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For $m_1 \geq m_2 \geq \dots m_k \geq 0$, a k-line partition of a positive integer N into k rows, each row having m_i non-zero parts, is a representation of the form

(1) $$N = \sum_{i=1}^{k} \sum_{j=1}^{m_i} a_{ij},$$

where each a_{ij} is a positive integer. Gordon and Houten obtained a generating function for k-line partitions of this form when $a_{ij} > a_{i,j+1}$ and $a_{ij} \geq a_{i+1,j}$. Gordon also found the generating function for $a_{ij} > a_{i,j+1}$ and $a_{ij} > a_{i+1,j}$. In this thesis, we shall obtain the generating function for k-line partitions of the form (1) with $a_{ij} > a_{i,j+1}$ and $a_{ij} \geq a_{i+1,j}-t$ for a non-negative integer t. For $m_i \geq m_{i+1}+k-i$, we shall also obtain the generating function in this case when $a_{ij} > a_{i,j+1}$ and $a_{ij} > a_{i+1,j}+1$. In the process, we shall establish Gordon's result by a different method.

A k-line slant partition of a positive integer N is a representation of the form $N = \sum_{i=1}^{k} \sum_{j=i}^{\infty} a_{ij}$, where each a_{ij} is a non-negative integer and $a_{ij} \geq a_{i,j+1}$ and $a_{ij} \geq a_{i+1,j}$. We shall show a one-to-one correspondence between partitions of this type and k-line partitions whose parts are decreasing along each column.

For $m_1 > m_2 \geq 0$, a slant partition of type a of a positive integer N into two rows, each row having m_i non-zero parts, is a representation of the form $N = \sum_{i=1}^{2} \sum_{j=i}^{m_i+i-1} a_{ij}$, where each a_{ij} is a positive integer such that $a_{ij} \geq a_{i,j+1}$ and $a_{ij} \geq a_{i+1,j}$. In this thesis, we shall obtain the generating function for partitions of this type for certain values of m_1 and m_2, conjecture a general form, and relate this form to the generating function for 2-line slant partitions.

Some Results on k-Line Partitions

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SOME RESULTS ON k-LINE PARTITIONS

I. INTRODUCTION

A partition of a positive integer N is any representation of N as a sum of non-negative integers. In this paper we shall discuss some aspects of partition theory.

DEFINITION 1.1. By a <u>one-line partition of a positive integer N</u>, we shall mean a representation $N = a_1 + a_2 + \ldots + a_m$, where the a_i's, called <u>parts</u>, are natural numbers. For example, the one-line partitions of 5 are 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1 and 1+1+1+1+1. The representation is usually written without the plus signs and such that $a_i \geq a_j$ for $i < j$. The above partitions of 5 are thus written 5, 41, 32, 311, 221, 2111, 11111. The number of one-line partitions of a positive integer N into parts chosen from the set of natural numbers is denoted by <u>$p(N)$</u>. In particular $p(5) = 7$. It is customary to define $p(0) = 1$ and call a function $P(x)$, such that $P(x) = \sum_{n=0}^{\infty} p(n)x^n$, the <u>generating function for p.</u>

Restricting the set from which the parts are chosen will in general affect the number of one-line partitions of a positive integer N and hence will affect the corresponding generating function. For example, suppose μ is an integer greater than zero. Let <u>$p_\mu(N)$</u> denote the number of one-line partitions of a positive integer N into parts chosen from the set of natural numbers less than or equal to μ. Then the generating function for p_μ is

(1) $$P_\mu(x) = \sum_{n=0}^{\infty} p_\mu(n)x^n = \prod_{m=1}^{\mu} (1-x^m)^{-1}$$

where $p_\mu(0)$ is defined to be 1.

To indicate the proof of (1), note that

$$\prod_{m=1}^{\mu} (1-x^m)^{-1} = (1+x+x^{1+1}+\ldots)(1+x^2+x^{2+2}+\ldots)\ldots(1+x^\mu+x^{\mu+\mu}+\ldots)$$

where each partition of n with no part larger than μ contributes exactly 1 to the coefficient of x^n. Hence the coefficient of x^n is $p_\mu(n)$. If the set from which the parts are chosen consists of all the natural numbers, that is $\mu = \infty$, then

$$P(x) = \lim_{\mu \to \infty} P_\mu(x) = \prod_{m=1}^{\infty} (1-x^m)^{-1}.$$

We can represent a one-line partition as an array of dots, where the number of dots in each line of the array represents a part. For example 221 is represented by

$$\begin{array}{ll} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \end{array}$$

Transposing the graph gives us

or 32. In this manner we can deduce a one-to-one correspondence between one-line partitions of a positive integer N whose largest part is less than or equal to μ and one-line partitions of N with at most μ parts. Hence $P_\mu(x)$ is also the generating function for the number of one-line partitions into at most μ parts chosen from the set of natural numbers.

We could also restrict the number of parts of a one-line partition of a positive integer N to be exactly μ. Let $\underline{r_\mu(N)}$ denote the number of such partitions of N. For $\mu = 3$, the partitions of 5 are 311 and 221, hence $r_3(t) = 2$. Note that $r_\mu(N) = 0$ for $0 \leq N < \mu$. Therefore

$$\sum_{n=0}^{\infty} r_\mu(n)x^n = x^\mu \sum_{n=0}^{\infty} f(n)x^n$$

where the function f is to be determined.

For each one-line partition of a positive integer N into exactly μ parts there corresponds a unique one-line partition of $N - \mu$ into no more than μ parts. Furthermore, the correspondence is one-to-one. As an example, let $N = 11$ and $\mu = 3$. The first row consists of all the partitions of 11 into exactly 3 parts and the second row is the corresponding partitions of $11 - 3 = 8$

into no more than 3 parts.

911	821	722	731	641	632	551	542	533	443
8	71	611	62	53	521	44	431	422	332

Therefore for any positive integer N, the one-line partitions enumerated by $r_{\mu}(N)$ are exactly those enumerated by $p_{\mu}(N-\mu)$ and hence,

$$\sum_{n=0}^{\infty} r_{\mu}(n)x^n = \sum_{n=0}^{\infty} p_{\mu}(n-\mu)x^n$$

$$= \sum_{n=\mu}^{\infty} p_{\mu}(n-\mu)x^n$$

$$= x^{\mu} \sum_{n-\mu=0}^{\infty} p_{\mu}(n-\mu)x^{n-\mu}$$

$$= x^{\mu} \sum_{m=0}^{\infty} p_{\mu}(m)x^m$$

$$= x^{\mu}P_{\mu}(x)$$

since $p_{\mu}(n-\mu) = 0$ for $n < \mu$.

Hence $f(n) = p_{\mu}(n)$.

DEFINITION 1.2. A <u>k-line partition of a positive integer N</u> is a representation of the form

$$N = \sum_{i=1}^{k} \sum_{j=1}^{\infty} a_{ij}$$

where each a_{ij} is a non-negative integer such that $a_{ij} \geq a_{i,j+1}$ and $a_{ij} \geq a_{i+1,j}$. By convention, we write the partition in a rectangular array with i indicating the row, j indicating the column and omitting zero summands and plus signs. As an example, consider all 3-line partitions of 5. Seven mentioned previously were 5, 41, 32, 311, 221, 2111, 11111. To this group we add the partitions

$$\begin{array}{ccccccccc} 4 & 3 & 31 & 22 & 21 & 211 & 21 & 1111 & 11 \\ 1 & 2 & 1 & 1 & 2 & 1 & 11 & 1 & 11 \end{array}$$

$$\begin{array}{ccccc} 3 & 2 & 21 & 111 & 11 \\ 1 & 2 & 1 & 1 & 11 \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

and we see that the number of 3-line partitions of 5 is 21.

Let $\underline{t_k(N)}$ represent the number of k-line partitions of a positive integer N. P.A. MacMahon (3) showed that

$$\sum_{n=0}^{\infty} t_k(n)x^n = \prod_{m=1}^{\infty} (1-x^m)^{-\min(m,k)} \tag{2}$$

where $t_k(0) = 1$.

Basil Gordon and Lorne Houten (1) considered k-line partitions of exactly m_i $(i = 1, \ldots, k)$ non-zero elements in each ith row such that the elements along each row are strictly decreasing. Letting $m_1 = m_2 = \ldots = m_k = \mu$, they established that the number of partitions of a positive integer n into k lines, the number of elements in each line not exceeding μ, is the coefficient of x^n of the generating function

$$(3) \qquad \frac{(1-x)^{k-1}(1-x^2)^{k-2}\ldots(1-x^{k-1})}{(1-x)^k(1-x^2)^k\ldots(1-x^\mu)^k(1-x^{\mu+1})^{k-1}\ldots(1-x^{\mu+k-1})}$$

Letting $\mu \to \infty$, yields the right side of equation (2).

In this paper we shall be looking first at k-line partitions with a specified number of non-zero elements in each row, the elements decreasing along each row. We shall, however, vary the restrictions on the elements along each column. In the second part we shall discuss mainly 2-line partitions with elements decreasing in each column with or without restrictions. We shall also indicate various approaches to equivalent problems.

We need to give some clarification of the notation to be used in this thesis.

(a) $\displaystyle\sum_{\epsilon_1, \ldots, \epsilon_k = 0}^{1} f(x; \epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ will denote a sume over all vectors $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ where each ϵ_i is either 1 or 0. As an example,

$$\sum_{\epsilon_1,\epsilon_2=0}^{1} f(x;\epsilon_1,\epsilon_2) = f(x;0,0) + f(x;0,1) + f(x;1,0) + f(x;1,1)$$

We shall use the following determinant notation.

(b) For $1 \leq s_1 < s_2 < \ldots < s_\ell = k$,

$$\begin{vmatrix} a_{ij}^{(1)} \\ a_{ij}^{(2)} \\ \vdots \\ a_{i,j}^{(\ell)} \end{vmatrix} \quad \begin{array}{l} i = 1,\ldots,s_1 \\ i = s_1+1,\ldots,s_2 \\ j = 1,\ldots,k \\ i = s_{\ell-1}+1,\ldots,s_\ell=k \end{array}$$

will denote the k by k determinant $|a_{ij}|$ where

$$a_{ij} = \begin{cases} a_{ij}^{(1)} & \text{for } i = 1,\ldots,s_1 \text{ and } j = 1,\ldots,k \\ a_{ij}^{(2)} & \text{for } i = s_1+1,\ldots,s_2 \text{ and } j = 1,\ldots,k \\ \vdots & \\ a_{ij}^{(\ell)} & \text{for } i = s_{\ell-1}+1,\ldots,s_\ell \text{ and } j = 1,\ldots,k \end{cases}$$

For example, let $\ell = 2$

$$\left|\begin{array}{l} \dfrac{b_{ij}}{x^{i-1}} \\ \dfrac{b_{ij}}{x^{s-1}} \end{array}\right| \begin{array}{l} i = 1, \ldots, s \\ j = 1, \ldots, k \\ i = s+1, \ldots, k \end{array} = \left|\begin{array}{cccc} b_{11} & b_{12} & \cdots & b_{1k} \\ \dfrac{b_{21}}{x} & \dfrac{b_{22}}{x} & \cdots & \dfrac{b_{2k}}{x} \\ \vdots & \vdots & & \vdots \\ \dfrac{b_{s1}}{x^{s-1}} & \dfrac{b_{s2}}{x^{s-1}} & \cdots & \dfrac{b_{sk}}{x^{s-1}} \\ \dfrac{b_{s+1,1}}{x^{s-1}} & \dfrac{b_{s+1,2}}{x^{s-1}} & \cdots & \dfrac{b_{s+1,k}}{x^{s-1}} \\ \vdots & \vdots & & \vdots \\ \dfrac{b_{k1}}{x^{s-1}} & \dfrac{b_{k2}}{x^{s-1}} & \cdots & \dfrac{b_{kk}}{x^{s-1}} \end{array}\right|$$

(c) We shall use $[\nu]$ to denote $(1-x^{\nu})$, where ν is a positive integer.

II. k-LINE PARTITIONS OF TYPE Q

We begin with a definition and some results from Gordon and Houten (1).

DEFINITION 2.1. For $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$, let $b(N; m_1, m_2, \ldots, m_k)$ denote the number of partitions of a positive integer N into k rows having m_i $(i = 1, \ldots, k)$ non-zero parts in each ith row, where the parts are strictly decreasing along each row and non-increasing along each column. Call such a partition one of type b and let

$$B(x; m_1, m_2, \ldots, m_k) = \sum_{n=0}^{\infty} b(n; m_1, m_2, \ldots, m_k) x^n$$

where we define $b(n; m_1, m_2, \ldots, m_k) = 1$ when $n = 0$ and $m_1 = m_2 = \ldots = m_k = 0$. Since $b(n; m_1, m_2, \ldots, m_k) = 0$ for $n > 0$ and $m_1 = m_2 = \ldots = m_k = 0$, $B(x; m_1, m_2, \ldots, m_k) = 1$ for $m_1 = m_2 = \ldots = m_k = 0$.

As an example of partitions of type b, we list all the partitions of 16 for $k = 3$, $m_1 = 3$ and $m_2 = m_3 = 2$.

721	631	541	532	621	531	432	521	521
21	21	21	21	31	31	31	41	41
21	21	21	21	21	21	21	21	21

521	431	431	431	421	421	421	321
31	41	32	31	42	41	32	32
31	21	21	31	21	31	31	32

Hence $b(16;3,2,2) = 17$.

For $m_1 \geq m_2 \geq \dots \geq m_k \geq 0$, Gordon and Houten proved that

$$(1) \qquad B(x;m_1,m_2,\dots,m_k) = x^{\Sigma_{i=1}^k m_i} \sum_{\epsilon_1,\dots,\epsilon_k=0}^{1} B(x;m_1-\epsilon_1,m_2-\epsilon_2,\dots,m_k-\epsilon_k).$$

Since the technique in showing (1) is used throughout Chapter II, we shall indicate the proof in detail. For $m_1 \geq m_2 \geq \dots \geq m_k \geq 0$, let

$$(2) \qquad \begin{array}{llll} a_{11} & a_{12} & \cdots & a_{1m_1} \\ a_{21} & a_{22} & \cdots & a_{2m_2} \\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{km_k} \end{array}$$

be a partition of a positive integer N of type b. Subtracting one from each element of the partition yields a partition of $N - \sum_{i=1}^{k} m_i$ also of type b. The number of non-zero elements in each ith row $(i = 1,\dots,k)$ of the resulting partition will either be $m_i - 1$ or m_i depending upon whether a 1 or not appears in the ith row of the partition of N. Hence to each partition of N of the form (2) there corresponds a unique partition of $N - \sum_{i=1}^{k} m_i$ of the form

$$(3)\qquad \begin{matrix} b_{11} & b_{12} & \cdots\cdots & b_{1,m_1-\epsilon_1} \\ b_{21} & b_{22} & \cdots\cdot & b_{2,m_2-\epsilon_2} \\ \vdots & \vdots & & \\ b_{k1} & b_{k2} & \cdots & b_{k,m_k-\epsilon_k} \end{matrix}$$

where $\epsilon_i = 1$ or 0. Also if m_i ones are added to each ith row of any partition of $N - \sum_{i=1}^{k} m_i$ of the form (3), the result is a unique partition of N of the form (2). Hence the correspondence is one-to-one and therefore

$$(4)\quad b(N;m_1, m_2,\ldots,m_k) = \sum_{\epsilon_1,\ldots,\epsilon_k=0}^{1} b(N-\sum_{i=1}^{k} m_i;m_1-\epsilon_1, m_2-\epsilon_2, \ldots, m_k-\epsilon_k)$$

where the right side is a summation over all vectors $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$.

