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Title: A Thin Codimension-one Decomposition of the Hilbert Cube

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Dennis J. Garity

For cell-like upper semicontinuous(usc) decompositions G of finite dimensional manifolds M, the decomposition space M/G turns out to be an ANR provided M/G is finite dimensional ([Dav07], page 129). Furthermore, if M/G is finite dimensional and has the Disjoint Disks Property (DDP), then M/G is homeomorphic to M ([Dav07], page 181). For an infinite dimensional M modeled on I^∞, we can construct cell-like usc decompositions G associated with defining sequences. But it is more complicated to check whether M/G is an ANR. We need an additional special property of the defining sequence. To check whether or not M/G is homeomorphic to M is even more difficult. We need M/G to be an ANR which has the DDP and which also satisfies the Disjoint Čech Carriers Property. We give a specific cell-like decomposition X of the Hilbert Cube Q with the following properties: The nonmanifold part N of X is complicated in the sense that it is homeomorphic to a Hilbert Cube of codimension 1 in Q. X is still a factor of Q because $X \times I^2 \cong Q$. If A is any closed subspace of N of codimension ≥ 1 in N, then the decomposition of Q over A is homeomorphic to Q. In particular, the nonmanifold nature of X is not detectable by examining closed subsets of codimension ≥ 1. This example is produced by combining mixing techniques for producing a nonmanifold space whose nonmanifold part is a Cantor set, with decompositions arising from a generalized Cantor function.

A Thin Codimension-one Decomposition of the Hilbert Cube

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Dean of the Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Aniruth Phon-On, Author

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1. OVERVIEW

1.1 Outline

To make it easier to navigate through the various sections, we start with an outline of the material contained in this thesis.

Chapter 1: This chapter introduces the background and definitions needed in the next chapters. The reader familiar with this terminology can skip this chapter and go directly to the next chapter.

- **Section 1.1**: In this section we go over history and theorems that are well-known and give motivation about this work.
- **Section 1.2**: In this section we summarize what we have in each chapter.
- **Section 1.3**: In this section we go over definitions, notation, and results related to **defining sequences, cellular sets, cell-like sets, and ANRs.** These will be needed in in the remaining chapters.
- **Section 1.4**: In this section we go over definitions and results related to **homological codimension and the disjoint discs property.** These will be needed in Chapter 3 and Chapter 5.
- **Section 1.5**: In this section we go over definitions and results related to **geometric centrality.** These will be needed in Chapter 2.

- **Section 1.6**: In this section we go over definitions and results related to **inverse limits and Čech homology.** These will be needed in section 3.3 and Chapter 5.

- **Section 1.7**: In this section we go over definitions and results related to **infinite codimension and Čech carriers.** These will be needed in section 3.3 and Chapter 5.

Chapter 2: In this chapter we construct special Cantor sets in the Hilbert cube. We generalize results from finite dimensions about detecting points in Cantor sets with triadic rational coordinates to infinite dimensions. The results in this chapter use definitions and theorems from sections 1.3, and1.6.

Chapter 3: In this chapter we construct a special decompostion of the Hilbert cube with nonmanifold part a Cantor set. This will be used in the main results in Chapter 5. The results in this chapter use definitions and theorems from section 1.3, 1.4, 1.5, 1.7, 1.8. Also, we use results from Chapter 2.

Chapter 4: This chapter goes over details about the Cantor function in finite and infinite dimensions. These are used to construct a decomposition that will be used in the construction in Chatpter 5. The results in this chapter use definitions and theorems from section 1.3.

Chapter 5: This chapter constructs the main example. The results in this chapter use definitions and theorems from section 1.3, 1.4, 1.8. Also we use results from chapter 3 and chapter 4.

Chapter 6: We summarize the results in the thesis and ask some questions about generalizations.

1.2 Historical Setting and Introduction

For the definitions of terms introduced in this section, see § 1.3. Let M and N be closed $n-$manifolds with $n \neq 3, 4$. Let $\mathcal{C}(M, N)$ denote the set of all continuous maps from M to N with the compact-open topology, $Hom(M, N)$ denote the subset of $\mathcal{C}(M, N)$ consisting all homeomorphisms from M to N, and $CEL(M, N)$ denote the subset of $\mathcal{C}(M, N)$ consisting all cell-like maps from M to N. Then $CEL(M, N)$ is precisely the closure of $Hom(M, N)$ in $\mathcal{C}(M, N)$, see [Lac77]. Hence after homeomorphisms, cell-like maps are considered to be the next simplest kinds of maps on manifolds. The cell-like concept has since been studied in great detail in finite dimensions. See Daverman's book [Dav07] for a large number of examples. One way to construct a cell-like map is using decomposition theory. In general, if $f : M \to X$ is an onto cell-map from $n-$manifold M to a topological space X, then M is not necessarily homeomorphic to X. For example, let Wh be the **Whitehead continuum** in $\mathbb{R}^3$. That is,

$$Wh = \cap_{i=1}^{\infty} T_i$$

where T_i is defined by the following: Let T_0 be a solid torus in $\mathbb{R}^3$. For $i \geq 1$, let $T_i = h(T_{i-1})$ where $h : \mathbb{R}^3 \to \mathbb{R}^3$ is the homeomorphism taking T onto W in Figure1.1. Note that $T_i \subset T_{i-1}$ for all i. Figure 1.1 demonstrates the first two stages of the construction of Wh.

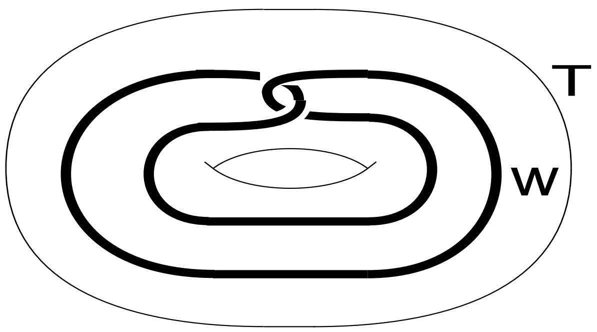


FIGURE 1.1: Whitehead Continuum

Then the map π from $\mathbb{R}^3$ to $\mathbb{R}^3/Wh$ is a cell-like map but $\mathbb{R}^3/Wh \ncong \mathbb{R}^3$ since any meridional disc in the larger torus must intersect Wh [Dav07]. In higher dimensions, the key property for determining when $M/G \cong M$ is the Disjoint Discs Property(DDP). The Cell-like Approximation Theorem [Edw80] is important and is used to prove that if G is a cell-like decomposition of an $n-$manifold M where $n \geq 5$, then $M/G \cong M$ if and only if M/G is an Absolute Neighborhood Retract(ANR) having the DDP. For a proof of this theorem, see [Dav07]. Daverman [Dav81] also provides a proof that if G is a cell-like decomposition of an $n-$manifold M then $M/G \times \mathbb{R}^2$ has the DDP, which implies that $M/G \times \mathbb{R}^2 \cong M \times \mathbb{R}^2$.

In the infinite dimensional setting, for a cell-like decomposition of an infinite dimensional manifold M, it is more complicated to check whether the space M/G is an ANR and even more complicated to check whether $M/G \cong M$. Not only do we need the DDP, but also the Disjoint Čech Carrier Property [DW81]. Thus, the motivation of this work is to produce a specific decomposition of the Hilbert Cube Q in order to investigate and to illustrate how complicated images of cell-like maps on Hilbert Cube Q can be.

The main goal of this work is to construct a specific cell-like, upper semi-continuous decomposition of the Hilbert Cube Q which yield a quotient space X that is not homeomorphic to Q and that has the property that any closed subspace A of the nonmanifold part N of X of codimension ≥ 1 induces a decomposition which is homeomorphic to Q. The nonmanifold part N of X is complicated in the sense that it is homeomorphic to a Hilbert Cube of codimension 1 in Q but X is still a factor of Q because $X \times I^2 \cong Q$.

In 1983, McCauley and Woodruff [MW83] produced such examples satisfying the above properties in $\mathbb{R}^3$.

In 1990, Garity [Gar91] generalized these examples in higher finite dimensions ($n \geq 5$).

We produce an example which is an infinite dimensional version of the finite dimensional example due to Garity,[Gar91]. In order to obtain this example in infinite dimensions it is necessary to discuss properties of the Cantor set that will be used in the construction of the examples and to express Cantor sets in Q as intersections in a special way. This is done in Chapter 2. In Chapter 2, we also describe the Cantor set C^∞ in Q in terms of a defining sequence such that we can detect when points in C^∞ have no triadic rational coordinates. This was previously done only in the finite dimensional case.

In Chapter 3, we produce a special decomposition H of Hilbert Cube Q. The decomposition, H has the non-manifold part of Q/H a Cantor set and has some additional special properties that we list in Chapter 3.

In Chapter 4, we produce an additional decomposition G of the Hilbert Cube Q using the generalized Cantor function.

In Chapter 5, we use the above decompositions G and H to construct the main example.

In this first Chapter we will give the basic definitions and notation needed in the remainder of the thesis.

1.3 Definitions and Notation

1.31 Topological Background

Throughout this thesis we assume all spaces X are separable metric spaces and the word map refers to a continuous function. For the readers who are not familiar with topology and algebraic topology, all basic topological terminology, notation, definitions, and theorems can be found in [Mun00], [May72].

Let B^2 be the unit disc in $\mathbb{R}^2$, I be the interval $[0,1]$. For each $n \geq 1$, we write

$$I^n = \prod_{i=1}^{n} I_i, \quad Q_{n+1} = \prod_{i=n+1}^{\infty} I_i$$

where $I_i = I$. After suitable parametrization, the **Hilbert Cube** is a countable product of I_i where for ease of notation, $I_1 = [-3,3]$, and $I_i = [0,1]$ for all $i \geq 2$, and is denoted by Q. That is,

$$Q = \prod_{i=1}^{\infty} I_i$$

Also, for each n we can write the Hilbert Cube as

$$Q = I^n \times Q_{n+1}.$$

We can define a metric $\rho : Q \times Q \to \mathbb{R}^+ \cup \{0\}$ on Q by

$$\rho\left(\left(x_i\right), \left(y_i\right)\right) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i} \quad \text{for all } \left(x_i\right), \left(y_i\right) \in Q$$

where (x_i) denotes the sequence of $x_i \in I_i$.

This metric generates the product topology on Q. The **pseudo interior** of Q is

$$s = \prod_{i=1}^{\infty} I_i^0,$$

where $I_1^0 = (-3,3)\,, I_i^0 = (0,1)$ for $i \geq 2$, and $Bd\,(Q) = Q - s$ is the **pseudo boundary** of Q.

Let $f, g : Q \to Y$, be maps, where Y is a metric space with a metric d. Then we define $\Psi\,(f,g)$ as

$$\Psi\,(f,g) = \sup\{d\,(f\,(x)\,, g\,(x))\,|x \in Q\}.$$

Definition 1.3.1. *[Lac77] A $Q-$manifold is a space which is locally homeomorphic to open subsets of Q.*

The following lemma is one of the facts about the Hilbert Cube that we use to prove some later theorems in this thesis.

Lemma 1.3.2. *(Tube Lemma)[Dav07] Suppose S is a compact subset of Q and U is an open subset of Q containing S. Then there exist an integer n and a ball B^n in I^n such that*

$$S \subset B^n \times Q_{n+1} \subset U.$$

A **decomposition** G of a space X is a partition of X. Explicitly, G is a subset of the power set of X, and its elements are pairwise disjoint nonempty sets that cover X. Associated with any decomposition G of a space X is the **decomposition space** denoted as X/G. Its topology is described by means of the **decomposition map** $\pi : X \to X/G$ sending each $q \in X$ to the unique element of G containing q. The topology on X/G is the quotient space topology induced by π. For any decomposition G of a space X we use H_G to denote the set of non-degenerate elements (elements of cardinality greater than 1) of G. We use N_G to denote the union of the elements of H_G. If π is the quotient map for the decomposition G, then we write $N_G = N_\pi$.

Definition 1.3.3. *The nonmanifold part of a decomposition space X/G where X is an n-manifold (the Hilbert cube) consists of those points in X/G that do not have neighborhoods homeomorphic to R^n (homeomorphic to the Hilbert Cube).*

Definition 1.3.4. *[Dav07] A decomposition G of X is said to be* **upper semicontinuous(usc)** *if every $g \in G$ is compact and the quotient map*

$$\pi \ : \ X \to X/G$$

is a closed map.

A basic property of an usc decomposition G of X is: given $g \in G$, $g \subset U$ where U is an open set in X. Then

$$V = \bigcup \{g \in G \mid g \subset U\}$$

is open.

Given a decomposition G of X and a closed subset A of X/G, we use $X/\pi^{-1}(A)$ to denote the **decomposition of X induced over** A. $X/\pi^{-1}(A)$ consists of all sets of the form $\pi^{-1}(a), a \in A$, and all singletons of $X - \pi^{-1}(A)$.

Theorem 1.3.5. *[Dav07](Realization) Suppose G is an upper semicontinuous decomposition of a space X and f is a closed map of X onto a space Y such that $G =$*

$\{f^{-1}(y \mid y \in Y)\}$. *Then* X/G *is homeomorphic to* Y.

Definition 1.3.6. *[Dav07] A decomposition* G *of a space* X *is* **realized by a pseudo-isotopy** *if there exists a pseudo-isotopy* Ψ_t *of* X *to* X *such that* $\Psi_0 = Id_X$ *and* $G = \{\Psi_1^{-1}(x) \mid x \in X\}$. *By a* **pseudo-isotopy** Ψ_t *of* X *to* X *we mean a homotopy* $\Psi_t : X \to X$ *such that* Ψ_t *is a homeomorphism for each* $t \in [0,1)$ *and* Ψ_1 *is a closed surjection. Similarly, by an* **isotopy** Ψ_t *of* X *to* X *we mean a homotopy* $\Psi_t : X \to X$ *such that* Ψ_t *is a homeomorphism for each* $t \in [0,1]$.

1.32 Defining Sequences for Decompositions

We need the following definitions related to defining sequences for decompositions before proceeding. Some of these definitions can be found in [Lay80] and [Dav07].

Definition 1.3.7. *Let* $\mathcal{S} = \{\mathcal{S}_i\}$ *be a sequence of collections of subsets of the Hilbert cube satisfying the following conditions:*

(1) Disjointness criterion: *For each* i, $\mathcal{S}_i$ *is a finite collection of compact subsets of* Q *with disjoint interiors.*

(2) Nesting criterion: *For every element* A *of* $\mathcal{S}_i$ *and for every* $j < i$ *there is a unique element* $Pre(A)$ *of* $\mathcal{S}_j$ *that contains* A.

(3) Boundary size criterion: *If* A *is an element of* $\mathcal{S}_i$ *and* x, y *are elements in* ∂A *then there is a* $j > i$ *such that no element of* $\mathcal{S}_j$ *contains both of* x *and* y.

(4) Null homotopy criterion: *For each* $i > 1$ *and each* $A \in \mathcal{S}_i$, *the inclusion map* $A \to Pre(A)$ *is null homotopic.*

Then the sequence $\mathcal{S} = \{\mathcal{S}_i\}$ *is called* **a defining sequence in Q.**

Definition 1.3.8. *Let* Q *be the Hilbert Cube and* $\mathfrak{M}$ *a collection of subsets of* Q, *not necessarily covering* Q. *Given an arbitrary set* Z *in* Q, *define its* **star in** $\mathfrak{M}$ *as*

$$St(Z, \mathfrak{M}) = Z \bigcup \left(\bigcup \{M \in \mathfrak{M} \mid M \cap Z \neq \emptyset\} \right),$$

also written as $St^1(Z, \mathfrak{M})$, *and, recursively for any integer* $k > 1$, *its* k*th* **star in** $\mathfrak{M}$ *as*

$$St^k(Z, \mathfrak{M}) = St\left(St^{k-1}(Z, \mathfrak{M}), \mathfrak{M}\right).$$

When $Z = \{x\}$, *we write* $St^k(\{x\}, \mathfrak{M})$ *simply as* $St^k(x, \mathfrak{M})$.

Definition 1.3.9. *Let* $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$ *be a defining sequence in* X. *Then the* **decomposition** G **associated with the defining sequence** $\mathcal{S}$ *is the relation prescribed by the rule: for each* $x \in X$,

$$G(x) = \bigcap_{i \geq 1} St^2(x, \mathcal{S}_i).$$

EXAMPLE 1.3.10. *Consider* $X = \mathbb{R}^2$. *Recall, in* $\mathbb{R}$, *that the Cantor set can be described as* $\bigcap \mathcal{S}_i$ *where* $\mathcal{S}_i$ *is a set with* 2^i *elements in it and each element is an interval of length* $\frac{1}{3^i}$. *For each* i, *and* $S = [a, b] \in \mathcal{S}_i$, *let* $U_S = \left[a - \frac{1}{3^{i+1}}, b + \frac{1}{3^{i+1}}\right]$. *Then let*

$$\mathcal{U}_i = \left\{U_S \times [-1, 1] \mid S \in \mathcal{S}_i\right\}.$$

It is easy to show that $\mathcal{U} = \left\{\mathcal{U}_0, \mathcal{U}_1, \ldots\right\}$ *is a defining sequence. Then the decomposition* G *associated with the defining sequence* $\mathcal{U}$ *consists of* $A = \left\{\{c\} \times [-1, 1] \mid c \in C\right\}$ *and the singletons from* $X - A$.

Lemma 1.3.11. *[Dav07] Let* G *be a decomposition of a space* X *associated with the defining sequence* $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$. *If*

$$x, y \in \partial\mathcal{S} = \cup\{\partial A \mid A \in \cup_i \mathcal{S}_i\}$$

such that $x \neq y$, *then* $\pi(x) \neq \pi(y)$.

From this lemma, π is one-to-one on $\partial\mathcal{S}$.

Before listing additional properties of X/G, we need to define cellularity and cell-likeness.

1.33 Cellular Sets and Cell-like Sets

Definition 1.3.12. *A closed set C in $\mathbb{R}^n$ or in an $n-$dimensional manifold is said to be* **cellular** *if there is a nested sequence $C_1, C_2, ...$ of n cells with C_{i+1} a subset of the interior of C_i and $C = \bigcap_i^\infty C_i$.*

EXAMPLE 1.3.13. *The closure of the graph of the $\sin\left(\frac{1}{x}\right)$ curve, $0 < x \leq 1$, in $\mathbb{R}^2$ is a cellular set, see Figure 1.2. This example shows that cellular sets need not be path connected.*

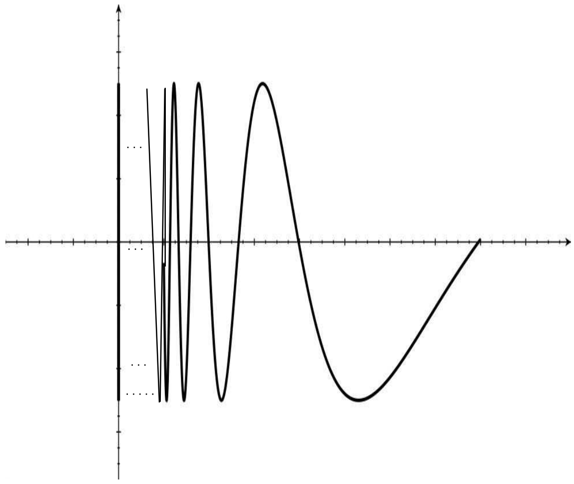


FIGURE 1.2: The closure of $\sin\left(\frac{1}{x}\right)$ curve, $0 < x \leq 1$, in $\mathbb{R}^2$.

Definition 1.3.14. *A compact subset C of a manifold X is* **cell-like in** *X if for each neighborhood U of C in X, C can be contracted to a point in U. A decomposition G of X is cell-like if each $g \in G$ is cell-like.*

Definition 1.3.15. *A mapping $f : X \to Y$ is* **cell-like** *if $f^{-1}(y)$ is a cell-like space for each $y \in Y$.*

The next theorem is a technical property called **approximate lifting** needed in what follows.

Theorem 1.3.16. *[Dav07] Let G be a cell-like decomposition of a space X, and $\pi : X \to X/G$ be a quotient map. Let K be a simplicial $n-$complex and L be a subcomplex of K.*

Let

$$f \ : \ K \to X/G$$

and $F_L \ : \ L \to X$ *be a map such that* $\pi \circ F_L = f|_L$ *and let* $\epsilon > 0$ *be given. Then there exists a map* $F \ : \ K \to X$ *such that* $\rho(f, \pi \circ F) < \epsilon$ *and* $F|L = F_L$.

See § 1.35 for the definition of ANR.

Theorem 1.3.17. *[Lac77] If* $X, Y,$ *and* Z *are locally compact ANR's and* $f : \ X \to Y$ *and* $g : \ Y \to Z$ *are proper cell-like maps then* $g \circ f$ *is cell-like.*

Definition 1.3.18. *Let* $f : \ X \to Y$ *be a map. Then* f *is called* **proper** *if* f *is closed map with compact point-inverses.*

Definition 1.3.19. *[Mun00] Two spaces* X *and* Y *are said to be* **homotopy equivalent,** *if there are maps*

$$f : \ X \to Y \quad \text{and} \quad g : \ Y \to X$$

such that $g \circ f \simeq i_X$ *and* $f \circ g \simeq i_Y$. *The maps* f *and* g *are often called* **homotopy equivalences.**

Theorem 1.3.20. *[Lac77] If* $f : X \to Y$ *is a proper map between locally compact ANR's, then the following are equivalent:*

(a) f *is cell-like;*

(b) For every open set $V \subset Y$, *the restriction* $f\big|_{f^{-1}(V)} \ : \ f^{-1}(V) \to V$ *is a proper homotopy equivalence.*

Theorem 1.3.21. *[Dav07] The decomposition* G *associated with with the defining sequence* $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$ *with properties as in definition 1.3.7 is a cell-like usc.*

1.34 Cellularity in the Hilbert Cube Q

Recall that the definition of a cellular set for a finite dimensional manifold is a nested intersection of $n-$cells [Dav07]. For an infinite dimensional manifold, the cellular sets are more difficult to characterize which explains why so many complications arise in attempts to find infinite dimensional analogues of some results about cellularity in finite dimensional manifolds.

Lemma 1.3.22. *([Dav07], page 120) A cellular subset of an n-manifold M is cell-like.*

Theorem 1.3.23. *([Dav07]) If g is cellular, then M/g is homeomorphic to M.*

Definition 1.3.24. *[Cha76] A closed subset A in a space X is said to be a $Z-$***set** *in X provided that for every open cover $\mathcal{U}$ of X there is a map f of X into $X - A$ which is $\mathcal{U}-$close to the identity map. That is, for each $x \in X$, there exists some element of $\mathcal{U}$ containing both x and $f(x)$.*

The following lemma gives a condition for a subset of the pseudo interior of Q to be a $Z-$set.

Lemma 1.3.25. *[Cha76] Any compact subset $A \subset Q$ with $A \subset s$, where s is the pseudo interior of Q, is a $Z-$set.*

Definition 1.3.26. *[Čer80] A closed subset K of a space M is a* **normal cube in** *M if K and the boundary $Bd(K)$ of K in M are homeomorphic to the Hilbert cube Q and $Bd(K)$ is a $Z-$set in K.*

Next we will define a cellularity in Q which is quite similar to the definition in finite dimensional case. Here we replace the term $n-$cells by normal cubes.

Definition 1.3.27. *[Čer80] Let X be a closed subset of Q. X is said to be a* **cellular** *subset of Q if $X = \cap_{i=1}^{\infty} K_i$ where $K_{i+1} \subset int(K_i)$ and K_i is a normal cube for all i.*

Alternately, a closed subset X of Q is cellular provided it has arbitrarily small open neighborhoods whose closures are normal cubes in Q.

Definition 1.3.28. *[Cha76] Let X be a compact space and Y be an ANR containing X. Then X is said to have* **trivial shape** *if for every neighborhood U of X in Y, X is contractible in U.*

It is obvious that every cell-like set has trivial shape.

Definition 1.3.29. *Let X be a subset of the Hilbert cube Q. Then $Q - X$ is* S^1-**trivial at** ∞ *provided for every open neighborhood U of X there is an open neighborhood V of X such that every map $f : S^1 \to V - X$ can be extended to a map from B^2 to $U - X$.*

The following lemma will be used to check whether or not a finite dimensional non-degenerate decomposition element is cellular.

Lemma 1.3.30. *[Čer80] Let A be a finite dimensional compactum in Q. Then the followings are equivalent.*

1. *A is cellular in Q.*
2. *A is a Z-set in Q and has trivial shape.*
3. *A has trivial shape and Q/A is a $Q-$manifold.*
4. *$Q - A$ is S^1-trivial at ∞.*
5. *$Q/A \cong Q$.*

1.35 ANRs

To later detect when Q/G is homeomorphic to Q we need to introduce ANRs. For additional information on ANRs, see [Dav07].

Definition 1.3.31. *A metric space Y is said to be an* **absolute neighborhood retract***(ANR) if for every closed subset A of a metric space X and for every such map $f : A \to Y$, there is a continuous extension $F : U \to Y$ defined on some neighborhood U of A in X.*

Definition 1.3.32. *Let $f : X \to Y$ be a map from a compact metric space X onto a compact metric space Y. Then f is said to be* **Approximately Right Invertible** *(ARI) if for each $\epsilon > 0$, there is a map $g : Y \to X$ such that $\rho(f \circ g, I_Y) < \epsilon$.*

Definition 1.3.33. *A defining sequence $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$ is said to be* **sharp** *if $\partial\mathcal{S} = \cup_{i\geq 1}\{\partial S : S \in \mathcal{S}_i\}$ is embedded in the decomposition space by the quotient map.*

Remark 1.3.34. *The defining sequences we use in definition 1.3.7 are sharp.*

The following theorem will be used to detect that certain decomposition spaces are ANR's.

Theorem 1.3.35. *([Koz81], page 21)Let $f : X \to Y$ be a cell-like map from a compact metric ANR X onto a compact metric space Y. If f is ARI, then Y is an ANR.*

The following theorem is from [Lay].

Theorem 1.3.36. *[Lay] If $\mathcal{S}$ is a sharp defining sequence for a decomposition G of the Hilbert Cube, then the quotient map is ARI and hence the space Q/G is an ANR.*

Theorem 1.3.37. *([Lay80], page 22)Let G be a cell-like decomposition of Q such that Q/G is an ANR. If A is a closed set in Q/G and $\pi^{-1}(A)$ is the decomposition of Q induced over A, then $Q/\pi^{-1}(A)$ is an ANR.*

1.4 Homological Codimension and Disjoint Discs Property

Definition 1.4.1. *Let A be a closed subset of Q. A is said to have* **codimension** $\geq n$ *in Q if for every open subset U and for every $i < n$*

$$H_i\left(U, U - A\right) \cong 0$$

where $H_i\left(A, B\right)$ is ith homology group of A modulo B and where i is an integer.

The subset A is said to have **codimension equal to** n if it has codimension $\geq n$ but does not have codimension $\geq n+1$. In other words, a closed subset A of Q is said to have codimension n if $H_i\left(U, U - A\right) \cong 0$ for all open sets U and for all $i < n$, and $H_n\left(V, V - A\right) \ncong 0$ for some open set V of Q. An arbitrary subset X of the Hilbert Cube Q is said to be of codimension n if any closed $C \subset X$ in Q has codimension n. (See [DW81].)

Example 1.4.2. *We will show that $F = \{0\} \times Q_2$ has codimension 1 in Q. First we will show that F has codimension ≥ 1. That is, we will show that $H_0\left(U, U - F\right) \cong 0$ for all open sets U in Q. To prove this, Without loss of generality, let U be an path connected open set in Q. Note that $U - F \neq \emptyset$. Consider the long exact sequence of the pair $\left(U, U - F\right)$:*

$$\cdots \to H_0\left(U - F\right) \xrightarrow{f} H_0\left(U\right) \xrightarrow{g} H_0\left(U, U - F\right) \xrightarrow{h} 0.$$

To show that $H_0\left(U, U - F\right) \cong 0$, it suffices to show that the map $H_0\left(U - F\right) \xrightarrow{f} H_0\left(U\right)$ is onto. Note that

$$H_0\left(U - F\right) \cong \widetilde{H}_0\left(U - F\right) \oplus \mathbb{Z}$$

$$H_0\left(U\right) \cong \mathbb{Z} \quad \text{since } U \text{ is path connected.}$$

It follows clearly that f is onto. This implies that

$$H_0\left(U\right) = Im\left(f\right) = ker\left(g\right)$$

and so

$$\{0\} = Im\,(g) = ker\,(h) = H_0\,(U, U - F)$$

Therefore, F has codimension ≥ 1 in Q.

Next we will show that F does not have codimension ≥ 2. Consider

$$U = (-1, 1) \times Q_2.$$

Then U is an open set in Q. Consider the long exact sequence

$$\cdots \to H_1\,(U - F) \to H_1\,(U) \to H_1\,(U, U - F) \xrightarrow{f} H_0\,(U - F) \xrightarrow{g} H_0\,(U) \xrightarrow{h} 0.$$

Note that

$$H_0\,(U - F) \cong \mathbb{Z} \oplus \mathbb{Z} \text{ since } U - F \text{ has 2 path components}$$

$$H_0\,(U) \cong \mathbb{Z} \text{ since } U \text{ is path connected.}$$

Thus, $H_0\,(U - F) \ncong H_0\,(U)$ which implies that $ker\,(g) \neq 0$. So, $Im\,(f) \neq 0$. Therefore, $H_1\,(U, U - F) \ncong 0$. This completes the proof.

Definition 1.4.3. *A closed subset A of Q is* **nowhere dense** *if for any nonempty open set U of Q, $U - A$ is not an empty set. That is every nonempty open set U is not a subset of A.*

The next result follows directly from the definitions. For completeness, we include a proof.

Lemma 1.4.4. *If A is a closed subset of Q which has codimension $\geq n$, with $n \geq 1$, then A is nowhere dense in Q.*

Proof. Let U be an non empty open set in Q. We will show that $U - A \neq \emptyset$. That is, we will show that $H_0\,(U - A) \ncong 0$. Clearly, $H_0\,(U) \ncong 0$ since U is not empty. Since A has codimension $\geq n$ with $n \geq 1$, $H_0\,(U, U - A) \cong 0$. Consider the long exact sequence of a pair $(U, U - A)$:

$$\cdots \to H_0\,(U - A) \xrightarrow{f} H_0\,(U) \xrightarrow{g} H_0\,(U, U - A) \xrightarrow{h} 0.$$

Since

$$Im\,(f) = ker\,(g)\,, \text{ and } Im\,(g) = ker\,(h) = H_0\,(U, U - A) \cong 0,$$

this forces $Im\,(f) = ker\,(g) = H_0\,(U) \neq 0$. Therefore, $H_0\,(U - A) \neq 0$ and so $U - A$ is not empty. Since U is arbitrary, it implies that A is nowhere dense.

