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It is the purpose of this study to examine some statistically-oriented considerations which may facilitate portfolio selection policies. Many of the preliminary topics discussed parallel and extend the notions of W. J. Baumol, H. M. Markowitz, and W. F. Sharpe.

The crux of the study introduces a quadratic programming algorithm which determines the optimal allocation of the investment dollar. Within the scope of the discussion, the optimal allocation is defined to be the one which minimizes the portfolio variance while maximizing the associated utility. It is shown how this allocation defines a portfolio which is efficient under all types of efficiency criteria.

Tolerance limits around the expected portfolio return are also introduced and discussed. Their importance lies

in the fact that a probability of occurrence may be associated with an interval estimate of a parameter but not with a point estimate of a parameter.

STATISTICAL CONSIDERATIONS IN PORTFOLIO SELECTION

by

JAMES ELROY REINMUTH

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[redacted]

Associate Professor of Statistics

In Charge of Major

[redacted]

Chairman of Department of Statistics

[redacted]

Dean of Graduate School

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STATISTICAL CONSIDERATIONS IN PORTFOLIO SELECTION

INTRODUCTION

As a manager for the funds of an individual, a trusteeship, a business enterprise, or any other organized group, the administrator of these funds usually desires to obtain a maximum "satisfaction," usually measured in terms of rate of return, while maintaining a minimal amount of uncertainty associated with that "satisfaction." Assuming he is a rational and prudent man, the administrator seeks to associate with his fund investment decisions a positive plan or program which has proven successful for pursuing the intended objectives dictated by the owners of the fund. If such a plan or program does not exist in association with his particular fund, it becomes necessary that the fund administrator devise a new policy or adapt an existent policy to meet the demands of the fund's owners. The resultant policy becomes the guide by which the fund administrator, from here on referred to as the portfolio administrator or the investor, constructs the portfolio to embody his assigned funds.

It is the purpose of this study to examine some statistically-oriented considerations which may facilitate policy development and, necessarily, determine our particular pattern of portfolio selection. This study will not tell us specifically what portfolio is optimum but will attempt to suggest certain policies which the portfolio administrator should follow in his pursuit of an optimum portfolio. Obviously, there are as many policies toward this pursuit as there are investors. Hence, we shall restrict our attention only to those policies related to security portfolio analysis of private funds.

Many of the preliminary topics discussed will parallel and extend the ideas of Harry M. Markowitz, William J. Baumol, and William F. Sharpe. The works of these men are considered classical in the realm of portfolio analysis and selection. Therefore, an introduction to some of the basic concepts presented by them is essential for the motivation of the extensions and developments to be presented later.

This paper shall introduce as the crux of its study a policy whereby a portfolio administrator might select an "optimum" portfolio from among many possible classes of securities represented within the portfolio. That is, a quadratic programming model will be established to represent these variables and show how one may optimally allocate his investment dollar among the various security classes. One should then be able to determine, within a certain degree of confidence, the portfolio with minimum variance at various levels of expected return.

PRELIMINARY PORTFOLIO CONSTRUCTION CONSIDERATIONS

Definition of a Portfolio

As used within the realm of this study, a portfolio is a group of investments. These investments may all be the same, such as investing all one's funds in savings bonds, or they may be diversified, say, investing some in government bonds and some in common stocks. This study will consider only diversified, security portfolios. That is, we shall examine portfolios which consist of diversified investments in stocks or bonds or both. However, if one is able to assign some meaningful index to the variables which represent other types of investments he can imply the generalizations we shall make about security portfolios to any type of portfolio.

Primary Policies of Investors

The portfolio administrator is an employee of the fund's owners. If he owns the fund, then you might say he is self-employed. Regardless of the relation which does exist, the portfolio administrator must adopt a policy or policies for portfolio selection which satisfy the specifications and demands of the fund's owners.

This limiting factor establishes the initial boundary lines for the portfolio administrator. His "field of play" may be quite broad or narrow depending on these dictates. These boundaries determine how far the portfolio administrator may venture in his decision of selecting from the available set of securities.

Depending on the severity of these boundaries, an investment policy may be classified as defensive or aggressive. The owners of a fund whose demands are quite conservative toward risk will establish narrow boundaries for the portfolio administrator. Consequently, a defensive policy will arise. Aggressive policies, in contrast, are just the opposite. The owners of these funds establish very wide boundaries. They are much more interested in obtaining a high rate of return than in protecting the stability of their investments.

Secondary Considerations

No fund ownership group can be classified as strictly defensive or aggressive. Such a group may have different attitudes toward various considerations at the same time.

For example, they may be very aggressive toward the pursuit of financial return but simultaneously desire to maintain a conservative attitude toward marketability of the portfolio.

The portfolio administrator must examine what the fund's owners consider risk factors. Certainly the owners have some aversion to financial risk. Also, the investor should attempt to measure the owners' aversion toward interest rate risk, purchasing power risk, marketability, and taxability considerations within the portfolio. This is a very delicate and complex matter since human beliefs and motives defy ordinary measurement procedures.

If the portfolio administrator can assign some meaningful value to all the apparent risk factors, he can then congregate them into the owners' utility function. This function will then represent the objectives of the owners or the boundaries for the portfolio administrator which must be satisfied. A study in the construction of such a utility function is an exhaustive task in itself. However, the reader may refer to the works of Markowitz on this topic (6).

Once the portfolio administrator has constructed a meaningful utility function, the development of the investment policy can be completed. The policy which gains a maximum in returns for a minimum of invested capital while satisfying the owners' utility is the optimum policy. The policy gives the investor a guide to the selection of securities. But, as is emphasized by Markowitz (6, p. 3), the investor should seek a balanced portfolio. That is, the dictates of the resultant policy should direct the investor's attention toward the activity of portfolio selection rather than security selection.

The object of the portfolio administrator's examination of construction considerations is to adopt a policy which will most likely satisfy the objectives of the fund's owners. The remainder of this study will attempt to construct a theoretical framework, embodying these considerations, which will facilitate policy formation and portfolio selection.

JUDGING THE MERIT OF AVAILABLE SECURITIES

Once the portfolio administrator has isolated the dominant risk factors associated with the fund owners' demands, he must devise some method to measure these risks as they exist within particular investments. First, he must define the meaning of "return" for his particular case. Return will have meaning as the opposite or contrasting factor to the inherent risk. Since the portfolio administrator is a prudent man, he desires this return to be as high as possible while still not violating the "overall risk" constraints. After defining and measuring this risk, the investor must then weight these factors against the inherent risk-return factors possessed by the available set of securities. Then he makes his portfolio selection decisions.