For example, the partitions of 16 for $m_1 = 3$ and $m_2 = m_3 = 2$ are listed after DEFINITION 2.1. The corresponding partitions of $16 - 7 = 9$ are listed as follows

61	52	43	421	51	42	321	41	41
1	1	1	1	2	2	2	3	21
1	1	1	1	1	1	1	1	1

41	32	32	32	31	31	31	21
2	3	21	2	31	3	21	21
2	1	1	2	1	2	2	21

Hence

$$b(16;3,2,2) = \sum_{\epsilon_1,\epsilon_2,\epsilon_3=0}^{1} b(9;3-\epsilon_1, 2-\epsilon_2, 2-\epsilon_3)$$

$$= b(9;3,2,2) + b(9;3,2,1) + b(9;3,1,2) + b(9;3,1,1)$$

$$+ b(9;2,2,2) + b(9;2,2,1) + b(9;2,1,2) + b(9;2,1,1)$$

$$= 0 + 0 + 0 + 1 + 1 + 4 + 0 + 11 = 17.$$

From (4) we see that

$$B(x;m_1,m_2,\ldots,m_k)$$

$$= \sum_{n=0}^{\infty} b(n;m_1,m_2,\ldots,m_k)x^n$$

$$= \sum_{n=0}^{\infty} \sum_{\epsilon_1,\ldots,\epsilon_k=0}^{1} b(n-\sum_{i=1}^{k} m_i;m_1-\epsilon_1,m_2-\epsilon_2,\ldots,m_k-\epsilon_k)x^n$$

$$= \sum_{\epsilon_1,\ldots,\epsilon_k=0}^{1} \sum_{n=0}^{\infty} b(n-\sum_{i=1}^{k} m_i;m_1-\epsilon_1,m_2-\epsilon_2,\ldots,m_k-\epsilon_k)x^n$$

$$= x^{\Sigma_{i=1}^k m_i} \sum_{\epsilon_1,\ldots,\epsilon_k=0}^{1} \sum_{n-\Sigma_{i=1}^k m_i=0}^{\infty} b(n-\sum_{i=1}^{k} m_i;m_1-\epsilon_1,m_2-\epsilon_2,\ldots,m_k-\epsilon_k)x^{n-\Sigma_{i=1}^k m_i}$$

$$= x^{\Sigma_{i=1}^k m_i} \sum_{\epsilon_1,\ldots,\epsilon_k=0}^{1} B(x;m_1-\epsilon_1,m_2-\epsilon_2,\ldots,m_k-\epsilon_k)$$

since $b(n-\sum_{i=1}^{k} m_i;m_1,m_2,\ldots,m_k) = 0$ for $n < \sum_{i=1}^{k} m_i$ and

$m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$.

$B(x;m_1, m_2, \ldots, m_k)$ is uniquely determined by the recursion (1) and the initial conditions

$$(5) \qquad B(x;m_1, m_2, \ldots, m_k) = \begin{cases} 1 & \text{if } m_1 = m_2 = \cdots = m_k = 0 \\ 0 & \text{if } m_{i+1} = m_i+1 \text{ for some } i \text{ such that } 1 \leq i \leq k-1 \end{cases}$$

The second condition in (5) is necessary when computing $B(x;m_1, m_2, \ldots, m_k)$ for $m_i = m_{i+1}$ for some i such that $1 \leq i \leq k-1$. Suppose π is a partition of a positive integer N of type b such that $m_i = m_{i+1}$ for some i such that $1 \leq i \leq k-1$. If a 1 appears in the ith row of π then a 1 must also appear in the $(i+1)$st row. Upon subtracting one from each element of π, no partition of $N - \sum_{i=1}^{k} m_i$ with $m_i - 1$ non-zero parts in the ith row and m_i non-zero parts in the $(i+1)$st row will result. Hence

$$b(n - \sum_{i=1}^{k} m_i; m_1 - \epsilon_1, \ldots, m_i - 1, m_{i+1}, \ldots, m_k - \epsilon_k) = 0$$

for all n and therefore

$$B(x;m_1 - \epsilon_1, \ldots, m_i - 1, m_{i+1}, \ldots, m_k - \epsilon_k) = 0.$$

Hence any expression for $B(x;m_1, m_2, \ldots, m_k)$ should indicate 0 for this case.

We shall need the following lemmas. Recall from the introduction that

$$P_\mu(x) = \sum_{n=0}^{\infty} p_\mu(n)x^n = \prod_{m=1}^{\mu} (1-x^m)^{-1}$$

is the generating function for the number of one-line partitions of a positive integer into at most μ parts.

LEMMA 2.2. For a positive integer μ,

$$B(x;\mu) = \frac{x^{S_\mu}}{(1-x^\mu)(1-x^{\mu-1})\ldots(1-x)}$$

where $S_\mu = \binom{\mu+1}{2}$.

Proof: For any positive integer N, the partitions enumerated by $b(N;\mu)$ are those obtained by superimposing one-line partitions with at most μ parts onto the line

$$\mu \quad \mu-1 \ \ldots \ 1$$

whose sum is S_μ. Thus $b(N;\mu) = p_\mu(N-S_\mu)$. Hence

$$
\begin{aligned}
B(x;\mu) &= \sum_{n=0}^{\infty} b(n;\mu)x^n \\
&= \sum_{n=0}^{\infty} p_\mu(n-S_\mu)x^n \\
&= x^{S_\mu} \sum_{n-S_\mu=0}^{\infty} p_\mu(n-S_\mu)x^{n-S_\mu} \\
&= x^{S_\mu} P_\mu(x) \\
&= \frac{s^{S_\mu}}{(1-x^\mu)(1-x^{\mu-1})\ldots(1-x)}
\end{aligned}
$$

since $p_\mu(n-S_\mu) = 0$ for $n < S_\mu$.

LEMMA 2.3. For a positive integer μ,

$$B(x;\mu) + B(x;\mu-1) = \frac{B(x;\mu)}{x^\mu}$$

Proof:

$$
\begin{aligned}
B(x;\mu) + B(x;\mu-1) &= \frac{x^{S_\mu}}{(1-x^\mu)(1-x^{\mu-1})\ldots(1-x)} + \frac{x^{S_{\mu-1}}}{(1-x^{\mu-1})(1-x^{\mu-2})\ldots(1-x)} \\
&= \frac{x^{S_{\mu-1}}(x^\mu+1-x^\mu)}{(1-x^\mu)(1-x^{\mu-1})\ldots(1-x)}
\end{aligned}
$$

$$= \frac{x^{S_\mu}}{x^{\mu}(1-x^{\mu})(1-x^{\mu-1})\ldots(1-x)}$$

$$= \frac{B(x;\mu)}{x^{\mu}}$$

LEMMA 2.4. For $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$,

$$B(x;m_1, m_2, \ldots, m_k) = |B(x;m_i+j-i)| \quad i, j = 1, \ldots, k$$

where $B(x;m_i+j-i) = 0$ for $m_i < i-j$.

Proof: The proof consists of showing that the above determinant satisfies the initial conditions (5) and the recursion (1).

$$|B(x;m_1)| = |1| = 1 \quad \text{for} \quad m_1 = 0.$$

If $m_{i+1} = m_i+1$ for some i such that $1 \leq i \leq k-1$,

$$\begin{aligned} B(x;m_i+j-1) &= B(x;m_{i+1}-1+j-i) \\ &= B(x;m_{i+1}+j-(i+1)) \end{aligned}$$

for all $j = 1, \ldots, k$. Hence the ith row is identical to the $(i+1)$st row and therefore the determinant has value zero. The initial conditions are satisfied.

To show that recursion (1) is satisfied, consider

$$x^{\Sigma_{i=1}^{k} m_i} \sum_{\epsilon_1, \ldots, \epsilon_k = 0}^{1} B(x; m_1 - \epsilon_1, m_2 - \epsilon_2, \ldots, m_k - \epsilon_k)$$

$$= x^{\Sigma_{h=1}^{k} m_h} \sum_{\epsilon_1, \ldots, \epsilon_k = 0}^{1} \left| B(x; m_i - \epsilon_i + j - i) \right| \quad i, j = 1, \ldots, k$$

$$= x^{\Sigma_{h=1}^{k} m_h} \sum_{\epsilon_1, \ldots, \epsilon_{k-1} = 0}^{1} \left\{ \begin{vmatrix} B(x; m_i + j - i - \epsilon_i) \\ B(x; m_k + j - k) \end{vmatrix} \begin{matrix} i = 1, \ldots, k-1 \\ j = 1, \ldots, k \end{matrix} \right.$$

$$\left. + \begin{vmatrix} B(x; m_i + j - i - \epsilon_i) \\ B(x; m_k + j - k - 1) \end{vmatrix} \begin{matrix} i = 1, \ldots, k-1 \\ j = 1, \ldots, k \end{matrix} \right\}$$

$$= x^{\Sigma_{h=1}^{k} m_h} \sum_{\epsilon_1, \ldots, \epsilon_{k-1} = 0}^{1} \begin{vmatrix} B(x; m_i + j - i - \epsilon_i) \\ B(x; m_k + j - k) + B(x; m_k + j - k - 1 \end{vmatrix} \begin{matrix} i=1,\ldots,k-1 \\ j=1,\ldots,k \end{matrix}$$

By LEMMA 2.3 we obtain from this

$$x^{\Sigma_{h=1}^{k} m_h} \sum_{\epsilon_1, \ldots, \epsilon_{k-1} = 0}^{1} \begin{vmatrix} B(x; m_i + j - i - \epsilon_i) \\ \dfrac{B(x; m_k - j - k)}{x^{m_k + j - k}} \end{vmatrix} \begin{matrix} i = 1, \ldots, k-1 \\ j = 1, \ldots, k \end{matrix}$$

$$= x^{\Sigma_{h=1}^{k} m_h} \sum_{\epsilon_1, \ldots, \epsilon_{k-2} = 0}^{1} \left\{ \begin{vmatrix} B(x; m_i + j - i - \epsilon_i) \\ B(x; m_{k-1} + j - k + 1) \\ \dfrac{B(x; m_k + j - k)}{x^{m_k + j - k}} \end{vmatrix} \begin{matrix} i = 1, \ldots, k-2 \\ j = 1, \ldots, k \end{matrix} \right.$$

$$+\left|\begin{array}{c} B(x;m_i+j-i-\epsilon_i) \\ B(x;m_{k-1}+j-k) \\ \dfrac{B(x;m_k+j-k)}{x^{m_k+j-k}} \end{array}\right| \quad \left.\begin{array}{l} i=1,\ldots,k-2 \\ j=1,\ldots,k \end{array}\right\}$$

Combining determinants and using LEMMA 2.3 we obtain

$$x^{\Sigma_{h=1}^k m_h} \sum_{\epsilon_1,\ldots,\epsilon_{k-2}=0}^{1} \left|\begin{array}{c} B(x;m_i+j-i-\epsilon_i) \\ \dfrac{B(x;m_{k-1}+j-k+1)}{x^{m_{k-1}+j-k+1}} \\ \dfrac{B(x;m_k+j-k)}{x^{m_k+j-k}} \end{array}\right| \quad \begin{array}{l} i=1,\ldots,k-2 \\ j=1,\ldots,k \end{array}$$

Hence after k applications the result is

$$x^{\Sigma_{h=1}^k m_h} \left|\frac{B(x;m_i+j-i)}{x^{m_i+j-i}}\right| \quad i,j=1,\ldots,k.$$

Multiplying each ith row $(i = 1,\ldots,k)$ by x^{m_i} we obtain

$$\left|\frac{B(x;m_i+j-i)}{x^{j-i}}\right| \quad i,j=1,\ldots,k.$$

Multiplying each ith row by x^{1-i} we obtain

$$x^{\Sigma_{h=1}^k (h-1)} \left|\frac{B(x;m_i+j-i)}{x^{j-1}}\right| \quad i,j=1,\ldots,k$$

and multiplying each jth column by x^{j-1} results in

$$|B(x;m_i+j-i)| \quad i,j = 1,\ldots,k.$$

Therefore the recursion (1) is satisfied and hence the lemma is proved.

DEFINITION 2.5. Let

$$\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_k \end{array}$$

be a column of a partition where each $a_i > 0$ $(i = 1,\ldots,k)$. For a non-negative integer t, define t-non-increasing along a column to mean $a_i \geq a_{i+1} - t$ for all $i = 1,\ldots,k-1$.

DEFINITION 2.6. For $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$ and for a non-negative integer t, let $q_t(N;m_1,m_2,\ldots,m_k)$ denote the number of partitions of a positive integer N into k rows having m_i $(i = 1,\ldots,k)$ non-zero parts in each ith row where the parts are strictly decreasing along each row and t-non-increasing along each column. Call such a partition one of type q_t and let $Q_t(x;m_1,m_2,\ldots,m_k) = \sum_{n=0}^{\infty} q_t(n;m_1,m_2,\ldots,m_k)x^n$ where we define $q_t(n;m_1,m_2,\ldots,m_k)$ to be 1 for $n = 0$ and

$m_1 = m_2 = \ldots = m_k = 0$. Since $q_t(n;m_1,m_2,\ldots,m_k) = 0$ for $n > 0$ and $m_1 = m_2 = \ldots = m_k = 0$, $Q_t(x;m_1,m_2,\ldots,m_k) = 0$ for $m_1 = m_2 = \ldots = m_k = 0$.

As an example of partitions of type q_t, we list the partitions of $N = 9$ for $t = 1$, $m_1 = 2$ and $m_2 = m_3 = 1$.

61	51	51	52	41	43	41	42	42	32	31	32	31	21
1	1	2	2	3	1	2	2	1	2	2	3	3	3
1	2	1	2	1	1	2	1	2	2	3	1	2	3

Hence $q_1(9; 2, 1, 1) = 14$.

We shall calculate $Q_1(x;m_1,m_2,\ldots,m_k)$ directly from the formula for $B(x;m_1,m_2,\ldots,m_k)$ as computed in LEMMA 2.4 and then extend the procedure for $t > 1$.

DEFINITION 2.7. For $m_1 \geq m_2 \geq \ldots \geq m_k \geq 0$, $k \geq 2$ and for an integer s such that $2 \leq s \leq k$, let $b_s(N;m_1,m_2,\ldots,m_k)$ be the number of partitions of a positive integer N into k rows having m_i $(i = 1,\ldots,k)$ non-zero parts in each ith row where the parts are strictly decreasing along each row and non-increasing along each column between rows s and k and 1-non-increasing along each column between rows 1 and s. Call such a partition one of type b_s and let

$$B_s(x;m_1,m_2,\ldots,m_k) = \sum_{n=0}^{\infty} b_s(n;m_1,m_2,\ldots,m_k)x^n$$

where we define $b_s(n;m_1,m_2,\ldots,m_k)$ to be 1 for $n=0$ and $m_1=m_2=\ldots=m_k=0$. Since $b_s(n;m_1,m_2,\ldots,m_k)=0$ for $n>0$ and $m_1=m_2=\ldots=m_k=0$, $B_s(x;m_1,m_2,\ldots,m_k)=1$ for $m_1=m_2=\ldots=m_k=0$.

