Abstract Approved  
(Major Professor)

This thesis deals with the problem of balancing the connecting rod linkage. After discussing the purposes of analysis of engine dynamics, and defining the scope of the treatment, the relations governing the substitution of lumped masses for the connecting rod are set forth and verified. The advantages of this substitution are explained.

Results of harmonic analysis of the forces and couples arising from the steady-state motion of the linkage are presented in several forms. Probably the most useful of these are the charts which show the values of all harmonics of forces and couples up to and including the fourth. The logarithmic scales of the one chart assure reasonable accuracy in reading the values of the higher harmonics. In addition to the charts, the values of the harmonic amplitudes are also given in two mathematical forms, viz. the closed form, and the open, or series, form. The latter is, in most cases, given in both general and specific form. It is explained that the open and closed forms are each useful in particular ranges of values of the connecting rod-crank ratio.
Following these results, the values of the harmonic amplitudes of inertia force are then given in closed form for the master connecting rod linkage, and it is shown how the charts and mathematical expressions previously given for the simple connecting rod may be used in this case also. The master connecting rod linkage is becoming of increasing importance, particularly in the design of aeroplane motors.

Balancing methods are then discussed, and typical problems are solved which indicate the specific uses that may be made of the preceding theoretical results. These problems illustrate the methods that may be applied in the analysis of single and multiple cylinder engines using the simple connecting rod, and also illustrate methods that may be used for the master connecting rod linkages.

In the appendices appear a description of the methods of analysis employed, and a discussion of the properties of a connecting rod linkage of limiting proportions, in which the lengths of the crank arm and connecting rod are equal. In general, the analysis requires the use of the elliptic integrals, which are the characteristic functions of the analysis. However, a few of the coefficients are exceptions, and require integration in the complex plane by means of the theorem of residues.

It is thought that the most useful contributions to the art of balancing presented in this thesis, in roughly their order of importance, are:

1. the charts for the harmonic amplitudes,
2. the determination of the harmonic amplitudes for the master connecting rod linkage, and the relating of these to the harmonics of the simple connecting rod linkage,

3. the assembling of a complete treatment of the problem in one publication, and

4. the derivation of the closed forms of the coefficients in the Fourier series.
BALANCING THE CONNECTING ROD LINKAGE

by

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The equation (8), while giving the correct value for the inertia torque arising from the oscillation of \( I_o \), does not give the correct value for the reaction on the frame of the machine, the reason being that this torque is divided and acts partly on the frame and partly on the flywheel. The correct value for the reaction on the frame is given by

\[
\frac{\mathcal{T}_1}{I_o \omega^2} = \frac{k (1-k^2) \sin \Theta}{\Delta^3} - \frac{k^2 (1-k^2) \sin 2\Theta}{2 \Delta^4}.
\]

The last term of the expression is the correction over the value previously given, and represents the amount of the torque transmitted to (or from) the flywheel. This term introduces even harmonics in the Fourier series for \( \mathcal{T}_1 \), so that equation (12) is incomplete, and should read:

\[
\frac{\mathcal{T}_1}{I_o \omega^2} = b_1 \sin \Theta - b_2 \sin 2\Theta - b_3 \sin 3\Theta + b_4 \sin 4\Theta + \ldots,
\]

where \( b_1 \) and \( b_3 \) have the same values as previously given, and the values of the even coefficients are:

\[
b_2 = k^2 (1-C_2), \quad \text{and} \quad b_4 = k^2 C_4.
\]
The values of $C_2$ and $C_4$ appear on the charts and are given in equations (18) and (20).

In the analysis of the master connecting rod linkage, the equation (29) is also in error for the same reason. The value of $T_1$, given there is only the inertia torque of the link, and does not represent the torque reaction on the frame. The value of $T_1$, as given by equation (29), divides into two parts, one reacting on the frame and the other on the crank shaft. The value of the reaction on the frame is given by:

$$\frac{T_1}{I_0 \omega^2} = c(l-c^2) \frac{\sin(\Theta+\mu)}{\Delta_2^3}$$

$$\quad \quad - \frac{R c(l-c^2)}{b \Delta_2^2} \frac{\sin(\Theta+\mu)}{\cos \Theta + \frac{a}{L} \cos(\Theta-\alpha)},$$

where the second term gives that portion of the torque acting on the crank shaft, and is the correction over the value previously given in equation (29).

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August 12, 1939
BALANCING THE CONNECTING ROD LINKAGE

The simple connecting rod linkage is one of the oldest in use. Its inherent simplicity and low friction losses makes it, to-day, the most commonly used linkage for converting linear motion to circular, and vice versa. Unfortunately, this simplicity is not so apparent when one attempts to analyze the forces induced by its motion. In spite of all that has been written on the subject, the technique of balancing this important linkage is by no means commonly understood by designers.

The purpose of this thesis, therefore, is to present, in a single publication, the results of a comprehensive study of the problem, and to present these results in such a fashion that they may be used by others without duplication of the analysis.
THE PURPOSE OF ANALYZING ENGINE DYNAMICS

The designer may have one or more purposes in mind when he analyzes the dynamic forces arising from the motion of the connecting rod linkage. These purposes may be any of the following:

Balancing The Primary Forces

The purpose may be to balance the primary reactions of the inertia forces and couples against the frame of the engine. This is usually done by attaching counterweights to the crank shaft. The word 'primary' is used to denote the first harmonic of a force or couple. When a shaft is revolving at constant speed, it is evident that only primary forces or couples may be balanced with counterweights. So when the designer wishes to determine the sizes of counterweights, he must know the magnitudes of the primary reactions on the frame of the machine. These reactions are usually computed with the assumptions of a constant shaft speed and of a rigid linkage.

Selecting the Cylinder Arrangement

The designer may have a choice of several cylinder arrangements, any of which will give him uniformly distributed torque pulsations. His final choice may be dictated by such factors as cost, cooling of cylinder walls (as in radial aeroplane engines), space requirements, ease of assembly or repair, and use of common parts in a group of engines of different sizes.
If none of these decides the arrangement, the designer may wish to select that cylinder arrangement which gives the least vibration. A part of this problem is the determination of the magnitudes of the principal harmonics of the inertia reactions in each of the cylinder arrangements under consideration. By 'principal harmonics' is meant those harmonics which have an appreciable influence on the vibration. It will be evident later that, with the usual connecting rod-crank ratios, harmonics higher than the fourth are so small in magnitude as to have negligible effects on the vibration provided the linkage is rigid. Different cylinder arrangements will, however, produce different magnitudes of all the harmonics of the net reactions. Therefore, the intelligent selection of cylinder arrangement requires a knowledge of the magnitudes of the higher harmonics of the reactions.

In some engine designs, counterweighted shafts, running at multiples of the crank shaft speed, have been added to the engine in order to balance the higher harmonics of the reaction.

Study of Resonance

With the practical, non-rigid linkage, vibration is also introduced through resonance at certain speeds. Analysis of this condition requires at least a knowledge of the frequencies of the higher harmonics, and also requires a knowledge of the torque reactions on the crank shaft, as differentiated from the reactions on the frame. As will be shown later, these reactions are not necessarily identical.
The treatment of this problem in the design influences both the disposition of counterweight and the cylinder arrangement, the tendency being to distribute integral counterweights along the shaft, and to select V type cylinder arrangements which give a short, rigid shaft.

Isolation of the Residual Vibration

When the designer has balanced the inertia reactions of a given engine (or compressor) as well as the permissible cost will allow, he usually gives some consideration to isolation of the residual forces from the foundation, room, chassis, or other structure on which the engine is mounted. This usually requires nothing more than a knowledge of the lowest frequencies with which he must deal. In exceptional cases, however, the problem of isolation may influence the sizes of counterweights selected. This may occur in single cylinder machines where the primary forces, which are influenced by the counterweights, are left partially unbalanced. In such designs, it is possible to select counterweights of such a size that the energy transmitted through the flexible mounting to the foundation is a minimum.

