Normal-Mode Analysis of a Baroclinic Wave-Mean Oscillation

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ABSTRACT

The stability of a time-periodic baroclinic wave-mean oscillation in a high-dimensional two-layer quasi-geostrophic spectral model is examined by computing a full set of time-dependent normal modes (Floquet vectors) for the oscillation. The model has 72 × 62 horizontal resolution and there are 8928 Floquet vectors in the complete set. The Floquet vectors fall into two classes that have direct physical interpretations: wave-dynamical (WD) modes and damped-advective (DA) modes. The WD modes (which include two neutral modes related to continuous symmetries of the underlying system) have large scales and can efficiently exchange energy and vorticity with the basic flow; thus, the dynamics of the WD modes reflects the dynamics of the wave-mean oscillation. These modes are analogous to the normal modes of steady parallel flow. On the other hand, the DA modes have fine scales and dynamics that reduce, to first order, to damped advection of the potential vorticity by the basic flow. While individual WD modes have immediate physical interpretations as discrete normal modes, the DA modes are best viewed, in sum, as a generalized solution to the damped advection problem. The asymptotic stability of the time-periodic basic flow is determined by a small number of discrete WD modes and, thus, the number of independent initial disturbances, which may destabilize the basic flow, is likewise small. Comparison of the Floquet exponent spectrum of the wave-mean oscillation to the Lyapunov exponent spectrum of a nearby aperiodic trajectory suggests that this result will still be obtained when the restriction to time periodicity is relaxed.

1. Introduction

The stability theory of geophysical flows has traditionally focused on stationary flows. While great strides toward understanding the variability of the atmosphere and oceans have been made by studying stationary flows (e.g., Charney 1947; Eady 1949), the assumption of stationarity is clearly an artifice since all physical flows exhibit time variability on a range of scales, from the subinertial to the millennial. It is thus desirable to relax the stationarity assumption and it is expected that a stability theory of time-dependent flows will allow similar fundamental insights into atmospheric and oceanic variability. A time-dependent stability theory is also of great interest to the forecasting community, as a significant portion of the forecast error at moderate lead times is ascribed to large-scale instabilities growing on the evolving flow (Toth and Kalnay 1993, 1997).

An important first step toward a stability theory of flows with arbitrary time dependence is the study of time-periodic flows. Time-periodic flows offer many of the same challenges—and potential for new insights—as flows with arbitrary time dependence, but are more computationally tractable since full information about the evolution of disturbances to a time-periodic flow may be obtained by model integration over a single period. A number of recent studies have examined periodic flows in geophysical models (Itoh and Kimoto 1996; Kazantsev 1998, 2001) and their stability (Samelson 2001b; Pedlosky and Thomson 2003; Poulin et al. 2003). Samelson and Wolfe (2003, hereafter SW03), reported some preliminary results concerning the model currently under study.

The present study gives the first complete numerical normal-mode analysis of the linear stability of a strongly nonlinear wave-mean oscillation in a two-layer channel model of the baroclinic instability. Although unstable, the basic wave-mean oscillation is periodic in time. The time-dependent normal modes (Floquet vectors) of a time-periodic flow can be obtained numerically using standard methods for the solution of linear differential equations with periodic coefficients.
(e.g., Coddington and Levinson 1955, chapter 3). As for steady parallel flow, these normal modes are intrinsic dynamical objects that can provide physical insight into the mechanisms of disturbance growth and decay.

The mathematical elements of stability theory for steady parallel flow generally involve solutions to ordinary differential equations and are well known for many geophysically relevant examples (e.g., Drazin and Reid 2004, chapter 4). Along with regular normal modes, they sometimes require singular neutral modes, which may fail to be continuously differentiable, for a complete description of the disturbance evolution. When the basic flow is both time periodic and nonparallel, the normal-mode problem is generally nonseparable. The corresponding mathematical theory for the Floquet analysis of partial differential equations is not as well established as that for ordinary differential equations, although some results are available (Kuchment 1993). For example, the possibility that analogs of singular neutral modes may exist in such flows has received limited attention. One goal of the present study is to examine this possibility in a specific geophysical model.

The format of the paper is as follows. In section 2, we discuss the model formulation and review some basic elements of Floquet theory. We then briefly describe the basic wave-mean oscillation and its relation to the general behavior of the model in section 3. Section 4 is devoted to a detailed discussion of the results of the Floquet analysis. The sensitivity of the results to changes in resolution is discussed in section 5. The significance of the results is discussed in section 6. Finally, we summarize our research in section 7.

2. Formulation

a. Model

The model studied here is the well known Phillips (1954) quasigeostrophic channel model and is described extensively in Pedlosky (1987, chapter 7). For the present study, the Coriolis parameter $f$ is constant, the equilibrium layer depths are equal, and the background flow is steady, uniform, and zonal. The evolution of disturbances to the background flow is governed by

$$\frac{\partial q_n'}{\partial t} + U_n \frac{\partial q_n'}{\partial x} + J(\psi_n, q_n) - (-1)^n FU_i \frac{\partial \psi_n}{\partial x} = -r N^2 \psi_n, \quad n = 1, 2, \quad (1)$$

where the $\psi_n$ and $q_n$ are the disturbance streamfunctions and potential vorticities (PV), respectively. The labeled terms in the above equation represent local change of disturbance PV (A), advection of disturbance PV by the background flow (B), advection of background PV (C), and Ekman dissipation of the disturbance flow (D). These labels will appear in the term-balance analysis of section 4. Equation (3) is solved using the same spectral decomposition used to solve Eq. (1).
Note that Eqs. (1) and (3) are invariant under the exchanges
\[
\begin{align*}
\psi_1(x) &\rightarrow -\psi_2(-x), \\
\psi_2(x) &\rightarrow -\psi_1(-x), \\
\psi'_1(x) &\rightarrow \pm \psi'_2(-x), \\
\psi'_2(x) &\rightarrow \pm \psi'_1(-x).
\end{align*}
\tag{4}
\]

The existence of these symmetries implies that there exist solutions to the evolution equations with the same properties. The nonlinear solutions discussed in section 3 exactly satisfy the symmetry (4), as does the basic cycle. Additionally, most of the Floquet vectors described in section 4 satisfy one of the two symmetries (5).