□

Definition 1.4.5. *Let X be a metric space. Let $f, g : B^n \to X$ be any two maps. Then the space X is said to have the* **disjoint** $n-$**disc property,** *abbreviated as DD^nP***,** *if for each ϵ there exist maps $f', g' : B^n \to X$ satisfying*

$$\rho\left(f, f'\right) < \epsilon \qquad \textit{and} \qquad \rho\left(g, g'\right) < \epsilon$$

and

$$f'\,(B^n) \cap g'\,(B^n) = \emptyset.$$

1.5 Geometrical Centrality

Definition 1.5.1. *[Dav07] Let N be an $n-$manifold with or without boundary and let $f : D^2 \to B^2 \times N$ with $f\left(\partial D^2\right) \subset \partial B^2 \times N$. The map f is said to be* **interior inessential***($I-$inessential) if there is a map $g : D^2 \to \partial B^2 \times N$ such that $f = g$ on ∂D^2. Otherwise, f is said to be $I-$***essential***. Let H be a disc with holes, $g : H \to M$ be a map with $g\,(\partial H) \subset \partial M$. The map g is said to be* **virtually I-essential** *if g extends to an $I-$essential map $f : B \to M$ with $f\,(B \setminus H) \subset \partial M$ where B is the unique 2-cell in B^2 with $H \subset B$ and with $\partial B \subset \partial H$.*

We will use this definition of $I-$essential to test the geometric centrality of a subset of a manifold. A subset A of $B^2 \times N$ is said to be **geometrically central** in $B^2 \times N$ if $f\left(D^2\right) \cap A \neq \emptyset$ for every $I-$essential map f. A collection $\mathcal{C}$ of subsets of $B^2 \times N$ is called a geometrically central family in $B^2 \times N$ if the union of elements of $\mathcal{C}$ is geometrically central in $B^2 \times N$. More details can be found in [DE87].

1.51 Standard Example of Geometrically Central Subset

EXAMPLE 1.5.2. *Let N be an n-manifold, and identify A as $\{0\} \times N$ in $B^2 \times N$. Suppose $f : D^2 \to B^2 \times N$ is an I-essential map with $f\left(D^2\right) \cap A = \emptyset$. Then $\partial B^2 \times N$ is a retract of $\left(B^2 \times N\right) - A$, so there is a map $g : D^2 \to \partial B^2 \times N$ satisfying $g|_{\partial D^2} = f|_{\partial D^2}$. This contradicts the fact that f was I-essential. This gives us that A is geometrically central in $B^2 \times N$.*

1.52 Results on Geometric Centrality

In order to construct the special Cantor set in the Hilbert Cube needed in our example, we need the following results.

Lemma 1.5.3. *[DE87] Given $B^2 \times I$ and given $\epsilon > 0$. Then there is a family*

$$\{C_1, C_2, \ldots, C_k\}$$

of subsets of $B^2 \times I$ so that $C_1 \cong C_k \cong B^2 \times I$ and for $i = 2, 3, \ldots, k-1$, $C_i \cong B^2 \times S^1$, so that the family is geometrically central in $B^2 \times I$, and so that for each C_i, the diameter of C_i is less than ϵ. Similarly, given $B^2 \times S^1$ and given $\epsilon > 0$, there is a geometrically central family

$$\{C_1, C_2, \ldots, C_k\}$$

so that for all $i = 1, 2, \ldots, k$, $C_i \cong B^2 \times S^1$ and the diameter of C_i is less than ϵ.

Figure 1.3 and 1.4 demonstrate Lemma 1.5.3 for $B^2 \times I$ and $B^2 \times S^1$, respectively.

For the proof of Lemma 1.5.3, see [DE87]. The proof of Lemma 1.5.3 relies on the following Lemma. For completeness, we also state this result. For definition of bicollared, see [Dav07].

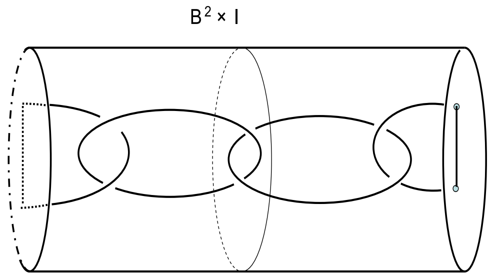


FIGURE 1.3: Geometrically Central Collection on $B^2 \times I$

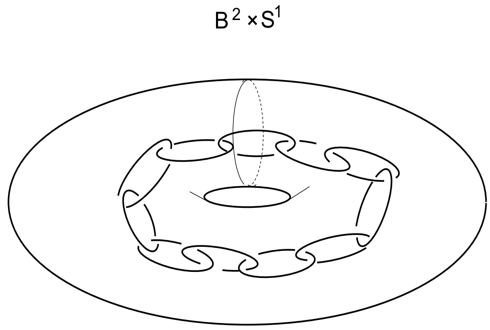


FIGURE 1.4: Geometrically Central Collection on $B^2 \times S^1$

Lemma 1.5.4. *[Lay80] Let D^2 be a disc with holes and $f: D^2 \to B^2 \times I$ be a map, and let P be a bicollared subset of $D^2 \times I$. Assume that $K = f^{-1}(P)$ is closed in D^2. If $F: K \times I \to P$ is a homotopy with $F_0 = f|_K$ and U is a neighborhood of $F(K \times I)$ in $B^2 \times I$, then there is a neighborhood V of K in D^2 and a map $g: D^2 \to B^2 \times I$ such that:*

1. $g|_{D^2-V} = f|_{D^2-V}$
2. $g|_K = F_1$
3. $g(V - K) \subset (U - P)$.

We also need the following result, first proved in [Ghi07]. For completeness, and because the proofs illustrate the concept of geometric centrality, we include the proof of this lemma and the following two lemmas.

Lemma 1.5.5. *[Ghi07] If N is a subset of an $n-$manifold $M \cong B^2 \times X$ which is geometrically central in M, then $N \times I$ is geometrically central in $M \times I \cong B^2 \times X \times I$. Similarly,* *if N is a subset of an $n-$manifold $M \cong B^2 \times X$ which is geometrically central in M, then $N \times S^1$ is geometrically central in $M \times I \cong B^2 \times X \times S^1$.*

Proof. Let D^2 be a disc with holes. If possible, let $f: D^2 \to M \times I$ be an I-essential map such that$f\left(D^2\right) \cap N \times I = \emptyset$. Decompose f into two factors f_M and f_I from D^2 to M and I respectively. We claim that f_M is I-essential into M. If not, there is a map $g: D^2 \to \partial B^2 \times X$ such that $g = f_M$ on ∂D^2. We can then define a map $h: D^2 \to \partial\left(M\right) \times I$ by $h = (g, f_I)$, then $f = g$ on ∂D^2. That contradicts that f was I-essential. Similar arguments prove that $N \times S^1$ is geometrically central in $M \times S^1$. □

What happens if we iterate the process of placing geometrically central sets in our construction? The following lemma from [Ghi07] shows how geometrical centrality is preserved.

Lemma 1.5.6. *[Ghi07] Let $A \cong B^2 \times X_1 \times X_2 \times \cdots \times X_n$, where each X_i is I or S^1. Let $\mathcal{C} = \{C_i: \ C_i \cong B^2 \times Y_{i1} \times Y_{i2} \times \cdots \times Y_{in}\}$, where Y_{ij} is I or S^1, be a finite collection of pairwise disjoint subsets of A which is geometrically central in A. Also, assume that for each C_i, there is a finite collection $\mathcal{D}_i = \{D_j: \ D_j \cong B^2 \times Z_{j1} \times Z_{j2} \times \cdots \times Z_{jn}\}$, where each Z_{jk} is I or S^1, of disjoint subsets of C_i, which is geometrically central in C_i. Then the collection $\mathcal{D} = \cup \mathcal{D}_i$ is geometrically central in A.*

Proof. Let D^2 be a disc with holes and $f: D^2 \to B^2 \times X_1 \times X_2 \times \cdots \times X_n$ be a virtually $I-$essential map. After a slight adjustment of f, we may consider $K = f^{-1}\{\mathcal{C}\}$ to be a 2-manifold in D^2 with a finite number of components. Hence $f^{-1}\left(\partial B^2 \times X_1 \times X_2 \cdots \times X_n\right)$ is a finite collection of simple closed curves in D^2. This implies that each component of $f^{-1}\left(C_i\right)$ is a disc with holes in D^2. Then K must have a component H such that f_H is a virtually $I-$essential into C_i for some i. If not, let H be given and let $f\left(H\right) \subset C_i$ for some fixed i. Also note that $f\left(\partial H\right) \subset \partial B^2 \times Y_{i1} \times \times \cdots \times Y_{in}$ for some $Y_{i1}, Y_{i2}, \ldots, Y_{in}$. Since $f|_H$ is not virtually $I-$essential, there is a map $g: H \to \partial B^2 \times Y_{i1} \times \times \cdots \times Y_{in}$ and $g = f|_H$

on ∂H and $C_i \cap C_j = \emptyset$ if $i \neq j$. Hence, we can push $g(H)$ off of $\partial B^2 \times X_1 \times X_2 \cdots \times X_n$ without intersecting any other C_j. Now repeating the same process with other components of K we can get a new map $h: D^2 \to \partial B^2 \times X_1 \times X_2 \times \cdots \times X_n$ that misses all C_i and $f = h$ on ∂D^2. That contradicts the fact that $\mathcal{C}$ is geometrically central in A. Hence $f|_H$ must be $I-$essential on some C_i for some component H and some i. But $\mathcal{D}_i$ is also geometrically central in C_i, and hence $f|_H$ must intersect some elements of $\mathcal{D}_i$, that is $f\left(D^2\right)$ must intersect $\mathcal{D}$. Therefore, $\mathcal{D}$ is geometrically central in A. □

We now give the generalization of Lemma 1.5.3 from [Ghi07]. That is, we will include more factors.

Lemma 1.5.7. *[Ghi07] Given $\epsilon > 0$ and $A = B^2 \times X_1 \times X_2 \times \cdots \times X_n$, where each $X_i = I$ or S^1. Then there is a finite geometrically central collection $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$, where $C_i \cong B^2 \times Y_1 \times Y_2 \times \cdots \times Y_n$ with each $Y_j = I$ or S^1, of disjoint subsets of A so that the diameter of C_i is less than ϵ for all i and for each $I-$essential map f, and for each C_i there is an $I-$essential map g with $f|_{\partial D^2} = g|_{\partial D^2}$ and $g\left(int\left(D^2\right)\right) \cap C_i \neq \emptyset$.*

Proof. We prove this lemma by induction over n. For $n = 1$, it holds by Lemma 1.5.3. Suppose the lemma is true for $n = k$. Let $A = B^2 \times X_1 \times X_2 \times \cdots \times X_k \times X_{k+1}$. Consider $B = B^2 \times X_1 \times X_2 \times \cdots \times X_k$. Then by assumption, there is a finite collection $\mathcal{C} = \{C_i : \; B^2 \times X_{i1} \times X_{i2} \times \cdots \times X_{ik}\}$ of disjoint subset of B, which is geometrically central in B and X_{ij} is I or S^1 and the diameter of $C_i < \epsilon$ for all i. By Lemma 1.5.5, the collection $\mathcal{D} = \{D_i : \; D_i := C_i \times X_{k+1}\}$ is geometrically central in A. Now switch the kth and $(k+1)$th components of D_i and let E_i be such an element. Consider the first k component of E_i. Then we will have a finite collection $\mathcal{F}_i$ such that $F_i \times X_k$ is geometrically central in E_i. Now switching back the kth and $(k+1)$th components of this collection and applying Lemma 1.5.5 we will have another collection G_i which is geometrically central in D_i. Consider $\mathcal{H}$ to be the union of these collections G_i. Then by Lemma 1.5.6, $\mathcal{H}$ is geometrically central in A. We may take enough components so that the diameter of each component is less than ϵ. $\square$

1.6 Inverse Limits and Čech Homology

Let (X, A) be a compact pair. Let $\Sigma(X)$ be the set of all finite open covers of X, and let G be a group. The only case we need is $G \cong \mathbb{Z}$, so we restrict to that case. To define the Čech homology of a compact pair (X, A), we begin with the concept of a **directed set.** These definitions and results are taken from [May72] and [Mun84].

Definition 1.6.1. *A* **directed set** J *is a set with a relation* $\leq$ *such that:*

(1) $\alpha \leq \alpha$ *for all* $\alpha \in J$.

(2) $\alpha \leq \beta$ *and* $\beta \leq \gamma$ *implies* $\alpha \leq \gamma$.

(3) Given α, β, *there exists* γ *such that* $\alpha \leq \gamma$ *and* $\beta \leq \gamma$.

We will show that $\sum(X)$ is a directed set. Let $\leq$ be defined by: If $\mathcal{U}, \mathcal{V} \in \sum(X)$, $\mathcal{U} \leq \mathcal{V}$ if and only if to every set $U \in \mathcal{U}$ there exists a set $V \in \mathcal{V}$ such that $V \subset U$. We can see that (1) and (2) are trivially satisfied. To show that (3) holds, let the covers $\mathcal{U}$ and $\mathcal{V}$ be given. Then the family $\mathcal{W}$ defined by

$$\mathcal{W} = \{U \cap V \mid U \in \mathcal{U},\ V \in \mathcal{V}\}$$

satisfies the requirements.

Definition 1.6.2. *An* **inverse system** *of abelian groups and homomorphisms, corresponding to the directed set* J, *is an indexed family* $\{G_\alpha\}_{\alpha \in J}$ *of abelian groups, along with a family of homomorphisms*

$$f_{\alpha\beta}:\ G_\beta \to G_\alpha,$$

defined for every pair of indices such that $\alpha \leq \beta$, *such that*

(1) For each $\alpha \in J$ there is a unique object G_α.

(2) For each pair $\alpha, \beta \in J$ there is a unique homomorphism

$$f_{\alpha\beta} : \ G_\beta \to G_\alpha.$$

(3) If $\alpha \leq \beta \leq \gamma$, then $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$.

(4) For all $\alpha \in J$, $f_{\alpha\alpha}$ is the identity.

The **inverse limit** *of an inverse system $\{G_\alpha\}_{\alpha \in J}$, denoted by $G_\infty = \varprojlim G_\alpha$, is the subset $\prod G_\alpha$ defined by the condition*

$$p = (p_\alpha) \in G_\infty \quad \text{ if } \quad f_{\alpha\beta}(p_\beta) = p_\alpha$$

where p_α is the αth component of the element $p \in \prod G_\alpha$.

Example 1.6.3. *[May72] Let D be the set of positive integers in their natural order and for each $n \in D$, let X_n be the set of real numbers. For each $m \leq n$, define $f_{mn} : \ X_n \to X_m$ by*

$$f_{mn}(x) = x - (n - m).$$

Then $X_\infty = \varprojlim (X_n)$ consists of all sequences of the form

$$(x, x+1, x+2, \ldots)$$

where x is real number.

After this preparation, we can define the following:

Definition 1.6.4. *Let (X, A) be a compact pair and let $\Sigma(X)$ be the family of finite open covers of X. Then the inverse limit*

$$\check{H}_p(X, A) = \varprojlim (H_p(\mathcal{U}, \mathcal{U}_A); \mathbb{Z})$$

*is the p***th Čech homology group of** *(X, A)* **over** *$\mathbb{Z}$ where $\mathcal{U} \in \Sigma(X)$ and $\mathcal{U}_A$ is the subfamily of U consisting of those sets which intersect A.*

1.7 Detecting $Q-$manifolds

In this section we discuss a characterization of $Q-$manifolds due to Daverman and Walsh [DW81].We begin with definitions and notation. This will be needed in the Chapter 3 and Chapter 5.

1.71 Infinite Codimension

Definition 1.7.1. *A closed subset F of an ANR X is said to have* **infinite codimension (in X)** *provided $H_q(U, U - F) = 0$ for all integers $q \geq 0$ and for all open subsets U of X. A set A in X is said to have* **infinite codimension** *if every closed subset of A has infinite codimension.*

The following result sets forth the basic characterization, a proof of which can be found in [Kro74].

Proposition 1.7.2. *A closed subset A of an ANR X is a $Z-$set if and only if A has infinite codimension.*

EXAMPLE 1.7.3. *For $i \geq 2$, let $J_i = \left[\frac{1}{4}, \frac{1}{2}\right] \subset I_i$. Let $A = \{0\} \times \prod_{i=2}^{\infty} J_i$. It is clear that A is closed and compact in Hilbert cube Q. Also, $A \subset s$, where s is a pseudo interior of Q. By Lemma 1.3.25 A is a $Z-$set and so by Proposition 1.7.2, A has infinite codimension. Note that this is true even though A is infinite dimensional.*

The following lemmas, taken directly from [DW81], concern infinite codimension.

Lemma 1.7.4. *[DW81] If all points in an ANR have infinite codimension, then so do finite dimensional subsets.*

Lemma 1.7.5. *[DW81] If $F_1, F_2, \ldots$ are closed subsets of an ANR Y such that each F_i has infinite codimension, then $F = \bigcup_{i\geq 1} F_i$ has infinite codimension.*

Lemma 1.7.6. *[DW81] Let Y be an ANR in which points have infinite codimension and let A be a closed subset of Y that can be expressed as the union of a finite dimensional set B and a set F having infinite codimension in Y. Then A has infinite codimension in Y.*

1.72 Čech Carriers

Definition 1.7.7. *[DW81] Let X be an ANR, let $U \supset V$ be open sets in X, and let $q \geq 0$. A* **Čech Carrier** *for an element $z \in H_q(U, V)$ is a compact pair $C \supset \partial C$ with $C \subset U$ and $\partial C \subset V$ such that*

$$z \in Im\{i_* : \ \check{H}_q(C, \partial C) \to H_q(U, V)\}$$

where i_* is the inclusion induced homomorphism. An ANR X is said to have the **Disjoint Čech Carriers Property** *provided for all open subsets $V_1 \subset U_1$ and $V_2 \subset U_2$ and elements $z_1 \in H_p(U_1, V_1)$ and $z_2 \in H_q(U_2, V_2)$ and for integers $p, q \geq 0$, there are Čech carriers $(C_1, \partial C_1)$ for z_1 and $(C_2, \partial C_2)$ for z_2 with $C_1 \cap C_2 = \emptyset$.*

Example 1.7.8. *Let $z \in H_q(U, V)$. Then*

$$z = \left[\sum_{i=1}^{k} n_i \sigma_i\right],$$

where each n_i is nonzero and each $\sigma_i : \ \Delta^q \to U$ is a map of the standard $q-$simplex. Let

$$C = \bigcup_{i=1}^{k} Im(\sigma_i) \quad \text{ and } \quad \partial C = \bigcup_{i=1}^{k} Im(\partial \sigma_i).$$

It is clear that $(C, \partial C) \subset (U, V)$. Also, we can see that

$$w = (z, z, z, \ldots, z, \ldots,) \in \check{H}_q(C, \partial C) = \varprojlim H_q(\mathcal{U}, \mathcal{U}_{\partial C})$$

and $i_*(w) = z$. Thus,

$$z \in Im\{i_* : \ \check{H}_q(C, \partial C) \to H_q(U, V)\}$$

which implies that $(C, \partial C)$ is a Čech carrier for z.

Definition 1.7.9. *Let X be an ANR. Then X has* **Čech carriers with infinite codimension** *if each $z \in H_q(U,V)$ has a Čech carrier $(C, \partial C)$ such that C has infinite codimension in X.*

The following Lemmas are taken from [DW81].

Lemma 1.7.10. *[DW81] A closed subset F of an ANR Y has infinite codimension if and only if, for each open pair $V \subset U$ and each $q \geq 0$, each element $z \in H_q(U,V)$ has a Čech carrier $(C, \partial C)$ with $C \cap F = \emptyset$.*

Lemma 1.7.11. *[DW81] For an ANR X, the Disjoint Čech Carriers Property is equivalent to X having Čech carriers with infinite codimension.*

Definition 1.7.12. *[Lay80] Let $A \subset X$. We say that X has* **disjoint Čech Carrier at** *A if for all open sets $U \supset V, U' \supset V'$, integers $q, q' \geq 0$, and $z \in H_q(U,V)$, $z' \in H_{q'}(U', V')$, there exist Čech carriers $(C, \partial C)$ for z and $(C', \partial C')$ for z' such that $C \cap C' \cap A = \emptyset$.*

The following Lemma and Proposition are taken from [Lay80].

Lemma 1.7.13. *[Lay80] Let Y be an ANR and let $A_1, A_2, \ldots$ be a collection of subsets of Y such that for each $k \geq 1$, Y has disjoint Čech carriers at A_k. Then Y has disjoint Čech carriers at $A = \cup_{k \geq 1} A_k$.*

Proposition 1.7.14. *[Lay80] Let Y be an ANR and let $A_1, A_2, \ldots$ be a collection of closed subsets of Y such that for each $i \geq 1$, Y has disjoint Čech carriers at A_i. Then for each open pair $U \supset V$ and integer $q \geq 0$, each $z \in H_q(U,V)$ has a Čech carrier $(C, \partial C)$ such that $C \cap (\cup_{i \geq 1} A_i)$ has infinite codimension in Y.*

Lemma 1.7.15. *[DW81] For an ANR X, the following statements are equivalent:*

(1) X satisfies the Disjoint Čech Carriers Property;

(2) X has Čech carriers of infinite codimension;

(3) X contains closed subsets $F_1, F_2, \ldots$ with each F_i having infinite codimension and each closed subset of $X - \cup F_i$ having infinite codimension;

(4) Points in X have infinite codimension and X has finite dimensional Čech carriers;

(5) $X \times I^2$ is a $Q-$manifold.

The following is the main theorem of this section. This powerful Theorem will be needed to characterize whether or not an ANR space satisfying the Disjoint Discs Property is $Q-$manifold. For more details, see [DW81], [Lay80].

Theorem 1.7.16. *[DW81] Let X be an ANR satisfying the Disjoint Discs Property. Then the following statements are equivalent:*

(1) X satisfies the Disjoint Čech Carriers Property;

(2) X has Čech carriers of infinite codimension;

(3) X contains closed subsets $F_1, F_2, \ldots$ with each F_i having infinite codimension and each closed subset of $X - \cup F_i$ having infinite codimension;

(4) Points in X have infinite codimension and X has finite dimensional Čech carriers;

(5) X is a $Q-$manifold.

2. A SPECIAL CANTOR SET IN THE HILBERT CUBE

2.1 Introduction

The standard Cantor set C in $[0,1]$ is obtained in the usual manner as $\cap_{i=1}^{\infty}\mathcal{S}_i$ where $\mathcal{S}_i = \{S(\sigma_1,\ldots,\sigma_i)\}$ is a set with 2^i elements in it. The σ_i are used to index the sets in $\mathcal{S}_i$. Here each $S(\sigma_1,\ldots,\sigma_i)$ is an interval of length $\frac{1}{3^i}$ and each σ_j is either a 1 or a 2. From this construction, each component at stage i has 2 components at stage $i+1$. Also, one can prove the following well-known theorem [Gar91].

Theorem 2.1.1. *If q is a point in C and $q = \bigcap_i S(\sigma_1,\ldots,\sigma_i)$, then $q = \sum_{j=1}^{\infty} \frac{q_j}{3^j}$ where $q_j = 2(\sigma_j - 1)$. Consequently, if the σ_i are not eventually constant, then q is not a triadic rational.*

Note that a triadic rational is a rational number that can be expressed as a fraction with denominator of the form 3^n. Also note that $C^k \subset I^k$, the product of k copies of the Cantor set is again a Cantor set.

In [Gar91], Garity showed a construction of a different sequence $\mathcal{S}_i$ for the Cantor set $C^k \subset I^k$ for $k \geq 2$.. In the construction, for technical reasons, he described a process where each component at stage i has 4 components at stage $i+1$ instead of 2 components and proved the following.

Theorem 2.1.2. *[Gar91] There exists a sequence $\mathcal{S}_i$ of collections of subsets of I^k so that $C^k = \bigcap_{i=1}^{\infty} \mathcal{S}_i$ and each component of $\mathcal{S}_i$ contains exactly 4 components of $\mathcal{S}_{i+1}$. If a point p in C^k is associated with a sequence $(\epsilon_1,\ldots,\epsilon_i,\ldots)$ where $\epsilon_n = (i_n, j_n)$, for i_n and j_n in $\{1,2\}$, and if there is N such that for all $i > N$ the first coordinates of the ϵ_i alternate, or the second coordinates of the ϵ_i alternate, then p has no triadic rational coordinates.*

For the construction and the proof of Theorem 2.1.2, see § 2.21.

Our main work in this Chapter is to generalize Theorem 2.1.2 to the infinite dimensional case. That is, we will construct and prove the following Theorem, in § 2.22.

Theorem 2.1.3. *There exists a sequence*

$$\mathcal{M} = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots\}$$

of collection subsets of Q satisfying the following properties:

- $\bigcap \mathcal{M} = C^\infty$ *where* $C^\infty = \prod_{i=1}^\infty C$ *and C is the Cantor set.*
- *Every point $p \in C^\infty$ is associated with a sequence $(\epsilon_1, \epsilon_2, \ldots)$ where $\epsilon_n = (i_n, j_n)$, for i_n and j_n in $\{1, 2\}$,*
- *If there is N such that for all $i > N$ the first coordinates of the ϵ_i alternate, or the second coordinates of the ϵ_i alternate, then p has no triadic rational coordinates.*

Also, in the next Chapter, we need constructions of the Cantor set in $[2, 3] \times Q_2$ and in $[-3, -2] \times Q_2$. This will be combined with the construction of Cantor set in the middle, $[-1, 1] \times Q_2$ to produce a decomposition of Q. We will show such constructions in sections 2.3 and 2.4

2.2 Cantor set in the Middle

2.21 Construction of the generalized Cantor set: C^k

Fixed k, we will use the standard representation of C to specifying the $k-$cells indexed as $N(\epsilon_1, \epsilon_2, \ldots, \epsilon_i)$ used in the description of C^k and then in the next section, we will generalize this idea to elements used in the description of C^∞.

From now, let $\epsilon_n = (i_n, j_n)$ where $i_n, j_n \in \{1, 2\}$ and

$$S[\epsilon_1, \epsilon_2, \ldots, \epsilon_n] = S(i_1, j_1, i_2, j_2, \ldots, i_n, j_n).$$

Let $\{4^{S_1}, 4^{S_2}, 4^{S_3}, \ldots\}$ be a sequence where $S_i = \sum_{j=1}^{i} kn_j$ and n_j is an even integer . The specific choice of k and n_j will become clear later. We want to choose a defining sequence $\mathcal{N} = \{\mathcal{N}_1, \mathcal{N}_2, \ldots\}$ in I^k in such a way that:

1. $\mathcal{N}_i$ has 4^{S_i} elements, each element of the form $S[\epsilon_1, \epsilon_2, \ldots, \epsilon_i]$;

2. $\bigcap \mathcal{N} = C^k$ is a Cantor set;

3. Every point p in C^k is associated with a sequence $(\epsilon_1, \epsilon_2, \epsilon_3, \ldots)$;

4. If there is some stage past which either the first coordinates of the ϵ_n alternate, or the second coordinates of the ϵ_i alternate, then p has no triadic rational coordinates.

To do this, let $\mathcal{N}_0 = \left\{I^k = I_1 \times I_2 \times \ldots \times I_k\right\}$. That is, $\mathcal{N}_0$ has a single element.

