (4.24) becomes

$$\frac{\partial \eta}{\partial x^j} + \frac{\partial \eta}{\partial u}u_{x^j} = \frac{d}{d\varepsilon}\bar{u}_{\bar{x}^k}\bigg|_{\varepsilon=0}\delta_j^{\ k} + u_{x^k}\Big(\frac{\partial \xi^k}{\partial x^j} + \frac{\partial \xi^k}{\partial u}u_{x^j}\Big)\ . \tag{4.25}$$

Note that the total derivatives of η, τ and ξ^k appear in the last equation, so (4.25) takes the form

$$\frac{d\eta}{dx^j} = \frac{d}{d\varepsilon}\bar{u}_{\bar{x}^j}\bigg|_{\varepsilon=0} + u_{x^k}\frac{d\xi^k}{dx^j}$$

from which we get

$$\frac{d}{d\varepsilon}\bar{u}_{\bar{x}^j}\bigg|_{\varepsilon=0} = \frac{d\eta}{dx^j} - u_{x^k}\frac{d\xi^k}{dx^j}\ . \tag{4.26}$$

Substitution of (4.23) and (4.26) into (4.16) produces

$$\begin{aligned}\frac{d\mathcal{L}}{d\varepsilon}\bigg|_{\varepsilon=0} &= \frac{\partial \mathcal{L}}{\partial t}\tau + \frac{\partial \mathcal{L}}{\partial x^k}\xi^k + \frac{\partial \mathcal{L}}{\partial u}\eta + \frac{\partial \mathcal{L}}{\partial u_t}\Big(\frac{d\eta}{dt} - u_t\frac{d\tau}{dt} - u_{x^k}\frac{d\xi^k}{dt}\Big)\\ &\quad + \frac{\partial \mathcal{L}}{\partial u_{x^k}}\Big(\frac{d\eta}{dx^k} - u_{x^j}\frac{d\xi^j}{dx^k}\Big) + \frac{\partial \mathcal{L}}{\partial z}\zeta\ .\end{aligned} \tag{4.27}$$

By inserting (4.27) into (4.15), we obtain the differential equation for the total variation ζ of the functional z under the transformation (4.3), namely,

$$\begin{aligned}\frac{d\zeta}{dt} &= \int_D\Big(\frac{\partial \mathcal{L}}{\partial t}\tau + \frac{\partial \mathcal{L}}{\partial x^k}\xi^k + \frac{\partial \mathcal{L}}{\partial u}\eta + \frac{\partial \mathcal{L}}{\partial u_t}\Big(\frac{d\eta}{dt} - u_t\frac{d\tau}{dt} - u_{x^k}\frac{d\xi^k}{dt}\Big)\\ &\quad + \frac{\partial \mathcal{L}}{\partial u_{x^k}}\Big(\frac{d\eta}{dx^k} - u_{x^j}\frac{d\xi^j}{dx^k}\Big) + \mathcal{L}\frac{d\xi^j}{dx^j}\Big)dx + \frac{d\tau}{dt}\int_D \mathcal{L}\,dx + \zeta\int_D \frac{\partial \mathcal{L}}{\partial z}dx\ .\end{aligned} \tag{4.28}$$

Its solution ζ evaluated at T is given by

$$\begin{aligned}\exp\Big(-\int_0^T\!\!\int_D \frac{\partial \mathcal{L}}{\partial z}\,dx\,d\theta\Big)\zeta - \zeta(0) &= \int_0^T \exp\Big(-\int_0^t\!\!\int_D \frac{\partial \mathcal{L}}{\partial z}\,dx\,d\theta\Big)\Big(\int_D\Big(\frac{\partial \mathcal{L}}{\partial t}\tau + \frac{\partial \mathcal{L}}{\partial x^k}\xi^k + \frac{\partial \mathcal{L}}{\partial u}\eta\\ &+ \frac{\partial \mathcal{L}}{\partial u_t}\Big(\frac{d\eta}{dt} - u_t\frac{d\tau}{dt} - u_{x^k}\frac{d\xi^k}{dt}\Big) + \frac{\partial \mathcal{L}}{\partial u_{x^k}}\Big(\frac{d\eta}{dx^k} - u_{x^j}\frac{d\xi^j}{dx^k}\Big) + \mathcal{L}\frac{d\xi^j}{dx^j}\Big)dx + \frac{d\tau}{dt}\int_D \mathcal{L}\,dx\Big)dt\end{aligned} \tag{4.29}$$

where T is the value of t at which the solution $z(t)$ of equation (4.1) was evaluated in order to obtain the functional $z[u;T]$. We note that $\zeta(0)=0$. Indeed, as explained earlier, in order to have a well-defined functional z of the function $u(x,t)$ we must evaluate the solution $z(t)$ of the equation (4.1) with the same fixed initial condition $z(0)$ independently of the function $u(x,t)$. So, $z(0)$ is independent of ε. Hence, the variation of z at $t=0$ is $\zeta(0)=d/d\varepsilon\, z[u;0,\varepsilon]\big|_{\varepsilon=0}=d/d\varepsilon\, z(0)\big|_{\varepsilon=0}=0$. Also, since by hypothesis the one–parameter group of transformations (4.3) leaves the functional $z=z[u(t,x);t,\varepsilon]$ stationary, we have $\zeta(t)=0$. Thus, (4.29) becomes

$$\int_0^T \exp\Big(-\int_0^t\!\!\int_D \frac{\partial \mathcal{L}}{\partial z}\,dx\,d\theta\Big)\Big(\int_D\Big(\frac{\partial \mathcal{L}}{\partial t}\tau+\frac{\partial \mathcal{L}}{\partial x^k}\xi^k+\frac{\partial \mathcal{L}}{\partial u}\eta+\frac{\partial \mathcal{L}}{\partial u_t}\Big(\frac{d\eta}{dt}-u_t\frac{d\tau}{dt}-u_{x^k}\frac{d\xi^k}{dt}\Big)$$
$$+\frac{\partial \mathcal{L}}{\partial u_{x^k}}\Big(\frac{d\eta}{dx^k}-u_{x^j}\frac{d\xi^j}{dx^k}\Big)+\mathcal{L}\,\frac{d\xi^j}{dx^j}\Big)dx+\frac{d\tau}{dt}\int_D\mathcal{L}\,dx\Big)\,dt=0\;. \qquad (4.30)$$

To shorten notation in what follows, we will use E for the exponent expression, according to (4.7). Now, the goal is to obtain the generalized Euler-Lagrange expressions

$$\frac{\partial \mathcal{L}}{\partial u}-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t}-\frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}}+\frac{\partial \mathcal{L}}{\partial u_t}\int_D\frac{\partial \mathcal{L}}{\partial z}\,dx$$

under the integral. Let us concentrate on the terms involving $d\eta/dt$ and $d\eta/dx^k$ in (4.30) and form a total divergence (see Definition 9 in the Appendix). We get

$$\int_0^T\!\!\int_D\Big(E\,\eta\Big(\frac{d\mathcal{L}}{du}-\frac{d}{dt}\frac{d\mathcal{L}}{du_t}-\frac{d}{dx^k}\frac{d\mathcal{L}}{du_{x^k}}+\frac{d\mathcal{L}}{du_t}\int_D\frac{\partial \mathcal{L}}{\partial z}\,dx\Big)+\frac{d}{dt}\Big(E\,\eta\,\frac{\partial \mathcal{L}}{\partial u_t}\Big)$$
$$+\frac{d}{dx^k}\Big(E\,\eta\frac{\partial \mathcal{L}}{\partial u_{x^k}}\Big)+E\Big(\frac{\partial \mathcal{L}}{\partial x^k}\,\xi^k-\frac{\partial \mathcal{L}}{\partial u_t}\,\frac{d\xi^k}{dt}u_{x^k}-\frac{\partial \mathcal{L}}{\partial u_{x^k}}\,\frac{d\xi^j}{dx^k}u_{x^j}+\mathcal{L}\,\frac{d\xi^j}{dx^j}\Big)\Big)\,dx\,dt$$
$$+\int_0^T E\,\frac{d\tau}{dt}\int_D\mathcal{L}\,dx\,dt+\int_0^T E\int_D\Big(\frac{\partial \mathcal{L}}{\partial t}\tau-\frac{\partial \mathcal{L}}{\partial u_t}\,\frac{d\tau}{dt}u_t\Big)\,dx\,dt=0$$