b. Floquet theory

When the basic flow \((\psi_n, q_n)\) is periodic in \(T\), Eq. (3) is a linear, homogeneous partial differential equation with \(T\)-periodic coefficients that, after expansion into zonal and meridional Fourier modes, is amenable to analysis using Floquet theory. Floquet’s theorem (Coddington and Levinson 1955, chapter 3) states that any solution to the truncated spectral expansion of Eq. (3) may be written as a fixed sum of the \(2N_xN_y\) Floquet eigenvectors \(\Phi_n(x, y, t)\), where (for nondegenerate systems) \(\Phi_n(x, y, t) = \phi_n(x, y, t)e^{\lambda t}\). The Floquet structure function \(\phi_n(x, y, t)\) is \(T\)-periodic (or \(2T\)-periodic) and affects the transient growth and decay of the Floquet vector, while the (possibly complex) Floquet exponent \(\lambda_n\) determines the asymptotic stability of the vector. If \(\text{Im}\{\lambda_n\} T/2\pi\) is a rational number of the form \(p/q\) (where \(p > 0\), and \(p\) and \(q\) are relatively prime), then the real and imaginary parts of \(\phi_n(x, y, t)e^{\text{Im}\{\lambda_n\} t}\) have period \(qT\). If \(\text{Im}\{\lambda_n\} T/2\pi\) is irrational, then the real and imaginary parts of \(\phi_n(x, y, t)e^{\text{Im}\{\lambda_n\} t}\) are quasi-periodic. Since Eq. (3) is unchanged by complex conjugation, complex Floquet vectors necessarily come in conjugate pairs.

The Floquet vectors were calculated by integrating a complete set of initial conditions to Eq. (3) over one period \(T\) of a periodic basic cycle to construct the monodromy matrix \(M\). In practice, \(M\) was nondegenerate. The eigenvectors and eigenvalues of \(M\) are the Floquet vectors \(\Phi_n(x, y, t)\) and the so-called Floquet multipliers \(\mu_n\) respectively. The Floquet exponents are given by \(\lambda_n = (1/T)\log\mu_n\). The number of operations required to compute \(M\) grows like \(N^5\), so that increasing the model resolution by a factor of 2 increases the computational burden of the Floquet calculation by a factor of 32. There exist methods for computing subsets of the Floquet vectors iteratively (e.g., Lust et al. 1998); a calculation that scales like \(N^3\) per the Floquet vector.

These methods were used to obtain some of the results discussed in SW03, however, our present interest in obtaining a complete set of Floquet vectors precludes the use of such methods.

As Floquet exponents characterize the asymptotic growth of disturbances to a periodic trajectory, they are intimately related to the Lyapunov exponents of that trajectory. Consider an arbitrary linear disturbance \(\xi_0\) to a trajectory \(x(t)\) of an \(N\)-dimensional dynamical system. Under fairly general conditions [primarily, that \(x(t)\) exists and is bounded as \(t \to \pm \infty\)], the limit
\[
\chi^+ = \lim_{t \to \pm \infty} \frac{1}{\log t} \ln \frac{\|\xi(t)\|}{\|\xi_0\|}
\tag{6}
\]
exists and is finite (Oseledec 1968). Furthermore, \(\chi^+ = \lambda_1^+\) (the leading forward Lyapunov exponent) independent of \(\xi_0\), unless \(\xi_0\) belongs to a set \(S_0\) of measure zero. If \(\xi_0 \in S_0\), then \(\chi^+ = \lambda_1^+ < \lambda_1^-\), where \(\lambda_1^-\) is the second forward Lyapunov exponent, unless—again—\(\xi_0\) belongs to a set \(S_0^+\) of measure zero relative to \(S_0^-\). This argument may be applied recursively to obtain \(M\) nested subspaces \(S_n\), each associated with a forward Lyapunov exponent \(\lambda_n\), where \(M \leq N\) and \(\lambda_n^+ > \lambda_{n+1}^+\). A similar set of \(M\) nested subspaces \(\{S_n\}\) is obtained as \(t \to -\infty\), with an associated set of backward Lyapunov exponents \(\lambda_n^-\), where \(\lambda_n^- = -\lambda_{M-n+1}^1\). If the intersection space \(S_n = S_n^+ \cap S_{M-n+1}^-\) is one dimensional, then \(S_n\) is a Lyapunov vector, which grows at the rate \(\lambda_n = \lambda_n^+\) as \(t \to +\infty\) and decays at the rate \(\lambda_n = \lambda_{M-n+1}^-\) as \(t \to -\infty\). If \(S_n\) is \(d > 1\), then \(d\) linearly independent Lyapunov vectors may be chosen arbitrarily from this space; in this case it is customary to assign \(d\) equal Lyapunov exponents to \(S_n\), so that the Lyapunov spectrum contains \(N\) (degenerate) exponents.

For a periodic trajectory, the Lyapunov exponents of the trajectory are given by the real parts of the Floquet exponents. Furthermore, if the Floquet exponent \(\lambda_i\) is real, then the \(i\)th Lyapunov vector is equal to the \(i\)th Floquet vector, and if the Floquet exponents \(\{\lambda_i, \lambda_{i+1}\}\) form a complex conjugate pair, then the \(i\)th and \((i + 1)\)th Lyapunov vectors lie in the subspace spanned by the real and imaginary parts of the \(i\)th Floquet vector. In this sense, Lyapunov vectors generalize Floquet vectors to trajectories of arbitrary time dependence.

3. Nonlinear wave-mean oscillations

a. Aperiodic trajectory

For the present study, we considered the most strongly supercritical set of parameters studied by Klein and Pedlosky (1986): \(\gamma = 0.20\) and \(\Delta = 45\), where
This corresponds here to \( U_x = 1, r \approx 0.4743, \) and \( F = 55.77. \) For these parameters, the zonally uniform solution \( \psi_1 = \psi_2 = 0 \) of (1) is unstable to six linear disturbances (Table 1). The most rapidly growing linear mode \((2, 1)\) dominates the early evolution of random disturbances to the zonally uniform state, but the \((k, l) = (1, 1)\) Fourier mode eventually dominates the subsequent evolution of the flow. The eventual dominance of a mode with larger scales than the most rapidly growing linear mode is often seen in nonlinear models of the baroclinic instability (Hart 1981; Pedlosky 1981).