Consideration of the objectives discussed above indicates the desirability of having data available on the magnitudes of the harmonics of inertia reactions.
THE SCOPE AND LIMITATIONS OF THIS TREATMENT

It will be evident that a complete treatment of the subject of balancing, with all it infers, would be very extensive. In fact, several complete books have been written on the subject. So it is desirable that the scope of this work be delineated before proceeding further. The following statements serve this purpose:

1. Two types of linkage are considered. One is the simple connecting-rod type, consisting of a crank shaft, connecting rod, piston, and stationary frame. The other is the master connecting rod type, illustrated in Figure 9, in which a master connecting rod has a number of connecting links pivotally attached to it. The master rod and each of the connecting links have pistons attached to them, the whole assembly giving a radial disposition of the cylinders.

2. The forces and couples considered are the reactions on the frame (or stationary link) of the forces that cause the accelerations. The torque pulsations on the crank shaft are not considered directly, since this problem has been fully treated in the papers of Frank M. Lewis and Frederick P. Porter. *

3. It is assumed that the crank shaft speed is constant. In practice, the fluctuations of speed are usually small enough so that their effect on the inertia forces or couples is

*See the references.
negligible.

4. Vibration caused by flexibility of the links and consequent resonance is not considered.

5. The treatment of the problem includes:
   a discussion of the lumped mass substitution for connecting rods or links,
   a presentation of the magnitudes of the harmonics in both analytical and chart form,
   a discussion of the methods of balancing these linkages,

appendices on analytical methods and on related problems.
THE LUMPED MASS SUBSTITUTION FOR THE CONNECTING ROD

The inertia forces and couples caused by acceleration of the connecting rod can, of course, be determined completely by considering the motion of and about its center of gravity. However, both the analysis and one's physical concepts are simplified by substituting for the connecting rod an equivalent dynamic system consisting of a lumped mass at each end, connected by a massless rod, and a massless moment of inertia. In making this substitution, the one mass is considered attached to the piston or crosshead, and the other to the crank pin, while the massless moment of inertia is considered to oscillate with the connecting rod, its location being immaterial.

The rules for proportioning these lumped masses are as follows:

1. The sum of the two masses must be equal to the total mass of the connecting rod.

2. Either mass is in direct proportion to the distance of the other mass from the center of gravity.

3. The massless moment of inertia is equal to the true moment of inertia of the connecting rod less the product of its mass by each of the distances from the center of gravity to the ends.

To express these rules by equations, see Fig. 1, and let

\[ M = \text{the total mass of the connecting rod,} \]
\[ M_1 \text{ and } M_2 = \text{the respective lumped masses to be substituted for } M, \]
\[ I = \text{the moment of inertia of the connecting rod about its center of gravity,} \]
\( I_o \) = the massless moment of inertia required in the substitution,
\( L \) = the distance between centers of the two bearings
(hereafter called the length of the connecting rod),
and \( L_1 \) and \( L_L \) = the respective distances from the center of gravity
to \( M_1 \) and \( M_2 \).

Then the relations are:
\[
M_1 = M \frac{L_2}{L} \quad (1)
\]
\[
M_2 = M \frac{L_1}{L} \quad (2)
\]
\[
I_o = I - M L_1 L_L \quad (3)
\]

These relations are easily proved by the principles of dynamics.
For the particular motion of a connecting rod, and assuming the conventional design, which is symmetrical about two planes, we know that the only dynamic constants which can influence the inertia forces and couples are

1. The total mass.
2. The location of the center of gravity.
3. The moment of inertia about the center of gravity.

Consequently, any arrangement of lumped masses and moment of inertia which, in the aggregate, duplicates these constants, will produce the same forces and couples when in motion in the same manner.
It will be found that the mass distribution given by equations 1, 2, and 3 duplicates the dynamic constants of the actual connecting rod.
Fig. 1. - Lumped Mass Substitution For Simple Connecting Rod.

Fig. 2. - Lumped Mass Substitution for A Master Connecting Rod.
It should be observed that the required value of \( I_0 \) may be negative. One's physical picture of the substitution need not be disturbed by this if it is realized that \( I_0 \) merely determines the magnitude and direction of a couple that would otherwise be disregarded in the two-mass substitution. Physically, a massless moment of inertia is approached by the rim of a flywheel whose radial thickness diminishes as the diameter is increased, the moment of inertia being fixed.

So far, we have discussed only the more conventional type of connecting rod, such as used with 'in-line' cylinder arrangements, and illustrated in Fig. 1. However, many designs are now using the master rod construction, with radial cylinder arrangements. In this type of construction, one of the pistons is linked to the crank shaft with the master rod, while the others are linked to the master rod. A simple master rod for two cylinders only is shown in Fig. 2, where one connecting link is attached at B.

The two-mass substitution may readily be used in this case by first substituting lumped masses for the links. The masses thus located on the master rod are then considered as part of it, and a two-mass substitution is made for this combination of the master rod with its equivalent link weights. In this manner a substitution of lumped masses may be made for any combination of connected links.

In designing a linkage of such a type, it is desirable so to distribute the mass of the master connecting rod that, after the equivalent link weights are added, the center of gravity of the combination
lies on the center line between the pin and crank bearings. If this can be done the treatment of the problem is no different than for an ordinary connecting rod. If, however, this is impractical, the two-mass substitution may still be made as shown in Fig. 2. In this case the mass \( M_1 \) is located at the wrist pin as before, but the mass \( M_2 \) is located on the line passing through the center of the wrist pin and the center of gravity, and at any convenient distance from \( M_1 \), but preferably at the intersection with the line through the center of the crank pin bearing at a right angle to the line of bearing centers. In this location, the motion of the mass \( M_2 \) and its contribution to the inertia forces are easily determined.
HARMONICS OF THE SIMPLE CONNECTING ROD LINKAGE

There are four basic forces or couples that arise from the motion of the simple connecting rod linkage at constant shaft speed. These are: the inertia force of the piston, the couple caused by the massless moment of inertia of the connecting rod, the couple caused by the piston inertia force, and the couple caused by the gravitational force of the piston.

The analytical expression will be given for each of these, followed by the corresponding Fourier series. Then expressions for the coefficients of the Fourier series will be given in both closed and open form.

The nomenclature necessary for this presentation is shown partly in Figure 3, and is as follows:

\( x \) = the distance from the center line of the crank shaft to the center line of the wrist pin, inches.
\( L \) = the length of the connecting rod, inches.
\( R \) = the length of the crank arm, inches.
\( \theta \) = the angle denoting the position of the crank arm, radians or degrees.
\( \beta \) = the angle defining the angular position of the connecting rod relative to the center line of the stationary link, radians or degrees.
\( \omega \) = the angular velocity of the crank shaft, radians per second.
\( \kappa = \frac{R}{L} \), the crank-connecting rod ratio.
\( \Delta = \cos \beta = \sqrt{1 - \kappa^2 \sin^2 \theta} \).
\[ K = \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{\Delta} = \text{the complete elliptic integral of the first kind.} \]

\[ E = \int_{0}^{\frac{\pi}{2}} \Delta d\theta = \text{the complete elliptic integral of the second kind.} \]

The force reactions are considered positive when acting in the direction of increasing \( x \), and couple reactions are positive when counterclockwise.

The Piston Inertia Force

Let the total mass at the wrist pin be \( M_p \). This is the total mass of all the reciprocating parts of one linkage, such as piston, piston rod, and cross-head, plus the equivalent mass \( M_1 \) of the wrist pin end of the connecting rod. The total mass \( M_p \) is considered to be concentrated at the center line of the wrist pin, and on the center line of the cylinder. It is assumed that the center of gravity of the reciprocating parts lies on the center line of the cylinder. If such is not the case, the fact must be considered in the process of balancing, although it makes no difference to the values of the harmonics.

The inertia force, \( F_p \), acting on the frame because of the acceleration of this mass is given by the equation

\[ \frac{F_p}{M_p R \omega^2} = -\cos \Theta + \frac{\mathcal{E}_1 \cos 2\Theta}{\Delta} + \frac{\mathcal{E}_2 \sin^2 2\Theta}{4 \Delta^3} \]  (4)
Fig. 3. - Schematic Diagram Of A Simple Connecting Rod Linkage.
This equation is, of course, derived by differentiation for the acceleration. By expanding the right side in a Fourier series, the following expression is obtained:

\[
\frac{F_p}{M_p R \omega^2} = -\cos \theta + a_2 \cos 2\theta - a_4 \cos 4\theta + \ldots
\]  \hspace{1cm} (5)

in which only the even harmonics appear after the first, and the signs alternate as shown. The magnitudes of the harmonics are, of course, the values of the coefficients \(a_2, a_4, \text{etc.}\). These values are given in the charts of Figures 4 and 5, together with the amplitudes of harmonics of the couples. In each chart, the amplitudes \(a_2, a_4, \text{etc.}\) are plotted against the ratio \(L/R = 1/k\). Figure 4 is a logarithmic plot covering the practical range of application, while Figure 5 is an arithmetic plot of the region near \(k = 1\), where the crank arm and connecting rod are nearly equal in length. It is thought that the charts will provide sufficiently accurate information for the majority of balancing problems. However, the analytical expressions for the harmonic coefficients (or amplitudes) will be given too, so that more accurate determination of the amplitudes may be made if desired. It will be noted that only the harmonics up to and including the fourth are given, because higher harmonics very rarely influence the vibration sufficiently to warrant their consideration unless resonance must be considered.

