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This paper is a theoretical approach to solve problems of horizontally curved beams Castigliano's theorem is applied to solve for the fixed end bending moments, maximum span moments and fixed end torsional moments. To calculate the vertical deflection at the mid-span of the curved beam the moment-area method is used. Equations are derived and solved by digital computer for different loading conditions and geometrical conditions. A moment distribution method is introduced for solving the problem of a multi-span horizontally curved beam.

HORIZONTALLY CURVED BEAM ANALYSIS AND DESIGN

by

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INTRODUCTION

It has been found advantageous to use horizontally curved beams or bow girders in building design and bridge design. Recently many architects and designers have become more interested in using them.

The difference in analysis and design between the beams curved in plan and the straight beams is mainly due to the presence of torsional movement induced by vertical load. Therefore, for such members, it is necessary to design for internal bending moment, and twisting moment as well as transverse shear.

The capability of resisting torsional moment is expressed by torsional rigidity which is defined as the torsional moment which, when applied to one end which is free to rotate, produces a unit angle of twist with respect to the other end assumed to be completely fixed (2). The greater the torsional rigidity, the greater the resistance to the torque. The value of torsional rigidity depends on the shape of the section. It was found that the box sections have comparatively large values of torsional rigidity (7) and are widely used in bridge design. However, the rectangular section is also commonly used.

Horizontally curved beams, either made of steel or reinforced concrete, can be continuous or monolithic at both ends. In this paper, equations for calculating bending moments, torsional moments, shearing stresses and deflections are derived. These equations are solved by digital computer for different types of loading conditions and geometrical conditions. For details of the computer programs, see Appendix I. For the designer's convenience, tables and charts are provided. To solve for the problem of continuous curved beams, a moment distribution method is introduced.

II. ASSUMPTIONS, NOTATIONS AND SIGN CONVENTIONS

Assumptions

In this analysis and design of horizontally curved beams, the following assumptions are used:

(1) Material is homogeneous and isotropic.

(2) The material has linear stress-strain relationships, so the principle of superposition is valid.

(3) The cross section of the beam is uniform and small compared with the radius of curvature.

Notations

M = Bending moment about radial axis. Its subscript indicates its location.

T = Torsional moment. Its subscript indicates its location.

F = Vertical force. Its subscript indicates its location.

r = radius of curvature of the beam.

ϕ = Angle spanned by the beam.

ϕ_o = Angle distance of the section, where the concentrated load acts, measured counterclockwise from the support.

θ = Angle distance of any section of the beam, measured counterclockwise from the support.

E = Modulus of elasticity in tension or compression.

G = Modulus of elasticity in shear. $G = E/2\ (1 + \mu)$

μ = Poisson's ratio.

I = Moment of inertia of the cross section of the beam about centroidal axis.

J = Polar moment of inertia of the cross section.

m = EI/GJ. See later explanation.

ω = Angle of rotation due to bending moment.

φ = Angle of rotation due to torsional moment.

Δ = Vertical deflection, its subscript indicates its location.

V = Shear force.

v = Shearing stress due to vertical force.

τ = Pure shear due to torsional moment.

P = Vertical concentrated load.

w = Vertical uniform load per unit length of the beam.

C_m = Fixed end bending moment coefficient.

C_t = Fixed end torsional moment coefficient.

C_f = Fixed end shear force coefficient.

Explanation of m:

The value of m, which depends on the material and shape of the section, can be calculated in the following manner:

$$m = \frac{EI}{GJ} = \frac{EI}{\frac{E}{2(1+\mu)}J} = \frac{2(1+\mu)I}{J}$$

For rectangular section, Figure 1a, $I = bh^3/12$. Where b, h are the dimension of the section parallel and perpendicular to the radial axis respectively. $J=\frac{1}{3}(h-0.63b)b^3$, where h is the long dimension and b is the short dimension of the rectangular section. If the section is so placed that the short side is parallel and the long side is perpendicular to the radial axis as shown in Figure 1a, $m=\frac{(1+\mu)h^3}{2b^2(h-0.63b)}$.

For sections composed of narrow rectangles, such as channels, I or T sections, the value of Jwill be $J=\frac{1}{3}\Sigma hb^3$. Fig.1b.

For box section as shown in Figure 1c. $J=\frac{4a^2}{\Sigma(ds/t)}$. (3)

In this equation, a is the shear area and equals to the product of b and h; ds is a short increment of length, for t1, t_2 as shown in Fig.1c ds=h, for t_3, t_4, ds=b.

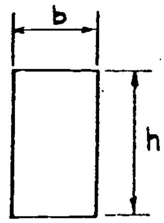


Fig. 1a

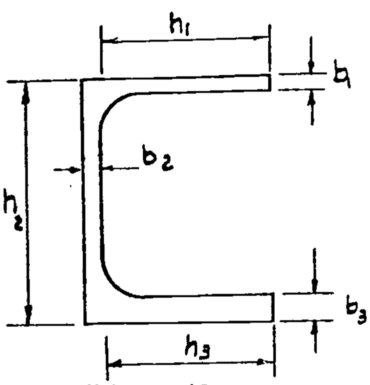


Fig. 1b

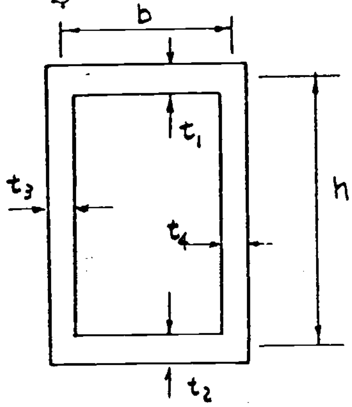


Fig. 1c

Sign Convention

Bending moment and torsional moment are expressed by moment vectors. Vertical force acting upward is represented by a solid circle ●. An open circle O represents the vertical force acting downward.

Bending moment will be taken as positive, if when looking outward from the center of the curvature it produces clockwise rotation about the radial axis. Torsional moments will be taken as positive, when looking along the tangent of the beam in a counterclockwise direction it produces a clockwise rotation. Vertical force will be taken as positive when it acts upward. Figure 2 shows the sign convention used in this analysis.

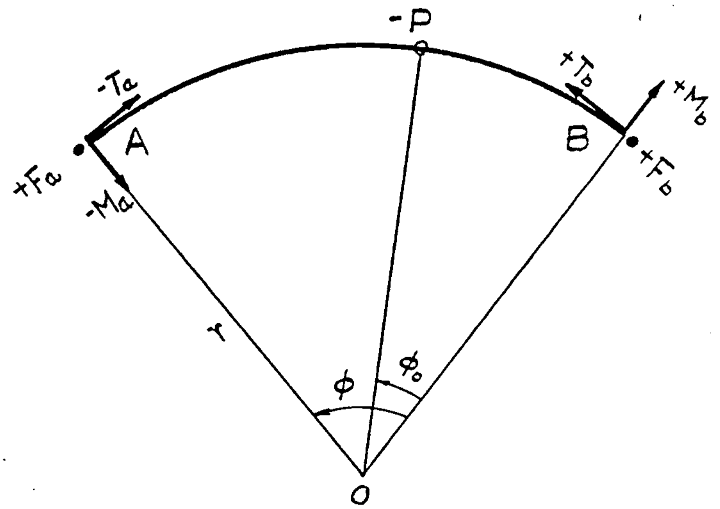


Figure 2. Sign convention.

III. HORIZONTALLY CURVED BEAM FIXED AT BOTH ENDS AND SUBJECTED TO CONCENTRATED LOAD

Statical analysis

Figure 3a shows a beam composed of two equal straight portions at right angles, rigidly connected at the point of intersection B and subjected to a force P which is perpendicular to the x-y plane.

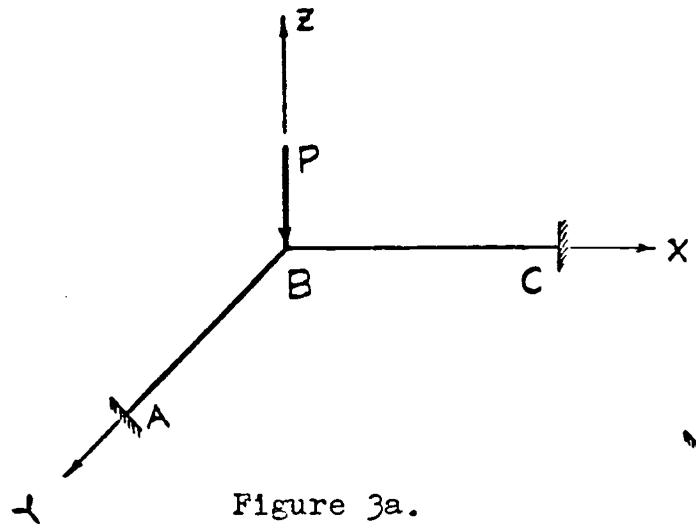


Figure 3a.

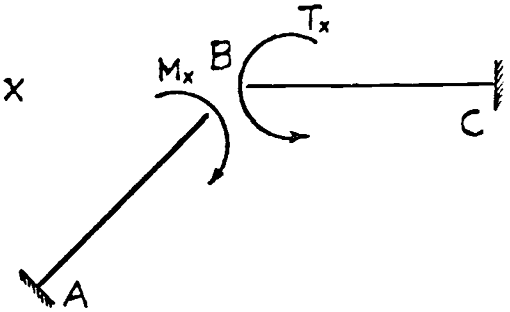


Figure 3b.

Load P will produce bending moment M_x and M_y at point B. The bending moment M_y about the y-axis will simultaneously cause a torsional moment T_y rotating about the axis of BA, i.e., y-axis. Similarly, the bending moment M_x about the x-axis also has a torsional effect T_x to the portion BC, i.e., the x-axis as shown in Figure 3b.

By the same token, in a horizontally curved beam if M_θ is the bending moment at any section θ about the radial axis at that section, it can have a bending moment component $M_{\theta x}$ and a torsional moment component $T_{\theta x}$ about the x-axis, as shown in Figure 3c.

In Fig. 3c Similarly, at the same section θ, a torsional moment T_θ also has a bending moment about the tangent of this section and a torsion component, as labeled $M_{\theta z}$ and $T_{\theta x}$, about the x-axis.

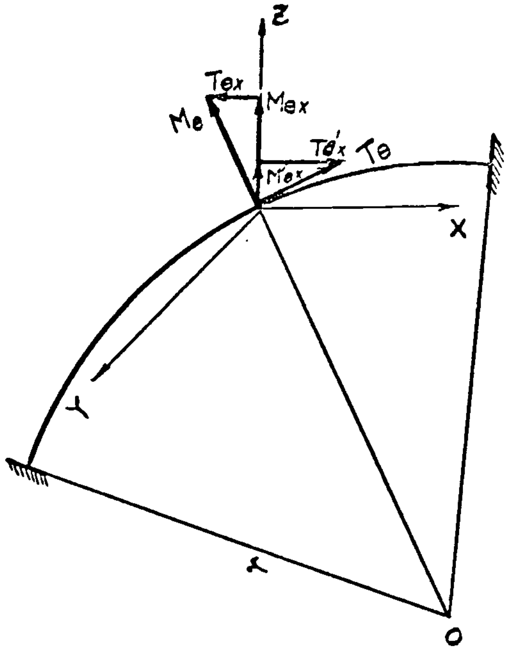


Figure 3c.

Therefore, if a bending moment M_{xy} on the x-y plane is expected to exist it must be accompanied by a torsional moment rotating about the z-axis and a certain amount of loading perpendicular to the x-z or y-z plane should be used.

In a transversely loaded curved beam, since there is no external load acting horizontally, there will not be any moment rotating about the z-axis. For such a loading condition in a horizontally curved beam the moments existing will be only those which bend or rotate in the z direction and perpendicular to the x-y plane.

Bending Moment and Torsional Moment

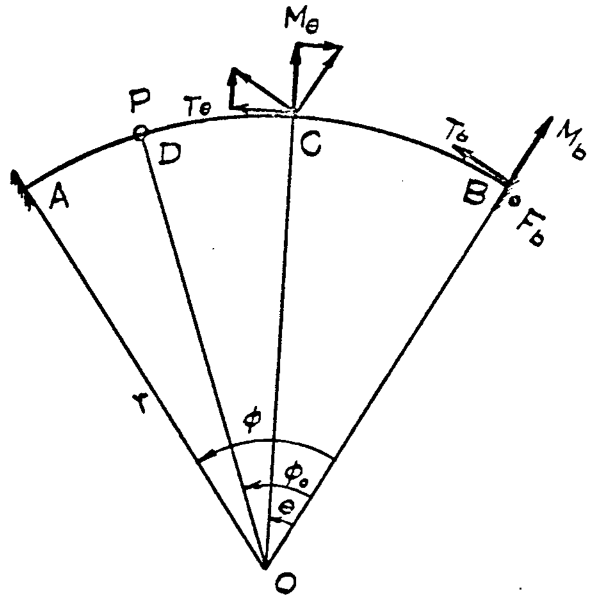


Figure 4.

As shown in Figure 4, a concentrated load acts at the point D with an angular distance ϕ_o from support B. M_θ and T_θ represent the bending moment and torsional moment at any section C with an angular distance θ from B.

For portion BC $(0 \leq \theta \leq \phi_o)$

$$M_\theta = M_b\cos\theta + T_b\sin\theta - F_b r\sin\theta \tag{1}$$

$$T_\theta = -M_b\sin\theta + T_b\cos\theta + F_b r(1 - \cos\theta) \tag{2}$$

For portion CA$(\phi_o \leq \theta \leq \phi)$

$$M_\theta = M_b\cos\theta + T_b\sin\theta - F_b r\sin\theta + Pr\sin(\theta - \phi_o) \tag{3}$$

$$T_\theta = -M_b\sin\theta + T_b\cos\theta + F_b r(1 - \cos\theta) - Pr(1-\cos(\theta-\phi_o)) \tag{4}$$

Applying Castigliano's theorem with U representing the strain energy;

$$U = \int \frac{(M_\theta)^2}{2EI} r d\theta + \int \frac{(T_\theta)^2}{2GJ} r d\theta$$

At support B $\omega_b = \varphi_b = \Delta_b = 0$

$$\frac{\partial U}{\partial M_b} = \frac{\partial U}{\partial T_b} = \frac{\partial U}{\partial F_b}$$

$$\frac{\partial U}{\partial M_b} = 0$$

$$\frac{1}{EI}\int \frac{M_\theta \ M_\theta}{M_b} + \frac{1}{GJ}\int \frac{T_\theta \ T_\theta}{M_b} = 0$$

With $m = \frac{EI}{GJ}$, $\frac{\partial M_\theta}{\partial M_b} = \cos\theta$, $\frac{\partial T_\theta}{\partial M_b} = -\sin\theta$

$$\frac{1}{EI}\Bigg[\int_0^{\phi}(M_b\cos\theta + T_b\sin\theta - F_b r\sin\theta)\cos\theta r d\theta + m\int_{\phi_o}^{\phi}$$

$$\Big[M_b\cos\theta + T_b\sin\theta - F_b r\sin\theta + Pr\sin(\theta-\phi)\Big]\cos\theta r d\theta$$

$$+m\int_0^{\phi_o}(-M_b\sin\theta + T_b\cos\theta + F_b r(1-\cos\theta))(-\sin\theta r d\theta)$$

$$+m\int_{\phi_o}^{\phi}((-M_b\sin\theta + T_b\cos\theta + F_b r(1-\cos\theta) - Pr(1-\cos(\theta-\phi_o))$$

$$(-\sin\theta r d\theta)\Bigg] = 0$$

$$\frac{r}{EI}\Bigg[M_b\int_0^{\phi}\cos^2\theta d\theta + mM_b\int_0^{\phi}\sin^2\theta d\theta + T_b\int_0^{\phi}\sin\theta\cos\theta d\theta$$

$$+mT_b\int_0^{\phi}(-\cos\theta\sin\theta d\theta) - F_b r\int_0^{\phi}\sin\theta\cos\theta d\theta$$

$$+F_b rm\int_0^{\phi}\cos\theta\sin\theta d\theta - F_b mr\int_0^{\phi}\sin\theta d\theta + Pr\int_{\phi_o}^{\phi}\sin(\theta-\phi_o)\cos\theta d\theta$$

$$+Prm\int_{\phi_o}^{\phi}\big(1-\cos(\theta-\phi_o)\big)\sin\theta d\theta = 0$$

$$\frac{r}{EI}\Big[M_b\Big(\frac{\phi}{2} + \frac{\sin 2\phi}{4} + m\frac{\phi}{2} - m\frac{\sin 2\phi}{4}\Big) + T_b\Big(-\frac{1}{4}(\cos 2\phi - 1)$$

$$+\frac{m}{4}(\cos(2\phi-1)\Big) + \frac{F_b r}{4}(1-m)\ (\cos 2\phi - 1) + F_b mr(\cos\phi - 1)\Big]$$

$$= \frac{Pr^2}{EI}(m-1)\ \cos\phi_o\Big(-\frac{1}{4}(\cos 2\phi - \cos 2\phi_o)\Big)$$

$$+ \frac{Pr^2}{EI}\sin\phi_o\Big[(\phi-\phi_o) + (m-1)\ \Big(\frac{\phi}{2}-\frac{\phi_o}{2}\Big) - \frac{\sin 2\phi - \sin 2\phi_o}{4}\Big]$$

$$+ \frac{mPr^2}{EI}(\cos\phi - \cos\phi_o)$$

$$\frac{r}{2EI}\Big[M_b(\phi(m+1) - \sin\phi\cos\phi(m-1)\Big] - \frac{r}{2EI}\ T_b(\sin^2\phi(m-1))$$

$$+\frac{r^2}{2EI}\ F_b\big(\sin^2\phi(m-1) + 2m(\cos\phi - 1)\big)$$

$$= \frac{Pr^2}{2EI}\Big[(m-1)\cos\phi_o(\sin^2\phi - \sin^2\phi_o) + \sin\phi_o(\phi-\phi_o)\ (m+1)$$

$$- \sin\phi_o\ \frac{(m-1)}{2}(\sin 2\phi - \sin 2\phi_o) + 2m(\cos\phi - \cos\phi_o)\Big] \qquad (5)$$