For large times, the solution approaches a chaotic attractor. The dominance of the \((1, 1)\) mode allows the trajectory to be conveniently represented in the \((A, \delta A)\) phase plane, where \( A \) is the amplitude of the \((1, 1)\) mode (Fig. 1). Numerical estimates based on a long integration \((T \approx 10^4, \) using a resolution of \(48 \times 40)\) indicate that the first three Lyapunov exponents, calculated using standard methods (Shimada and Nagashima 1979; Bennetin et al. 1980), are positive, confirming that the attractor is chaotic (Fig. 2). The resulting Kaplan–Yorke dimension (Grassberger and Procaccia 1983) for this attractor is \(7.008 \pm 0.004.\)

b. Periodic basic cycle

Chaotic attractors, such as that discussed above, are often accompanied by a set of unstable periodic cycles that “fill out” (more precisely, and under specific technical conditions, which may or may not hold in the present case, are dense in) the attractor. Certain mean properties of the flow on the chaotic attractor (e.g., Lyapunov exponents, heat flux, etc.) may be recovered by constructing suitable averages over unstable periodic cycles (Cvitanović et al. 2005). Often, surprisingly good results are obtained using just a few low-order cycles (Kazantsev 1998, 2001).

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**Table 1.** Zonal and meridional wavenumbers \( k \) and \( l \), respectively, and growth rates of the six unstable linear disturbances to the zonally uniform state.

<table>
<thead>
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<th>((k, l))</th>
<th>Growth rate</th>
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<tr>
<td>((2, 1))</td>
<td>1.6502</td>
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<td>((1, 1))</td>
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<td>((2, 2))</td>
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<td>0.6804</td>
</tr>
<tr>
<td>((1, 3))</td>
<td>0.0596</td>
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</tbody>
</table>

\[ \Delta = F - \pi^2 - 4r^2, \quad (7a) \]

\[ \gamma = r \sqrt{\frac{8}{\Delta}}, \quad (7b) \]

The basic cycle and the methods used to obtain it were described in detail in SW03; a brief recapitulation is provided here for completeness. A low-order unstable periodic cycle associated with the above attractor was found by first searching for near recurrences in a long aperiodic trajectory, and then refining the resulting first-guess initial condition using the Newton–Picard iteration code “PDEcont” (Lust et al. 1998).
orbit obtained by this code returns to its initial condition with a relative error of less than $10^{-4}$. The periodic orbit so obtained forms the basis for the following Floquet analysis and will henceforth be referred to as “the basic cycle.” It has a period of $T \approx 38.488$ and, like the attractor, is dominated by the $(1, 1)$ Fourier components and so may be plotted in the same $(A, \partial_A)$ phase plane as the attractor (Fig. 1).

The basic cycle begins as a nearly zonal flow with a small superimposed perturbation (Fig. 3, top panels). This perturbation grows into a pair of eddies, which grow in amplitude as they advect heat (proportional to $\partial_1$) downgradient, across the channel. By $t/T = 15/50$, the cross-channel heat flux reduces the background potential vorticity gradient sufficiently to halt and then reverse the growth of the eddies (Fig. 3, middle panels). Toward the end of the decay phase, the weakening eddies advect heat upgradient, extracting energy from the wave and reestablishing the nearly zonal initial state, now shifted down channel by one-half the channel length (Fig. 3, bottom panels). After passing through a second growth and decay phase the flow returns to its initial state.

As expected, the leading Lyapunov exponents of the cycle are good approximations to the leading Lyapunov vectors of the chaotic attractor (Fig. 2), with a relative error in the Lyapunov multipliers $e^h$ of less than 10%. Furthermore, the time-averaged heat flux for the basic cycle $\overline{F_T} = 0.1554$, where

$$\overline{F_T} = \frac{F}{2} \lim_{T \to \infty} \frac{1}{T} \int_0^T \int \frac{\partial_1}{\partial x} \psi_1 \, dx \, dy \, dt,$$

which approximates that of the chaotic attractor ($\overline{F_T} = 0.1531 \pm 0.0004$) to better than 2%. This suggests that mean quantities may be obtained fairly accurately by averaging over the basic cycle rather than long trajectories on the attractor.

Fig. 3. Contours of the (left) upper- and (right) lower-layer streamfunction vs $x$ (horizontal axis) and $y$ (vertical axis) during the evolution of the unstable periodic orbit. Negative contours are dashed. For $t/T > 25/50$, the cycle repeats itself shifted half-way down the channel.
4. Time-dependent normal modes

a. Overview

The time-dependent normal modes for linear disturbances to the basic cycle described in section 3b are the Floquet vectors (FVs), described in section 2b, which completely characterize the evolution of these disturbances. The spatiotemporal characteristics and asymptotic stability of the FVs thus determine if and how the basic cycle is unstable. In the present case, three of the FVs are unstable and two are neutral, indicating that the basic cycle is, in fact, unstable. The number of unstable and neutral modes found here is independent of resolution and thus is a characteristic of the basic cycle only. The rest of the Floquet spectrum is completed by a large number of decaying modes, the exact number of which depends on the resolution. As in Samelson (2001b), the FVs in the present study fall into two physically meaningful classes, described below.

The real parts of the Floquet exponents are equal to the Lyapunov exponents of the basic cycle to within the accuracy of the Lyapunov exponents for all the cases that were checked (Fig. 2). That this must necessarily be the case follows from the discussion of the relationship between Lyapunov and Floquet exponents in section 2b. Thus, the numerical equality between these two quantities provides a useful check of the consistency of the numerics.

The majority of the 8928 Floquet vectors of the 72 × 62 model have exponents whose real parts that lie near, but slightly above, the dissipation rate $r$, while a small number of vectors have exponents whose real parts which are significantly greater (the “leading” vectors), or less (the “trailing” vectors), than the dissipation rate (Fig. 4a). Thus, the bulk of the Floquet vectors are stable and decay at rates near the dissipation rate of the model. The leading vectors either grow or decay weakly while the trailing vectors decay much more rapidly than the dissipation rate.