which on solutions of the generalized Euler-Lagrange equations becomes

$$\int_0^T\!\!\int_D\Big(\frac{d}{dt}\Big(E\,\eta\frac{\partial \mathcal{L}}{\partial u_t}\Big)+\frac{d}{dx^k}\Big(E\,\eta\frac{\partial \mathcal{L}}{\partial u_{x^k}}\Big)\Big)\,dx\,dt$$
$$+\int_0^T\!\!\int_D E\Big(\frac{\partial \mathcal{L}}{\partial x^k}\xi^k-\frac{\partial \mathcal{L}}{\partial u_t}\,\frac{d\xi^k}{dt}u_{x^k}-\frac{\partial \mathcal{L}}{\partial u_{x^k}}\,\frac{d\xi^j}{dx^k}u_{x^j}+\mathcal{L}\,\frac{d\xi^j}{dx^j}\Big)\,dx\,dt$$
$$+\int_0^T E\,\frac{d\tau}{dt}\int_D\mathcal{L}\,dx\,dt+\int_0^T E\int_D\Big(\frac{\partial \mathcal{L}}{\partial t}\tau-\frac{\partial \mathcal{L}}{\partial u_t}\,\frac{d\tau}{dt}u_t\Big)\,dx\,dt=0\;. \qquad (4.31)$$

Next, let us concentrate on the second integral in (4.31) which involves ξ^k. The goal is to obtain the generalized Euler-Lagrange expressions and total divergences.

$$\begin{aligned}&\int_0^T\!\!\int_D\Big(E\,\xi^k\frac{\partial\mathcal{L}}{\partial x^k}-\frac{d}{dt}\Big(E\,\xi^k u_{x^k}\frac{\partial\mathcal{L}}{\partial u_t}\Big)-\frac{d}{dx^k}\Big(E\,\xi^j u_{x^j}\frac{\partial\mathcal{L}}{\partial u_{x^k}}\Big)+\frac{d}{dx^j}(E\,\mathcal{L}\,\xi^j)\\&+E\,\xi^k\frac{\partial\mathcal{L}}{\partial u_t}u_{tx^k}+E\,\xi^k u_{x^k}\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial u_t}-E\,\xi^k u_{x^k}\frac{\partial\mathcal{L}}{\partial u_t}\int_D\frac{\partial\mathcal{L}}{\partial z}\,dx+E\,\xi^j\frac{\partial\mathcal{L}}{\partial u_{x^k}}u_{x^jx^k}\\&+E\,\xi^j u_{x^j}\frac{d}{dx^k}\frac{\partial\mathcal{L}}{\partial u_{x^k}}-E\,\xi^j\Big(\frac{\partial\mathcal{L}}{\partial x^j}+\frac{\partial\mathcal{L}}{\partial u}u_{x^j}+\frac{\partial\mathcal{L}}{\partial u_{x^k}}u_{x^kx^j}+\frac{\partial\mathcal{L}}{\partial u_t}u_{tx^j}\Big)\Big)\,dx\,dt\\&+\int_0^T E\,\frac{d\tau}{dt}\int_D\mathcal{L}\,dx\,dt+\int_0^T\!\!\int_D\Big(\frac{d}{dt}\Big(E\,\eta\frac{\partial\mathcal{L}}{\partial u_t}\Big)+\frac{d}{dx^k}\Big(E\,\eta\frac{\partial\mathcal{L}}{\partial u_{x^k}}\Big)\Big)\,dx\,dt\\&+\int_0^T E\int_D\Big(\frac{\partial\mathcal{L}}{\partial t}\tau-\frac{\partial\mathcal{L}}{\partial u_t}\frac{d\tau}{dt}u_t\Big)\,dx\,dt=0\,.\end{aligned}\tag{4.32}$$

We cancel three pairs of terms and factor the generalized Euler-Lagrange expressions to obtain:

$$\begin{aligned}&\int_0^T\!\!\int_D\Big(\frac{d}{dt}\Big(E\,\eta\frac{\partial\mathcal{L}}{\partial u_t}\Big)+\frac{d}{dx^k}\Big(E\,\eta\frac{\partial\mathcal{L}}{\partial u_{x^k}}\Big)-\frac{d}{dt}\Big(E\,\xi^k u_{x^k}\frac{\partial\mathcal{L}}{\partial u_t}\Big)-\frac{d}{dx^k}\Big(E\,\xi^j u_{x^j}\frac{\partial\mathcal{L}}{\partial u_{x^k}}\Big)\\&+\frac{d}{dx^j}(E\,\mathcal{L}\,\xi^j)-E\,\xi^j u_{x^j}\Big(\frac{d\mathcal{L}}{du}-\frac{d}{dt}\frac{d\mathcal{L}}{du_t}-\frac{d}{dx^k}\frac{d\mathcal{L}}{du_{x^k}}+\frac{d\mathcal{L}}{du_t}\int_D\frac{\partial\mathcal{L}}{\partial z}\,dx\Big)\Big)\,dx\,dt\\&+\int_0^T E\int_D\Big(\frac{\partial\mathcal{L}}{\partial t}\tau-\frac{\partial\mathcal{L}}{\partial u_t}\frac{d\tau}{dt}u_t\Big)\,dx\,dt+\int_0^T E\,\frac{d\tau}{dt}\int_D\mathcal{L}\,dx\,dt=0\,.\end{aligned}\tag{4.33}$$

Thus, on solutions of the generalized Euler-Lagrange equations, (4.33) becomes

$$\begin{aligned}&\int_0^T\!\!\int_D\Big(\frac{d}{dt}\Big(E\,\eta\frac{\partial\mathcal{L}}{\partial u_t}\Big)+\frac{d}{dx^k}\Big(E\,\eta\frac{\partial\mathcal{L}}{\partial u_{x^k}}\Big)-\frac{d}{dt}\Big(E\,\xi^k u_{x^k}\frac{\partial\mathcal{L}}{\partial u_t}\Big)-\frac{d}{dx^k}\Big(E\,\xi^j u_{x^j}\frac{\partial\mathcal{L}}{\partial u_{x^k}}\Big)\\&+\frac{d}{dx^j}(E\,\mathcal{L}\,\xi^j)\Big)\,dx\,dt+\int_0^T\!\!\int_D E\Big(\frac{\partial\mathcal{L}}{\partial t}\tau-\frac{\partial\mathcal{L}}{\partial u_t}u_t\frac{d\tau}{dt}+\frac{d\tau}{dt}\mathcal{L}\Big)\,dx\,dt=0\,.\end{aligned}\tag{4.34}$$

The last integral in (4.34), involving τ, is transformed as follows:

$$\begin{aligned}&\int_0^T\!\!\int_D E\Big(\frac{\partial\mathcal{L}}{\partial t}\tau-\frac{\partial\mathcal{L}}{\partial u_t}u_t\frac{d\tau}{dt}+\frac{d\tau}{dt}\mathcal{L}\Big)\,dx\,dt\\&=\int_0^T\!\!\int_D\Big(E\tau\frac{\partial\mathcal{L}}{\partial t}-\frac{d}{dt}\Big(E\Big(\frac{\partial\mathcal{L}}{\partial u_t}u_t-\mathcal{L}\Big)\tau\Big)-E\Big(\frac{\partial\mathcal{L}}{\partial u_t}u_t-\mathcal{L}\Big)\tau\int_D\frac{\partial\mathcal{L}}{\partial z}\,dx\\&+E\tau\Big(u_t\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial u_t}+\frac{\partial\mathcal{L}}{\partial u_t}u_{tt}-\frac{d\mathcal{L}}{dt}\Big)\Big)\,dx\,dt\\&=\int_0^T\!\!\int_D\Big(E\tau\Big(u_t\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial u_t}-\frac{\partial\mathcal{L}}{\partial u}u_t-\frac{\partial\mathcal{L}}{\partial u_{x^k}}u_{x^kt}-\frac{\partial\mathcal{L}}{\partial z}\int_D\mathcal{L}\,dx\\&+\mathcal{L}\int_D\frac{\partial\mathcal{L}}{\partial z}dx-\frac{\partial\mathcal{L}}{\partial u_t}u_t\int_D\frac{\partial\mathcal{L}}{\partial z}dx\Big)-\frac{d}{dt}\Big(E\tau\Big(\frac{\partial\mathcal{L}}{\partial u_t}u_t-\mathcal{L}\Big)\Big)\Big)\,dx\,dt\,.\end{aligned}\tag{4.35}$$