Similarly

By $\frac{\partial U}{\partial T_b} = 0$ we obtain

$$- \frac{r}{2EI}\ M_b\sin^2\phi(m-1) + \frac{r}{2EI}\ \Big[\ \phi(m+1) + \sin\phi\cos\phi(m-1)\Big]$$

$$+ \frac{r^2 F_b}{2EI}\Big[2m\sin\phi - \phi(m+1) - \sin\phi\cos\phi(m-1)\Big]$$

$$= \frac{Pr^2}{2EI}\Big[-\cos\phi_o(\phi-\phi_o)\ (m+1) - \frac{1}{2}\cos\phi_o(m-1)\ (\sin 2\phi - \sin 2\phi_o)$$

$$-(m-1)\sin\phi_o(\sin^2\phi - \sin^2\phi_o) + 2m\ (\sin\phi - \sin\phi_o)\Big] \qquad (6)$$

By $\frac{\partial U}{\partial F_b} = 0$ we obtain

$$\frac{r^2}{2EI}\ M_b\Big[\sin^2\phi(m-1) + 2m(\cos\phi - 1)\Big] + \frac{r^2 T_b}{2EI}\Big[2m\sin\phi - \phi(m+1)$$

$$- \sin \phi\cos\phi(m-1)\Big] + \frac{r^3F_b}{2EI}\Big[\phi(m+1) + \sin \phi\cos\phi(m-1)$$

$$- 4m\sin\phi + 2m\phi \Big] = \frac{Pr^3}{2EI}\Big[(\frac{m-1}{2})\cos\phi_o(\sin2\phi-\sin2\phi)$$

$$- 2m(1+\cos\phi_o)(\sin\phi - \sin \phi_o) + 2m\sin\phi_o(\cos\phi-\cos\phi_o)\Big] \tag{7}$$

Let

$$a_1 = \phi(m+1) - \sin\phi\cos\phi(m-1)$$

$$b_1 = \sin^2\phi(m-1)$$

$$c_1 = \sin^2\phi(m-1) + 2m(\cos\phi-1)$$

$$a_o = (m-1)\cos\phi_o(\sin^2\phi - \sin^2\phi_o) + \sin \phi_o(\phi-\phi_o)(m+1)$$
$$- \sin\phi_o(\frac{m-1}{2})(\sin2\phi - \sin2\phi_o) + 2m(\cos\phi-\cos\phi_o)$$

$$a_2 = b_1$$

$$b_2 = \phi(m+1) + \sin\phi\cos\phi(m-1)$$

$$c_2 = 2m\sin\phi - \phi(m+1) - \sin\phi\cos\phi(m-1)$$

$$b_o = -\cos\phi_o(\phi-\phi_o)(m+1) - \tfrac{1}{2}\cos\phi_o(m-1)(\sin2\phi-\sin2\phi_o)$$
$$-(m-1)\sin\phi_o(\sin^2\phi - \sin^2\phi_o) + 2m(\sin\phi - \sin\phi_o)$$

$$a_3 = c_1$$

$$b_3 = c_2$$

$$c_3 = \phi(m+1) + \sin\phi\cos\phi(m-1) - 4m\sin\phi + 2m\phi$$

$$c_o = \frac{(m-1)}{2}\cos\phi_o(\sin2\phi - \sin2\phi_o) + (\phi-\phi_o)(\cos\phi_o(m+1)+2m)$$
$$+ (m-1)\sin\phi_o(\sin^2\phi - \sin^2\phi_o) - 2m(1+\cos\phi_o)(\sin\phi$$
$$- \sin\phi_o) + 2m\sin\phi_o(\cos\phi - \cos\phi_o) \tag{8}$$

Substitute the coefficients in equation 8 into equations 5, 6, and 7 and we obtained the following simultaneous equations

$$a_1 M_b - b_1 T_b + C_1 F_b r = a_o Pr$$

$$-b_1 M_b + b_2 T_b + C_2 F_b r = b_o Pr$$

$$C_1 M_b + C_2 T_b + C_3 F_b r = C_o Pr \qquad (9)$$

Placing equation 9 in matrix form, the following results;

$$|A| = \begin{vmatrix} a_1 & -b_1 & C_1 \\ -b_1 & b_2 & C_2 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$M_b = \frac{1}{|A|}\begin{vmatrix} a_o Pr & -b_1 & C_1 \\ b_o Pr & b_2 & C_2 \\ C_o Pr & C_2 & C_3 \end{vmatrix} = PrC_m$$

$$T_b = \frac{1}{|A|}\begin{vmatrix} a_1 & a_o Pr & C_1 \\ -b_1 & b_o Pr & C_2 \\ C_1 & C_o Pr & C_3 \end{vmatrix} = PrC_t$$

$$F_b r = \frac{1}{|A|}\begin{vmatrix} a_1 & -b_1 & a_o Pr \\ -b_1 & b_2 & b_o Pr \\ C_1 & C_2 & C_o Pr \end{vmatrix} = PrC_f$$

$$F_b = PC_f \qquad (10)$$

Solving equation 10 by digital computer, see Appendix I, and substituting the values of M_b, T_b and F_b calculated from equation 10 into equations 1 through 4, we can calculate the bending moment and torsional moment at any section θ, i.e., M_θ and T_θ.

From equation 10 we understand that M_b, T_b and F_b vary with ϕ, ϕ_o, m, and r. These variations are expressed by coefficients C_m, C_t and C_f. The relationships between the angle $\emptyset$ and the values of C_m and C_t for the load acting at the mid-span are shown in Figure 5 and 6. To obtain the maximum bending moment and torsional moment we differentiate equations 1 through 4. Obviously the maximum bending moment occurs at the section of $\theta = \emptyset_o$.

$$M_{max} = M\emptyset_o = M_b\cos\emptyset_o + T_b\sin\emptyset_o - F_br\sin\emptyset_o.$$

$$= PrC_{mm} \qquad (11)$$

The values of C_{mm} varying with $\emptyset$ and m are shown in Figure 7.

Differentiating equations 2 and 4 and letting $\frac{\partial T_\theta}{\partial \theta} = 0$

$$-M_b\cos\theta - T_b\sin\theta + F_br\sin\theta = 0$$

$$\text{Tan}\theta = \frac{M}{(F_br - T_b)} \quad (0 \leqslant \theta \leqslant \emptyset_o) \qquad (12)$$

$$-M_b\cos\theta - T_b\sin\theta + F_br\sin\theta - \cos\emptyset_o\sin\theta Pr$$

$$+\sin\emptyset_o\cos\theta Pr = 0$$

$$\text{Tan}\theta = \frac{F_br - T_b - \cos\emptyset_o Pr}{M_b - Pr\sin\emptyset_o} \quad (\emptyset_o \leqslant \theta \leqslant \emptyset) \qquad (13)$$

Deflection

Neglecting the rotation of the section due to the torsional moment and the effect of deflection caused by transverse shear, we can derive the equation of the vertical deflection for a horizontally curved beam by the moment area method.

$$EI\Delta_\theta = \int_o^\theta M_\theta r\sin(\frac{\emptyset}{2} - \theta)ds + m\int_o^\theta T_\theta r(1-\cos(\frac{\emptyset}{2}-\theta))ds \qquad (14)$$

To solve for the vertical deflection under the load which is placed at the mid-span of the beam, we substitute equations 1 and 2 into equation 14 and let $ds=rd\theta$, $\theta=\phi/2$

$$\Delta_{\phi/2} = \frac{1}{EI}\int_0^{\phi/2} M_\theta r^2 \sin(\frac{\phi}{2}-\theta)d\theta + \frac{1}{GJ}\int_0^{\phi/2} T_\theta r^2 (1-\cos(\frac{\phi}{2}-\theta))d\theta$$

$$= \frac{r^2}{EI}\left[\frac{M_b}{4}(1-\cos\phi) + (T_b-F_b r)(\frac{\phi}{4} - \frac{\sin\phi}{4})\right]$$

$$+ \frac{mr^2}{EI}\Big[M_b(\cos\frac{\phi}{2} - 1) + (T_b-F_b r)(\sin\frac{\phi}{2}) + F_b r(\frac{\phi}{2})$$

$$- \cos\frac{\phi}{2}(-\frac{M_b}{4}(1-\cos\phi) + (T_b-F_b r)(\phi+\sin\phi)/4$$

$$+ F_b r\sin\frac{\phi}{2}) - \sin\frac{\phi}{2}(\frac{-M_b}{4}(\phi-\sin\phi) + \frac{T_b-F_b r}{4}(1-\cos\phi)$$

$$- F_b r(\cos\frac{\phi}{2} - 1))\Big]$$

$$= \frac{Pr^3}{EI} C_d \qquad (15)$$

The vertical deflection calculated from equation 15 will be the maximum deflection only if the load is applied at the mid-span of the beam. The downward deflection will be taken as positive. The values of C_d are plotted into curves as shown in Figure 8.

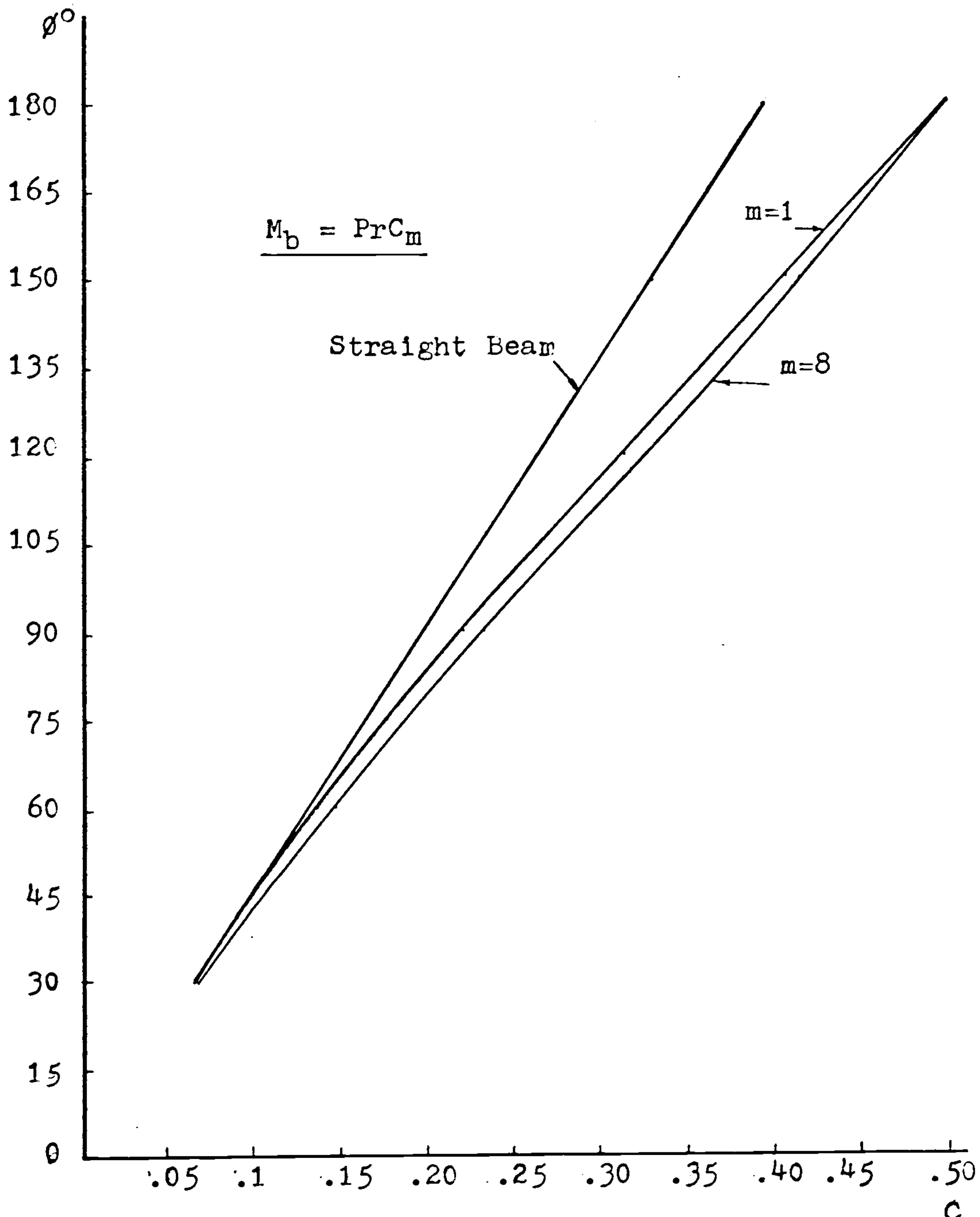


Figure 5. Variation of fixed end bending moment coefficients with span angle for curved beams loaded with concentrated load.

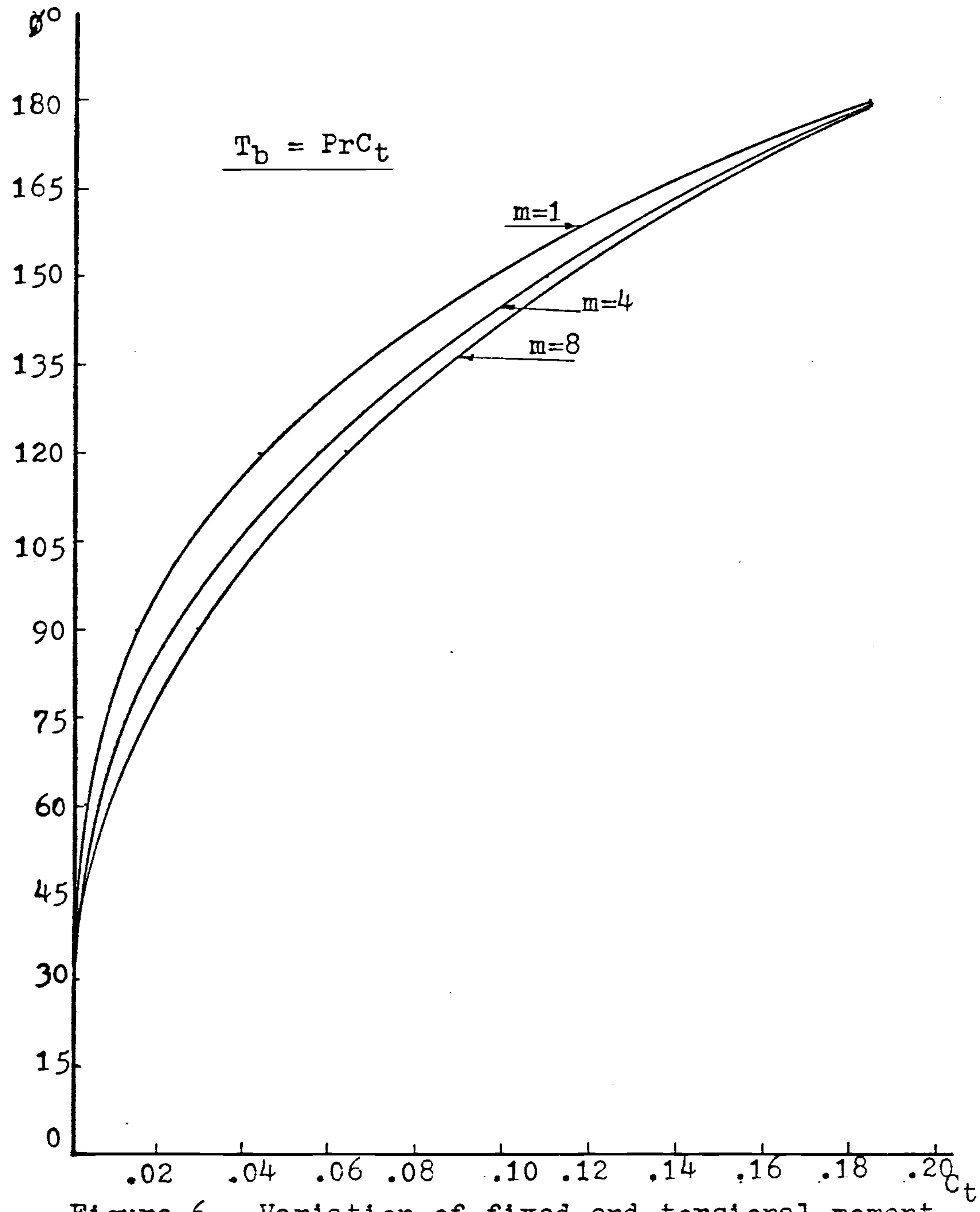


Figure 6. Variation of fixed end torsional moment coefficients with span angle for curved beams loaded with concentrated load.