Floquet vectors with decay rates well separated from the dissipation rate tend to be dominated by disturbances with large scales, while those with decay rates near the dissipation rate have much smaller scales, where the scale is measured by the time-averaged mean wavenumber $\bar{K}$ (Fig. 4), defined by

\[
\langle \bar{K}, l \rangle = \sum \frac{\pi(|k|, |l|) |A_{k,l,n}|^2}{\sum |A_{k,l}|^2},
\]

where the sums are taken over all possible values of $k$, $l$, and $n$. These two classes will be referred to as the “wave-dynamical” and “damped-advective” classes, respectively, and will be discussed in sections 4b, d. Those Floquet vectors with large scales have—with few exceptions—purely real Floquet exponents (i.e., they are frequency locked to the basic cycle). Two Floquet vectors have exponents exactly equal to 0 (Fig. 4a). These neu-

\[
\bar{K} = \frac{1}{T} \int_0^T (\bar{K}^2 + \bar{l}^2)^{1/2} dt,
\]
cral modes, which are a subset of the wave-dynamical class, arise from continuous symmetries of the basic system and are described in section 4c. While the large scale FVs are dominated by a small number of Fourier components, the fine scale FVs tend to contain significant contributions from many Fourier components. These FVs have complex Floquet exponents with real parts that lie near the dissipation rate and imaginary parts that are distributed approximately uniformly between \( \pm \pi/T \). For most of the complex Floquet exponents, \( \text{Im}[\lambda_j]T/2\pi \) is not well approximated (to within \( 10^{-6} \)) by any rational number with denominator smaller than 100. The associated Floquet vector structure functions \( \phi'_n \) thus either have long periods (\( \approx 100T \)) or may be quasi-periodic.

The transition from wave-dynamical behavior to damped-advective behavior is gradual and there are modes at intermediate wavenumbers (\( 3\pi \leq k \leq 4\pi \)) that show characteristics of both classes. Nevertheless, the modes with \( k < 3\pi \) show characteristics that clearly place them in the wave-dynamical class whereas those with \( k > 4\pi \) clearly belong to the damped-advective class. These dividing lines are shown in Fig. 4. The intermediate class will not be discussed in a separate section, since these modes do not have any features that clearly distinguish them from the modes in the other classes. There are 62 wave-dynamical, 831 intermediate, and 8035 damped-advective modes at the resolution considered here (72 \( \times \) 62).

b. Wave-dynamical modes

The wave-dynamical (WD) Floquet vectors are characterized by large-scale, quasi-stationary wave patterns, decay rates well separated from the model dissipation rate (Fig. 4), and large transient amplitude fluctuations. These modes depend on the vertical shear of the background flow for their growth and maintenance. The characteristics of the leading 10 and trailing 9 WD modes are summarized in Table 2.

The leading 12 WD modes, of which the first 3 are unstable, are described in SW03. That study used an approximate monodromy matrix constructed from and projected into the subspace spanned by the gravest 36 zonal and 40 meridional Fourier modes of a 48 \( \times \) 40 resolution model to obtain the Floquet vectors, instead of the full monodromy matrix at 72 \( \times \) 62 resolution. The approximate method was only able to obtain the leading 12 WD vectors accurate to within a 10% first-return error. In contrast, the first 12 WD modes obtained by the current method return to their initial conditions to within \( 10^{-12} \). Despite the relative inaccuracy of the method and lower resolution used by SW03, the large-scale structures of the leading WD modes obtained using the two methods are remarkably similar. In general, the leading WD modes appear to be robust to changes in resolution or computational method. Modes obtained with one resolution or method can be identified in the spectrum determined using another resolution or method with only modest changes in structure or Floquet exponent.

Since the leading modes obtained in the present study are similar to those described in SW03, only the leading WD mode will be described here as an example (Fig. 5, cf. SW03’s Fig. 5). This mode is dominated by the second along-channel Fourier component, which is also the most unstable normal mode of the spatially homogeneous system. Comparison with the basic cycle (Fig. 3) shows that this disturbance grows and decays in phase with the basic cycle and represents an intensification and down-channel shift of one eddy and a weakening and up-channel shift of the other. The net effect is to narrow and intensify one cross-channel jet while broadening and weakening the other.

The other growing and weakly decaying WD modes show a similar pattern. They are dominated by large scales and grow and decay nearly in phase with the basic cycle (Fig. 6). In spatial structure, the third growing mode is similar to the first, as it is dominated by the second along-channel Fourier component (as well as the second cross-channel Fourier component), while the second growing mode has the (1, 1) Fourier com-

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Fig. 5. Contours of the (left) upper- and (right) lower-layer streamfunction $\psi_*(x, y)$ during the evolution of the Floquet structure function $\phi^f$ for which $\lambda = 0.0277$. Negative contours are dashed.
ponent as its primary component and thus has a spatial structure similar to that of the basic cycle (Table 2).

While the leading WD mode structure function $\phi^1$ is growing during the basic cycle growth phase, the perturbation dynamics are dominated by the exchange of PV with the mean flow [Fig. 7; terms B and C in (3)]. During this time, the perturbation PV flux is strongly down the background flow PV gradient (Fig. 8b), as measured by the normalized perturbation enstrophy production:

$$\int \int \mathbf{u}' q' \cdot \nabla q \, dx \, dy \frac{1}{\left[ \int \int \| \mathbf{u}' q' \| \, dx \, dy \int \int \| \nabla q \| \, dx \, dy \right]^{1/2}}.$$  

(10)

where $\mathbf{u}'$ and $q'$ are computed from the Floquet structure functions. Between growth phases, the leading Floquet vectors are advected by the background flow and eroded by Ekman dissipation (Fig. 7b). During this time, the perturbation PV flux is only weakly downgradient. The leading FVs do not appear to undergo an

Fig. 6. Total wave energy $E$ as a function of normalized time $t/T$. (top) Basic cycle, (middle) 10 most rapidly growing Floquet vectors, and (bottom) 9 most rapidly decaying Floquet vectors. The legends give the FV index of each mode.
The growth-phase countergradient PV flux is largest in the growing FVs, smaller in the weakly decaying FVs, and insignificant for $n$, $n > 18$. The behavior of the perturbation heat flux (not shown) is similar.