Measuring Risk

Many methods of assessing the risk within securities or groups of securities are available. A few possible alternatives one may pursue are:

1) Subjective prediction - Predictions about deviation or risk from the initial value are based

partly or wholly on judgement.

2) Range method - The high and low value over some past period are examined. A wide range would signify an aggressive security, a narrow range, a defensive one.

3) Deviation from initial value - The dispersion or variance from the return value of a security at a particular past period is calculated.

4) Deviation from the expected value - This concept considers future expectations based on past performances and estimated future performances.

Other methods are available but are of trivial importance.

For discussion, we shall define the random variable X to be the return from portfolio P_{χ}. Therefore, the expected value of the random return from portfolio P_{χ} is E(X). We then note the deviation or variance of X from E(X) and call this our risk.

Assuming that squared deviation is the appropriate measure to denote dispersion, then the "root mean square from the expected value" method is obviously superior to the others. This is true because $E(X - a)^2$ is minimized by $a=E(x)$. Using the expected value of return as the reference point gives us a weighted estimate of central tendency. It takes into consideration equal aspects of probability and frequency of occurance of each expected outcome. But since the expected value alone does not say anything about the dispersion of possible values, we measure this factor by the standard deviation.

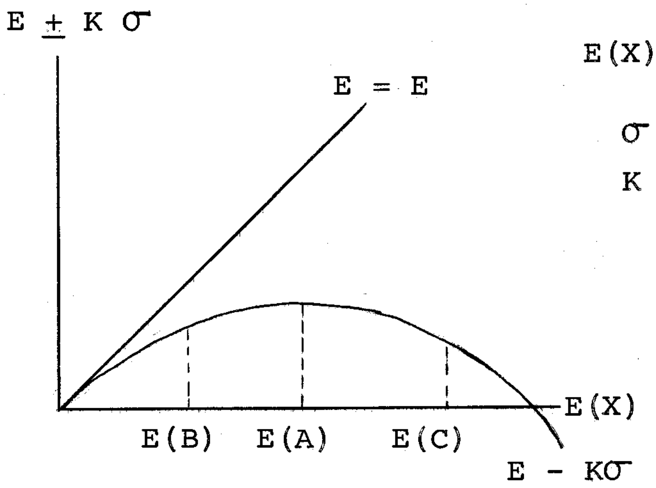
However, if the array of outcomes is linear, then one would choose as his measure of dispersion $E\left|X - a\right|$. This is minimized by a = median (X). Therefore, the measure of central tendency is dependent upon what we choose, or are forced to choose because of the distribution of our outcomes, as our measure of dispersion.

Extensions of the Risk Concept

Markowitz (6, p. 154) says that when the expected value of return E(X) is constant, we should choose the portfolio with the least σ, and when σ is constant we should choose the one with the greatest E(X). He says that this method gives us an efficient set of portfolios. From this (E, σ) efficient set we then choose the one which best satisfies the owner's indifference restrictions expressed by his utility function.

Baumol (1) suggests that we carry this procedure one step further. He contends that we should seek not necessarily a moderate standard deviation but should seek a highest possible "floor" on the expected return. This floor is the lower tolerance limit ($E - K\sigma$) on the realized return which is analogous to the "max-min" of elementary decision theory.

Consider the diagram below:



$E(X)$ = expected return of portfolio P_χ.
σ = standard deviation
K = positive constant

Figure 1

Baumol contends that a rational portfolio administrator would accept portfolio P_a over portfolio P_b in this case since both $E(A) > E(B)$ and $E(A) - K\sigma(A) > E(B) - K\sigma(B)$. He would not, however, accept portfolio P_c. Even though $E(C) > E(A) > E(B)$, the floor $E(C) - K\sigma(C)$ is much less than the floor of the other two portfolios.

This approach is more cautious toward risk than the

Markowitz approach since Baumol attempts to shy away as much as possible from the consequences of an unfavorable outcome. It assumes that the portfolio administrator is more concerned with maximizing the lower limit with a constant expected return than he is with minimizing the standard deviation. Assuming that the distributions of probable outcomes of various portfolios are normal, we can place more meaning on Baumol's results. Looking at the tolerance intervals about the actual outcome, we see:

$$\Pr\left[E(X) - \sigma(X) < X < E(X) + \sigma(X)\right] = 0.68$$
$$\Pr\left[E(X) - 2\sigma(X) < X < E(X) + 2\sigma(X)\right] = 0.95$$
$$\Pr\left[E(X) - 3\sigma(X) < X < E(X) + 3\sigma(X)\right] = 0.996$$

where $E(X)$ = expected return of portfolio P_χ.

$\sigma(X)$ = standard deviation of portfolio P_χ.

X = actual return of portfolio P_χ.

This tells us that under Baumol's criteria we can determine what the probable floor on our expected outcome is for a particular constant K*. Then we can use this factor and judge the available set of portfolios by $(E, E - K\sigma)$ efficiency as opposed to Markowitz's (E, σ) efficiency.

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* That is, if we say $\Pr\left[E(X) - 2\sigma(X) < X < E(X) + 2\sigma(X)\right] = 0.95$, we can be sure with a probability of 0.975 that the floor on our actual return will not be less than $E(X) - 2\sigma(X)$.

Baumol's Paradox. Baumol (1, p. 180) attempts to show by the following diagram how the acceptability of his criteria depends on the value of K and the degree of defensiveness of the investing group.

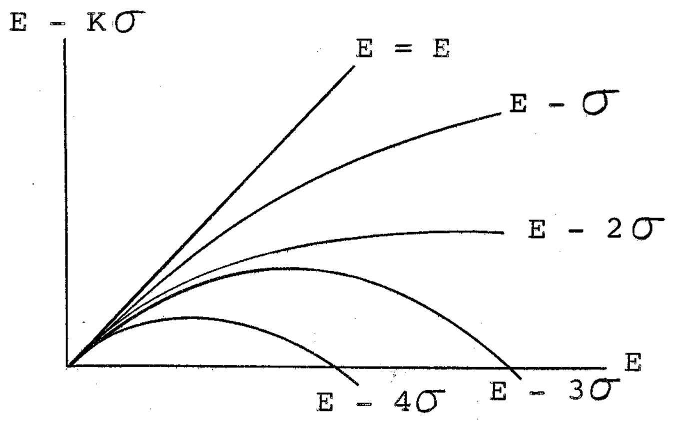

Figure 2

A very aggressive investor is willing to accept a much lower floor on possible returns for a greater expected return. In contrast, a defensive policy demands a much greater return for a small decrease of the lower expected return limit.