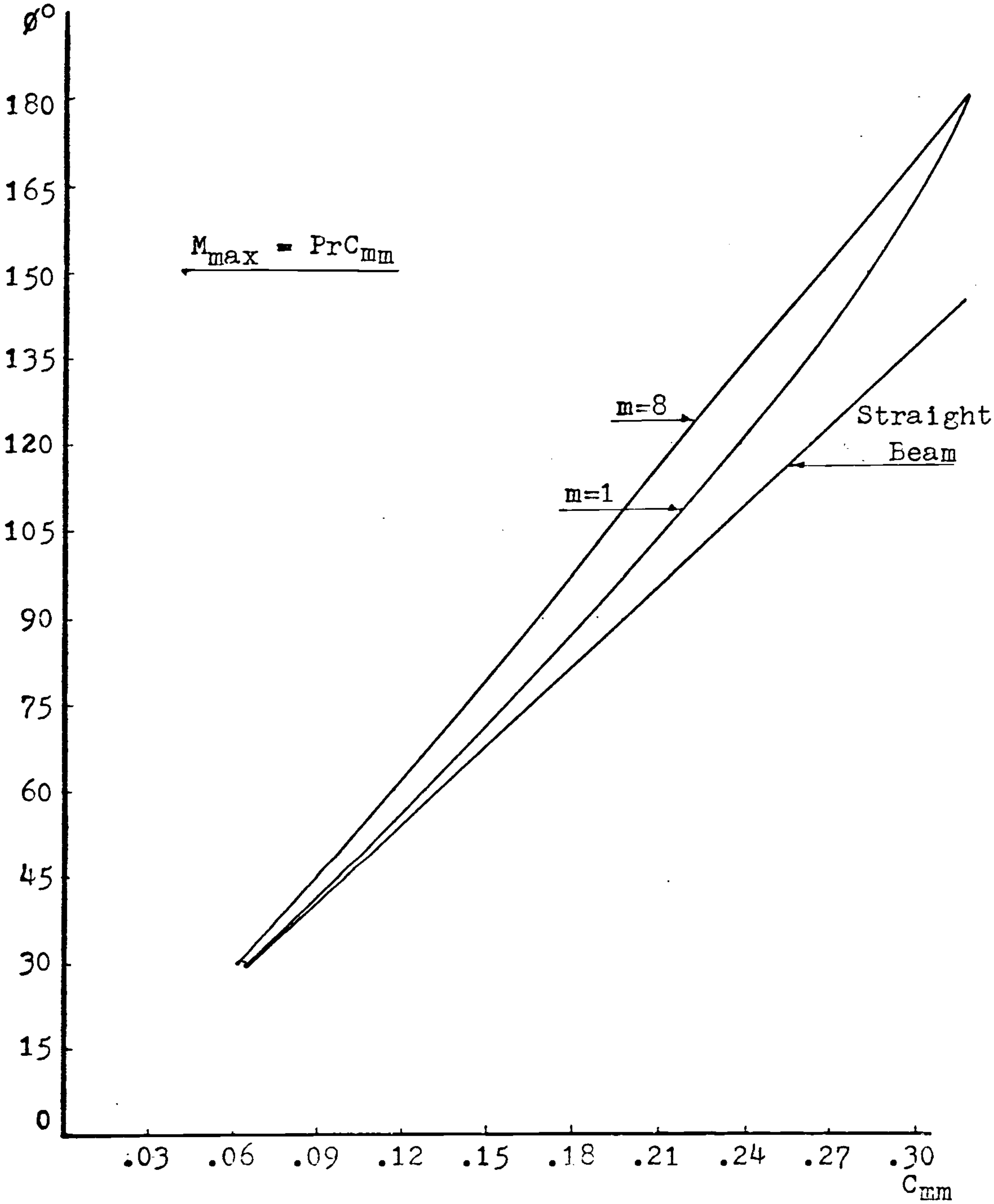


Figure 7. Variation of maximum span moment coefficients with span angle for curved beams loaded with concentrated load.

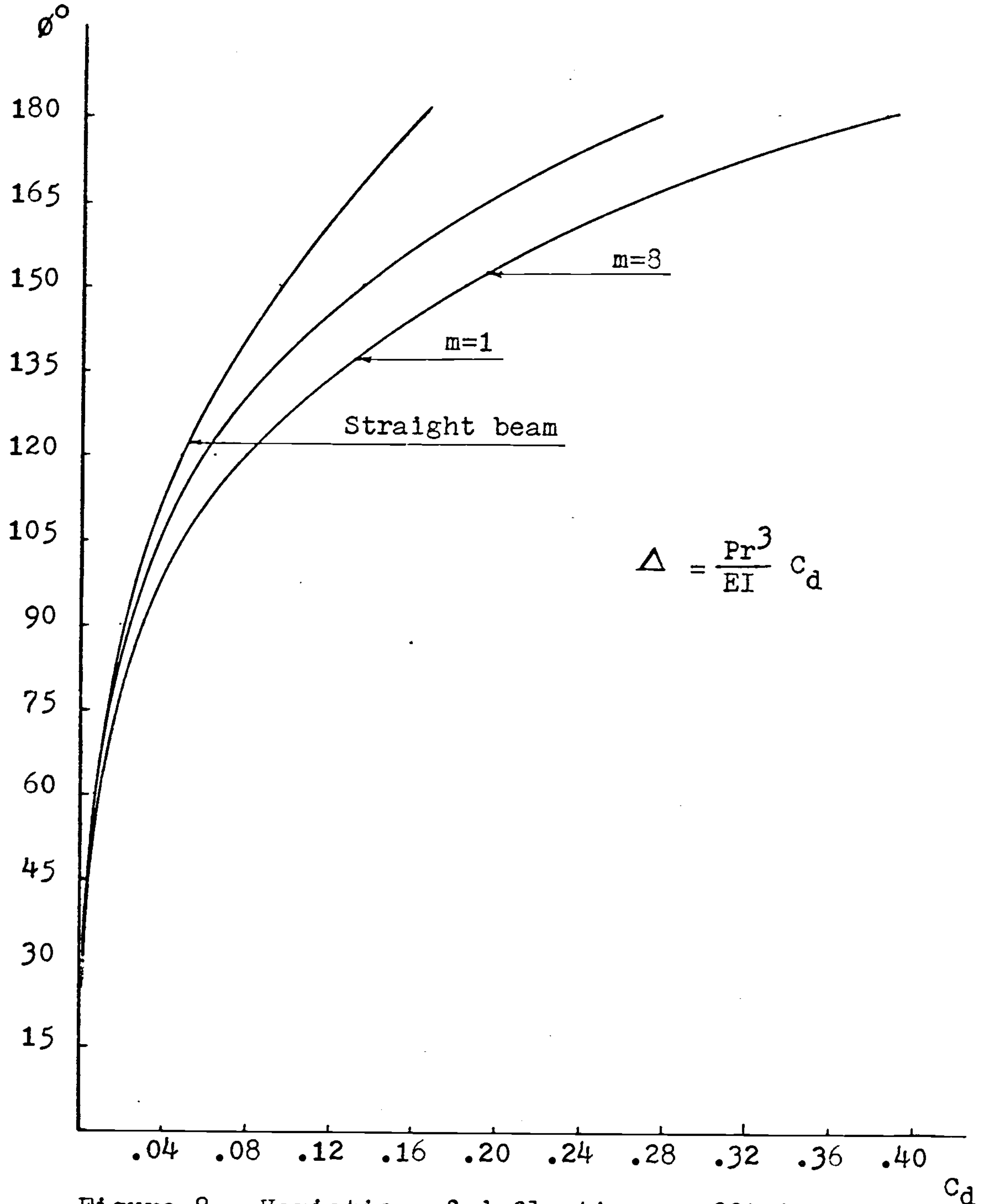


Figure 8. Variation of deflection coefficients with span angle for curved beams loaded with concentrated load.

IV. HORIZONTALLY CURVED BEAM FIXED AT BOTH ENDS AND SUBJECTED TO UNIFORM LOAD

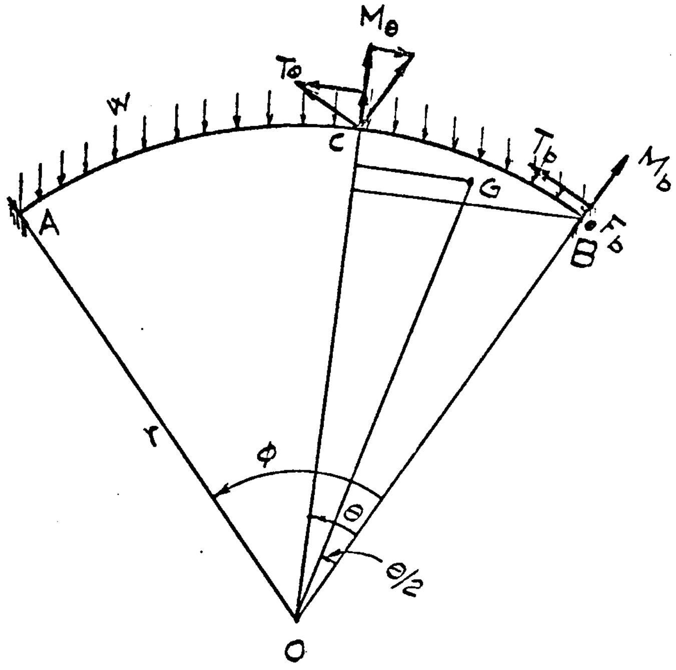


Figure 9

Bending Moment and Torsional Moment

Because of the symmetry of the loading, the vertical reaction F_b at support B is equal to:

$$F_b = \frac{w\, r\, \phi}{2}$$

In Figure 9 the distance $\overline{OG}$ from the center of the curvature to the center of gravity of arc BC is

$$\overline{OG} = \frac{r \sin\frac{\theta}{2}}{\theta/2}$$

$$\widehat{BC} = r\theta$$

At any section C

$$M_\theta = M_b\cos\theta + T_b\sin\theta - F_b r\sin\theta + wr\theta\left(\frac{r\sin\theta/2}{\theta/2}\sin\theta/2\right)$$

$$= M_b\cos\theta + T_b\sin\theta - \left(\frac{wr\theta}{2}\right)r\sin\theta + wr^2(1-\cos\theta) \quad (16)$$

$$T_\theta = T_b\cos\theta - M_b\sin\theta + F_b r(1-\cos\theta)$$

$$- wr\theta\left(r - \frac{r\sin\theta/2}{\theta/2}\cos\theta/2\right) = T_b\cos\theta - M_b\sin\theta$$

$$+ \frac{wr^2\phi}{2}(1-\cos\theta) - wr^2(\theta-\sin\theta) \quad (17)$$

Applying Castigliano's theorem

$$U = \int\left(\frac{M_\theta^2}{2EI}\right)rd\theta + \int\left(\frac{T_\theta^2}{2GJ}\right)rd\theta$$

$$\frac{\partial M_\theta}{\partial M_b} = \cos\theta \quad \frac{\partial T_\theta}{\partial M_b} = (-\sin\theta) \quad m = \frac{EI}{GJ} \quad \frac{\partial M_\theta}{\partial T_b} = \sin\theta$$

$$\frac{\partial T_\theta}{\partial T_b} = \cos\theta$$

$$\frac{\partial U}{\partial M_b} = \omega_b = 0$$

$$\frac{r}{EI}\int_0^\phi\left[M_b\cos\theta + T_b\sin\theta - \frac{wr^2}{2}\sin\theta + wr^2(1-\cos\theta)\right]\cos\theta d\theta$$

$$- \frac{m}{EI}r\int_0^\phi\left[T_b\cos\theta - M_b\sin\theta + \frac{wr^2\phi}{2}(1-\cos\theta)\right.$$

$$\left.- wr^2(\theta-\sin\theta)\right]\sin\theta d\theta = 0$$

which gives

$$\frac{1}{2}M_b\left[\phi(m+1) - \sin\phi\cos\phi(m-1)\right] - \frac{1}{2}T_b\sin^2\phi(m-1)$$

$$-\frac{1}{2}wr^2\left[\phi(m+1) + \phi m(1+\cos\phi) - \frac{1}{2}\phi\sin^2\phi(m-1)\right.$$

$$\left.-2\sin\phi(m+1) - \sin\phi\cos\phi(m-1)\right] = 0 \quad (18)$$

Similarly, $\frac{\partial U}{\partial T_b} = \varphi_b = 0$

$$\frac{r}{EI}\int_0^\phi\left[M_b\cos\theta + T_b\sin\theta - \frac{wr\phi}{2}r\sin\theta + wr^2(1-\cos\theta)\right]\sin\theta d\theta$$

$$+\frac{mr}{EI}\int_0^\phi\left[T_b\cos\theta - M_b\sin\theta + \frac{wr\phi}{2}(1-\cos\theta) - wr^2(\theta-\sin\theta)\right]\cos\theta d\theta$$

$$= 0$$

which gives

$$- \frac{1}{2}M_b \sin^2\phi(m-1) + \frac{1}{2}T_b\left[\phi(m+1) + \sin\phi\cos\phi(m-1)\right]$$

$$- \frac{1}{2}r^2\left[\frac{1}{2}\phi^2(m+1) + \frac{1}{2}\phi\sin\phi\cos\phi(m-1) - \sin^2\phi(m-1)\right.$$

$$\left. + m\phi\sin\phi - 2(m-1)\ (1-\cos\phi)\right] = 0 \qquad (19)$$

Cancelling the 1/2 and substituting as follows

$$a = \phi(m+1)$$

$$b = \sin^2\phi(m-1)$$

$$c = \sin\phi\cos\phi(m-1)$$

$$d = \phi(m+1) + m\phi\ (1+\cos\phi) - \frac{\phi}{2}\sin^2\phi(m-1) - 2\sin\phi(m+1)$$

$$-\sin\phi\ \cos\phi(m-1)$$

$$e = \frac{1}{2}\phi^2(m+1) + \frac{\phi}{2}\sin\phi\ \cos\phi(m-1) - \sin^2\phi\ (m-1)$$

$$+ m\phi\sin\phi - 2(m+1)\ (1-\cos\phi)$$

Equations 18 and 19 become

$$M_b(a-c) - T_b b - wr^2 d = 0$$

$$-M_b(b) + T_b(a+c) - wr^2 e = 0 \qquad (20)$$

Solving for M_b and T_b

$$M_b = -wr^2\ \frac{be + d(a+c)}{-a^2 + b^2 + c^2}$$

$$T_b = -wr^2\ \frac{bd + e(a-c)}{-a^2 + b^2 + c^2}$$

After substituting the values of the various terms for a, b, c, d, e, the fixed end bending moment and torsional moment become

$$M_b = M_{fb} = -wr^2\left|\frac{2(m+1)\ \sin\phi - m\phi(1 + \cos\phi)}{\phi(m+1) - \sin\phi(m-1)} - 1\right]$$

$$= wr^2\ C_m \qquad (21)$$

$$T_b = T_{fb} = -wr^2 \left| \frac{2(m+1)\ (1-\cos\phi) - m\phi\ \sin\phi}{\phi(m+1) - \sin\phi\ (m-1)} - \frac{\phi}{2} \right]$$

$$= wr^2\ C_t \qquad (22)$$

Substituting the values of M_b and T_b into equations 17 and 18, M_θ and T_θ cna be expressed as follows

$$M_\theta = wr^2\ C_{m\theta}$$

$$T_\theta = wr^2\ C_{t\theta} \qquad (23)$$

The values of M_b T_b and thus the values of M_θ and T_θ are functions of m, ϕ for a certain value of radius r. The values of C_m and C_t varying with different m, ϕ, are plotted into curves as shown in Figure 10 and Figure 11 respectively.

The maximum bending moment occurs at the middle of the span, that is, at the section of $\theta = \phi/2$

$$M_{max} = M(\phi/2) = M_b\cos\frac{\phi}{2} + T_b\sin\frac{\phi}{2} - \frac{wr^2\phi}{2}\ \sin\frac{\phi}{2}$$

$$+ wr^2(1-\cos\frac{\phi}{4}) = wr^2 \left[C_m\cos\frac{\phi}{2} + C_t\sin\frac{\phi}{2} - \frac{\phi}{2}\sin\frac{\phi}{2}\right.$$

$$\left. + (1-\cos\frac{\phi}{4})\right] = wr^2\ C_{mm} \qquad (24)$$

The values of C_{mm} are shown in Figure 12.

By differentiating equation 17 with respect to θ and setting

$$\frac{dT_\theta}{d\theta} = 0$$

$$- T_b\sin\theta - M_b\cos\theta + \frac{wr^2\phi}{2}\sin\theta - wr^2 + wr^2\cos\theta = 0$$

$$(-T_b + \frac{wr^2\phi}{2})\ \sin\theta - (M_b - wr^2)\ \cos\theta - wr^2 = 0$$

$$\operatorname{Tan}\theta = \frac{(M_b - wr^2) + \frac{wr^2}{\cos\theta}}{(\frac{wr^2\phi}{2} - T_b)} = \frac{(C_m - 1) + \frac{1}{\cos\theta}}{(\frac{\phi}{2} - C_t)} \qquad (25)$$

Deflection

By the moment-area method the vertical deflection can be calculated as follows:

$$\Delta_\theta = \frac{1}{EI}\int_0^\theta M_\theta r \sin(\frac{\emptyset}{2}-\theta)ds + \frac{1}{GJ}\int_0^\theta T_\theta (ds) r (1-\cos(\frac{\emptyset}{2}-\theta)$$

Under uniform load the maximum deflection occurs at the middle of the span, i.e., when $\theta=\emptyset/2$. Substituting equations 16 and 17 into the above equation the maximum deflection will be:

$$\max = \frac{wr^4}{EI}\Big[\sin\frac{\emptyset}{2}((\emptyset+\sin\emptyset)(C_m-1)/4 + (C_t-\frac{\emptyset}{2})(\frac{1-\cos\emptyset}{4}$$

$$+ m - m(1-\cos\emptyset)/4) - \cos\frac{\emptyset}{2}(C_m(1-\cos\emptyset)(1+m)/4$$

$$- (1-\cos\emptyset)(1-m)/4 + (C_t-\frac{\emptyset}{2})(\emptyset-\sin\emptyset+m\emptyset+m\sin\emptyset)/4$$

$$+ (1-\cos\frac{\emptyset}{2})(1+m)) + m ((\cos\frac{\emptyset}{2}-1)(C_m-1) + \frac{\emptyset^2}{8})$$

$$+ \sin\frac{\emptyset}{2}(1-m+m(C_m-1)(\emptyset-\sin\emptyset)/4)\Big]$$

$$= \frac{wr^4}{EI} C_d \qquad (26)$$

In equation 26, C_m and C_t are bending moment and torsional coefficients respectively; the values of C_m and C_t are shown in Figures 10 and 11. Upward deflection is negative and downward deflection is positive. The values of C_d are plotted into curves as shown in Figure 13.