In addition to the growing and weakly decaying modes, there is a complementary set of inviscidly damped WD modes (Fig. 9). These modes have spatial scales similar to the leading Floquet vectors, but decay at rates much greater than the frictional dissipation rate. Samelson (2001b) found a similar inviscidly damped mode in a weakly nonlinear model of the baroclinic instability. SW03 could not compute these modes because of limitations in their computational method.

The spatial scales of the three most rapidly decaying Floquet vectors follow a pattern similar to the leading three Floquet vectors, with the structure functions $\phi^{8926}$ and $\phi^{8928}$ dominated by the second along-channel Fourier component and $\phi^{8927}$ dominated by the first along-channel Fourier component (Table 2). These trailing modes are nearly completely out of phase with the basic flow (Fig. 6) and obtain their maximum amplitude while the basic cycle is most zonally homogeneous. Thus, the leading modes grow while the basic cycle is growing, and the trailing modes grow while the basic cycle is either decaying or at very low amplitude. For $t/T \approx 25/50$, $\phi^{8926}$ and $\phi^{8928}$ repeat their growth–decay cycle shifted halfway down channel; $\phi^{8927}$ repeats its growth–decay cycle shifted halfway down channel with the opposite sign. These disturbances have a slight eastward phase shift with height, which is unfavorable for baroclinic growth, and decay rapidly as the basic cycle enters its growth phase. Note that the eastward phase shift of the modes shown in Fig. 9 is not as dramatic as the westward phase shift of the leading WD mode shown in Fig. 5. This is in fact consistent with the classical Phillips model with dissipation: growing modes must have a significant westward phase shift merely to maintain themselves against dissipation. A small eastward phase shift can result in a large decay rate because the inviscid decay created by the unfavorable phase shift is added to the already large viscous decay rate of the mode. The apparent north–south phase shift seen in Fig. 9 is likely due to the north–south asymmetry of the basic cycle.

In contrast to the leading FVs, the peak in PV exchange [terms B and C in (3)] for the trailing FVs occurs before the growth phase of the basic cycle, while the basic cycle is most zonally uniform (Fig. 7d shows $\phi^{8928}$; the other trailing FVs are similar), and results in strong up-gradient flux of PV (Fig. 8). The up-gradient PV flux is at a minimum when the basic cycle reaches its maximum amplitude, then rebounds to moderate levels during the decay phase of the basic cycle. The trailing modes are thus inviscidly damped throughout most of the basic cycle.

In summary, the leading WD modes grow by advecting PV and heat downstream while the basic cycle is growing and are able to continue to do so even when the background flow is highly zonally inhomogeneous (Figs. 6 and 8). The downgradient PV and heat flux

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**Fig. 8.** (top) Time evolution of basic cycle potential enstrophy $Q$. (bottom) Correlation of perturbation PV flux with the background PV gradient ($u'q'^*\nabla q$) for WD modes $\phi^1$ (solid), $\phi^{8928}$ (dash–dot), and DA modes $\phi^{8924}$ (dotted) as a function of time. The vertical dashed lines pick out the cycle extrema.
becomes weak during the decay phase of the basic cycle, during which time the disturbances amplified during the growth phase are advected and distorted by the background flow. The trailing FVs reach their largest amplitude when the background flow is most zonal and advect PV and heat up gradient, leading to rapid inviscid decay (Figs. 6 and 8). Dissipation is never a leading-order effect (Fig. 7), but acts continuously and

Fig. 9. Contours of the upper- and lower-layer streamfunction vs $x$ (horizontal axis) and $y$ (vertical axis) of the structure functions of the three most rapidly damped FVs at (left) the beginning of the cycle and (right) at $t/T = 15/50$. Negative contours are dashed.
is sufficient to reduce the growth rates of the three leading FVs and stabilize the others. Note, however, that, since the basic flow has neither temporal nor spatial symmetry, a physically consistent quadratic disturbance quantity cannot be defined, and the rigor of this interpretation of the wave-mean interaction is necessarily limited.

c. Neutral modes

Two of the wave-dynamical modes are neutral, with the Floquet exponent exactly equal to zero (Table 2). These modes exist as a consequence of the two continuous symmetries—time and zonal translation—of the nonlinear evolution Eq. (1). The temporal neutral mode has larger scales than the zonal-translation neutral mode. The spatiotemporal structure of these modes is described in SW03, where the temporal and zonal-translation modes are referred to as $\Phi^t$ and $\Phi^s$, respectively. SW03, however, did not identify $\Phi^s$ as a neutral mode.

To see how these neutral modes arise from continuous symmetries of the nonlinear evolution equation, represent a general nonlinear equation for the evolution of the state vector $\psi$ by

$$\psi_t = \mathcal{N}(\psi),$$  \hspace{1cm} (11)

where $\mathcal{N}$ is the nonlinear evolution operator and $\psi$ is either steady, time periodic, or uniformly bounded in time $t$ and may depend on space $x$. Linear disturbances $\psi'$ to $\psi$ satisfy

$$\psi'_t = L_\phi \psi',$$  \hspace{1cm} (12)

where $L_\phi$ is the linearization of $\mathcal{N}$ about $\psi$. Particular interest attaches to “normal-mode-type” solutions to Eq. (12), that is, solutions of the form

$$\psi'(x, t) = \phi(x, t)e^{\lambda t},$$  \hspace{1cm} (13)

where the structure function $\phi$ is steady, time periodic, or uniformly bounded in time depending on the time dependence of $\psi$. The evolution of $\phi$ is governed by the (linear) equation

$$\phi_t + \lambda \phi = L_\phi \phi.$$  \hspace{1cm} (14)

Suppose now that $\mathcal{N}$ is invariant to changes in the continuous variable $\xi$, which may be, for example, $x$ or $t$. Then, if $\phi(\xi)$ is a solution to (11), also so is $\phi(\xi + \delta \xi)$ for any value of $\delta \xi$. (The dependence of $\psi$ on variables other than $\xi$ has been suppressed.) If $\delta \xi$ is small, then

$$\psi(\xi + \delta \xi) = \psi(\xi) + \delta \xi \psi(\xi) + O(\delta \xi^2).$$  \hspace{1cm} (15)

Substitution into (11) gives

$$\psi_t + \delta \xi \psi_{\xi} = \mathcal{N}(\psi + \delta \xi \psi) + O(\delta \xi^2),$$  \hspace{1cm} (16a)

$$= \mathcal{N}(\psi) + \delta \xi L_\phi \psi + O(\delta \xi^2),$$  \hspace{1cm} (16b)

which simplifies to

$$(\psi_\xi)_t = L_\phi \psi_\xi$$  \hspace{1cm} (17)

after the limit $\delta \xi \to 0$ is taken. Note that $\psi_\xi$ is steady, time periodic, or uniformly bounded in time if $\psi$ is as well.