Let $A_0^1 = \left\{I^k = I_1 \times I_2 \times \ldots \times I_k\right\}$. The set A_1^1 will be obtained from A_0^1 by subdividing the first interval factor of each element into 4 equal subintervals. That is, each component of A_1^1 is of the form

$$N(\epsilon_1) = S[\epsilon_1] \times I_2 \times I_3 \times \ldots \times I_k$$

Next, the set A_2^1 will be obtained from A_1^1 by subdividing the second interval factor into 4 equal subintervals. That is, each component of A_2^1 is of the form

$$N(\epsilon_1, \epsilon_2) = S[\epsilon_1] \times S[\epsilon_2] \times I_3 \times \ldots \times I_k.$$

This process will continue until the first k factors have each been subdivided into 4 equal subintervals. The process will then continue by subdividing the kth factor again and then working backwards towards the first factor. From this process, after $2k$ times, we have

$$N(\epsilon_1) = S[\epsilon_1] \times I_2 \times I_3 \times \ldots \times I_k \in A_1^1,$$

$$N(\epsilon_1, \epsilon_2) = S[\epsilon_1] \times S[\epsilon_2] \times I_3 \times \ldots \times I_k \in A_2^1,$$

$$N(\epsilon_1, \epsilon_2, \epsilon_3) = S[\epsilon_1] \times S[\epsilon_2] \times S[\epsilon_1] \times \ldots \times I_k \in A_3^1,$$

$$N(\epsilon_1, \epsilon_2, \ldots, \epsilon_k) = S[\epsilon_1] \times S[\epsilon_2] \times \ldots \times S[\epsilon_1] \in A_k^1,$$

$$N(\epsilon_1, \ldots, \epsilon_k, \epsilon_{k+1}) = S[\epsilon_1] \times \ldots \times S[\epsilon_k, \epsilon_{k+1}] \in A_{k+1}^1,$$

$$N(\epsilon_1, \ldots, \epsilon_k, \epsilon_{k+1}, \epsilon_{k+2}) = S[\epsilon_1] \times \ldots \times S[\epsilon_{k-1}, \epsilon_{k+2}] \times S[\epsilon_k, \epsilon_{k+1}] \in A_{k+2}^1,$$

$$N(\epsilon_1, \ldots, \epsilon_k, \epsilon_{k+1}, \ldots, \epsilon_{2k}) = S[\epsilon_1, \epsilon_{2k}] \times \ldots \times S[\epsilon_{k-1}, \epsilon_{k+2}] \times S[\epsilon_k, \epsilon_{k+1}] \in A_{2k}^1$$

The process will be repeated again until we have $A_{kn_1}^1$. That is, we will repeat this process $\frac{n_1}{2}$ times where n_1 is even. Thus let $\mathcal{N}_1 = A_{kn_1}^1$. In fact, each element of $\mathcal{N}_1$ is of the form

$$N(\epsilon_1, \ldots, \epsilon_{S_1}) = \prod_{i=1}^{k} S[(\epsilon_{i_l})]$$

where (ϵ_{i_l}) is the subsequence of $E = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_1}\}$, $i = 1, 2, \ldots, k$, $l = 1, 2, \ldots, n_1$, and for each i, the index

$$i_l = \begin{cases} 2k - i + 1 + (l-2)k & \text{if } l \text{ even} \\ i + (l-1)k & \text{if } l \text{ odd.} \end{cases}$$

To get $\mathcal{N}_2$, we will consider $A_0^2 = \mathcal{N}_1$. The set A_1^2 will be obtained from A_0^2 by subdividing the first interval factor of each element into 4 equal subintervals. That is. the element of the set A_1^2 are of the form

$$N(\epsilon_1, \ldots, \epsilon_{S_1}, \epsilon_{S_1+1}) = S[(\epsilon_{i_l}), \epsilon_{S_1+1}] \times \prod_{i=2}^{k} S[(\epsilon_{i_l})]$$

Next, the set A_2^2 will be obtained from A_1^2 by subdividing the second interval factor into 4 equal subintervals. That is the element of the set A_2^2 are of the form

$$N(\epsilon_1, \ldots, \epsilon_{S_1}, \epsilon_{S_1+1}, \epsilon_{S_1+2}) = S[(\epsilon_{1_l}), \epsilon_{S_1+1}] \times S[(\epsilon_{2_l}), \epsilon_{S_1+2}] \times \prod_{i=3}^{k} S[(\epsilon_{i_l})].$$

This process will continue until each interval in the first k factors has been subdivided into 4 equal subintervals. The process will then continue by subdividing the kth factor again and then working backwards towards the first factor. For convenience, for each n, let

$$\overline{\epsilon}_n = \left(\epsilon_{S_{n-1}+1}, \epsilon_{S_{n-1}+2}, \ldots, \epsilon_{S_n}\right),$$

Again, we can see from this process that, after $2k$ times,

$$N(\overline{\epsilon}_1, \epsilon_{S_1+1}) = S[(\epsilon_{1_l}), \epsilon_{S_1+1}] \times \prod_{i=2}^{k} S[(\epsilon_{i_l})] \in A_1^2,$$

$$N(\overline{\epsilon}_1, \epsilon_{S_1+1}, \epsilon_{S_1+2}) = S[(\epsilon_{1_l}), \epsilon_{S_1+1}] \times S[(\epsilon_{2_l}), \epsilon_{S_1+2}] \times \prod_{i=3}^{k} S[(\epsilon_{i_l})] \in A_2^2,$$

$$N(\overline{\epsilon}_1, \epsilon_{S_1+1}, \ldots, \epsilon_{S_1+k}) = \prod_{i=1}^{k} S[(\epsilon_{i_l}), \epsilon_{S_1+i}] \in A_k^2,$$

$$N(\overline{\epsilon}_1, \epsilon_{S_1+1} \ldots, \epsilon_{S_1+k}, \epsilon_{S_1+k+1}) = \prod_{i=1}^{k-1} S[(\epsilon_{i_l}), \epsilon_{S_1+i}] \times S[(\epsilon_{k_l}), \epsilon_{S_1+k}, \epsilon_{S_1+k+1}] \in A_{k+1}^2,$$

$$N(\overline{\epsilon}_1, \epsilon_{S_1+1}, \ldots, \epsilon_{S_1+2k}) = \prod_{i=1}^{k} S[(\epsilon_{i_l}), \epsilon_{S_1+i}, \epsilon_{S_1+2k-i+1}] \in A_{2k}^2.$$

The process will be repeated again until we have $A_{kn_2}^2$. That is, we will repeat this process $\frac{n_2}{2}$ times. Then let $\mathcal{N}_2 = A_{kn_2}^2$. Then $\mathcal{N}_2$ has 4^{S_2} elements and each element is of the form

$$N(\epsilon_1, \ldots, \epsilon_{S_2}) = \prod_{i=1}^{k} S[(\epsilon_{i_l})]$$

where (ϵ_{i_l}) is the subsequence of $E = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_2}\}$, $i = 1, 2, \ldots, k$, $l = 1, 2, \ldots, \sum_{i=1}^{2} n_i$, and for each i, the index is given by

$$i_l = \begin{cases} 2k - i + 1 + (l-2)k & \text{if } \ l \ \text{ even} \\ i + (l-1)k & \text{if } \ l \ \text{ odd.} \end{cases}$$

For $m \geq 3$, we inductively define $\mathcal{N}_m$ as follows. Inductively assume we have $\mathcal{N}_m$

with 4^{S_m} elements. And assume that each element of $\mathcal{N}_m$ is of the form

$$N(\epsilon_1, \ldots, \epsilon_{S_m}) = \prod_{i=1}^{k} S[(\epsilon_{i_l})]$$

where (ϵ_{i_l}) is the subsequence of $E = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_m}\}$, $i = 1, 2, \ldots, k$, $l = 1, 2, \ldots, \sum_{j=1}^{m} n_j$, and for each i, the index is given by

$$i_l = \begin{cases} 2k - i + 1 + (l-2)k & \text{if } l \text{ even} \\ i + (l-1)k & \text{if } l \text{ odd} \end{cases}.$$

To define $\mathcal{N}_{m+1}$, let $A_0^{m+1} = \mathcal{N}_m$. The set A_1^{m+1} will be obtained from A_0^{m+1} by subdividing the first interval factor of each element into 4 equal subintervals. That is the element of A_1^{m+1} is of the form

$$N(\bar{\epsilon}_1, \ldots, \bar{\epsilon}_m, \epsilon_{S_m+1})$$

Next, the set A_2^{m+1} will be obtained from A_1^{m+1} by subdividing the second interval factor into 4 equal subintervals. That is, the element of the set A_2^{m+1} are of the form

$$N(\bar{\epsilon}_1, \ldots, \bar{\epsilon}_m, \epsilon_{S_m+1}, \epsilon_{S_m+2}).$$

This process will continue until the first k factors have each been subdivided into 4 equal subintervals. The process will then continue by subdividing the kth factor again and then working backwards towards the first factor. After this, the process will be repeated again until we have $A_{kn_{m+1}}^{m+1}$. That is, we will repeat this process $\frac{n_{m+1}}{2}$ times. Thus, let $\mathcal{N}_{m+1} = A_{kn_{m+1}}^{m+1}$ which has $4^{S_{m+1}}$ elements, and each element of the set $\mathcal{N}_{m+1}$ is of the form

$$N(\epsilon_1, \ldots, \epsilon_{S_{m+1}}) = \prod_{i=1}^{k} S[(\epsilon_{i_l})]$$

where (ϵ_{i_l}) is the subsequence of $E = \left\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_{m+1}}\right\}$, $i = 1, 2, \ldots, k$, $l = 1, 2, \ldots, \sum_{j=1}^{m+1} n_j$, and for each i, the index is given by

$$i_l = \begin{cases} 2k - i + 1 + (l-2)k & \text{if } l \text{ even} \\ i + (l-1)k & \text{if } l \text{ odd.} \end{cases}$$

From the above construction, we have the following facts:

Remark 2.2.1. *1. $\mathcal{N} = \{\mathcal{N}_1, \mathcal{N}_2, \ldots\}$ is a defining sequence,*

2. $\bigcap_i \mathcal{N}_i = C^k$, and

3. each point $p = (p_1, \ldots, p_k)$ in C^k corresponds to a sequence, say $E = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{i-1}, \epsilon_i, \ldots)$.

4. for each $1 \leq i \leq k$, $\pi^i(p) = p_i$ corresponds to a subsequence of E, say E_i, where π^i is the projection onto the i^{th} coordinate of p. Indeed, we can find the subsequence $E_i = (\epsilon_{i_l})_{l=1}^{\infty}$ from the Table 2.1.

i^{th} coor. of p	$E_i = (\epsilon_{i_l})_{l=1}^{\infty}$					
1	ϵ_1	ϵ_{2k}	$\cdots$	ϵ_{S_1-2k+1}	ϵ_{S_1}	$\cdots$
2	ϵ_2	ϵ_{2k-1}	$\cdots$	ϵ_{S_1-2k+2}	ϵ_{S_1-1}	$\cdots$
3	ϵ_3	ϵ_{2k-2}	$\cdots$	ϵ_{S_1-2k+3}	ϵ_{S_1-2}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$k-1$	ϵ_{k-1}	ϵ_{k+2}	$\cdots$	ϵ_{S_1-k-1}	ϵ_{S_1-k+2}	$\cdots$
k	ϵ_k	ϵ_{k+1}	$\cdots$	ϵ_{S_1-k}	ϵ_{S_1-k+1}	$\cdots$

TABLE 2.1: Subsequence of E_i corresponding to i^{th} coordinate of p in I^k

From Table 2.1, the subsequence E_i can be written as: $E_i = (\epsilon_{i_l})_{l=1}^{\infty}$ where the index is given by

$$i_l = \begin{cases} 2k - i + 1 + (l-2)k & \text{if } l \text{ even} \\ i + (l-1)k & \text{if } l \text{ odd} \end{cases}.$$

5. The number of stages strictly between the consecutive elements in each E_i is even.

Next, we will show some examples to illustrate certain elements in some stages of this construction.

EXAMPLE 2.2.2. *Let* $k = 2$. *Then*

$A_0^1 = \{I_1 \times I_2\}$*(See Figure 2.1),*

$A_1^1 = \{S[\epsilon_1] \times I_2\}$*(See Figure 2.2(a)),*

$A_2^1 = \{S[\epsilon_1] \times S[\epsilon_2]\}$*(See Figure 2.2(b))),*

$A_3^1 = \{S[\epsilon_1] \times S[\epsilon_2, \epsilon_3]\}$*(See Figure 2.2(c)),*

$A_4^1 = \{S[\epsilon_1, \epsilon_4] \times S[\epsilon_2, \epsilon_3]\}$*(See Figure 2.2(d)).*

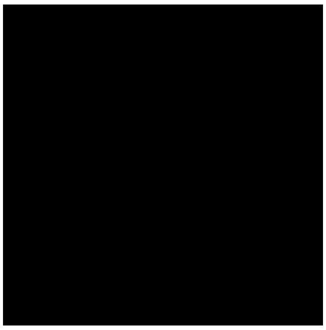


FIGURE 2.1: The element in A_0^1

Also, if

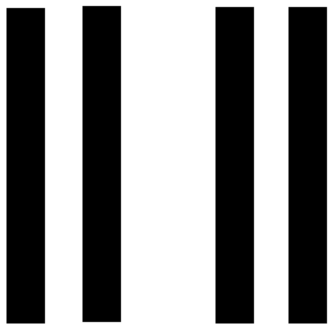
$$\begin{aligned} p = (p_1, p_2) &= \bigcap_j N(\epsilon_1, \ldots, \epsilon_{S_j}) \\ &= N(\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_j}, \ldots) \\ &= S[\epsilon_1, \epsilon_4, \epsilon_5, \ldots] \times S[\epsilon_2, \epsilon_3, \epsilon_6, \ldots], \end{aligned}$$

then

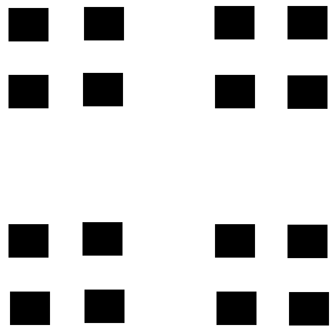
$$E_1 = (\epsilon_1, \epsilon_4, \epsilon_5, \ldots) = (i_1, j_1, i_4, j_4, i_5, j_5, \ldots),$$

$$E_2 = (\epsilon_2, \epsilon_3, \epsilon_6, \ldots) = (i_2, j_2, i_3, j_3, i_6, j_6, \ldots)$$

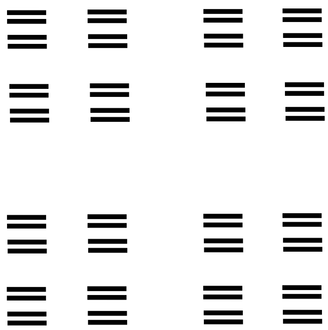
correspond to p_1 *and* p_2*, respectively. Furthermore, assume there exists a positive integer* N *so that: for all* $i \geq 0$, $\epsilon_{N+2i} = (1, j_{N+2i})$ *and* $\epsilon_{N+2i+1} = (2, j_{N+2i+1})$*, then the first coordinates of the elements in each* E_i *after the stage* N *must alternate between ones and twos. This implies that* p *has no triadic rational coordinates. Similarly, if there exists a*



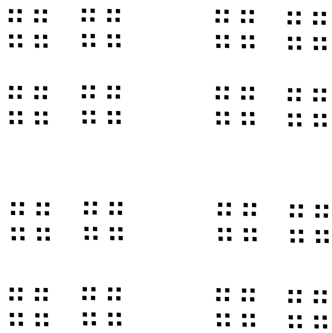
(a) elements in A_1^1



(b) elements in A_2^1



(c) elements in A_3^1



(d) elements in A_4^1

FIGURE 2.2: elements in $A_i^1, i = 1, 2, 3, 4$

positive integer N *so that: for all* $i \geq 0$, $\epsilon_{N+2i} = (j_{N+2i}, 1)$ *and* $\epsilon_{N+2i+1} = (j_{N+2i+1}, 2)$, p *then has no triadic rational coordinates.*

From Example 2.2.2, we can generalize this to any $k \geq 1$ so that the following Theorem is true. See [Gar91] for more details.

Theorem 2.2.3. *Let* $\mathcal{N} = \{\mathcal{N}_1, \mathcal{N}_2, \ldots\}$ *be defined as above. Then* $\mathcal{N}$ *is a defining sequence in* I^k *so that the* $C^k = \bigcap \mathcal{N}$ *is a Cantor set and if* $p = \bigcap_{i=1}^{\infty} N(\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_i})$ *where* $S_i = \sum_{j=1}^{i} kn_j$ *and* n_j *is even and if there is* N *so that: either for all* $i \geq 0$, $\epsilon_{N+2i} = (1, j_{N+2i})$ *and* $\epsilon_{N+2i+1} = (2, j_{N+2i+1})$, *or for all* $i \geq 0$, $\epsilon_{N+2i} = (j_{N+2i}, 1)$ *and* $\epsilon_{N+2i+1} = (j_{N+2i+1}, 2)$, *then* p *has no triadic rational coordinates.*

Proof. It is obvious that $\mathcal{N}$ is a defining sequence and

$$p = \bigcap_{i=1}^{\infty} N(\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_i})$$

is in C^k. Since either the first or the second coordinates of the ϵ_α from the sequence of p alternate between ones and twos past stage N and the number of stages between the consecutive elements in each E_i is even by the Remark 2.2.1(5), it follows that for each $1 \leq i \leq k$, the first or the second coordinates of the ϵ_{i_l} also alternate between ones and twos. This implies that each coordinate of p does not have a triadic expansion that is eventually constant and hence p has no triadic rational coordinates. $\square$

2.22 Construction of the generalized Cantor set: C^∞

Next we will use the idea of the previous section to construct a sequence $\{\mathcal{M}_i\}$ in $Q = I^3 \times Q_4$ so that:

1. Each $\mathcal{M}_i$ has 4^{S_i} elements, where $S_i = \sum_{j=1}^{i}(j+1)n_j$ and n_j is even, and each element of $\mathcal{M}_i$ is of the form $\{0\} \times S[\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_i}] \times Q_{i+3}$ where $S[\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_i}] \subset I_2 \times \cdots \times I_{i+2}$;

2. $\bigcap \mathcal{M} = \{0\} \times C^\infty$ is a Cantor set;

3. Every point p in C^∞ is associated with a sequence $(\epsilon_1, \epsilon_2, \epsilon_3, \ldots)$;

4. If there is some stage past which either the first coordinates of the ϵ_n alternate, or the second coordinates of the ϵ_i alternate, then p has no triadic rational coordinates.

We start the construction viewing the copy of Q as $B^1 \times I_1' \times I_2' \times Q_4$, where for each i, $I_i' = I_{i+1}$. Let $\mathcal{M}_0 = \{\{0\} \times I_1' \times I_2'\} \times Q_4$ be the starting element of the sequence.

To find $\mathcal{M}_1$, consider the set $A_0^1 = \{I_1' \times I_2'\}$. The set A_{1k}^1 will be obtained from A_0^1 by subdividing the first interval factor into 4 equal subintervals. The set A_2^1 will be obtained from A_1^1 by subdividing the second interval factor into 4 equal subintervals.

This process will continue until the last interval factor has been subdivided into 4 equal subintervals. The process will then continue by subdividing the last interval factor again and then working backwards towards the first factor. We can see that, after 4 times, we have

$$\begin{aligned}
&M(\epsilon_1) = S[\epsilon_1] \times I_2' \in A_1^1,\\
&M(\epsilon_1, \epsilon_2) = S[\epsilon_1] \times S[\epsilon_2] \in A_2^1,\\
&M(\epsilon_1, \epsilon_2, \epsilon_3) = S[\epsilon_1] \times S[\epsilon_2, \epsilon_3] \in A_3^1,\\
&M(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = S[\epsilon_1, \epsilon_4] \times S[\epsilon_2, \epsilon_3] \in A_4^1,
\end{aligned}$$

The process will be repeated again until we have $A^1_{2n_1}$. That is, we will repeat this process $\frac{n_1}{2}$ times. Thus, let

$$\mathcal{M}_1 = \{\{0\} \times M(\epsilon_1, \epsilon_2, \ldots, \epsilon_{2n_1})\} \times Q_4$$

where

$$M(\epsilon_1, \ldots, \epsilon_{S_1}) = \prod_{i=1}^{2} S[(\epsilon_{i_l})]$$

where (ϵ_{i_l}) is a subsequence of $E = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_1}\}$, $i = 1, 2$, $l = 1, 2, \ldots, n_1$, and for each i, the index from Table 2.2 is given by

$$i_l = \begin{cases} 2l - i + 1 & \text{if } l \text{ even} \\ 2l + i - 1 & \text{if } l \text{ odd} \end{cases}.$$

So, we can see that there are 4^{S_1} elements in $\mathcal{M}_1$.

For the element $\mathcal{M}_2$ of the sequence, we consider

$$A_0^2 = A^1_{2n_1} \times I_3' = \left\{ \prod_{i=1}^{2} S[(\epsilon_{i_l})] \times I_3' \right\},$$

where (ϵ_{i_l}) is the subsequence of $E = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_1}\}$, $i = 1, 2$, $l = 1, 2, \ldots, n_1$, and for each i,

$$i_l = \begin{cases} 2l - i + 1 & \text{if } l \text{ even} \\ 2l + i - 1 & \text{if } l \text{ odd} \end{cases}.$$

The set A_1^2 will be obtained from A_0^2 by subdividing the first interval factor into 4 equal subintervals. The set A_2^2 will be obtained from A_1^2 by subdividing the second interval factor into 4 equal subintervals. This process will continue until the last interval factor has been subdivided into 4 equal subintervals. The process will then continue by subdividing the last interval factor again and then working backwards towards the first factor. So, after 6 times, we have

$$M(\bar{\epsilon}_1, \epsilon_{S_1+1}) = S[(\epsilon_{1_l}), \epsilon_{S_1+1}] \times S[(\epsilon_{i_l})] \times I_3' \in A_1^2,$$

$$M(\bar{\epsilon}_1, \epsilon_{S_1+1}, \epsilon_{S_1+2}) = S[(\epsilon_{1_l}), \epsilon_{S_1+1}] \times S[(\epsilon_{2_l}), \epsilon_{S_1+2}] \times I_3' \in A_2^2,$$

$$M(\bar{\epsilon}_1, \epsilon_{S_1+1}, \ldots, \epsilon_{S_1+3}) = \prod_{i=1}^{2} S[(\epsilon_{i_l}), \epsilon_{S_1+i}] \times S[\epsilon_{S_1+3}] \in A_3^2,$$

$$M(\bar{\epsilon}_1, \epsilon_{S_1+1} \ldots, \epsilon_{S_1+3}, \epsilon_{S_1+4}) = \prod_{i=1}^{2} S[(\epsilon_{i_l}), \epsilon_{S_1+i}] \times S[\epsilon_{S_1+3}, \epsilon_{S_1+4}] \in A_4^2,$$

$$M(\bar{\epsilon}_1, \epsilon_{S_1+1}, \ldots, \epsilon_{S_1+5}) = S[(\epsilon_{1_l}), \epsilon_{S_1+1}] \times S[(\epsilon_{2_l}), \epsilon_{S_1+2}, \epsilon_{S_1+5}] \times S[\epsilon_{S_1+3}, \epsilon_{S_1+4}] \in A_5^2$$

$$M(\bar{\epsilon}_1, \epsilon_{S_1+1}, \ldots, \epsilon_{S_1+6}) = \prod_{i=1}^{2} S[(\epsilon_{i_l}), \epsilon_{S_1+i}, \epsilon_{S_1+7-i}] \times S[\epsilon_{S_1+3}, \epsilon_{S_1+4}] \in A_5^2.$$

The process will be repeated again until we have $A_{3n_2}^2$. That is, we will repeat this process $\frac{n_2}{2}$ times. Thus, let

$$\mathcal{M}_2 = \{\{0\} \times M(\epsilon_1, \ldots, \epsilon_{S_1}, \ldots, \epsilon_{S_2})\} \times Q_5.$$

Then $\mathcal{M}_2$ has 4^{S_2} elements and each element is of the form

$$\{0\} \times M(\epsilon_1, \ldots, \epsilon_{S_2}) \times Q_5 = \{0\} \times \prod_{i=1}^{3} S\left[(\epsilon_{i_l})_{l>n_{i-2}}^{\infty}\right] \times Q_5$$

where (ϵ_{i_l}) is the subsequence of $E = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_2}\}$, $i = 1, 2, 3$, $l = 1, 2, \ldots, \sum_{i=1}^{2} n_i$, and assume that $n_{-1} = 0 = n_0$ and $S_0 = 0$.

If $l = 1, \ldots, n_1$, then the index for $i = 1, 2$

$$i_l = \begin{cases} 2l - i + 1 & \text{if } l \text{ even} \\ 2l + i - 2 & \text{if } l \text{ odd} \end{cases}$$

and if $l = n_1 + 1, \ldots, n_1 + n_2$, then the indice for $i = 1, 2, 3$

$$i_l = \begin{cases} S_1 + (l - n_1)3 - i + 1 & \text{if } l \text{ even} \\ S_1 + i + 3(l - n_1 - 1) & \text{if } l \text{ odd} \end{cases}$$

So, we can see that there are 4^{S_2} elements in $\mathcal{M}_2$.

Using induction and the same process above to obtain the following Remark 2.2.4.

Remark 2.2.4. *From this construction, we see that:*

1. $\mathcal{M} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ *is a defining sequence.*

2. $\bigcap_i \mathcal{M}_i = \{0\} \times C^\infty$

3. *Each point* $p = \{0\} \times p'$ *where* $p' = (p_1, p_2, \ldots, p_k, \ldots)$ *in* $\{0\} \times C^\infty$ *corresponds to a sequence, say* $E = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{i-1}, \epsilon_i, \ldots)$. *Furthermore, for each* $i \geq 1$ $\pi^i(p') = p_i$ *corresponds to a subsequence of* E, *say* E_i *Indeed, we can find the subsequence* $E_i = (\epsilon_{i_l})_{l > n_{i-2}}^\infty$ *from the following table. From the table, we can write the subsequence* E_i *into the general form:* $E_i = (\epsilon_{i_l})_{l > n_{i-2}}^\infty$ *where for all* $j \in \mathbb{N} \cup \{0\}$, *for* $n_j < l \leq n_{j+1}$ *with* $n_{-1} = 0 = n_0, S_0 = 0$, *the indice*

$$i_l = \begin{cases} S_j + (j+2)(l - n_j) - i + 1 & \textit{if } l \textit{ even} \\ S_j + i + (j+2)(l - n_j - 1) & \textit{if } l \textit{ odd} \end{cases}.$$

4. *The number of stages between the consecutive elements in each* E_i *is even. This guarantees that if there exists an* N *so that: either for all* $r \geq 0$, $\epsilon_{N+2r} = (1, j_{N+2r})$ *and* $\epsilon_{N+2r+1} = (2, j_{N+2r+1})$, *or for all* $r \geq 0$, $\epsilon_{N+2r} = (j_{N+2r}, 1)$ *and* $\epsilon_{N+2r+1} =$

i^{th}co. of p'	$E_i = (\epsilon_{i_l})_{l>n_{i-2}}^{\infty}$									
1	ϵ_1	ϵ_4	$\ldots$	ϵ_{S_1}	ϵ_{S_1+1}	ϵ_{S_1+6}	$\cdots$	ϵ_{S_2-5}	ϵ_{S_2}	$\cdots$
2	ϵ_2	ϵ_3	$\ldots$	ϵ_{S_1-1}	ϵ_{S_1+2}	ϵ_{S_1+5}	$\cdots$	ϵ_{S_2-4}	ϵ_{S_2-1}	$\cdots$
3					ϵ_{S_1+3}	ϵ_{S_1+4}	$\cdots$	ϵ_{S_2-3}	ϵ_{S_1-2}	$\cdots$
$\vdots$										

TABLE 2.2: Subsequence of E_i corresponding to i^{th} coordinate of p in Q

$(j_{N+2r+1}, 2)$, *then for each* $i \geq 1$ *either the first coordinates of the* ϵ_{i_l} *in* E_i *must alternate, or the second coordinates of the* ϵ_{i_l} *in* E_i *must alternate. This again implies that p has no triadic rational coordinates. So, one has the following Theorem.*

Theorem 2.2.5. *Let* $\mathcal{M} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ *be defined as above. Then* $\mathcal{M}$ *is a defining sequence in* Q *so that the* $\bigcap \mathcal{M} = \{0\} \times C^\infty$ *is a Cantor set and if* $p = \{0\} \times p'$ *where* $p' = \cap_{i=1}^{\infty} M(\epsilon_1, \epsilon_2, \ldots, \epsilon_{S_i})$, $S_i = \sum_{j=1}^{i}(j+1)n_j$, *and* n_j *is even and if there exists an* N *so that: either for all* $i \geq 0$, $\epsilon_{N+2i} = (1, j_{N+2i})$ *and* $\epsilon_{N+2i+1} = (2, j_{N+2i+1})$, *or for all* $i \geq 0$, $\epsilon_{N+2i} = (j_{N+2i}, 1)$ *and* $\epsilon_{N+2i+1} = (j_{N+2i+1}, 2)$, *then* p' *has no triadic rational coordinates.*

Proof. The point p is obviously in $\{0\} \times C^\infty$. By assumption and by the Remark 2.2.4(4), one can show that none of the coordinates of p' have a triadic expansion that is eventually constant. Thus, this implies that p' has no triadic rational coordinates. $\square$

Remark 2.2.6. *This result is new and generalizes a previous result about Cantor sets in* I^k *to Cantor sets in* Q.

2.3 Cantor set on the top ($[2,3] \times Q_2 \subset Q$)

Before proceeding with the construction in this section, we need some additional results about geometric centrality that can be obtained by using a method known as ramification. This is described below.

Let $A = B^2 \times X_1 \times X_2 \times \cdots X_n$, where $X_i = I$ or S^1. Let $C = \{C_1, C_2 \ldots, C_k\}$, where each $C_i \cong B^2 \times Y_1 \times Y_2 \times \cdots Y_n$ with $Y_i = I$ or S^1, be a finite collection of disjoint subsets of A such that C is geometrically central in A. For each i, in order to ramify $C_i = B^2 \times Y_1 \times Y_2 \times \cdots Y_n$, we take a finite number of subdiscs $D_1, D_2, \ldots, D_n$ on B^2. Then each ramified copy $D_j \times Y_1 \times Y_2 \times \cdots Y_n$ of C_i is called a **parallel interior manifold**, see [DG82]. We will use the following lemma about parallel interior manifolds in many places.

Lemma 2.3.1. *[DG82] Let $M = B^2 \times N$ be an $m-$manifold. Let $D_1, D_2, \ldots, D_n$ be disjoint subdiscs in B^2. Then each parallel manifold $D_i \times N$ of M is geometrically central in M.*

We will modify the construction below of the Cantor set by ramifying the manifolds in the defining sequence and modifying the choice of factors so as to get a different and more useful way of determining when $p \in C^\infty$ has no triadic rational coordinates. This will be needed in the next Chapter.

We will construct a Cantor set in $[2,3] \times Q_2 \subset Q$ by constructing a defining sequence, $\mathcal{T} = \{\mathcal{T}_1, \mathcal{T}_2, \ldots\}$ in $[2,3] \times Q_2 \subset Q$, in such a way that

$$\mathcal{T}_1, \mathcal{T}_2 \subset [2,3] \times I^2 \times Q_4$$

$$\mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5 \subset [2,3] \times I^3 \times Q_5$$

and in general for $k > 2$,

$$\mathcal{T}_{\frac{k(k-1)}{2}+i} \subset [2,3] \times I^k \times Q_{k+2}$$

for $i = 0, \ldots, k-1$ and so that $\bigcap \mathcal{T}_i$ is a Cantor set.

2.31 Zero and First Stages of Construction: $k = 2$

$\underline{\mathcal{T}_0}$: Let $\mathcal{T}_0 = \mathcal{W}_0 \times Q_4$ where

$$\mathcal{W}_0 = \left\{ W_0 = B^2 \times I_2 \right\}$$

and $B^2 \subset [2,3] \times I_3 \subset I_1 \times I_3$.

$\underline{\mathcal{T}_1 :}$ By Lemma 1.5.7, there is a geometrically central family

$$\mathcal{W}_{11} = \{W_1, W_2, \ldots, W_{2^{n_1}}\},$$

and $W_i \cong B^2 \times X_1$ with $X_1 \cong I$ or S^1, are disjoint subsets of W_0 such that the diameter of W_i in I^3 is less than $\frac{1}{2}$ for every $i = 1, \ldots, 2^{n_1}$ with $2^{n_1} \geq 4$.

Let $\mathcal{W}_1 = W_{11}$ and ramify each component of $\mathcal{W}_1$ 2^{n_1} times and denote each ramified element by $T(\epsilon_1)$ where $\epsilon_1 = (\sigma_{11}, \sigma_{12})$ with

$$\sigma_{1i} = (i_1, i_2, \ldots, i_{n_1})\,; \quad i_l \in \{1, 2\}\,.$$

Then let

$$\mathcal{T}_1 = \mathcal{W}^1 \times Q_4,$$

where $\mathcal{W}^1 = \{T(\epsilon_1)\}$. Figure 2.3 demonstrates the zero stage and the first stage of construction in $[2,3] \times I^2$ in the case of 2 times ramification of a $B^2 \times I$.