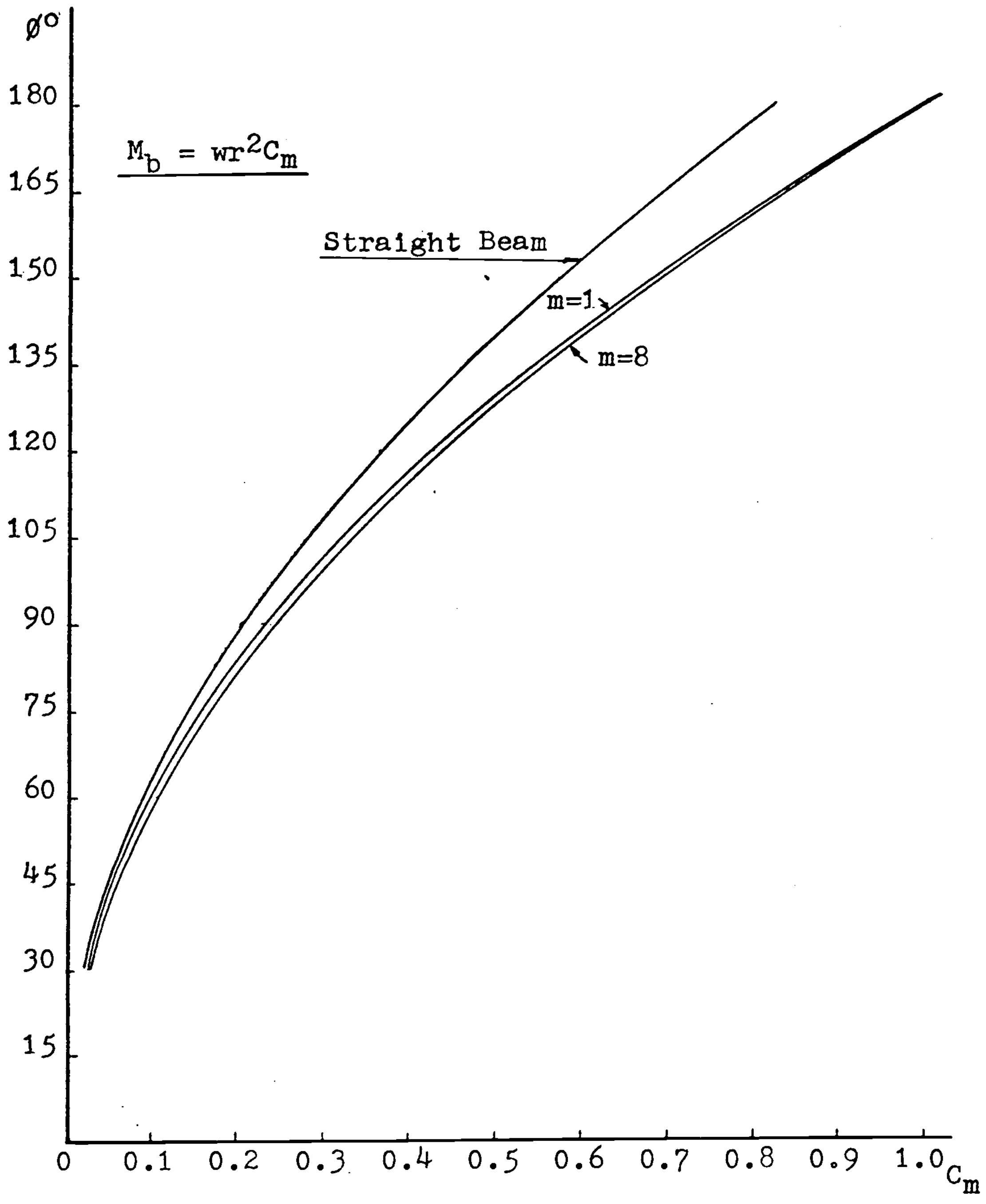


Figure 10. Variation of fixed end bending moment coefficients with span angle for uniformly loaded curved beams.

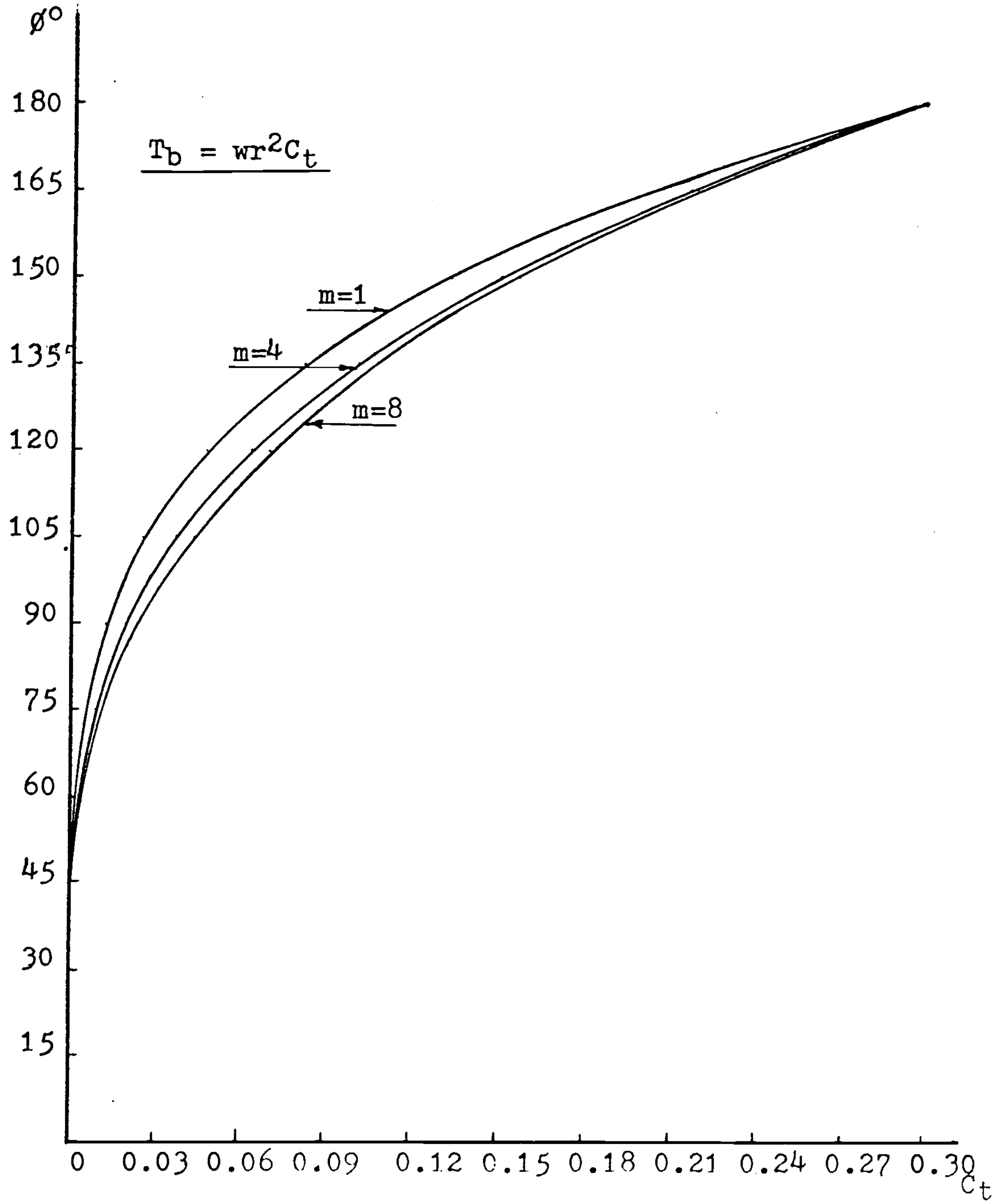


Figure 11. Variation of fixed end torsional moment coefficients with span angle for uniformly loaded curved beams.

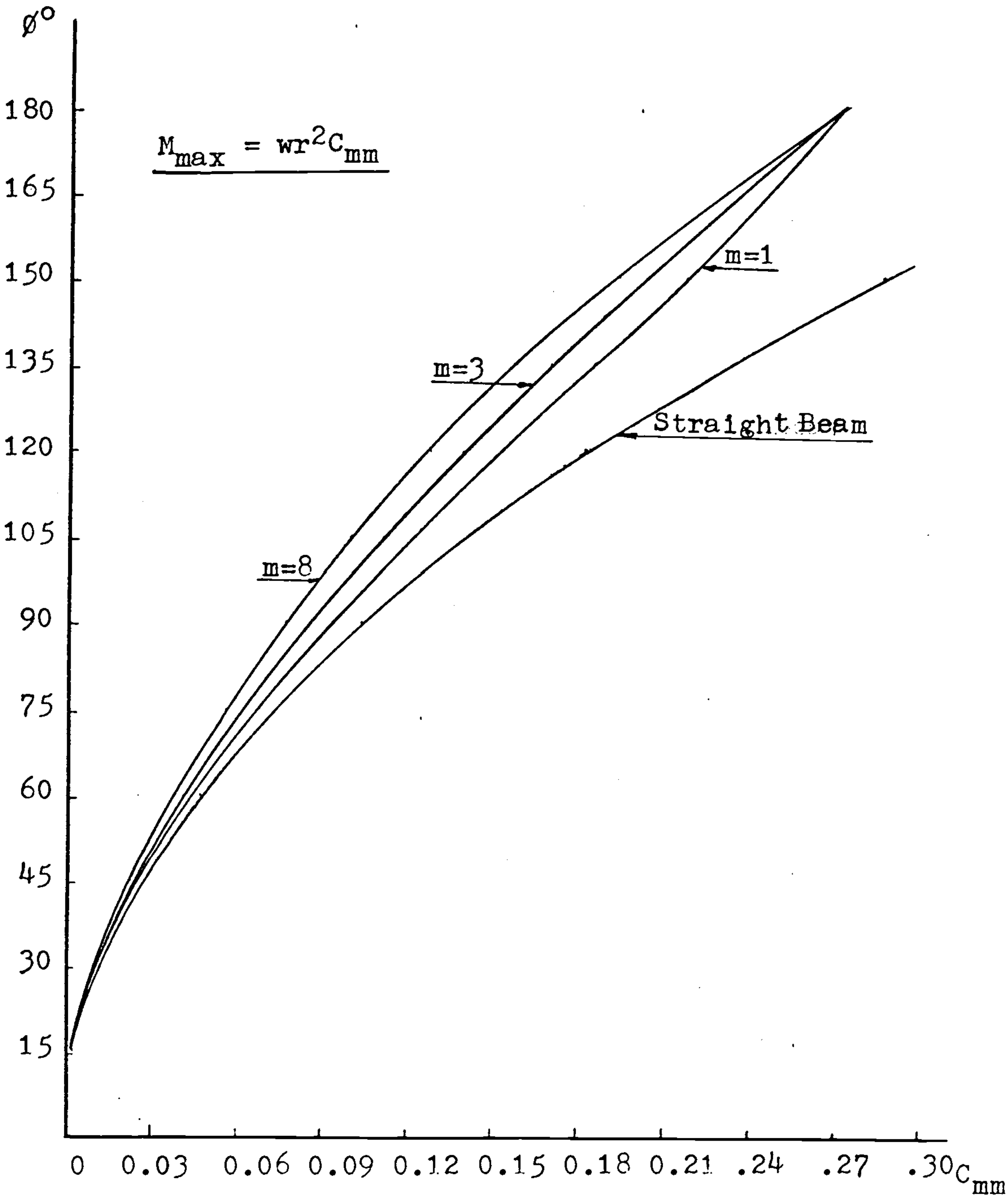


Figure 12. Variation of maximum span moment coefficients with span angle for uniformly loaded curved beam.

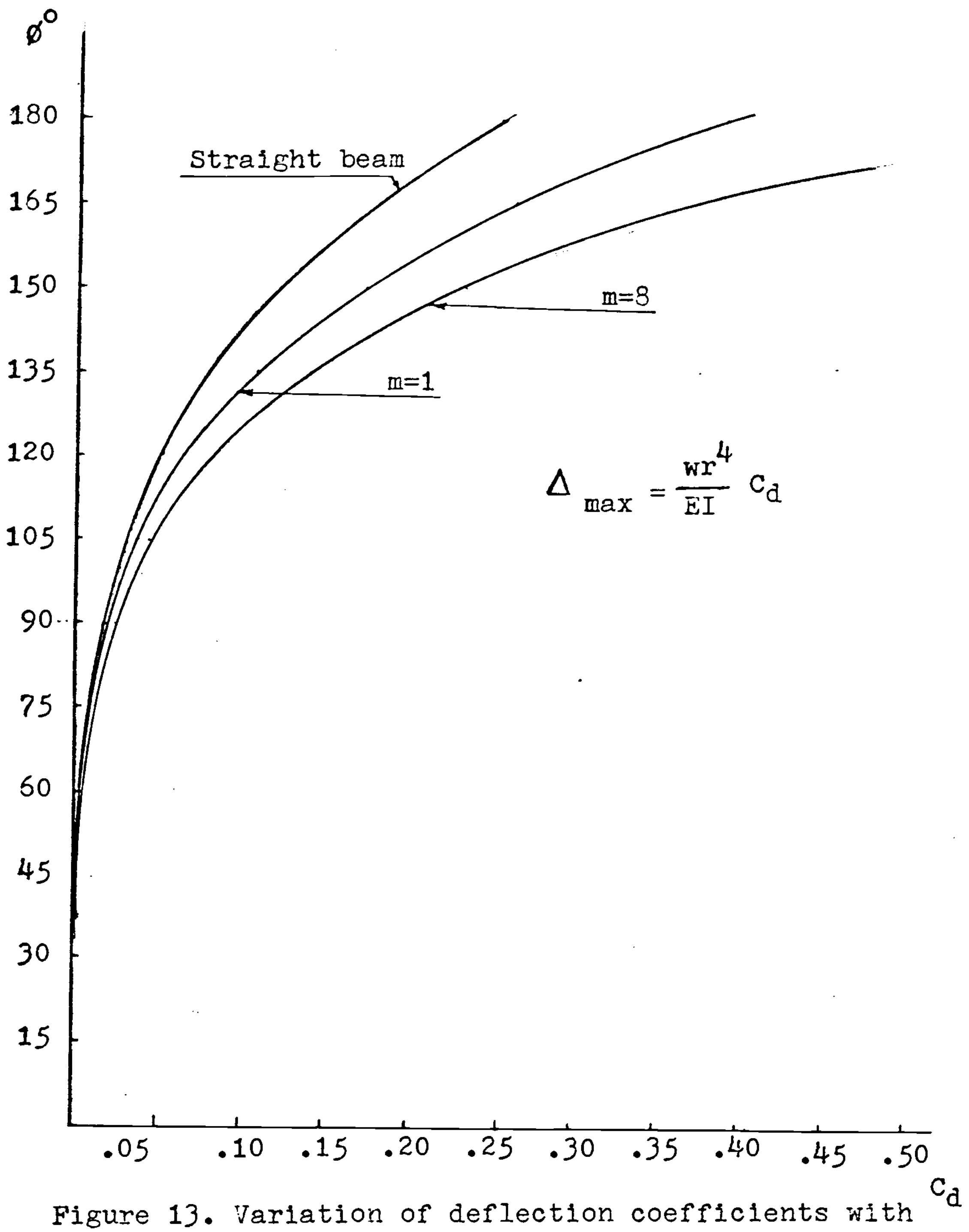


Figure 13. Variation of deflection coefficients with span angle for uniformly loaded curved beams.

V. ANALYSIS OF SHEARING STRESS IN HORIZONTALLY CURVED BEAM

The existance of torsional moment makes the distribution of shearing stress on a section of the curved beam more complicated than that of a straight beam. Generally speaking, two types of sections are used for horizontally curved beams: (a) rectangular sections and (b) box or closed sections. On either type of section the shearing stresses are eventually composed of two parts; i.e., those caused by vertical shear and those caused by torsion. This analysis will be concerned only with these two types of shearing stresses.

Shearing Stress due to Vertical Shear

This type of shear can be calculated by the formula

$$v = \frac{VQ}{Ib}$$

where $Q = \int_{y_1}^{c_1} y\,dA$

The variations of shearing stress on a rectangular and a box section are shown in Figure 14.

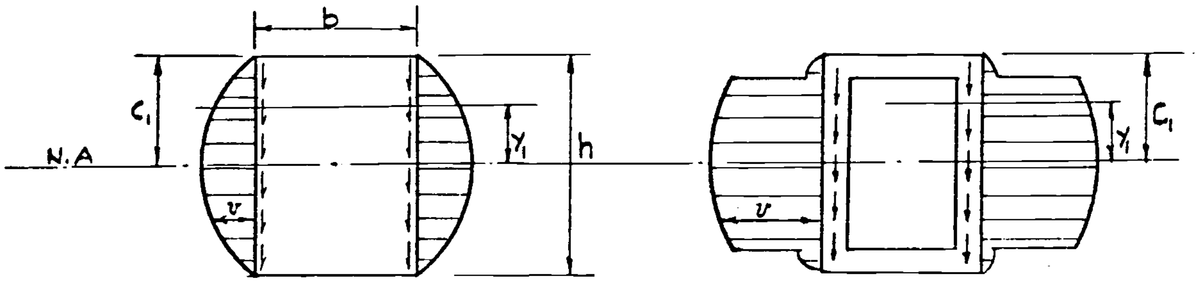


Figure 14

The maximum shearing stress distributes uniformly along the width at neutral axis. The shearing stress is signified as positive when it acts upward and as negative when it acts downward. On another hand, the positive shearing force produces positive shearing stress. This is illustrated in Figure 15.

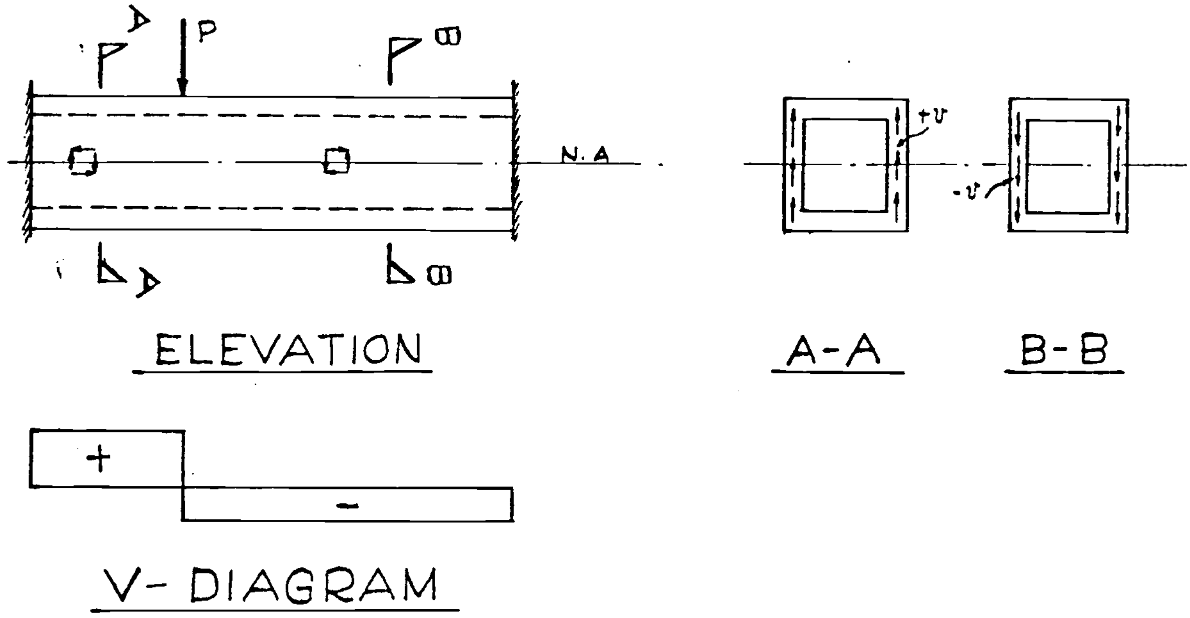


Figure 15.