The analytical expressions for the harmonic coefficients of the piston inertia force, which appear in equation (5) are these:-
\[ a_2 = \frac{16}{3\pi^2 \kappa^3} \left[ (2 - \kappa^2) E - 2 (1 - \kappa^2) K \right] \]  

(6)

\[ = 8 \kappa \sum_{n=2}^{\infty} \frac{n}{(n-1)(n+2)} \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots (n)}^2 \kappa^{n-2} \]  

\[ (n = 2, 4, 6, \ldots) \]  

(6a)

\[ = \kappa \left( 1 + \frac{1}{4} \kappa^2 + \frac{15}{128} \kappa^4 + \frac{35}{512} \kappa^6 + \frac{795}{16, 384} \kappa^8 + \cdots \right) \]  

(6b)

\[ a_4 = \frac{64}{15 \pi^3 \kappa^5} \left[ (16 - 16 \kappa^2 + \kappa^4) E - 8 (2 - \kappa^2)(1 - \kappa^2) K \right] \]  

(7)

\[ = 32 \kappa^3 \sum_{n=8}^{\infty} \frac{(n-5)(n-6)}{n(n-2)(n-4)} \frac{1 \cdot 3 \cdots (n-7)}{2 \cdot 4 \cdots (n-6)}^2 \kappa^{n-8} \]  

\[ (n = 8, 10, 12, \ldots) \]  

(7a)

\[ = \kappa^3 \left( \frac{1}{4} + \frac{1}{16} \kappa^2 + \frac{35}{256} \kappa^4 + \frac{105}{1, 024} \kappa^6 + \frac{105, 395}{131, 072} \kappa^8 + \cdots \right) \]  

(7b)

Of these expressions, the closed forms were first derived, and the series forms were then derived by substituting the known series for the elliptic integrals. For computation, the last series form is the most convenient for all usual values of \( k \). The first series form is useful for extending the series when this is required for...
Fig. 4. - Amplitudes Of The Harmonics Of Forces And Couples Of The Connecting Rod Linkage.
Fig. 5. - Amplitudes of the harmonics in the region near L/R = 1.
accuracy. While the series in each case converge when $k^2 < 1$, they are not very practical for computation when $k^2$ is greater than $1/2$. It is somewhat easier in such a case to use the closed forms, taking the values of the elliptic integrals directly from Legendre's tables.* Conversely, the closed forms are not practical for computations when $k^2$ is much less than $1/2$ because the expressions involve small differences of large quantities. Since the usual connecting rod design uses a value of $k (= R/L)$ of $1/3$ or less, it is evident that the series forms will be used in most practical cases.

The Inertia Couple of the Connecting Rod

This is the couple arising from the oscillation of the massless moment of inertia $I_o$ which is used in the lumped mass substitution for the connecting rod. Denoting this couple by $\tau_i$, its value is given by the equation

$$\frac{\tau_i}{I_o \omega^2} = \frac{k \sin \theta}{\Lambda} - \frac{k^3 \cos \theta \sin^2 \theta}{2 \Lambda^3}$$

(8)

It should be noted that this couple does not appear in the same form at the crank shaft. Consequently, this is not the correct expression to use if one is studying the pulsations of crank shaft torque. This fact is most clearly realized by observing the energy

relations involved. The kinetic energy resident in the oscillating moment of inertia $I_o$ is

$$T = \frac{1}{2} I_o \left(\frac{d\beta}{dt}\right)^2.$$  \hfill(9)

For small values of $k$, $d\beta/dt$ is essentially the simple harmonic $\omega \cdot k \cdot \cos \theta$; so for such a case,

$$T = \frac{1}{4} I_o \omega^2 k^2 (1 + \cos 2\theta),$$ \hfill(10)

or essentially a second harmonic function. Now this kinetic energy must be imparted by the crank shaft at the constant speed $\omega$. Consequently, the torque on the crank shaft is

$$T = \frac{dT}{d\theta} = -\frac{1}{2} I_o \omega^2 k^2 \sin 2\theta,$$ \hfill(11)

and is essentially a second harmonic. Similarly, the first harmonic piston inertia force produces a second harmonic pulsation on the crank shaft, but in this case the torque reaction on the frame is also essentially a second harmonic. This illustration serves to show why one must be careful not to assume that couples appearing at the crank shaft necessarily have opposite reactions on frame.

Returning now to equation 8, the Fourier series for this expression takes the form:

$$\frac{T}{I_o \omega^2} = b_1 \sin \theta - b_3 \sin 3\theta + b_5 \sin 5\theta - \ldots$$ \hfill(12)
The coefficients $b_1$ and $b_3$ are plotted on the charts of Figures 4 and 5. The analytical expressions for them are:

$$b_1 = \frac{4}{7-k} \left[ E - (1-k^2)K \right]$$

$$= k + 2k \sum_{n=4}^{\infty} \frac{1}{n} \left( \frac{1}{2} \ldots \frac{1}{n-2} \right) k^{n-2}$$

$$= k \left( 1 + \frac{1}{8} k^2 + \frac{3}{64} k^4 + \frac{25}{1,024} k^6 + \frac{245}{16,384} k^8 + \cdots \right)$$

$$b_3 = \frac{4}{7-k} \left[ (8-7k^2)E - (8-3k^2)(1-k^2)K \right]$$

$$= 18k^3 \sum_{n=6}^{\infty} \frac{1}{n(n-2)} \left( \frac{1}{2} \ldots \frac{1}{n-4} \right) k^{n-6}$$

$$= 18k^3 \left( \frac{1}{48} + \frac{3}{256} k^2 + \frac{15}{2,048} k^4 + \frac{245}{49,152} k^6 + \frac{945}{262,144} k^8 + \cdots \right)$$

Regarding the use of these expressions, the same remarks apply as were previously made for the coefficients $a_2$ and $a_4$.

Except for unusual designs, it will be found that the couple $T_1$ is relatively small in magnitude. Often it is neglected altogether.

The Couple Caused by the Piston Inertia Force

It will be evident from a study of the linkage that the piston inertia force cannot react directly on the frame, but must be
transmitted through the crank shaft, and thus give rise to a couple which exerts itself between the flywheel and the frame. The magnitude of this couple is proportional to the total reciprocating mass, \( M_p \). Denoting the couple by \( T_z \), its magnitude is given by the equation:

\[
\frac{T_z}{M_p R^2 \omega} = \sin \Theta - \frac{d}{\Delta} \sin \Theta \cos^2 \Theta - \frac{d}{\Delta^3} \sin \Theta (\cos 2\Theta + \frac{R}{\Delta} \sin 4\Theta) - \frac{1 - \frac{R^2}{\Delta^4}}{\Delta^2} \sin \Theta. \tag{15}
\]