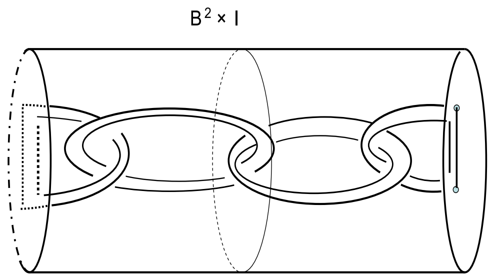


FIGURE 2.3: Ramified copies of component of $\mathcal{W}_1$ on $B^2 \times I$

We observe the following Remark [Gar91] on the first construction:

Remark 2.3.2. *If $f : H \to T = B^2 \times I$ is any virtually $I-$essential map of a disc with holes H, then there exists an σ'_{11} so that f is virtually $I-$essential with respect to all $T((\sigma'_{11}, \sigma_{12}))$. Moreover, if $f : H \to T$ is any virtually $I-$essential map of a disc with holes H and the choice of σ'_{11} is fixed, then there exists a map $g : H \to T$ so that $g\Big|_{\partial H} = f\Big|_{\partial H}$ and $g(H) \cap T((\sigma_{11}, \sigma_{12})) \neq \emptyset$ if and only if $\sigma_{11} = \sigma'_{11}$.*

For $\mathcal{T}_2$, consider $T(\epsilon_1) \in \mathcal{W}^1$. By Lemma 1.5.7 there is a geometrically central family

$$\mathcal{W}_{2i} = \left\{W_{1i}, W_{2i}, \ldots, W_{2^{n_2)}i}\right\},$$

where $W_{ji} \cong B^2 \times X_1$ with $X_1 \cong I$ or S^1, are disjoint subsets of W_i such that the diameter of W_{ji} is less than $\frac{1}{2^2}$ in $[2,3] \times I^2$ for every $j = 1, \ldots, 2^{n_2}$ with $2^{n_2} \geq 4$. Let $\mathcal{W}_2 = \cup \mathcal{W}_{2i}$.

Let us ramify each element in $\mathcal{W}_2$ 2^{n_2} times and denote each ramified element by $T(\epsilon_1, \epsilon_2)$ Thus, $\mathcal{T}_2 = \mathcal{W}^2 \times Q_5$, where $\mathcal{W}^2 = \{T(\epsilon_1, \epsilon_2)\}$ and $\epsilon_l = (\sigma_{l1}, \sigma_{l2})$ with

$$\sigma_{li} = (i_1, i_2, \ldots, i_{n_2})\,; \quad l = 1, 2; i_k, i \in \{1, 2\}\,; i = 1, 2.$$

2.32 k^{th} Stage of Construction :$k \geq 3$

For $k \geq 3$, we use induction to describe $\mathcal{T}_{\frac{(k-1)(k)}{2}+i}$ for $i = 0, \ldots, k-1$ as follows. Let $p = \frac{(k-1)(k)}{2}$. Assume that we have

$$\mathcal{T}_{p-1} = \mathcal{W}^{p-1} \times Q_{k+1} = \{T(\epsilon_1, \ldots, \epsilon_{p-1}) \times Q_{k+1}\} \subset [2,3] \times I^{k-1} \times Q_{k+1}$$

so that:

1. $T(\epsilon_1, \ldots, \epsilon_{p-1}) = B^2 \times X_1 \times X_2 \times \cdots \times X_{k-2};\ X_j \cong I,$ or $X_j \cong S^1$

2. The diameter of $T(\epsilon_1, \ldots, \epsilon_{p-1})$ is less than $\frac{1}{2^{p-1}}$ in $[2,3] \times I^{k-1}$.

We will define $\mathcal{T}_{p+i}$ for $i = 0, \ldots, k-1$ as follows: For $i = 0$, let

$$A = T(\epsilon_1, \ldots, \epsilon_{p-1}) = B^2 \times X_1 \times X_2 \times \cdots \times X_{k-2} \in \mathcal{W}^{p-1},$$

where $X_j \cong S^1$ or I and the diameter of A_i is less than $\frac{1}{2^{p-1}}$ in $[2,3] \times I^{k-1}$. Consider

$$B_1 = A_i \times I_{k+1} = \left\{B^2 \times X_1\right\} \times C_1$$

where $C_1 = X_2 \times \cdots \times X_{k-2} \times I_{k+1}$. Then by Lemma 1.5.7, there is a finite geometrically central family

$$\mathcal{W}_{(p+1)i} = \{W_{1i}, W_{2i}, \ldots, W_{2^{n_p}i}\}$$

in $B^2 \times X_1$ so that each W_{ji} in $\mathcal{W}_{(p+1)i}$ is $B^2 \times X_{11}$ with $X_{11} \cong S^1$ or I and the diameter of W_{ji} is less than $\frac{1}{2^p}$ in $[2,3] \times I^k$ for all $j = 1, \ldots, 2^{n_p}$ with $2^{n_p} \geq 4$. Now let

$$\mathcal{W}_p = \cup(\mathcal{W}_{(p+1)i} \times C_1).$$

Ramify each element in $\mathcal{W}_p$ 2^{n_p} times and denote each ramified element by $T(\epsilon_1, \ldots, \epsilon_p)$.

Therefore, let $\mathcal{T}_p = \mathcal{W}^p \times Q_{k+2}$ where

$$\mathcal{W}^p = \{T(\epsilon_1, \epsilon_2, \ldots, \epsilon_p)\},$$

for each $\epsilon_l = (\sigma_{l1}, \sigma_{l2})$ with

$$\sigma_{li} = \left(i_1, i_2, \ldots, i_{n_p}\right) \quad l = 1, \ldots, p; i_m, i \in \{1, 2\}.$$

Assume that for $i = i_0$, we have $T_{p+i_0} = W^{p+i_0} \times Q_{k+2}$ such that

1. $W^{p+i_0} = \{T(\epsilon_1, \ldots, \epsilon_{p+i_0})\}$

2. $T = T(\epsilon_1, \ldots, \epsilon_{p+i_0}) = \left\{B^2 \times X_{i_0 1}\right\} \times C_{i_0}$ where

$$C_{i_0} = X_{11} \times \cdots \times X_{(i_0-1)1} \times X_{i_0+1} \times \cdots \times X_{k-2} \times I_{k+1} \in \mathcal{W}^{p+i_0},$$

   with $X_i \cong S^1$ or I and the diameter of T is less than $\frac{1}{2^{p+i_0}}$ in $[2,3] \times I^k$.

To define $\mathcal{T}_{p+i_0+1}$, let $A_i = T \in \mathcal{T}_{p+i_0}$. Then $T = \left\{B^2 \times X_{i_0+1}\right\} \times C_{i_0+1}$, where

$$C_{i_0+1} = X_{11} \times \cdots \times X_{(i_0)1} \times X_{i_0+2} \times \cdots \times X_{k-2} \times I_{k+1},$$

satisfies the condition (2) above. Then by Lemma 1.5.7, there is a finite geometrically central family

$$\mathcal{W}_{(p+i_0+1)i} = \left\{W_1, W_2, \ldots, W_{2^{n_{p+i_0+1}}}\right\}$$

in T so that each W_j is $B^2 \times X_{(i_0+1)1}$ with $X_i \cong S^1$ or I and the diameter of W_j is less than $\frac{1}{2^{p+i_0+1}}$ in $[2,3] \times I^k$ for all $j = 1, \ldots, 2^{n_{p+i_0+1}}$ with $2^{n_{p+i_0+1}}$. Now let

$$\mathcal{W}_{p+i_0+1} = \bigcup \left(\mathcal{W}_{(p+i_0+1)i}\right) \times C_{i_0+1}.$$

Ramify each element in $\mathcal{W}_{p+i_0+1}$ $2^{n_{p+i_0+1}}$ times and denote each ramify element by $T(\epsilon_1, \ldots, \epsilon_{p+i_0+1})$.

Therefore, let $\mathcal{T}_{p+i_0+1} = \mathcal{W}^{p+i_0+1} \times Q_{k+2}$ where

$$\mathcal{W}^{p+i_0+1} = \{T(\epsilon_1, \epsilon_2, \ldots, \epsilon_{p+i_0+1})\},$$

for each $\epsilon_l = (\sigma_{l1}, \sigma_{l2})$ with

$$\sigma_{li} = \left(i_1, i_2, \ldots, i_{n_{p+i_0+1}}\right) \quad l = 1, \ldots, p+i_0+1; i_m, i \in \{1, 2\}.$$

This completes the inductive description of $\mathcal{T}_{p+i}$ for $i = 0, 1, \ldots, k-1$ at stage k.

Remark 2.3.3. *From the construction, $n_p, n_{p+1}, \ldots, n_{p+k-1}$ can be chosen to be equal.*

Lemma 2.3.4. *If $f : H \to T = T(\epsilon_1, \ldots, \epsilon_n)$ is any virtually $I-$essential map of a disc with holes H, then there exists an $\sigma'_{(n+1)1}$ so that f is virtually $I-$essential with respect to all $L(\epsilon_1, \ldots, \epsilon_1, (\sigma'_{(n+1)1}, \sigma_{(n+1)2}))$. Moreover, if $f : H \to T$ is any virtually $I-$essential map of a disc with holes H and the choice of $\sigma'_{(n+1)1}$ is fixed, then there exists a map $g : H \to T$ so that $g\Big|_{\partial H} = f\Big|_{\partial H}$ and $g(H) \cap T(\epsilon_1, \ldots, \epsilon_n, (\sigma_{(n+1)1}, \sigma_{(n+1)2})) \neq \emptyset$ if and only if $\sigma_{(n+1)1} = \sigma'_{(n+1)1}$.*

Proof. The proof can be found in [Gar91], [Dav07],[GD83]. $\square$

2.33 Construction of the Cantor set

Let us recall some definitions and theorems concerning about the Cantor set.

Definition 2.3.5. *A space X is* **totally disconnected** *if the only nonempty connected subsets of X are the one-point sets.*

Definition 2.3.6. *A set A in a space X is* **perfect** *in X if A is closed and dense in itself; i.e., each point of A is a limit point of A.*

From these definitions, we have the following Theorem.

Theorem 2.3.7. *[Wil70] A compact set X is homeomorphic to the standard Cantor set C if and only if X is totally disconnected and perfect.*

From the construction above, we have the following theorem.

Theorem 2.3.8. *Let $\mathcal{T} = \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \ldots\}$ be a sequence in Q such that $\mathcal{T}_k = \mathcal{W}^k \times Q_{k+3}$ where $\mathcal{W}^k = \{T(\epsilon_1, \ldots, \epsilon_i)\}$ defined as above. Then $C = \bigcap (\bigcup \mathcal{T}_i)$ is a Cantor set in Q.*

Proof. Since the intersection is of closed and compact sets with the non-empty finite intersection property, the intersection itself is compact and since the size of the components is going to zero, this implies the total disconnectedness of the intersection. Also, since each component at stage i has more than 2 components at stage $i + 1$, this implies that every point in C is a limit point. Hence C is perfect. Thus, by Theorem 2.3.7 C is a Cantor set in Q, completing the proof. □

2.4 Cantor set on bottom ($[-3, -2] \times Q_2 \subset Q$)

By reflecting the construction of the defining sequence $\mathcal{T} = \{\mathcal{T}_1, \mathcal{T}_2, \ldots\}$ about $\{0\} \times Q_2$, we have a defining sequence $\mathcal{D} = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots\}$ which each $\mathcal{D}_i$ is a homeomorphic copy of $\mathcal{T}_i$ contained in $[-3,-2]\times Q_2 \subset Q$ so that $\mathcal{D} = \bigcap \mathcal{D}_k$ is a Cantor set, where $\mathcal{D}_i = \{D(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)\} \times Q_{m(i)+2}$ where $\epsilon_l = (\sigma_{l1}, \sigma_{l2})$ with

$$\sigma_{li} = (i_1, i_2, \ldots, i_{n_l})\,; \quad l = 1, \ldots, l; i_m, i \in \{1, 2\}\,.$$

2.5 Modification of the Construction of the Cantor set in the Middle

We will modify the construction of the defining sequence in the middle so that each element in the new defining sequence is matched with an element on the top and an element on the bottom. To do this, we will find a sequence $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots\}$ in such a way that

$$\mathcal{B}_1, \mathcal{B}_2 \subset \{0\} \times I^2 \times Q_4$$

$$\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5 \subset \{0\} \times I^3 \times Q_5$$

and for $k > 2$,

$$\mathcal{B}_{\frac{k(k-1)}{2}+i} \subset \{0\} \times I^k \times Q_{k+2}$$

for $i = 0, \ldots, k-1$ and $\bigcap (\bigcup \mathcal{B}_i)$ is a Cantor set.

2.51 Zero and First stage of Construction: $k = 2$

We start the construction viewing the copy of Q as $B^1 \times I_1' \times I_2' \times Q_4$, where for each i, $I_i' = I_{i+1}$. Let $\mathcal{B}_0 = \{\{0\} \times I_1' \times I_2'\} \times Q_4$ be the starting element of the sequence.

To define $\mathcal{B}_1$, consider the set $B^0 = \{I_1' \times I_2'\}$. The set B^1 will be obtained from B^0 by subdividing the first interval factor into 2^{2n_1} equal subintervals. Denote each element in B^1 by $M(\epsilon_1)$ where $\epsilon_1 = (\sigma_{11}, \sigma_{12})$ with

$$\sigma_{1i} = (i_1, i_2 \ldots, i_{n_1})\,; \quad i_l, i \in \{1, 2\}\,.$$

Thus, $\mathcal{B}_1 = B^1 \times Q_4$

To define $\mathcal{B}_2$, the set B^2 will be obtained from B^1 by subdividing the second interval factor into 2^{2n_2} equal subintervals. Denote each element in B^2 by $B(\epsilon_1, \epsilon_2)$ where $\epsilon_l =$

$(\sigma_{l1}, \sigma_{l2})$ with

$$\sigma_{li} = (i_1, i_2 \ldots, i_{n_l})\,; \quad l = 1,2; i_l, i \in \{1,2\}\,.$$

Thus, $\mathcal{B}_2 = B^2 \times Q_4$

Remark 2.5.1. *n_2 can be chosen to be equal n_1 so that I_1', I_2' will be subdivided into same number of subintervals.*

2.52 k^{th} Stage of Construction :$k \geq 3$

For $k \geq 3$, we use induction to describe $\mathcal{B}_{\frac{(k-1)(k)}{2}+i}$ for $i = 0, \ldots, k-1$ as follows. Let $p = \frac{(k-1)(k)}{2}$. Assume that we have

$$\mathcal{B}_{p-1} = \mathcal{B}^{p-1} \times Q_{k+1} = \{B(\epsilon_1, \ldots, \epsilon_{p-1}) \times Q_{k+1}\} \subset \{0\} \times I^{k-1} \times Q_{k+1}$$

where $\epsilon_l = (\sigma_{l1}, \sigma_{l2})$ with

$$\sigma_{li} = (i_1, i_2 \ldots, i_{n_l})\,; \quad i \in \{1,2\}\,, i_r \in \{1,2\}\,, r = 1, \ldots, n_l.$$

We will define $\mathcal{B}_{p+i}$ for $i = 0, \ldots, k-1$ as follows: For $i = 0$, let $B^{p-1} = \{B(\epsilon_1, \ldots, \epsilon_{p-1})\}\,.$ Then the set B^p will be obtained from B^{p-1} by subdividing the first interval factor into 2^{2n_p} equal subintervals. Denote each element in B^1 by $B(\epsilon_1, \epsilon_2, \ldots, \epsilon_p)$ where $\epsilon_l = (\sigma_{l1}, \sigma_{l2})$ with

$$\sigma_{li} = (i_1, i_2 \ldots, i_{n_l})\,; \quad i \in \{1,2\}\,, i_r \in \{1,2\}\,, r = 1, \ldots, n_l.$$

Thus, $\mathcal{B}_p = B^p \times Q_{k+2}$. Assume that for $i = i_0$, we have $T_{p+i_0} = B^{p+i_0} \times Q_{k+2}$ where $B^{p+i_0} = \{B(\epsilon_1, \ldots, \epsilon_{p+i_0})\}\,.$ To define $\mathcal{T}_{p+i_0+1}$, let the set B^{p+i_0+1} will be obtained from B^{p+i_0} by subdividing the i_0+1th interval factor into $2^{2n_{p+i_0+1}}$ equal subintervals. Denote each element in B^{p+i_0+1} by $B(\epsilon_1, \epsilon_2, \ldots, \epsilon_{p+i_0+1})$ where $\epsilon_l = (\sigma_{l1}, \sigma_{l2})$ with

$$\sigma_{li} = (i_1, i_2 \ldots, i_{n_l})\,; \quad i \in \{1,2\}\,, i_r \in \{1,2\}\,, r = 1, \ldots, n_l.$$

Thus, $\mathcal{B}_{p+i_0+1} = B^{p+i_0+1} \times Q_{k+2}$.

This completes the inductive description of $\mathcal{T}_{p+i}$ for $i = 0, 1, \ldots, k-1$ at the stage k.

Remark 2.5.2. *each n_i can be chosen so that after this stage all factors $I_1', I_2', \ldots, I_k'$ have same number of subintervals.*

Using the induction and the same process above to obtain the following Remark 2.5.3.

Remark 2.5.3. *From this construction,first let Let $\bar{1}$ and $\bar{2}$ represent the collections of all finite sequences of the form $(1, 1, \ldots, 1)$ and $(2, 2, \ldots, 2)$, respectively. Then we see that:*

1. *$\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots\}$ is a defining sequence.*
2. *$\bigcap_i \mathcal{B}_i = \{0\} \times C^\infty$*
3. *Each point $p = \{0\} \times p'$ where $p' = (p_1, p_2, \ldots, p_k, \ldots)$ in $\{0\} \times C^\infty$ corresponds to a sequence, $E = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{i-1}, \epsilon_i, \ldots)$. Furthermore, if there is some stage N so that for $i > n$ either all the first components or all the second components of $\epsilon_i = (\sigma_{i1}, \sigma_{i2})$ are not in $\bar{1} \cup \bar{2}$, then $p \in \{0\} \times C^\infty$ has no triadic rational coordinates.*

Theorem 2.5.4. *Let $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \ldots\}$ be defined as above. Then $\mathcal{B}$ is a defining sequence in Q so that the $\bigcap \mathcal{B} = \{0\} \times C^\infty$ is a Cantor set and if $p = \{0\} \times p'$ where $p' = \cap_{i=1}^\infty B(\epsilon_1, \epsilon_2, \ldots, \epsilon_i)$, and if there exists an N so that for all $i > N$, either all the first components or all the second components of ϵ_i are not in $\bar{1} \cup \bar{2}$, then $p \in \{0\} \times C^\infty$ has no triadic rational coordinates in C^∞.*

Proof. The point p is obviously in $\{0\} \times C^\infty$. By assumption and by Remark 2.5.3(3), one can show that none of the coordinates of p' has a triadic expansion that is eventually constant. Thus, this implies that p' has no triadic rational coordinates. $\square$

3. A DECOMPOSITION WITH NONMANIFOLD PART A CANTOR SET

3.1 Introduction

The aim of this chapter is to construct a decomposition H of Q which satisfies the following properties:

- (P1) For each nondegenerate element h of H,

$$h \cap \{0\} \times Q_2$$

  is a single point in $\{0\} \times C^\infty$.

- (P2) Let f_1 and f_2 be maps from B^2 into Q/H and let A be any dense subset of C^∞. Then f_1 and f_2 are approximable by maps g_1 and g_2 satisfying:

  (i) $g_1(B^2) \cap g_2(B^2) \subset \pi_H(A)$, and

  (ii) if $p = \{0\} \times p'$ is a point of $\{0\} \times C^\infty$ with $\pi_H(p) \in (g_1(B^2) \setminus g_2(B^2)) \cup (g_2(B^2) \setminus g_1(B^2))$, then p' has no triadic rational coordinates.

- (P3) Q/H has nonmanifold part equal to $\pi_H(\{0\} \times C^\infty) \cong \{0\} \times C^\infty$.

3.11 Preview of Our Plan for Constructing the Decomposition H of Q

Recall from the previous Chapter,

$$\mathcal{T} = \{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots, \},$$

$$\mathcal{D} = \{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \ldots\},$$

$$\mathcal{B} = \{\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots\}$$

are defining sequences in $[2, 3] \times Q_2$, $\{0\} \times Q_2$, and $[-3, -2] \times Q_2$, respectively. To produce the decomposition H, we will construct a defining sequence $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, \ldots, \}$ so that each component of $\mathcal{L}_i$ consists of a single component of $\left[-\frac{1}{2^i}, \frac{1}{2^i}\right] \times B \in \left[-\frac{1}{2^i}, \frac{1}{2^i}\right] \times \mathcal{B}_i$, a component of $\mathcal{T}_i$, a component of $\mathcal{D}_i$, together with tubes (regular neighborhoods of arcs) joining the top of $\left[-\frac{1}{2^i}, \frac{1}{2^i}\right] \times B$, $\{\frac{1}{2^i}\} \times B$, and the bottom of $\left[-\frac{1}{2^i}, \frac{1}{2^i}\right] \times B$, $\{-\frac{1}{2^i}\} \times B$ to these components.

3.2 A Defining Sequence associated with the Decomposition H

To construct the defining sequence

$$\mathcal{L} = \{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots\}$$

of Q, we will modify the defining sequences $\mathcal{T}, \mathcal{B}, \mathcal{D}$ constructed in the previous Chapter and we describe the construction inductively.

We start the construction viewing $Q = I^3 \times Q_4$ where $I_1 = [-3, 3]$. We write $I^k = I_1 \times I_2 \times \cdots \times I_k$ and $B^2 = I_1 \times I_3$. Similarly, we write B^{n-1} as an embedded copy of $I_2 \times \cdots \times I_n$. An $n-$tube is a homeomorphic copy of $B^{n-1} \times [0, 1]$.

Let

$$\mathcal{T} = \{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots, \},$$

$$\mathcal{D} = \{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \ldots\},$$

$$\mathcal{B} = \{\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots\}$$

be defining sequences from the previous Chapter so that the diameter of $\left[-\frac{1}{2^j}, \frac{1}{2^j}\right] \times B_{\frac{(j+2)(j+1)}{2}-1}$ is less than $\frac{1}{j}$ in I^{j+2} for $j > 1$.

3.21 Stage Zero of Construction

Stage 0 : $\mathcal{L}_0$ has a single element $L_0 \times Q_4$ where L_0 consists of $[-1, 1] \times B_0$ and 3-tubes joining the top of $[-1, 1] \times B_0$ to $T_0 = D_1 \times I_2$ and the bottom of $[-1, 1] \times B_0$ to $D_0 = D_2 \times I_2$ where $D_1 \subset [2, 3] \times I_2$ and $D_2 \subset [-3, -2] \times I_2$. This joining is done in such a way that

$$L_0 \times Q_4 \cap \Big([-1,1] \times I^2 \times Q_4 \Big) = [-1,1] \times B_0 \times Q_4.$$

Figure 3.1 shows this stage of the construction in I^3.

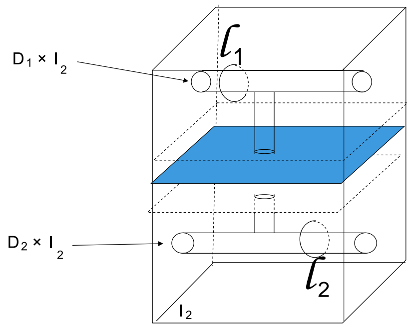


FIGURE 3.1: Stage Zero of construction

Let D_1' and D_2' be slightly bigger discs in $I_1 \times I_3$ containing D_1 and D_2, respectively, as shown in Figure 3.1. Let $l_1 = \partial(D_1' \times \{pt\})$ and $l_2 = \partial(D_2' \times \{pt\})$ be loops and let $\epsilon > 0$ be such that

$$\rho(l_1, \partial(D_1 \times I_2)) > \epsilon \quad \text{and} \quad \rho(l_2, \partial(D_2 \times I_2)) > \epsilon.$$

We can assume that $l_1 \times \{0\}$ and $l_2 \times \{0\}$ in $I^3 \times Q_4$ are loops in Q such that for any contraction f_1 of l_1 and f_2 of l_2, in general position respect to the boundary component of $\mathcal{L}_0$, there exist discs with holes K_1 and K_2 so that $f_1\Big|_{K_1}$ is virtually $I-$essential in $D_1 \times I_2$ and $f_2\Big|_{K_2}$ is virtually $I-$essential in $D_2 \times I_2$, and so that for any maps $g_1, g_2 : B^2 \to Q$ in general position with respect to $\mathcal{L}_0$ and

$$\rho(l_1, g_1(\partial B^2)) < \frac{\epsilon}{2} \quad \text{and} \quad \rho(l_2, g_2(\partial B^2)) < \frac{\epsilon}{2},$$

there are discs with holes H_1 and H_2 such that $g_1\Big|_{H_1}$ is virtually $I-$essential in $D_1 \times I_2$ and $g_2\Big|_{H_2}$ is $I-$essential in $D_2 \times I_2$. This follows from the geometric centrality argument earlier.

3.22 First Stage of Construction

Stage 1 : Inside L_0, join the top of $\left[-\frac{1}{2}, \frac{1}{2}\right] \times B(\epsilon_1)$ to $T(\epsilon_1)$ and the bottom of $\left[-\frac{1}{2}, \frac{1}{2}\right] \times B(\epsilon_1)$ to $D(\delta_1)$ with $3-$tubes where $\delta_n = (\sigma_{n2}, \sigma_{n1})$ if $\epsilon_n = (\sigma_{n1}, \sigma_{n2})$. These tubes should run straight through the tube joining T_0 and D_0 from the previous stage. Let $L(\epsilon_1)$ be the resulting element. The joining should be done in such a way that

$$L(\epsilon_1) \times Q_4 \cap \left(\left[-\frac{1}{2}, \frac{1}{2}\right] \times I^2 \times Q_4 \right) = \left[-\frac{1}{2}, \frac{1}{2}\right] \times B(\epsilon_1) \times Q_4.$$

Then

$$\mathcal{L}_1 = \{L(\epsilon_1)\} \times Q_4.$$

Figure 3.2 demonstrates a representation of the construction with 4 components in $\mathcal{L}_1$. The four components in the cylinder at the top come from $\mathcal{T}_1$, The four components in the cylinder at the bottom come from $\mathcal{D}_1$, and the four components in the middle come from $\mathcal{B}_1$. The arrows indicate how some of the components are connected by tubes.

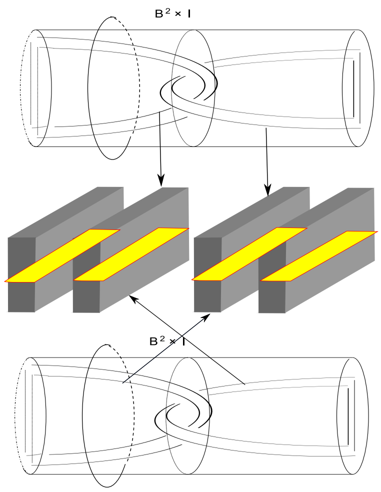


FIGURE 3.2: First Stage of construction $\mathcal{L}$ with 4 components

We may observe the following facts about the first two constructions.

1. For every $x \in Q$, there is a $3-$cell B^3 in I^3 such that

$$st(x, \mathcal{L}_1) \subset B^3 \times Q_4 \subset \mathcal{L}_0.$$

2. For each element L of $\mathcal{L}_i$, $i = 0, 1$, we can get a $\frac{1}{i+1}-$map from L to a $1-$complex.

3. For any $x \in \partial\mathcal{L}_0$ with x not in $\left[-\frac{1}{2}, \frac{1}{2}\right] \times Q_2$, x is not in L_j for every L_j in $\mathcal{L}_1$.

4. For every contraction f_1 and f_2 of any loops within ϵ of the loops l_1 and l_2, respectively, there is a component $T(\epsilon_1, \epsilon_2)$ of $\mathcal{W}^2$ and $D(\epsilon_1, \epsilon_2)$ of $\mathcal{D}^2$ and discs D_1' and D_2' with holes such that $f_1\Big|D_1'$ is virtually $I-$essential on $T(\epsilon_1, \epsilon_2)$ and $f_2\Big|D_2'$ is virtually $I-$essential on $D(\epsilon_1, \epsilon_2)$.

For the higher dimensional construction, assume that $\mathcal{L}_{r-1}, r \geq 1$ has been constructed. We need the following inductive hypotheses to be true for $j = r - 1$ in $I^k \times Q_{k+1}, k \geq 3$ as in [DG82].

IH1 For each element L of $\mathcal{L}_j$ there is $\frac{1}{j+1}-$map from L to a 1-complex.

IH2 The diameter of each $\left[-\frac{1}{2^j}, \frac{1}{2^j}\right] \times B(\epsilon_1, \epsilon_2, \ldots, \epsilon_j) \times Q_{j+3} < \frac{1}{j}$

IH3 If x is a point in Q in ∂L_s for some L_s in $\mathcal{L}_{j-1}$ and not in $\left[-\frac{1}{2^{j-1}}, \frac{1}{2^{j-1}}\right] \times Q_2$, then x is not in L_i for every L_i in $\mathcal{L}_j$.

IH4 Let $\delta > 0$. For any contraction f_1 and f_2 of any loops within δ of loops l_1 and l_2 into I^k, there is an element $B = \left[-\frac{1}{2^j}, \frac{1}{2^j}\right] \times B(\epsilon_1, \epsilon_2, \ldots, \epsilon_j)$ and a component $T = T(\epsilon_1, \epsilon_2, \ldots, \epsilon_j)$ of $\mathcal{T}_j$ and a component $D = D(\epsilon_1, \epsilon_2, \ldots, \epsilon_j)$ of $\mathcal{D}_j$ and discs D_1 and D_2 with holes such that B is connected to T and $f_1|_{D_1}$ is virtually I-essential into T and B is connected to D and $f_2|_{D_2}$ is virtually I-essential into D.

Next we will construct the r^{th} state of construction.

3.23 $r^{\mathbf{th}}$ Stage of Construction

Stage $r-1$: In addition to the inductive hypotheses on the previous page, inductively assume:

(i) Each component $L = L(\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1}) \times Q_{m(r-1)+3}$ of $\mathcal{L}_{r-1}$ is a manifold so that $L(\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1})$ is composed of the $B = \left[-\frac{1}{2^{r-1}}, \frac{1}{2^{r-1}}\right] \times B(\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1})$, together with the tubes joining the top of B to $T(\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1})$ and the bottom of M to $D(\delta_1, \delta_2, \ldots, \delta_{r-1})$ where $\delta_n = (\sigma_{n2}, \sigma_{n1})$ if $\epsilon_n = (\sigma_{n1}, \sigma_{n2})$.

(ii)

$$L \cap \left[-\frac{1}{2^{r-1}}, \frac{1}{2^{r-1}}\right] \times I^r \times Q_{m(r-1)+3} = \left[-\frac{1}{2^{r-1}}, \frac{1}{2^{r-1}}\right] \times B(\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1}) \times Q_{m(r-1)+3}.$$

Consider a component L of $\mathcal{L}_{r-1}$. Suppose

$$L = L(\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1}) \times Q_{r+2}).$$

Then by (i), $L(\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1})$ consists of $B = \left[-\frac{1}{2^{r-1}}, \frac{1}{2^{r-1}}\right] \times B(\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1})$, together with the tubes joining B to $T(\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1})$ and the bottom of B to $D(\delta_1, \delta_2, \ldots, \delta_{r-1})$.