As an example, in a horizontally curved beam of rectangular section

$$v_{max} = \frac{3}{2} \frac{V}{bh} \,*$$

Where V is value of shear force which can be determined as follows

(1) When the beam is subjected to uniform load

$$-v_\theta = F_b - wr\theta.$$

(2) When the beam is subjected to concentrated load

$$-v_\theta = F_b \qquad (0 \leq \theta \leq \phi_o)$$

$$-v_\theta = F_b - P \qquad (\phi_o \leq \theta \leq \phi)$$

*This formula is used only for analysis. In practical design $v = \frac{V}{bdJ}$ or $v = \frac{V}{bd}$ (1) for reinforced concrete rectangular beams.

Shearing Stress Caused by Torsion

Torsional moment produces pure shear which distributes around the whole section and theoretically varies with the polar distance measured from the centroid of the section for an isotropic material.

On rectangular section. Figure 16 shows the pattern of shearing stress distribution on a rectangular cross section when the beam is subjected to torsion (9).

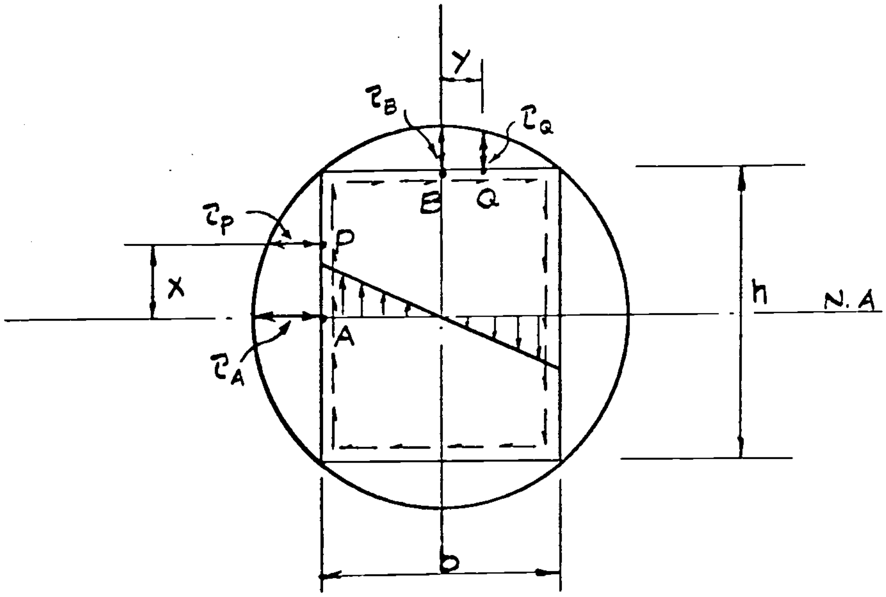


Figure 16

The maximum values of shearing stress will occur at the center of each side. The shearing stress at the mid-point of the longer side, such as τ_A, has the maximum value:

$$\tau_A = \left(\frac{1}{\alpha}\right) \left(\frac{T}{b^2 h}\right) \tag{27}$$

In equation 27 the values of α are listed in Table I.*

TABLE I. VALUES OF α.

h/b =	1.0	1.5	2.0	2.5	3.0	4.0	6.0	10.0	∞
α =	0.208	0.231	0.246	0.256	0. 267	0.282	0.299	0.312	0.333

*The values of α in this table are extracted from reference (9), p. 271.

The shearing stress at any point P on the longer side of the section with a distance x from the mid-point A is calculated by the formula

$$\tau_p = \tau_A \left(1 - (\frac{2\,x}{h})^2\right) \qquad (28)$$

The shearing stress at point B will be proportional to A and equal to

$$\tau_B = (\frac{b}{h})\ (\tau_A) = (\ \frac{1}{\alpha})\ (\ \frac{T}{b\ h^2}) \qquad (29)$$

The shearing stress at any point Q on the shorter side of the section with a distance y from the mid-point B is

$$\tau_Q = \tau_B \left((1 - (\frac{2y}{b})^2\right)$$
$$= \frac{b}{h} \left((1 - (\frac{2y}{b})^2\right)\tau_A \qquad (30)$$

The pure shear will be signified as positive when it is produced by positive torsional moment. That is, when looking along the tangent of the beam in a counterclock-wise sense it flows in a clockwise direction.

<u>On Box Section</u>

The shearing stress on a box section, or on a hollow tube of rectangular cross section, can be analyzed by means of the membrane or "soap-film" analogy theory (9).

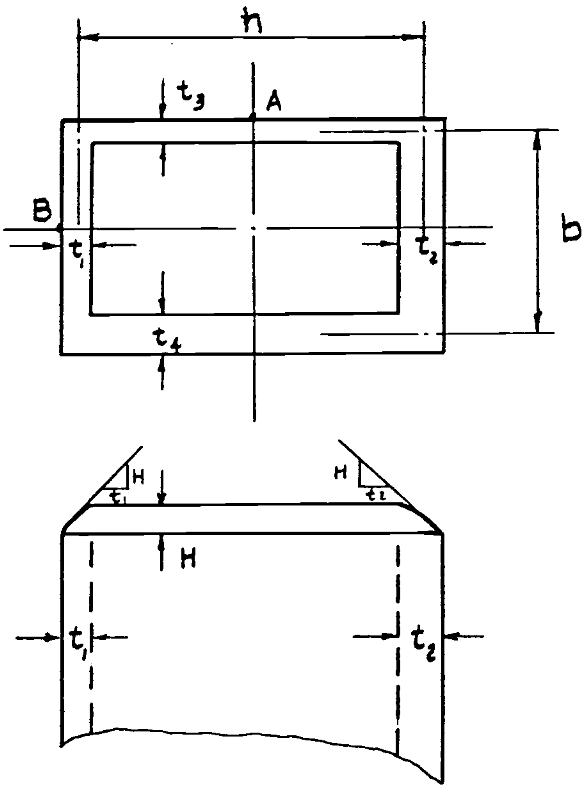


Figure 17

$$T = 2\,a\,H \qquad H = \frac{T}{2\,a}$$

where

T = the torsional moment

a = the area enclosed by mean perimeter of the wall of the tube. For instance, the section in Figure 17 will have the value of 'a' equal to bxh.

H = the height of the flat part of the surface along the hollow portion of the cross section.

The shearing stress will be

$$\left.\begin{aligned} \tau &= \frac{H}{t} = \frac{T}{2\,a\,t} \\ \tau_A &= \frac{T}{2\,b\,h\,t_3} \\ \tau_B &= \frac{T}{2\,b\,h\,t_2} \end{aligned}\right\} \qquad (31)$$

From equations 31 for a tube section the maximum shearing stress is inversely proportional to the thickness of the tube 't'.

Combination of Shearing Stresses due to Vertical Shear Force and Torsional Moment

Figure 18 shows the general pattern of bending moment and torsional moment for a horizontally curved beam subjected to uniform load.

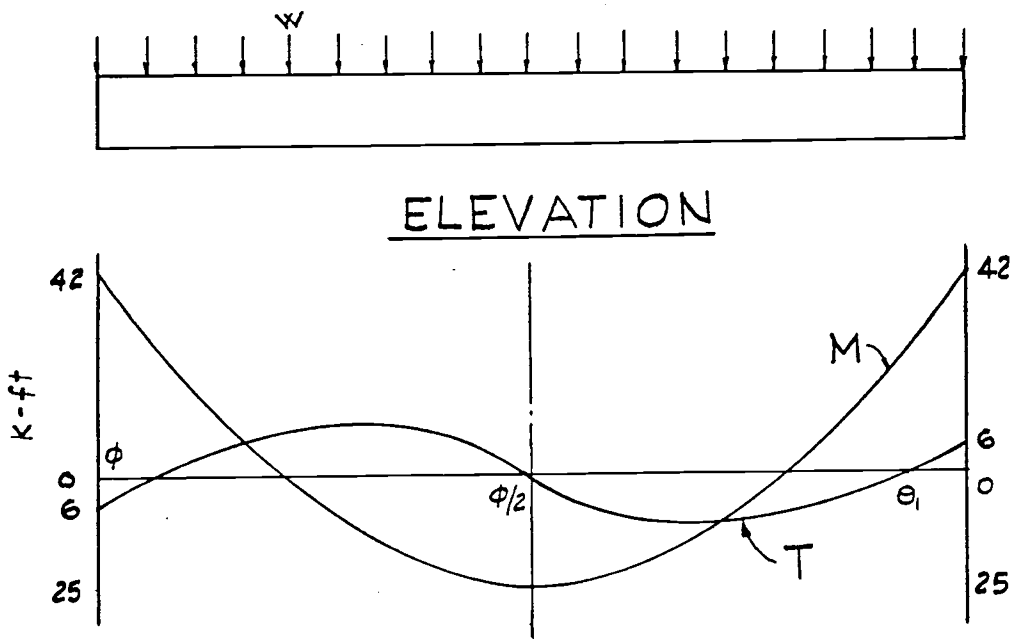


Figure 18 M and T - Diagram

We assume that at θ_1 the torsional moment is equal to zero in the portion $0 \leq \theta \leq \theta_1$. T_θ is positive and produces a positive shear flow around the section that is, the shear will act upward on the inside face of the section and downward on the outside face. In the portion $\theta_1 \leq \theta \leq \phi/2$ the torsional moment is negative and causes a negative shear flow, that is, the shear will act downward on the inside face and upward on the outside face of the section.

As both the bending moment and torsional moment vary with θ, the combination of the shearing stresses will vary with θ.

$$\tau' = v + \tau \tag{32}$$

where

τ' = combined shearing stress

v = vertical shearing stress due to vertical shear

τ = pure shear due to torsional moment

The values of τ' can be summarized as follows.

TABLE II.

Portion	Sign v	Sign τ	Combined shearing stress τ': inside face	Combined shearing stress τ': outside face
$0 \leq \theta \leq \theta_1$	-	+	$\tau' = \tau + v$	$\tau' = \tau - v$
$\theta_1 \leq \theta \leq \phi/2$	-	-	$\tau' = \tau - v$	$\tau' = \tau + v$
$\phi/2 \leq \theta \leq (\phi - \theta_1)$	+	+	$\tau' = \tau - v$	$\tau' = \tau + v$
$(\phi - \theta_1) \leq \theta \leq \phi$	+	-	$\tau' = \tau + v$	$\tau' = \tau - v$

In the same way the combination of shearing stresses for different loading conditions can be obtained.

Combination of Shearing Stress and Bending Stress

On the top or bottom of the section, the vertical shearing stress equals zero, but the shearing stress due to torsional moment is not zero and can be calculated by equation 29 and 31. This type of shear can be combined with the fiber stress. The combined stresses are expressed as follows

$$\sigma_n = \frac{\sigma_\theta}{2} \pm \sqrt{\left(\frac{\sigma_\theta}{2}\right)^2 + \tau_\theta^2} \qquad (33)$$

where

σ_n = the combined stress, tension is positive and compression is negative

σ_θ = fiber stress due to bending moment at θ, tension is positive and compression is negative.

τ = pure shear due to torsional moment at θ.

As the value of σ_θ and τ_θ vary with θ, the combined stress varies with θ.

VI. ANALYSIS OF CONTINUOUS HORIZONTALLY CURVED BEAM

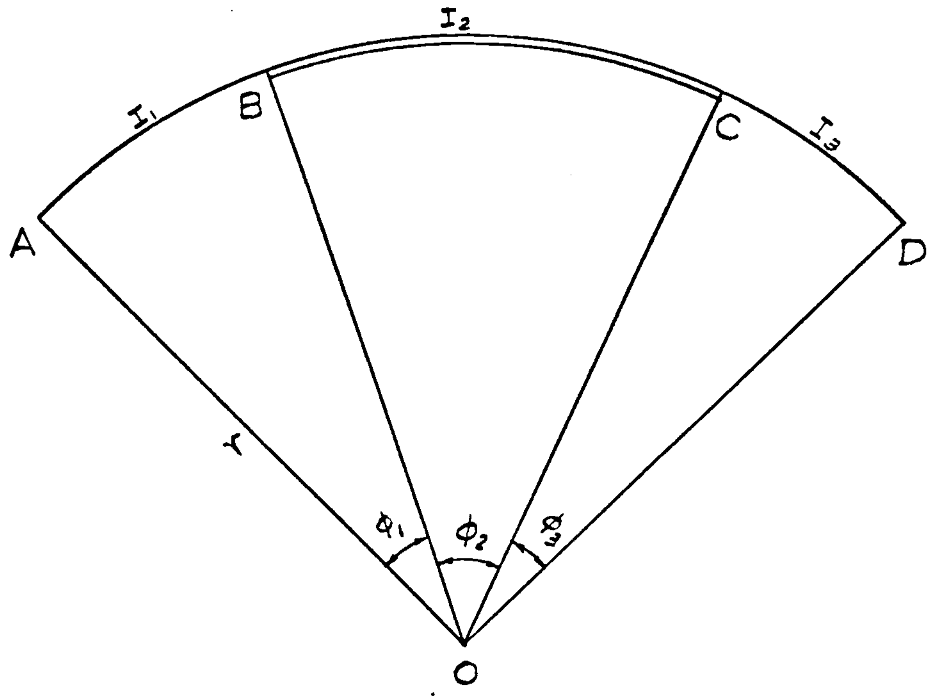


Figure 19. Continuous horizontally curved beam

A moment distribution method for the analysis of continuous horizontally curved beam is here introduced. Because of the presence of torsional moment there will be two sets of distribution and carry over factors, one for bending and one for torsion. The stiffness coefficients of a curved beam for bending moment or torsional moment are interpreted as the bending moment or the torsional moment that produces a unit rotation about the axis it rotates. Applying Castigliano's theory and assuming that deformations at supports due to vertical force are negligible we can derive the following equations.

$$\frac{\partial U}{\partial M_b} = \omega_b = 1$$

$$\frac{\partial U}{\partial T_b} = \varphi_b = 0$$

$$\frac{\partial U}{\partial F_b} = \Delta_b = 0 \qquad (34)$$

and

$$\frac{\partial U}{\partial M_b} = \omega_b = 0$$

$$\frac{\partial U}{\partial T_b} = \varphi_b = 1$$

$$\frac{\partial U}{\partial F_b} = \Delta_b = 0 \qquad (35)$$

Substituting the coefficients in equation 8 into 34 and 35 we obtain

$$a_1 M_b - b_1 T_b + C_1 F_b r = \omega_b \frac{2\ EI}{r} = \frac{2\ EI}{r}$$

$$- b_1 M_b + b_2 T_b + C_2 F_b r = \varphi_b \frac{2\ EI}{r} = 0$$

$$C_1 M_b + C_2 T_b + C_3 F_b r = \Delta_b \frac{2\ EI}{r} = 0 \qquad (36)$$

and

$$a_1 M_b - b_1 T_b + C_1 F_b r = \omega_b \frac{2\ EI}{r} = 0$$

$$- b_1 M_b + b_2 T_b + C_2 F_b r = \varphi_b \frac{2\ EI}{r} = \frac{2\ EI}{r}$$

$$C_1 M_b + C_2 T_b + C_3 F_b r = \Delta_b \frac{2\ EI}{r} = 0 \qquad (37)$$

Solve equation 35 for M_b T_b and F_b r.

$$T_b = \frac{b_1 C_3 + C_1 C_3}{b_2 C_3 - C_2^2} M_b \qquad (38)$$

$$F_b r = -\frac{(b_1C_2 + b_2C_1)}{C_3b_2 - C_2^2} M_b \qquad (39)$$

$$M_b \, a_1 - \frac{b_1(b_1C_3 - C_1C_3)}{b_2C_3 - C_2^2} - \frac{C_1(b_2C_1 + b_1C_2)}{C_3b_2 - C_2^2} = \frac{2EI}{r}$$

$$M_b = \left(\frac{2\,(b_2C_3 - C_2^2)}{a_1(b_2C_3 - C_2^2) - b_1(b_1C_3 + C_1C_2) - C_1(b_2C_1 + b_1C_2)}\right) \frac{EI}{r} \qquad (40)$$

Solve equation 37 for M_b, T_b and $F_b r$.

$$M_b = \frac{b_1C_3 + C_2C_1}{a_1C_3 - C_1^2} T_b \qquad (41)$$

$$F_b r = \frac{-(b_1C_1 + a_1C_2}{a_1C_3 - C_1 2} T_b \qquad (42)$$

$$T_b = \frac{2\,(a_1C_3 - C_1^2)}{b_2(a_1C_3 - C_1^2) - b_1(b_1C_3 + C_2C_1) - C_2(b_1C_1 + a_1C_2)} \frac{EI}{r} \qquad (43)$$

With M_b, T_b and F_b known, the bending moment and torsional moment at any angle θ from support B can be found by equations 1 and 2 listed as follows

$$M_\theta = M_b\cos\theta + T_b\sin\theta - F_b r\sin\theta$$

$$T_\theta = -M_b\sin\theta + T_b\cos\theta + F_b r(1-\cos\theta)$$

By making $\theta = \phi$, $M\phi = M_a$, $T\phi = T_a$ will be the bending moment and torsional moment at another support.