After integrating the term $-u_{x^k t}\,\partial\mathcal{L}/\partial u_{x^k}$ by parts, (4.35) becomes

$$\int_0^T\!\!\int_D\Big(E\,\tau\,u_t\Big(-\frac{d\mathcal{L}}{du}+\frac{d}{dt}\frac{d\mathcal{L}}{du_t}+\frac{d}{dx^k}\frac{d\mathcal{L}}{du_{x^k}}-\frac{d\mathcal{L}}{du_t}\int_D\frac{\partial\mathcal{L}}{\partial z}\,dx\Big)-E\tau\frac{d}{dx^k}\Big(\frac{\partial\mathcal{L}}{\partial u_{x^k}}u_t\Big)$$

$$+E\tau\Big(\mathcal{L}\int_D\frac{\partial\mathcal{L}}{\partial z}\,dx-\frac{\partial\mathcal{L}}{\partial z}\int_D\mathcal{L}\,dx\Big)-\frac{d}{dt}\Big(E\tau\Big(\frac{\partial\mathcal{L}}{\partial u_t}u_t-\mathcal{L}\Big)\Big)\Big)\,dx\,dt\ . \tag{4.36}$$

We next observe that

$$\int_D\Big(\mathcal{L}\int_D\frac{\partial\mathcal{L}}{\partial z}dx-\frac{\partial\mathcal{L}}{\partial z}\int_D\mathcal{L}\,dx\Big)dx=\int_D\frac{\partial\mathcal{L}}{\partial z}\,dx\int_D\mathcal{L}\,dx-\int_D\mathcal{L}\,dx\int_D\frac{\partial\mathcal{L}}{\partial z}\,dx=0\ .$$

So on solutions of the generalized Euler-Lagrange equations, (4.36) becomes

$$-\int_0^T\!\!\int_D\Big(\frac{d}{dt}\Big(E\tau\Big(\frac{\partial\mathcal{L}}{\partial u_t}u_t-\mathcal{L}\Big)\Big)+\frac{d}{dx^k}\Big(E\tau\frac{\partial\mathcal{L}}{\partial u_{x^k}}u_t\Big)\Big)dx\,dt\ .$$

Thus, for the last integral in (4.34) we have

$$\begin{aligned}&\int_0^T\!\!\int_D E\Big(\frac{\partial\mathcal{L}}{\partial t}\tau-\frac{\partial\mathcal{L}}{\partial u_t}\frac{d\tau}{dt}u_t+\frac{d\tau}{dt}\mathcal{L}\Big)\,dx\,dt\\&=-\int_0^T\!\!\int_D\Big(\frac{d}{dt}\Big(E\tau\Big(\frac{\partial\mathcal{L}}{\partial u_t}u_t-\mathcal{L}\Big)\Big)+\frac{d}{dx^k}\Big(E\tau\frac{\partial\mathcal{L}}{\partial u_{x^k}}u_t\Big)\Big)dx\,dt\end{aligned}$$

which we substitute into (4.34) to obtain

$$\int_0^T\!\!\int_D\Big(\frac{d}{dt}\Big(E\,\eta\frac{\partial\mathcal{L}}{\partial u_t}\Big)+\frac{d}{dx^k}\Big(E\,\eta\frac{\partial\mathcal{L}}{\partial u_{x^k}}\Big)-\frac{d}{dt}\Big(E\,\xi^k u_{x^k}\frac{\partial\mathcal{L}}{\partial u_t}\Big)-\frac{d}{dx^k}\Big(E\,\xi^j u_{x^j}\frac{\partial\mathcal{L}}{\partial u_{x^k}}\Big)$$

$$+\frac{d}{dx^j}(E\,\mathcal{L}\,\xi^j)-\frac{d}{dt}\Big(E\tau\Big(\frac{\partial\mathcal{L}}{\partial u_t}u_t-\mathcal{L}\Big)\Big)-\frac{d}{dx^k}\Big(E\tau\frac{\partial\mathcal{L}}{\partial u_{x^k}}u_t\Big)\Big)\,dx\,dt=0\ . \tag{4.37}$$

Since T is arbitrary, it follows from the above equation that the identity

$$\begin{aligned}&\int_D\Big(\frac{d}{dt}\Big(E\,\Big(\Big(\frac{\partial\mathcal{L}}{\partial u_t}u_t-\mathcal{L}\Big)\tau+\frac{\partial\mathcal{L}}{\partial u_t}\big(\xi^j u_{x^j}-\eta\big)\Big)\Big)\\&+\frac{d}{dx^k}\Big(E\,\Big(\frac{\partial\mathcal{L}}{\partial u_{x^k}}u_t\,\tau-\mathcal{L}\,\xi^k+\frac{\partial\mathcal{L}}{\partial u_{x^k}}\big(\xi^j u_{x^j}-\eta\big)\Big)\Big)\Big)dx=0\end{aligned}$$

holds on solutions of the generalized Euler-Lagrange equations where D is any subdomain of Ω, including Ω itself, with closure D^{cl} contained in the closure of Ω. This completes the proof of the theorem.

Corollary 4.2. *Theorem 4.1 reduces to the classical first Noether theorem when the generalized variational principle with several independent variables reduces to the classical variational principle with several independent variables.*

Proof: Consider the defining integro-differential equation

$$\frac{dz}{dt} = \int_{\Omega} \mathcal{L}(t, x, u(t,x), u_t, u_x, z)\, dx \tag{4.38}$$

for the functional z, where $x = (x^1, \ldots, x^n)$ and $u_x = (u_{x^1}, \ldots, u_{x^n})$. The generalized Euler–Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}} + \frac{\partial \mathcal{L}}{\partial u_t}\int_{\Omega}\frac{\partial \mathcal{L}}{\partial z}\, dx = 0 \tag{4.39}$$

provide the extrema of z. Identity (4.6), which Theorem 4.1 provides, holds on solutions of the generalized Euler-Lagrange equations, where D is *any* subdomain of Ω, including Ω itself, with closure $D^{cl} \subset \Omega^{cl}$. If $\mathcal{L}$ does not depend on z the integrand in (4.6) is independent of D. Then, the arbitrariness of D in the identity (4.6) implies that the integrand is identically zero. Moreover, in this case, $E = 1$ as seen from (4.7). Hence, the conservation law

$$\begin{aligned}
&\frac{d}{dt}\left(\left(\frac{\partial \mathcal{L}}{\partial u_t}u_t - \mathcal{L}\right)\tau + \frac{\partial \mathcal{L}}{\partial u_t}\left(\xi^j u_{x^j} - \eta\right)\right) \\
&+ \frac{d}{dx^k}\left(\frac{\partial \mathcal{L}}{\partial u_{x^k}}u_t\,\tau - \mathcal{L}\,\xi^k + \frac{\partial \mathcal{L}}{\partial u_{x^k}}\left(\xi^j u_{x^j} - \eta\right)\right) = 0\,.
\end{aligned} \tag{4.40}$$

holds on solutions of the classical Euler-Lagrange equations. Since $\mathcal{L}$ does not depend on z, (4.38) reduces to the classical definition of a functional by an integral and (4.40) is precisely the conservation law provided by the classical first Noether theorem.

5. CONSERVED QUANTITIES IN DISSIPATIVE AND GENERATIVE SYSTEMS WITH SEVERAL INDEPENDENT VARIABLES

Physical systems described by the generalized Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}} + \frac{\partial \mathcal{L}}{\partial u_t}\int_{\Omega}\frac{\partial \mathcal{L}}{\partial z}\, dx = 0 \tag{5.1}$$

with several independent variables are not in general conservative. The physical field which evolves according to these equations is represented by the function $u = u(t, x)$. It can describe an electromagnetic or a gravitational field, a temperature distribution of a body, a flow of a gas or liquid, and so on. In this section we show how Theorem 4.1 can be used to find conserved quantities in such systems. To carry out this procedure, we must find the symmetries of the functional z defined by

$$\frac{dz}{dt} = \int_\Omega \mathcal{L}(t, x, u(t,x), u_t, u_x, z)\, dx \ . \tag{5.2}$$

Each one-parameter symmetry provides one conserved quantity. To test whether a one-parameter group of transformations of the independent and dependent variables is a symmetry group of the functional z we use the infinitesimal criterion which Proposition 2.3 provides.