Let $L(\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1}, \epsilon_r)$ be the element obtained by joining

$$B_1 = \left[-\frac{1}{2^r}, \frac{1}{2^r}\right] \times B(\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1}, \epsilon_r),$$

together with tubes joining the top of B_1 to

$$T(\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1}, \epsilon_r)$$

and the bottom of M_1 to

$$D(\delta_1, \delta_2, \ldots, \delta_{r-1}, \delta_r)$$

where $\delta_n = (\sigma_{n2}, \sigma_{n1})$ if $\epsilon_n = (\sigma_{n1}, \sigma_{n2})$. The joining should be done in such a way that condition (ii) of the inductive hypothesis and **IH4** are satisfied.

This completes the construction.

Remark 3.2.1. *Note that by the inductive construction, for each $L_i \in \mathcal{L}_i$, there exists $Pre(L_i) \in \mathcal{L}_{i-1}$ such that $L_i \subset Pre(L_i)$ and the inclusion map $L_i \to Pre(L_i)$ is null homotopic. This implies that each element of the decomposition H associated with the defining sequence $\mathcal{L}$ is cell-like.*

Theorem 3.2.2. *Let $\mathcal{L} = \{\mathcal{L}_2, \mathcal{L}_5, \mathcal{L}_9, \ldots\}$ be defined as above and satisfy all the conditions* **IH 1** *through* **IH4**. *Then $\mathcal{L}$ is a defining sequence. Let H be the decomposition associated with $\mathcal{L}$. Then H is an upper semicontinuous decomposition of Q, and for each nondegenerate element h of H, $h \cap \{0\} \times Q_2$ is a single point in $\{0\} \times C^\infty$.*

Proof. From the construction, we clearly see that $\mathcal{L}$ is a defining sequence, and hence by Theorem 1.3.21 the decomposition H is upper semicontinuous. Also the induction hypothesis IH2 implies that for each nondegenerate $h \in H$, $h \cap \{0\} \times Q_2$ is contained in $\{0\} \times C^\infty$. □

Theorem 3.2.3. *Let H be the decomposition associated with $\mathcal{L} = \{\mathcal{L}_2, \mathcal{L}_5, \mathcal{L}_9, \ldots\}$ in Theorem 3.2.2. Then H satisfies the following properties:*

(1) If h is a non-degenerate decomposition element on H, and U is any open set in Q containing h, then there is a $n-$ball B^n such that $h \subset B^n \times Q_{n+1} \subset U$.

(2) Each non-degenerate element of H has dimension one.

(3) Let $\pi_H : Q \to Q/H$ be the quotient map. Let L be any element of $\mathcal{L}_r$ for some r. Then π_H is one to one on the boundary of L.

(4) The set $\{0\} \times Q_2$ is mapped homeomorphically by the quotient map π_H.

(5) Every element of H is cellular.

Proof. For (1), this is true by the Tube Lemma from section 1, 1.3.2.

For (2), the connectedness of each non-generate element h of the decomposition H implies that the dimension of h is ≥ 1. Since by the condition **IH1** for every L of $\mathcal{L}_n$ there is a $\frac{1}{n}-$map from L to a 1 complex, by a result from dimension theory ([HW48], page 73) the dimension of h is ≤ 1. Hence, each non-degenerate element of H has dimension one.

Condition (3) follows from Lemma 1.3.11.

For (4), it suffices to show that π_H is one-to-one on $\{0\} \times Q_2$. Let $p = \{0\} \times p_1, q = \{0\} \times q_1 \in \{0\} \times Q_2$ be such that $\pi_H(p) = \pi_H(q)$. Then there is $h \in H$ so that $p, q \in h$. Thus, $p \in \cap St^2(q, \mathcal{L}_i)$ and $q \in \bigcap_{i=1}^{\infty} St^2(p, \mathcal{L}_i)$. Since elements P, Q in $\mathcal{L}_i$ are pairwise disjoint, it follows that $p, q \in \cap_{i=1}^{\infty} St(p, \mathcal{L}_i)$. If there is index i so that $p \notin L$ for all $L \in \mathcal{L}_i$, then $p = q$ since the map π_H is one-to-one on the complement of the elements of $\mathcal{L}_i$. Suppose that for each i there is $L_i \in \mathcal{L}_i$ such that $p \in L_i$. Then $q \in L_i$ for all i. Since $p, q \in \{0\} \times Q_2$, this forces $p, q \in \left[-\frac{1}{2^i}, \frac{1}{2^i}\right] \times B(\epsilon_1, \ldots, \epsilon_{\frac{(i)(i+1)}{2}-1})$ for all i. Since the limit of diameter of each $\left[-\frac{1}{2^i}, \frac{1}{2^i}\right] \times B(\epsilon_1, \ldots, \epsilon_{\frac{(i)(i+1)}{2}-1})$ is zero by **IH2**, this implies that $p = q$ and hence π_H is one-to-one on $\{0\} \times Q_2$.

For (5),(argument modified from [Ghi07]) let h be a nondegenerate decomposition element of H. We consider two cases.

Case 1 : If h does not intersect any face of the Hilbert cube Q, then $h \subset s$, where s is the pseudo interior of Q. By Lemma 1.3.25, h is a $Z-$set and so by Lemma 1.3.30, h is cellular.

Case 2 : Assume that h intersects the pseudo boundary of Q. To show that h is cellular, it suffices to show that $Q - h$ is S^1-trivial at ∞. Let U be a neighborhood of h and let $U = V$ and $f : S^1 \to V - h$. Since h is compact and $f(S^1) \cap h = \emptyset$, there is a distance $\epsilon > 0$ between $f(S^1)$ and h. By the tube lemma, there is an integer $n > 0$ such that

$$h \subset B^n \times Q_{n+1} \subset U \quad \text{and} \quad f(S^1) \subset U - B^n \times Q_{n+1}.$$

Let F be any contraction of f in general position with respect to the boundary of I^n. Since h intersects the pseudo boundary of Q, this implies that B^n intersects ∂I^n in a finite number of disjoint components. Removing these components from $\partial I^n \cong S^{n-1}, n \geq 4$ leaves a simply connected component. Let us write the map F as $(F_n, F_{Q_{n+1}})$ where $F_n : B^2 \to B^n$ and $F_{Q_{n+1}} : B^2 \to Q_{n+1}$. Consider $F_n^{-1}(\partial B^n)$. Notice that $F_n^{-1}(\partial B^n)$ is a collection of a finite number of closed curves. That is,

$$F_n^{-1}(\partial B^n) = \{J_i \mid J_i \text{ is a closed curve in } B^2,\, i = 1, \ldots, r\}$$

Let $J \in F_n^{-1}(\partial B^n)$. Let H_J be a disc in B^2 with boundary J. Then we can extend the map $F_n|_J$ to a map $F_n'|_{H_j}$ on ∂B^n. Taking the innermost components of $F_n^{-1}(\partial B^n)$ one at a time, we can adjust the map F_n to a map G_n on ∂B^n. In this manner we can extend the map f to the map G from B^2 to $U - h$. This shows that $Q - h$ is S^1-trivial at ∞. Then by Lemma 1.3.30, h is cellular in Q. $\square$

Lemma 3.2.4. *Let $L = B \cup T \cup D$ together with the tubes joining these be a fixed element in $\mathcal{L}_r$ for some r. Then for any virtually $I-$essential maps $f_1 : D_1 \to T$ and $f_2 : D_1 \to D$, there is an $\delta > 0$ so that if f_1', f_2' are maps within δ of f_1 and f_2 respectively and in general position with respect to $\mathcal{L}_{r+1}$, then there is a component $L \supset L' = B' \cup T' \cup D'$ together with the tubes joining these and discs with holes $D_1' \subset D_1, D_2' \subset D_2$ such that f_1' and f_2' are virtually $I-$essential into T' and D' respectively.*

Proof. This desired result follows inductively by the condition **IH4**. For more details, see [Gar91]. $\square$

Corollary 3.2.5. *Let f and g be maps from B^2 to $[2,3] \times Q_2$ and B^2 to $[-3,-2] \times Q_2$, respectively, with $f(\partial B^2) = l_1$ and $g(\partial B^2) = l_2$ and $\epsilon > 0$. If f' and g' are such that $f'(\partial B^2)$ and $g'(\partial B^2)$ are ϵ approximations to $f(\partial B^2)$ and $g(\partial B^2)$, then both $f'(\partial B^2)$ and $g'(\partial B^2)$ intersect a common element h of H.*

Proof. This follows from Lemma 3.2.4. $\square$

Lemma 3.2.6. *(argument modified from [Gar91]) Let H be as in Theorem 3.2.2, and let A be a dense subset of $\{0\} \times C^\infty$. Let L be a fixed component of some $\mathcal{L}_k$. Let f and g be maps from B^2 into Q, in general position with respect to $\mathcal{L}_k$, transverse to $\{0\} \times Q_2$, with*

$(f(B^2) \cup g(B^2)) \cap \{0\} \times Q_2)$ contained in the complement of $\mathcal{L}_k$. Then there exist maps f' and g' from B^2 into Q with

$$f'|_{B^2\setminus(f^{-1}(L))} = f|_{B^2\setminus(f^{-1}(L))} \quad , \quad g'|_{B^2\setminus(g^{-1}(L))} = g|_{B^2\setminus(g^{-1}(L))}$$

and satisfying the following conditions:

(i) If $h \in H, h \subset L$ and both $g'(B^2)$ and $f'(B^2)$ intersect h, then $h \cap \{0\} \times C^\infty \in A$.

(ii) If $h \in H, h \subset L$ and $g'(B^2)$ intersects h and $f'(B^2)$ does not intersect h, then $h \cap \{0\} \times C^\infty$ has no triadic rational coordinates in C^∞.

(iii) If $h \in H, h \subset L$ and $f'(B^2)$ intersects h and $g'(B^2)$ does not intersect h, then $h \cap \{0\} \times C^\infty$ has no triadic rational coordinates in C^∞.

Proof. The result will be obtained by successively pushing f and g off certain components of $\mathcal{L}_i$ for $i > k$. Note that in higher dimension, if maps f and g intersect an element $L \in \mathcal{L}_i$ where L consists of B joining T at the top and joining D at the bottom with regular tubes G_1 and G_2, then the intersection of f with L and the intersection of g with L can be adjusted so as to miss the tubes G_1 and G_2.

Let $L = L(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ be as in the construction above. The maps f' and g' will be inductively constructed.

If both $f(B^2)$ and $g(B^2)$ intersect L, replace f and g by maps f_1 and g_1 so that

$$f_1|_{B^2\setminus(f^{-1}(L))} = f|_{B^2\setminus(f^{-1}(L))} \text{ and } g_1|_{B^2\setminus(g^{-1}(L))} = g|_{B^2\setminus(g^{-1}(L))},$$

and so that $f_1(B^2) \cap L$ is contained in

$$(*)\ \bigcup T\big(\epsilon_1, \ldots, \epsilon_k, \left(\sigma^1_{(k+1)1}, \sigma_{(k+1)2}\right)\big) \bigcup \big(\bigcup D\big(\delta_1, \ldots, \delta_k, \left(\sigma^1_{(k+1)1}, \sigma_{(k+1)2}\right)\big)\big)$$

and that $g_1(B^2) \cap M$ is contained in

$$(**)\ \bigcup T\big(\epsilon_1, \ldots, \epsilon_k, \left(\sigma^2_{(k+1)1}, \sigma_{(k+1)2}\right)\big) \bigcup \big(\bigcup D\big(\delta_1, \ldots, \delta_k, \left(\sigma^2_{(k+1)1}, \sigma_{(k+1)2}\right)\big)\big)$$

where $\sigma^m_{(k+1)1} = (m, \tau_2, \ldots, \tau_{n_{k+1}}) \notin \bar{1} \cup \bar{2}$ for $m, \tau_i \in \{1, 2\}$.

(i) If both $f_1(B^2)$ and $g_1(B^2)$ intersect an element N of $\mathcal{L}_{k+1}$ contained in L, with $N = L(\epsilon_1, \epsilon_2, \ldots, \epsilon_{k+1})$, let $T_1 = T(\epsilon_1, \epsilon_2, \ldots, \epsilon_{k+1})$, and $D_1 = D(\delta_1, \delta_2, \ldots, \delta_{k+1})$. By (*) and (**), it then follows that either $f_1(B^2) \cap N$ is contained in T_1 and $g_1(B^2) \cap N$ is contained in D_1, or $f_1(B^2) \cap N$ is contained in D_1 and $g_1(B^2) \cap N$ is contained in T_1. At this point fix an element $h \in H$ contained in N with $h \cap \{0\} \times C^\infty \subset A$.

For each such N, maps f_i and g_i, $i \geq 2$, can be defined inductively so that at stage j, the only element of $\mathcal{L}_{k+j}$ contained in N that both $f_j(B^2)$ and $g_j(B^2)$ intersect is the element containing h, so that

$$f_j|_{B^2 \setminus (f^{-1}_{j-1}(\cup \mathcal{L}_{k+j-1}))} = f_{j-1}|_{B^2 \setminus (f^{-1}_{j-1}(\cup \mathcal{L}_{k+j-1}))}$$

and so that

$$g_j|_{B^2 \setminus (g^{-1}_{j-1}(\cup \mathcal{L}_{k+j-1}))} = g_{j-1}|_{B^2 \setminus (g^{-1}_{j-1}(\cup \mathcal{L}_{k+j-1}))}.$$

The fact that the diameters of the components of $\mathcal{T}_i$ and $\mathcal{D}_i$ go to zero allows one to define $f^{''}$ and $g^{''}$ as the limits of the maps f_i and g_i respectively. The maps thus obtained satisfy condition (i).

(ii) If at some stage $j \geq k$, $f^{''}(B^2)$ intersects an element N of $\mathcal{L}_j$, and $g^{''}$ does not, then a map f' will be inductively constructed so that f' and $g^{''}$ satisfy both conditions (i) and (ii).

Case 1: If $f^{''}(B^2)$ intersects the element N of $\mathcal{L}_j$ both in $I^n_+ = [0, 3] \times I^{n-1}$ and in $I^n_- = [-3, 0] \times I^{n-1}$ for all n, fix an element h of H so that $h \in N$ and so that $h \cap \{0\} \times C^\infty$ has no triadic rational coordinates in C^∞. The map $f^{''}$ will be replaced by f' so that if $f'(B^2)$ intersects any element of H contained in N both in I^n_+ and in I^n_- for all n, then this element must be in $H(f'')$. As a first approximation, define f_1 so that

$$f_1|_{B^2 \setminus (f^{-1}(N))} = f^{''}|_{B^2 \setminus (f^{-1}(N))}$$

and so that $f_1(B^2) \cap N$ is contained in

$$(\bigcup T(\epsilon_1, \ldots, \epsilon_j, \big(\sigma_{(j+1)1}, \sigma_{(j+1)2}\big)) \bigcup (\bigcup D(\delta_1, \ldots, \delta_j, \big(\sigma'_{(j+1)1}, \sigma_{(j+1)2}\big)))$$

Here $\sigma_{(j+1)1}$ and $\sigma'_{(j+1)1}$ are chosen to correspond to the element of $\mathcal{L}_{j+1}$ that contains h. It then follows that the element of $\mathcal{L}_{j+1}$ contained in N that f_1 intersects both in I^n_+ and in I^n_- for all n, is the element containing $h \in H(f'')$.

We can now continue inductively, defining maps f_i, $i \geq 2$, so that at stage p, the element of $\mathcal{L}_{j+p}$ contained in N that $f_p(B^2)$ intersects both in I^n_+ and in I^n_- for all n, is the element containing $h \in H(f'')$, and so that

$$f_p|_{B^2 \setminus (f^{-1}_{p-1}(\cup \mathcal{L}_{j+p-1}))} = f_{p-1}|_{B^2 \setminus (f^{-1}_{p-1}(\cup \mathcal{L}_{j+p-1}))}.$$

The map f' will be defined as the limit of the maps f_i.

Case 2: Suppose $f^{''}(B^2)$ intersects the element N of $\mathcal{L}_j$ only in I^n_+ or in I^n_- for some n. For each such N, maps f_i, $i \geq 2$, can be defined inductively so that if $f_i(B^2)$ intersects an element $L(\epsilon_1, \epsilon_2, \ldots, \epsilon_{j+i})$ of $\mathcal{L}_{j+i}$ contained in N, then the first components or the second components of $\epsilon_{j+i} = (\sigma_{(j+2p)1}, \sigma_{(j+2p)1})$ are not in $\bar{1} \cup \bar{2}$. Moreover, we can require that

$$f_i|_{B^2 \setminus (f^{-1}_{i-1}(\cup \mathcal{L}_{j+i-1}))} = f_{i-1}|_{B^2 \setminus (f^{-1}_{i-1}(\cup \mathcal{L}_{j+i-1}))}.$$

The fact that the diameters of the components of $\mathcal{T}_i$ and $\mathcal{D}_i$ is going to zero allows one to define $f^{'}$ as the limits of the maps f_i. Theorem 2.5.4 implies that if f' intersects an element h of H, then $h \cap \{0\} \times C^\infty$ has no triadic rational coordinates in C^∞.

(iii) A map g' can be constructed just as f' was constructed above, so that f' and g' satisfy conditions (i), (ii), and (iii). This completes the proof. $\square$

Theorem 3.2.7. *The decomposition H described above is a cell-like decomposition of Q satisfying four properties, (P1), (P2), and (P3) listed as follows:*

- *(P1) For each nondegenerate element h of H,*

$$h \cap \{0\} \times Q_2$$

is a single point in $\{0\} \times C^\infty$

- *(P2) Let f_1 and f_2 be maps from B^2 into Q/H and let A be any dense subset of C^∞. Then f_1 and f_2 are approximable by maps g_1 and g_2 satisfying:*

  (i) $g_1(B^2) \cap g_2(B^2) \subset \pi_H(A)$, and

  (ii) if $p = \{0\} \times p'$ is a point of $\{0\} \times C^\infty$ with

$$\pi_H(p) \in (g_1(B^2) \setminus g_2(B^2)) \cup (g_2(B^2) \setminus g_1(B^2)),$$

  then p' has no triadic rational coordinates.

- *(P3) Q/H has nonmanifold part equal to $\pi_H(\{0\} \times C^\infty) \cong \{0\} \times C^\infty$.*

Proof. Let h be a nondegenerate element of H. Then by Lemma 3.2.2

$$h \cap \{0\} \times Q_2$$

is a single point in $\{0\} \times C^\infty$. So (P1) is satisfied.

For (P2), let f_1 and f_2 be maps from B^2 into Q/H and let A be any dense subset of $\{0\} \times C^\infty$. Then by approximate lifting there are lifts f'' and g'' from B^2 to Q so that the following are satisfied:

(1) $f''(B^2) \cap g''(B^2) = \emptyset$;

(2) f'' and g'' are transverse with respect to $\{0\} \times Q_2$;

(3) $f''(B^2) \cap \{0\} \times C^\infty = \emptyset = g''(B^2) \cap \{0\} \times C^\infty$;

(4) $\rho(\pi_H \circ f'', f_1) < \frac{\epsilon}{3}$ and $\rho(\pi_H \circ g'', f_2) < \frac{\epsilon}{3}$.

Then by (2), there is a stage $\mathcal{L}_{r_0}$ in the defining sequence for H so that $(f^{''}(B^2) \cup g^{''}(B^2)) \cap \{0\} \times Q_2$ is contained in the complement of $\mathcal{L}_{r_0}$ and so that the diameter of $\pi_H(L)$ is less than $\frac{\epsilon}{3}$ for all $L \in \mathcal{L}_{r_0}$. A further general position adjustment puts $f^{''}$ and $g^{''}$ in general position with respect to all elements of $\mathcal{L}_{r_0}$. By Lemma 3.2.6 we can find maps f' and g' so that $\pi_H \circ f'$ and $\pi_H \circ g'$ are the desired approximations.

Finally, let p be any point of $\pi_H(\{0\} \times C^\infty)$ and U be an ϵ neighborhood of p. By (P1), it is clear that the image of the nondegenerate element of H is contained in $\pi_H(\{0\} \times C^\infty)$ and so it implies that the nonmanifold part of Q/H is contained in $\pi_H(\{0\} \times C^\infty)$. To complete the proof, it suffices to show that the disjoint discs property fails in U[Edw80]. Given $L = B \cup T \cup D$ together with the tubes joining these of some $\mathcal{L}_k$ so that $\pi_H(L)$ is contained in the $\frac{\epsilon}{4}$ neighborhood of p. Choose $f_1, f_2 \; : \; B^2 \to Q$ so that f_1 is an $I-$essential map into T and f_2 is an $I-$essential into D. Let $f_1' = \pi_H \circ f_1$ and $f_2' = \pi_H \circ f_2$. If $f_1'(B^2) \cap f_2'(B^2) = \emptyset$, then there are approximate lifts $f_1^{'''}, f_2^{'''} \; : \; B^2 \to Q$ so that $f_1^{'''}\Big|_{\partial B^2}$ and $f_2^{'''}\Big|_{\partial B^2}$ are ϵ' approximations to $f_1\Big|_{\partial B^2}$ and $f_2\Big|_{\partial B^2}$, and so that $\pi_H \circ f_1^{'''}(B^2) \cap \pi_H \circ f_2^{'''}(B^2) = \emptyset$. This leads to a contradiction by Lemma 3.2.4. Assume that $f_1'(B^2) \cap f_2'(B^2) \neq \emptyset$. Then there is a δ so that if $f_1^{''}$ and $f_2^{''}$ are lifts of $\frac{\delta}{2}$ approximations of f_1' and f_2' with $\pi_H \circ f_1^{''}$ and $\pi_H \circ f_2^{''}$ within δ of f_1' and f_2', then $f_1^{''}|_{\partial B^2}$ and $f_2^{''}|_{\partial B^2}$ are within ϵ' of $f_1|_{\partial B^2}$ and $f_2|_{\partial B^2}$. It follows by Lemma 3.2.4 that any $\frac{\delta}{2}$ approximations of f_1' and f_2' have a common point of U in their image.

□

Corollary 3.2.8. *Let H be the decomposition of Hilbert Cube Q defined in theorem 3.2.2. Then Q/H does not satisfy the DDP, and hence $Q/H \ncong Q$.*

Proof. This follows from (P3). □

In the next section we will investigate further properties of the decomposition Q/H.

3.3 Additional Properties of Q/H

Recall that [DW81] and [Lay80] contain information on Čech carriers. The construction of the decomposition space Q/H of the Hilbert cube associated with the defining sequence $\mathcal{L}$ was done in the previous section, and at the end of section we showed that Q/H does not satisfy the DDP and so $Q/H \ncong Q$. However, Q/H does satisfy the Disjoint Čech Carrier Property, see the Definition 1.7.7. Thus, the main goal of this section is to prove that the decomposition Q/H satisfies the following properties:

1. Q/H satisfies the Disjoint Čech Carrier Property.
2. Q/H is an ANR.
3. $Q/H \times I^2 \cong Q$.

Lemma 3.3.1. *Let $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots\}$ be the defining sequence defined in Theorem 3.2.2. Then $\mathcal{L}$ is sharp.*

Proof. By properties (3) and (4) of Theorem 3.2.3, the defining sequence $\mathcal{L}$ is sharp. □

Lemma 3.3.2. *Let H be the decomposition of Hilbert Cube Q defined in Theorem 3.2.2. Then Q/H is an ANR*

Proof. By Lemma 3.3.1 and Lemma 1.3.36, Q/H is an ANR. □

Lemma 3.3.3. *[Lay80] Let $\mathcal{L}$ be the defining sequence of Q defined as above. Let $k \geq 1$ be fixed and $L \in \mathcal{L}_k$. Then for each $j > k$, if W is compact subset of Q with $W \subset Int(L)$, then*

$$W \cap \pi^{-1}(\pi(\partial L) \quad \text{is 1-dimensional.}$$

Proof. (modified from [Lay80]) Fix $k \geq 1$ and $L \in \mathcal{L}_k$. Let $r > k$, then $L_j = L(\epsilon_1, \epsilon_2, \ldots, \epsilon_r)\times$ $Q_{m(r)+3}$ of $\mathcal{L}_r$ is a manifold so that $L(\epsilon_1, \epsilon_2, \ldots, \epsilon_r)$ is composed of the $B_j = \left[-\frac{1}{2^r}, \frac{1}{2^r}\right] \times$

$B(\epsilon_1, \epsilon_2, \ldots, \epsilon_r)$, together with the tubes joining the top of B to $T_j(\epsilon_1, \epsilon_2, \ldots, \epsilon_r)$ and the bottom of B_j to $D_j(\delta_1, \delta_2, \ldots, \delta_r)$. That is,

$$L_j = (T_j \cup B_j \cup D_j \cup B^{r+1} \times I) \times Q_{r+3}$$

where $diam(T_j) < \frac{1}{j}$ and $diam(D_j) < \frac{1}{j}$. If $L_j \cap \partial L \neq \emptyset$, then

$$(B_j \times Q_{m(r)+3}) \cap \partial L \neq \emptyset.$$

Let W be a compact subset contained in $Int(L)$. Note that

$$\pi_H^{-1}(\pi_H(L)) = A_1 \cup A_2$$

where

$$A_1 = \{x \in \partial L \mid \pi_H^{-1}(\pi_H)(x) = \{x\}\}$$

and

$$A_2 = \cup\{\pi_H^{-1}(\pi_H)(x) \mid \pi_H^{-1}(\pi_H)(x) \neq \{x\},\ x \in \partial L\}.$$

Since $W \subset Int(L)$, it follows that $W \cap A_1 = \emptyset$. Thus, $W \cap \pi_H^{-1}(\pi_H(L)) = W \cap A_2$. Note also that for all $x \in \partial L \setminus A_1$,

$$\pi_H^{-1}(\pi_H)(x) = \cap St(x, \mathcal{L}_i).$$

Since elements in each $\mathcal{L}_i$ are pairwise disjoint, for all i, $St(x, \mathcal{L}_i) = L_{xi}$ for some $L_{xi} \in \mathcal{L}_i$. Since $W \cap \partial L = \emptyset$, it follows that there exists i_0 so that for each $i > i_0$

$$\pi_H^{-1}(\pi_H(\partial L)) \subset \cup_{x \in \partial L \setminus A_1} L_{xi}$$

and

$$W \cap \pi_H^{-1}(\pi_H(\partial L)) = W \cap A_2 \subset \cup_{x \in \partial L \setminus A_1} \Big(T_{xi} \cup D_{xi} \cup B^{r+1} \times I\Big) \times Q_{m(r)+3}.$$

It is clear that $\big\{\big(T_{xi} \cup D_{xi} \cup B^{r+1} \times I\big)\big\}_{x \in \partial L \setminus A_1}$ is finite and each admits $\frac{1}{i+1}-$ map to a $1-$complex. Therefore, $W \cap A_2$ is 1-dimensional. $\square$

Lemma 3.3.4. *Let $\mathcal{L} = \{\mathcal{L}_2, \mathcal{L}_5, \mathcal{L}_9, \ldots\}$ be the defining sequence constructed in the previous section, let H be the decomposition associated with $\mathcal{L}$, and let $\pi_H : Q \to Q/H$ be the natural quotient map. Let $\partial\mathcal{L} = \cup_{i\geq 1}\{\partial L : L \in \mathcal{L}_i\}$. Then $\pi_H(N_{\pi_H}) - \pi_H(\partial\mathcal{L})$ is 0-dimensional.*

Proof. We will show that $\pi_H(N_{\pi_H}) - \pi_H(\partial\mathcal{L})$ is $0-$dimensional at each point $S \in \pi_H(N_{\pi_H}) - \pi_H(\partial\mathcal{L})$. Let $S \in \pi(N_{\pi_H}) - \pi_H(\partial\mathcal{L})$ and U be an open neighborhood of S in Q/H. Then $V = \pi^{-1}(U)$ is open in Q. Let

$$V^* = \cup\{g \in H \mid g \subset V\}.$$

Since H is upper semicontinuous, V^* is saturated open in Q. Then $\pi(V^*)$ is open in Q/H which implies that $W = \pi(V^*) \cap \Big(\pi_H(N_{\pi_H}) - \pi_H(\partial\mathcal{L})\Big)$ is open in $\pi_H(N_{\pi_H}) - \pi_H(\partial\mathcal{L})$. Clearly, $\partial W = \emptyset$ and $W \subset U$. Then it follows that $\pi_H(N_\pi) - \pi_H(\partial\mathcal{L})$ is $0-$dimensional at point S. Since S is arbitrary, $\pi_H(N_{\pi_H}) - \pi_H(\partial\mathcal{L})$ is $0-$dimensional. $\square$

Recall that $H_*, \check{H}_*$ denote the singular and Čech homology respectively with respect to integer coefficients.

Corollary 3.3.5. *For the decomposition H associated with the defining sequence $\mathcal{L}$, $\pi^{-1}(x)$ has infinite codimension in Q for every $x \in Q/H$.*

Proof. Let $x \in Q/H$. If $x \notin \pi_H(N_{\pi_H})$, then $\pi_H^{-1}(x)$ is a singleton in Q and hence it has infinite codimension in Q. Assume that $x \in \pi_H(N_{\pi_H})$. It follows from Theorem 3.3.4 that x is one dimensional. Then by Lemma 1.7.4[DW81], $\pi_H^{-1}(x)$ has infinite codimension in Q. $\square$

We will use the following theorem to show that points in Q/H have infinite codimension.

Theorem 3.3.6. *[Beg50](Vietoris-Begle Mapping Theorem) Let X and Y be compact metric spaces, and let $f : X \to Y$ be surjective and continuous. Suppose*

$$H_i^{\#}(f^{-1}(x)) \cong 0, \qquad i = 0, 1, \ldots, n-1, \quad \textit{for} \;\; x \in Y.$$

Then $H_i^{\#}(X) \cong H_i^{\#}(Y)$.

Note that this also can apply to the homology of pairs and the conditions on $f^{-1}(x)$ are satisfied for cell-like sets.