Comparing Eq. (17) to Eq. (14) shows that $\phi = \psi_\xi$ is a solution to Eq. (14) with $\lambda = 0$. Thus, if $\mathcal{N}$ is invariant to changes in $\xi$, $\psi_\xi$ can be identified as the structure function of a neutral normal mode. If the nonlinear solution $\psi$ is time periodic, then so is the structure function $\psi_\xi$ and we can thus identify $\psi_\xi$ as a neutral Floquet vector. Since (1) is invariant to translations in time $t$ and the zonal coordinate $x$, neutral modes will exist that are proportional to temporal and along-channel derivatives of the basic cycle. The former corresponds to an infinitesimal shift in time of the basic wave-mean oscillation while the later corresponds to an infinitesimal along-channel shift in the position of the oscillation.

In practice, neither of the numerically determined neutral modes have Floquet exponents exactly equal to zero due to numerical errors. For the temporal neutral mode, small departures of the basic cycle from exact periodicity break the time translation invariance of the basic cycle leading to small departures of the mode from neutrality. The departures from exact periodicity are small ($<10^{-6}$) so the magnitude of the corresponding exponent is less than $10^{-8}$.

The precision with which the zonal-translation neutral mode is determined is limited both by the ability of the space-differencing scheme to resolve the fourth derivative of the basic cycle (three from the gradient of the potential vorticity and one from the neutral mode itself) and by the fact that imposing a numerical grid transforms the continuous zonal symmetry into a discrete symmetry. The basic cycle contains enough power at high wavenumbers and the numerical grid is course enough even at the relatively high resolution of $72 \times 62$ that the magnitude of the numerically determined Floquet exponent is greater than $10^{-3}$. Numerical experiments that computed a subset of the Floquet vectors show that, when the resolution is increased to $128 \times 64$, $\Phi^s$ converges to neutrality to within $10^{-4}$ (almost 100 times smaller than the next largest Floquet exponent). For consistency with the rest of the calculations, the solution of $\Phi^s$ has been held at $72 \times 62$, but the numerically determined zonal-translation neutral vector has been replaced by the along-channel derivative of
the basic cycle, and the corresponding exponent has been set to zero.

d. Damped-advective modes

The vast majority (~95% at 72 × 62 resolution) of the Floquet spectrum is made up of vectors with small, irregular spatial features that are advected by the mean flow and decay nearly at the dissipation rate $r$; they are thus called damped-advective (DA) modes (Fig. 4). Samelson (2001b) found a related set of damped modes in the weakly nonlinear problem. SW03 did not find a set of DA in the fully nonlinear problem due to limitations of their numerical method.

The evolution of the structure function $\phi^{4464}$, which is representative of this class, is characterized by the creation, decay, and advection of incoherent finescale eddies (Fig. 10). These modes do not appear to have any coherent phase shift with height. The perturbation PV flux of this mode is uncorrelated with the background PV gradient (Fig. 8); the perturbation heat flux (not shown) is similarly uncorrelated with the background temperature gradient. This mode is thus unable to effectively exchange vorticity or energy with the background flow and can only decay by Ekman dissipation. Consistent with this fact, analysis of the term balance for this mode shows that the dynamics reduces to passive advection of the disturbance PV by the background flow (Fig. 7c). Although the dissipation term (D) remains small throughout the evolution of this mode, it remains larger than the only term that can affect growth (term C). The dominance of terms A and B reflects the leading-order passive advective dynamics of the mode. The net effect of the relatively small, continuous viscous decay is a large reduction in the amplitude of the mode over the basic cycle.

The other DA modes are similar: they have complex exponents, little or no phase shift with height or correlation between the perturbation PV (heat) fluxes with the background PV (temperature) gradient, and term balances dominated by passive advection. For a given mode amplitude, the dissipation term for the DA modes is generally larger than for the WD modes, because the DA modes have smaller scales. A few DA modes with very fine spatial scales (off the right-hand side of Fig. 4) decay significantly slower than the dissipation rate ($\text{Re}[\lambda] \sim -0.4$) due to the fact that the Runge–Kutta time-stepping algorithm anomalously amplifies high wavenumber Fourier modes (e.g., Durran 1998, chapter 2). In a subset of numerical solutions using the Adams–Bashforth scheme, which does not amplify these Fourier modes, these vectors had the same spatial structures as those found with the Runge–Kutta scheme, but decayed with rates near $r$ like the other DA modes.

In addition to having small spatial scales (i.e., large values of $\overline{K}$), the DA modes typically have broad wavenumber spectra (Fig. 11). For example, the potential enstrophy wavenumber spectrum of $\phi^{4464}$ is essentially flat (Fig. 11b), as is the case for almost all of the DA modes, as evidenced by the fact that the energy and potential enstrophy wavenumber spectra averaged over all DA modes with $4\pi < \overline{K} < 8\pi$ (8291 modes) are

![Fig. 10. Contours of the (left) upper- and (right) lower-layer streamfunction vs $x$ (horizontal axis) and $y$ (vertical axis) during the evolution of the Floquet structure function $\phi^{4464}$ for which $\lambda \sim -0.9865r - i0.3367T$. Negative contours are dashed.](image-url)
almost identical to the spectra of $\phi^{4464}$ (Fig. 11). This result is not specific to the choice of $\phi^{4464}$, as nearly all of the DA modes are statistically similar to each other. Note that, if the enstrophy wavenumber spectra remain flat in the continuum limit, the integral of the spectra (the total enstrophy) will be unbounded.