Consider the following example. Suppose two portfolios were under consideration and had the following characteristics. For a particular amount of invested capital, $E(A) = 12$, $E(B) = 9$, $\sigma(A) = 4$, $\sigma(B) = 2$. Baumol says that under Markowitz's criteria neither portfolio dominates the other. However, he would contend that under his considerations portfolio P_a would dominate portfolio P_b since

a) $E(A) > E(B)$, $(12 > 9)$

b) $E(A) - \sigma(A) > E(B) - \sigma(B)$, $(8 > 7)$

It does not seem that what was presented in the previous paragraph is adequate to judge the merits of these two portfolios. Suppose we extend the floors on the expected returns as follows:

	a	b
E	12	9
σ	4	2
$E - \sigma$	8	7
$E - 2\sigma$	4	5
$E - 3\sigma$	0	3

According to Baumol, it is obvious that when E(X) is constant a rational portfolio administrator would choose the same portfolio under both sets of criteria. But as E(X) varies, this may not apply. Under Baumol's criteria, portfolio P_a would dominate portfolio P_b when K = 1, but neither would dominate the other when $K > 1.5$. Therefore, Baumol is not telling us any more than is Markowitz since an investor does not accept a particular K value as Baumol suggests. Not just one but many K values are associated with every portfolio. Under the assumption that the underlying distribution of outcomes is normal, the tolerance limits for each limit $(E, E - K\sigma)$ have similar meanings in terms of probability of occurrence. Since each pair of portfolios which are equally attractive under Markowitz's criteria are also equally attractive (one is not dominant) for some K under Baumol's criteria, both sets of considerations are identical.

<u>Definition</u>. If $\{(E, \sigma)\}$ is a Markowitz efficient set of portfolios, then for any $E' > E$ representing a portfolio outside the Markowitz efficient set, $\sigma' > \sigma$.

If $\{(E, \sigma, K)\}$ is a Baumol efficient set of portfolios, then for any $E' > E$ representing a portfolio outside the Baumol efficient set, $E' - K\sigma' < E - K\sigma$ for some $K > 0$.

<u>Theorem</u>. A portfolio or set of portfolios is Markowitz efficient if and only if it is Baumol efficient.

(Proof): To show Baumol efficiency implies Markowitz efficiency; suppose that we have a Baumol efficient set of portfolios (E, σ, K) for $K > 0$. Consider any other portfolio (E', σ') outside the Baumol efficient set where $E' > E$. By Baumol efficiency, for some $K > 0$,

$$E' - K\sigma' < E - K\sigma.$$

This implies

$$(E' - E) < K\sigma' - K\sigma = K(\sigma' - \sigma).$$

Now, since $E' > E$,

$$(E' - E) > 0 \text{ and, therefore,}$$

$$0 < (E' - E) < K(\sigma' - \sigma) \text{ and}$$

$$(\sigma' - \sigma) > 0 \text{ since } K > 0.$$

This implies that

$$\sigma' > \sigma.$$

Now, notice that this satisfies the criterion for Markowitz efficiency. Then we may say that any set (E', σ', K) of portfolios which is outside the Baumol efficient set is also outside the Markowitz efficient set. Consequently, any set (E, σ, K) which is Baumol efficient is also Markowitz efficient.

To show Markowitz efficiency implies Baumol efficiency, suppose that we have a Markowitz efficient set of portfolios (E, σ). Consider any other portfolio (E', σ') where $E' > E$ outside the Markowitz efficient set. Then, by the Markowitz efficiency criterion, $\sigma' > \sigma$. Now, we wish to find a K such that

$$E' - K\sigma' < E - K\sigma .$$

Rearranging the above expression, we get

$$E' - E < K(\sigma' - \sigma), \text{ or}$$

$$K > \frac{(E' - E)}{(\sigma' - \sigma)} > 0.$$

Necessarily, then, $K > 0$ since

$$E' - E > 0 \text{ and } \sigma' - \sigma > 0.$$

This implies that there exists a $K > 0$ such that

$$E' - K\sigma' < E - K\sigma \text{ when } E' > E \text{ and } \sigma' > \sigma .$$

We notice that the foregoing satisfies the criterion for Baumol efficiency. Therefore, we can conclude that for any set (E', σ') of portfolios outside the Markowitz set of efficient portfolios, there exists a $K > 0$ such that (E', σ') is also outside the Baumol set of efficient portfolios. Therefore, any set (E, σ) of portfolios which is Markowitz efficient is also Baumol efficient.

Since Baumol efficiency implies Markowitz efficiency and since Markowitz efficiency implies Baumol efficiency, then, Baumol efficiency and Markowitz efficiencies are equivalent, and the theorem is proved.

Further Extensions of Efficiency Considerations. Perhaps Baumol's general notion may be best interpreted by examining the probabilities of occurrence of the lower tolerance limits. With the assumption of normally distributed outcomes, an aggressive investor may say that he desires only a probability of 0.16 that the lower tolerance limit not drop below a point L. But a very conservative investor may say that he desires that the probability be less than 0.0025 that the lower tolerance limit not drop below L. Therefore, the former has a $K = 1$ and the latter has a $K = 3$ since, where

$L = E(X) - K \quad (X)$

$$Pr\left[E(X) - 1\sigma(X) < X < E(X) + 1\sigma(X)\right] = 0.68$$

$$Pr\left[E(X) - 3\sigma(X) < X < E(X) + 3\sigma(X)\right] = 0.995$$

under assumptions of normality. It may be far more meaningful to first look at the probability of exceeding a particular limit and then finding the associated K-value than vice-versa as does Baumol.

From the preceding ideas, we may extend the following notions. First, a particular K-value, when associated with a pair of acceptable portfolios, gives the investor some criteria about when to accept each particular portfolio depending on his risk aversion.