Expressing T_b, $F_b r$ in terms of M_b

$$M_a = M_b\cos\phi + T_b\sin\phi - F_b r\sin\phi = M_b\Big(\cos\phi + \frac{b_1C_3 + C_1C_2}{b_2C_3 - C_2^2}\sin\phi + \frac{(b_1C_2 + b_2C_1)}{C_3b_2 - C_2^2}\sin\phi\Big)$$

$$= M_b m_{ba} \qquad (44)$$

Expressing M_b, F_br in terms of T_b

$$M_a = T_b \left(\frac{b_1C_3 + C_2C_1}{a_1C_3 - C_1^2} \cos\phi + \sin\phi + \frac{b_1C_1 + a_1C_2}{a_1C_3 - C_1^2} \sin\phi\right)$$

$$= T_b \, t_b \, m_a \qquad (45)$$

Expressing M_a in terms of T_a from equation 41

$$M_a = \frac{b_1C_3 + C_2C_1}{a_1C_3 - C_1^2} T_a = T_a \, t_a \, m_a \qquad (46)$$

Expressing M_b, F_br in terms of T_b, from equation 2

$$T_a = T_b\left[\frac{b_1C_3 + C_2C_1}{a_1C_3 - C_1^2}(-\sin\phi) + \cos\phi - \frac{a_1C_2 + b_1C_1}{a_1C_3 - C_1^2}(1-\cos\phi)\right] = T_b \, t_{ba} \qquad (47)$$

Expressing T_a in terms of M_a from equation 38

$$T_a = \frac{b_1C_3 + C_1C_2}{b_2C_3 - C_2^2} M_a = M_a \, m_{a}t_{a} \qquad (48)$$

Expressing T_b and F_br in terms of M_b from equation 2

$$T_a = M_b\left[-\sin\phi + \left(\frac{b_1C_3 + C_1C_2}{b_2C_3 - C_2^2}\right)\cos\phi - \left(\frac{b_1C_2 + b_2C_1}{C_3b_2 - C_2^2}\right)(1-\cos\phi)\right] = M_bM_bt_a \qquad (49)$$

Using subscript 1 indicating the left-hand support and subscript 2 indicating the right-hand support of the curved beam, we may summarize the stiffness factors and carry over factors as follows.

(1) Stiffness factors

for bending moment

$$S_m = \frac{2\,(b_2C_3 - C_2^2)}{a_1(b_2C_3 - b_3C_2) - b_1(b_1C_3 + C_1C_2) + C_1(b_2C_1 + b_1b_3)} \frac{EI}{r} \qquad (50)$$

for torsional moment

$$S_t = \frac{2\ (a_1C_3 - C_1^2)}{b_2(a_1C_3 - C_1^2) - b_1(b_1C_3 + C_1C_2) - C_2(b_1C_1 + a_1C_2)} \frac{EI}{r} \quad (51)$$

(2) Carry over factors

$$M_{21} = (\cos\emptyset + \frac{b_1C_3 + C_1C_2}{b_2C_3 - C_2^2}\sin\emptyset + \frac{b_1C_2 + b_2C_1}{C_3b_2 - C_2^2}\sin\emptyset) \quad (52)$$

$$t_2m_1 = \frac{b_1C_3 + C_2C_1}{a_1C_3 - C_1^2}\cos\emptyset + \sin\emptyset + \frac{b_1C_1 + a_1C_2}{a_1C_3 - C_1^2}\sin\emptyset \quad (53)$$

$$t_1m_1 = \frac{b_1C_3 + C_2C_1}{a_1C_3 - C_1^2} \quad (54)$$

$$t_{21} = \frac{b_1C_3 + C_2C_1}{a_1C_3 - C_1^2}\sin\emptyset + \cos\emptyset - \frac{a_1C_2 + b_1C_1}{a_1C_3 - C_1^2}(1 - \cos\emptyset) \quad (55)$$

$$m_1t_1 = \frac{b_1C_3 + C_1C_2}{b_2C_3 - C_2^2} \quad (56)$$

$$m_2t_1 = \frac{b_1C_3 + C_1C_2}{b_2C_3 - C_2^2}\cos\emptyset - \sin\emptyset - \frac{b_2C_2 + b_2C_1}{C_3b_2 - C_2^2}(1 - \cos\emptyset) \quad (57)$$

$$M_1 = M_2m_{21}$$

$$M_1 = T_2t_2m_1$$

$$M_1 = T_1t_1m_1$$

$$T_1 = T_2t_{21}$$

$$T_1 = M_1m_1t_1$$

$$T_1 = M_2m_2t_1 \quad (58)$$

$$M_2 = M_1\ m_{12}$$

$$M_2 = T_1\ t_1\ m_1$$

$$M_2 = T_2\ t_2\ m_2$$

$$T_2 = T_1\ t_{12}$$

$$T_2 = M_2\ m_2\ t_2$$

$$T_2 = M_1\ m_1\ t_2 \qquad (59)$$

To apply the moment distribution method to the horizontally curved beam design it would be necessary to review the sign convention. Figure 20 shows the positive sign convention that could be applied to the end of any horizontally curved beam.

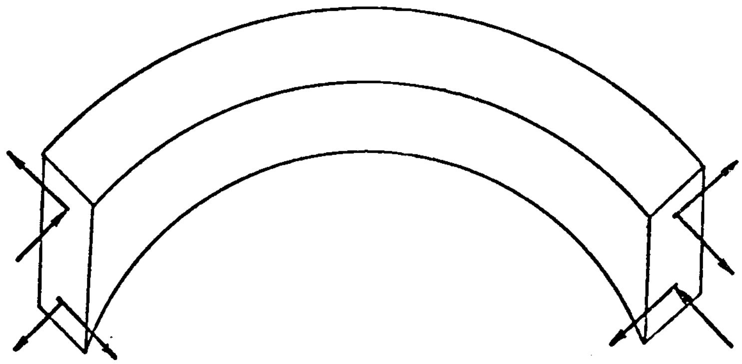


Figure 20. Positive sign convention.

By using this convention the following relationships are obvious.

$$m_{12} = m_{21}$$

$$t_{12} = t_{21}$$

$$t_1\ m_1 = -t_2\ m_2$$

$$m_1\ t_1 = -m_2\ t_2$$

$$m_1 t_2 = -t_2 m_1$$

$$t_1 m_2 = -m_2 t_1 \qquad (60)$$

Figure 21 shows the application of the above carry-over factors. The sign before the factor indicates its proper sign. The stiffness factors and carry-over factors for different values of θ and m are calculated by digital computer; see appendix, and listed in Table III.

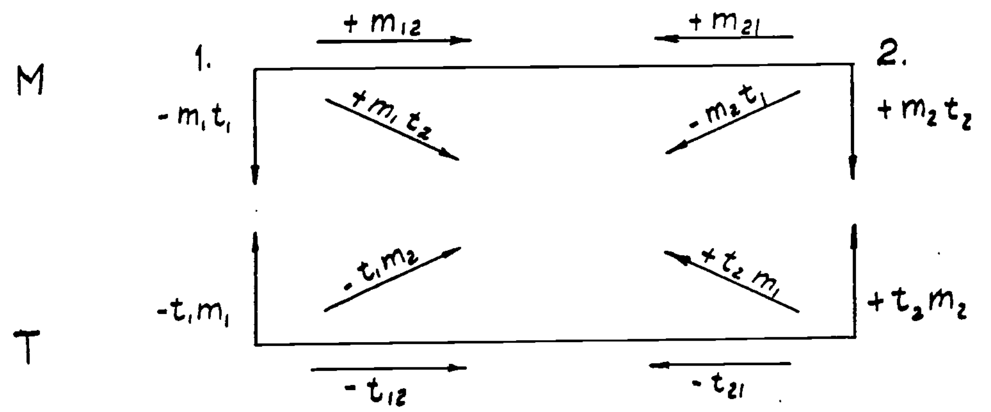


Figure 21. Proper signs for carry-over factors.

TABLE III. STIFFNESS AND CARRY-OVER COEFFICIENTS.

	m	S_m	S_t	M_{21}	t_2m_1	t_1m_1	t_{21}	m_1t_1	m_2t_1
$\phi = 15°$	1	15.175	3.907	-0.505	-0.001	.254	.967	.065	.000
	2	15.140	1.996	-0.507	-0.003	.496	.935	.065	.000
	3	15.106	1.359	-0.508	-0.007	.726	.905	.065	.001
	4	15.073	1.041	-0.510	-0.011	.945	.877	.065	.001
	5	15.039	.849	-0.512	-0.016	1.154	.850	.065	.001
	6	15.006	.722	-0.513	-0.021	1.354	.825	.065	.001
	7	14.973	.630	-0.515	-0.028	1.545	.800	.065	.001
	8	14.940	.562	-0.517	-0.035	1.727	.777	.065	.001
$\phi = 30°$	1	7.433	2.081	-0.521	-0.009	.467	.877	.131	.002
	2	7.368	1.123	-0.527	-0.023	.855	.777	.130	.004
	3	7.305	.802	-0.533	-0.042	1.182	.695	.130	.005
	4	7.244	.640	-0.539	-0.065	1.462	.626	.129	.006
	5	7.184	.542	-0.545	-0.090	1.706	.567	.129	.007
	6	7.127	.476	-0.551	-0.116	1.920	.517	.128	.008
	7	7.070	.428	-0.556	-0.144	2.110	.473	.128	.009
	8	7.015	.391	-0.561	-0.173	2.281	.435	.127	.010
$\phi = 45°$	1	4.789	1.524	-0.546	-0.026	.617	.755	.196	.008
	2	4.700	.878	-0.560	-0.064	1.040	.596	.194	.012
	3	4.616	.657	-0.572	-0.107	1.353	.484	.193	.015
	4	4.537	.543	-0.583	-0.154	1.595	.403	.191	.018
	5	4.463	.472	-0.594	-0.201	1.791	.342	.190	.021
	6	4.392	.423	-0.604	-0.249	1.953	.294	.188	.024
	7	4.324	.386	-0.613	-0.297	2.090	.257	.187	.027
	8	4.260	.358	-0.622	-0.344	2.210	.227	.186	.029
$\phi = 60°$	1	3.426	1.279	-0.582	-0.053	.700	.627	.261	.020
	2	3.320	.781	-0.603	-0.118	1.093	.438	.257	.028
	3	3.225	.604	-0.621	-0.186	1.352	.327	.253	.035
	4	3.138	.509	-0.638	-0.252	1.541	.256	.250	.041
	5	3.057	.448	-0.652	-0.316	1.687	.209	.247	.046
	6	2.983	.404	-0.665	-0.378	1.806	.176	.245	.051
	7	2.914	.371	-0.676	-0.437	1.905	.152	.242	.056
	8	2.849	.344	-0.687	-0.493	1.991	.136	.240	.060
$\phi = 75°$	1	2.580	1.153	-0.627	-0.089	.729	.509	.326	.040
	2	2.466	.735	-0.655	-0.180	1.067	.319	.318	.054
	3	2.367	.579	-0.678	-0.266	1.275	.225	.312	.065
	4	2.279	.492	-0.697	-0.345	1.422	.174	.307	.074
	5	2.200	.434	-0.713	-0.418	1.535	.143	.302	.082
	6	2.128	.391	-0.727	-0.486	1.626	.125	.299	.089
	7	2.062	.358	-0.739	-0.548	1.702	.115	.295	.095
	8	2.001	.332	-0.749	-0.606	1.767	.109	.293	.100

TABLE III (Continued)

	m	S_m	S_t	M_{21}	t_2m_1	t_1m_1	t_{21}	m_1t_1	m_2t_1
∅ = 90°	1	1.999	1.080	-0.681	-0.130	.720	.411	.389	.070
	2	1.884	.708	-0.714	-0.243	1.003	.239	.377	.091
	3	1.787	.562	-0.739	-0.340	1.171	.169	.368	.107
	4	1.703	.477	-0.758	-0.426	1.290	.136	.361	.119
	5	1.629	.419	-0.774	-0.501	1.380	.121	.355	.129
	6	1.562	.377	-0.786	-0.569	1.453	.115	.351	.137
	7	1.502	.344	-0.797	-0.629	1.515	.115	.347	.144
	8	1.447	.317	-0.806	-0.684	1.567	.117	.344	.150
∅ = 105°	1	1.574	1.033	-0.743	-0.175	.685	.335	.450	.115
	2	1.463	.687	-0.776	-0.303	.923	.191	.433	.142
	3	1.372	.545	-0.800	-0.406	1.063	.144	.422	.161
	4	1.294	.461	-0.817	-0.492	1.162	.128	.414	.175
	5	1.227	.404	-0.831	-0.566	1.237	.125	.407	.186
	6	1.167	.361	-0.841	-0.629	1.298	.128	.402	.195
	7	1.113	.328	-0.850	-0.685	1.349	.134	.398	.202
	8	1.064	.301	-0.857	-0.733	1.392	.141	.394	.208
∅ = 120°	1	1.252	.997	-0.809	-0.222	.636	.282	.507	.176
	2	1.149	.666	-0.839	-0.359	.839	.169	.486	.208
	3	1.065	.526	-0.859	-0.463	.958	.142	.473	.229
	4	.995	.443	-0.872	-0.546	1.043	.139	.464	.243
	5	.934	.386	-0.882	-0.614	1.107	.145	.457	.253
	6	.881	.343	-0.890	-0.671	1.159	.154	.452	.262
	7	.834	.311	-0.896	-0.719	1.202	.165	.448	.268
	8	.792	.284	-0.901	-0.761	1.238	.175	.444	.273
∅ = 135°	1	1.003	.964	-0.876	-0.268	.580	.249	.557	.258
	2	.908	.643	-0.899	-0.410	.755	.166	.535	.290
	3	.833	.505	-0.912	-0.510	.860	.156	.522	.309
	4	.769	.422	-0.921	-0.587	.934	.163	.513	.322
	5	.716	.366	-0.928	-0.648	.990	.175	.506	.332
	6	.669	.325	-0.932	-0.698	1.034	.188	.501	.338
	7	.628	.292	-0.936	-0.739	1.070	.201	.498	.344
	8	.593	.267	-0.939	-0.774	1.100	.212	.495	.348
∅ = 150°	1	.808	.931	-0.937	-0.314	.520	.234	.598	.362
	2	.722	.618	-0.950	-0.455	.675	.179	.578	.389
	3	.653	.482	-0.957	-0.549	.768	.182	.566	.404
	4	.597	.401	-0.961	-0.618	.833	.196	.559	.414
	5	.550	.345	-0.964	-0.670	.882	.212	.553	.421
	6	.510	.305	-0.967	-0.712	.919	.227	.550	.426
	7	.476	.274	-0.969	-0.746	.950	.240	.547	.430
	8	.445	.249	-0.970	-0.775	.975	.252	.545	.433

TABLE III (Continued)

	m	S_m	S_t	M_{21}	t_2m_1	t_1m_1	t_{21}	m_1t_1	m_2t_1
Ø = 165°	1	.655	.894	-0.982	-0.358	.459	.237	.626	.489
	2	.576	.589	-0.986	-0.494	.599	.204	.612	.505
	3	.514	.456	-0.988	-0.579	.682	.217	.605	.514
	4	.465	.378	-0.989	-0.639	.739	.236	.601	.519
	5	.424	.324	-0.990	-0.683	.782	.253	.598	.522
	6	.390	.285	-0.991	-0.718	.814	.269	.595	.525
	7	.361	.255	-0.991	-0.746	.840	.282	.594	.527
	8	.335	.231	-0.992	-0.768	.861	.293	.592	.528
Ø = 180°	1	.535	.854	-1.000	-0.399	.399	.254	.637	.637
	2	.462	.558	-1.000	-0.526	.526	.240	.637	.637
	3	.406	.430	-1.000	-0.601	.601	.259	.637	.637
	4	.362	.354	-1.000	-0.652	.652	.280	.637	.637
	5	.327	.302	-1.000	-0.689	.689	.298	.637	.637
	6	.298	.265	-1.000	-0.716	.716	.313	.637	.637
	7	.274	.236	-1.000	-0.738	.738	.326	.637	.637
	8	.253	.213	-1.000	-0.756	.756	.336	.637	.637

VII. ILLUSTRATED EXAMPLES

Example 1. Application of the Moment Distribution Method

A continuous beam loaded uniformly is shown in Figure 13. The loading and geometrical conditions are given as follows. Apply the moment distribution method to calculate bending and torsional moments at each support.

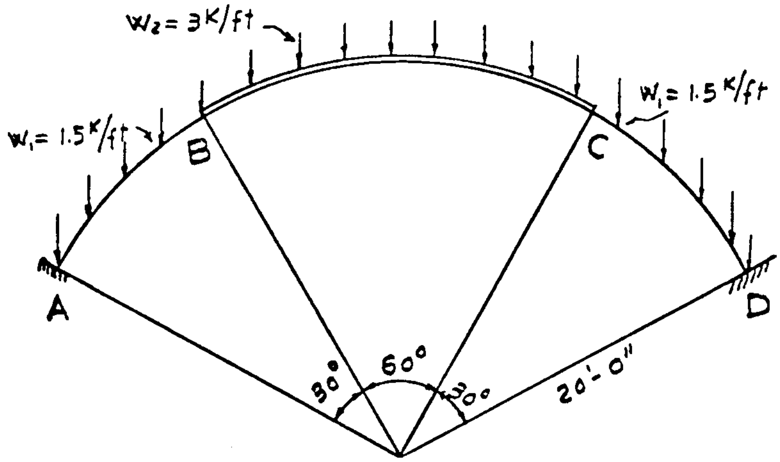


Figure 22.

Given:

$I_1 = 1$

$I_2 = 2\ I_1 = 2$

$m = \frac{EI}{GJ}$, $m_{AB} = m_{CD} = 1$

$m_{BC} = 2$

$E_{AB} = E_{BC} = E_{CD}$

Req'd: M_A, M_B, M_C, M_D, T_A, T_B, T_C, T_D.