For example let $N=9$, $s=2$, $m_1=2$ and $m_2=m_3=1$. The partitions of 9 of type b_2 are

61	51	52	41	43	41	42	32	32	31	21
1	2	1	3	1	2	2	2	3	3	3
1	1	1	1	1	2	1	2	1	2	3

Hence $b_s(9;2,1,1)=11$.

THEOREM 2.8. For $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$ and for an integer s such that $2 \leq s \leq k$,

$$B_s(x;m_1,m_2,\ldots,m_k) = \begin{vmatrix} \dfrac{B(x;m_i+j-i)}{x^{(i-1)(j-i)}} \\ \\ \dfrac{B(x;m_i+j-i)}{x^{(s-1)(j-i)}} \end{vmatrix} \quad \begin{array}{l} i=1,\ldots,s-1 \\ j=1,\ldots,k \\ i=s,\ldots,k \end{array}$$

where $B(x;m_i+j-i)=0$ for $m_i < i-j$.

Proof: The proof is by induction on s. Let

$$(6)\qquad \begin{array}{cccc} a_{11} & a_{12} & \cdots\cdots & a_{1m_1} \\ a_{21} & a_{22} & \cdots\cdots & a_{2m_2} \\ \vdots & \vdots & & \\ a_{k1} & a_{k2} & \cdots & a_{km_k} \end{array}$$

be any partition of a positive integer of type b_2. If one is subtracted from each element of the rows 2 through k, the result is a partition of $N - \sum_{i=2}^{k} m_i$ of type b of the form

$$(7)\qquad \begin{array}{cccc} a_{11} & a_{12} & \cdots\cdots & a_{1,m_1} \\ b_{12} & b_{22} & \cdots\cdots & b_{2,m_2-\epsilon_2} \\ \vdots & \vdots & & \\ b_{k1} & b_{k2} & \cdots & b_{k,m_k-\epsilon_k} \end{array}$$

where each ϵ_i $(i = 2,\ldots,k)$ is 1 or 0 depending upon whether a 1 or not appears in the ith row of (1). Hence to each partition of N of type b_2 of the form (1) there exists a unique partition of $N - \sum_{i=2}^{k} m_i$ of type b of the form (2) and the correspondence is one-to-one. Therefore

$$b_2(N;m_1,m_2,\ldots,m_k) = \sum_{\epsilon_2,\ldots,\epsilon_k=0}^{1} b(N-\sum_{i=2}^{k} m_k;m_1,m_2-\epsilon_2,\ldots,m_k-\epsilon_k)$$

Hence

$$B_2(x;m_1,m_2,\dots,m_k)$$

$$= \sum_{n=0}^{\infty} b_2(n;m_1,m_2,\dots,m_k)x^n$$

$$= \sum_{n=0}^{\infty} \sum_{\epsilon_2,\dots,\epsilon_k=0}^{1} b(n-\sum_{i=2}^{k} m_i;m_1,m_2-\epsilon_2,\dots,m_k-\epsilon_k)x^n$$

$$= \sum_{\epsilon_2,\dots,\epsilon_k=0}^{1} \sum_{n=0}^{\infty} b(n-\sum_{i=2}^{k} m_i;\ m_1,m_2-\epsilon_2,\dots,m_k-\epsilon_k)x^n$$

$$= x^{\Sigma_{i=2}^{k} m_i} \sum_{\epsilon_2,\dots,\epsilon_k=0}^{1} \sum_{n-\Sigma_{i=2}^{k} m_i=0}^{\infty} b(n-\sum_{i=2}^{k} m_i;m_1,m_2-\epsilon_2,\dots,m_k-\epsilon_k)x^{n-\Sigma_{i=2}^{k} m_i}$$

$$= x^{\Sigma_{i=2}^{k} m_i} \sum_{\epsilon_2,\dots,\epsilon_k}^{1} B(x;m_1,m_2-\epsilon_2,\dots,m_k-\epsilon_k)$$

since $b(n-\sum_{i=2}^{k};m_1,m_2,\dots,m_k) = 0$ for $n < \sum_{i=2}^{k} m_i$. Again, if $m_i = m_{i+1}$ for some i such that $2 \leq i \leq k$, $B(x;m_1,m_2-\epsilon_2,\dots,m_i-1,m_{k+1},\dots,m_k-\epsilon_k) = 0$. Hence by LEMMA 2.4

$$B_2(x;m_1,m_2,\dots,m_k)$$

$$= x^{\Sigma_{i=2}^{k} m_i} \sum_{\epsilon_2,\dots,\epsilon_k=0}^{1} \begin{vmatrix} B(x;m_1+j-1) \\ B(x;m_i+j-i-\epsilon_i) \end{vmatrix} \begin{matrix} j = 1,\dots,k \\ i = 2,\dots,k \end{matrix}$$

$$= x^{\Sigma_{h=2}^{k} m_h} \sum_{\epsilon_2,\ldots,\epsilon_{k-1}=0}^{1} \left\{ \left| \begin{matrix} B(x;m_1+j-1) \\ B(x;m_i+j-i-\epsilon_i) \\ B(x;m_k+j-k) \end{matrix} \right| \begin{matrix} j = 1,\ldots,k \\ i = 2,\ldots,k-1 \end{matrix} \right.$$

$$\left. + \left| \begin{matrix} B(x;m_1+j-1) \\ B(x;m_i+j-i-\epsilon_i) \\ B(x;m_k+j-k-1) \end{matrix} \right| \begin{matrix} j = 1,\ldots,k \\ i = 2,\ldots,k-1 \end{matrix} \right\}$$

$$= x^{\Sigma_{h=2}^{k} m_h} \sum_{\epsilon_2,\ldots,\epsilon_{k-1}=0}^{1} \left| \begin{matrix} B(x;m_1+j-1) \\ B(x;m_i+j-i-\epsilon_i) \\ \dfrac{B(x;m_k+j-k)}{x^{m_k+j-k}} \end{matrix} \right| \begin{matrix} j = 1,\ldots,k \\ i = 2,\ldots,k-1 \end{matrix}$$

After $k-1$ applications of combining determinants and using LEMMA 2.3 we obtain

$$x^{\Sigma_{h=2}^{k} m_h} \left| \begin{matrix} B(x;m_1+j-1) \\ \dfrac{B(x;m_i+j-i)}{x^{m_i+j-i}} \end{matrix} \right| \begin{matrix} j = 1,\ldots,k \\ i = 2,\ldots,k \end{matrix}$$

Multiplying each ith row $(i = 2,\ldots,k)$ by x^{m_i} we obtain

$$(8) \qquad \left| \begin{matrix} B(x;m_1+j-1) \\ \dfrac{B(x;m_i+j-i)}{x^{j-i}} \end{matrix} \right| \begin{matrix} j = 1,\ldots,k \\ i = 2,\ldots,k \end{matrix}$$

If $m_i+1 = m_{i+1}$ for some i such that $2 \leq i \leq k-1$, the ith row

$$\frac{xB(x;m_i+j-i)}{x^{j-i}} = \frac{B(x;m_{i+1}+j-(i+1))}{x^{j-(i+1)}}$$

for all $j = 1, \ldots, k$, where the right side is identical to the $(i+1)$st row and hence (8) indicates that $B_2(x;m_1, m_2, \ldots, m_k) = 0$ for $m_i+1 = m_{i+1}$.

Assume that for $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$ and for some s such that $2 \leq s \leq k$,

$$B_{s-1}(x;m_1, m_2, \ldots, m_k) = \begin{vmatrix} \dfrac{B(x;m_i+j-i)}{x^{(i-1)(j-i)}} \\ \\ \dfrac{B(x;m_i+j-i)}{x^{(s-2)(j-i)}} \end{vmatrix} \begin{array}{l} i = 1, \ldots, s-2 \\ \\ j = 1, \ldots, k \\ \\ i = s-1, \ldots, k \end{array}$$

Note again the above formula indicates that $B_s(x;m_1,m_2,\ldots,m_k) = 0$ when $m_i+1 = m_{i+1}$ for some i such that $s-1 \leq i \leq k-1$. That is

$$\frac{x^{s-2}(Bx;m_i+j-i)}{x^{(s-2)(j-i)}} = \frac{B(x;m_{i+1}+j-(i+1))}{x^{(s-2)(j-(i+1))}}$$

for all $j = 1, \ldots, k$.

Given any partition of a positive integer N of type b_s $(2 \leq s \leq k)$ subtract one from each of the elements in rows s through k. The result is a partition of $N - \sum_{i=s}^{k} m_i$ of type b_{s-1}

and therefore as in the case when $s = 2$

$$b_s(N;m_1,m_2,\ldots,m_k)$$

$$= \sum_{\epsilon_s,\ldots,\epsilon_k=0}^{1} b_{s-1}\Big(N-\sum_{i=s}^{k};m_1,m_2,\ldots,m_{s-1},m_s-\epsilon_s,m_{s+1}$$

$$-\epsilon_{s+1},\ldots,m_k-\epsilon_k\Big)$$

Hence,

$$B_s(x;m_1,m_2,\ldots,m_k)$$

$$= x^{\Sigma_{i=s}^{k} m_i} \sum_{\epsilon_s,\ldots,\epsilon_k=0}^{1} B_{s-1}(x;m_1,\ldots,m_{s-1},m_s-\epsilon_s,\ldots,m_k-\epsilon_k)$$

$$= x^{\Sigma_{h=a}^{k} m_h} \sum_{\epsilon_s,\ldots,\epsilon_k=0}^{1} \begin{vmatrix} \dfrac{B(x;m_i+j-i)}{x^{(i-1)(j-i)}} \\ \dfrac{B(x;m_{s-1}+j-s+1)}{x^{(s-2)(j-s+1)}} \\ \dfrac{B(x;m_i+j-i-\epsilon_i)}{x^{(s-2)(j-i)}} \end{vmatrix} \quad \begin{matrix} i = 1,\ldots,s-2 \\ j = 1,\ldots,k \\ i = s,\ldots,k \end{matrix}$$

$$= x^{\Sigma_{h=s}^{k} m_h} \sum_{\epsilon_s,\ldots,\epsilon_k=0}^{1} \begin{vmatrix} \dfrac{B(x;m_i+j-i)}{x^{(i-1)(j-i)}} \\ \dfrac{B(x;m_i+j-i-\epsilon_i)}{x^{(s-2)(j-i)}} \end{vmatrix} \quad \begin{matrix} i = 1,\ldots,s-1 \\ j = 1,\ldots,k \\ i = s,\ldots,k \end{matrix}$$

Summing as in the case when $s = 2$ we obtain

$$x^{\Sigma_{h=s}^{k} m_h} \begin{vmatrix} \dfrac{B(x;m_i+j-i)}{x^{(i-1)(j-i)}} \\ \\ \dfrac{B(x;m_i+j-i)}{x^{m_i+j-i}\, x^{(s-2)(j-i)}} \end{vmatrix} \qquad \begin{array}{l} i = 1, \ldots, s-1 \\ j = 1, \ldots, k \\ i = s, \ldots, k \end{array}$$

Multiplying each ith row $(i = s, \ldots, k)$ by x^{m_i} we obtain

$$\begin{vmatrix} \dfrac{B(x;m_i+j-i)}{x^{(i-1)(j-i)}} \\ \\ \dfrac{B(x;m_i+j-i)}{x^{(s-1)(j-i)}} \end{vmatrix} \qquad \begin{array}{l} i = 1, \ldots, s-1 \\ j = 1, \ldots, k \\ i = s, \ldots, k \end{array}$$

Hence the theorem is proved.

COROLLARY 2.9. For $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$,

$$(9) \qquad Q_1(x;m_1, m_2, \ldots, m_k) = \begin{vmatrix} \dfrac{B(x;m_i+j-i)}{x^{(i-1)(j-i)}} \end{vmatrix} \qquad i,j = 1, \ldots, k$$

where $B(x;m_i+j-i) = 0$ for $m_i < i-j$.

Proof:

$$Q_1(x;m_1, m_2, \ldots, m_k) = B_k(x;m_1, m_2, \ldots, m_k)$$

Setting $s = k$ in THEOREM 2.8 we obtain

$$\left| \begin{array}{c} \dfrac{B(x;m_i+j-i)}{x^{(i-1)(j-i)}} \\ \\ \dfrac{B(x;m_k+j-k)}{x^{(k-1)(j-k)}} \end{array} \right| \quad \begin{array}{l} i = 1,\ldots,k-1 \\ j = 1,\ldots,k \end{array}$$

$$= \left| \frac{B(x;m_i+j-i)}{x^{(i-1)(j-i)}} \right| \quad i,j = 1,\ldots,k$$

REMARK: Equation (9) does not indicate zero for $m_{i+1} = m_i+1$ for some i such that $1 \leq i \leq k-1$. In fact if we compute $Q(x;m_1,m_2,\ldots,m_k)$ when $m = m_1$, $m_2 = m_1+1$, $m_3 = m_2+1,\ldots, m_k = m_{k-1}+1$, we obtain an interesting result.

$$Q_1(x;m,\ m+1,\ldots,m+k-1) = \left| \frac{B(x;m+j-1)}{x^{(i-1)(j-i)}} \right| \quad i,j = 1,\ldots,k$$

$$= \prod_{h=1}^{k} B(x;m+h-1) \left| \frac{1}{x^{(i-1)(j-i)}} \right| \quad i,j = 1,\ldots,k.$$

Multiplying each ith row $(i = x,\ldots,k)$ by $x^{(i-1)(k-i)}$, the result is

$$\frac{\prod_{h=1}^{k} B(x;m+h-1)}{x^{\Sigma_{\ell=1}^{k}(\ell-1)(k-\ell)}} \left| x^{(i-1)(k-j)} \right| \quad i,j = 1,\ldots,k.$$

The determinant

$$\left| x^{(i-1)(k-j)} \right| \quad i,j = 1,\ldots,k$$

is the Vandermonde determinant

$$\left| z_j^{i-1} \right| \begin{array}{l} i = 1, \ldots, k \\ j = 0, \ldots, k-1 \end{array} = \prod_{1 \le i < j \le k} (z_j - z_i)$$

where $z_o = 1$ and $z_j = x^{k-j}$. Hence

$$Q_1(x; m_1, m_2, \ldots, m_k) = \frac{\prod_{h=1}^{k} B(x; m+h-1)}{x^{\Sigma_{\ell=1}^{k} (\ell-1)(k-\ell)}} \prod_{1 \le i < j \le k} (x^{k-j} - x^{k-i}).$$

Dividing each term $(x^{k-j} - x^{k-i})$ in the product by x^{k-j}, we obtain

$$\frac{\prod_{h=1}^{k} B(x; m+h-1) \; x^{\Sigma_{\lambda=1}^{k} (k-\lambda)(\lambda-1)}}{x^{\Sigma_{\ell=1}^{k} (\ell-1)(k-\ell)}} \prod_{1 \le i < j \le k} (1 - x^{j-i})$$

$$= \prod_{h=1}^{k} B(x; m+h-1) \prod_{1 \le i < j \le k} (1 - x^{j-i})$$

$$= \frac{x^{\Sigma_{h=1}^{k} S_{m+h-1}}}{[m+k-1][m+k-2]^2 \ldots [m]^k \ldots [2]^k [1]^k} \prod_{1 \le i < j \le k} (1 - x^{j-i})$$

where we recall that $S_\mu = \binom{\mu+1}{2}$ and $[\mu] = (1 - x^\mu)$ for a positive integer μ. Hence

$Q(x;m, m+1, \ldots, m+k-1)$

$$= \frac{x^{\Sigma_{h=1}^{k} S_{m+h-1}}}{[m+k-1][m+k-2]^2 \ldots [m]^k \ldots [k]^k [k-1]^{k-1} \ldots [2]^2 [1]}$$

$$= x^{\Sigma_{h=1}^{k} S_{m+h-1}} T_k^m(x)$$

where $T_k^m(x)$ denotes the generating function for the number of k-line partitions where the number of non-zero parts in each row does not exceed m.