Many of the features of the DA modes are similar to those of the continuum modes, which are frequently encountered in the study of scalar advection and the stability of shear flow (e.g., Drazin and Reid 2004, chapter 4). In models with continuous shear—such as the model discussed here—the spectrum of discrete normal modes is generally not complete and must be supplemented by a set of continuum normal modes in order to describe the evolution of arbitrary initial disturbances (Orr 1907). Such continuum modes are “weak” solutions to the governing equations because, while their streamfunction fields are continuous, their vorticity fields contain singular structures and the associated enstrophy is unbounded. From one point of view, the continuum modes appear as a consequence of applying the normal-mode formalism to a system that may be alternatively formulated as an advective solution to an initial value problem (Orr 1907; Case 1960). The two formalisms are equivalent for steady parallel shear flow, but the physical interpretation of the advective solution is intuitively more appealing.

When represented by a finite-dimensional numerical model, the continuous spectrum is artificially truncated and rendered discrete. The spatial structures of the resulting numerically determined continuum modes are generally sensitive to the exact nature of the numerical model, but the wavenumber spectra may often contain significant contributions at high wavenumbers; in particular, if the PV wavenumber spectrum is nearly flat, it may indicate that the numerical model is attempting to resolve singular structures in the PV. Thus, it is tempting to view the DA modes obtained here as representing generalizations of the classical singular modes to time-dependent background flows. The DA modes appear to have unbounded enstrophies in the continuum limit and, consistent with the expected sensitivity of continuum modes to the details of the numerical model, they are not robust to changes in resolution. Floquet spectra calculated at slightly different resolutions produce sets of DA modes, which are statistically similar, but there is no direct correspondence between the individual vectors produced at different resolutions (see section 5). However, such a view is necessarily speculative, since it is based in part on the lack of convergence of the numerical results to individually well-defined modes, and since it is not known whether the time-dependent basic flow should possess an analogous set of singular continuum modes. Consequently, the DA modes are best viewed as representing, as a class, a generalized solution to the damped-advective initial value problem. This point is further discussed in section 6.

5. Convergence of the numerical method

The continuous limit of the basic cycle possesses, in principle, an infinite set of Floquet vectors. It would be
of interest to determine, in the continuous limit, the number of FVs in each class and whether either class contains a continuum of modes. There are few theoretical results, which would constrain the number of discrete normal modes in the continuous limit, so estimates of this number must be obtained as part of the numerical solution.

Full Floquet spectra were obtained for resolutions of $24 \times 22, 36 \times 32, 48 \times 40, 54 \times 45,$ and $72 \times 62$. As resolution increases, the corresponding distributions of $\lambda$ and $\overline{K}_\phi$ rapidly approach limiting distributions dominated by the DA modes (Figs. 12 and 13). The spike at $\text{Im}[\lambda] = 0$ (Fig. 12b) and its disappearance as the resolution is increased indicates that the number of wave-dynamical modes with $\text{Im}[\lambda] = 0$, $N_{\text{Re}}$, increases much more slowly than the total number of Floquet vectors. The rate at which $N_{\text{Re}}$ increases becomes slower at high resolutions (Fig. 13). This suggests that $N_{\text{Re}}$ may be finite in the continuous limit.

The distribution of potential vorticity–mean wavenumber $\overline{K}_q$ (Fig. 12d) does not converge to a limiting distribution, but takes the form of a series of spikes at progressively higher wavenumbers. The positions of the maxima are consistent with a white enstrophy spectra

![Figure 12. Normalized histograms of (a) Floquet vector growth rate Re[\lambda], (b) imaginary part of the Floquet vector Im[\lambda], (c) mean total wavenumber $\overline{K}_\phi$ of the streamfunction $\phi$, and (d) mean total wavenumber $\overline{K}_q$ of the potential vorticity $q$. The line style gives the model resolution: $24 \times 22$ (dash-dot black), $36 \times 32$ (dashed gray), $48 \times 40$ (dashed black), $54 \times 45$ (solid gray), and $72 \times 62$ (solid black). In (a), the dashed vertical line gives the frictional dissipation rate $-r$. In (b), the dashed vertical lines are at $\pm \pi/T$. In (d), the dashed vertical lines are at $K_{\text{max}}/2$. The PDFs have negligible amplitude outside the ranges shown. Fifty bins were used in all cases.](image-url)
for the DA modes (see Fig. 11), for which the expected value of the mean wavenumber is \( K_q = K_{\text{max}}/2 \), where \( K_{\text{max}} \) is the maximum resolved wavenumber (Fig. 12d).

The total number of WD modes (i.e., those with \( K < 3\pi \)), \( N_{\text{WD}} \), fluctuate at low resolutions, but appear to have reached a limiting value \( N_{\text{WD}} = 62 \) at the highest two resolutions considered (Fig. 13). It is not clear whether this represents the true number of WD modes in the continuum limit or whether more WD modes will appear at higher resolution. Nevertheless, this result strongly suggests that the number of WD modes is bounded in the limit of an infinitely finely resolved model. In contrast to \( N_{\text{WD}} \) and \( N_{\text{Re}} \), the number of intermediate and DA modes increases rapidly as resolution increases. Whether the DA modes eventually merge into a continuum is an open question.

6. Discussion

We have obtained a complete set of Floquet vectors for a baroclinic wave-mean oscillation at high resolution. The Floquet vectors fall into two classes that have direct physical interpretations, the WD and DA modes.

The WD class (which includes the two neutral modes) consists of two groups, one that contains vectors that grow or weakly decay (the leading vectors) and one that contains vectors that rapidly decay (the trailing vectors). The two groups are distinguished primarily by their phase shift with height and their growth-phase relationship with the basic cycle: the leading vectors have a westward phase shift and grow while the basic cycle is growing while the trailing vectors are tilted slightly eastward and decay during the basic cycle growth phase. The leading vectors are important for determining the asymptotic stability of the oscillation and, indeed, the existence of three WD modes with \( \text{Re}[\lambda] > 0 \) confirms that the basic cycle is unstable.

The DA class has many more members than the wave-dynamical class, but they are all similar in spatial structure and decay rate. Due to their dominance of the Floquet spectrum, any initial disturbance that contains small-scale features will excite a large number of DA modes. These modes will then describe most of the variance at small scales before the asymptotic growth of the leading wave-dynamical modes begins to dominate. This occurs very quickly since the e-folding decay time of each DA mode is \( T_{\text{dec}} \approx 0.05T \). The dynamics of the DA modes are, to first order, advection of the PV field by the background flow. These modes share many characteristics with the singular modes of parallel shear flow, including flat PV wavenumber spectra and sensitivity to the details of the model configuration.