The Fourier series corresponding to this expression is:

\[
\frac{T_z}{M_p R^2 \omega} = c_1 \sin \Theta + c_2 \sin 2\Theta - c_3 \sin 3\Theta + c_4 \sin 4\Theta + \ldots, \tag{16}
\]

and the values of the coefficients are:

\[
c_1 = \frac{1}{4} a_2 \quad \text{(see equation 6)} \tag{17}
\]

\[
c_2 = 1 - \frac{4 \sqrt{1 - \frac{R^2}{\Delta^2}}}{\frac{R^2}{\Delta^2}} \left( 1 - \frac{1}{2} \frac{R^2}{\Delta^2} - \sqrt{1 - \frac{R^2}{\Delta^2}} \right) \tag{18}
\]

\[
= \frac{1}{2} + \frac{1}{32} \frac{R^4}{\Delta^2} + \frac{1}{32} \frac{R^6}{\Delta^2} + \frac{7}{256} \frac{R^8}{\Delta^4} + \ldots \tag{18a}
\]

\[
c_3 = \frac{4}{5 \pi \Delta} \left[ (32 - 22 \frac{R^2}{\Delta^2} - 3 \frac{R^4}{\Delta^2}) E - 2 (16 - 3 \frac{R^2}{\Delta^2}) (1 - \frac{R^2}{\Delta^2}) K \right] \tag{19}
\]

\[
= \frac{3}{4} \frac{R}{\Delta} + 18 \frac{R}{\Delta} \sum_{k=8}^{\infty} \frac{n-5}{n(n-2)} \left( \frac{1 \cdot 3 \cdot \ldots \cdot (n-7)}{2 \cdot 4 \cdot \ldots \cdot (n-6)} \right)^2 \frac{n-6}{\Delta^6} \tag{19a}
\]

\( (n = 8, 10, 12, \ldots) \)
\[ c_4 = \frac{4\sqrt{1-k^2}}{R^2} \left[ 8(1-k^2) - 4(2-k^2)\sqrt{1-k^2} + k^4 \right] \]  
\[ = \frac{1}{4} R^2 \left( 1 + \frac{1}{2} k^2 + \frac{1}{4} k^4 + \frac{1}{16} k^6 + \frac{23}{256} k^8 + \cdots \right) \]  

The values of these coefficients are, of course, plotted on the charts with the exception of \( c_1 \), which is expressed in terms of \( a_2 \), the amplitude of the second harmonic of the piston inertia force. It will be observed that only the odd coefficients in this series involve the elliptic integrals, the even coefficients involving only irrational algebraic functions. Since these algebraic functions are readily expanded by use of the binomial theorem, it was thought unnecessary to give the expansions in general terms as was done for those expressions containing elliptic integrals.

**The Couple Caused by the Gravitational Force of the Piston**

If the piston of an engine acts in a vertical line, the crank shaft exerts a torque in overcoming the gravitational force, and this torque reacts on the frame. If, on the other hand, the piston acts horizontally, the variation of its distance from the crank shaft center line causes a fluctuation in the torque it exerts. When, as
often is the case, the piston acts on some line inclined to the vertical, as in the V type of cylinder arrangement, both the horizontal and vertical components of the piston weight will exert their respective torques. For this problem, let

\[ W = \text{the piston weight (including part of the connecting rod),} \]
\[ \text{corresponding to } M_p. \ (W = M_p g) \]
\[ \sigma = \text{the angle of inclination of the cylinder bore with the vertical, positive when measured counterclockwise.} \]
\[ d = \text{the distance from the wrist pin to the center of gravity of the piston weight, } W, \text{ measured along the center line of the cylinder, and away from the center line of the crank shaft.} \]
\[ \tau_3 = \text{the gravitational couple acting on the frame.} \]

The value of \( \tau_3 \) is given by the equation:

\[
\frac{\tau_3}{WR} = \cos \sigma \left( \sin \Theta - \frac{k}{2 \Delta} \sin 2\Theta \right) + \sin \sigma \left( \frac{d}{R} + \frac{A}{R} - \cos \Theta \right) \quad (21)
\]

The Fourier series corresponding to this expression is:

\[
\frac{\tau_3}{WR} = \cos \sigma \left( \sin \Theta - \frac{1}{2} a_2 \sin 2\Theta + \frac{1}{4} a_4 \sin 4\Theta - \ldots \right) + \sin \sigma \left( -\cos \Theta + \frac{1}{2} a_2 \cos 2\Theta - \frac{1}{4} a_4 \cos 4\Theta + \ldots \right), \quad (22)
\]
in which the constant term from equation (21) is dropped, since it has no influence on the vibration. In equation (22), the coefficients \( a_2, a_4, \) etc. are the same as those given in equation (5) for the piston inertia force.
Ordinarily, the gravitational torque will be negligible compared to the inertia torques, and the torques resulting from gas pressure in the cylinders, but they may be an appreciable factor in heavy, slow speed engines.

These expressions give all the forces acting on the frame because of the acceleration of the links of a single linkage of the simple type. The Fourier series of each force shows, through its coefficients, the magnitudes of the various harmonics of these forces and couples, and the charts of Figures 4 and 5 make these easily available for use. In using these data for multi-cylinder engines it is only necessary to add the forces or couples in their proper phase relations of time and space. This will later be illustrated by example.
HARMONICS OF THE MASTER CONNECTING ROD LINKAGE

The master connecting rod linkage was mentioned previously. It is illustrated in Figure 9. Its most common use is in the complete radial engine, as exemplified in the construction of many aeroplane motors, but it is also used at times in the V-type of cylinder arrangement, where two pistons are linked to one crank pin. It lends itself to a radial disposition of cylinders in one or more planes. This cylinder disposition may be required for air cooling of the cylinders, or may be merely a matter of obtaining a short, rigid crank shaft, or perhaps of meeting space requirements. In addition to the advantages it offers in cylinder arrangement, this linkage usually has a lower friction loss than that of the simple connecting rod type, because of the elimination of a number of bearings. In Figure 6 is shown a schematic diagram of the master rod linkage, the master rod being shown in the vertical position, with one auxiliary cylinder inclined at an angle \( \alpha \) to it.

When designing a linkage of this type, it usually is desirable to make the extreme distance of each wrist pin from the shaft center identical, so that the cylinders will all be at the same distance from center. With several auxiliary cylinders, it is also desirable to have the connecting links (denoted as 'b' in the sketch) all the same length. This will, in general, lead to a different distance 'a' for the point of attachment of each link. The correct value of 'a' may be determined either graphically, or mathematically by applying the
condition that when $\frac{dx}{d\theta} = 0$ the maximum value of $x$ must equal the sum $R + L$. It will be observed that the distance 'a' may (within practical limits) be selected arbitrarily provided 'a' and 'b' together satisfy the condition just mentioned. Moreover, the angle $\gamma$ subtended in the master rod by the arms $a$ and $L$ is not necessarily equal to $\alpha$, the angle between the corresponding cylinders.

Thus it is evident that several variables appear, making it impossible to present the data for all possible combinations in simple chart form.

However, a simplification of the analysis occurs in the case where $\gamma = \alpha$. Since these angles can usually be made equal, the results presented here are restricted to this case. In general, these results will be sufficiently accurate even when $\gamma$ and $\alpha$ are not equal, since they will usually be within a few degrees of each other.

In the expressions which follow, most of the symbols refer to distances or angles of the linkage as shown in Fig. 6. The forces, couples, masses, etc., all have the same notation as used previously, but it should be understood that these now refer to the auxiliary piston and the link connecting it to the master rod. Other notation found necessary is the following:
Fig. 6. - Schematic Diagram Of A Master Connecting Rod Linkage.
\[ R = \frac{R}{L} \quad \text{(as before)} \]

\[ \Delta_1 = \sqrt{1 - \frac{R^2 \sin^2 (\theta - \alpha)}{L^2}} \quad (23) \]

\[ c = \frac{R}{\Delta} \sqrt{(1 - \frac{a}{L} \cos \alpha)^2 + \left( \frac{a}{L} \sin \alpha \right)^2} \quad (24) \]

\[ \mu = \tan^{-1} \left( \frac{\frac{a}{L} \sin \alpha}{1 - \frac{a}{L} \cos \alpha} \right) \quad (25) \]

\[ \Delta_2 = \sqrt{1 - \frac{c^2 \sin^2 (\theta + \mu)}{L^2}} \quad (26) \]