The result then would indicate that the partitions enumerated by $q_1(n;m, m+1, \ldots, m+k-1)$ are those obtained by superimposing k-line partitions, whose non-zero parts in each row is less than m, onto the array

$$\begin{array}{ccccc} m & m-1 & \ldots & 1 & \\ m+1 & m & \ldots & 2 & 1 \\ \vdots & \vdots & & & \\ m+k-1 & m+k-2 & \ldots\ldots\ldots & & 1 \end{array}$$

whose sum is $\sum_{h=1}^{k} S_{m+h-1}$.

In order to compute $Q_t(x;m_1, m_2, \ldots, m_k)$ for $t > 1$, we shall define a more general "intermediate" partition and calculate the corresponding generating function.

DEFINITION 2.10. For $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$, $k \geq 3$ and for integers r and s such that $2 \leq s \leq k-1$ and $0 \leq r \leq t$, let ${}^r q_t^s(N; m_1, m_2, \ldots, m_k)$ denote the number of partitions of a positive integer N into k rows having m_i $(i = 1, \ldots, k)$ non-zero parts in each ith row, the parts strictly decreasing along each row, t-non-increasing along each column between rows 1 and s, r-non-increasing along each column between rows s and $s+1$ and non-increasing along each column between $s+1$ and k. Call such a partition one of type ${}^r q_t^s$ and let

$$ {}^r Q_t^s(x; m_1, m_2, \ldots, m_k) = \sum_{n=0}^{\infty} {}^r q_t^s(n; m_1, m_2, \ldots, m_k) x^n $$

where we define ${}^r q_t^s(n; m_1, m_2, \ldots, m_k)$ to be 1 for $n = 0$ and $m_1 = m_2 = \ldots = m_k = 0$. Since ${}^r q_t^s(n; m_1, m_2, \ldots, m_k) = 0$ for $n > 0$ and $m_1 = m_2 = \ldots = m_k = 0$, ${}^r Q_t^s(x; m_1, m_2, \ldots, m_k) = 1$ for $m_1 = m_2 = \ldots = m_k = 0$.

As an example, let $N = 121$. The following array is an example of a partition of type ${}^2 q_3^4$ where $k = 7$.

$$
\begin{array}{rrrr}
5 & 4 & 3 & 1 \\
6 & 5 & 1 & \\
9 & 3 & 2 & \\
10, & 6 & 5 & \\
12, & 8 & 6 & \\
12, & 8 & 5 & \\
5 & 4 & 1 &
\end{array}
$$

OBSERVATIONS: For $t \geq 1$ and $2 \leq s \leq k-1$,

(a) ${}^{o}Q_1^{s}(x;m_1,m_2,\ldots,m_k) = B_s(x;m_1,m_2,\ldots,m_k)$

(b) ${}^{t}Q_t^{s}(x;m_1,m_2,\ldots,m_k) = {}^{o}Q_t^{s+1}(x;m_1,m_2,\ldots,m_k)$

(c) ${}^{t}Q_t^{k-1}(x;m_1,m_2,\ldots,m_k) = Q_t(x;m_1,m_2,\ldots,m_k)$

LEMMA 2.11. For $m_1 \geq m_2 \geq \ldots \geq m_k \geq 0$, $k \geq 3$, and $t \geq 1$.

$$ {}^{o}Q_t^{2}(x;m_1,m_2,\ldots,m_k) = \begin{vmatrix} B(x;m_1+j-1) \\ \dfrac{B(x;m_i+j-i)}{x^{t(j-i)}} \end{vmatrix} \quad \begin{array}{l} j = 1,\ldots,k \\ i = 2,\ldots,k \end{array} $$

where $B(x;m_i+j-i) = 0$ for $m_i < i-j$.

Proof: The proof is by induction on t.

$$ {}^{o}Q_1^{2}(x;m_1,m_2,\ldots,m_k) = B_2(x;m_1,m_2,\ldots,m_k) $$

$$ = \begin{vmatrix} B(x;m_1+j-1) \\ \dfrac{B(x;m_i+j-i)}{x^{(j-i)}} \end{vmatrix} \quad \begin{array}{l} j = 1,\ldots,k \\ i = 2,\ldots,k \end{array} $$

by observation (a) following DEFINITION 2.10. For $t \geq 2$ assume

$${}^{o}Q^{2}_{t-1}(x;m_1,m_2,\ldots,m_k) = \begin{vmatrix} B(x;m_1+j-1) \\ \dfrac{B(x;m_i+j-i)}{x^{(t-1)(j-i)}} \end{vmatrix} \qquad \begin{array}{l} j = 1,\ldots,k \\ i = 2,\ldots,k \end{array}$$

Note that the form indicates that if $m_i+1 = m_{i+1}$ for some i such that $2 \leq i \leq k-1$, ${}^{o}Q^{2}_{t-1}(x;m_1,m_2,\ldots,m_k) = 0$. That is

$$\frac{x^{t-1}B(x;m_i+j-i)}{x^{(t-1)(j-i)}} = \frac{B(x;m_{i+1}+j-(i+1))}{x^{(t-1)(j-(i+1))}}$$

for all $j = 1,\ldots,k$.

Let any partition of a positive integer N of type ${}^{o}q^{2}_{t}$ with m_i $(i = 1,\ldots,k)$ non-zero parts in each ith row be given. Subtract one from each part in the ith row $(i = 2,\ldots,k)$. The result is a partition of $N - \sum_{i=2}^{k} m_i$ of type ${}^{o}q^{2}_{t-1}$ with $m_i - \epsilon_i$ non-zero parts in each ith row $(i = 2,\ldots,k)$. Here as before $\epsilon_i = 1$ or 0 depending upon whether the partition of N has a 1 or not in the ith row. The correspondence is one-to-one and hence

$${}^{o}q^{2}_{t}(N;m_1,m_2,\ldots,m_k) = \sum_{\epsilon_2,\ldots,\epsilon_k=0}^{1} {}^{o}q^{2}_{t-1}(N-\sum_{i=2}^{k} m_i;m_1,m_2-\epsilon_2,\ldots,m_k-\epsilon_k) ,$$

therefore, as in the proof of THEOREM 2.8,

$${}^{o}Q_t^2(x;m_1,m_2,\ldots,m_k)$$

$$= x^{\Sigma_{i=2}^k m_i} \sum_{\epsilon_2,\ldots,\epsilon_k=0}^{1} {}^{o}Q_{t-1}^2(x;m_1,m_2-\epsilon_2,\ldots,m_k-\epsilon_k)$$

$$= x^{\Sigma_{h=2}^k m_h} \sum_{\epsilon_2,\ldots,\epsilon_k=0} \left| \begin{array}{c} B(x;m_1+j-1) \\ \dfrac{B(x;m_i+j-i-\epsilon_i)}{x^{(t-1)(j-i)}} \end{array} \right| \quad \begin{array}{l} j=1,\ldots,k \\ i=2,\ldots,k \end{array}$$

by the induction hypothesis. Summing the determinants and using LEMMA 2.3 we obtain

$$x^{\Sigma_{h=2}^k m_h} \left| \begin{array}{c} B(x;m_1+j-1) \\ \dfrac{B(x;m_i+j-i)}{x^{m_i+j-i}\, x^{(t-1)(j-i)}} \end{array} \right| \quad \begin{array}{l} j=1,\ldots,k \\ i=2,\ldots,k \end{array}$$

Multiplying each ith term $(i = 2,\ldots,k)$ by x^{m_i} completes the proof of the lemma.

THEMOREM 2.12. For $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$ and $t \geq 1$,

$$Q_t(x;m_1,m_2,\ldots,m_k) = \left| \frac{B(x;m_i+j-i)}{x^{t(i-1)(j-i)}} \right| \quad i,j = 1,\ldots,k$$

where $B(x;m_i+j-i) = 0$ for $m_i < i-j$.

Proof: The proof consists of an induction argument on r and s computing ${}^{r}Q_t^s(x;m_1,m_2,\ldots,m_k)$ where $k \geq 3$, $0 \leq r \leq t$ and

$2 \leq s \leq k-1$. The result follows when $r = t$ and $s = k-1$ by observation (c) following DEFINITION 2.10.

LEMMA 2.11 is the case where $r = 0$ and $s = 2$. For r and s such that $0 \leq r \leq t-1$ and $2 \leq s \leq k-1$, assume

$$ {}^{r}Q_t^{s-1}(x;m_1,m_2,\ldots,m_k) = \left| \begin{array}{c} \dfrac{B(x;m_i+j-i)}{x^{t(i-1)(j-i)}} \\ \\ \dfrac{B(x;m_i+j-i)}{x^{(t(s-2)+r)(j-i)}} \end{array} \right| \begin{array}{l} i = 1,\ldots,s-1 \\ j = 1,\ldots,k \\ i = s,\ldots,k \end{array} $$

Again note that if $m_{i+1} = m_i+1$ for some i such that $s \leq i \leq k$, the form indicates zero. That is

$$ \frac{x^{t(s-2)+r}B(x;m_i+j-i)}{x^{(t(s-2)+r)(j-i)}} = \frac{B(x;m_{i+1}+j-(i+1))}{x^{(t(s-2)+r)(j-(i+1))}} \quad \text{for all } j = 1,\ldots,k. $$

Let a partition of a positive integer N of type ${}^{r+1}q_t^{s-1}$ with m_i $(i = 1,\ldots,k)$ non-zero parts in each ith row be given. Subtract one from each ith row $(i = s,\ldots,k)$, thereby obtaining a partition $N - \sum_{i=s}^{k} m_i$ of type ${}^{r}q_t^{s-1}$ with $m_i - \epsilon_i$ non-zero parts in each ith row $(i = s,\ldots,k)$. Since the correspondence is one-to-one,

$$ {}^{r+1}q_t^{s-1}(N;m_1,m_2,\ldots,m_k) \ldots $$

$$ = \sum_{\epsilon_s,\ldots,\epsilon_k=0}^{1} {}^{r}q_t^{s-1}(N-\sum_{i=s}^{k} m_i;m_1,m_2,\ldots,m_{s-1},m_s-\epsilon_s,\ldots,m_k-\epsilon_k) $$

and hence as in the proof of THEOREM 2.8,

$$ {}^{r+1}Q_t^{s-1}(x;m_1m,{}_2,\ldots,m_k) $$

$$ = x^{\Sigma_{i=s}^{k} m_i} \sum_{\epsilon_s,\ldots,\epsilon_k=0}^{1} {}^{r}Q_q^{s-1}(x;m_1,m_2,\ldots,m_{s-1},m_s-\epsilon_s,\ldots,m_k-\epsilon_k) $$

$$ = x^{\Sigma_{h=s}^{k} m_h} \sum_{\epsilon_s,\ldots,\epsilon_k=0} \left| \begin{array}{c} \dfrac{B(x;m_i+j-i)}{x^{t(i-1)(j-i)}} \\ \\ \dfrac{B(x;m_i+j-i-\epsilon_i)}{x^{(t(s-2)+r)(j-i)}} \end{array} \right| \begin{array}{l} i = 1,\ldots,s-1 \\ \\ j = 1,\ldots,k \\ \\ i = s,\ldots,k \end{array} $$

by the induction hypothesis. Summing the determinants and using LEMMA 2.3 we obtain

$$ x^{\Sigma_{h=s}^{k} m_h} \left| \begin{array}{c} \dfrac{B(x;m_i+j-i)}{x^{t(i-1)(j-i)}} \\ \\ \dfrac{B(x;m_i+j-i)}{x^{m_i+j-i}\, x^{(t(s-2)+r)(j-i)}} \end{array} \right| \begin{array}{l} i = 1,\ldots,s-1 \\ \\ j = 1,\ldots,k \\ \\ i = s,\ldots,k \end{array} $$

$$= \left| \begin{array}{c} \dfrac{B(x;m_i+j-i)}{x^{t(i-1)(j-i)}} \\ \\ \dfrac{B(x;m_i+j-i)}{x^{(t(s-2)+r+1)(j-i)}} \end{array} \right| \quad \begin{array}{l} i = 1,\ldots,s-1 \\ \\ j = 1,\ldots,k \\ \\ i = s,\ldots,k \end{array}$$

Setting $r = t$, we have by observation (b) following DEFINITION 2.10,

$${}^{o}Q_t^{s+1}(x;m_1,m_2,\ldots,m_k) = {}^{t}Q_t^{s}(x;m_1,m_2,\ldots,m_k)$$

$$= \left| \begin{array}{c} \dfrac{B(x;m_i+j-i)}{x^{t(i-1)(j-i)}} \\ \\ \dfrac{B(x;m_i+j-i)}{x^{(t(s-1)+t)(j-i)}} \end{array} \right| \quad \begin{array}{l} i = 1,\ldots,s \\ \\ j = 1,\ldots,k \\ \\ i = s+1,\ldots,k \end{array}$$

$$= \left| \begin{array}{c} \dfrac{B(x;m_i+j-i)}{x^{t(i-1)(j-i)}} \\ \\ \dfrac{B(x;m_i+j-i)}{x^{ts(j-i)}} \end{array} \right| \quad \begin{array}{l} i = 1,\ldots,s \\ \\ j = 1,\ldots,k \\ \\ i = s+1,\ldots,k \end{array}$$

Hence for $s = k-1$,

$$Q_t(x;m_1,m_2,\ldots,m_k) = {}^{t}Q_t^{k-1}(x;m_1,m_2,\ldots,m_k)$$

$$= \left| \dfrac{B(x;m_i+j-i)}{x^{t(i-1)(j-i)}} \right| \quad i,j = 1,\ldots,k$$

and the theorem is proved for $k \geq 3$. For $k = 2$,

$$Q_1(x;m_1,m_2) = B_1(x;m_1,m_2)$$

$$= \begin{vmatrix} B(x;m_1) & B(x;m_1+1) \\ xB(x;m_2-1) & B(x;m_2) \end{vmatrix}$$

By induction on t following the same procedure as in the proof of LEMMA 2. 11, it follows that

$$Q_t(x;m_1,m_2) = \begin{vmatrix} B(x;m_1) & B(x;m_1+1) \\ x^t(B(x;m_2-1) & B(x;m_2) \end{vmatrix}$$

Hence the theorem is proved.