Corollary 5.1. *Let*

$$\begin{aligned} \bar{t} &= \phi(t, \varepsilon) \\ \bar{x}^k &= \varphi^k(t, x, u, \varepsilon), \qquad k = 1, \dots n \\ \bar{u} &= \psi(t, x, u, \varepsilon) \ , \end{aligned} \tag{5.3}$$

be a symmetry group for the functional z defined by (5.2), with infinitesimal generator

$$\mathbf{v} = \tau(t)\frac{\partial}{\partial t} + \xi^i(t, x, u)\frac{\partial}{\partial x^i} + \eta(t, x, u)\frac{\partial}{\partial u} \ . \tag{5.4}$$

If

$$\frac{\partial \mathcal{L}}{\partial u_{x^k}} u_t \tau - \mathcal{L}\xi^k + \frac{\partial \mathcal{L}}{\partial u_{x^k}}\Big(\xi^j u_{x^j} - \eta\Big) = 0 \ , \quad k = 1, \dots, n$$

on the boundary $\partial\Omega$ of Ω, then the quantity

$$\int_\Omega E\bigg(\Big(\frac{\partial \mathcal{L}}{\partial u_t} u_t - \mathcal{L}\Big)\tau + \frac{\partial \mathcal{L}}{\partial u_t}\Big(\xi^j u_{x^j} - \eta\Big)\bigg) dx \tag{5.5}$$

is conserved on solutions of the generalized Euler-Lagrange equations (5.1) of the functional z, where E is

$$E \equiv \exp\Big(-\int_0^t \int_\Omega \frac{\partial \mathcal{L}}{\partial z}\, dx\, d\theta\Big) \ . \tag{5.6}$$

Proof: Consider the identity (4.6) which Theorem 4.1 provides. Since this identity holds for every subdomain D of Ω, including Ω itself, we have

$$\int_{\Omega}\left(\frac{d}{dt}\left(E\left(\left(\frac{\partial\mathcal{L}}{\partial u_t}u_t-\mathcal{L}\right)\tau+\frac{\partial\mathcal{L}}{\partial u_t}\left(\xi^j u_{x^j}-\eta\right)\right)\right)\right.$$
$$\left.+\frac{d}{dx^k}\left(E\left(\frac{\partial\mathcal{L}}{\partial u_{x^k}}u_t\tau-\mathcal{L}\xi^k+\frac{\partial\mathcal{L}}{\partial u_{x^k}}\left(\xi^j u_{x^j}-\eta\right)\right)\right)\right)dx=0$$

on solutions of the generalized Euler-Lagrange equations (5.1). An application of the Gauss theorem yields

$$\frac{d}{dt}\int_{\Omega}E\left(\left(\frac{\partial\mathcal{L}}{\partial u_t}u_t-\mathcal{L}\right)\tau+\frac{\partial\mathcal{L}}{\partial u_t}\left(\xi^j u_{x^j}-\eta\right)\right)dx$$
$$+\int_{\partial\Omega}E\left(\frac{\partial\mathcal{L}}{\partial u_x}u_t\tau-\mathcal{L}\xi+\frac{\partial\mathcal{L}}{\partial u_x}\left(\xi^j u_{x^j}-\eta\right)\right)\cdot N dA=0\ , \tag{5.7}$$

where

$$\frac{\partial\mathcal{L}}{\partial u_x}u_t\tau-\mathcal{L}\xi+\frac{\partial\mathcal{L}}{\partial u_x}\left(\xi^j u_{x^j}-\eta\right)$$

stands for the vector with components

$$\frac{\partial\mathcal{L}}{\partial u_{x^k}}u_t\tau-\mathcal{L}\xi^k+\frac{\partial\mathcal{L}}{\partial u_{x^k}}\left(\xi^j u_{x^j}-\eta\right)\ ,\quad k=1,\ldots,n\ ,$$

N is the unit normal vector to the boundary $\partial\Omega$ and dA is an element of $\partial\Omega$. Since by hypothesis the quantities

$$\frac{\partial\mathcal{L}}{\partial u_{x^k}}u_t\tau-\mathcal{L}\xi^k+\frac{\partial\mathcal{L}}{\partial u_{x^k}}\left(\xi^j u_{x^j}-\eta\right)\ ,\quad k=1,\ldots,n$$

are all zeros when evaluated on the boundary $\partial\Omega$, (5.7) becomes

$$\frac{d}{dt}\int_{\Omega}E\left(\left(\frac{\partial\mathcal{L}}{\partial u_t}u_t-\mathcal{L}\right)\tau+\frac{\partial\mathcal{L}}{\partial u_t}\left(\xi^j u_{x^j}-\eta\right)\right)dx=0\ .$$

The statement of the corollary follows immediately from the above equation.

The following four corollaries are specific cases in which Corollary 5.1 holds.

Corollary 5.2. *Let the group of transformations*

$$\bar{t}=\phi(t,\varepsilon)$$
$$\bar{x}^k=\varphi^k(t,x,u,\varepsilon),\qquad k=1,\ldots n$$
$$\bar{u}=\psi(t,x,u,\varepsilon)\ ,$$

with infinitesimal generator

$$\mathbf{v} = \tau(t)\frac{\partial}{\partial t} + \xi^i(t,x,u)\frac{\partial}{\partial x^i} + \eta(t,x,u)\frac{\partial}{\partial u}$$

be a symmetry group of the functional z *, defined by*

$$\frac{dz}{dt} = \int_{\mathbb{R}^n} \mathcal{L}(t,x,u(t,x),u_t,u_x,z)\,dx \ .$$

If

$$\left|\int_{\mathbb{R}^n} \mathcal{L}(t,x,u(t,x),u_t,u_x,z)\,dx\right| < \infty \tag{5.8}$$

then the quantity

$$\int_{\mathbb{R}^n} E\left(\left(\frac{\partial \mathcal{L}}{\partial u_t}u_t - \mathcal{L}\right)\tau + \frac{\partial \mathcal{L}}{\partial u_t}\left(\xi^j u_{x^j} - \eta\right)\right)dx$$

is conserved on solutions of the generalized Euler-Lagrange equations (5.1) of the functional z *defined by (5.2), where* $x \equiv (x^1,\ldots,x^n)$ *and* E *is*

$$E \equiv \exp\left(-\int_0^t\int_{\mathbb{R}^n} \frac{\partial \mathcal{L}}{\partial z}\,dx\,d\theta\right) .$$

Proof: The requirement (5.8) implies that $\lim_{|x|\to\infty} \mathcal{L} = 0$. Then $\partial\mathcal{L}/\partial u_{x^i}$, $i = 1,\ldots,n$, are also zero at infinity. Thus, the hypotheses of Corollary 5.1 are satisfied.

Corollary 5.3. *Let the functional* z *defined by the equation*

$$\frac{dz}{dt} = \int_\Omega \mathcal{L}(x,u(t,x),u_t,u_x,z)\,dx$$

be invariant with respect to translations in time $\bar{t} = t+\varepsilon$, $\bar{x}^i = x^i$, $i = 1,\ldots,n$, $\bar{u} = u$. *Then the quantity*

$$\int_\Omega \exp\left(-\int_0^t\int_\Omega \frac{\partial \mathcal{L}}{\partial z}\,dx\,d\theta\right)\left(\frac{\partial \mathcal{L}}{\partial u_t}u_t - \mathcal{L}\right)dx$$

is conserved on solutions of the generalized Euler-Lagrange equations (5.1) with the boundary condition $u = 0$ *on* $\partial\Omega$.