Now, the inertia force of the auxiliary piston is given by the equation

\[
\frac{F_p}{M_p R \omega^2} = -\cos \theta + \frac{a}{L} \left\{ \frac{R \cos 2(\theta - \alpha)}{\Delta_1} + \frac{R^3 \sin^2 2(\theta - \alpha)}{4 \Delta_1^3} \right\} \\
+ \frac{bc}{R} \left\{ \frac{c \cos 2(\theta + \mu)}{\Delta_2} + \frac{c^3 \sin^2 2(\theta + \mu)}{4 \Delta_2^3} \right\}, \quad (27)
\]

and this expression corresponds to equation 4, which expresses the corresponding piston inertia force for the simple connecting rod linkage. Comparison of equations 27 and 4 shows that the expressions in braces in equation 27 are both similar to the last two terms of equation 4. Consequently, the right side of equation 27 may be expanded in a Fourier series by analogy. By relegating the phase angles \( \alpha \) and \( \mu \) to the coefficients, one obtains for this expansion,
\[
\frac{F_p}{M_p R \omega^2} = -\cos \theta + d_2 \cos 2\theta - d_4 \cos 4\theta + \ldots \\
+ p_2 \sin 2\theta - p_4 \sin 4\theta + \ldots 
\] (28)

where the coefficients are:

\[
d_2 = \left( \frac{a}{L} \cos 2\alpha \right) a_2 + \left( \frac{b c}{R} \cos 2\mu \right) a'_2 
\] (28a)

\[
d_4 = \left( \frac{a}{L} \cos 4\alpha \right) a_4 + \left( \frac{b c}{R} \cos 4\mu \right) a'_4 
\] (28b)

\[
p_2 = \left( \frac{a}{L} \sin 2\alpha \right) a_2 - \left( \frac{b c}{R} \sin 2\mu \right) a'_2 
\] (28c)

\[
p_4 = \left( \frac{a}{L} \sin 4\alpha \right) a_4 - \left( \frac{b c}{R} \sin 4\mu \right) a'_4 
\] (28d)

Succeeding terms may be written down by following the sequence.

In these expressions for the coefficients, \(a_2\), \(a_4\), etc., have the same values as previously given in equations 6 and 7, or in the charts of Figures 4 and 5. They are determined by the value of \(k\). The values of \(a'_2\), \(a'_4\), etc., are also determined from the charts or from the equations 6 and 7, except that the value of 'c', (equation 24) is substituted for \(k\). It will be observed that when the value of 'a' (see Figure 6) becomes zero, the Fourier series in equation 28 reduces to an identity with that of equation 5 for the simple linkage, as it should.

It is evident, then, that the harmonics of the piston inertia forces are easily determined for the case where \(\delta = \alpha\). One of the important effects of the master rod linkage, it will be noted, is the
introduction of sine terms in the Fourier series. Because of this, and the fact that the magnitudes of the harmonics vary among the different cylinders (because of the variation of $\alpha$), the cancellation of the harmonics is not so easily accomplished as with the simple connecting rod linkage.

Harmonic analysis of the couples $T_1$, $T_2$, and $T_3$ of an auxiliary linkage of the master rod linkage is not so easily made. The equations for these three couples are:

$$\frac{T_1}{I_o \omega^2} = \frac{c}{\Delta_2} \sin(\Theta + \mu) - \frac{c^3}{2 \Delta_2^3} \cos(\Theta + \mu) \sin \lambda (\Theta + \mu) .$$  \hspace{1cm} (29)

$$\frac{T_2}{M_p R \omega^2} = \frac{c}{\Delta_2} \sin(\Theta + \mu) \cdot \left( \cos \Theta - \frac{a}{KL} \Delta_1 + \frac{b}{R} \Delta_2 \right) \frac{F_p}{M_p R \omega^2} .$$  \hspace{1cm} (30)

$$\frac{T_3}{W R} = c \cos \sigma \cdot \sin(\Theta + \mu) \cdot \left( \frac{b}{R} + \frac{a}{R} \Delta_1 - \frac{\cos \sigma}{\Delta_2} \right)$$

$$+ \sin \sigma \left( \frac{d}{R} + \frac{b}{R} \Delta_2 + \frac{a}{R} \Delta_1 - \cos \sigma \right) .$$  \hspace{1cm} (31)

In equation 31, the distance between the wrist pin and the center of gravity of the piston is denoted by '$d$', as before.

Of these three equations, the first, giving the inertia torque of the $I_o$ of the connecting link is easily written in the Fourier series form by analogy with equation 8. The expansions of equations 30 and 31, however, can only be obtained with great difficulty, because of the products or quotients of the radicals $\Delta_1$ and $\Delta_2$. 
Since it is usually necessary to plot the torque arising from gas pressure, it seems advisable to add these functions to the gas torque, and plot the sum. The Fourier coefficients may then be obtained for the composite torque reaction either by numerical or graphical integration. This is often done for the gas torque alone; so the addition of the inertia torques does not appreciably complicate the procedure, not detract from the accuracy of the completed solution. The determination of the magnitudes of the harmonics of the inertia forces, of course, can be obtained as accurately as desired from the equation 28.

While the mathematical expressions for the coefficients of the Fourier series corresponding to equations 30 and 31 are difficult to obtain, the difficulty is caused mainly by the tediousness of the process and the cumbersome nature of the expression obtained, and not by any inherent mathematical difficulty. The most obvious method of obtaining the expressions for the coefficients is that of expanding the factors of each expression, multiplying the resulting series termwise, and then collecting terms so as to obtain the coefficients of each of the respective harmonics. However, it appears that the amount of work required is scarcely justified, and that if obtained, the expressions for the coefficients would be so cumbersome as to make a graphical analysis more attractive.
BALANCING METHODS

The Single Cylinder Machine

With single cylinder engines, it is common practice to balance the crankshaft dynamically, considering the equivalent mass $M_2$ of the connecting rod attached to the crank pin, and then to add to the counterweights sufficient mass to balance, in addition, one-half of the first harmonic of the piston inertia force. Whatever the method, if one is restricted to the use of counterweights on the shaft it is obvious that some compromise is necessary.

However, the compromise just mentioned is by no means the only one available. By permitting the addition of odd harmonic couples, it is possible to balance the inertia forces completely, and to make the couple $T_2$ of the piston inertia force vanish. Referring to Figure 1, it is apparent that by adding sufficient weight to the connecting rod cap, the mass $M_1$ can be made negative, and of equal magnitude to the mass of the piston, so that, in effect, the reciprocating mass is made to vanish, and only the mass $M_2$ on the crank pin remains to be balanced. Since this is traveling in a circular path with the crank pin, it is readily balanced with rotating counterweights. In doing this, the inertia couple $T_2$, whose principal component is a second harmonic, is also made to vanish since its magnitude is proportional to $M_2$. However, the couple $T_1$, whose principal component is a first harmonic, is increased to a considerable magnitude.
The principal effect of this method, then, is to eliminate the primary inertia force and the principle second harmonic inertia couple and to substitute for these a first harmonic inertia torque. The effect of this inertia torque cannot be properly evaluated without adding it vectorially to the gas torque, since it may be partially cancelled, but in general a couple is much less objectionable than a force, so that this method of balancing usually produces satisfying results. However, it is limited in application to small machines, because of space limitations, and it requires fairly close accuracy in the dimensions of the counterweights in order to maintain accuracy of balance. This is so because of the relatively large masses involved.

In some cases it may be desirable to eliminate the first harmonic couple, perhaps for the purpose of avoiding excitation of a resonant vibration. Ordinarily this cannot be done without adding additional links to the mechanism. In exceptional cases, however, the elimination of the first harmonic may be feasible. From the equations 3, 12, 16 and 22, the value of the first harmonic is

\[ b_1(I - ML, l_1)\omega^2 + c_1M_p R^2\omega^2 + M_p g R \cos\sigma + f_1 \] \[ + (g_1 - M_p g R \sin\sigma) \cos\Theta \], \hspace{1cm} (32)

where \( f_1 \) and \( g_1 \) are the coefficients of \( \sin\Theta \) and \( \cos\Theta \), respectively, in the Fourier series for the torque exerted by gas pressure in the cylinder. Obviously, the possibility of reducing the first harmonic
to zero depends upon the signs and magnitudes of \( f_1 \) and \( g_1 \). Some effect may be secured by varying the magnitudes of \( I \) and \( M \), which depend on the shape and size of the connecting rod, but ordinarily the range of possible effect is small.

Small single-cylinder compressors are often spring mounted, and in such cases the prime objective in balancing may be to procure the minimum transmission of vibrational energy through the spring mounting. This requires a study of the relative transmission of couples and forces, taking into account the characteristics of the spring mounting. This subject has been discussed by the author in a previous paper.*

Summarizing the case of the single-cylinder engine, there are four general methods that may be used. These are:

1. By using an overhung connecting rod, cause the reciprocating force and its couple to vanish. This is done at the expense of introducing a primarily first harmonic couple.