Expanding the above we obtain

$$\frac{x^{S_{m_1}+S_{m_2}}}{[m_1]\ldots[m_2+1][m_2]^2\ldots[1]^2} - \frac{x^{S_{m_1+1}+S_{m_2-1}+t}}{[m_1+1]\ldots[m_2][m_2-1]^2\ldots[1]^2}$$

$$= \frac{x^{S_{m_1}+S_{m_2}}(1-x^{m_1+1}-x^{t+m_1+1-m_2}(1-x^{m_2}))}{[m_1+1]\ldots[m_2+1][m_2]^2\ldots[1]^2}$$

Letting $m_1 = m_2 = m$, we have

$$Q_t(x;m,m) = \frac{x^{2S_m}(1-x^{m+1}-x^{t+1}(1-x^m))}{[m+1][m]^2\ldots[1]^2}$$

as $t \to \infty$,

$$Q_t(x, m, m) \to \frac{x^{2S_m}}{[m]^2 \ldots [1]^2}$$

$$= B(x;m)^2$$

$$= x^{2S_m} P_m(x)^2$$

where $P_m(x)$ is the generating function for the number of one-line partitions into at most m parts. As $m \to \infty$,

$$\frac{B(x;m)^2}{x^{2S_m}} \to P(x)^2$$

which is the generating function for the number of partitions into 2 distinct lines, each line independent from the other. This is to be expected since letting $t \to \infty$, removes the dependency between the rows for 2-line partitions of type q_t.

III. k-LINE PARTITIONS OF TYPE F

DEFINITION 3.1. For $m_1 \geq m_2 \geq \dots \geq m_k \geq 0$ and for a positive integer t, let $f_t(N;m_1,m_2,\dots,m_k)$ denote the number of partitions of a positive integer N into k rows having m_i $(i = 1,\dots,k)$ non-zero parts in each ith row, where the parts are strictly decreasing along each row and decreasing by at least t along each column. Call such a partition one of type f_t and let $F_t(x;m_1,m_2,\dots,m_k) = \sum_{n=0}^{\infty} f_t(n;m_1,m_2,\dots,m_k)x^n$ where we define $f_t(n;m_1,m_2,\dots,m_k)$ to be 1 for $n = 0$ and $m_1 = m_2 = \dots = m_k = 0$. Since $f_t(n;m_1,m_2,\dots,m_k) = 0$ for $n > 0$ and $m_1 = m_2 = \dots = m_k = 0$, $F_t(x;m_1,m_2,\dots,m_k) = 1$ for $m_1 = m_2 = \dots = m_k = 0$.

As an example, let $N = 16$, $m_1 = 3$ and $m_2 = 2$. The partitions of f_2 are

931	841	832	751	742	652	643	831	741	732
21	21	21	21	21	21	21	31	31	31

651	642	543	731	641	542	731	641	632
31	31	31	32	32	32	41	41	41

REMARK: For $t \geq 2$, a 1 may appear in any ith row $(i = 1,\dots,k-1)$ of a partition of type f_t as long as $m_i > m_{i+1}$. A 1 may also appear in the kth row.

For $m_1 > m_2 > \ldots > m_k$, let the array

$$
(1) \qquad \begin{array}{cccc}
a_{11} & a_{12} & \cdots\cdots & a_{1m_1} \\
a_{12} & a_{22} & \cdots & a_{2m_2} \\
\vdots & \vdots & & \\
a_{k1} & a_{k2} & \cdots & a_{km_k}
\end{array}
$$

be a partition of a positive integer N of type f_1. Upon subtracting 1 from each part of the partition, we obtain a partition of $N - \sum_{i=1}^{k} m_i$ of type f_1 of the form

$$
\begin{array}{cccc}
b_{11} & b_{12} & \cdots\cdots & b_{1,m_1-\epsilon_1} \\
b_{21} & b_{22} & \cdots & b_{2,m_2-\epsilon_2} \\
\vdots & \vdots & & \\
b_{k1} & b_{k2} & \cdots & b_{k,m_k-\epsilon_k}
\end{array}
$$

where again $\epsilon_i = 1$ or 0 depending upon whether $a_{im_i} = 1$ or not. Each partition of N of type f_1 with m_i non-zero parts in each ith row gives rise to a partition of $N - \sum_{i=1}^{k} m_i$ also of type f_1 with $m_i - \epsilon_i$ non-zero parts in each ith row. Since the correspondence is one-to-one, the total number of partitions of N is the sum over all vectors $(\epsilon_1, \ldots, \epsilon_k)$ of the number of partitions of

$N - \sum_{i=1}^{k} m_i$. Hence

$$(2)\quad f_1(N;m_1,m_2,\dots,m_k) = \sum_{\epsilon_1,\dots,\epsilon_k=0}^{1} f_1(N-\sum_{i=1}^{k};m_1-\epsilon_1,m_2-\epsilon_2,\dots,m_k-\epsilon_k)$$

Consider the array (1) and let $m_s = m_{s+1} = \dots = m_{s+t}$ for some s and t such that $1 \leq s \leq k$ and $1 \leq t \leq k-s$. Hence $a_{sm_s} > a_{s+1,m_{s+1}} > \dots > a_{s+t,m_{s+t}}$. Subtracting 1 from each part yields a partition of the form

$$\begin{array}{llll}
b_{11} & b_{12} & \cdots\cdots & b_{1,m_1-\epsilon_1} \\
\vdots & \vdots & & \\
b_{s-1,1} & b_{s-1,2} & \cdots\cdots & b_{s-1,m_{s-1}-\epsilon_{s-1}} \\
b_{s1} & b_{s2} & \cdots\cdots & b_{sm_s} \\
\vdots & \vdots & & \vdots \\
b_{s+t-1,1} & b_{s+t-1,2} & \cdots\cdots & b_{s+t-1,m_{s+t-1}} \\
b_{s+t,1} & b_{s+t,2} & \cdots & b_{s+t,m_{s+t}-\epsilon_{s+t}} \\
\vdots & \vdots & & \\
b_{k1} & b_{k2} & \cdots & b_{k,m_k-\epsilon_k}
\end{array}$$

Since the correspondence is one-to-one, the total number of partitions of N of type f_1 is the sum over all vectors $(\epsilon,\dots,\epsilon_{s-1},\epsilon_{s+t},\dots,\epsilon_k)$ of the number of partitions of $N - \sum_{i=1}^{k} m_i$ of type f_1. Hence

$$f_1(N;m_1,m_2,\ldots,m_k)$$

$$= \sum_{\epsilon_1,\ldots,\epsilon_{s-1},\epsilon_{s+t},\ldots,\epsilon_k=0}^{1} f_1(N-\sum_{i=1}^{k} m_i;m_1-\epsilon_1,\ldots,m_{s-1}-\epsilon_{s-1},m_s,\ldots,$$

$$m_{s+t-1},m_{s+t}-\epsilon_{s+t},\ldots,m_k-\epsilon_k)$$

For example, the partitions of 11 of type f_1 with $m_1 = 2$ and $m_2 = m_3 = 1$ are

71	62	53	61	52	43	51	42	51
2	2	2	3	3	3	3	3	4
1	1	1	1	1	1	2	2	1

The corresponding partitions of $11 - 4 = 7$ are

6	51	42	5	41	32	4	31	4
1	1	1	2	2	2	2	2	3
						1	1	

Hence,

$$f_1(11;2,1,1) = \sum_{\epsilon_1,\epsilon_3=0}^{1} f_1(7;2-\epsilon_1,1,1-\epsilon_3)$$

$$= f_1(7;2,1,1) + f_1(7;1,1,1) + f_1(7;2,1) + f(7;1,1)$$

$$= 1 + 1 + 4 + 3 = 9.$$

The same procedure would extend to a partition of a positive integer N with any number of blocks of rows, each row in a given block having the same number of non-zero elements. Hence, if there are r blocks of rows of equal length, each block containing t_h+1 rows $(h = 1, \ldots, r)$, we have

(3)

$$
\begin{aligned}
&f_1(N; m_1, m_2, \ldots, m_k) \\
&= \sum_{\epsilon_j=0}^{1} f_1\Big(N - \sum_{i=1}^{k} m_i; m_1 - \epsilon_1, \ldots, m_{s_1-1} - \epsilon_{s_1-1}, m_{s_1}, \ldots, m_{s_1+t_1-1}, \\
&\qquad m_{s_1+t_1} - \epsilon_{s_1+t_1}, \ldots, m_{s_2-1} - \epsilon_{s_2-1}, m_{s_2}, \ldots, m_{s_2+t_2-1}, \\
&\qquad m_{s_2+t_2} - \epsilon_{s_2+t_2}, \ldots, m_{s_r-1} - \epsilon_{s_r-1}, m_{s_r}, \ldots, m_{s_r+t_r-1}, \\
&\qquad m_{s_r+t_r} - \epsilon_{s_r+t_r}, \ldots, m_k - \epsilon_k\Big)
\end{aligned}
$$

where $j = 1, \ldots, s_1-1, s_1+t_1, \ldots, s_2-1, s_2+t_2, \ldots, s_r-1, s_r+t_r, \ldots, k.$

Equation (2) gives

$$
\begin{aligned}
&F_1(x; m_1, m_2, \ldots, m_k) \\
&= \sum_{n=0}^{\infty} f_1(n; m_1, m_2, \ldots, m_k) x^n \\
&= \sum_{n=0}^{1} \sum_{\epsilon_1, \ldots, \epsilon_k=0} f_1\Big(n - \sum_{i=1}^{k} m_i; m_1 - \epsilon_1, m_2 - \epsilon_2, \ldots, m_k - \epsilon_k\Big) x^n
\end{aligned}
$$

$$= \sum_{\epsilon_1, \ldots, \epsilon_k = 0}^{1} \sum_{n=0}^{\infty} f_1(n - \sum_{i=1}^{k} m_i; m_1 - \epsilon_1, m_2 - \epsilon_2, \ldots, m_k - \epsilon_k) x^n$$

$$= x^{\Sigma_{i=1}^k m_i} \sum_{\epsilon_1, \ldots, \epsilon_k = 0}^{1} \sum_{n - \Sigma_{i=1}^k m_i = 0}^{\infty} f_1(n - \sum_{i=1}^{k} m_i; m_1 - \epsilon_1, m_2 - \epsilon_2, \ldots, m_k - \epsilon_k) x^{n - \Sigma_{i=1}^k m_i}$$

$$= x^{\Sigma_{i=1}^k m_i} \sum_{\epsilon_1, \ldots, \epsilon_k = 0}^{1} F_1(x; m_1 - \epsilon_1, m_2 - \epsilon_2, \ldots, m_k - \epsilon_k)$$

Since $f(n - \sum_{i=1}^{k}; m_1, m_2, \ldots, m_k) = 0$ for $n < \sum_{i=1}^{k} m_i$. Repeating the process for Equation (3) we are led to the following theorem.

THEOREM 3.2. For $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$,

$$F_1(x; m_1, m_2, \ldots, m_k)$$

$$= x^{\Sigma_{i=1}^k m_i} \sum_{\epsilon_1, \ldots, \epsilon_k = 0}^{1} F_1(x; m_1 - \epsilon_1, m_2 - \epsilon_2, \ldots, m_k - \epsilon_k)$$

if $m_1 > m_2 > \ldots > m_k$, and

$$F_1(x; m_1, m_2, \ldots, m_k)$$

$$= x^{\Sigma_{i=1}^k m_i} \sum_{\epsilon_j = 0}^{1} F_1(x; m_1 - \epsilon_1, \ldots, m_{s_1 - 1} - \epsilon_{s_1 - 1}, m_{s_1}, \ldots, m_{s_1 + 1},$$

$$m_{s_1 + t_1} - \epsilon_{s_1 + t_1}, \ldots, m_{s_2 - 1} - \epsilon_{s_2 - 1}, m_{s_2}, \ldots, m_{s_2 + t_2 - 1},$$

$$m_{s_2+t_2}-\epsilon_{s_2+t_2}, \ldots, m_{s_r-1}-\epsilon_{s_r-1}, m_{s_r}, \ldots, m_{s_r+t_r-1},$$

$$m_{s_r+t_r}-\epsilon_{s_r+t_r}, \ldots, m_k-\epsilon_k)$$

if $m_{s_h} = m_{s_h+1} = \ldots m_{s_h+t_h}$ for all $h = 1, \ldots, r$,

where

$$j = 1, \ldots, s_1-1, s_1+t_1, \ldots, s_2-1, s_2+t_2, \ldots, s_r-1, s_r+t_r, \ldots, k.$$

The function $F_1(x;m_1, m_2, \ldots, m_k)$ is uniquely determined by the above recursion and the initial condition, $F_1(x;m,m_2,\ldots,m_k) = 1$ for $m_1 = m_2 = \ldots = m_k = 0$.

THEOREM 3.3. For $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$,

$$F_1(x;m_1, m_2, \ldots, m_k) = \left| \frac{B(x;m_i+j-i)}{x^{(k-i)(j-i)}} \right| \quad i, j = 1, \ldots, k$$

where $B(x;m_i+j-i) = 0$ for $m_i < i-j$.