Proof: By Proposition 2.3, of this Chapter, follows that $\partial\mathcal{L}/\partial t = 0$. The infinitesimal generator of the group of time translations is $\partial/\partial t$. Apply Corollary 5.1 with

$$\tau = \frac{d\bar{t}}{d\varepsilon}\Big|_{\varepsilon=0} = 1\,, \quad \xi^i = \frac{d\bar{x}^i}{d\varepsilon}\Big|_{\varepsilon=0} = 0\,, \quad i=1,\ldots,n\,, \quad \eta = \frac{d\bar{u}}{d\varepsilon}\Big|_{\varepsilon=0} = 0\,.$$

Since $u = 0$ and $u_t = 0$ on $\partial\Omega$ the hypotheses of Corollary 5.1 are satisfied.

More generally, we have

Corollary 5.4. *Let the functional z defined by the equation*

$$\frac{dz}{dt} = \int_\Omega \mathcal{L}(t,x,u(t,x),u_t,u_x,z)\,dx$$

be invariant with respect to the transformation $\bar{t} = \phi(t,\varepsilon)$, $\bar{x}^i = x^i$, $i=1,\ldots,n$, $\bar{u} = u$. Then the quantity

$$\int_\Omega \exp\Big(-\int_0^t\int_\Omega \frac{\partial\mathcal{L}}{\partial z}\,dx\,d\theta\Big)\Big(\frac{\partial\mathcal{L}}{\partial u_t}u_t - \mathcal{L}\Big)\Big(\frac{\partial\phi}{\partial\varepsilon}\Big|_{\varepsilon=0}\Big)\,dx$$

is conserved on solutions of the generalized Euler-Lagrange equations (5.1) with a boundary condition $u = 0$ on $\partial\Omega$.

Proof: Apply Corollary 5.1 with

$$\xi^i = \frac{d\bar{x}^i}{d\varepsilon}\Big|_{\varepsilon=0} = 0\,, \quad i=1,\ldots,n\,, \quad \eta = \frac{d\bar{u}}{d\varepsilon}\Big|_{\varepsilon=0} = 0\,.$$

Since $u = 0$ and $u_t = 0$ on $\partial\Omega$ the hypotheses of Corollary 5.1 are satisfied.

Example:

Consider the two-dimensional damped wave equation

$$u_{tt} + u_t = u_{xx} + u_{yy} \tag{5.9}$$

on a subdomain Ω of $\mathbb{R}^2$.

This equation is the generalized Euler-Lagrange equation obtained from the generalized variational principle with several independent variables with Lagrangian $\mathcal{L} = (u_x^2 + u_y^2 - u_t^2)/2 + \alpha(x,y)z$, where the function α satisfies the conditions

$$\iint_\Omega \alpha(x,y)\,dx\,dy = -1 \qquad \text{and} \qquad \alpha = 0 \ \text{ on } \partial\Omega\,.$$

Since the Lagrangian $\mathcal{L}$ is independent of time, we know from Proposition 2.3, in this chapter, that time-translations form a symmetry group of the functional z defined by the equation

$$\frac{dz}{dt} = \int_{\Omega} \Big(\frac{1}{2}(u_x^2 + u_y^2 - u_t^2) + \alpha(x,y)z \Big)\, dx \ .$$

Then Corollary 5.3 asserts that the quantity

$$e^t \int_{\Omega} \Big(\frac{\partial \mathcal{L}}{\partial u_t} u_t - \mathcal{L} \Big) dx = -\frac{1}{2} e^t \int_{\Omega} (u_t^2 + u_x^2 + u_y^2)\, dx\, dy + e^t z$$

is conserved on solutions of the damped wave equation with a boundary condition $u = 0$ on $\partial\Omega$.

6. FURTHER APPLICATIONS OF THE FIRST NOETHER–TYPE THEOREM FOR THE GENERALIZED VARIATIONAL PRINCIPLE WITH SEVERAL INDEPENDENT VARIABLES

Consider the equation

$$\nabla^2 u - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} + G(uu^*)u = 0 \tag{6.1}$$

for the real or complex field $u = u(x,t)$, where u^* denotes the complex conjugate of u, G is a non-constant differentiable function and v is a constant. This equation is known in physics as the Nonlinear Klein-Gordon equation. It plays an important role in relativistic field theories. The linear version of the same equation, which is obtained from (6.1) when G = constant, is the Klein-Gordon equation. It is the basic equation in early relativistic quantum mechanics. The one-dimensional version of (6.1) with real u and $G(u^2)u = \sin u$ is the well known Sine-Gordon equation in Soliton Theory. Its space-localized solutions are solitons.

Any non-linear wave equation of the form (6.1) for the real or complex field $u = u(x,t)$ can be derived from the Lagrangian

$$\mathcal{L}(u, u_t, \nabla u) = \nabla u \cdot \nabla u^* - \frac{1}{v^2} \frac{\partial u}{\partial t} \frac{\partial u^*}{\partial t} - F(uu^*) \tag{6.2}$$

where

$$\frac{dF(\rho)}{d\rho} = G(\rho) \quad \text{and} \quad F(0) = 0 \ .$$

We consider (physically meaningful) only those solutions of (6.1) for which

$$\left| \int_\Omega \mathcal{L}(t, x, u, u_t, u_x)\, dx \right| < \infty .$$

In the case of an unbounded domain Ω, motivated by physical considerations, we only consider those solutions of (6.1) which have no singularities, i.e., for which there is no point $y \in \Omega$ such that $\lim_{x \to y} |u(t,x)| = \infty$.

The fact that equation (6.1) is derivable from a Lagrangian which does not depend on time explicitly shows that all processes described with an equation of the form (6.1) are conservative.

In physics and engineering one is also interested in non-conservative processes involving fields. The simplest modification of (6.1) which makes it non-conservative is to add a term proportional to the time-derivative of the field. Thus, a non-conservative version of (6.1) is

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} + G(uu^*)u = 0 , \tag{6.3}$$

where k is a constant. When $k > 0$ the process described by (6.3) is generative, and when $k < 0$ it is dissipative.

Equations of the form (6.3) cannot be derived from a classical time-independent variational principle. However, if u is a real field, $\partial F(\rho)/\partial \rho = G(\rho)$, and $\alpha = \alpha(x)$ is a given function of the coordinates $x = (x^1, \ldots, x^n)$ with

$$\left| \int_\Omega \alpha(x)\, dx \right| < \infty ,$$

then (6.3) can be derived via the generalized variational principle with several independent variables from the Lagrangian

$$\mathcal{L} = \nabla u \cdot \nabla u - \frac{1}{v^2}\left(\frac{\partial u}{\partial t}\right)^2 - F(u^2) + \alpha(x)\, z . \tag{6.4}$$

Indeed, insert the Lagrangian (6.4) into the generalized Euler-Lagrange equations (3.2):

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}} + \frac{\partial \mathcal{L}}{\partial u_t}\int_\Omega \frac{\partial \mathcal{L}}{\partial z}\, dx \\
&= -2\, u\,\frac{\partial F}{\partial u^2} + \frac{2}{v^2}\frac{\partial^2 u}{\partial t^2} - 2\,\nabla u \cdot \nabla u - \frac{2}{v^2}\frac{\partial u}{\partial t}\int_\Omega \alpha(x)\, dx = 0 .
\end{aligned}$$

The last expression is the same as (6.3) with

$$k = \frac{1}{v^2} \int_\Omega \alpha(x)\, dx = \text{constant} \; . \tag{6.5}$$

Consequently, we may apply the first Noether-type theorem 4.1 to obtain conserved quantities. In particular, observing that the Lagrangian (6.4) is invariant under time-translations we may apply Corollary 5.3 to obtain the conserved quantity

$$\int_\Omega \exp\Big(-\int_0^t \int_\Omega \frac{\partial \mathcal{L}}{\partial z}\, dx\, d\theta \Big) \Big(\frac{\partial \mathcal{L}}{\partial u_t} u_t - \mathcal{L} \Big)\, dx = \text{constant}$$

which, after inserting (6.4) for $\mathcal{L}$, becomes

$$\exp(-kv^2 t) \int_\Omega \Big(\frac{1}{v^2} \Big(\frac{\partial u}{\partial t} \Big)^2 + \nabla u \cdot \nabla u - F(u^2) + \alpha(x) z \Big)\, dx = \text{constant} \; , \tag{6.6}$$

where z is the solution of the defining equation (4.1).