2. The connecting rod may be made of conventional proportions, in which case the rotating masses will be dynamically balanced, and the reciprocating mass may be partially balanced by counterweights, leaving unbalanced forces acting both in line with, and at a right angle to, the

line of piston action. The proportion of piston weight matched by counterweight will depend on the particular results desired.

3. In spring mounted machines, the method of balancing will depend in part on the isolating characteristics of the spring mounting.

4. In unusual cases, the balancing method may have for its object the elimination of a particular harmonic of torque.

The Multiple-Cylinder Engine With Simple Connecting Rods

Because of the great number of possible cylinder arrangements, it is impractical to attempt a compilation of data on the unbalanced forces and couples of multiple-cylinder engines. However, the problem of determining the magnitudes of these forces is a relatively simple one when the data for the single-cylinder linkage are available, and the same general method is applicable to any machine.

Machines of this type present one with these two balancing problems:

1. The moments of the counterweights required for balancing the first harmonic forces and couples must be determined.

2. The magnitudes of the residual harmonics of forces and couples must usually be ascertained. Sometimes the magnitudes of the residual harmonics influence the choice of cylinder arrangement, and occasionally the designer
provides a special linkage for the balancing of certain harmonics.

The object here is to show, by example, how these problems are solved by using the data previously given for a single connecting rod linkage. The cylinder arrangement chosen for this example is the four cylinder, 90°-V type. The reason for this choice is that this type combines the essential problems of both the 'in-line' and radial cylinder arrangements. But note that the results obtained are not the same as would be obtained with the master connecting rod linkage, since the harmonics of these two linkages differ in both magnitudes and phase relations.

Referring to Fig. 7, it will be seen that

$c =$ the offset between pistons that are linked to the same crank pin. This is usually equal to the width of the connecting rod bearing.

$f =$ the distance between a pair of cylinders in one bank.

$h =$ the distance between the effective center lines of the counterweights.

$\phi =$ the angle of position of the front crank.

$\chi =$ the gravitational moment of one counterweight, $= M \bar{x}$ in the expression $F_c = M \bar{x} \omega^2$, where $F_c$ is the centrifugal force.

$R$ and $L$ are the respective lengths of crank arm and connecting rod, as before.
To begin, assume that the equivalent masses for the connecting rods have been determined. It is then an easy matter to compute the counterweight mass needed to balance the couple caused by the masses \( M_z \) on the crank pins. Assume this to be done, and let us consider only the forces and couples arising from the reciprocating masses and from the \( I_x \)'s of the connecting rods. It will be understood that the counterweight moments hereafter referred to are only those necessary for balancing these forces or couples.

Note that the frame of reference for the machine consists of three planes, mutually at right angles, denoted by \( XZ, YZ, \) and \( XY \), and that the center line of cylinder No. 1 is in the \( XY \) plane. The fact that the center lines of two banks of cylinders fall in the \( XZ \) and \( YZ \) planes should be considered incidental to the particular cylinder arrangement. Three radial banks, equally spaced, would not give this coincidence. Also note that the symbol \( \phi \) is chosen to represent the crank position rather than \( \Theta \), so that all forces or couples may be expressed as functions of this one angle without confusion. The symbol \( \Theta \) will still be used to denote the crank position from lower dead center for each individual cylinder.

The general method of analysis consists of determining the net forces or couples acting in each plane or about each axis. The positive directions of couples and forces are indicated in Fig. 7 by arrows.
Fig. 7. - Schematic Diagram Of The Linkage Of A Four-Cylinder 90°-V Type Engine.

Fig. 8. - Lumped Mass Substitution Applied To A Four-Cylinder Master Connecting Rod Linkage.
Consider now the summation of forces in the XZ plane. Because of the symmetry of the design, it will be apparent that the counterweights need produce only a couple. Consequently there are no forces from this source, and only the inertia forces of the piston masses need be considered. The Fourier series for these forces is given in equation 5, in which, for cylinder No. 1,

\[ \theta = \phi \]

and for cylinder No. 3,

\[ \theta = \phi + \pi. \]

Making these substitutions in equation 5, and using subscripts to relate the forces to their respective cylinders,

\[
\frac{F_{p1}}{M_p R \omega^2} = -\cos \phi + a_{2} \cos 2\phi - a_{4} \cos 4\phi + \cdots .
\]

(33)

\[
\frac{F_{p2}}{M_p R \omega^2} = + \cos \phi + a_{2} \cos 2\phi - a_{4} \cos 4\phi + \cdots .
\]

(34)

By adding these two, the net force in the XZ plane is obtained. Thus,

\[
\frac{F_{p1} + F_{p3}}{M_p R \omega^2} = 2a_{2} \cos 2\phi - 2a_{4} \cos 4\phi + \cdots ,
\]

(35)

and it is seen that the first harmonics cancel, while all higher harmonics add. Similarly, it is found that the net force in the YZ plane is given by

\[
\frac{F_{p2} + F_{p4}}{M_p R \omega^2} = -2a_{2} \cos 2\phi - 2a_{4} \cos 4\phi - \cdots .
\]

(36)
Next consider the moments about the OY and OX axes caused by the piston inertia forces and by the counterweights. About the OY axis, the couple caused by the piston inertia forces is

\[-F_{p3}f = -M_p R \omega^2 f (\cos \phi + a_2 \cos 2\phi - a_4 \cos 4\phi + \cdots).\] (37)

That caused by the component of the counterweight couple about OY is

\[M \omega^2 h \cos \phi.\] (38)

It is now evident that by choosing a value of \(M\) such that

\[M h = M_p R f,\] (39)

the first harmonic is cancelled. This gives the solution to the first part of the problem. The remaining couple about OY is:

\[-M_p R \omega^2 f (a_2 \cos 2\phi - a_4 \cos 4\phi + \cdots).\] (40)

Similarly, with counterweights so chosen, one finds the residual couple about OX to be

\[-M_p R \omega^2 (f + 2e) (a_2 \cos 2\phi + a_4 \cos 4\phi + \cdots).\] (41)

It now becomes evident that by shifting the XY plane of reference positively along the Z axis a distance \(\frac{1}{2} (f + e)\), which usually will bring it near the center of gravity of the machine, these two residual couples reduce to:

\[M_p R \omega^2 e (a_2 \cos 2\phi - a_4 \cos 4\phi + \cdots)\] about OY. (42)
and \(- M_p R \omega e (a_2 \cos 2\phi + a_4 \cos 4\phi + \ldots)\) about CX. \(\ldots\) (43)

There are now only the couples about the Z axis to consider. The summations for these are obtained in exactly similar fashion, by substituting the proper value for \(\phi\) in terms of \(\phi\) for each cylinder in equations 12, 16 and 22. From the summations, one finds the net value of \(T_1\), the torque caused by the \(I_o\)'s of the connecting rods, is zero. The net value of \(T_2\), the torque caused by the piston inertia forces, is:

\[
T_2 = M_p R^2 \omega^2 \left(4 c_4 \sin 4\phi + \ldots\right)
\]  

(44)

and that caused by the effect of gravity on the piston, is

\[
T_3 = WR \left(a_4 \cos \frac{\pi}{4} \sin 4\phi + \ldots - a_2 \sin \frac{\pi}{4} \cos 2\phi + \ldots\right)
\]  

(45)

Therefore, the sum of all inertia and weight torques about the Z axis is:

\[
T_1 + T_2 + T_3 = -WR a_2 \sin \frac{\pi}{4} \cos 2\phi
\]

\[+ (4M_p R^2 \omega^2 c_4 + WR a_4 \cos \frac{\pi}{4}) \sin 4\phi + \ldots\]  

(46)

With the ordinary proportions it will be found this torque is fairly small compared to the corresponding harmonics of the torque arising from gas pressure in the cylinders, although it may still be in significant proportion.

Summarizing the results of this problem, one sees that the most significant of the residual forces and couples are the forces given by equations 35 and 36, which are of appreciable magnitude, and in which all even harmonics appear. The couples caused by these forces
(equations 42 and 43) are small, provided the displacement between banks is small, as it usually is, and that the center of gravity is not too far from the assumed position. In any actual problem, of course, the axes should be chosen with the origin at the center of gravity of the machine.