Corollary 3.3.7. *Points in Q/H have infinite codimension in Q/H.*

Proof. The result follows from Corollary 3.3.5 and Theorem 3.3.6. For more details, see [Lay80]. $\square$

Lemma 3.3.8. *[Lay80] For each $i \geq 1$, let $L \in \mathcal{L}_i$. Q/H has disjoint Čech carriers at $\pi_H(\partial L)$. Consequently, Q/H has disjoint Čech carriers at $\pi_H(\partial \mathcal{L})$.*

Proof. (modified from [Lay80]) Let $V_1 \subset U_1$ and $V_2 \subset U_2$ be open pairs in Q/H. Let $q_1, q_2 \geq 0$ and let

$$z_1 \in H_{q_1}(U_1, V_1), z_2 \in H_{q_2}(U_2, V_2).$$

Then choose

$$\bar{z}_1 \in H_{q_1}(\pi_H)^{-1}(U_1), \pi_H^{-1}(V_1)), \bar{z}_2 \in H_{q_2}(\pi_H^{-1}(U_2), \pi_H^{-1}(V_2))$$

such that $(\pi_H)_*(\bar{z}_1) = z_1$ and $(\pi_H)_*(\bar{z}_2) = z_2$. Since Q is a $Q-$manifold, it follows that there exist Čech carriers $(C_1, \partial C_1)$ for $\bar{z}_1$ and $(C_2, \partial C_2)$ for $\bar{z}_2$ such that $C_1 \cap C_2 = \emptyset$. Let $(W_1, \partial W_1)$ be a compact neighborhood of $(C_1, \partial C_1)$ such that $(W_1, \partial W_1) \subset (\pi_H^{-1}(U_1), \pi_H^{-1}(V_1))$ and such that $W_1 \cap C_2 = \emptyset$. Then by Lemma 3.3.3,

$$\dim(\pi_H^{-1}(\pi_H(C_2 \cap \partial L)) \cap W_1) \leq 1.$$

This implies that

$$\pi_H^{-1}(\pi_H(C_2 \cap \partial L)) \cap W_1$$

has infinite codimension in Q. Consider $(Int(W_1), Int(\partial W_1))$, then since $\pi_H^{-1}(\pi_H(C_2 \cap \partial L)) \cap W_1$ has infinite codimension, we can choose a Čech carrier $(C_1', \partial C_1')$ for $\bar{z}_1$ with

$$(C_1', \partial C_1') \subset (Int(W_1), Int(\partial W_1))$$

and $C_1' \cap \pi_H^{-1}(\pi_H(C_2 \cap \partial L)) = \emptyset$. By a similar argument, we can find a Čech carrier C_2' such that

$$C_2' \cap \pi_H^{-1}(\pi_H(C_1' \cap \partial L)) = \emptyset = C_1' \cap \pi_H^{-1}(\pi_H(C_2' \cap \partial L)).$$

Now obtain disjoint compact neighborhoods $(\bar{W}_i, \partial \bar{W}_i)$ of $(C_i', \partial C_i'), i = 1, 2$, such that $(\bar{W}_i, \partial \bar{W}_i) \subset (\pi_H^{-1}(U_i), \pi_H^{-1}(V_i))$ and such that

$$\bar{W}_1 \cap \pi_H^{-1}(\pi_H(\bar{W}_2 \cap \partial L)) = \emptyset = \bar{W}_2 \cap \pi_H^{-1}(\pi_H(\bar{W}_1 \cap \partial R)).$$

The set

$$W = \pi_H^{-1}\Big(\pi_H(\bar{W}_1) \cap \pi_H(\bar{W}_2) \cap \pi_H(\partial L)\Big) \cap \partial L$$

is a compact set disjoint from $\bar{W}_1 \cup \bar{W}_2$. Find Čech carriers $(\bar{C}_i, \partial \bar{C}_i)$ for $\bar{z}_i, i = 1, 2$, such that $(\bar{C}_i, \partial \bar{C}_i) \subset (\bar{W}_i, \partial \bar{W}_i)$ and $\pi_H^{-1}(\pi_H(W)) \cap \bar{C}_i = \emptyset$. It follows that $(\pi_H(\bar{C}_i), \pi_H(\partial \bar{C}_i))$ is a Čech carrier for $z_i, i = 1, 2$, and that $\pi_H(\bar{C}_1) \cap \pi_H(\bar{C}_2) \cap \pi_H(\partial A) = \emptyset$. Thus, Q/H has disjoint Čech carriers at $\pi_H(\partial L)$. By Lemma 1.7.13, Q/H has disjoint Čech carriers at $\pi_H(\partial \mathcal{L})$. □

Lemma 3.3.9. *Let A be a closed subset in $\pi_H(Q - N_{\pi_H})$. Then Q/H has Disjoint Čech carriers at A.*

Proof. Let $V_1 \subset U_1$ and $V_2 \subset U_2$ be open pairs in Q/H. Let $q_1, q_2 \geq 0$ and let

$$z_1 \in H_{q_1}(U_1, V_1), z_2 \in H_{q_2}(U_2, V_2).$$

Then choose

$$\bar{z}_1 \in H_{q_1}(\pi_H)^{-1}(U_1), \pi_H^{-1}(V_1)), \bar{z}_2 \in H_{q_2}(\pi_H^{-1}(U_2), \pi_H^{-1}(V_2))$$

such that $(\pi_H)_*(\bar{z}_1) = z_1$ and $(\pi_H)_*(\bar{z}_2) = z_2$. Since Q is a $Q-$manifold, it follows that there exist Čech carriers $(C_1, \partial C_1)$ for $\bar{z}_1$ and $(C_2, \partial C_2)$ for $\bar{z}_2$ such that $C_1 \cap C_2 = \emptyset$. Since $A \subset \pi_H(Q - N_{\pi_H})$ and π_H is one-to-one on $Q - N_{\pi_H}$, it follows that $\pi_H \pi_H^{-1}(A) = A$. Thus consider $W_1 = \pi_H(C_1), W_2 = \pi_H(C_2)$. Then W_1 and W_2 are Čech Carriers for z_1 and z_2 respectively, and $W_1 \cap W_2 \subset \pi_H(N_\pi)$ and so $W_1 \cap W_2 \cap A = \emptyset$. Therefore, Q/H has Disjoint Čech Carriers at A. □

Theorem 3.3.10. *Q/H has the Disjoint Čech Carrier Property.*

Proof. We will show that Q/H has Čech Carrier with inifinite codimension and then apply Lemma 1.7.11. First note that

$$Q/H = \pi(\partial\mathcal{L}) \bigcup \Big(\pi_H(N_{\pi_H}) - \pi(\partial\mathcal{L})\Big) \bigcup \cup A_i$$

where $\cup A_i = \pi_H(Q - N_{pi_H})$ with A_i closed. By Lemma 3.3.8 and Lemma 3.3.9 Q/H has Disjoint Čech Carriers at $\pi\partial\mathcal{L}$ and A_i, respectively, and so at $U = \pi(\partial\mathcal{L}) \bigcup \cup A_i$ by Lemma 1.7.13. Then by Lemma 1.7.14,Q/H has Disjoint Čech Carrier C whose intersections with the set U have infinite codimension. But we know that

$$C = (C \cap U) \bigcup \Big(C \cap (\pi_H(N_{\pi_H} - \pi(\partial\mathcal{L})\Big)$$

which is the union of a set with infinite codimension (in Q/H) and a finite dimensional set, and so by Lemma 1.7.6, C has infinite codimension. Therefore, by Lemma 1.7.11, Q/H has the Disjoint Čech Carriers Property, completing the proof. □

To prove that $Q/H \times I^2$ is a $Q-$manifold, we use Lemma 1.7.15. Now we prove the following theorem.

Theorem 3.3.11. *Let H be a decomposition of Q defined as above. Then $Q/H \times I^2$ is a $Q-$manifold.*

Proof. First note that Q/H is an ANR. By Lemma 3.3.10, we have Q/H has Disjoint Čech Carriers Property, and hence by Lemma 1.7.15, $Q/H \times I^2$ is a $Q-$manifold. □

4. A DECOMPOSITION FROM THE CANTOR FUNCTION

The aim of this chapter is to produce another decomposition that will be used in obtaining the main results in the next section. This decomposition will be constructed using the Cantor function.

4.1 Product of Cantor Sets

Definition 4.1.1. *Let $n \in \mathbb{Z}^{+}$. Define an interval $\Theta(k_1, \ldots, k_n)$ by*

$$\Theta(k_1, \ldots, k_n) = \left[\sum_{i=1}^{i=n} \frac{k_i}{3^i}, \sum_{i=1}^{i=n} \frac{k_i}{3^i} + \frac{1}{3^n} \right] \text{ for } k_i \in \{0, 1, 2\}$$

Note: it is clear by the definition that the length of $\Theta(k_1, \ldots, k_n)$ is $\frac{1}{3^n}$.

Let

$$C_n = \{\Theta(k_1, k_2, \ldots, k_n) \mid k_i \in \{0, 2\}\}.$$

Let $C_1^c = \{\Theta(1)\}$, and for $n \geq 2$, define

$$C_n^c = \{\Theta(k_1, \ldots, k_{n-1}, 1) \mid k_i \in \{0, 2\}\}.$$

Let ∂C_n be the set of all end points of the intervals in C_n. If we unravel the notation of C_n, we see that

$$C_1 = \{\Theta(0), \Theta(2)] = \left\{ \left[0, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right] \right\}$$

$$\begin{aligned} C_2 &= \{\Theta(0,0), \Theta(0,2), \Theta(2,0), \Theta(2,2)] \\ &= \left\{ \left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{3}{9}\right], \left[\frac{6}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right] \right\} \end{aligned}$$

and

$$C_1^c = \left\{ \left[\frac{1}{3}, \frac{2}{3} \right] \right\}$$

$$C_2^c = \{\Theta(0,1), \Theta(2,1)] = \left\{ \left[\frac{1}{9}, \frac{2}{9} \right], \left[\frac{7}{9}, \frac{8}{9} \right] \right\}$$

$$\begin{aligned} C_3^c &= \{\Theta(0,0,1), \Theta(0,2,1), \Theta(2,0,1), \Theta(2,2,1)] \\ &= \left\{ \left[\frac{1}{27}, \frac{2}{27} \right], \left[\frac{7}{27}, \frac{8}{27} \right], \left[\frac{19}{27}, \frac{20}{27} \right], \left[\frac{25}{27}, \frac{26}{27} \right] \right\}. \end{aligned}$$

Note that for each n

$$[0,1] = \cup C_n \cup \left(\cup_{j=1}^{n} C_j^c \right)$$

Definition 4.1.2. *For each $\Theta = \Theta(k_1, \ldots, k_n) \in C_n \cup C_n^c$ with $n \geq 2$, define*

$$J(\Theta) = \begin{cases} 3\Theta & \text{if } k_1 = 0 \\ 3\Theta - 2 & \text{if } k_1 = 2. \end{cases}$$

We can see from the definition that

$$3\Theta(0,1) = 3\left[\frac{1}{9}, \frac{2}{9} \right] = \left[\frac{1}{3}, \frac{2}{3} \right] \in C_1^c$$

$$3\Theta(2,1) - 2 = 3\left[\frac{7}{9}, \frac{8}{9} \right] - 2 = \left[\frac{1}{3}, \frac{2}{3} \right] \in C_1^c.$$

The following lemma tells us that this is true for all $n \geq 2$.

Lemma 4.1.3. *If $\Theta = \Theta(k_1, \ldots, k_n) \in C_n \cup C_n^c$ with $n \geq 2$, then $J(\Theta) \in C_{n-1} \cup C_{n-1}^c$.*

Proof. Let $\Theta \in C_n \cup C_n^c$.

Case I: $\Theta \in C_n^c$. Assume that $\Theta = \Theta(k_1, \ldots, k_{n-1}, 1) \in C_n^c$ for some $k_1, \ldots, k_{n-1} \in \{0, 2\}$. If $k_1 = 0$, then

$$\begin{aligned} J(\Theta) &= 3\Theta \\ &= 3\Theta(k_1, \ldots, k_{n-1}, 1) \\ &= \left[3\left(\sum_{i=1}^{i=n-1} \frac{k_i}{3^i} + \frac{1}{3^n} \right), 3\left(\sum_{i=1}^{i=n-1} \frac{k_i}{3^i} + \frac{2}{3^n} \right) \right] \\ &= \left[\sum_{i=1}^{i=n-1} \frac{k_i}{3^{i-1}} + \frac{1}{3^{n-1}}, \sum_{i=1}^{i=n-1} \frac{k_i}{3^{i-1}} + \frac{2}{3^{n-1}} \right] \\ &= \Theta(k_2, \ldots, k_{n-1}, 1) \in C_{n-1}^c. \end{aligned}$$

Similarly, if $k_2 = 2$, then

$$J(\Theta) = 3\Theta - 2 = \Theta(k_2, \ldots, k_{n-1}, 1) \in C_{n-1}^c.$$

Case II: $\Theta \in C_n$

Assume that $\Theta = \Theta(k_1, \ldots, k_{n-1}, k_n) \in C_n^c$ for some $k_1, \ldots, k_n \in \{0, 2\}$. If $k_1 = 0$, then

$$\begin{aligned} J(\Theta) &= 3\Theta \\ &= 3\Theta(k_1, \ldots, k_{n-1}, k_n) \\ &= \left[3\left(\sum_{i=1}^{i=n} \frac{k_i}{3^i}\right), 3\left(\sum_{i=1}^{i=n} \frac{k_i}{3^i} + \frac{1}{3^n}\right)\right] \\ &= \left[\sum_{i=1}^{i=n} \frac{k_i}{3^{i-1}}, \sum_{i=1}^{i=n} \frac{k_i}{3^{i-1}} + \frac{1}{3^{n-1}}\right] \\ &= \Theta(k_2, \ldots, k_{n-1}, k_n) \in C_{n-1}. \end{aligned}$$

Similarly, if $k_2 = 2$, then

$$J(\Theta) = 3\Theta - 2 = \Theta(k_2, \ldots, k_{n-1}, k_n) \in C_{n-1}.$$

This completes the proof. $\square$

Now we will use the definition of C_n to define the Cantor set. That is, the Cantor set defined as:

$$C = \bigcap_{n=0}^{\infty} \bigcup C_n.$$

One can show that each element in C can be written as a ternary representation consisting entirely of zeros or twos.

The standard Cantor map $f : I \to I$ is also defined as a constant on the closure of each component of $I \setminus C$ and on C is defined by:

$$f\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i}\right) = \sum_{i=1}^{\infty} \frac{a_i}{2^{i+1}},$$

where a_i is either zero or two.

Remark 4.1.4. *Let D be the set of all dyadic rationals in the closed unit interval. That is,*

$$D = \{\frac{m}{2^n} \in [0,1] \mid m, n \in \mathbb{Z}\}$$

1. *If $d \in I$, the*

$$f^{-1}(d) = \begin{cases} 1 - cell & \text{if } d \in D \\ singleton & \text{if } d \notin D. \end{cases}$$

2. *$f|_C$ is two-to-one over the dyadic rationals in D;*

3. *$f|_C$ is one-to-one over the complement of dyadic rationals D;*

4. *f itself is one-to-one over the complement of dyadic rationals D.*

By Remark 4.1.4, for $p \in C$,

p is triadic rational if and only if $f(p)$ is dyadic rational.

Thus, if $f(p) = \frac{m}{2^n}$ for some m, n, then by Remark 4.1.4(2), $p = \frac{2k}{3^n}$ or $p = \frac{2k+1}{3^n}$ for some k.

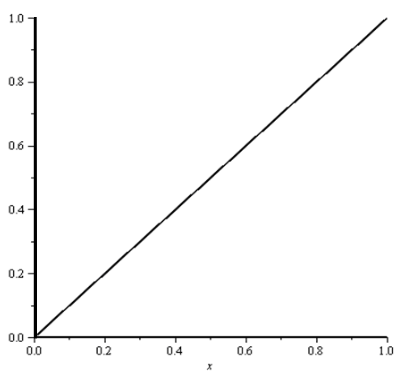
In [Cha91], Chalice proved that there is a sequence of step functions that converges to the Cantor function. Properties of the Cantor function as a limit of continuous function are also well known, see for example [Dob96]. Also see [DMRV06] for a survey of results about the Cantor function. For completeness, below we define a sequence $\{f_n\}$ of continuous functions on the unit interval that converges to the Cantor function and prove the standard results about these functions.

Let $f_0(x) = x$. Then, for each $n \in \mathbb{Z}^+$, the function f_{n+1} is defined in terms of function f_n as follows:

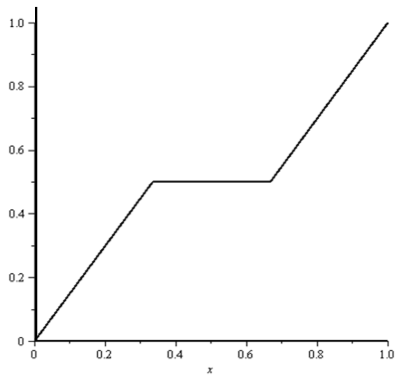
$$f_{n+1} = \begin{cases} \frac{1}{2} f_n(3x) & \text{for} \quad 0 \le x \le \frac{1}{3} \\ \frac{1}{2} & \text{for} \quad \frac{1}{3} \le x \le \frac{2}{3} \\ \frac{1}{2} + \frac{1}{2} f_n(3x - 2) & \text{for} \quad \frac{2}{3} \le x \le 1. \end{cases}$$

Because $f_n(0) = 0$ and $f_n(1) = 1$ for all n, by induction, the function f_n is continuous.

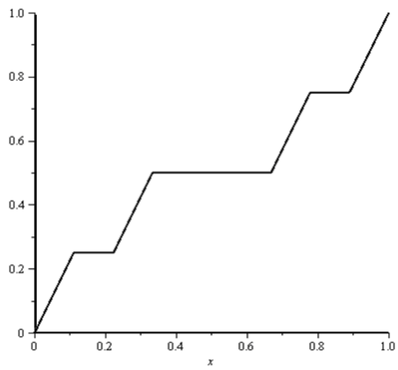
Figure 4.1 below demonstrates the graphs of f_n for $n = 0, \ldots, 3$.



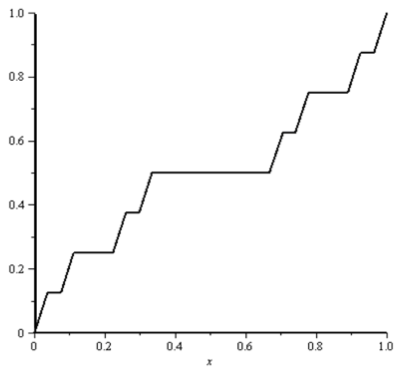
(a) $n = 0$



(b) $n = 1$



(c) $n = 2$



(d) $n = 3$

FIGURE 4.1: The graph of f_n for $n = 0, 1, 2, 3$.

Also, we have the following lemma.

Lemma 4.1.5. *For each n, let $\Theta \in C_n \cup \cup_{j=1}^{n} C_j^c$. Then we have the following:*

1.

$$f_n(\Theta) = \begin{cases} f_{n-1}(\Theta) & \text{if} \quad \Theta \in \cup_{j=1}^{n-1} C_j^c \\ \sum_{i=1}^{n-1} \frac{k_i}{2^{i+1}} + \frac{1}{2^n} & \text{if} \quad \Theta \in C_n^c. \end{cases}$$

2. f_n on Θ is a linear map joining $(l, f_n(l))$ and $(r, f_n(r))$ with slope $\left(\frac{3}{2}\right)^n$ if $\Theta \in C_n$ and l, r are the left and the right end points of Θ.

Proof. Let $\Theta \in C_n \cup \cup_{j=1}^{n} C_j^c$. We consider two cases.

Case I: $\Theta \in \cup_{j=1}^{n} C_j^c$, corresponding to the statement 1. So, we will show the statement 1 by induction on n.

For $n = 1$, $\Theta \in C_1^c = \left\{ \left[\frac{1}{3}, \frac{2}{3} \right] \right\}$. Thus,$\Theta = \Theta(1)$ by definition of the function f_1,

$$f_1(\Theta) = \frac{1}{2}$$

For $n = 2$,

$$\begin{aligned} f_2(\Theta(0,1)) &= \frac{1}{4} = \frac{0}{4} + \frac{1}{2^2} \\ f_2(\Theta(2,1)) &= \frac{3}{4} = \frac{2}{4} + \frac{1}{2^2} \\ f_2(\Theta(1)) &= f_1(\Theta(1)). \end{aligned}$$

Next, assume that for $n = k$, for $\Theta \in \cup_{j=1}^{k} C_j^c$,

$$f_k(\Theta) = \begin{cases} f_{k-1}(\Theta) & \text{if} \quad \Theta \in \cup_{j=1}^{k-1} C_j^c \\ \sum_{i=1}^{k-1} \frac{k_i}{2^{i+1}} + \frac{1}{2^k} & \text{if} \quad \Theta = \Theta(k_1, \ldots, k_{k-1}, 1) \in C_k^c \end{cases}$$

$\Theta(k_1, \ldots, k_k, 1)$.

We will show that it is also true for $n = k+1$. Let $\Theta \in \cup_{j=1}^{k+1} C_j^c$. If $\Theta \in \cup_{j=1}^{k} C_j^c$, then $\Theta = \Theta(k_1, \ldots, k_{l-1}, 1) \in C_l^c$ for some $l \leq k$. So, $J(\Theta) \in C_{l-1}^c$. By the definition of f_k and by the inductive hypothesis,

$$f_k(\Theta) = \begin{cases} \frac{1}{2} f_{k-1}(J(\Theta)) = \frac{1}{2} f_k(J(\Theta)) & \text{if} \quad k_1 = 0 \\ \frac{1}{2} + \frac{1}{2} f_{k-1}(J(\Theta)) = \frac{1}{2} + \frac{1}{2} f_k(J(\Theta)) & \text{if} \quad k_1 = 2. \end{cases}$$

Then by the definition of f_{k+1} in either case we have

$$f_{k+1}(\Theta) = f_k(\Theta)$$

Assume that $\Theta \in C_{k+1}^c$. Then $\Theta = \Theta(k_1, \ldots, k_k, 1)$ for some $k_1, \ldots, k_k \in \{0, 2\}$. So, $J(\Theta) = \Theta(k_2, \ldots, k_k, 1) \in C_k^c$. By the inductive hypothesis, we have

$$f_k(J(\Theta)) = \begin{cases} \sum_{i=1}^{k} \frac{k_i}{2^i} + \frac{1}{2^k} & \text{if} \quad k_1 = 0 \\ \sum_{i=2}^{k} \frac{k_i}{2^i} + \frac{1}{2^k} & \text{if} \quad k_1 = 2. \end{cases}$$

It follows that

$$f_{k+1}(\Theta) = \begin{cases} \frac{1}{2} f_k (J(\Theta)) = \frac{1}{2}\left(\sum_{i=1}^{k} \frac{k_i}{2^i} + \frac{1}{2^k}\right) = \sum_{i=1}^{k} \frac{k_i}{2^{i+1}} + \frac{1}{2^{k+1}} & \text{if } \ k_1 = 0 \\ \frac{1}{2} + \frac{1}{2} f_k (J(\Theta)) = \frac{1}{2} + \frac{1}{2}\left(\sum_{i=2}^{k} \frac{k_i}{2^i} + \frac{1}{2^k}\right) = \sum_{i=1}^{k} \frac{k_i}{2^{i+1}} + \frac{1}{2^{k+1}} & \text{if } \ k_1 = 2. \end{cases}$$

Case II: $\Theta \in C_n$. We will show that f_n is a linear map joining the left and the right end points of Θ with the slope $\left(\frac{3}{2}\right)^n$. Again we will prove this by induction on n.

For $n = 1$,

$$f_1(x) = \frac{1}{2}(3x) \quad \text{for all } x \in \Theta(0) = \left[0, \frac{1}{3}\right]$$
$$f_1(x) = \frac{1}{2} + \frac{1}{2}(3x - 2) \quad \text{for all } x \in \Theta(0) = \left[\frac{2}{3}, 1\right].$$

Assume that for $n = k$, if $\Theta \in C_k$, then f_k is a linear map joining the left and the right end points of Θ with the slope $\left(\frac{3}{2}\right)^n$. We will show that this statement in also true for $n = k+1$. Let $\Theta \in C_{k+1}$. Then $\Theta = \Theta(k_1, \ldots, k_{k+1})$ for some $k_1, \ldots, k_{k+1} \in \{0, 2\}$. So, $J(\Theta) \in C_k$. By the inductive hypothesis, f_k is a linear function joining the right and left end point of $J(\Theta)$ with the slope $\left(\frac{3}{2}\right)^k$. Thus, by the definition of function f_{k+1}, on Θ,

$$f_{k+1}(x) = \begin{cases} \frac{1}{2} f_k(3x) & \text{if } \ k_1 = 0 \text{ and } 3x \in J(\Theta) \\ \frac{1}{2} + \frac{1}{2} f_k(3x - 2) & \text{if } \ k_1 = 2 \text{ and } 3x - 2 \in J(\Theta). \end{cases}$$

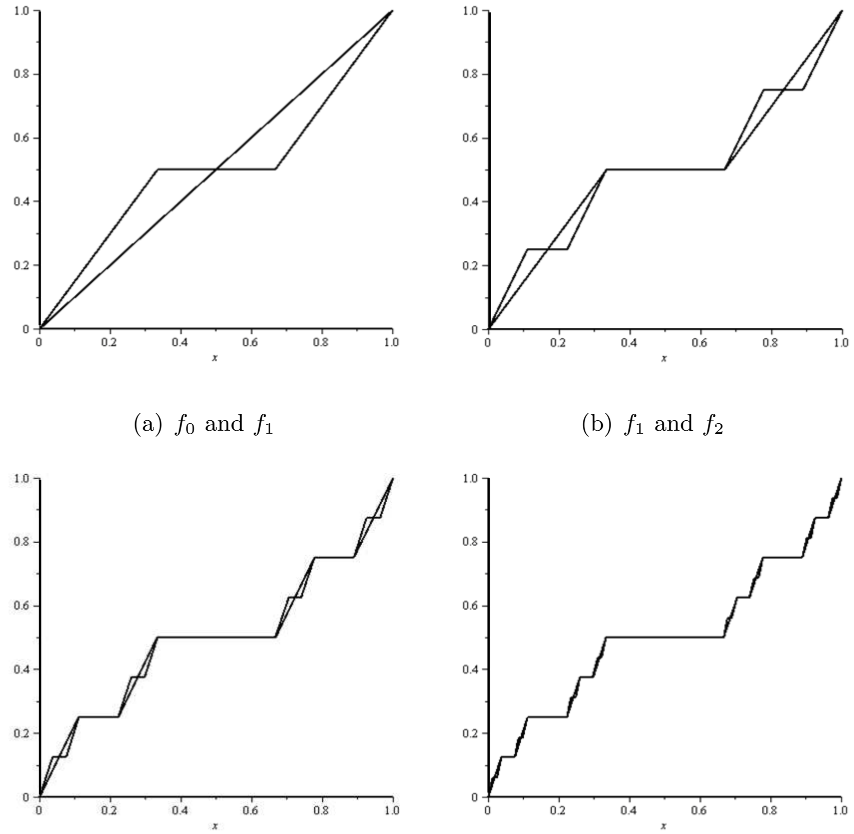
Since f_k is a linear function on $J(\Theta)$, it is clear that f_{k+1} is a linear function on Θ joining the left and the right end points of I with the slope $\left(\frac{3}{2}\right)^{k+1}$. $\square$

The following Figure 4.2 shows the graphs of f_n and f_{n-1} on the same axis.

Let $g : I \to I$ be defined by

$$g(x) = \lim_{n \to \infty} f_n(x).$$

Later in Lemma 4.1.9 we will show that g exists by showing that f_n is uniformly Cauchy on $[0, 1]$.



(a) f_0 and f_1 (b) f_1 and f_2

(c) f_2 and f_3 (d) f_3 and f_4

FIGURE 4.2: The graph of f_n and f_{n-1} for $n = 0, \ldots, 4$ on the same axis

Lemma 4.1.6. *Let TR be the set of all triadic rationals in the unit interval. That is,*

$$TR = \left\{ \frac{m}{3^n} \in [0,1] \;\middle|\; m, n \in \mathbb{Z} \right\}.$$

Then f and g agree on the set TR.

Proof. If $p = 0$ or $p = 1$, then it is obvious that $f(p) = g(p)$. Next assume that $p \neq 0, 1$.

Let $p = \frac{m}{3^n} \in TR$ for some m, n. Then $p \in C_n \cup \left(\cup_{j=1}^{n} C_j^c \right)$.

Case I: $p \in C_n$. Then $p \in \Theta(k_1, \ldots, k_n) \in C_n$ for some $k_1, \ldots, k_n \in \{0, 2\}$. That is, $p \in \left[\sum_{i=1}^{n} \frac{k_i}{3^i}, \sum_{i=1}^{n} \frac{k_i}{3^n} + \frac{1}{3^n} \right]$. Since $\Theta(k_1, \ldots, k_n)$ has length $\frac{1}{3^n}$ and $p \in \Theta(k_1, \ldots, k_n)$, this implies $p = \sum_{i=1}^{n} \frac{k_i}{3^i}$ or $p = \sum_{i=1}^{n} \frac{k_i}{3^i} + \frac{1}{3^n}$.

Next we will show that $p \in \cup_{j=1}^{n} C_j^c$. If $p = \sum_{i=1}^{n} \frac{k_i}{3^i}$, then one can write p as $p = \sum_{i=1}^{l} \frac{k_i}{3^i}$ where $l \leq n$, $k_l \neq 0$ but $k_j = 0$ for $j = l+1, \ldots, n$. It follows that

$$p = \sum_{i=1}^{l-1} \frac{k_i}{3^i} + \frac{2}{3^l} \in \Theta\left(k_1, \ldots, k_{l-1}, 1\right) \in C_l^c.$$

Thus, by Lemma 4.1.4,

$$f_{i+1}\left(p\right) = f_i\left(p\right) = \sum_{i=1}^{l-1} \frac{k_i}{2^{i+1}} + \frac{1}{2^l} \quad \text{for all } i \geq l$$

and so

$$g\left(p\right) = \lim_{i \to \infty} f_i\left(p\right) = f_l\left(p\right) = \sum_{i=1}^{l-1} \frac{k_i}{2^{i+1}} + \frac{1}{2^l} = f\left(\sum_{i=1}^{l-1} \frac{k_i}{3^i} + \frac{2}{3^l}\right) = f\left(p\right).$$

If $p = \sum_{i=1}^{n} \frac{k_i}{3^i} + \frac{1}{3^n}$, we consider two cases. For the first case, if $k_n = 0$, then $p = \sum_{i=1}^{n-1} \frac{k_i}{3^i} + \frac{1}{3^n} \in \Theta\left(k_1, \ldots, k_{n-1}, 1\right) \in C_n^c$. By Lemma 4.1.4,

$$f_{i+1}\left(p\right) = f_i\left(p\right) = \sum_{i=1}^{n-1} \frac{k_i}{2^{i+1}} + \frac{1}{2^n} \quad \text{for all } i \geq n$$

and so

$$g\left(p\right) = \lim_{i \to \infty} f_i\left(p\right) = f_n\left(p\right) = \sum_{i=1}^{n-1} \frac{k_i}{2^{i+1}} + \frac{1}{2^n} = f\left(\sum_{i=1}^{n-1} \frac{k_i}{3^i} + \frac{1}{3^n}\right) = f\left(p\right).$$

For the second case, assume that $k_n = 2$. Since $p \neq 1$, then $k_i = 0$ for some $i \in \{1, 2, \ldots, n\}$. Pick i_0 so that for all $i > i_0$, $k_i = 2$. Thus, we can write p as

$$p = \sum_{i=1}^{i_0-1} \frac{k_i}{3^i} + \frac{1}{3^{n-(n-i_0)}} = \sum_{i=1}^{i_0-1} \frac{k_i}{3^i} + \frac{1}{3^{i_0}}$$

and we can see that $p \in \Theta\left(k_1, \ldots, k_{i_0-1}, 1\right) \in C_{i_0}^c$. By Lemma 4.1.4,

$$f_{i+1}\left(p\right) = f_i\left(p\right) = \sum_{i=1}^{i_0-1} \frac{k_i}{2^{i+1}} + \frac{1}{2^{i_0}} \quad \text{for all } i \geq i_0$$

and so

$$g\left(p\right) = \lim_{i \to \infty} f_i\left(p\right) = f_{i_0}\left(p\right) = \sum_{i=1}^{i_0-1} \frac{k_i}{2^{i+1}} + \frac{1}{2^{i_0}} = f\left(\sum_{i=1}^{i_0-1} \frac{k_i}{3^i} + \frac{1}{3^{i_0}}\right) = f\left(p\right).$$

Case II: $p \in \cup_{j=1}^{n} C_j^c$. Then

$$p \in \Theta = \Theta\left(k_1, \ldots, k_{l-1}, 1\right) = \left[\sum_{i=1}^{i=l-1} \frac{k_i}{3^i} + \frac{1}{3^l}, \sum_{i=1}^{i=l-1} \frac{k_i}{3^i} + \frac{2}{3^l}\right]$$

for some $k_1, k_2, \ldots, k_{l-1} \in \{0, 2\}$. By Lemma 4.1.5,

$$f_{i+1}(p) = f_i(p) = \sum_{i=1}^{l-1} \frac{k_i}{2^{i+1}} + \frac{1}{2^l} \quad \text{for all } i \geq l$$

and since p and $\sum_{i=1}^{i=l-1} \frac{k_i}{3^i} + \frac{1}{3^l}$ are in the same Θ, by the definition of the Cantor set f, we have

$$g(p) = \lim_{i \to \infty} f_i(p) = f_l(p) = \sum_{i=1}^{l-1} \frac{k_i}{2^{i+1}} + \frac{1}{2^l} = f\left(\sum_{i=1}^{l-1} \frac{k_i}{3^i} + \frac{1}{3^l}\right) = f(p).$$

□

For each n, define $g_n : I \to I$ by

$$g_n(x) = f_n(x) - f_{n-1}(x) \quad \text{for all} \quad x \in I.$$

It is clear that g_n is continuous.