In accordance with the conservative case, the quantity

$$\int_\Omega \Big(\frac{\partial \mathcal{L}}{\partial u_t} u_t - \mathcal{L} \Big)\, dx \equiv - \int_\Omega \Big(\frac{1}{v^2} \Big(\frac{\partial u}{\partial t} \Big)^2 + \nabla u \cdot \nabla u - F(u^2) + \alpha(x) z \Big)\, dx \tag{6.7}$$

can be interpreted as the total energy of the field $u(t,x)$. Similarly, the integrand of (6.7) can be interpreted as the total energy density of the field $u(t,x)$. We see that these quantities are not constant, they increase or decrease exponentially in time as seen from (6.6).

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APPENDIX

APPENDIX

In this Appendix we state a few well known definitions and theorems for the sake of quick reference.

Definition 0.1 An *r-parameter Lie group* is a group G which also carries the structure of an r-dimensional manifold in such a way that both the group operation

$$m : G \times G \to G, \qquad m(g,h) = g \cdot h, \qquad g,h \in G,$$

and the inversion

$$i : G \to G, \qquad i(g) = g^{-1}, \qquad g \in G,$$

are smooth maps between manifolds.

Let $TM|_x$ denote the tangent space to a manifold M at the point x.

Definition 0.2 A *vector field* $\mathbf{v}$ on a manifold M is a map which assigns a tangent vector $\mathbf{v}|_x \in TM|_x$ to each point $x \in M$, with $\mathbf{v}|_x$ varying smoothly from point to point. In local coordinates $(x^1, \ldots, x^n)$, a vector field has the form

$$\mathbf{v}|_x = \xi^1(x)\frac{\partial}{\partial x^1} + \ldots + \xi^n(x)\frac{\partial}{\partial x^n}\ , \tag{0.1}$$

where each $\xi^i(x)$ is a smooth function of x.

Definition 0.3 An *integral curve* of a vector field $\mathbf{v}$ is a smooth parametrized curve $x = \phi(\varepsilon)$ whose tangent vector at any point coincides with the value of $\mathbf{v}$ at the same point:

$$\frac{d\phi(\varepsilon)}{d\varepsilon} = \mathbf{v}|_{\phi(\varepsilon)}$$

for all ε.

In local coordinates, $x = \phi(\varepsilon) = (\phi^1(\varepsilon), \ldots, \phi^n(\varepsilon))$ must be a solution to the autonomous system of ordinary differential equations

$$\frac{dx^i}{d\varepsilon} = \xi^i(x)\ , \quad i = 1, \ldots, n, \tag{0.2}$$

where the $\xi^i(x)$ are the coefficients of $\mathbf{v}$ at x. For $\xi^i(x)$ smooth, the standard existence and uniqueness theorems for systems of ordinary differential equations guarantee that there is a unique solution to this system for each set of initial data

$$\phi(0) = x_0\ . \tag{0.3}$$

This implies the existence of a unique maximal integral curve $\phi : I \to M$ passing through a given point $x_0 = \phi(0) \in M$. See Olver, [5].

Definition 0.4 If $\mathbf{v}$ is a vector field, we denote the parametrized maximal integral curve passing through x in M by $\Psi(\varepsilon, x)$ and call Ψ the *flow* generated by $\mathbf{v}$.

Thus for each $x \in M$, and ε in some interval I_x containing 0, $\Psi(\varepsilon, x)$ will be a point on the integral curve passing through x in M. The flow of a vector field has the basic properties:

$$\Psi(\delta, \Psi(\varepsilon, x)) = \Psi(\delta + \varepsilon, x), \quad x \in M, \tag{0.4}$$

for all real δ and ε such that both sides of the equation are defined,

$$\Psi(0, x) = x \ , \tag{0.5}$$

and

$$\frac{d}{d\varepsilon}\Psi(\varepsilon, x) = \mathbf{v}|_{\Psi(\varepsilon,x)} \tag{0.6}$$

for all ε where defined. Here (0.6) states that $\mathbf{v}$ is tangent to the curve $\Psi(\varepsilon, x)$ for a fixed x, and (0.5) gives the initial conditions for this integral curve.

The flow generated by a vector field is the same as a local group action of the Lie group $\mathbb{R}$ on the manifold M, often called a *one-parameter group of transformations*. The vector field $\mathbf{v}$ is called the *infinitesimal generator* of the action since by Taylor's theorem, in local coordinates

$$\Psi(\varepsilon, x) = x + \varepsilon\xi(x) + O(\varepsilon^2) \ ,$$

where $\xi = (\xi^1, \ldots, \xi^n)$ are the coefficients of $\mathbf{v}$. The orbits of the one-parameter group action are the maximal integral curves of the vector field $\mathbf{v}$. Conversely, if $\Psi(\varepsilon, x)$ is any one-parameter group of transformations acting on M, then its infinitesimal generator is obtained by specializing (0.6) at $\varepsilon = 0$:

$$\mathbf{v}|_x = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Psi(\varepsilon, x) \ . \tag{0.7}$$

Uniqueness of solutions to (0.2), (0.3) guarantees that the flow generated by $\mathbf{v}$ coincides with the given local action of $\mathbb{R}$ on M on the common domain of definition. Thus, there is a one-to-one correspondence between local one-parameter groups of transformations and their infinitesimal generators.

The computation of the flow or *one-parameter group* generated by a given vector field $\mathbf{v}$ (in other words, solving the system of ordinary differential equations) is often referred to as *exponentiation* of the vector field. The notation is

$$\exp(\varepsilon\mathbf{v})x \equiv \Psi(\varepsilon, x) \ .$$

In terms of this notation properties (0.4), (0.5) and (0.6) can be written as

$$\exp[(\delta+\varepsilon)\mathbf{v}]x = \exp(\delta\mathbf{v})\exp(\varepsilon\mathbf{v})x$$

whenever defined,

$$\exp(0\mathbf{v})x = x \ ,$$

and

$$\frac{d}{d\varepsilon}[\exp(\varepsilon\mathbf{v})x] = \mathbf{v}|_{\exp(\varepsilon\mathbf{v})x}$$

for all $x \in M$. These properties mirror the properties of the usual exponential function, which justifies the notation.

Let $\mathbf{v}$ be a vector field on M and $f : M \to \mathbb{R}$ a smooth function. We are interested in seeing how f changes under the flow generated by $\mathbf{v}$, meaning we look at $f(\exp(\varepsilon\mathbf{v}))$ as ε varies. In local coordinates, if $\mathbf{v} = \Sigma\, \xi^i(x)\partial/\partial x^i$, then using the chain rule and (0.6) we find

$$\begin{aligned}\frac{d}{d\varepsilon}f(\exp(\varepsilon\mathbf{v})x) &= \sum_{i=1}^{n} \xi^i(\exp(\varepsilon\mathbf{v})x)\frac{\partial f}{\partial x^i}(\exp(\varepsilon\mathbf{v})x) \\ &\equiv \mathbf{v}(f)[\exp(\varepsilon\mathbf{v})x] \ .\end{aligned}$$

In particular, at $\varepsilon = 0$,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(\exp(\varepsilon\mathbf{v})x) = \sum_{i=1}^{n} \xi^i(x)\frac{\partial f}{\partial x^i}(x) = \mathbf{v}(f)(x) \ ,$$

i.e., the vector field $\mathbf{v}$ acts as a first order partial differential operator on real-valued functions $f(x)$ on M. Furthermore, by Taylor's theorem, $f(\exp(\varepsilon\mathbf{v})x) = f(x) + \varepsilon\,\mathbf{v}(f)(x) + O(\varepsilon^2)$, so $\mathbf{v}(f)$ gives the *infinitesimal change* in the function f under the flow generated by $\mathbf{v}$.

A symmetry group of a system of differential equations is a maximal local group of transformations acting on the independent and dependent variables of the system with the property that it transforms solutions of the system to other solutions.

Definition 0.5 Let be a system of differential equations. A *symmetry group* of the system is a local group of transformations G acting on an open subset M of the space of independent and dependent variables for the system with the property that whenever $u = f(x)$ is a solution of , and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system.