Proof: The proof consists of showing that this formula fits the initial condition and satisfies both parts of the recursion of THEOREM 3.2.

$$F_1(x;m_1) = \left| \frac{B(x;m_1)}{x^0} \right| = |1| = 1 \quad \text{for} \quad m_1 = 0$$

Assume $m_s = m_{s+1} = m_{s+2} = \dots = m_{s+t}$ for some s and t such that $1 \le s \le k$ and $1 \le t \le k-s$. Consider

$$x^{\Sigma_{i=1}^k m_i} \sum_{\epsilon_1,\dots,\epsilon_{s-1},\epsilon_{s+t},\dots,\epsilon_k=0}^{1} F_1(x;m_1-\epsilon_1,\dots,m_{s-1}-\epsilon_{s-1},m_s,\dots,$$

$$m_{s+t-1},m_{s+t}-\epsilon_{s+t},\dots,m_k-\epsilon_k)$$

$$= x^{\Sigma_{h=1}^k m_h} \sum_{\epsilon_1,\dots,\epsilon_{s-1},\epsilon_{s+t},\dots,\epsilon_k=0}^{1} \begin{vmatrix} \dfrac{B(x;m_i+j-i-\epsilon_i)}{x^{(k-i)(j-i)}} \\ \dfrac{B(x;m_i+j-i)}{x^{(k-i)(j-i)}} \\ \dfrac{B(x;m_i+j-i-\epsilon_i)}{x^{(k-i)(j-i)}} \end{vmatrix} \begin{array}{l} i = 1,\dots,s-1 \\ j = 1,\dots,k \\ i = s,\dots,s+t-1 \\ i = s+t,\dots,k \end{array}.$$

Summing by combining determinants and using LEMMA 2.3, we obtain

$$x^{\Sigma_{h=1}^k m_h} \begin{vmatrix} \dfrac{B(x;m_i+j-i)}{x^{m_i+j-i}\,x^{(k-i)(j-i)}} \\ \dfrac{B(x;m_i+j-i)}{x^{(k-i)(j-i)}} \\ \dfrac{B(x;m_i+j-i)}{x^{m_i+j-i}\,x^{(k-i)(j-i)}} \end{vmatrix} \begin{array}{l} i = 1,\dots,s-1 \\ j = 1,\dots,k \\ i = s,\dots,s+t-1 \\ i = s+t,\dots,k \end{array}.$$

Multiplying each ith row by x^{m_i} $(i = 1,\dots,s-1,s+t,\dots,k)$, we obtain

$$(4) \qquad x^{\Sigma_{h=s}^{s+t-1} m_h} \left| \begin{array}{c} \dfrac{B(x;m_i+j-i)}{x^{j-i}x^{(k-i)(j-i)}} \\ \dfrac{B(x;m_i+j-i)}{x^{(k-i)(j-i)}} \\ \dfrac{B(x;m_i+j-i)}{x^{j-i}x^{(k-i)(j-i)}} \end{array} \right| \quad \begin{array}{l} i = 1, \ldots, s-1 \\ j = 1, \ldots, k \\ i = s, \ldots, s+t-1 \\ i = s+t, \ldots, k \end{array}$$

Consider the $(s+t)$th row,

$$\frac{B(x;m_{s+t}+j-s-t)}{x^{j-s-t}x^{(k-s-t)(j-s-t)}}, \qquad j = 1, \ldots, k$$

$$= \frac{B(x;m_{s+t-1}+j-s-t)}{x^{(k-s-t+1)(j-s-t)}}, \qquad j = 1, \ldots, k$$

since $m_{s+t} = m_{s+t-1} = \ldots = m_s$.

Hence by LEMMA 2.3 we obtain

$$\frac{1}{x^{(k-s-t+1)(j-s-t)}} \left\{ \frac{B(x;m_{s+t-1}+j-s-t+1)}{x^{m_{s+t-1}+j-s-t+1}} - B(x;m_{s+t-1}+j-s-t+1) \right\},$$

$$j = 1, \ldots, k$$

Multiplying the $(s+t)$th row by $\dfrac{1}{x^{k-s-t+1}}$, we have

$$\frac{1}{x^{(k-s-t+1)(j-s-t+1)}} \left\{ \frac{B(x;m_{s+t-1}+j-s-t+1)}{x^{m_{s+t-1}+j-s-t+1}} - B(x;m_{s+t-1}+j-s-t+1) \right\},$$

$$j = 1, \ldots, k$$

Adding this row to the $(s+t-1)$st row, the $(s+t-1)$st row becomes

$$\frac{B(x; m_{s+t-1}+j-s-t+1)}{x^{m_{s+t-1}} x^{j-s-t+1} x^{(k-s-t+1)(j-s-t+1)}}, \quad j = 1, \ldots, k.$$

Thus if we multiply the $(s+t-1)$st row by $x^{m_{s+t-1}}$, the expression (4) equals

$$x^{\Sigma_{h=s}^{s+t-2} m_h} \begin{vmatrix} \dfrac{B(x;m_i+j-i)}{x^{j-i} x^{(k-i)(j-i)}} \\ \dfrac{B(x;m_i+j-i)}{x^{(k-i)(j-i)}} \\ \dfrac{B(x;m_i+j-i)}{x^{j-i} x^{(k-i)(j-i)}} \end{vmatrix} \quad \begin{array}{l} i = 1, \ldots, s-1 \\ j = 1, \ldots, k \\ i = s, \ldots, s+t-2 \\ i = s+t-1, \ldots, k \end{array}$$

Repeating the process t times yields

$$\left| \frac{B(x;m_i+j-i)}{x^{j-i} x^{(k-i)(j-i)}} \right| \quad i, j = 1, \ldots, k.$$

Multiplying each ith row by x^{1-i} we obtain

$$x^{\Sigma_{h=1}^{k}(h-1)} \left| \frac{B(x;m_i+j-i}{x^{j-1} x^{(k-i)(j-i)}} \right| \quad i, j = 1, \ldots, k.$$

Multiplying each jth column by x^{j-1} yields

$$\left| \frac{B(x;m_i+j-i)}{x^{(k-i)(j-i)}} \right| \qquad i,j = 1,\ldots,k$$

The result for $m_{s_h} = m_{s_h+1} = \ldots = m_{s_h+t_h}$ for all $h = 1,\ldots,r$ follows as a generalization of the previous procedure. Hence the second part of the recursion is satisfied.

For $m_1 > m_2 > \ldots > m_k$,

$$x^{\Sigma_{i=1}^k m_i} \sum_{\epsilon_1,\ldots,\epsilon_k=0}^{1} F_1(x;m_1-\epsilon_1, m_2-\epsilon_2, \ldots, m_k-\epsilon_k)$$

$$= x^{\Sigma_{h=1}^k m_h} \sum_{\epsilon_1,\ldots,\epsilon_k=0}^{1} \left| \frac{B(x;m_i+j-i-\epsilon_i)}{x^{(k-i)(j-i)}} \right| \qquad i,j = 1,\ldots,k$$

Combining determinants and using LEMMA 2.3, we obtain

$$x^{\Sigma_{h=1}^k m_h} \left| \frac{B(x;m_i+j-i)}{x^{m_i+j-i}\, x^{(k-i)(j-i)}} \right| \qquad i,j = 1,\ldots,k.$$

Multiplying each ith row by x^{m_i}, we have

$$\left| \frac{B(x;m_i+j-i)}{x^{j-i}\, x^{(k-i)(j-i)}} \right| \qquad i,j = 1,\ldots,k.$$

Multiplying each row by x^{1-i} and each column by x^{j-1}, the result is

$$\left| \frac{B(x;m_i+j-i)}{x^{(k-i)(j-i)}} \right| \qquad i,j = 1,\ldots,k$$

Hence the theorem is proved.

If we transpose the above determinant, the result may also be written

$$\left| \frac{B(x;m_j+i-j)}{x^{(k-j)(i-j)}} \right| \qquad i,j = 1,\ldots,k$$

$$= \left| \frac{B(x;m_j+i-j)}{x^{(2ki-2kj-2ji+2j^2)/2}} \right| \qquad i,j = 1,\ldots,k$$

$$= \left| \frac{B(x;m_j+i-j)}{x^{(2ki-i^2-k^2-2kj+j^2+k^2-2ji+i^2+j^2)/2}} \right| \qquad i,j = 1,\ldots,k$$

$$= \left| \frac{B(x;m_j+i-j)}{x^{-(i-k)^2/2+(j-k)^2/2+(i-j)^2/2}} \right| \qquad i,j = 1,\ldots,k.$$

Multiplying each jth column $(j = 1,\ldots,k)$ by $x^{((j-k)^2+j)/2}$, we obtain

$$x^{-\Sigma_{h=1}^{k}\frac{(h-k)^2+h}{2}} \left| \frac{B(x;m_j+i-j)}{x^{-(i-k)^2/2+(i-j)^2/2-j/2}} \right| \qquad i,j = 1,\ldots,k.$$

Multiplying each ith row $(i = 1,\ldots,k)$ by $x^{-((i-k)^2+i)/2}$, gives us

$$\left| \frac{B(x;m_j+i-j)}{x^{(i-j)^2/2+(i-j)/2}} \right| \quad i,j = 1,\ldots,k$$

$$= \left| \frac{B(x;m_j+i-j)}{x^{(i-j)(i-j+1)/2}} \right| \quad i,j = 1,\ldots,k$$

$$= \left| \frac{x^{-(i-j)(i-j+1)/2}\, x^{S_{m_j+i-j}}}{[m_j+i-j][m_j+i-j-1]\ldots[1]} \right| \quad i,j = 1,\ldots,k$$

This form agrees with that of Gordon's (2, p. 95), computed by a different method.

We shall now compute $F_2(x;m_1,m_2,\ldots,m_k)$ subject to the conditions,

$$\begin{aligned} m_1 &\geq m_2+k-1 \\ m_2 &\geq m_3+k-2 \\ &\vdots \\ m_{k-1} &\geq m_k+1 \geq 1 \end{aligned}$$

DEFINITION 3.4. For $m_1 \geq m_2 \geq \cdots \geq m_k$ and for an integer s such that $2 \leq s \leq k$, let $f_2^s(N;m_1,m_2,\ldots,m_k)$ be the number of partitions of a positive integer N into k rows having m_i $(i = 1,\ldots,k)$ non-zero parts in each ith row, where the parts are strictly decreasing along each row and strictly decreasing along each column between rows 1 and s and decreasing by at least 2 between rows s and k. Call such a parition one of type

f_2^s and let $F_2^s(x;m_1,m_2,\ldots,m_k) = \sum_{n=0}^{\infty} f_2^s(n;m_1,m_2,\ldots,m_k)x^n$ where we define $f_2^s(n;m_1,m_2,\ldots,m_k)$ to be 1 for $n = 0$ and $m_1 = m_2 = \ldots = m_k = 0$. Since $f_2^s(n;m_1,m_2,\ldots,m_k) = 0$ for $n > 0$ and $m_1 = m_2 = \ldots = m_k = 0$, $F_2^s(x;m_1,m_2,\ldots,m_k) = 1$ for $m_1 = m_2 = \ldots = m_k = 0$.

THEOREM 3.5. For $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$ and $m_i \geq m_{i+1}+k-i > 1$ $(i = 1,\ldots,k-1)$,

$$F_2(x;m_1,m_2,\ldots,m_k)$$

$$= x^{\Sigma_{i=1}^{k-1}(k-i)m_i} \sum_{\epsilon_1^1=0}^{1} x^{-\epsilon_1^1} \sum_{\epsilon_1^2,\epsilon_2^2=0}^{1} x^{-\epsilon_1^1-\epsilon_1^2-\epsilon_2^2} \cdots$$

$$\sum_{\epsilon_1^{k-2},\ldots,\epsilon_{k-2}^{k-2}=0}^{1} x^{-\Sigma_{\ell=1}^{k-2}\Sigma_{i=1}^{\ell}\epsilon_i^{\ell}}$$

$$\sum_{\epsilon_1^{k-1},\ldots,\epsilon_{k-1}^{k-1}=0}^{1} F_1(x;m_1-\sum_{i=1}^{k-1}\epsilon_1^i, m_2-\sum_{i=2}^{k-1}\epsilon_2^i,\ldots,m_{k-1}-\epsilon_{k-1}^{k-1},m_k)$$

Proof: Let a partition of a positive integer N of type f_2 with m_i $(i = 1,\ldots,k)$ non-zero parts in each ith row be given.

Subtracting 1 from each part in the first row yields a partition of $N - m_1$ of type f_2^2 with $m_1 - \epsilon_1^1$ non-zero parts in the first row, where $\epsilon_1^1 = 1$ or 0 depending upon whether a 1 or not appears in the first row of the partition of N. The correspondence is one-to-one, hence

$$f_2(N;m_1,m_2,\ldots,m_k) = \sum_{\epsilon_1^1=0}^{1} f_2^2(N-m_1,m_1-\epsilon_1^1,m_2,\ldots,m_k)$$

Therefore as in the proof of THEOREM 3.2,

$$(4) \qquad F_2(x;m_1,m_2,\ldots,m_k) = x^{m_1}\sum_{\epsilon_1^1=0}^{1} F_2^2(x;m_1-\epsilon_1^1,m_2,\ldots,m_k)$$

Given a partition of $N - m_1$ of type f_2^2 with $m_1-\epsilon_1^1,m_2,\ldots,m_k$ non-zero parts in each ith row, subtract 1 from each element of the first 2 rows. The result is a partition of $N - m_1 - (m_1-\epsilon_1^1+m_2)$ of type f_2^3 with $m_1-\epsilon_1^1-\epsilon_1^2$ non-zero parts in the first row and $m_2-\epsilon_2^2$ non-zero parts in the second row, where $\epsilon_i^2 = 1$ or 0 $(i = 1, 2)$. Hence,

$$f_2^2(N-m_1;m_1-\epsilon_1^1,m_2,\ldots,m_k) = \sum_{\epsilon_1^2,\,\epsilon_2^2=0}^{1} f_2^3(N-m_1-m_1+\epsilon_1^1-m_2;m_1-\epsilon_1^1-\epsilon_1^2, m_2-\epsilon_2^2,m_3,\ldots,m_k)$$

Therefore as before,

$$
\begin{aligned}
&F_2^2(x; m_1 - \epsilon_1^1, m_2, \ldots, m_k) \\
&= \sum_{n=0}^{\infty} f_2^2(n; m_1 - \epsilon_1^1, m_2, \ldots, m_k) x^n \\
&= \sum_{n=0}^{\infty} \sum_{\epsilon_1^2, \epsilon_2^2 = 0}^{1} f_2^3(n - m_1 + \epsilon_1^1 - m_2; m_1 - \epsilon_1^1 - \epsilon_1^2, m_2 - \epsilon_2^2, m_3, \ldots, m_k) x^n \\
&= \sum_{\epsilon_1^2, \epsilon_2^2 = 0}^{1} \sum_{n=0}^{\infty} f_2^3(n - m_1 + \epsilon_1^1 - m_2; m_1 - \epsilon_1^1 - \epsilon_1^2, m_2 - \epsilon_2^2, m_3, \ldots, m_k) x^n \\
&= x^{m_1 - \epsilon_1^1 + m_2} \sum_{\epsilon_1^2, \epsilon_2^2 = 0}^{1} \sum_{n - (m_1 - \epsilon_1^1 + m_2) = 0}^{\infty} f_2^3(n - m_1 + \epsilon_1^1 - m_2; m_1 - \epsilon_1^1 - \epsilon_1^2, m_2 - \epsilon_2^2, \\
&\qquad\qquad m_3, \ldots, m_k) x^{n - (m_1 - \epsilon_1^1 + m_2)} \\
&= x^{m_1 - \epsilon_1^1 + m_2} \sum_{\epsilon_1^2, \epsilon_2^2 = 0}^{1} F_2^3(x; m_1 - \epsilon_1^1 - \epsilon_1^2, m_2 - \epsilon_2^2, m_3, \ldots, m_k) .
\end{aligned}
$$

Substituting into (4), it follows that

$$F_2(x;m_1,m_2,\ldots,m_k)$$

$$= x^{m_1}\sum_{\epsilon_1^1=0}^{1} x^{m_1-\epsilon_1^1+m_2} \sum_{\epsilon_1^2,\,\epsilon_2^2=0}^{1} F_2^3(x;m_1-\epsilon_1^1-\epsilon_1^2,m_2-\epsilon_2^2,m_3,\ldots,m_k)$$

$$= x^{2m_1+m_2}\sum_{\epsilon_1^1=0}^{1} x^{-\epsilon_1^1} \sum_{\epsilon_1^2,\,\epsilon_2^2=0}^{1} F_2^3(x;m_1-\epsilon_1^1-\epsilon_1^2,m_2-\epsilon_2^2,m_3,\ldots,m_k)\ .$$

Repeating the process, we have

$$F_2(x;m_1,m_2,\ldots,m_k)$$

$$= x^{3m_1+2m_2+m_3}\sum_{\epsilon_1^1=0}^{1} x^{-\epsilon_1^1} \sum_{\epsilon_1^2,\,\epsilon_2^2=0}^{1} x^{-\epsilon_1^1-\epsilon_1^2-\epsilon_2^2}$$

$$\sum_{\epsilon_1^3,\,\epsilon_2^3,\,\epsilon_3^3=0}^{1} F_2^4(x;m_1-\epsilon_1^1-\epsilon_1^2-\epsilon_1^3,m_2-\epsilon_2^2-\epsilon_2^3,m_3-\epsilon_3^3,m_4,\ldots,m_k)$$

After $k-1$ applications of the same procedure, it follows that

$F_2(x;m_1, m_2, \ldots, m_k)$

$$= x^{\Sigma_{i=1}^{k-1}(k-i)m_i} \sum_{\epsilon_1^1=0}^{1} x^{-\epsilon_1^1} \sum_{\epsilon_1^2, \epsilon_2^2=0}^{1} x^{-\epsilon_1^1-\epsilon_1^2-\epsilon_2^2} \cdots$$

$$\sum_{\epsilon_1^{k-2}, \ldots, \epsilon_{k-2}^{k-2}=0}^{1} x^{-\Sigma_{\ell=1}^{k-2} \Sigma_{i=1}^{\ell} \epsilon_i^\ell} \sum_{\epsilon_1^{k-1}, \ldots, \epsilon_{k-1}^{k-1}=0} F_2^k(x; m_1 - \sum_{i=1}^{k-1} \epsilon_1^i, m_2 - \sum_{i=2}^{k-1} \epsilon_2^i, \ldots, m_{k-1} - \epsilon_{k-1}^{k-1}, m_k) .$$

$$F_2^k(x;m_1, m_2, \ldots, m_k) = F_1(x;m_1, m_2, \ldots, m_k)$$

implies the result.