The following Figure 4.3 illustrates the graph of g_n for $n = 1, 2, 3, 4$.

Lemma 4.1.7. *For each $n \geq 2$,*

$$\max_{x \in \Theta(k_1, \ldots, k_{n-1})} |g_n| = \frac{1}{3 \cdot 2^n} \quad \textit{for all } \Theta\left(k_1, \ldots, k_{n-1}\right) \in C_{n-1}.$$

Consequently,

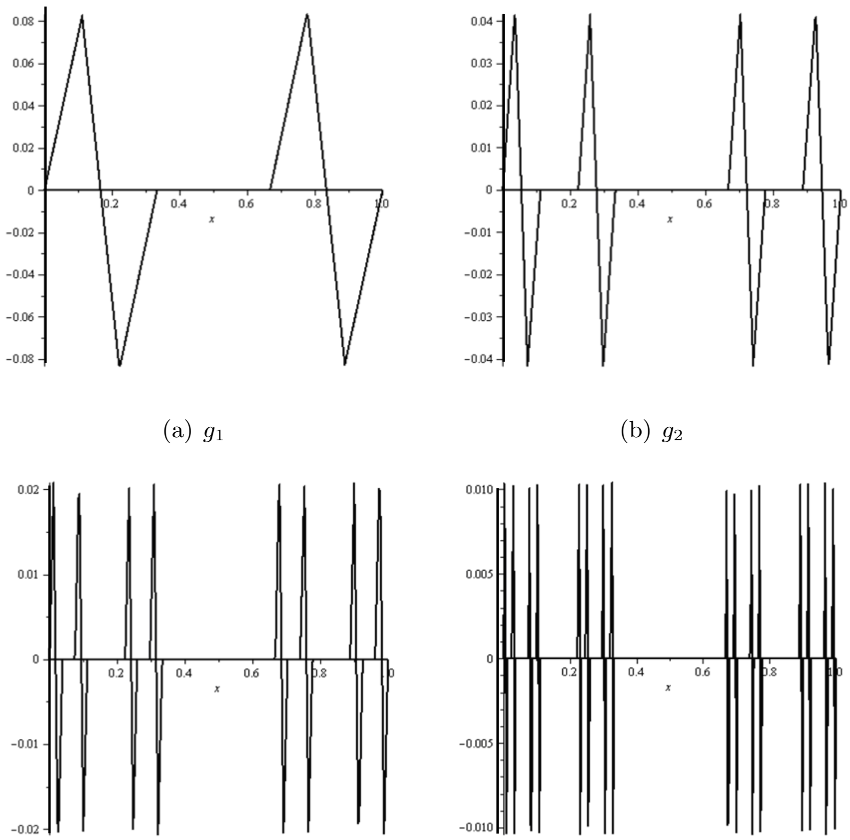
$$\max_{x \in \cup C_{n-1}} |g_n| = \frac{1}{3 \cdot 2^n}.$$

Proof. Let $n \in \mathbb{N}$. Let $\Theta\left(k_1, \ldots, k_{n-1}\right) = \left[\sum_{i=1}^{n-1} \frac{k_i}{3^i}, \sum_{i=1}^{n-1} \frac{k_i}{3^i} + \frac{1}{3^{n-1}}\right] \in C_{n-1}$. Denote

$$\Theta_1 = \left[\sum_{i=1}^{n-1} \frac{k_i}{3^i}, \sum_{i=1}^{n-1} \frac{k_i}{3^i} + \frac{1}{3^n}\right],$$

$$\Theta_2 = \left[\sum_{i=1}^{n-1} \frac{k_i}{3^i} + \frac{1}{3^n}, \sum_{i=1}^{n-1} \frac{k_i}{3^i} + \frac{2}{3^n}\right],$$

$$\Theta_3 = \left[\sum_{i=1}^{n-1} \frac{k_i}{3^i} + \frac{2}{3^n}, \sum_{i=1}^{n-1} \frac{k_i}{3^i} + \frac{1}{3^{n-1}}\right].$$



(a) g_1 (b) g_2

(c) g_3 (d) g_4

FIGURE 4.3: The graph of function g_n for $n = 1, \ldots, 4$.

as shown in Figure 4.4. Clearly, $\Theta(k_1, \ldots, k_{n-1}) = \Theta_1 \cup \Theta_2 \cup \Theta_3$. Define

$$G_1(x) = g_n|_{\Theta_1}, \quad G_2(x) = g_n|_{\Theta_2}, \quad G_3(x) = g_n|_{\Theta_3}.$$

By Lemma 4.1.5,

$$G_1\left(\sum_{i=1}^{n-1} \frac{k_i}{3^i}\right) = 0, \quad G_3\left(\sum_{i=1}^{n-1} \frac{k_i}{3^i} + \frac{1}{3^{n-1}}\right) = 0.$$

Also, the slope of G_1 and G_3 is $\frac{1}{2}\left(\frac{3}{2}\right)^{n-1}$. Thus, we can express the functions G_1 and G_3

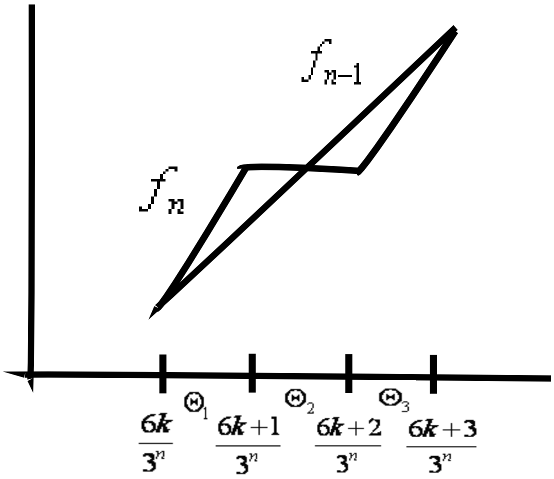


FIGURE 4.4: The graph of f_n and f_{n-1} on interval $\Theta\left(k_1, \ldots, k_{n-1}\right)$

in the form:

$$G_1(x) = \frac{1}{2}\left(\frac{3}{2}\right)^{n-1}\left(x - \sum_{i=1}^{n-1}\frac{k_i}{3^i}\right)$$

$$G_3(x) = \frac{1}{2}\left(\frac{3}{2}\right)^{n-1}\left(x - \left(\sum_{i=1}^{n-1}\frac{k_i}{3^i} + \frac{1}{3^{n-1}}\right)\right).$$

We can see that both G_1 and G_3 are increasing functions and for each $x \in \Theta_3$,

$$\begin{aligned}
-G_1\left(-x + 2\left(\sum_{i=1}^{n-1}\frac{k_i}{3^i}\right) + \frac{1}{3^{n-1}}\right) &= -\frac{1}{2}\left(\frac{3}{2}\right)^{n-1}\left(-x + 2\left(\sum_{i=1}^{n-1}\frac{k_i}{3^i}\right) + \frac{1}{3^{n-1}} - \sum_{i=1}^{n-1}\frac{k_i}{3^i}\right) \\
&= -\frac{1}{2}\left(\frac{3}{2}\right)^{n-1}\left(-x + \sum_{i=1}^{n-1}\frac{k_i}{3^i} + \frac{1}{3^{n-1}}\right) \\
&= \frac{1}{2}\left(\frac{3}{2}\right)^{n-1}\left(x - \left(\sum_{i=1}^{n-1}\frac{k_i}{3^i} + \frac{1}{3^{n-1}}\right)\right) \\
&= G_3.
\end{aligned}$$

On Θ_2 the slope of G_2 is $-\left(\frac{3}{2}\right)^{n-1}$ and so G_2 is a decreasing function on Θ_2. This implies that

$$\max_{x\in\Theta_1}\left(|G_2|\right) = G_2\left(\sum_{i=1}^{n-1}\frac{k_i}{3^i} + \frac{1}{3^n}\right) = G_1\left(\sum_{i=1}^{n-1}\frac{k_i}{3^i} + \frac{1}{3^n}\right).$$

Thus,

$$\max_{x\in\Theta_3}\left(|G_3|\right) = \max_{x\in\Theta_2}\left(|G_2|\right) = \max_{x\in\Theta_1}\left(|G_1|\right) = G_1\left(\sum_{i=1}^{n-1}\frac{k_i}{3^i} + \frac{1}{3^n}\right) = \frac{1}{3\cdot 2^n}.$$

This shows that

$$\max_{x\in\Theta(k_1,\ldots,k_{n-1})}(|g_n|)=\frac{1}{3\cdot 2^n}.$$

Consequently, $\max_{x\in\cup C_{n-1}}|g_n|=\frac{1}{3\cdot 2^n}$. □

Corollary 4.1.8. *For each $n\geq 2$,*

$$\max_{x\in[0,1]}|g_n|=\frac{1}{3\cdot 2^n}.$$

Proof. Let $n\geq 2$. Let $x\in\cup C_{n-1}$. Then by Lemma 4.1.7,

$$\max_{x\in\cup C_{n-1}}|g_n|=\frac{1}{3\cdot 2^n}.$$

On the other hand, for $x\notin\cup C_{n-1}$, by Lemma 4.1.5, $f_n(x)=f_{n-1}(x)$, and so $g_n(x)=0$. Therefore,

$$\max_{x\in[0,1]}|g_n|=\max_{x\in[0,1]}|f_n(x)-f_{n-1}(x)|=\frac{1}{3\cdot 2^n}.$$

□

For each n, define

$$D_n=\left\{\frac{b}{2^n}\;\middle|\;b\in\mathbb{Z}^+\text{ and }0\leq b\leq 2^n-1\right\}$$

and so

$$D=\cup_{n=0}^{\infty}D_n.$$

Note that the following properties hold. Fix n.

1. If $d\in\Theta$, the

$$f_n^{-1}(d)=\begin{cases}1-\text{cell} & \text{if } d\in D_n\\ \text{singleton} & \text{if } d\notin D_n.\end{cases}$$

2. $f_n|_C$ is two-to-one over the dyadic rationals in D_n;

3. $f_n|_C$ is one-to-one over the complement of dyadic rationals D_n;

4. f_n itself is one-to-one over the complement of dyadic rationals D_n.

The next lemma guarantees that the sequence $\{f_n\}$ converges to the Cantor function f.

Lemma 4.1.9. *Let $\{f_n\}$ be the sequence of continuous functions defined as above. Then $\{f_n\}$ converges uniformly to the Cantor function f.*

Proof. Let $g(x) = \lim_{n\to\infty} f_n(x)$. Claim that $g = f$. First we will prove that g is continuous by showing that f_n converges uniformly to g. Given $\epsilon > 0$. There is N such that $\frac{1}{2^N} < \epsilon$. By Lemma 4.1.8, for each $n \geq 0$

$$\max_{x\in[0,1]} |f_{n+1}(x) - f_n(x)| = \frac{1}{3\cdot 2^{n+1}} \leq \frac{1}{2^{n+1}}$$

Without loss of generality, assume that $m > n > N$. It follows that

$$\begin{aligned}
\max_{x\in[0,1]} |f_m(x) - f_n(x)| &\leq \sum_{k=n}^{k=m-n-1} \max_{x\in[0,1]} |f_{k+1}(x) - f_k(x)| \\
&\leq \sum_{k=n}^{k=m-n-1} \frac{1}{2^{k+1}} \\
&= \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots + \frac{1}{2^m} \\
&= \frac{1}{2^n}\left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{m-n}}\right) \\
&< \frac{1}{2^n} < \epsilon.
\end{aligned}$$

This implies that f_n is a uniformly Cauchy sequence and hence the function $g(x) = \lim_{n\to\infty} f_n(x)$ is well-defined. Also, by the Cauchy Criterion, f_n converges uniformly on $[0,1]$ which implies g is continuous. It remains to show that $g = f$. By Lemma 4.1.6, g and f agree on the set TR. It follows that f and g agree on $[0,1]$ since TR is dense in $[0,1]$. This gives the desired result. □

Let $C^k \subset I^k$ be the product of k copies of C. For each n, let $f_n^k : I^k \to I^k$ be defined by:

$$f_n^k(x) = (f_n(x_1), f_n(x_2), \ldots, f_n(x_k)).$$

Let $f^k : I^k \to I^k$ be defined as the limit of a sequence f_n^k. That is,

$$\begin{aligned} f^k(x) &= \lim_{n\to\infty} f_n^k(x) \quad \text{for all } \ x \in I^k \\ &= \left(f(x_1), f(x_2), \ldots, f(x_k)\right). \end{aligned}$$

Note that f^k is continuous since each component is continuous. For example $f_2^2(x_1, x_2) = (f(x_1), f(x_2))$ where f is the Cantor map. The following Figure 4.5 shows how the map f_2^2 sends certain squares to points.

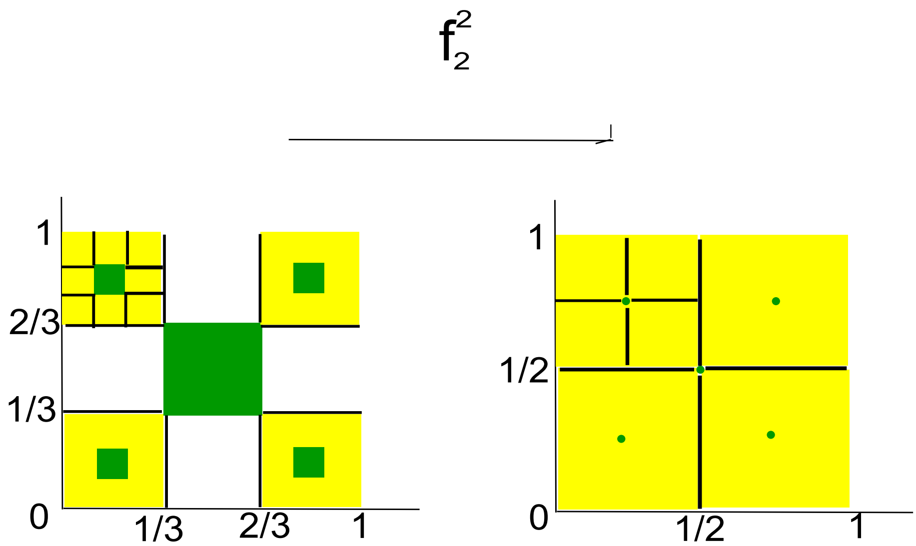


FIGURE 4.5: The function f_2^2.

Lemma 4.1.10. *Let $n \in \mathbb{Z}^+$ and $p \in I^k$. Then $\left(f_n^k\right)^{-1}(p)$ is either a point or a $l-$cell where l corresponds to the number of dyadic rational coordinates that p has, and hence $\left(f^k\right)^{-1}(p)$ is either a point or a $l-$cell.*

Proof. Fix n and let $p = (x_1, \ldots, x_k) \in I^k$. If p has no dyadic rational coordinates, then each $x_i \notin D$ and so obviously $x_i \notin D_n$. Thus, $(f_n)^{-1}(x_i)$ is just a point in I which implies that $\left(f_n^k\right)^{-1}(p)$ is a point in I^k. Next assume that the number of dyadic rational coordinates of p is l. Denote each b_i the dyadic rational coordinates of p for $i = 1, \ldots, l$. Then each $(f_n)^{-1}(b_i)$ is a 1-cell in I so $\left(f_n^k\right)^{-1}(p)$ is a $l-$cell in I^k. $\square$

We will generalize this idea to obtain the function $f^{\infty} : Q^2 \to Q^2$ where $Q^2 = \{0\} \times Q_2$. First for each n and k define $g_n^k : Q \to Q$ by

$$g_n^k\left((0, x_2, \ldots, x_k, \ldots)\right) = f_n^k\left((0, x_2, \ldots, x_k)\right) \times Id_{Q_{k+1}}\left(x_{k+1}, \ldots\right).$$

Thus, the function $f^{\infty} : Q^2 \to Q^2$ is defined by

$$\begin{aligned} f^{\infty}(x) &= \lim_{k\to\infty}\left(\lim_{n\to\infty} g_n^k(x)\right) \\ &= \left(0, f(x_2), f(x_3), \ldots\right). \end{aligned}$$

Since Q^2 is compact, it is obvious that f^{∞} is a closed map.

Lemma 4.1.11. *For each point $p \in Q^2$, $(f^{\infty})^{-1}(p)$ is either a point, a cell or a copy of Q^2 and the dimension of these sets corresponds to the number of dyadic rational coordinates that p has.*

Proof. If p has no dyadic rational coordinates, it is clear that $(f^{\infty})^{-1}(p)$ is just a point in Q^2. If p has l dyadic rational coordinates, then $(f^{\infty})^{-1}(p)$ is a $l-$cell in Q^2. If p has infinitely many dyadic rational coordinates, then $(f^{\infty})^{-1}(p)$ is a copy of Q^2. □

By Remark 4.1.4, it is clear that if p is a point of $\{0\} \times C^{\infty}$ with no triadic rational coordinates in C^{∞}, then $(f^{\infty})^{-1} \circ f^{\infty}(p) = p$.

Lemma 4.1.12. *If A is a nowhere dense subset of Q^2, then there exists a dense subset P of $\{0\} \times C^{\infty}$ so that $f^{\infty}(P) \cap A = \emptyset$ and $(f^{\infty})^{-1}(f^{\infty}(d)) = d$ for each d in P.*

Proof. Let $E \equiv \left\{x \in \{0\} \times C^{\infty} \mid (f^{\infty})^{-1}(f^{\infty}(x)) = x\right\}$.
Let $P \equiv \left(\{0\} \times C^{\infty} \setminus (f^{\infty})^{-1}(A)\right) \cap E$. We will show that P is dense in $\{0\} \times C^{\infty}$. Given any point $c \in \{0\} \times C^{\infty}$ and U a neighborhood of c in Q_2. Note that $U = U' \times Q_{k+1}$ for some k. Then $f^{\infty}(U' \times Q_{k+1}) = f^k(U') \times Q_{k+1}$ contains an open set $V \times Q_{k+1}$ in Q^2. Since A is nowhere dense, $V \times Q_{k+1}$ contains an open set $W \times Q_{k+1}$ in Q^2 with $W \times Q_{k+1} \cap A = \emptyset$. Choose a point p in $V \times Q_{k+1}$ for which none of its coordinates is dyadic rational. Then

$(f^{\infty})^{-1}(p)$ is a singleton, say $\{x\}$. We can see that x is in $U' \times Q_{k+1} \cap \{0\} \times C^{\infty}$ and so it is in P. $\square$

Let $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots\}$ be the defining sequence in Q defined for the Cantor set in the middle slice of Q in Chapter 2.

The following sets will be used in the chapter 5. For each j and $M(\epsilon_1, \ldots, \epsilon_j) \in \mathcal{M}_j$, define

$$P = \left[-\frac{1}{2^j}, \frac{1}{2^j}\right] \times f_{N(j)}^{m(j)+1}(M)$$

where $N(j) = \sum_{i=1}^{j} n_i$ and each n_i is defined in Chapter 2 and $m(0) = 2$ and for $j \geq 1$ $m(j) = k$ if there is an integer k such that $j = \frac{k(k-1)}{2} + i$ for some $i \in \{0, 1, \ldots, k-1\}$.

4.2 Construction of the Decomposition G

We now view f^{∞} as a map from $Q_2 \to Q_2$. To construct a decomposition G on Q, first we will use the function f^{∞} to define the decomposition G_f on Q^2. Explicitly,

$$G_f = \left\{\{0\} \times (f^{\infty})^{-1}(p) \mid p \in Q_2\right\}.$$

Theorem 4.2.1. *The decomposition G_f defined as above is upper semicontinuous.*

Proof. This follows from the fact that $\pi_{G_f} = \{0\} \times f^{\infty}$. Moreover, by the Lemma 4.1.11, G_f is cellular.

$\square$

Next will show that the decomposition G_f is realized by a pseudo-isotopy.

Lemma 4.2.2. *The decomposition G_f is realized by a pseudo-isotopy.*

Proof. Recall $f^{\infty} : Q_2 \to Q_2$ is a generalized Cantor function in which each component is the Cantor function $f : [0,1] \to [0,1]$. To show that the decomposition G_f is realized by a pseudo-isotopy, it suffices to show that there exists a pseudo-isotopy Ψ_t of $I^\infty \to I^\infty$ such that Ψ_0 is the identity Id_{I^∞} and $G_f = \left\{\Psi_1^{-1}(c) \mid c \in I^\infty\right\}$. For $t \in [0,1]$, define $\Psi_t : I^\infty \to I^\infty$ by

$$\Psi_t(x) = (1-t)x + tf^\infty(x).$$

It is clear that $\Psi_1 = f^\infty$ which is a closed surjection. For $t < 1$, Ψ_t is onto since each component is onto by the Intermediate Value Theorem. Also, it is continuous, and hence Ψ_t^{-1} is continuous since Ψ_t is a closed map. It remains to show that for $t < 1$, Ψ_t is one-to-one. Let $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots) \in I^\infty$ be such that $\Psi_t(x) = \Psi_t(y)$. Then

$$(1-t)x + tf^\infty(x) = (1-t)y + tf^\infty(y)$$

which implies that $(1-t)(x-y) = t(f^\infty(y) - f^\infty(x))$. If $x \neq y$, then there is i such that $x_i \neq y_i$. Without loss of generality, assume that $x_i < y_i$. We know that $(1-t)(x_i - y_i) = t(f(y_i) - f(x_i))$ Then the left hand side of equation is negative but the right hand side of equation is non-negative since the Cantor function f is non-decreasing function. This leads to a contradiction. Thus Ψ_t is one-to-one. Also, we can see that

$$G_f = \left\{\{0\} \times \Psi_1^{-1}(p) \mid p \in Q_2\right\}.$$

□

The next lemma follows from the fact that G_f is realized by a pseudo-isotopy.

Lemma 4.2.3. *Let G_f be the decomposition of $Q^2 = \{0\} \times Q_2$ induced by the map f^∞. Then π_{G_f} from $\{0\} \times Q_2$ to $(\{0\} \times Q_2)/G_f$ is approximable by homeomorphisms.*

Lemma 4.2.4. *The decomposition G_f is cellular.*

Proof. This follows from Lemma 4.1.11. □

Next we will define a decomposition G on Q. Let G be the partition consisting of $G_f = \left\{\{0\} \times (f^\infty)^{-1}(c) \mid c \in Q_2\right\}$ and all singletons in $Q - \{0\} \times Q_2$. It is clear that G is a usc decomposition of Q by a similar idea as shown in Lemma 4.2.2. Next we also show that G is realized by a pseudo-isotopy. First, define $K^1 : [0,3] \times Q_2 \times \to [0,1] \times Q_2$ by

$$K^1(s,x) = \left(\frac{s}{3}, x\right),$$

and for each $t \in [0,1]$, define $K_t^2 \;:\; [0,1] \times Q_2 \to [0,1] \times Q_2$ by

$$K_t^2(s,x) = (s, sx + (1-s)\Psi_t(x)) \quad \text{where} \quad \Psi_t(x) = (1-t)x + tf^\infty(x),$$

and define $K^3 : [0,1] \times Q_2 \to [0,3] \times Q_2$ by

$$K^3(s,x) = (3s, x).$$

We can see that K^1 and K^3 are homeomorphisms. Let

$$K_t(s,x) = K^3 \circ K_t^2 \circ K^1 : [0,3] \times Q_2 \to [0,3] \times Q_2.$$

Claim that for $t < 1$, K_t is homeomorphism. Clearly, K_t is onto, continuous and K_t^{-1} is continuous since K^1, K^2, and K^3 are. It remains to show that K_t is one-to-one. Since K^1, K^3 are one-to-one, it suffices to show that K_t^2 is one-to-one Suppose that $K_t^2(a,x) = K_t^2(b,y)$ for some $(a,x), (b,y) \in [0,1] \times Q_2$. Then by the definition of K_t^2 we have $a = b$ which implies that

$$ax + (1-a)\Psi_t(x) = ay + (1-a)\Psi_t(y).$$

By simplifying, we have

$$\begin{aligned}
\Psi_{t(1-a)}(x) &= (1 - t(1-a))x + t(1-a)f^\infty(x) \\
&= ax + (1-a)\Psi_t(x) \\
&= ay + (1-a)\Psi_t(y) \\
&= (1 - t(1-a))y + t(1-a)f^\infty(y) \\
&= \Psi_{t(1-a)}(y).
\end{aligned}$$

Since $t(1-a) \neq 1$, this yields $\Psi_{t(1-a)}$ is one-to-one and hence $x = y$. Therefore, K_t^2 is one-to-one. Moreover, we can see that for each $(s,x) \in (0,3] \times Q_2$, $K_1^{-1}(s,x)$ is singleton and if $(s,x) \in \{0\} \times Q_2$, $K_t(s,x) = \{0\} \times (f^\infty)^{-1}(x) \in G_f$. Thus,

$$G_1 = \left\{ K_1^{-1}(c) \mid c \in [0,3] \times Q_2 \right\} = S \cup G_f$$

where S is the set of all singleton in $(0,3] \times Q_2$.

Similarly, we can define a pseudo-isotopy $L_t : [-3,0] \times Q_2 \to [-3,0] \times Q_2$ by $L_t(s,x) = K_t(-s,x)$ so that

$$G_2 = \left\{ L_1^{-1}(c) \mid c \in [-3,0] \times Q_2 \right\} = R \cup G_f$$

where R is the set of all singleton in $[-3,0) \times Q_2$. Therefore, we see that $G = G_1 \cup G_2$ is realized by pseudo-isotopies K and L. Thus, we have the following lemma.

Lemma 4.2.5. *The decomposition G is realized by pseudo-isotopy.*

The result of Lemma 4.2.5 gives the following Theorem.

Theorem 4.2.6. *Let G be the decomposition defined as above. Then π_G from Q to Q/G is approximable by homeomorphisms.*

Lemma 4.2.7. *The decomposition G is cellular.*

Proof. This follows from Lemma 4.1.11. $\square$

5. MAIN RESULTS

The main result of this chapter is Theorem 5.4.1 where we show that if GH is the decomposition of Q associated with the defining sequence $\mathcal{R}$, then GH satisfies the following three properties:

- GH is cellular
- The nonmanifold part of Q/GH is homeomorphic to a copy of Q whose codimension is 1.
- If A is any closed subspace of X of codimension ≥ 1 in $\pi(N_\pi)$, then the decomposition of Q induced over A is shrinkable. That is, $Q/\pi^{-1}(A) \cong Q$.

Thus the manifold nature of Q/GH can not be detected by looking at finite dimensional subsets or even at infinite dimensional subsets of codimension ≥ 2 in Q/GH.

We apply a characterization of $Q-$manifolds due to Daverman and Walsh in Theorem 1.7.16[DW81]. Their results imply that $Q/\pi^{-1}(A)$ is a $Q-$manifold provided $Q/\pi^{-1}(A)$ is an ANR satisfying the Disjoint Discs Property(DDP) and $Q/\pi^{-1}(A)$ has Čech carriers with infinite codimension. Since Q/GH is an ANR, by Theorem 1.3.37, $Q/\pi^{-1}(A)$ is an ANR. Our discussion now mainly focuses on the Čech carriers property. Not only do we need to show that $Q/\pi^{-1}(A)$ has Čech carriers with infinite codimension in order to apply Daverman-Walsh characterization, but also we have to show that $Q/\pi^{-1}(A)$ satisfies DDP.

We will assume the notation and occasionally refer to certain steps and conditions in the construction in the previous chapters.

5.1 Constructing the Decomposition GH

To construct the decomposition GH, there are several approaches. One can construct the decomposition GH by defining the equivalence relation generated by both decompositions G and H. Also, one can approach this construction by producing a defining sequence and then generating that sequence to get the decomposition GH. We will start with the first approach.

Let G be the cellular decomposition of Q induced by the map $f^\infty : Q_2 \to Q_2$ in Chapter 4 and π_G be the quotient map. Let H be the cellular decomposition of Q described in the Chapter 3 and π_H be the quotient map. Let, for $T \subset Q$,

$$\begin{aligned} G^*(T) &= \cup\{g \in G \ | g \cap T \neq \emptyset\} \\ H^*(T) &= \cup\{h \in H \ | h \cap T \neq \emptyset\} \end{aligned}.$$

Before we define an equivalence relation on GH, we will list some properties induced from both G and H in the following lemma.