Given a smooth function $u = f(x)$, $f\colon X \to U$, where X and U are the spaces of the independent and dependent variables, there is an induced function $u^{(m)} = pr^{(m)} f(x)$, called the *m-th prolongation of f*. The function $pr^{(m)} f(x)\colon X \to U^{(m)}$ and for each x in X, $\mathrm{pr}^{(m)} f(x)$ is a vector whose entries represent the values of f and all its derivatives up to order m at the point x. For example, in the case $u = f(x,y)$, the second prolongation $u^{(2)} = \mathrm{pr}^{(2)} f(x,y) = (u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$, all evaluated at (x,y).

A smooth solution of the system of differential equations

$$\Delta_\nu(x, u^{(m)}) = 0 \,, \qquad \nu = 1, \ldots, l \,,$$

involving $x = (x^1, \ldots, x^n)$, $u = (u^1, \ldots, u^q)$ and the derivatives of u up to order m, where the functions $\Delta(x, u^{(m)}) = (\Delta_1(x, u^{(m)}), \ldots, \Delta_l(x, u^{(m)}))$ are assumed to be smooth, is a smooth function $u = f(x)$ such that

$$\Delta_\nu(x, \mathrm{pr}^{(m)} f(x)) = 0 \,, \qquad \nu = 1, \ldots, l \,,$$

whenever x lies in the domain of f. This is just a restatement of the fact that the derivatives of f must satisfy the algebraic constraints imposed by the system of differential equations. This condition is equivalent to the statement that the graph $\Gamma_f^{(m)}$ of the prolongation $\mathrm{pr}^{(m)} f(x)$ must lie entirely within the subvariety $\mathcal{S}_\Delta$ determined by the system:

$$\Gamma_f^{(m)} \equiv \{(x, \mathrm{pr}^{(m)} f(x))\} \subset \mathcal{S}_\Delta = \{(x, u^{(m)}) : \Delta(x, u^{(m)}) = 0\} .$$

Suppose G is a local group of transformations acting on an open subset M of the space of independent and dependent variables. There is an induced local action of G on the m jet space $M^{(m)}$, called the *m-th prolongation* of G, (or more correctly, the m-th prolongation of the action of G on M) and denoted $\mathrm{pr}^{(m)} G$. This prolongation is defined so that it transforms the derivatives of functions $u = f(x)$ into the corresponding derivatives of the transformed function $\tilde{u} = \tilde{f}(\tilde{x})$. If g is an element of G sufficiently near the identity, the transformed function $g \cdot f$ is defined in a neighborhood of the corresponding point $(\tilde{x}_0, \tilde{u}_0) \equiv g \cdot (x_0, u_0)$, with $u_0 = f(x_0)$ being the zeroth order components of $u_0^{(m)}$. We then determine the action of the prolonged group transformation $\mathrm{pr}^{(m)} g$ on the point $(x_0, u_0^{(m)})$ by evaluating the derivatives of the transformed function $g \cdot f$ at $\tilde{x}_0$; explicitely

$$\mathrm{pr}^{(m)} g \cdot (x_0, u_0^{(m)}) = (\tilde{x}_0, \tilde{u}_0^{(m)})$$

where $\tilde{u}_0^{(m)} \equiv \mathrm{pr}^{(m)} (g \cdot f)(\tilde{x}_0)$.

Definition 0.6 Let M be an open subset of the independent and dependent variables and suppose $\mathbf{v}$ is a vector field on M with corresponding (local) one-parameter group $\exp(\varepsilon \mathbf{v})$. The m-th *prolongation* of $\mathbf{v}$, denoted $\mathrm{pr}^{(m)} \mathbf{v}$, will be a vector field on the m-jet space $M^{(m)}$, and is defined to be the infinitesimal generator of the corresponding prolonged one-parameter group $\mathrm{pr}^{(m)}[\exp(\varepsilon \mathbf{v})]$. In other words,

$$\mathrm{pr}^{(m)} \mathbf{v}|_{(x, u^{(m)})} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{pr}^{(m)} [\exp(\varepsilon \mathbf{v})](x, u^{(m)}) \qquad (0.8)$$

for any $(x, u^{(m)}) \in M^{(m)}$.

The explicit formula for the m-th prolongation of a vector field

$$\mathbf{v} = \xi^i(x, u) \frac{\partial}{\partial x^i} + \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

is given by

$$\mathrm{pr}^{(m)}\mathbf{v} = \mathbf{v} + \phi_\alpha^J(x, u^{(m)})\frac{\partial}{\partial u_J^\alpha} \ , \tag{0.9}$$

with the summation convention on repeated indices assumed to hold here and for the rest of the appendix, the second summation being over all (unordered) multi-indices J. The coefficient functions ϕ_α^J of $\mathrm{pr}^{(m)}\mathbf{v}$ are given by the following formula:

$$\phi_\alpha^J(x, u^{(m)}) = D_J(\phi_\alpha - \xi^i u_i^\alpha) + \xi^i u_{J,i}^\alpha$$

where $u_i^\alpha = \partial u^\alpha / \partial x^i$ and $u_{J,i}^\alpha = \partial u_J^\alpha / \partial x^i$.

Let Ω be an open, connected subset of the space of independent variables $x = (x^1, \ldots, x^n)$ with smooth boundary $\partial\Omega$, and let $u = (u^1, \ldots, u^l)$ be the dependent variables.

Definition 0.7 The *classical variational problem* consists of finding the extrema (minima or maxima) of a functional defined by the integral

$$[u] = \int_\Omega L(x, u^{(m)})\, dx \tag{0.10}$$

in some class of functions $u = f(x)$ defined over Ω. The integrand $L(x, u^{(m)})$ called the Lagrangian of the variational problem , is a smooth function of x, u and various derivatives of u up to order m.

Theorem 0.1 If $u = f(x)$ is a smooth extremal of the variational problem (0.10) then it is a solution of the classical Euler-Lagrange equations

$$E_\alpha \equiv -D_J \frac{\partial L}{\partial u_J^\alpha} = 0 \ , \qquad \alpha = 1, \ldots, l \ , \tag{0.11}$$

with the sum extending over all multi-indices $J = (j_1, \ldots, j_k)$ with $1 \leq j_k \leq n, \ \ k \geq 0$.

Here D_J denotes the total derivative with respect to the independent variables $x^{j_1}, \ldots, x^{j_k}$, with $J = (j_1, \ldots, j_k)$.

Definition 0.8 A local group of transformations G, acting on an open subset M of the space of independent and dependent variables, is a *variational symmetry group* of the functional (0.10) if, whenever D is a subdomain with closure $D^{cl} \subset$

Ω^{cl}, $u = f(x)$ is a smooth function defined over D whose graph lies in M, and $g \in G$ is such that $\tilde{u} = \tilde{f}(\tilde{x}) = g \cdot f(\tilde{x})$ is a single-valued function defined over $\tilde{D} \subset \Omega$, then

$$\int_{\tilde{D}} L(\tilde{x}, \mathrm{pr}^{(m)} \tilde{f}(\tilde{x}))\, d\tilde{x} = \int_{D} L(x, \mathrm{pr}^{(m)} f(x))\, dx \ .$$

Definition 0.9 If $P(x, u^{(m)}) = (P_1(x, u^{(m)}), \ldots, P_n(x, u^{(m)}))$ is an n-tuple of smooth functions of $x = (x^1, \ldots, x^n)$, u and the derivatives of u, we define the *total divergence* of P to be the function

$$\mathrm{Div\, P} = \frac{dP_1}{dx^1} + \cdots + \frac{dP_n}{dx^n} \ ,$$

where dP_i/dx^i denotes the total derivative of P_i with respect to x^i.

The following theorem provides the infinitesimal criterion for the invariance of the classical variational problem under a group of transformations. See Olver, [5].

Theorem 0.2 A connected group of transformations G acting on an open subset M of the independent and dependent variables is a variational symmetry group of the functional (0.10) if and only if

$$\mathrm{pr}^{(m)} \mathbf{v}(L) + L\, \mathrm{Div}\, \xi = 0 \tag{0.12}$$

for all $(x, u^{(m)}) \in M^{(m)}$ and every infinitesimal generator

$$\mathbf{v} = \xi^i(x, u) \frac{\partial}{\partial x^i} + \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

of G. In (0.12) $\mathrm{Div}\, \xi$ denotes the total divergence of the n-tuple $\xi = (\xi^1, \ldots, \xi^n)$.