Assume, as before, that E(X) is the expected value of return from portfolio P_X. Now, suppose we have two portfolios which are Markowitz acceptable and $L(A) = E(A) - K\,\sigma(A)$ and $L(B) = E(B) - K\,\sigma(B)$. Equating these two limits, we get

$$K = \frac{E(A) - E(B)}{\sigma(A) - \sigma(B)}.$$

Consider our previous example where both portfolios were Markowitz acceptable and

	a	b
E	12	9
σ	4	2

Solving for K, we find that K = 1.5. This would tell us that under Baumol's criterion there is a probability of 0.067 that portfolio P_a will not dominate portfolio P_b, under the assumption of normally distributed outcomes. This implies that an investor who is willing to accept a probability of 0.067 that the floor on the possible outcomes may drop below L = 6 will accept portfolio P_a. But one who wishes this probability to be smaller would consider these portfolios equally attractive or unattractive. If the K-value equating the lower tolerance limits of the two portfolios was high, say 3, we could say that an investor would have to be extremely defensive if he considered both portfolios equally attractive since

$$Pr \left[X < E(X) - 3\sigma(X) \right] = 0.0025.$$

It is, in the opinion of this writer, much more meaningful to examine Baumol's notions from a probability sense than from associating a particular K value with an investor. The latter approach gives little regard to the probability of occurrence of the associated random outcome and, consequently, is useless as a practical tool for judging the merits of available securities.

Another notion of both Markowitz and Baumol which should be explored concerns the dominance of portfolios. According to both men, if for three portfolios E(A) = E(B) = E(C) then any rational investor would accept the portfolio with the smallest variance. The one with the smallest variance would also have the highest lower tolerance limit and satisfy Baumol's criteria.

By making such an assumption, it seems that they are classifying the "rational investor" as strictly defensive. They are not even considering an aggressive investor who may be much more interested in maximizing his probability of a high return than he is in maximizing the lower tolerance limit of probable returns to protect his invested capital.

Suppose for a particular amount invested two portfolios exist such that

	a	b
E	10	10
σ	2	5

Both Markowitz and Baumol would say it is obvious that the investor would accept portfolios P_a. But let us look at the upper tolerance limits of expected return.

Portfolio P_a: $E(A) + K\sigma(A) = 10 + 2K$

Portfolio P_b: $E(B) + K\sigma(B) = 10 + 5K$

The upper tolerance limit of portfolio P_b is higher than that for portfolio P_a for any $K > 0$. Consequently, portfolio P_b would be much more attractive to an aggressive investor. Assuming normality of outcomes, we can examine their respective probabilities of occurrence.

For $K = 1$, $\Pr\ (0(A) > 10 + 2K = 12) = 0.16$

$\Pr\ (0(B) > 10 + 5K = 15) = 0.16$

For $K = 2$, $\Pr\ (0(A) > 10 + 2K = 14) = 0.025$

$\Pr\ (0(B) > 10 + 5K = 20) = 0.025$

where $0(X)$ = actual outcome of portfolio P_X and probability expressions for each K are interpreted with a meaning similar to that expressed in the preceding section.

Obviously, an aggressive investor who is seeking to maximize his upper limit would for any $K > 0$, contrary to both Markowitz and Baumol, select portfolio P_b. The risk of a greater variance or lower possible floor on expected returns is incidental to such an investor. Consequently, we must say that portfolio P_a and portfolio P_b are both admissible unless something is known of the degree of defensiveness or aggressiveness possessed by the fund ownership group.

No definite assumption about dominance of portfolios or other investments applies to all investors. Each investor has a different aversion toward risk than any other. Therefore, the Markowitz and Baumol dominance criteria for portfolios with equal expected returns are invalid when the investing group is aggressive.

Basic Utility Considerations

In the preceding pages much has been mentioned about determining a knowledge of the fund owners' utility toward risk before evaluating the available set of investments. Maximizing the expected return or minimizing the expected variance of the portfolio has little meaning when not considering utility. This was shown quite clearly in the preceding section.

According to Markowitz and Sharpe, the goal of a rational investor is to maximize his utility. Each alternative available investment is assigned some value by the utility function which indicates the fund owners' degree of satisfaction with that alternative. The investor then judges the available set of alternatives only with an eye toward the one with the greatest expected utility.

To see what this means in a practical sense, consider the following approach. Suppose the investor knew the utility function associated with the convictions of his fund's owners. Assume, as is the usual case in reality, that the utility function is quadratic where the constants a, b, and c are known. Such a utility function, defined on a specific interval, would appear as below,

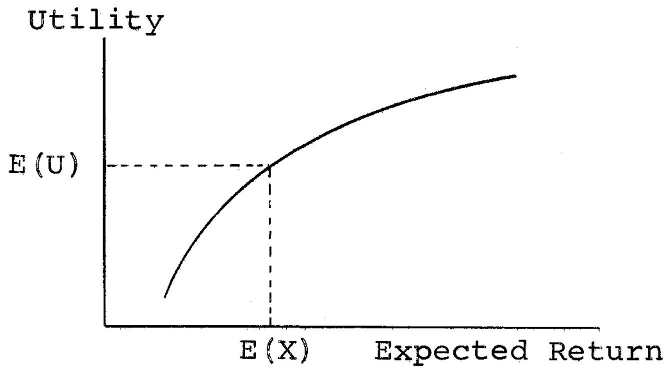

Figure 3

where the utility function $U = a + b(X) + c(X^2)$, and

X = outcome of alternative x.

Now, the expected utility is

$$E(U) = a + B(E(X)) + c(E(X^2)).$$

Naturally, $E(X)$ is the expected outcome or the expected value of return of a particular alternative, but the second moment $E(X^2)$ has no meaning as such. However, the variance of the outcome or return is expressed by

$$\text{var}(X) = E(X^2) - (E(X))^2.$$

Therefore, we can say

$$E(U) = a + b(E(X)) + c(\text{var}(X) + (E(X))^2).$$

This tells us just what we had hoped it would. Namely, this says that when considering the relative merits of available alternatives, we have sufficient information by knowing their means and variances. We can apply our previous discussion to the determination of expected value and risk and then associate the resultant values with our particular utility curve. Then, the process of portfolio selection from the available set becomes trivial.