The couples about the \( Z \) axis, given by equation 46, are usually insignificant, but in any case their effect cannot be properly evaluated without a knowledge of the couples caused by gas pressure in the cylinders.

The residual forces, it will be noted, can only be reduced by

1. reducing the crank-connecting rod ratio, \( k \), or

2. reducing the mass \( M_p \). This may be done by using a light alloy for the pistons, by overhanging the connecting rod, or by doing both things, or

3. by using a special linkage of some sort.

Of these, the most common practice is the use of light alloys for the pistons.

It is obvious that the method of determining the magnitudes of the residual harmonics of forces and couples is simple when the data for a single linkage are known, and may readily be applied to any cylinder arrangement.

The Multiple-Cylinder Machine With Master Rod Construction

The purpose of this example is to illustrate the use of the lumped mass substitution. No harmonic analysis will be made, because the
method is just the same as that explained in the previous example, except that the proper harmonic coefficients, as given in equation 28, must be used in place of those for the simple rod for all the auxiliary pistons.

In Fig. 8 is shown a radial construction with four cylinders, each 45° apart. This construction is chosen deliberately for this example to illustrate the fact that the master rod construction is not restricted to an equal spacing of cylinders around the crank shaft. Of course, this construction will not give equal intervals between the torque pulsations, but it has the advantage of retaining a crankcase under the cylinders. In the figure, the blacked-in circles indicate equivalent lumped masses. A small white spot in the center of a 'mass' indicates a pivoting point, and the heavy lines indicate parts of the linkage. The arc joining the three arms 'a' with the master rod (cylinder No. 3) shows that these are rigidly connected together.

Balancing of this linkage is accomplished by means of a simple counterweight on the shaft, which is capable of balancing all first harmonic forces. It is obvious, having selected the cylinder arrangement, that nothing can be done with such a counterweight towards balancing either the couples or the higher harmonics of the forces. The balancing procedure may be carried out in the following logical order:

a. Determine the equivalent masses \( M_1 \) and \( M_2 \) of the links as described previously. Neglect the value of \( I_0 \), since nothing can be done to balance its couple.
To $M_1$ add the mass of the piston, and call this sum $M_p$. It is located at the center of the wrist pin. Remember that the location of the center of gravity of the piston is immaterial so long as it lies on the center line of the cylinder bore. Consider $M_2$ as a part of the master rod.

b. Each of the three connecting links should be identical in weight. After completing (a) there will be three masses $M_2$ attached to the master rod as shown in Fig. 8. The next step, then, is to counterbalance the master rod so that its equivalent lumped mass at the wrist pin (of cylinder No. 3 in this case) is exactly the same as that of each connecting link. This will result in making each of the four masses $M_p$ at the wrist pins identical. The counterweight is preferably an integral part of the master connecting rod (or of the bearing cap, if one is used) and should be so placed that the center of gravity of the master rod (always including the masses $M_2$) lies on its own line of bearing centers through crank pin and wrist pin. However, this is not absolutely essential, so long as the value of $M_p$ is correct. If the center of gravity is to one side of the center line, it will result in introducing unbalanced couples and forces that usually are comparatively small. In Fig. 8, the mass $M_3$ represents the counterweight added to the master rod so as to obtain the proper value of $M_p$ for cylinder No. 3.
c. Next determine the equivalent lumped mass of the master rod at the crank pin. This is simply the total mass of the master rod (including the three masses $M_2$ and the counterweight $M_4$) less the mass $M_p$ which is concentrated on the wrist pin. This mass at the crank pin is shown as $M_c$.

d. The last step is the determination of the counterweight moment, $M_c r$, required. It will be noted that pistons No. 1 and No. 3 act at a right angle to one another, as do No. 2 and No. 4. It is well-known that when two pistons act 90° apart in phase with simple harmonic motion, their combined inertia forces are the same as though the mass of one piston were concentrated on the crank pin. Equations 5 and 28 show that the first harmonics of the piston inertia forces have the same magnitude for each piston and have a phase relation corresponding to the cylinder spacing. Consequently, the rule for 90° spacing holds in this case, and since there are two such groups, we must consider the mass concentrated on the crank pin as $2M_p + M_4$. In order that this mass shall be balanced by the counterweight, we must have

$$M_c r = (2M_p + M_4) R,$$

and this determines the size of the counterweight.
This procedure could have been modified, if desired, by dispensing with the counterweight on the master rod and adding weight to piston No. 1 so as to match the mass \( M_p \) on No. 3. This method generally is objectionable because it requires one odd piston.

In Fig. 9 is a picture of an actual four-cylinder radial type linkage, showing the manner in which the master connecting rod cap was counterweighted. This example illustrates very nicely the ease with which the two mass substitutions will handle the balancing problem of a fairly complicated linkage.

**ADDITIONAL REFERENCES**


Fig. 9. - A Master Connecting Rod Linkage

Fig. 10. - Schematic Diagram Of A Connecting Rod Linkage In Which $k = 1$. 
APPENDIX I  METHOD OF ANALYSIS FOR THE HARMONICS

In making an analysis for the harmonics, it is necessary first of all to have the analytical expressions for the forces and couples under consideration. The equation 4 is an example. The derivation of such expressions is an elementary matter and needs no discussion. Having these expressions, several methods may be used to obtain the corresponding Fourier series. These are stated and discussed below.

1. Graphical integration for the Fourier coefficients may be used. With this method, it is difficult to obtain the desired accuracy for the higher harmonics, and since separate integrations must be made for each of a series of values of the modulus $k$, a considerable amount of work is involved.

2. Numerical integration may be used. The desired accuracy may be secured by this method with painstaking work, but the amount of work involved almost prohibits the compilation of extensive data in the desired chart form.

3. Direct expansions of the radicals by means of the binomial theorem may be used. This results in power series of trigonometric functions, which may be converted into Fourier series by reducing the power terms to functions of multiple angles. This method is the one that has been most commonly used. It is accurate, but has the
disadvantage of producing only the series forms of the coefficients. These series forms are not suitable for computation when the crank-connecting rod ratio, k, approaches unity, although this rarely happens except in very special machinery.

4. The preferred method of obtaining the coefficients is, of course, a direct integration giving the expressions for the coefficients in closed form. The closed forms may then be expanded into series forms when this form is easier to compute. This method is used here. It has the advantage of disclosing the characters of the functions with which one is working, and also of disclosing the preferred form of the general term in the power series for any coefficient.

When the functions are continuously analytical throughout the range of integration, as is the case with inertia forces and couples, it is permissible to perform a required differentiation termwise on a Fourier series. This simplifies the integrations in some cases. For example, we may express the distance x as a Fourier series in \( \Theta \), and later determine the value of the acceleration, \( \frac{d^2x}{dt^2} \), by differentiating the series termwise.

If \( f(\Theta) \) is a function to be expressed as a Fourier series in the interval \(-\pi\) to \(+\pi\), the coefficients, as determined from the theory of normal functions, for the cosine and sine terms, respect-
ively, are:

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta , \quad (48) \]

and

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta , \quad (49) \]

where \( n \) denotes the order of the harmonic. After writing such an expression one may quickly determine, by inspecting the character of the integrand, whether the sine terms or cosine terms vanish, and whether only odd, only even, or mixed harmonics remain.

The functions characteristic of this particular problem usually involve the elliptic integrals. When this is the case, those parts of the integrands which yield elliptic integrals can be expressed entirely in terms of radicals of the form \( \Delta = \sqrt{1 - k^2 \sin^2 \theta} \) and these will appear in odd powers of \( \Delta \). An example is the function.

\[ \frac{\cos^2 \theta}{\Delta} = \frac{c_1}{\Delta} + c_2 \Delta + c_3 \Delta^3 , \quad (50) \]

where \( c_1, c_2, \) and \( c_3 \) are constants involving \( k \).

Integrals that appear in this form may at once be expressed as elliptic integrals by means of the following formulas:

\[ * \]

\begin{align*}
\int_0^{\pi/2} \frac{d\theta}{\Delta^3} &= \frac{E}{1-k^2} \\
\int_0^{\pi/2} d\theta &= K \\
\int_0^{\pi/2} \Delta d\theta &= E \\
\int_0^{\pi/2} \Delta^3 d\theta &= \frac{4-2k^2}{3}E - \frac{1-k^2}{3}K \\
\int_0^{\pi/2} \Delta^{m+3} d\theta &= \frac{m+2}{m+3} \left(2-k^2\right) \int_0^{\pi/2} \Delta^{m+1} d\theta - \frac{m+1}{m+3} \left(1-k^2\right) \int_0^{\pi/2} \Delta^{m-1} d\theta
\end{align*}

Equation 55 is a reduction formula by means of which integrals involving powers of \(\Delta\) higher than the third may be evaluated.