Theorem 0.3 If G is a variational symmetry group of the functional defined by the integral (0.10), then G is a symmetry group of the classical Euler-Lagrange equations (0.11).

Definition 0.10 Consider a system of differential equations $\Delta(x, u^{(m)}) = 0$. A *conservation law* is a divergence expression

$$\mathrm{Div} P = 0$$

which vanishes for all solutions $u = f(x)$ of the given system. Here $P(x, u^{(m)}) = (P_1(x, u^{(m)}), \ldots, P_n(x, u^{(m)}))$ is an n-tuple of smooth functions of x, u and the derivatives of u.

The general principle relating symmetry groups and conservation laws was first determined by E. Noether. The next theorem presents her result, known as the classical frst Noether theorem.

Theorem 0.4 *Suppose G is a (local) one-parameter group of symmetries of the variational problem $\mathcal{L} = \int L(x, u^{(m)})\, dx$. Let*

$$\mathbf{v} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

be the infinitesimal generator of G, and let $Q_\alpha(x,u) \equiv \phi_\alpha - \xi^i u_i^\alpha$, $u_i^\alpha \equiv \partial u^\alpha / \partial x^i$. Then there is a n-tuple $P(x, u^{(m)}) = (P_1, \ldots, P_n)$ such that

$$\mathrm{Div} P = Q_\alpha\, E_\alpha(L)$$

is a conservation law for the Euler-Lagrange equations $E(L) = 0$.

The next result is the second Noether theorem. For the proof see Logan, [1].

Theorem 0.5 *Suppose that the infinite continuous group of transformations*

$$\begin{aligned}
\bar{t} &= \phi(t, x, p(t), p^{(1)}(t), \ldots, p^{(r)}(t)) \\
\bar{x}^k &= \psi^k(t, x, x, p(t), p^{(1)}(t), \ldots, p^{(r)}(t))\ , \qquad k = 1, \ldots, n,
\end{aligned} \tag{0.13}$$

is a symmetry group of the functional

$$I\,[x] = \int_a^b \mathcal{L}(t, x, \dot{x})\, dt\ ,$$

where $p \in C^{r+2}[a,b]$ is the parameter function. Let

$$\begin{aligned}
\bar{t} &= t + U(p) \\
\bar{x}^k &= x^k + X^k(p)\ , \qquad k = 1, \ldots, n,
\end{aligned}$$

be the infinitesimal group of transformations which corresponds to (0.13), where U and X^k are linear differential operators. Then the identity

$$\tilde{X}^k(Q_k) - \tilde{U}(\dot{x}^k Q_k) = 0$$

holds, where $\tilde{X}^k$ *and* $\tilde{U}$ *are the adjoints of the operators* X^k *and* U *and* Q_k *are the Euler-Lagrange expressions*

$$Q_k \equiv \frac{\partial L}{\partial x^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} .$$

A refference for the remaining part of this Appendix is Guenther, [1].

Definition 0.11 Let $z = z(x^1, \ldots, x^n)$. A point, $(x^1, \ldots, x^n, z, p^1, \ldots, p^n)$, in 2n+1 dimensional Cartesian space, with $p^i = \partial z / x^i$ is called an element. A continuously differentiable, one-to-one transformation defined on a domain in $(x^1, \ldots, x^n, z, p^1, \ldots, p^n)$ space with range in $(X^1, \ldots, X^n, Z, P^1, \ldots, P^n)$ space, which may or may not coincide with the original space, given by the functions

$$X = X(x, z, p), \quad Z = Z(x, z, p), \quad P = P(x, z, p),$$

is called an *element transformation.* We assume that the Jacobian of the transformation is distinct than zero.

Definition 0.12 An element transformation which maps one-to-one some domain D in $(x^1, \ldots, x^n, z, p^1, \ldots, p^n)$ space onto a domain $\tilde{D}$ in $(X^1, \ldots, X^n, Z, P^1, \ldots, P^n)$ space is called a *contact transformation* if

$$p^i\, dx^i - dz = 0 \quad \text{implies} \quad P^i\, dX^i - dZ = 0 .$$

Theorem 0.6 *An element transformation is a contact transformation if and only if there is a function* $\rho = \rho(x^1, \ldots, x^n, z, p^1, \ldots, p^n) \neq 0$ *such that*

$$p^i\, dx^i - dz = \rho\,(P^i\, dX^i - dZ) .$$

Consider the generalized variational principle of Herglotz, i.e., let the functional z be defined by the differential equation

$$\dot{z} = L(t, x, \dot{x}, z) \tag{0.14}$$

with corresponding generalized Euler-Lagrange equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_i} = 0 , \qquad i = 1, \ldots, n , \tag{0.15}$$

which provide the extrema of the functional z .

If we apply the Legendre transformation

$$H(t,x,\dot{x},z) = p_j\dot{x}_j - L(t,x,\dot{x},z),$$
$$p_j = \frac{\partial L}{\partial \dot{x}_j}\ , \qquad\qquad j = 1,\ldots,n$$

to the system (0.14), (0.15) we obtain the system

$$\begin{aligned}
\dot{x}_j &= \frac{\partial H}{\partial p_j} \\
\dot{z} &= p_i\,\frac{\partial H}{\partial p_i} - H \\
\dot{p}_j &= -\Big(\frac{\partial H}{\partial x_j} + p_j\frac{\partial H}{\partial z}\Big)\ , \qquad j = 1,\ldots,n,
\end{aligned} \tag{0.16}$$

which Herglotz calls *canonical equations.* When H does not depend on z this system reduces to the Hamiltonian system and so can be reffered to as *generalized Hamiltonian system.*

Theorem 0.7 *Let* L *be as in* (0.14). *Suppose that* $\det(\partial^2 L/\partial\dot{x}_i\partial\dot{x}_j)\neq 0$ *and that the system (0.16) has initial conditions* $x^0 = x(0)$ *,* $p^0 = p(0)$ *and* $z^0 = z(0)$ *. Then the transformation*

$$\begin{aligned}
x &= x(t,x^0,p^0,z^0) \\
p &= p(t,x^0,p^0,z^0) \\
z &= z(t,x^0,p^0,z^0)
\end{aligned} \tag{0.17}$$

is a one-parameter family of contact transformations which contains the identity. If L *is independent of* t *, the family is a one-parameter group.*

Consider the system of 2n+1 differential equations for the 2n+1 unknowns $X = (X_1,\ldots,X_n),\ Z,\ P = (P_1,\ldots,P_n)$

$$\dot{X} = \xi(X,Z,P,t)\ , \qquad \dot{Z} = \zeta(X,Z,P,t)\ , \qquad \dot{P} = \pi(X,Z,P,t)\ , \tag{0.18}$$

which satisfy the initial conditions

$$X = x, \quad Z = z, \quad P = p, \quad \text{when} \quad t = 0\ . \tag{0.19}$$

The functions $\xi = (\xi_1, \ldots, \xi_n)$, ζ, and $\pi = (\pi_1, \ldots, \pi_n)$ are all assumed to be continuously differentiable. The solutions

$$X = X(x, z, p, t), \quad Z = Z(x, z, p, t), \quad P = P(x, z, p, t), \tag{0.20}$$

to (0.18) and (0.19) determine a family of transformations $S_t : (x, z, p) \to (X, Z, P)$.

Theorem 0.8 *In order for the solution (0.20) of the system (0.18) to represent a one-parameter family of contact transformations containing the identity, it is necessary that there exists a function, H=H(X,Z,P,t), such that the system (0.18) has the form*

$$\begin{aligned} \frac{d}{dt}X_j &= \frac{\partial H}{\partial P_j} \\ \frac{d}{dt}Z &= P_i \frac{\partial H}{\partial P_i} - H \\ \frac{d}{dt}P_j &= -\Big(\frac{\partial H}{\partial X_j} + P_j \frac{\partial H}{\partial Z}\Big) , \qquad j = 1, \ldots, n . \end{aligned} \tag{0.21}$$