Splitting the Investment Dollar

The available set of alternatives, in our case the set of available securities or portfolios, may be independent of each other, correlated, or perfectly correlated. Assuming our portfolio under consideration consists of two alternative securities S_1 and S_2, then the expected value and variance of this portfolio are

$$E(P) = X_1 E_1 + X_2 E_2$$

$$var(P) = X_1^2 \, var_1 + X_2^2 \, var_2 + 2X_1X_2cov_{12}$$

where X_1 = percentage of each investment dollar devoted to S_1, $X_2 = 1 - X_1$.

Since the correlation between S_1 and S_2 is $\rho_{12} = \dfrac{cov_{12}}{\sigma_1 \sigma_2}$,

then for independence between alternatives

$$var(P) = X_1^2 \, var_1 + X_2^2 \, var_2 ,$$

for correlation between alternatives

$$var(P) = X_1^2 \, var_1 + X_2^2 \, var_2 + 2\rho_{12}\sigma_1 \sigma_2 \, (X_1X_2) ,$$

for perfect correlation between alternatives

$$var(P) = X_1^2 \, var_1 + X_2^2 \, var_2 + 2(1)\sigma_1\sigma_2(X_1X_2) , \text{ or}$$

$$var(P) = (X_1 \sigma_1 + X_2\sigma_2)^2.$$

<u>Independent Returns</u>. For the first case, consider an investment consisting of a risk-bearing security and a non risk-bearing security. Therefore,

$$\rho_{12} = \text{var}_2 = 0$$

$$E(P) = X_1E_1 + X_2E_2$$

$$\text{var}(P) = X_1^2 \ \text{var}_1.$$

The overall variance of the portfolio would decrease for a smaller amount X_1 invested in security S_1. But this decrease in variance would most surely be accompanied by a decrease in expected value E(P) such as below.

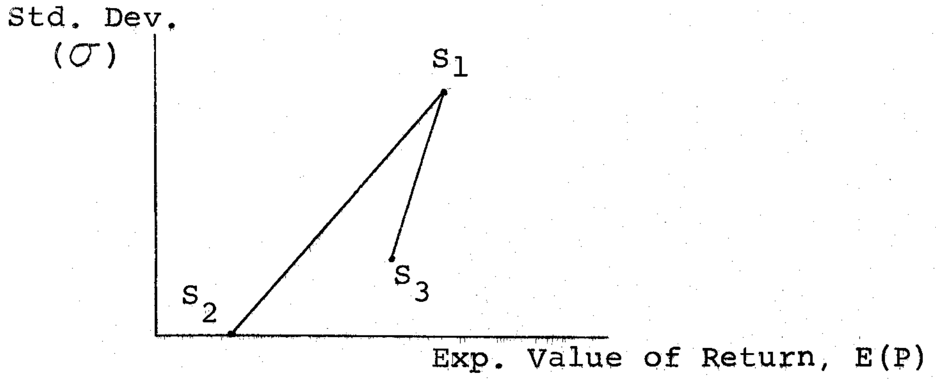

Figure 4

The line between each pair of securities indicates the range of possible (E, σ) combinations for a portfolio where the investment dollar is split between the two securities.

Now consider another available risk-bearing security S_3 such that $\rho_{13} = 0$. The variance of a portfolio containing S_1 and S_3 would be

$$\text{var}(P) = X_1^2 \ \text{var}_1 + X_3^2 \ \text{var}_3.$$

The range of possible (E, σ) combinations with this portfolio are quite different from those between S_1 and S_2. Some combinations of one portfolio will dominate certain combinations of the other in each case. The acceptable combination which we, as a rational and prudent investor, should accept is the one from the available set which maximizes our respective expected utility.

Correlated Returns. The correlation, ρ_{12}, between two variables may range from -1 to +1. When the correlation is positive, we say that the variables, in our case, the securities, vary in a similar direction. That is, when one security gains in price appreciation another security which is positively correlated with the first security also gains. When the correlation between two securities is negative, the relation is like that between a government bond and a corporate stock. If one gains in price appreciation, the other loses in price appreciation.

As was noted earlier, a portfolio with two correlated securities has the following properties:

$$E(P) = X_1 E_1 + X_2 E_2$$

$$\text{var}(P) = X_1^2 \text{ var}_1 + X_2^2 \text{ var}_2 + 2\rho_{12}\sigma_1\sigma_2(X_1X_2)$$

This gives us a situation such as follows:

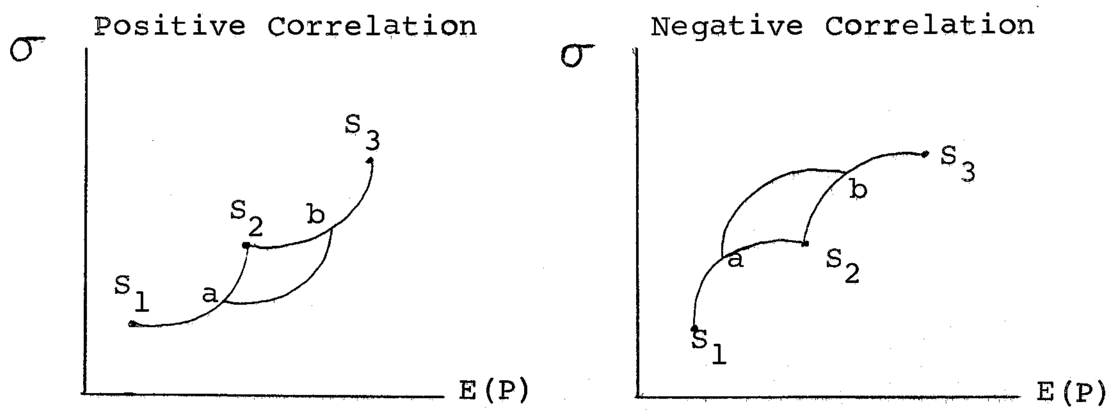

Figure 5

When we split our investment dollar between two correlated securities the resultant set of available (E, σ) combinations is quadratic. The optimal combination would be determined by a quadratic programming problem. We would construct some equation for our portfolio, say $Y = E - K\sigma$ where σ is quadratic and K is fixed. We would then maximize Y for particular slopes determined by K which is dependent on the associated utility. Then, by a process of quadratic programming, the optimal combination which maximizes the expected utility can be found.

Assume now another correlated security S_3 belongs to the available set. Suppose we wish to obtain the combination for our portfolio from the three securities which maximizes our expected utility. That is, we wish to find the optimum way to divide our investment dollar three ways.