Once an expression has been integrated in the form of elliptic integrals, an easy way of obtaining the series form is to use the known expansions of the elliptic integrals. This method was followed in deriving the results given in this thesis. It has the advantage of producing relatively simple expressions for the general terms in the series, as given in Equation 6a, for example.
It will be observed that the coefficients $c_2$, $c_4$, etc. of the even harmonics in the Fourier series for torque caused by the piston inertia force are exceptions, in that they do not involve the elliptic integrals. In these cases, the integrands of the integral expressions for the coefficients involved only even powers of the radical $\Delta$. To illustrate, the integral for $c_2$ is given by the equation:

$$1 - c_2 = \frac{1 - k^2}{2\pi} \int_0^{2\pi} \frac{\sin^2 \theta}{\Delta^4} \, d\theta.$$  \hspace{1cm} (56)

Integrals of this sort are more difficult to resolve than those which produce the elliptic integrals. However, the difficulty is not great if the integration is transferred to the complex plane. In using this method, the path of integration is transferred from the real axis to a circular path of unit radius in the complex plane. This transfer is effected by substituting in equation 56

$$\Theta = i \log z,$$ \hspace{1cm} (57)

in which $z$ is the complex variable $x + iy$, and $\Theta$ is the amplitude of $z$. The path of the independent variable $z$ is thus a circle of unit radius, with its center at the origin of the complex plane. The integrand in equation 56 then becomes a function of $z$ and this function is integrated by using the theorem of residues.
The equation 57 gives us the following direct substitutions for the equation 56:

\[
\sin \theta = -\frac{i}{2} i (z - \frac{1}{z}) \quad (58)
\]
\[
\cos \theta = \frac{1}{2} (z + \frac{1}{z}) \quad (59)
\]
\[
d\theta = -i \frac{dz}{z} \quad (60)
\]

By substituting these expressions in equation 56, one obtains the following integral:

\[
1-c_L = \frac{2i (1-k^2)}{\pi k^4} \int_C \frac{(z-1)^2 dz}{Z (z-z_1)(z-z_2)(z^2 + \lambda)^2}, \quad (61)
\]

where the path of integration, \(C\), is the unit circle, \(z_1\) and \(z_2\) are roots of the denominator, and

\[
\lambda = \frac{1}{k^2} \left[ (2-k^2) + 2\sqrt{1-k^2} \right]. \quad (62)
\]

Obviously, there are four second order and one first order zeros in the denominator. However, the roots \(\pm i \sqrt{\lambda}\) fall outside of the circle of integration. Hence one is only concerned with the three roots which provide residues within the circle. These roots occur at

\[
Z = 0, \quad (63)
\]
\[
Z = z_1 = \frac{i}{k} \sqrt{2-k^2 - 2\sqrt{1-k^2}}, \quad (64)
\]

and

\[
Z = z_2 = -z_1. \quad (65)
\]
It is proved in the theory of the complex variable that
\[ \int_C f(z) \, dz = 2\pi i \sum R_n, \tag{66} \]
where \( R_n \) is the coefficient of \( (z-z_n)^{-1} \) in the Laurent expansion of \( f(z) \) about the pole at \( z = z_n \), and the summation includes all such residues of poles enclosed by the path of integration, \( C \).

Returning to the integral of equation 61, let \( f(z) \) denote the integrand, and let \( f_n(z) \) denote a function derived from \( f(z) \) by deleting the factor \( (z-z_n)^m \) from the denominator. Also let \( f'_n(z) \) denote the derivative of \( f_n(z) \). Then, following the definition of a residue as the coefficient of \( (z-z_n)^{-1} \) in the Laurent expansion of \( f(z) \), it is evident that the residues of \( f(z) \) are:

\[ R_0 = f_0(0) = 1, \tag{67} \]

where
\[ f_0(z) = \frac{(z^4-1)^2}{(z-z_1)^3(z-z_2)(z^2+\alpha)^2}. \]

\[ R_1 = f'_1(z_1) = -\frac{2-\sqrt{1-\alpha}^2}{4\sqrt{1-\alpha}^2}, \tag{68} \]

where
\[ f_1(z) = \frac{(z^4-1)^2}{z(z-z_1)^3(z^2+\alpha)^2}. \]

\[ R_\omega = f'_\omega(z_\omega) = -\frac{2-\sqrt{1-\alpha}^2}{4\sqrt{1-\alpha}^2}, \tag{69} \]

where
\[ f_\omega(z) = \frac{(z^4-1)^2}{z(z-z_\omega)^3(z^2+\alpha)^2}. \]
Substituting \(2\pi (R_0 + R_1 + R_2)\) for the integral in equation 61, one obtains the result:

\[
c_2 = 1 - \frac{4\sqrt{1 - \frac{k^2}{k^4}}}{k^4} \left(1 - \frac{1}{2} \frac{k^2}{k^4} - \sqrt{1 - \frac{k^2}{k^4}}\right), \tag{70}
\]

which was previously given in equation 18. The determination of \(c_4\) was made in similar fashion.
APPENDIX II  THE LIMITING CASE WHEN $R = L$

When the connecting rod length is equal to that of the crank arm, the ratio $k$ is, of course, unity. In this case the linkage is not a practical one except in very odd or special designs, and the purpose of studying it is only to verify the limiting values of the harmonic coefficients as $k$ approaches unity.

By referring to Figure 10, it will be apparent that three types of motion are theoretically possible. First, if the design permits, the wrist pin center may permanently coincide with the shaft center, in which case the connecting rod merely rotates about the wrist pin with the crank arm. Secondly, the wrist pin may oscillate through the shaft center, giving double the stroke obtained with the conventional motion. For this case, it is evident from Figure 10 that

$$\beta = \pi - \Theta, \quad \frac{d\Theta}{dt} = -\omega, \quad \text{and} \quad \frac{d^2\Theta}{dt^2} = 0.$$ 

Also, $\frac{d^2x}{dt^2} = 2R\omega^2 \cos \Theta$. These relations show that for this motion there is no angular acceleration of the connecting rod, and that it rotates at the same angular speed as, but in the opposite direction from, the crank shaft. They also show that the linear acceleration of the piston is a simple harmonic form, with no higher harmonics present. This type of motion is physically possible, and is the type that would tend to occur when $k = 1$.

The third possible motion is that of the limiting case of the ordinary connecting rod motion as the value of $k$ approaches unity.
When \( k = 1 \), this motion is physically impossible without shock because it involves a discontinuity of the velocity of the wrist pin. However, it may be approached closely if the coefficient of friction at each joint of the linkage is sufficiently low. If the friction coefficient is appreciable, the linkage will become self locking. It is evident, therefore, that this motion with \( k = 1 \) is purely theoretical. Nevertheless, the harmonic coefficients are easily derived either by taking the limiting values of the expressions previously given as \( k \) approaches unity, or by a direct Fourier expansion for this particular case. Some of these coefficients are:

\[
\begin{align*}
a_n &= \frac{1}{5\pi}, \quad a_0 = \frac{6\pi}{15\pi}, \quad \cdots \quad \lim_{n \to \infty} a_n = \frac{2}{\pi} \quad (n \text{ even}) \\
b_1 &= \frac{4}{\pi}, \quad b_3 = \frac{4}{\pi}, \quad \cdots \quad b_n = \frac{4}{\pi} \quad (n \text{ odd}) \\
c_2 &= 1, \quad c_3 = \frac{2\pi}{5\pi}, \quad c_4 = 0, \quad \cdots
\end{align*}
\]

It must be understood, of course, that while these coefficients are finite, the resulting series for the forces or couples can not converge on finite values at the values of crank angle where the shocks occur. Moreover, the convergence will be slow when the value of \( k \) is near to, but less than unity.

The values of these coefficients were used-in making the chart of Figure 5. This chart shows that the values of most of the coefficients drop sharply for a relatively small increase in the ratio of \( L \) to \( R \) when this ratio is near unity.