THEOREM 3.6. For $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$ and $m_i \geq m_{i+1}+k-i \geq 0 \quad (i = 1, \ldots, k-1)$,

$$F_2(x;m_1, m_2, \ldots, m_k) = \left| \frac{B(x;m_i+j-i)}{x^{2(k-i)(j-i)}} \right| \qquad i,j = 1, \ldots, k$$

where $B(x;m_i+j-i) = 0$ for $m_i < i-j$.

Proof: Since

$$F_1(x;m_1, m_2, \ldots, m_k) = \left| \frac{B(x;m_i+j-i)}{x^{(k-i)(j-i)}} \right| \qquad i,j = 1, \ldots, k$$

for $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$,

$$F_2(x;m_1,m_2,\ldots,m_k)$$

$$= x^{\Sigma_{h=1}^{k-1}(k-h)m_h} \sum_{\epsilon_1^1=0}^{1} x^{-\epsilon_1^1} \sum_{\epsilon_1^2,\epsilon_2^2=0}^{1} x^{-\epsilon_1^1-\epsilon_1^2-\epsilon_2^2} \cdots$$

$$\sum_{\epsilon_1^{k-2},\ldots,\epsilon_{k-2}^{k-2}=0}^{1} x^{-\Sigma_{\ell=1}^{k-2}\Sigma_{h=1}^{\ell}\epsilon_h^{\ell}} \sum_{\epsilon_1^{k-1},\ldots,\epsilon_{k-1}^{k-1}=0}^{1} \left| \frac{B(x;m_i-\Sigma_{\ell=i}^{k-1}\epsilon_i^{\ell}+j-i)}{x^{(k-i)(j-i)}} \right| \quad i,j=1,\ldots,k$$

where $\sum_{\ell=i}^{k-1} \epsilon_i^{\ell} = 0$ for $i > k-1$.

Summing the determinants

$$\left| \begin{array}{c} \dfrac{B(x;m_i-\Sigma_{\ell=i}^{k-2}\epsilon_i^{\ell}+j-i-\epsilon_i^{k-1})}{x^{(k-i)(j-i)}} \\ B(x;m_k+j-k) \end{array} \right| \quad \begin{array}{l} i=1,\ldots,k-1 \\ j=1,\ldots,k \end{array}$$

over all vectors $(\epsilon_1^{k-1},\ldots,\epsilon_{k-1}^{k-1})$, we obtain

$$x^{\Sigma_{h=1}^{k-1}(k-h)m_h} \sum_{\epsilon_1^1=0}^{1} x^{-\epsilon_1^1} \cdots \sum_{\epsilon_1^{k-2},\ldots,\epsilon_{k-2}^{k-2}=0}^{1} x^{-\Sigma_{\ell=1}^{k-2}\Sigma_{h=1}^{\ell}\epsilon_h^{\ell}}$$

$$\left| \begin{matrix} \dfrac{B(x;m_i-\Sigma_{\ell=i}^{k-2}\epsilon_i^{\ell}+j-i)}{x^{m_i-\Sigma_{\ell=i}^{k-2}\epsilon_i^{\ell}+j-i}\, x^{(k-i)(j-i)}} \\ B(x;m_k+j-k) \end{matrix} \right| \quad \begin{matrix} i = 1,\ldots,k-1 \\ j = 1,\ldots,k \end{matrix}$$

where $\sum_{\ell=i}^{k-2} \epsilon_i^{\ell} = 0$ for $i > k-2$.

Multiplying each ith row $(i = 1,\ldots,k-2)$ by $x^{-\Sigma_{\ell=i}^{k-2}\epsilon_i^{\ell}}$ and noting that $\sum_{\ell=1}^{k-2}\sum_{h=1}^{\ell}\epsilon_h^{\ell} = \sum_{i=1}^{k-2}\sum_{\ell=i}^{k-2}\epsilon_i^{\ell}$, we obtain

$$x^{\Sigma_{h=1}^{k-1}(k-h)m_h} \sum_{\epsilon_1^1=0}^{1} x^{-\epsilon_1^1} \cdots$$

$$\sum_{\epsilon_1^{k-2},\ldots,\epsilon_{k-2}^{k-2}=0}^{1} \left| \begin{matrix} \dfrac{B(m_i+j-i-\Sigma_{\ell=i}^{k-2}\epsilon_i^{\ell})}{x^{m_i+j-i}\, x^{(k-i)(j-i)}} \\ B(m_k+j-k) \end{matrix} \right| \quad \begin{matrix} i = 1,\ldots,k-1 \\ j = 1,\ldots,k \end{matrix}$$

Summing the above determinants over the vectors $(\epsilon_1^{k-2},\ldots,\epsilon_{k-2}^{k-2})$ yields,

$$x^{\Sigma_{h=1}^{k-1}(k-h)m_h} \sum_{\epsilon_1^1=0}^{1} x^{-\epsilon_1^1} \cdots$$

$$\sum_{\epsilon_1^{k-3},\ldots,\epsilon_{k-3}^{k-3}=0}^{1} x^{-\Sigma_{\ell=1}^{k-3}\Sigma_{h=1}^{\ell}\epsilon_h^\ell} \left| \begin{array}{c} \dfrac{B(x;m_i+j-i-\Sigma_{\ell=i}^{k-3}\epsilon_i^\ell)}{x^{2(m_i+j-i)-\Sigma_{\ell=i}^{k-3}\epsilon_i^\ell}\, x^{(k-i)(j-i)}} \\[2ex] \dfrac{B(x;m_{k-1}+j-k+1)}{x^{m_k+j-k+1}\, x^{j-k+1}} \\[2ex] B(x;m_k+j-k) \end{array} \right| \begin{array}{l} i=1,\ldots,k-2 \\[2ex] j=1,\ldots,k \end{array}$$

where $\sum_{\ell=i}^{k-3} \epsilon_i^\ell = 0$ for $i > k-3$.

Multiplying each ith row $(i = 1, \ldots, k-3)$ by $x^{\Sigma_{\ell=1}^{k-3}\epsilon_i^\ell}$, we obtain

$$x^{\Sigma_{h=1}^{k-1}(k-h)m_h} \sum_{\epsilon_1^1=0}^{1} x^{-\epsilon_1^1} \cdots$$

$$\sum_{\epsilon_1^{k-3},\ldots,\epsilon_{k-3}^{k-3}=0}^{1} \left| \begin{array}{c} \dfrac{B(x;m_i+j-i-\Sigma_{\ell=i}^{k-3}\epsilon_i^\ell)}{x^{2(m_i+j-i)}\, x^{(k-i)(j-i)}} \\[2ex] \dfrac{B(x;m_{k-1}+j-k+1)}{x^{m_{k-1}+j-k+1}\, x^{j-k+1}} \\[2ex] B(x;m_k+j-k) \end{array} \right| \begin{array}{l} i = 1, \ldots, k-2 \\[2ex] j = 1, \ldots, k \end{array}$$

Repeating the process $k-1$ times, results in

$$x^{\Sigma_{h=1}^{k-1}(k-h)m_h} \left| \frac{B(x;m_i+j-i)}{x^{(k-i)(m_i+j-i)}\,x^{(k-i)(j-i)}} \right| \qquad i,j = 1,\ldots,k.$$

Multiplying each ith row $(i = 1,\ldots,k-1)$ by $x^{(k-i)m_i}$, we obtain

$$\left| \frac{B(x;m_i+j-i)}{x^{2(k-i)(j-i)}} \right| \qquad i,j = 1,\ldots,k.$$

Hence the theorem is proved.

REMARK: For $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$ and $m_i < m_{i+1}+k-i$ $(i = 1,\ldots,k-1)$, the above formula is not correct. For example, let $k = 2$, then $m_1 < m_2+1$ or $m_1 = m_2 = m$. The above formula is thus

$$F_2(x;m,m) = \begin{vmatrix} B(x;m) & \dfrac{B(x;m+1)}{x^2} \\ B(x;m-1) & B(x;m) \end{vmatrix}$$

$$= B(x;m)^2 - \frac{B(x;m-1)B(x;m+1)}{x^2}$$

$$= \frac{x^{2S_m}}{[m]^2[m-1]^2\ldots[1]^2} - \frac{x^{S_{m+1}+S_{m-1}-2}}{[m+1][m][m-1]^2[m-2]^2\ldots[1]^2}$$

(5) $$= \frac{x^{2S_m}(1-x^{m+1}-x^{-1}(1-x^m))}{[m+1][m]^2[m-1]^2\ldots[1]^2}$$

However let a partition of a positive integer N of type f_2 with m non-zero elements in the first and second row be given. Subtract 1 from each element in the first row. Since a 1 cannot appear in the first row, the result is a partition of $N - m$ of type f_1 with m non-zero parts in each row. Hence $f_2(N;m,m) = f(N-m;m,m)$ and therefore

$$F_2(x;m,m) = x^m F_1(x;m,m)$$

$$= \frac{x^{2S_m + 2m}}{[m+1][m]^2 \ldots [2]^2[1]}$$

which does not agree with (5).

In summary,

$$F_2(x;m_1,m_2) = \begin{cases} \left| \dfrac{B(x;m_i+j-i)}{x^{2(2-i)(j-i)}} \right| & i,j = 1,2 \quad \text{if} \quad m_1 > m_2 \\ x^{m_1} F_1(x;m_1,m_2) & \text{if} \quad m_1 = m_2 \end{cases}$$

IV. 2-LINE PARTITIONS DECREASING ALONG EACH COLUMN

We shall now look at k-line partitions where the parts decrease along each column. As an example, the 2-line partitions of 5 of this type are 5, 41, 32, 221, 331, 2111, 11111, $\begin{matrix}4\\1\end{matrix}$, $\begin{matrix}3\\2\end{matrix}$, $\begin{matrix}31\\1\end{matrix}$, $\begin{matrix}22\\1\end{matrix}$, $\begin{matrix}211\\1\end{matrix}$.

DEFINITION 4.1. For a positive integer N, let $c_k(N)$ be the number of k-line partitions of N whose parts are strictly decreasing along each column. Call such a partition one of type c_k and let $C_k(x) = \sum_{n=0}^{\infty} c_k(n)x^n$ where we define $c_k(0)$ to be 1.

DEFINITION 4.2. For a positive integer N, a k-line slant partition of N is a representation of the form

$$N = \sum_{i=1}^{k} \sum_{j=i}^{\infty} a_{ij}$$

where each a_{ij} is a non-negative integer such that $a_{ij} \geq a_{i+1,j}$ and $a_{ij} \geq a_{i,j+1}$. The partition is written as an array without plus signs and zero terms. Hence N has the form

$$\begin{array}{cccccc}
a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1h_1} & \\
 & a_{22} & a_{23} & \cdots\cdots & a_{2h_2} & \\
 & & a_{33} & \cdots\cdots & a_{3h_3} & \\
 & & & \ddots & & \\
 & & & a_{\ell\ell} & \cdots & a_{\ell h_\ell}
\end{array}$$

where ℓ is a positive integer less than or equal to k and each ith row $(i = 1, \ldots, \ell)$ contains $h_i - i + 1$ non-zero elements. For example the 2-line slant partitions of 5 are 5, 41, 32, 311,

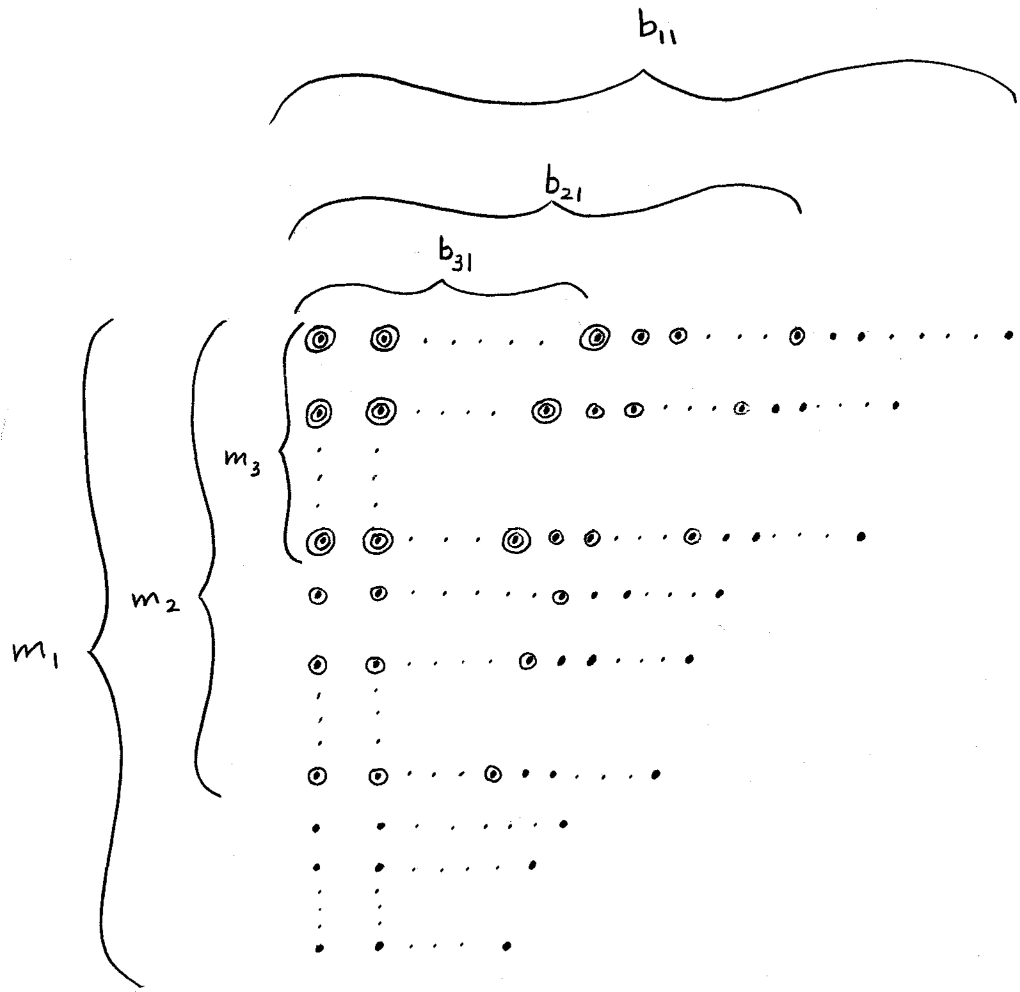
$$\begin{array}{llllllll} 31, & 221, & 22, & 2111, & 211, & 11111, & 1111, & 111. \\ \ \ 1 & & \ \ 1 & & \ \ 1 & & \ \ 1 & \ \ 11 \end{array}$$

LEMMA 4.3. For any positive integer N, there is a one-to-one correspondence between the partitions of N of type c_k and the k-line slant partitions of N.