This is also a quadratic programming problem. If we take some point "a" on the arc of available combinations between S_1 and S_2 and some point "b" on the arc of available combinations between S_2 and S_3, then the available combinations among S_1, S_2, and S_3 can be represented as shown in Figure 5. As can be seen, this combination produces in most cases portfolios with a more efficient (E, σ) criteria when the securities are positively correlated than either the S_1, S_2 or S_2, S_3 combinations. Moreover, the admissible set of Markowitz efficient (E, σ), undominated portfolios is the envelope of all possible divisions of the investment dollar among S_1, S_2, and S_3. This envelope forms a convex set with positive correlation and a concave set with negative correlation. The optimal point which maximizes the expected utility can then be determined by a quadratic programming problem similar in design to that described for the two security case.

Perfect Correlation. When the correlation between two securities equals one, we say they are perfectly correlated. Price fluctuations, market stimuli, or any other contributing factor has an equal effect on two securities that are perfectly correlated. Perfect positive correlation between securities reacts the same as combined investment in a single security.

For two securities S_1 and S_2 which are perfectly correlated,

$$E(P) = X_1 E_1 + X_2 E_2$$

$$var(P) = (X_1 \sigma_1 + X_2 \sigma_2)^2.$$

The variance of the portfolio for perfect correlation is greater than the variance of the portfolio for partial correlation. In fact, the variance of the portfolio for perfect correlation is the upper bound for the variance of the portfolio containing any two securities which are correlated as can be seen from Figure 6.

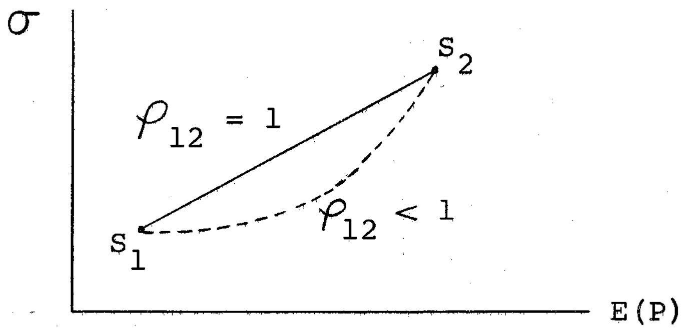

Figure 6

Since $var(P) = (X_1 \sigma_1 + X_2 \sigma_2)^2$ then $\sigma(P) = X_1 \sigma_1 + X_2 \sigma_2$. This says we can assume that both E(P) and σ(P) are linear. Consequently, our problem of determining the optimum combination between two perfectly correlated securities becomes a linear programming problem. We vary our objective equation $Y = E - K\sigma$ for a fixed K until we maximize Y. Since this objective equation was constructed to meet the constraints imposed by the utility function, maximizing Y will maximize the expected utility.

Summary

Section II attempted to present some statistical considerations in evaluating risk. Whatever risk may be to a particular investor, he usually wishes that it be minimized.

Some of the ideas of Markowitz and Baumol on criterion of efficiency were examined and extended. Also, the notions of Markowitz and Sharpe on the relation of utility to risk and correlation considerations were explored.

However, this writer believes that one underlying assumption of all these pursuits creates an overly restrictive environment. They all assume that when expected returns are equivalent, a rational investor selects the portfolio with the smallest variance. By so doing this minimizes the tolerance limits around the expected return and penalizes the highly aggressive investor.

If we wish to create a broad pattern of considerations for all types of investors, we must not restrict, by specific assumptions, some particular subgroup of these investors. The author has attempted to present and extend the preceding notions with reference to the fact that all decisions of dominance or efficiency must be in harmony with one's utility criterion. Therefore, the only valid assumption should be that we wish to select a portfolio from the available set which maximizes the expectation of one's utility.

A QUADRATIC PROGRAMMING MODEL FOR PORTFOLIO SELECTION

According to Markowitz, the investor selects his portfolio by considering three sets of criteria:

1) Probabilistic estimates about future performances of securities within each portfolio of the available set,
2) Determining the efficient set of portfolios,
3) Selecting from this set the portfolio which best satisfies the investor's assigned utility function.

Much was said in the preceding sections about how an investor might examine the latter two considerations. This section, though, will attempt to extend the first notion of Markowitz through a quadratic programming model and show how such a model facilitates Markowitz's second and third portfolio selection criteria.

The theme of this section is to show that the process of portfolio selection may be accomplished by a statistically oriented, step-wise procedure. The first step is to measure the return, variance of return, and correlation of returns between security classes for a particular amount of invested capital. Then, for a given expected return, we attempt to find the optimal allocation among the various security classes which minimizes the overall portfolio variance.

Throughout the course of this discussion, it will be assumed that all dollar units expressed in regard to security returns are directly proportional to personal utility units. Therefore, an allocation of the investment dollar among the various security classes which minimizes the portfolio variance at some level of return simultaneously maximizes the associated utility.

Construction and Examination of Security Classes

The choice of particular security classes to embody the available set of securities from which we shall select the components of our portfolio is dependent upon the fund's owners utility or indifference function. The utility function, the summation of the fund's owners feelings toward all types of risk, is our guide to all security selection decisions. This fact has been repeated continuously throughout this paper, but its importance cannot be overemphasized. However, a knowledge of the demands and restrictions implied by a particular utility function does not specify which particular security issues the investor should select to comprise his portfolio. To accomplish this end, the investor must translate these constraints into a form which may be related to conditions existing within the market and among various securities.

Since the construction of a utility function for a particular group of fund owners comprises the feelings of many people, one cannot say that the amount of allowable risk at any level of expected return is precisely some specific value. That is, the investor reasons about his associated risk restrictions in terms of an interval estimate as opposed to a point estimate. Interval estimates are much more meaningful than point estimates since, by the implications of the Central Limit Theorem, one may assign a probability to the chance that a particular interval contains the actual risk parameter.

When we speak of a security class, we refer to a group of security issues which are related with respect to the business nature of the issuing firms. Examples of security classes are equity issues of a specific industry, issues of particular utility groups, or certain grades of corporate debt issues. Also, the class of risk-free investments, which is uncorrelated with a risk-bearing security class, must be considered by the investor.

Class Returns - Definition. As used within the scope of this chapter, the return or expected return of a security class is defined to be the combined effect of the market price appreciation and dividend or interest accrual. This gain will be measured for a particular amount of invested capital over an annual time period.