The existence of these two classes of normal-mode solutions to the numerical Floquet eigenvalue problem indicates a dynamical splitting of the linear disturbance problem for this time- and space-dependent baroclinic flow. On the one hand, a small set of discrete, large-scale, growing or decaying normal-mode structures is easily identified. These modes have immediate physical interpretations and are analogous to the familiar normal modes of steady parallel flow. They appear naturally as intrinsic time-dependent eigenmodes of the linear disturbance flow. In contrast, the individual members of the large set of DA modes should be interpreted to represent, in sum, the frictionally damped advection of small-scale potential vorticity anomalies by the basic flow. From a physical point of view, the solution of this portion of the linear-disturbance initial value problem might be most appropriately conceptualized, and perhaps even quantitatively computed, by using the method of characteristics to follow initial disturbances along Lagrangian fluid trajectories. This would be a natural extension of Orr’s (1907) characteristic-based solution for advective motions in parallel shear flow, which provides an intuitive alternative to the singular neutral mode description that arises from the corresponding normal-mode eigenvalue problem (Case 1960).

As discussed in SW03, the unstable WD modes have intriguing similarities to the unstable modes of the spa-
tially homogeneous flow $\psi_0 = 0$. In particular, the dominant Fourier component of these modes corresponds, respectively, to the first three normal modes of the spatially homogeneous flow (cf. Tables 1 and 2). Both occur in the order $(2, 1), (1, 1), (2, 2)$ although the leading WD modes also contain strong contributions from other Fourier components. The correspondence of the three most rapidly decaying WD modes with the three most stable modes of the homogeneous flow (which occur in the reverse order of the unstable modes) is even stronger, since the trailing WD modes have their largest amplitude when the background flow is most zonally uniform. The growth (decay) rates of the leading (trailing) WD modes are greatly reduced from those of the homogeneous, steady flow. This is evidently primarily due to a reduction in the time-mean vertical shear from its undisturbed value of $U^0_z = 1$ and the introduction of a time-mean barotropic shear due to modifications to the mean flow by the basic cycle. The former effect reduces the potential energy available for growing disturbances, while the latter reduces the ability of disturbances to maintain the proper phase shift necessary to extract energy [i.e., the “barotropic governor” effect; James (1987)].

It is interesting to note that the spatially homogeneous state $\psi_0 = 0$ appears to be a better predictor of the spatial structure and ordering of the leading WD modes than the time and zonal mean of the basic cycle. The latter state is unstable to a single linear disturbance (not shown), which is dominated by the $(1, 1)$ Fourier component and closely resembles WD mode $\phi^2$. The growth rate of this mode is 0.0763, only slightly greater than the growth rate of $\phi^2$ (0.0215). Disturbances to the time- and zonal-mean flow that resemble WD modes $\phi^i$ and $\phi^j$ are both stable, with the analog of $\phi^1$ more stable than that of $\phi^2$. The analogs to the trailing WD modes are also disordered, with the first, second, and third most stable disturbances to the time- and zonal-mean flow corresponding, respectively, to the second, third, and first mode stable WD modes. While an analysis of the time mean of a nonstationary flow often does not provide a good estimate of the stability characteristics of the nonstationary flow, it is perhaps surprising that this estimate is even worse, with regard to mode structure and ordering, than that provided by the analysis of the $\psi_0 = 0$ state.

While the addition of high-order (proportional to $\nabla^n \psi$, with $n \geq 4$) dissipation is known to remove singular modes from the spectrum of fluid stability problems (Case 1960), the Ekman dissipation—which reduces to Rayleigh damping of the PV at high wavenumbers—used here is compatible with the existence of singular modes. Numerical experiments at low resolution show that the addition of weak damping proportional to $\nabla^4 \psi$ produced no systematic change in the general spatial structure of the DA modes. The decay rates of the DA modes were enhanced, but the mean PV spectra of the DA modes remained white.

With the reasonable assumption that averages over the basic cycle provide useful estimates of averages over the chaotic attractor (Samelson 2001a), we may cautiously generalize several of the results. The scales of the unstable modes on the attractor should be similar to the scales of the background flow and have PV fluxes that are strongly correlated to the background PV gradient, especially when the background PV gradient is large. Disturbances that have scales significantly smaller than those of the background flow will tend to project onto the generalizations of the DA modes; thus, these disturbances will be simply advected and rapidly damped. As a consequence, an arbitrary initial disturbance should become dominated by large-scale structures correlated with the background PV gradient in a time that is short compared to the time scales of the background flow. This would greatly reduce the size of the space that must be searched to find initial disturbances with a large impact on the flow at moderate lead times.

7. Summary

The Floquet vectors (FVs) obtained here are the time-dependent normal modes for linear disturbances to the time-periodic background flow. The FVs split into two dynamical classes that have direct physical interpretations: wave-dynamical (WD) modes and damped-advective (DA) modes. The WD modes are dominated by large-scale disturbances, which are frequency locked to the basic cycle. These vectors grow and decay via the same mechanisms as the basic cycle, by advecting heat (vorticity) across the background temperature (PV) gradient. The number of WD modes is much smaller than the total number of numerically determined FVs and these modes appear to form a discrete set. By contrast, the DA modes dominate the numerical FV spectrum and the DA class, taken as a whole, appears to represent a generalized solution to the DA problem, but individual DA modes do not appear to have natural physical interpretations. These modes have fine scales and decay at or near the frictional damping rate, but the detailed spatial structures of individual modes are not stable with respect to small changes in resolution or numerical method.

Accurate ensemble forecasting requires that the ensembles be initialized in such a way that their subsequent evolution is representative of the possible future
states of the atmosphere or ocean. This initialization should also be economical, so that the broadest possible set of future states is achieved by the smallest possible ensemble. In the present case, the asymptotic stability of the basic cycle is determined by the leading WD modes so these modes are a natural choice for the ensemble initial conditions. These modes have the advantage that they are few in number and that they are fluid instabilities, which are related, in a straightforward manner, to the background flow. This implies that the modes that are most interesting from the standpoint of geophysical fluid dynamical instability theory are also natural choices for ensemble initial conditions. Extensions of this work are in progress to address these questions also from the point of view of the optimal disturbance theory.

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