Solution:

From Table III the stiffness factors and carry-over factors are listed as follows:

Span AB, CD		Span BC	
ϕ_1	$= 30^\circ \quad m = 1$	ϕ_2	$= 60^\circ \quad m = 2$
s_m	$= 7.433$	s_m	$= 3.320$
s_t	$= 2.081$	s_t	$= 0.781$
M_{21}	$= 0.521$	M_{21}	$= 0.603$
$t_2\ m_1$	$= 0.009$	$t_2\ m_1$	$= 0.118$
$t_2\ m_2$	$= 0.467$	$t_2\ m_2$	$= 1.093$
t_{21}	$= -0.877$	t_{21}	$= -0.438$
$m_2\ t_2$	$= 0.131$	$m_2\ t_2$	$= 0.257$
$m_2\ t_1$	$= -0.002$	$m_2\ t_1$	$= -0.028$

Distribution factor for bending moment

member	Sm		I/r		D. F.
AB	7.433	x	1/20	= 0.37165	0.527
BC	3.320	x	2/20	= 0.3320	0.473
			Total	= 0.70365	1.000

Distribution factor for torsional moment

member	S_t		I/r		D. F.
AB	2.081	x	1/20	= 0.10405	0.571
BC	0.781	x	2/20	= 0.0781	0.429
			Total	= 0.18215	1.000

fixed end moments from Figure 10.

$$M_{AB} = -M_{BA} = w\,r^2\,C_m = 1.5 \times (20)^2 \times 0.0231 = -13.9 \text{ k - ft.}$$

$$M_{BC} = -M_{CB} = 3.0 \times (20)^2 \times 0.0971 = -116.5 \text{ k - ft.}$$

fixed end torsional moments from Figure 11.

$$T_{AB} = T_{BA} = r^2 C_t = 1.5 \times 20^2 \times 0.0005 = 0.3 \text{ k - ft.}$$

$$T_{BC} = T_{CB} = 3.0 \times 20^2 \times 0.00232 = 2.78 \text{ k - ft.}$$

Because of the symmetry of the beam, we only operate the method for the left-half.

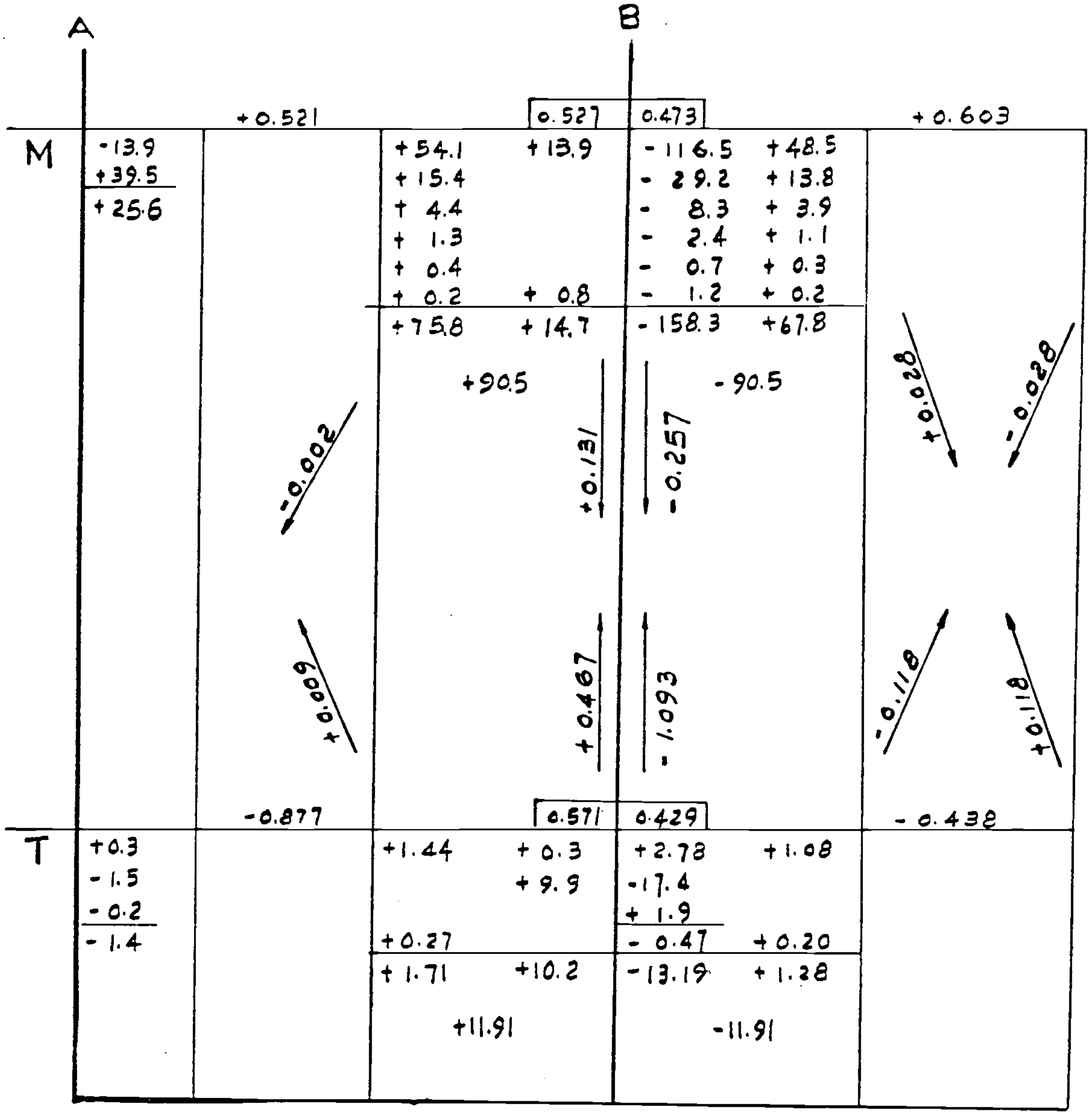


Figure 23. Moment distribution method

In Figure 23 the distribution and carry-over of the bending moments are shown at the top and are primarily carried out in a manner similar to that for a straight beam. The distribution and carry-overs of the torsional moments are shown at the bottom of the figure. To illustrate how to carry out the distribution and carry-overs, we consider joint B.

The unbalanced moment is - 116.5 + 13.9 = - 102.6 k - ft.

The distributions are:

$M'_{BA} = 0.527 \times (+ 102.6) = + 54.1$

$M'_{BC} = 0.473 \times (+ 102.6) = + 48.5$

The first carry-over from joint C is (- 48.5) x 0.603 = 29.2.

The distributions of this carry-over are:

$M'_{BA} = 0.527 \times (+29.2) = +15.4$

$M'_{BC} = 0.473 \times (+ 29.2) = + 13.8.$

Then the next carry-over from joint C is calculated and distributed and so on.

The total distributed bending moments M_{BA} and M_{BC} are added up and equal to 75.6 k - ft. and 67.6 k - ft respectively. They are then carried over to the torsional moment area and recorded.

These carry-overs are:

from $M_{BA}' = 0.131 \times (+ 75.6) = + 9.9$

from $M_{BC}' = (-0.257) \times (+67.6) = -17.4$.

Carry over from joint C $M_{CB}' = (-0.028) \times (-67.6)$

$= +1.9$.

The first distributions for torsion are

$T_{BA}' = -(0.571) \times (0.3 + 9.9 + 2.78 - 17.4 + 1.9)$

$= +1.44$

$T_{BC}' = (0.429) \times (2.52) = +1.08$.

The first carry-over from joint C is

$(-0.438) \times (+1.08) = -0.47$.

which is distributed to spans BA and BC.

$T_{BA}' = (0.571)(+0.47) = +0.27$

$T_{BC}' = (0.429)(+0.47) = +0.19$.

Then the total distributed torsional moments are carried over back to the bending moment area. One cycle of distribution and carry-over is completed. For different accuracy requirements, these cycles can be carried out any number of times. But it is found that because the values converge rapidly, not more than two cycles will give sufficient accuracy for design purposes.

Then, the total distributed bending moment and torsional moment are carried over to the end support A as follows:

M_A:

fixed end moment		13.9
carry over from M_{BA}'	$= 0.521 \times (+75.8) =$	+39.5
carry over from T_{BA}'	$0.009 \times (+1.71) =$	0.0
	total =	+25.6

T_A:

fixed end torsional moment	+0.3
carry over from M'_{BA} = (-0.002) x (+ 75.8) =	-1.5
carry over from T'_{BA} = (-0.877) x (+ 1.71) =	-0.2
Total =	-1.4

EXAMPLE 2.

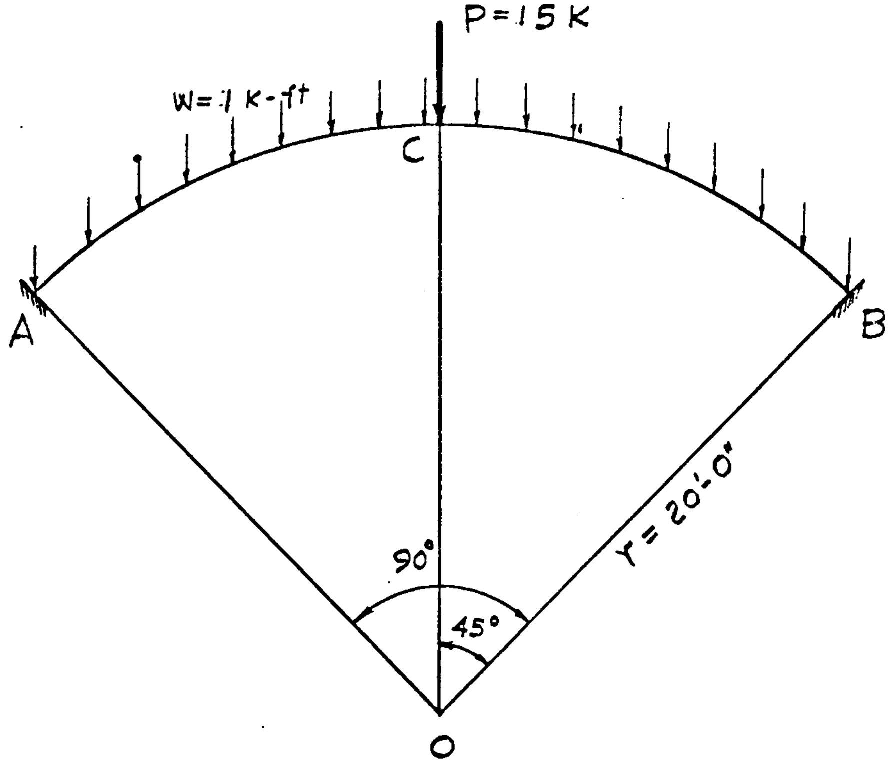


Figure 24.

A reinforced concrete bow girder is designed to resist the loads shown in Figure 24. Select a proper section and determine the required reinforcement. Poisson's ratio for concrete is 0.125, E = 2.25G, f_s = 20000 psi, f'_c = 3000 psi.

Solution

By Working Stress Design method

1. Bending moments and torsional moments

assume $d/b = 2.0$

$$I = \frac{bd^3}{12} = \frac{2}{3} b^4$$

$$J = \frac{1}{3} (d - 0.63\ b)\ b^3 = \frac{b^4}{3} (1.37) = 0.457\ b^4$$

From equations 10, 22 and 23

$M_b = P\ r\ C_m$ $\quad T_b = P\ r\ c_t$ $\quad M_{max} = P\ r\ C_{mm}$

$M_b = w r^2\ C_m$ $\quad T_b = w\ r\ C_t$ $\quad M_{max} = w r^2\ C_{mm}$

From figures 5,6,7,10, 11, and 12, with $m = 3.285$

For uniform load

$C_m = 0.233$ $\quad C_t = 0.018$ $\quad C_{mm} = 0.085$

For concentrated load

$C_m = 0.229$ $\quad C_t = 0.022$ $\quad C_{mm} = 0.176$

Try $d = 28$ in., $b = 14$ in., the overall depth $= 28 + 2$

$= 30$ in.

$$D.L. = \frac{30 \times 14}{144} \times 150 = 425 \text{ lb/ft.} = 0.425\text{k/ft.}$$

Fixed end bending moment

$$M_b = -\ (0.425 + 1) \times 20^2 \times 0.233 + 15 \times 20 \times 0.299$$

$= -\ (132 + 89.6)$

$= -\ 221.6$ ft - kips.

Mid-span bending moment

$$M_c = 1.425 \times 20^2 \times 0.085 + 15 \times 20 \times 0.176$$
$$= 48.6 + 52.8$$
$$= 101.4 \text{ ft-kips.}$$

Fixed end torsional moment

$$T_B = -T_A = 1.425 \times 20^2 \times 0.018 + 15 \times 20 \times 0.022$$
$$= 10.2 + 6.7$$
$$= 16.9 \text{ ft-kips.}$$

Check the assumed d value

$$d = \sqrt{\frac{M_b}{jkbf_c\ /2}}$$

$f_c = 1350$ psi $k = 0.383$ $j = 0.872$

$$d = \sqrt{\frac{221600 \times 12}{236 \times 14}} = 28.2 \text{ in.}$$

Use d = 29 in. and b = 14 in.

2. Determine the reinforcement required to resist bending moments.

(1) at support

$$A_s = \frac{M}{f_s jd} = \frac{221.6 \times 12}{20 \times 0.872 \times 29} = 5.14 \text{ in.}^2$$

Use 2#8 and 2#11 round bars, $A_s = 5.12$ in^2.

(2) at mid-span

$$A_s = 5.14 \times \frac{101.4}{221.6} = 2.4 \text{ in}^2.$$

Use 3 #8 round bars, $A_s = 2.37$ in^2.

3. Shearing stresses

(1) at support

From Table I. with $\frac{h}{b} = \frac{29}{14} = 2.1 \qquad \alpha = 0.25$

$$\tau = \frac{1}{\alpha}\left(\frac{T}{b^2h}\right) = \frac{1}{0.25}\left(\frac{16.9 \times 12000}{14^2 \times 29}\right)$$

$$= 217 \text{ psi.}$$

$$V = 0.5P + \frac{wr\phi}{2}$$

$$= 0.15 \times 15 + 1.425 \times 20 \times \frac{\pi}{4}$$

$$= 7.5 + 22.4 = 29.9 \text{ kips.}$$

$$v = \frac{V}{bd} = \frac{29900}{14 \times 29} = 73.7 \text{ psi.}$$

The combined shearing $\tau' = \tau + v = 290.7$ psi.

The allowable shearing stress

$$v_c = 1.1\sqrt{f'_c} = 60 \text{ psi}$$

The shearing stress carried by the web reinforcement will be

$$v' = \tau' - v_c = 290.7 - 60 = 230.7 \text{ psi.}$$

#4 stirrups will be used for the web reinforcement

$$A_v = 2 \times 0.20 = 0.40 \text{ in}^2.$$

The required stirrup spacing is

$$s = \frac{A_v f_v d}{v' bd} = \frac{A_v f_v}{v' \, b} = \frac{0.44 \times 20000}{230.7 \times 14} = 2.7 \text{ in.}$$

Use s = 3 in.

(2) in span $0 \leq \theta \leq \phi/2$ measured from support B.

From equation 27, $\tau = \frac{12000T_\theta}{0.248 \times 29 \times 14^2} = 8.6T_\theta$

From equations 2 and 16

$$T_\theta = -M_b\sin\theta + T_b\cos\theta + F_b r(1-\cos\theta) - wr^2(\theta-\sin\theta)$$

$$T_\theta = -(221.6-1.425 \times 20^2)\sin\theta + 16.9\cos\theta$$
$$+ 29.9 \times 20(1-\cos\theta) - 1.425 \times 20^2\theta$$
$$= 348.4\sin\theta + 16.9\cos\theta + 598 - 598\cos\theta - 570\theta$$
$$= 348.4\sin\theta - 581.1\cos\theta + 598 - 570\theta$$

$$\tau_\theta = 8.6\, T_\theta$$
$$= 2980\sin\theta - 4990\cos\theta + 5130 - 4890\theta$$

$$v_\theta = \frac{V}{bd} = \frac{1000}{14 \times 29} V = 2.46\, V$$

$$V = F_b - wr\theta$$
$$= 29.9 - 1.425 \times 20\theta = 29.9 - 28.5\,\theta$$

$$v_0 = 2.46\, V = 2.46(29.9 - 28.5\,\theta) = 73.6 - 70.2\,\theta$$

The combined shearing stress will be

$$\tau' = \tau + v = 2980\sin\theta - 4990\cos\theta + 5204 - 4960\,\theta$$

The maximum shearing stress will occur at $\theta = 27^\circ 54'$.

$$\tau'_{max} = 2980 \times 0.4679 - 4990 \times 0.8838 + 5204$$
$$- 4960 \times 0.4806 = -195 \text{ psi.}$$

$$v' = \tau' - v_c = 195 - 60 = 135$$

\# 4 stirrup spacing is

$$s = \frac{0.44 \times 20000}{135 \times 14} = 4.65 \text{ in.} \quad \text{Use } 4.5 \text{ in.}$$

For details of reiforcement see Figure 26.

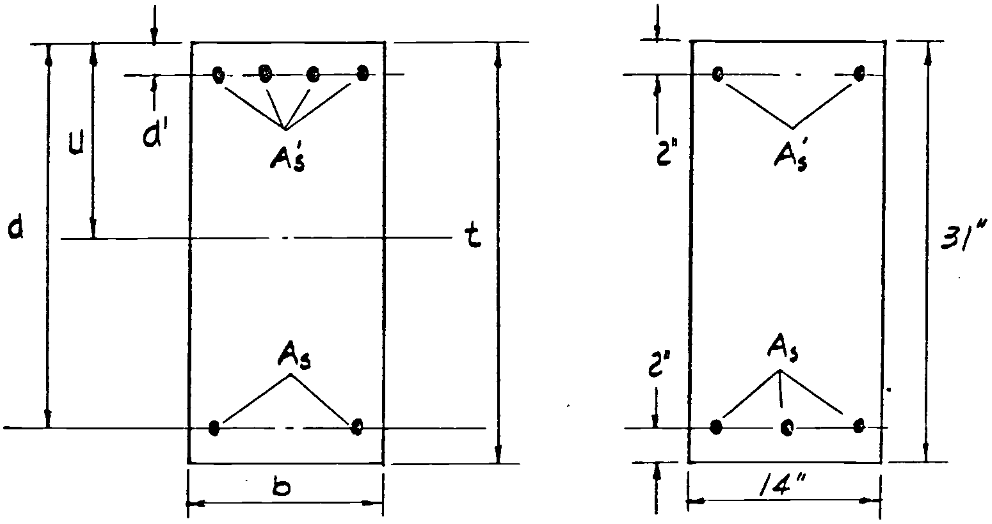
4. Deflection

$$E_c = 3.3\, W^{1.5}\sqrt{f'_c} = 3,160,000 \text{ psi}$$

E_s =29,000,000 psi   n= 29/3.16 = 9.2

Use n =9

The moment of inertia for the transformed section is calculated as follows.