Price appreciation has different meanings with respect to debt and equity issues. In a debt issue, the price appreciation or negative price appreciation is a certain percentage of its par value which is also dependent upon the grade classification of the bond. For example, suppose BBB industrial bonds were selling at an average yield of 4.55% and A bonds were selling at an average yield of 3.84%. A debt issue which has a 4% coupon due in twenty years would be worth 92.85% of par if classified as BBB and would be worth 102.10% of par if classified as A (8). Therefore, it is important that a debt issue be properly classified according to grade to avoid discrepancies in recording the correct return.

In general, the annual return for a dollar invested in debt security class D in year j is

$$R_{D,j} = \frac{(V_{D,j} - V_{D,j-1}) + I_{D,j}\ P_{D,j}}{V_{D,j-1}}$$

where $V_{D,j}$ = value of a security from debt class D at the end of year j,

$V_{D,j-1}$ = value of a security from debt class D at the end of year j-1, therefore, the purchase price of the security,

$I_{D,j}$ = average interest for a security from class D in year j,

$P_{D,j}$ = average par value of a security from class D in year j.

These figures should represent the weighted average of the values for the members of each class so the returns for respective classes may be comparable.

The choice of boundaries for the classes of equity issues is a more involved process. However, one simple method, which is the one which will be adopted for this discussion, is available. This procedure uses the classes as defined by Standard and Poor, Dow - Jones, or Moody's Investor Service. For instance, Moody lists under the "Tire and Rubber" class, Firestone Tire, Goodrich, Goodyear, and U. S. Rubber. Other tire and rubber manufacturers exist but those listed as class representatives dominate them with respect to their effect on the class as a whole.

Suppose the investor adopts the 48 listed classes in (7). He should then determine the total price appreciation for each class over a series of annual periods. This class appreciation or return should here as well as in debt classes denote the weighted average of return with respect to size of only those firms in each class listed as "class representatives."

Therefore, the total return per dollar invested in equity class K for year j may be expressed as follows:

$$R_{K,j} = \frac{M_{K,j} \sum_{i=1}^{n} d_{i,j} \dfrac{G_i}{\sum_{i=1}^{n} G_i} + (M_{K,j} - M_{K,j-1})}{M_{K,j-1}}$$

where $M_{K,j}$ = average market price for a share from class K at the end of year j,

$M_{K,j-1}$ = average market price for a share from class K at the end of year j-1, which is then the purchase price of the security,

$d_{i,j}$ = dividend per share for firm i of class K in year j,

G_i = gross revenue (to denote size) for firm i of class K in year j.

Recapitulating, debt classes were defined to be those groups of bond issues bearing a similar grade, ranging from AAA down to C according to Moody's ratings and a similar range according to other sources. Equity classes were accepted as determined by Moody or other investor service groups. The annual return per dollar invested in each class for some year j was defined to be $R_{D,j}$ for debt classea and $R_{K,j}$ for equity classes where these values have meaning in the sense of previously discussed factors. In addition, we must include the annual return $R_{F,j}$ of risk-free investments for some year j per dollar invested. This value is simply

$$R_{F,j} = I_{F,j} \ (100)$$

where $I_{F,j}$ = the average interest rate for risk-free class F in year j.

An alternate method of determining the annual security class return is presented in the appendix. This notion considers a regression estimate as suggested by Sharpe.

<u>Class Returns - Correlation</u>. Consider the array of all returns $R_{D,j}$, $R_{K,j}$, $R_{F,j}$ for all classes over a series of n-years as discussed in the previous section. For each pair of classes C_g, C_h, the correlation of returns over the n-year period is

$$\rho_{C_g, C_h} = \frac{\text{cov}(C_g, C_h)}{\sigma_{C_g} \sigma_{C_h}} ,$$

where $\text{cov}(C_g, C_h) = \sum_{j=1}^{n} (R_{g,j} - \bar{R}_g)(R_{h,j} - \bar{R}_h)$

and $\sigma_{Cg} = \sqrt{\sum_{j=1}^{n} \frac{(R_{gj} - \bar{R}_g)^2}{n - 1}}$,

for years j 1,2,...,n. However, when one of the classes is the risk-free class, $\rho_{C_g, C_F} = 0$ since the covariance between the returns of the risk-free class and the returns of any risk-bearing class is always zero.

From the previous chapter, we have sets of criteria with which to judge the importance and meaning of our resultant pairs of correlations. For instance, an investor who is quite defensive would, according to this criteria, split his investment dollar between securities with a low, positive correlation. On the other hand, a gambler would seek a portfolio with a high correlation between securities.

Correlation actually measures the degree of diversification existing within the portfolio. A portfolio whose securities are highly correlated with each other would consist of securities from the same or highly related industries. Consequently, the portfolio would have little if any diversification among security classes represented within the portfolio. In contrast, a portfolio with very low correlation between securities would be highly diversified with regard to related security classes.

Markowitz (6) extensively discusses the merits of a diversified portfolio under various conditions. The essence of his discussion supports what we have emphasized in this section. That is, the variance or risk on return of a portfolio can be measured by the correlation between and among its securities.

Minimizing Risk at a Given Expected Return

Let us assume that the array of covariances between security class returns over the n-year period may be represented by the matrix

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} \cdots & \sigma_{2m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \sigma_{m1} & \sigma_{m2} \cdots & \sigma_{mm} \end{bmatrix}$$

The matrix Σ is an mxm symmetric matrix with the diagonal elements $\sigma_{11}, \sigma_{22}, \ldots, \sigma_{mm}$ representing the variance of returns within a class. Since Σ is a covariance matrix, it is positive definite, and the quadratic form $\underline{X}' \Sigma \underline{X}$ is strictly convex.

Suppose we let the m x 1 vector $\underline{X}$ represent the proportions of our investment dollar allocated to each class. Then the quadratic form $\underline{X}' \Sigma \underline{X}$ becomes the variance of return of a portfolio defined by the allocation vector

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix}.$$

Now that we can measure the variance of return of a portfolio, it may be useful to examine the variance associated with various levels of expected return. By means of quadratic programming, the investor may then determine the point of minimum variance associated with each level of expected return E(P). The set of all minimum variance points forms the envelope of the convex set of all (E, σ^2) points. The portfolio of minimum variance which maximizes the investor's assigned utility would then define the optimal allocation of the investor's investment dollar.