Lemma 5.1.1. *Fix $h \in H_H$ where H_H is the set of all nondegenerate elements in H. Let $g_0 \in H_G$ be such that $g_0 \cap h \neq \emptyset$. Then if $g \cap h \neq \emptyset$ and $g \in H_G$, then $g = g_0$.*

Proof. Assume that $g \cap h \neq \emptyset$. Suppose that $g \neq g_0$. Then $g \cap g_0 = \emptyset$. We can see that by Theorem 3.2.2, $h \cap \{0\} \times Q_2$ is a single point in $\{0\} \times Q_2$. Let x_0 be such an element. Since $g_0 \subset \{0\} \times Q_2$, it follows that $g_0 \cap h = \{x_0\}$. Thus, if $g \cap h \neq \emptyset$ and we know also that $g \subset \{0\} \times Q_2$, then $g \cap h = \{x_0\}$ which implies that $g \cap g_0 \neq \emptyset$, a contradiction. Thus, $g \cap h = \emptyset$. □

Lemma 5.1.1 has the following Corollaries.

Corollary 5.1.2. *Let $x \in Q$. Then*

$$G^*(H^*(G^*(H^*(x)))) = H^*(G^*(H^*(x))).$$

Proof. Note that

$$H^*(x) = \cup\{h \in H_H \mid x \in h\} \cup \{x\}$$

$$G^*(H^*(x)) = \cup\{g \in H_G \mid g \cap H^*(x) \neq \emptyset\} \cup H^*(x)$$

$$H^*(G^*(H^*(x))) = \cup\{h \in H_H \mid h \cap G^*(H^*(x)) \neq \emptyset\} \cup G^*(H^*(x))$$

$$G^*(H^*(G^*(H^*(x)))) = \cup\{g \in H_G \mid g \cap H^*(G^*(H^*(x))) \neq \emptyset\} \cup H^*(G^*(H^*(x))).$$

It is obvious that $H^*(G^*(H^*(x))) \subset G^*(H^*(G^*(H^*(x))))$. Next we will show that

$$G^*(H^*(G^*(H^*(x)))) \subset H^*(G^*(H^*(x))).$$

It suffices to show that $T = \{g \in G \mid g \cap H^*(G^*(H^*(x)))\}$ is a subset of $H^*(G^*(H^*(x)))$. Let $g \in T$. If g is singleton, we are done. Assume that g is a nondegenerate element. We will show that $g \subset G^*(H^*(x))$. Since $g \cap H^*(G^*(H^*(x))) \neq \emptyset$, there is a nondegenerate element $h \in \{h \in H \mid h \cap G^*(H^*(x)) \neq \emptyset\}$ such that $g \cap h \neq \emptyset$. This implies that by Lemma 5.1.1 $g \in G^*(H^*(x))$. Therefore,

$$G^*(H^*(G^*(H^*(x)))) = H^*(G^*(H^*(x))).$$

□

Let $\sim_G$ and $\sim_H$ be equivalence relations on G and H. Then we can define the equivalence relation generated by $\sim_G \cup \sim_H$ as the smallest equivalence relation containing $\sim_G \cup \sim_H$, denoted by $\sim_{GH}$.

Lemma 5.1.3. *Let $a, b \in Q$. Then $a \sim_{GH} b$ if and only if there exist $x_1, x_2, \ldots, x_n \in Q$ such that $x_1 = a$, $x_n = b$ in Q and either $(x_{i-1}, x_i) \in \sim_H$ or $(x_{i-1}, x_i) \in \sim_G$.*

Proof. This follows from the definition of an equivalence relation generated by the relations G and H. □

Remark 5.1.4. *Since $\sim_G$ and $\sim_H$ are equivalence relations, we can choose x_i such that if $(x_{i-1}, x_i) \in \sim_H$, then $(x_i, x_{i+1}) \in \sim_G$ or such that if $(x_{i-1}, x_i) \in \sim_G$, then $(x_i, x_{i+1}) \in \sim_H$.*

Lemma 5.1.5. *Let $a, b, c, d, e \in Q$. If $a \sim_G b \sim_H c \sim_G d$, then $c = d$. Consequently, If $a \sim_H b \sim_G c \sim_H d \sim_G e$, then $a \sim_H b \sim_G c \sim_H d$.*

Proof. This follows from Lemma 5.1.1 and Theorem 3.2.2. □

Next we will show that $\sim_{GH}$ is equivalent to an equivalence relation defined as follows:

Let GH be the decomposition of Q given by

$$\pi(x) = \pi(y) \text{ if and only if } \pi_G \circ \pi_H^{-1} \circ \pi_H(x) \bigcap \pi_G \circ \pi_H^{-1} \circ \pi_H(y) \neq \emptyset.$$

That is, we will show the following lemma.

Lemma 5.1.6. *$x \sim_{GH} y$ if and only if $\pi_G \circ \pi_H^{-1} \circ \pi_H(x) \cap \pi_G \circ \pi_H^{-1} \circ \pi_H(y) \neq \emptyset$.*

Proof. Assume that $x \sim_{GH} y$. Then there exist $a, b \in Q$ such that $x \sim_H a \sim_G b \sim_H y$. Note that $G^*(H^*(x)) = \pi_G(\pi_H^{-1}(\pi_H(x)))$ and $G^*(H^*(y)) = \pi_G(\pi_H^{-1}(\pi_H(y)))$. Since $a \sim_G b$, it follows that $\pi_G(a) = \pi_G(b)$. Since $a \sim_H x$ and $b \sim_H y$, it follows that $\pi_G(a) \subset G^*(H^*(x))$ and $\pi_G(b) \subset G^*(H^*(y))$. Therefore $G^*(H^*(x)) \cap G^*(H^*(y)) \neq \emptyset$ which implies that $\pi_G \circ \pi_H^{-1} \circ \pi_H(x) \cap \pi_G \circ \pi_H^{-1} \circ \pi_H(y) \neq \emptyset$.

Conversely, if $\pi_G \circ \pi_H^{-1} \circ \pi_H(x) \cap \pi_G \circ \pi_H^{-1} \circ \pi_H(y) \neq \emptyset$, then there exist $a, b \in Q$ such that $a \in \pi_H^{-1}(\pi_H(x)), b \in \pi_H^{-1}(\pi_H(y))$ such that $\pi_G(a) = \pi_G(b)$. Hence $\pi_H(a) = \pi_H(x)$ and $\pi_H(b) = \pi_H(y)$ and so $x \sim_H a \sim_G b \sim_H y$ which implies that $x \sim_{GH} y$. This completes the proof. □

Next we will define $\pi_1 : Q/G \to Q/GH$ and $\pi_2 : Q/H \to Q/GH$ such that the following diagram commutes. For $x, y \in Q/G$ define $\pi_1 : Q/G \to Q/GH$ by

$$\pi_1(x) = \pi_1(y) \text{ if and only if } \pi_G \circ \pi_H^{-1} \circ \pi_H \circ \pi_G^{-1}(x) \cap \pi_G \circ \pi_H^{-1} \circ \pi_H \circ \pi_G^{-1}(y) \neq \emptyset.$$

For $x, y \in Q/H$ define $\pi_2 : Q/H \to Q/GH$ by

$$\pi_2(x) = \pi_2(y) \text{ if and only if } \pi_G \circ \pi_H^{-1}(x) \cap \pi_G \circ \pi_H^{-1}(y) \neq \emptyset.$$

$$\begin{tikzcd}
 & Q \arrow[dl, "\pi_G"'] \arrow[dr, "\pi_H"] \arrow[dd, "\pi"] & \\
Q/G \arrow[dr, "\pi_1"] & & Q/H \arrow[dl, "\pi_2"'] \\
 & Q/GH &
\end{tikzcd}$$

Since $Q/G \cong Q$, to define a defining sequence in Q for GH it suffices to construct a defining sequence in Q/G. From the above lemma, we can define a defining sequence for Q/G as follows:

Let $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, \ldots\}, \mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots\}$ be the defining sequences for H and for the Cantor set in the middle as defined as in Chapters 2 and 3. Then define the sequence $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots\}$ for Q/G by

$$\mathcal{R}_i = \{\pi_G(L_j \times Q_{m(i)+3}) \ \mid \ L_j \times Q_{m(i)+3} \in \mathcal{L}_i\}$$

Note that $\pi_G(L_j \times Q_{m(i)+3})$ can be viewed as $K_j \times Q_{m(i)+3}$ where $K_j = T_j \cup P_j \cup D_j$ together with the tubes joining these where P_j is homeomorphic to $[-\frac{1}{2^i}, \frac{1}{2^i}] \times f_{N(j)}^{m(j)+1})(B_j)$. Then $\mathcal{R}$ is a defining sequence for Q/G. All inductive hypotheses **IH1,**..., **IH4** in the construction of defining sequence $\mathcal{L}$ from Chapter 3 are still true for the defining sequence $\mathcal{R}$. Figure 5.1 shows the first stage of construction with 4 components. The four components in the cylinder at the top come from $\mathcal{T}_1$. The four components in the cylinder at the bottom come from $\mathcal{D}_1$, and the four components in the middle come from $\mathcal{P}_1$. The tubes indicate how some of the components are connected.

Remark 5.1.7. *Each $R_i \in \mathcal{R}_i$, there exists $Pre(R_i) \in \mathcal{R}_{i-1}$ such that $R_i \subset Pre(R_i)$ and the inclusion map $R_i \to Pre(R_i)$ is null homotopic. This implies that each element of the decomposition GH associated with defining sequence $\mathcal{R}$ is cell-like.*

Lemma 5.1.8. *Let $R = T \cup P \cup D$ together with the tubes joining these be a fixed element in $\mathcal{R}_r$ for some r. Then for any virtually $I-$essential maps $f_1 : D_1 \to T$ and $f_2 : D_1 \to D$,*

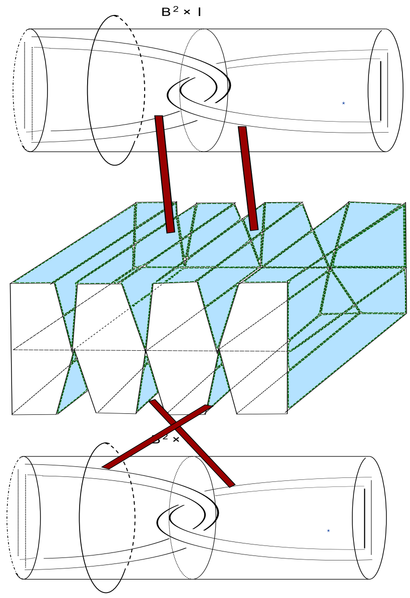


FIGURE 5.1: First Stage of construction $\mathcal{R}$ with 4 components

there is an $\delta > 0$ so that if f_1', f_2' are maps within δ of f_1 and f_2 respectively and in general position with respect to $\mathcal{R}_{r+1}$, then there is a component $R' = B' \cup T' \cup D'$ together with the tubes joining these where $R' \subset R$ and discs with holes $D_1' \subset D_1, D_2' \subset D_2$ such that f_1' and f_2' are virtually $I-$essential into T' and D' respectively.

Proof. This follows directly from condition **IH4**. For more details, see [Gar91]. ☐

Lemma 5.1.9. *Let $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots\}$ of Q be the defining sequence defined as above. Then $\mathcal{R}$ is sharp.*

Proof. By Remark 1.3.34, the defining sequence $\mathcal{R}$ is sharp. ☐

5.2 Properties of Q/GH.

Lemma 5.2.1. *Let $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots\}$ be the defining sequence constructed in the previous section, let GH be the decomposition associated with $\mathcal{R}$, and let $\pi : Q \to Q/GH$ be the natural quotient map. Let $\partial\mathcal{R} = \cup_{i\geq 1}\{\partial R : R \in \mathcal{R}_i\}$. Then*

(1) π is cell-like.

(2) Each non-degenerate element of GH has dimension one.

(3) $\pi(N_\pi) - \pi(\partial\mathcal{R})$ is 0-dimensional

(4) Q/GH is an ANR

(5) If A is a closed subset in Q/GH, then $Q/\pi^{-1}(A)$ is an ANR.

Proof. Condition (1) holds by Remark 5.1.7 and Theorem 1.3.21.

For (2), the connectedness of each non-generate element h of the decomposition GH gives the dimension of h is ≥ 1 and since for every R of $\mathcal{R}_n$ there is a $\frac{1}{n}-$map from R to 1 complex by the condition **IH1** for the defining sequence $\mathcal{R}$. Then by a result in dimension theory, [HW48], (page 73), the dimension of h is ≤ 1. Hence, each non-degenerate element of H has dimension one.

For (3), we will show that $\pi(N_\pi) - \pi(\partial\mathcal{R})$ is $0-$dimensional at each point $P \in \pi(N_\pi) - \pi(\partial\mathcal{R})$. Let $P \in \pi(N_\pi) - \pi(\partial\mathcal{R})$ and U be an open neighborhood of P in Q/GH. Then $V = \pi^{-1}(U)$ is open in Q. Let

$$V^* = \cup\{g \in GH \mid g \subset V\}.$$

Since GH is upper semicontinuous, V^* is saturated and open in Q. Then $\pi(V^*)$ is open in Q/GH which implies that $W = \pi(V^*) \cap \Big(\pi(N_\pi) - \pi(\partial\mathcal{R})\Big)$ is open in $\pi(N_\pi) - \pi(\partial\mathcal{R})$. Clearly, $\partial W = \emptyset$ and $W \subset U$. Then it follows that $\pi(N_\pi) - \pi(\partial\mathcal{R})$ is $0-$dimensional at the point P. Since P is arbitrary, $\pi(N_\pi) - \pi(\partial\mathcal{R})$ is $0-$dimensional.

For (4), since $\pi(\partial\mathcal{R})$ contains the boundary of a basis for the topology of $\pi(N_\pi)$ and from Lemma 1.3.11 π is one-to-one on $\partial\mathcal{R}$, it follows from Theorem 1.3.36 that

$$Q/GH \text{ is an ANR}$$

since $\mathcal{R}$ is sharp.

For (5), since Q/GH is an ANR by (5), Theorem 1.3.37 from [Lay80] shows that for each closed set A in Q/GH,

$$Q/\pi^{-1}(A) \quad \text{is an ANR.}$$

□

The proof of Lemma 5.2.2 is similar to the arguments on page 36 and on page 41 in [Lay80]

Lemma 5.2.2. *[Lay80] Let $\mathcal{L}$ be the defining sequence of Q. Let $k \geq 1$ be fixed and $R \in \mathcal{R}_k$. Then for each $j > k$, if W is compact subset of Q with $W \subset Int(R)$, then*

$$W \cap \pi^{-1}(\pi(\partial R) \quad \textit{is 1-dimensional.}$$

Proof. The proof is similar to Lemma 3.3.3 □

Now we will discuss the Čech Carrier Property of Q/GH

Corollary 5.2.3. *For the decomposition GH associated with the defining sequence $\mathcal{R}$, $\pi^{-1}(x)$ has infinite codimension in Q for every $x \in Q/GH$.*

Proof. Let $x \in Q/GH$. If $x \notin \pi(N_\pi)$, then $\pi^{-1}(x)$ is a singleton in Q and hence it has infinite codimension in Q. Assume that $x \in \pi(N_\pi)$. It follows from Lemma 5.2.1 that x is one dimensional. Then by Lemma 1.7.4 [DW81], $\pi^{-1}(x)$ has infinite codimension in Q. □

We will use Theorem 3.3.6 to show that points in Q/H have infinite codimension.

Corollary 5.2.4. *Points in Q/GH have infinite codimension in Q/GH.*

Proof. The result follows from Theorem 3.3.6. For more details, see the argument in [Lay80]. □

This next argument is similar to that for Q/H in Chapter 3.

Lemma 5.2.5. *[Lay80] For each $i \geq 1$, let $R \in \mathcal{R}_i$. Then Q/GH has disjoint Čech carriers at $\pi(\partial R)$. Consequently, Q/GH has disjoint Čech carriers at $\pi(\partial \mathcal{R})$.*

Proof. The proof is similar to that for Q/H in Chapter 3. $\square$

In the next two Theorems, the proof is similar to that of previous argument in Chapter 3. So, we omit the proof here.

Lemma 5.2.6. *Let A be a closed subset in $\pi(Q - N_\pi)$. Then Q/GH has Disjoint Čech carrier at A.*

Theorem 5.2.7. *Q/GH has Disjoint Čech Carriers Property.*

Theorem 5.2.7 gives the following Corollary.

Corollary 5.2.8. *If A is a closed subset of Q/GH, then $Q/\pi^{-1}(A)$ has Disjoint Čech Carriers Property.*

Proof. This follows from Theorem 5.2.7. $\square$

5.3 Properties of $Q/\pi^{-1}(A)$

Let $\mathcal{R}$ be the defining sequence defined above. Let $A \subset Q/GH$. Then the sub-defining sequence $\mathcal{R}' = \{\mathcal{R}'_1, \mathcal{R}'_2, \ldots\}$ of $\mathcal{R}$ is defined as follows: For $i \geq 1$,

$$\mathcal{R}'_i = \{R \in \mathcal{R}_i \mid R \cap g \neq \emptyset \text{ for some } \ g \in A\}.$$

We now state and prove the main results of this section.

Theorem 5.3.1. *Let A be a closed subset of Q/GH where $A \subset \pi(N_\pi)$ has codimension ≥ 1 in $\pi(N_\pi)$. Then $Q/\pi^{-1}(A)$ satisfies the DDP.*

Proof. We modified the proof from an argument in [Gar91]. Let A be a closed subset of Q/GH. To show that $Q/\pi^{-1}(A)$ satisfies the DDP, it suffices to show the following condition. For each $\epsilon > 0$ and for each pair maps $f, g : B^2 \to Q/GH$, there are approximating maps $\tilde{f}, \tilde{g} : B^2 \to Q/GH$ such that

$$d(f, \tilde{f}) < \epsilon, \quad d(g, \tilde{g}) < \epsilon$$

with $\tilde{f}(B^2) \cap \tilde{g}(B^2) \cap A = \emptyset$.

First, by Lemma 4.1.12, we can choose a dense subset P in $\{0\} \times C^\infty$ so that $f^\infty(P) \cap A = \emptyset$ and $(f^\infty)^{-1}(f^\infty(p)) = p$ for each p in P.

Next, given ϵ, f and g as in the condition above, by Theorem 1.3.16 and since Q/G satisfying the DDP, we can choose approximate lifts $f_1, g_1 : B^2 \to Q/G$ of f and g respectively and $\delta > 0$ so that $f_1(B^2) \cap g_1(B^2) = \emptyset$ and so that if $\tilde{f}, \tilde{g} : B^2 \to Q/G$ satisfies

$$\rho(f_1, \tilde{f}) < \delta \quad \text{and} \quad \rho(g_1, \tilde{g}) < \delta,$$

then

$$\rho(\pi_1 \circ \tilde{f}, f) < \frac{\epsilon}{3} \quad \text{and} \quad \rho(\pi_1 \circ \tilde{g}, g) < \frac{\epsilon}{3},$$

and $\tilde{f}(B^2) \cap \tilde{g}(B^2) = \emptyset$.

Next, by Theorem 1.3.16 again, we can choose approximate lifts $f_2, g_2 : B^2 \to Q$ of f_1 and g_1, respectively so that $f_2(B^2) \cap g_2(B^2) = \emptyset$, so that f_2 and g_2 are transverse to $\{0\} \times Q_2$, with $(f_2(B^2) \cup g_2(B^2)) \cap (\{0\} \times Q_2)$ missing $\{0\} \times C^\infty$, and so that

$$\rho(\pi_G \circ f_2, f_1) < \delta \quad \text{and} \quad \rho(\pi_G \circ g_2, g_1) < \delta.$$

It follows that

$$\rho(\pi \circ f_2, f) = \rho((\pi_1 \circ \pi_G) \circ f_2, f) = \rho(\pi_1 \circ (\pi_G \circ f_2), f) < \frac{\epsilon}{3}$$

and

$$\rho(\pi \circ g_2, g) = \rho((\pi_1 \circ \pi_G) \circ g_2, g) = \rho(\pi_1 \circ (\pi_G \circ g_2), g) < \frac{\epsilon}{3}.$$

That is, $\pi \circ f_2$ and $\pi \circ g_2$ are $\frac{\epsilon}{3}$ approximations to f and g respectively. Finally, choose k so that if R is any component of $\mathcal{R}_k$, then $diam(\pi(R)) < \frac{\epsilon}{3}$, and so that $(f_2(B^2) \cup g_2(B^2)) \cap \{0\} \times Q_2)$ is contained in the complement of $\mathcal{R}_k$. After a small general position adjustment, by using Lemma 3.2.6, there are approximations $f_3, g_3 : B^2 \to Q$ to f_2 and g_2 respectively so that

$$f_3\Big|_{B^2\setminus(f_2^{-1}(R))} = f_2\Big|_{B^2\setminus(f_2^{-1}(R))} \quad , \quad g_3\Big|_{B^2\setminus(g_2^{-1}(R))} = g_2\Big|_{B^2\setminus(g_2^{-1}(R))},$$

and satisfying the following conditions:

(i) If $h \in H, h \subset R$ and both $g_3(B^2)$ and $f_3(B^2)$ intersect h, then $h \cap \{0\} \times C^\infty \in P$.

(ii) If $h \in H, h \subset R$ and $g_3(B^2)$ intersects h and $f_3(B^2)$ does not intersect h, then $h \cap \{0\} \times C^\infty$ has no triadic rational coordinates.

(iii) If $h \in H, h \subset R$ and $f_3(B^2)$ intersects h and $g_(B^2)$ does not intersect h, then $h \cap \{0\} \times C^\infty$ has no triadic rational coordinates.

We claim that $F = \pi \circ f_3$ and $G = \pi \circ g_3$ are the required approximations to f and g. First we can see that

$$\rho(\pi \circ f_3, f) \leq \rho(\pi \circ f_3, \pi \circ f_2) + \rho(\pi \circ f_2, f) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}$$
$$\rho(\pi \circ g_3, g) \leq \rho(\pi \circ g_3, \pi \circ g_2) + \rho(\pi \circ g_2, g) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.$$

Next we will show that

$$F(B^2) \cap G(B^2) \cap A = \emptyset.$$

If $F(B^2) \cap G(B^2) = \emptyset$, then we are done. Now suppose that

$$F(B^2) \cap G(B^2) \neq \emptyset.$$

Since $\pi_G \circ f_3 \cap \pi_G \circ g_3 = \emptyset$, the points in $F(B^2) \cap G(B^2)$ can arise in one of the following ways:

(a) Such points can arise from an element h of H that intersects both $f_3(B^2)$ and $g_3(B^2)$, or

(b) they can arise from the elements h_1 of H that $f_3(B^2)$ intersects and h_2 of H that $g_3(B^2)$ intersects where $\pi_G(h_1) \cap \pi_G(h_2) \neq \emptyset$.

Suppose elements of the second type exist, and let $(f^\infty)^{-1}(y) \in \pi_G(h_1) \cap \pi_G(h_2)$. That is, there are $x_1 \in h_1$ and $x_2 \in h_2$ with $x_1 \neq x_2$ so that $f^\infty(x_1) = y = f^\infty(x_2)$ which implies that x_1 and x_2 have triadic rational coordinates. Since D consist only of points with no triadic rational, it follows that $x_1, x_2 \notin D$. But by conditions (ii) and (iii) above, $x_1, x_2 \in D$, So, this leads to a contradiction. Thus, there are no points of the latter type.

The condition (i) above shows that any point of the first type lies in the complement of A. That is,

$$F(B^2) \cap G(B^2) \cap A = \emptyset.$$

This completes the proof of the Lemma.

□

5.4 Main Theorem

Theorem 5.4.1. *Let GH be the decomposition of Q defined as the previous section. Then GH satisfies the following four properties:*

(a) The nonmanifold part of Q/GH is homeomorphic to a copy of Q whose codimension is 1.

(b) $Q/GH \ncong Q$

(c) If A is any closed subspace of X where $A \subset \pi(N_\pi)$ of codimension ≥ 1 in $\pi(N_\pi)$, then the decomposition of Q induced over A is shrinkable. That is, $Q/\pi^{-1}(A) \cong Q$.

(d) GH is cellular

(e) $Q/GH \times I^2 \cong Q$

Proof. For (a), we can see that first the nonmanifold part of Q/GH is contained in $\pi(Q^2)$ where $Q^2 = \{0\} \times Q_2$. Note that Q^2 has codimension 1 in Q by Example 1.4.2. Claim that $\pi(Q^2)$ also has codimension 1. Let U be any open set in Q/GH. Consider $H_0(U, U - \pi(Q^2))$. Since π is cell-like map and Q^2 has codimension 1, by Theorem 1.3.20, it implies that

$$H_0(U, U - \pi(Q^2)) \cong H_0(\pi^{-1}(U), \pi^{-1}(U) - Q^2) \cong 0.$$

Let U be a saturated open set such that $H_1(U, U - Q^2) \ncong 0$. Note that $\pi(U)$ is open in Q/GH. Then again by Theorem 1.3.20,

$$H_1(\pi(U), \pi(U) - \pi(Q^2)) \cong H_1(U, U - Q^2) \ncong 0.$$

Therefore, we have $\pi(Q^2)$ has codimension 1 in Q/GH. It remains to show that $\pi(Q^2)$ is the nonmanifold part of Q/GH. Given $\epsilon > 0$. Let $p \in \pi(Q^2)$ and U be a ϵ neighborhood of p in $\pi(Q^2)$. We will show that the disjoint disc property fails in U. Choose an element R of some $\mathcal{R}_k$ so that $\pi(R)$ is contained in the $\frac{\epsilon}{4}$ neighborhood of p.

Given $R = P \cup T \cup D$ together with the tubes joining these of some element in $\mathcal{R}_k$ so that $\pi_H(R)$ is contained in the $\frac{\epsilon}{4}$ neighborhood of p. Choose $f, g : B^2 \to Q$ so that f is an $I-$essential map into T and g is an $I-$essential into D. Let $\pi \circ f, \pi \circ f : B^2 \to Q/GH$. Then by Theorem 1.3.16, there is $\delta > 0$ so that if f'' and $g'' : B^2 \to Q$ are approximate lifts of $\frac{\delta}{2}$ approximations of f' and g' respectively with

$$\rho(\pi \circ f'', \pi \circ f) < \delta \quad \text{and} \quad \rho(\pi \circ g'', \pi \circ g) < \delta,$$

then

$$\rho\left(f''\Big|_{\partial B^2}, f\Big|_{\partial B^2}\right) < \epsilon' \quad \text{and} \quad \rho\left(g''\Big|_{\partial B^2}, g\Big|_{\partial B^2}\right) < \epsilon',$$

It follows by Lemma 5.1.8 that if f_1 and g_1 are any $\frac{\delta}{2}$ approximations of $\pi_H \circ f$ and $\pi_H \circ g$ respectively, then

$$f_1(B^2) \cap g_1(B^2) \neq \emptyset.$$

Thus U does not satisfy the DDP and hence the manifold part of Q/GH is $\pi(Q^2)$.

For (b), this follows from (a).

For (c), notice that $Q/\pi^{-1}(A)$ is an ANR by Lemma 5.2.1(5) and has DDP by Lemma 5.3.1. By Corollary 5.2.8, $Q/\pi^{-1}(A)$ has Disjoint Čech Carriers Property. It follows that by Theorem 1.7.16, $Q/\pi^{-1}(A)$ is a $Q-$manifold. That is, $Q/\pi^{-1}(A) \cong Q$.

For (d), the map π is cell-like by Lemma 5.2.1(1). It implies that each non-degenerate element h has trivial shape. By the condition (c), for $h \in N_\pi$, $Q/\pi^{-1}(h) \cong Q$. Thus, by Theorem 1.3.30 h is cellular. That is, GH is a cellular decomposition.

For (e), first note that Q/GH is an ANR. By Lemma 5.2.7, we have Q/GH has Disjoint Čech Carriers Property, and hence by Lemma 1.7.15, $Q/GH \times I^2$ is a $Q-$manifold.

□

6. CONCLUSION

The main results of this work is the existence of a cell-like decomposition GH of the Hilbert Cube Q such that Q/GH is nonmanifold. The nonmanifold part of Q/GH is homeomorphic to a copy of Q whose codimension is 1. Also, if A is any closed subspace of the nonmanifold part of codimension ≥ 1 in the nonmanifold part, then the decomposition of Q induced over A is shrinkable. That is, $Q/\pi^{-1}(A) \cong Q$. However, Q/GH is still a factor of Q since $Q/GH \times I^2 \cong Q$.

The principal new results in this thesis are:

- (Theorem 2.5.4) There exists a sequence

$$\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \ldots\}$$

  of collection subsets of Q satisfying the following properties:

  1. $\bigcap \mathcal{B} = \{0\} \times C^\infty$ where $C^\infty = \prod_{i=1}^{\infty} C$ and C is the Cantor set.
  2. Every point $p \in \{0\} \times C^\infty$ is associated with a sequence $(\epsilon_1, \epsilon_2, \ldots)$ where $\epsilon_l = (\sigma_{l1}, \sigma_{l2})$, and where σ_{l1} and σ_{l2} are n_l-tuples whose components are in $\{1, 2\}$,
  3. If there is N such that for all $i > N$ either all the first coordinates of the ϵ_i or all the second coordinates of the ϵ_i is not in $\bar{1} \cup \bar{2}$, then $p \in \{0\} \times C^\infty$ has no triadic rational coordinates.

- (Theorem 3.2.7) There exists a decomposition H of Q which satisfies the following properties:

  1. For each nondegenerate element h of H,

$$h \cap \{0\} \times Q_2$$

is a single point in $\{0\} \times C^\infty$.

2. Let f_1 and f_2 be maps from B^2 into Q/H and let A be any dense subset of C^∞. Then f_1 and f_2 are approximable by maps g_1 and g_2 satisfying:

   (i) $g_1(B^2) \cap g_2(B^2) \subset \pi_H(A)$, and

   (ii) if $p = \{0\} \times p'$ is a point of $\{0\} \times C^\infty$ with

   $\pi_H(p) \in (g_1(B^2) \setminus g_2(B^2)) \cup (g_2(B^2) \setminus g_1(B^2))$,

   then p' has no triadic rational coordinates.

3. Q/H has nonmanifold part equal to $\pi_H(\{0\} \times C^\infty) \cong \{0\} \times C^\infty$.

4. $Q/H \times I^2 \cong Q$.

- After combining the decomposition H in Chapter 3 and the decomposition G in Chapter 5, we have a decomposition GH satisfying the following three properties (Theorem 5.4.1):

  1. GH is cellular

  2. The nonmanifold part of Q/GH is homeomorphic to a copy of Q whose codimension is 1.

  3. $Q/GH \times I^2 \cong Q$.

  4. If A is any closed subspace of X of codimension greater than ≥ 1 in $\pi(N_\pi)$, then the decomposition of Q induced over A is shrinkable. That is, $Q/\pi^{-1}(A) \cong Q$.

Note that the example in this work yields the nonmanifold part of codimension 1. A question for further investigation is the following:

Question: Does there exist a similar example of a decomposition of the Hilbert cube having the nonmanifold part of codimension k for any positive integer k?

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