Section at support   Section at center

Figure 25.

In figure 25 let

A_s = area of steel near tension surface

A_s' = area of steel near compressive surface

$P = \frac{A_s}{bt}$ $P' = \frac{A_s'}{bt}$

U = distance from compressive surface to gravity axis of transformed section

I_t = moment of inertia of transformed section

$$I_t = I_c + (n-1)I_s$$

$$U = \frac{t/2 + P(n-1)d + P'(n-1)d'}{1 + P(n-1) + P'(n-1)}$$

$$I_c = \frac{1}{3}\, b\left[U^3 + (t-U)^3\right]$$

$$I_s = A_s(d-U)^2 + A_s'(U-d')^2$$

At support the moment of inertia in denoted as I_{t1}.

$P = \dfrac{1.58}{14 \times 31} = 0.00376 \qquad P' = \dfrac{5.12}{14 \times 31} = 0.0121$

$d' = 2$ in. $\quad d = 29$ in. $\quad (n-1) = 8$

$$U = \frac{14.5 + 0.87 + 0.194}{1 + 0.03 + 0.0968} = 13.8$$

$I_c = 14\ (2640 + 5100)/3 = 36100$ in^4.

$I_s = 1.58\ (15.2)^2 + 5.12\ (11.8)^2$

$\quad = 1071$ in^4.

$I_{t1} = 36100 + 8 \times 1071 = 44170$ in^4.

At the center, the moment of inertia is I_{t2}.

$P = \dfrac{2.37}{14 \times 31} = 0.0056 \qquad P' = \dfrac{1.58}{14 \times 31} = 0.00376$

$$U = \frac{14.5 + 1.3 + 0.06}{1 + 0.0436 + 0.03} = 14.8$$

$I_c = \dfrac{1}{3} \times 14\ (3220 + 4290) = 35500$ in^4.

$I_s = 2.37 \times 200 + 1.58 \times 164 = 733$ in^4.

$I_{t2} = 35500 + 8 \times 733$

$\quad = 41360$ in^4.

The moment of inertia for calculating the deflection at the center will be the average value of I_{t1} and I_{t2}.

$I_t = (I_{t1} + I_{t2})/2 = 42765$ in^4.

From equations 15 and 16 $\Delta = \dfrac{Pr^3}{EI} C_{d1} + \dfrac{wr^4}{EI} C_{d2}$

From Figures 8 and 13

for concentrated load $\qquad C_{d1} = 0.0253$

for uniform load $\qquad C_{d2} = 0.0198$

$$= \left(\frac{20^3 \times 12^3}{3160 \times 42765}\right) \times 15 \times 0.0253$$

$$+ \left(\frac{1.425 \times 20^4 \times 12^4}{3160 \times 42765}\right) \times 0.0198$$

$$= 0.039 + 0.7$$

$$= 0.739 \text{ in.} < L/360. \quad \text{O.K.}$$

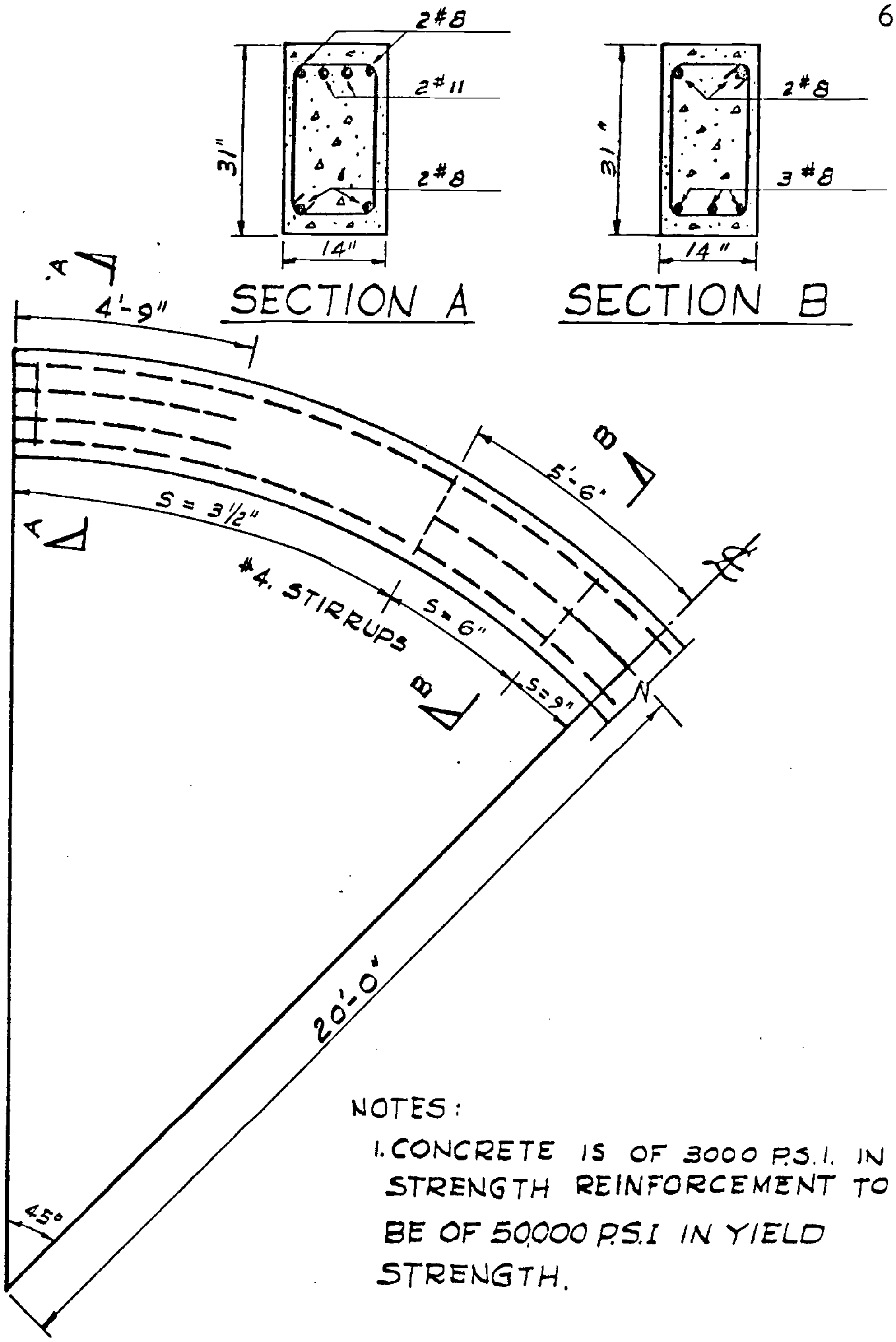


Figure 26. Horizontally curved girder.

VIII. CONCLUSIONS

The main difference in the analysis and design between a horizontally curved beam and a straight beam subjected to vertical loads is the presence of torsional moments, which, to some extent, will have influence on the bending, shear and deflection. The magnitude of torsion in a horizontally curved beam is a function of the span angle ϕ and it is also noticeably influenced by the shape of the section. This influence can be espressed by the ratio of flexural rigidity to the torsional rigidity, i.e., by the term $m = \frac{EI}{GJ}$.

From the results of the theoretical analysis, see Figures 5 through 8 and Figures 10 through 13, the following conclusions are obtained.

1. As the ratio m increases, the fixed end bending moment, the fixed end torsional moment and the vertical deflection increase, but the maximum span moment decreases.

2. Bending moments, torsional moments and deflection increase as the span angle ϕ increases up to 180°. The rates of increase become larger as the span angle ϕ increases.

3. Comparing the straight beam of a length equal to ϕr the following statements are made. When both straight beam and curved beam are subjected to the same amount of vertical load, a straight beam has smaller fixed end bending moment, smaller vertical deflection, and greater maximum span

moment. When the span angle $\emptyset$ of a curved beam is small (less than 15°), all the variables such as M_b, T_b, M_{max} and will be close to that of a straight beam.

4. The analysis of multi-span continuous beams curved in plan can be made without difficulty by the introduced moment distribution method, if both fixed end bending moments and torsional moments are known. The mathematical operations are not difficult but attention should be paid to the signs of the carry-over factors.

As long as the material is homogeneous and isotropic as assumed, the formulas derived in this paper can be used. In the preparation of this paper it was found that there were many factors affecting the actual behaviour of the curved beam, such as the degree of curvature, the degree of fixity at support, the shape of section and so on. This analysis is only expected to be a theoretical approach to solving the problem of a horizontally curved beam and should be verified by more experiments.

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APPENDIX

The following notations used in the programs represent those used in the text.

Programs	Text
TX	ϕ
TXZ	ϕ_o
P,M	m
CTD, DTXZ	C_d
CMC	$C_{m\theta}$
CTC	$C_{t\theta}$

PROGRAM 1. CONCENTRATED LOADED CURVED BEAM

```
      PROGRAM WON
      REAL M,MMAX
C     CONCENTRATE LOAD
      DIMENSION A(3,3),X(3),B(3)
      EQUIVALENCE (X,B)
      KCOUNT=1
      WRITE(10,10)
   10 FORMAT(1H1,////////)
      DO 60 III=30,180,30
      TX=III*0.01745329
      DO 60 MM=1,8
      M=MM
      DO 70 JJ=15,180,30
      TXZ=JJ*0.01745329
      IF (TXZ.GT.TX) GO TO 70
C     CALCULATION OF CONSTANTS
      A(1,1)=TX*(M+1.)-SIN(TX)*COS(TX)*(M-1.)
      A(2,1)=SIN(TX)**2*(M-1.)
      A(3,1)=SIN(TX)**2*(M-1.)+2.*M*(COS(TX)-1.)
      A(1,2)=A(2,1)
      A(2,2)=TX*(M+1.)+SIN(TX)*COS(TX)*(M-1.)
      A(3,2)=2.*M*SIN(TX)-TX*(M+1.)-SIN(TX)*COS(TX)*(M-1.)
      A(1,3)=A(3,1)
      A(2,3)=A(3,2)
      A(3,3)=TX*(M+1.)+SIN(TX)*COS(TX)*(M-1.)-4.*M*SIN(TX)
     1+2.*M*TX
      B(1)=(M-1.)*COS(TXZ)*(SIN(TX)**2
     2-SIN(TXZ)**2)+SIN(TXZ)*(TX-TXZ)
     3*(M+1.)-(SIN(TXZ)*(M-1.)/2.)*(SIN(2.*TX)-SIN(2.*TXZ))
     4+2.*M*(COS(TX)-COS(TXZ))
      B(2)=-COS(TXZ)*(TX-TXZ)*(M+1.)-COS(TXZ)*(M-1)*
     5(SIN(2.*TX)-SIN(2.*TXZ))/2.-(M-1.)*SIN(TXZ)*(SIN(TX)
     6**2-SIN(TXZ)**2)+2.*M*(SIN(TX)-SIN(TXZ))
      B(3)=(M-1.)*COS(TXZ)*(SIN(2.*TX)-SIN(2.*TXZ))/2.
     7+(TX-TXZ)*(COS(TXZ)*(M+1.)+2.*M)+(M-1.)*SIN(TXZ)*
     8(SIN(TX)**2-SIN(TXZ)**2)-2.*M*(1.+COS(TXZ))*(SIN(TX)
     9-SIN(TXZ))+2.*M*SIN(TXZ)*COS(TX)-COS(TXZ))
      CALL SIMQ(A,B,3,0.00001,0)†
      MMAX=X(1)*COS(TXZ)+X(2)*SIN(TXZ)-X(3)*SIN(TXZ)
      WRITE(10,101)III,JJ,M,(X(I),I=1,3),MMAX
      IF(TXZ.EQ.TX/2.) 40,70
   40 DTXZ=SIN(TX/2.)*(X(1)*(TX+SIN(TX))/4.+(X(2)-X(3))*
     1(1.-COS(TX))/4.)-COS(TX/2.)*(X(1)*(1.-COS(TX))/4.
     2+(X(2)-X(3))*(TX/4.-SIN(TX)/4.))
     3+M*(X(1)*(COS(TX/2.)-1.)+(X(2)-X(3))*SIN(TX/2.)+X(3)
     4*(TX/2.))-M*COS(TX/2.)*(-X(1)*(1.-COS(TX))/4.+(X(2)-
     5X(3))*(TX+SIN(TX))/4.+X(3)*SIN(TX/2.))-M*SIN(TX/2.)
```

```
   6*(-X(1)*(TX-SIN(TX))/4.+(X(2)-X(3))*
   7(1.-COS(TX))/4.-X(3)*(COS(TX/2.)-1.))
    WRITE (10,101) III,JJ,M,(X(I), I=1,3), MMAX, DTXZ
    KCOUNT=KCOUNT+1
    IF(KCOUNT.EQ.49) 50, 70
 50 WRITE( 10, 10 )
    KCOUNT=1
101 FORMAT(I 12, I5.1, 3F7.4, 2F9.4)
 70 CONTINUE
 60 CONTINUE
    END
```

†SIMQ is a subroutine program for solution of simultaneous linear equation. For detail of this program see No. F4-INDU-SIMQ, Computer library, Oregon State University.

PROGRAM 2. UNIFORMLY LOADED CURVED BEAM

```
      PROGRAM WON
C     UNIFORM LOAD
      KCOUNT=1
      WRITE(10,10)
   10 FORMAT(1H,///////)
      DO 40 I=15,180,15
      TX-I*0.0174533
      DO $) J=1,8
      P=J
      DO 30 K=16,181,15
      KK=(K-1)
      TXC=KK*0.0174533
      IF(TXC.GT.TX) GO TO 40
      CM=-((2.*(P+1.)*SIN(TX)-P*TX*(1.+COS(TX)))/
     1(TX*(P+1.)-SIN(TX)*(P-1.))-1.)
      CT=-((2.*(P+1.)*(1.-COS(TX))-P*TX*SIN(TX))/
     2(TX*(P+1.)-SIN(TX)*(P-1.))-TX/1.)
      CMC=CM*COS(TXC)+CT*SIN(TXC)-TX*SIN(TXC)/2.+(1.-COS(TXC))
      CMM=CM*COS(TX/2.)+CT*SIN(TX/2.)-(TX/2.)*SIN(TX/2.)
     3+(1.-COS(TX/2.))
      CTC=CT*COS(TXC)-CM*SIN(TXC)+TX*(1.-COS(TXC))/2.
     4-(TXC-SIN(TXC))
      CTD=SIN(TX/2.)*((TX+SIN(TX))*(CM-1.)/4.+(CT-TX/2.)*
     1((1.-COS(TX))/4.+P-P*(1.-COS(TX))/4.)+SIN(TX/2.)*
     2(1.+P)+P*(CM-1.)*(TX-SIN(TX))/4.-TX*P/2.)-COS(TX/2.)
     3*(CM*(1.-COS(TX))*(1.-P)/4.-(1.-COS(TX))*(1.-P)/4.+
     4(CT-TX/2.)*(TX-SIN(TX)+P*TX+P*SIN(TX))/4.+(1.-
     5COS(TX/2.))*(1.+P))+P*((COS(TX/2.)-1.)*(CM-1.)
     6+TX**2/8.
      WRITE(10,20) I,P,KK,CM,CT,CMC,CTC,CTD
      KCOUNT=KCOUNT+1
      IF(KCOUNT.EQ.51) 50,30
   50 WRITE(10,10)
      KCOUNT=1
   20 FORMAT(I5,F5.1,I5,6F9.4)
   30 CONTINUE
   40 CONTINUE
      STOP
      END
```

PROGRAM 3. MOMENT DISTRIBUTION FACTORS FOR HORIZONTALLY CURVED BEAM

```
      PROGRAM WON
      REAL M21,M1T1,M2T2
C     MOMENT DISTRIBUTION FACTORS
      KCOUNT=1
      WRITE(10,10)
   10 FORMAT(1H1,////////)
      DO 30 I=15,180,15
      TX=I*0.0174533
      DO 30 J=1,8
      P=J
      A1=TX*(P+1.)-SIN(TX)*COS(TX)*(P-1.)
      B1=SIN(TX)**2*(P-1.)
      C1=SIN(TX)**2*(P-1.)+2.*P*(COS(TX)-1.)
      A2=B1
      B2=TX*(P+1.)+SIN(TX)*COS(TX)*P-1.)
      C2=2.*P*SIN(TX)-TX*(P+1.)-SIN(TX)*COS(TX)*(_-1.)
      A3=C1
      B3=C2
      C3=TX*(P+1.)+SIN(TX)*COS(TX)*(P-1.)-4.*P*SIN(TX)+
     2.*P*TX
      SM=2.*(B2*C3-C2**2)/(A1*(B2*C3-B3*C2)-B1*(B1*C3+C1*C2)
     1-C1*(B2*C1+B1*C2))
      ST=2.*(A1*C3-C1**2)/(B2*(A1*C3-C1**2)
     2-B1*(B1*C3+C1*C2)-C2*(B1*C1+A1*C2))
      M21=COS(TX)+(B1*C3+C1*C2)*SIN(TX)/(B2*C3 C2**2)
     3+(B1*C2+B2*C1)*SIN(TX)/(C3*B2-C2**2)
      T2M1=SIN(TX)+(B1*C3+C2*C1)*COS(TX)/(A1*C3-Ca**2)
     4+(B1*C1+A1*C2)*SIN(TX)/(A1*C3-C1**2)
      T1M1=(B1*C3+C2*C1)/(A1*C3-C1**2)
      T21=COS(TX)-(B1*C3+C2*C1)*SIN(TX)/(A1*C3-C1**2)
     5-(A1*C2+B1*C1)*(1.-COS(TX))/(A1*C3-C1**2)
      M1T1=(B1*C3+C1*C2)/(B2*C3-C2**2)
      M2T1=-SIN(TX)+(B1*C3+C1*C2)*COS(TX)/(B2*C3-C2**2)
     6-(B1*C2+B2*C1)*(1.-COS(TX))/(C3*B2-C2**2)
      WRITE(10,20) I,P,SM,ST,M21,T2M1, T1M1, T21,M1T1,M2T1
      KCOUNT=KCOUNT+1
      IF (KCOUNT.EQ.46) 50,30
   50 WRITE(10,10)
      KCOUNT=1
   20 FORMAT(19,F5,1,8F6.3)
   30 CONTINUE
      STOP
      END
```