Assume that class returns are defined as expressed earlier. The expected return of a portfolio over all possible classes is then

$$E(P) = \underline{X}'\hat{\mu} = \sum_{i=1}^{m} X_i \bar{R}_i$$

where X_i = the proportion of the investment dollar allocated to the $i^{\underline{th}}$ security class,

and $\bar{R}_i$ = the average class return over an n-year period of class i.

Since the convex set of all (E, σ^2) points (Figure 7) is dense, there are many allocation vectors associated with any given level of E(P). Since this allocation vector for E(P) is not unique, the investor must have considerations other than E(P) by which to select the optimum vector.

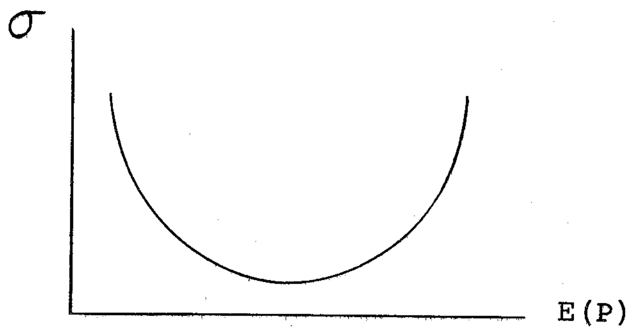

Figure 7

As can be seen from Figure 7, the point of minimum variance describes a unique point at each level of E(P). However, the allocation vector $\underline{X}$ which defines this point need not be unique.

Tolerance Interval Estimates. As was mentioned earlier, the investor does not seek a point estimate of the risk or return parameter. He seeks an interval estimate of these parameters so he may associate a probability to the chance that the interval contains the parameter under consideration.

Suppose at a certain level of return E(P) per dollar invested desired by an investor, the minimum variance associated with this portfolio is

$$\mathrm{var}_m\ (P) = \underline{X}_m' \textstyle\sum \underline{X}_m$$

where $\underline{X}_m$ denotes the allocation of his investment dollar to attain var_m (P). Therefore, the investor can be sure with a probability of $1 - \alpha$ that the true portfolio return described by allocation vector $\underline{X}_m$ is contained within the interval below. That is,

$$\Pr\left[E(P) - Z_{\alpha/2}\sqrt{\mathrm{var}_m\ (P)} < P < E(P) + Z_{\alpha/2}\sqrt{\mathrm{var}_m\ (P)}\right] = 1 - \alpha$$

where P = actual return of portfolio P described by $\underline{X}$,

α = significance level (probability of Type I error),

and $Z_{\alpha/2}$ = normal distribution tabulated value.

Quadratic Programming Solution Technique. Given an array of security class returns with covariance matrix Σ, the quadratic form $\underline{X}' \Sigma \underline{X}$ is a convex expression, and our problem is one of minimization with respect to linear constraints.

Therefore, with respect to the linear constraints

$$E(P) = \underline{X}'\hat{\underline{\mu}} = K_0 \text{ (a given value)}$$

$$\sum_{i=1}^{m} X_i = 1$$

$$\text{all } X_i \geq 0$$

we minimize the quadratic expression $\underline{X}' \Sigma \underline{X}$.

The computational method of the quadratic programming algorithm will not be discussed here. For assistance in this endeavor, the reader may refer to Dantzig (4) or Vajda (10).

Conclusion

Earlier in this section, the author listed Markowitz's three selection considerations. It was the intention of this section to show that, by using the quadratic programming method as discussed, the study of one of these considerations involves the study of the others.

The initial step in any portfolio selection program must be to associate some reasonable values to the expected return of the available securities. Perhaps the method adopted within this discussion to accomplish the aforementioned result is inadequate, but let it suffice until a better method appears.

From the array of security class returns, we develop a model which gives us, at a given level of expected return for the portfolio, the allocation among the classes describing the portfolio of minimum variance. Since this method gives the envelope of the convex set of (E, σ^2) points, the process of obtaining the efficient set of portfolios is accomplished. Furthermore, the point at the level of maximum allowable variance and its associated expected return defines the allocation vector describing the portfolio of maximum utility. Therefore, we might say that the quadratic programming method is ideal for satisfying Markowitz's selection criteria provided our initial estimates of future performances are accurate.

It has been said that next to the temperament of a woman nothing is less predictable than security price movements. However, this author does not intend to let the latter problem lie unresolved. He will attempt to discuss in future studies how security price movements seem to follow a Markovian or stochastic process. These notions will hopefully be presented and extended in his doctoral dissertation.

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APPENDIX

As was mentioned earlier, a better method of measuring the annual security class appreciation may be to use a regression estimate. Sharpe (9) suggests that this annual appreciation may be related to such factors as GNP and the market average as a whole.

Extending these notions, an appropriate regression model might be

$$\mu_{ki} = B_0 + B_1 X_1 + B_2 X_2 + B_3 X_3 + e_{ki}$$

where μ_{ki} = regression estimate of the return per dollar invested in security class K during year i,

X_1 = class return R as defined earlier,

X_2 = average GNP during year i,

X_3 = average market value for year i,

e_{Ki} = random error associated with the return from security class K in year i.

The regression coefficients B_0, B_1, B_2, and B_3 may be estimated by the method of least squares assuming $E(e_{Ki}) = 0$ and $E(e_{Ki}, e_{Kj}) = \sigma^2$, for all K, i combinations. If this is not a valid assumption, then a weighted regression model may be desired.

With either model, we may make probabilistic interval estimates about the outcome μ_{Ki}. That is,

$$Pr\left[\bar{Y}_{Ki} - Z_{\alpha/2}\sqrt{\frac{\sigma^2}{r}} < \mu_{Ki} < \bar{Y}_{Ki} + Z_{\alpha/2}\sqrt{\frac{\sigma^2}{r}}\right] = 1 - \alpha$$

at given values for X_1, X_2, X_3

where $\bar{y}_{Ki}$ = sample regression return value for security class K in year i,

$Z_{\alpha/2}$ = normal distribution tabulated value,

α = significance level (probability of Type I error).

r = number of observed returns sampled within security class K in year i.

Therefore, with a probability of $1 - \alpha$, we can be sure that the preceding interval contains the actual outcome μ_{Ki}.