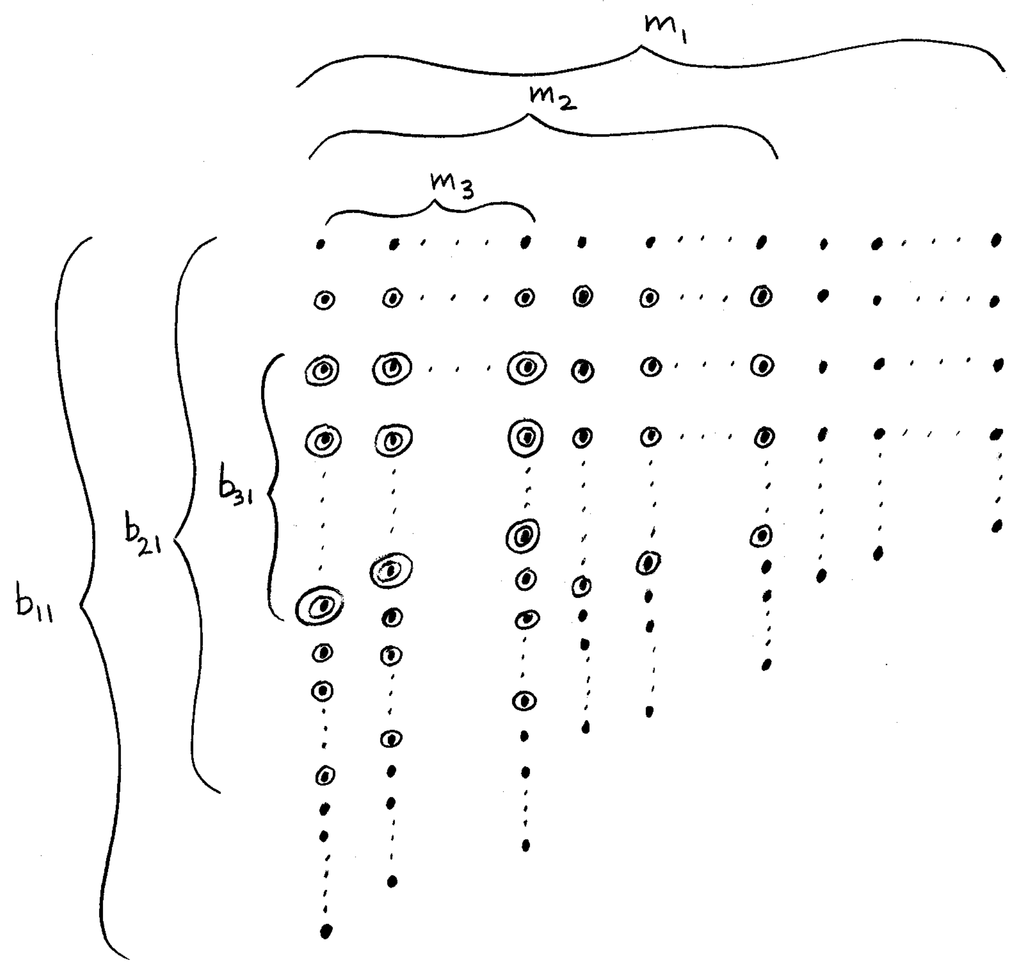
Proof: We shall illustrate the correspondence for $\ell = 3 \leq k$. Let

$$\begin{array}{llll} b_{11} & b_{12} & \cdots\cdots & m_{1m_1} \\ b_{21} & b_{22} & \cdots\cdot & b_{2m_2} \\ b_{31} & b_{23} & \cdots & b_{3m_3} \end{array}$$

be any partition of type c_k where m_i indicates the number of non-zero elements in each ith row $(i = 1, 2, 3)$. A graphical representation is given on the following page where the dots represent the graph of the first line of the array, the smaller circles represent the graph of the second line and the larger circles represent the graph of the third line. Since $b_{1j} > b_{2j}$ for all $j = 1, \ldots, m_1$, each smaller circle can move one position to the right and not travel beyond the dots. Likewise since $b_{2j} > b_{3j}$ for all $j = 1, \ldots, m_2$, each larger circle can move 2 positions to the right and not travel



beyond the dots or the smaller circles. Transposing the resultant graph, we obtain.



This is the graphical representation of the k-line slant partition

$$m_1 = a_{11} \quad a_{12} \quad a_{13} \quad \cdots\cdots \quad a_{1b_{11}}$$

$$m_2 = a_{22} \quad a_{23} \quad \cdots\cdot \quad a_{2,b_{21}+1}$$

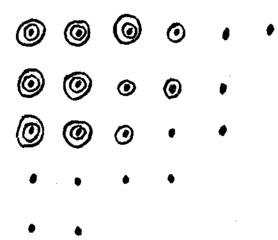
$$m_3 = a_{33} \quad \cdots \quad a_{3,b_{31}+2}$$

Reversing the procedure, that is starting with a k-line slant partition, guarantees that the resultant partition has the property that the parts are decreasing along each column.

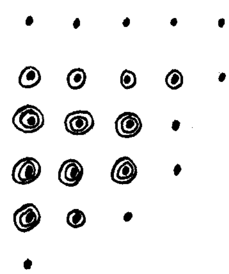
As an example, the graph of the partition

$$\begin{array}{ccccc} 6 & 5 & 5 & 4 & 2 \\ 4 & 4 & 3 & 1 & \\ 3 & 2 & 2 & & \end{array}$$

is



Transforming this graph, we have



which is the graphical representation of the 3-line slant partition

$$\begin{array}{cccccc} 5 & 5 & 4 & 4 & 3 & 1 \\ & 4 & 3 & 3 & 2 & \\ & & 3 & 3 & 1 & \end{array}$$

We shall now discuss k-line slant partitions with a particular number of non-zero elements in each ith row $(i = 1, \ldots, k)$.

DEFINITION 4.4. For $m_1 > m_2 > \ldots > m_k \geq 0$, let $a(N; m_1, m_2, \ldots, m_k)$ be the number of slant partitions of a positive integer N into k rows each row containing m_i $(i = 1, \ldots, k)$

non-zero parts, where the parts are strictly decreasing along each row. Call such a partition one of type a and let

$A(x;m_1,m_2,\ldots,m_k) = \sum_{n=0}^{\infty} a(n;m_1,m_2,\ldots,m_k)x^n$ where we define $a(n;m_1) = 1$ for $n = 0$ and $m_1 = 0$. Since $a(n;m_1) = 0$ for $n > 0$ and $m_1 = 0$, $A(x;m_1) = 1$ for $m_1 = 0$.

LEMMA 4.5. For $m_1 > m_2 > \ldots > m_k \geq 0$,

$$(1) \qquad A(x;m_1,m_2,\ldots,m_k) = x^{\Sigma_{i=1}^{k} m_i} \sum_{\epsilon_1,\ldots,\epsilon_k=0}^{1} A(x;m_1-\epsilon_1,m_2-\epsilon_2,\ldots,m_k-\epsilon_k)$$

Proof: The proof is identical with the proof of recursion (1) as given in Chapter II.

$A(x;m_1,m_2,\ldots,m_k)$ is uniquely determined by recursion (1) and the initial conditions

$$(2) \qquad A(x;m_1,m_2,\ldots,m_k) = \begin{cases} 0 \text{ if } m_i = m_{i+1} > 0 \text{ for some } i \text{ such} \\ \qquad\qquad \text{that } 1 \leq i \leq k-1 \\ 1 \text{ if } m_1 = 0 \end{cases}$$

Let $m_1 = \mu, m_2 = \mu-1, \ldots, m_k = \mu-k+1$, where μ is a positive integer greater than or equal to k. The slant partitions of a positive integer N enumerated by $a(N;m_1,m_2,\ldots,m_k)$ are exactly those found by superimposing k-line slant partitions with no more than $m_i = \mu-i+1$ $(i = 1,\ldots,k)$ elements in each ith row onto the array

$$\begin{array}{cccccc}
\mu & \mu-1 & \mu-2 & \cdots\cdots\cdots & & 1 \\
 & \mu-1 & \mu-2 & \cdots\cdots\cdots & & 1 \\
 & & \mu-2 & \cdots\cdots\cdots & & 1 \\
 & & & \ddots & & \\
 & & & \mu-k+1 & \cdots & 1
\end{array}$$

whose sum is $\sum_{h=\mu-k+1}^{\mu} S_h$, where again $S_h = \binom{h+1}{2}$. Hence, if $a_\mu^k(M)$ denotes the number of k-line slant partitions of a positive integer M with no more than $\mu-i+1$ non-zero elements in each ith row $(i = 1,\ldots,k)$, $a(N;\mu,\mu-1,\ldots,\mu-k+1) = a_\mu^k(N-\sum_{h=\mu-k+1}^{\mu} S_h)$.

Therefore

$$\begin{aligned}
&A(x;\mu,\mu-1,\ldots,\mu-k+1) \\
&= \sum_{n=0}^{\infty} a(n;\mu,\mu-1,\ldots,\mu-k+1)x^n \\
&= \sum_{n=0}^{\infty} a_\mu^k(n-\sum_{h=\mu-k+1}^{\mu} S_h)x^n \\
&= x^{\Sigma_{h=\mu-k+1}^{\mu} S_h} \sum_{n=\Sigma_{h=\mu-k+1}^{\mu} S_h}^{\infty} a_\mu^k(n-\sum_{h=\mu-k+1}^{\mu} S_h)x^{n-\Sigma_{h=\mu-k+1}^{\mu} S_h} \\
&= x^{\Sigma_{h=\mu-k+1}^{\mu} S_h} \sum_{m=0}^{\infty} a_\mu^k(m)x^m
\end{aligned}$$

since

$$a_{\mu}^{k}\Big(n-\sum_{h=\mu-k+1}^{\mu} S_h\Big) = 0 \quad \text{for} \quad n < \sum_{h=\mu-k+1}^{\mu} S_h .$$

Hence

$$\sum_{n=0}^{\infty} a_{\mu}^{k}(n)x^n = \frac{A(x;\mu,\mu-1,\ldots,\mu-k+1)}{x^{\Sigma_{h=\mu-k+1}^{\mu} S_h}}$$

Letting $\mu \to \infty$, we obtain the generating function for the number of k-line slant partitions. By LEMMA 3.3, this is also the generating function for the number of k-line partitions whose parts decrease along each column. Therefore,

$$(3) \qquad \sum_{n=0}^{\infty} c_k(n)x^n = \lim_{\mu\to\infty} \sum_{n=0}^{\infty} a_{\mu}^{k}(n)x^n$$

$$= \lim_{\mu\to\infty} \frac{A(x;\mu,\mu-1,\ldots,\mu-k+1)}{x^{\Sigma_{h=\mu-k+1}^{\mu} S_h}}$$

Thus, if $A(x;m_1,m_2,\ldots,m_k)$ were known, $\sum_{n=0}^{\infty} c_k(n)x^n$ could be calculated.

We shall look at the case $k = 2$, compute $A(x;m_1,m_2)$ for particular values of m_1 and m_2 and conjecture a general formula for $A(x;m_1,m_2)$.

LEMMA 4.6. For $m_1 \geq 0$,

$$A(x;m_1) = \frac{x^{S_{m_1}}}{[m_1][m_1-1]\ldots[1]} ,$$

where $[\nu] = (1-x^{\nu})$.

Proof: Since $A(x,m_1) = B(x;m_1)$, the lemma is proved by LEMMA 2.2.

THEOREM 4.7. For $m_1 = m < m_2$ and $m_2 = 1,2,3,4$,

$$\text{(a)}\quad A(x;m,1) = \frac{x^{S_m+1}(1-x^{m-1})}{[m+1][m]\ldots[2][1]^2}$$

$$\text{(b)}\quad A(x;m,2) = \frac{x^{S_m+3}(1-x^{m-1}+x^m-x^{m+2})(1-x^{m-2})}{[m+2][m+1]\ldots[3][2]^2[1]^2}$$

(c) $A(x;m,3)$

$$= \frac{x^{S_m+6}\{(1-x^{m-2}+x^m-x^{m+2})(1-x^{m+3})+x^{2m-3}(1-x)(1-x^2)\}(1-x^{m-3})}{[m+3][m+2]\ldots[4][3]^2[2]^2[1]^2}$$

(d) $A(x;m,4)$

$$= \frac{x^{S_m+10}}{D_m^4}\{(1-x^{m-3}+x^m-x^{m+2})(1-x^{m+3})(1-x^{m+4})$$
$$+x^{2m-5}(1-x^{m-1}+x^m-x^{m+4})(1-x^2)(1-x^3)\}(1-x^{m-4})$$

where $$D_m^4 = [m+4][m+3]\ldots[5][4]^2[3]^2[2]^2[1]^2$$

Proof: $A(x;m) = 1$ for $m = 0$ by DEFINITION 4.4.

$A(x;m_1,m_2) = 0$ for $m = m_2$ in Equations (a), (b), (c) and (d).

Hence initial conditions (2) are satisfied. By LEMMA 4.5,

$$A(x;m,1) = x^{m+1}\{A(x;m,1) + A(x;m-1,1) + A(x;m) + A(x;m-1)\}$$

or

$$A(x;m,1) - x^{m+1}A(x;m,1)$$

$$= x^{m+1}\{A(x;m-1,1) + A(x;m) + A(x;m-1)\}$$

hence,

$$A(x;m,1) = \frac{x^{m+1}}{[m+1]}\{a(x;m-1,1) + A(x;m) + A(x;m-1)\}$$

By LEMMA 4.6 and assuming (a) for $m_1 = m-1$, it follows that

$$A(x;m,1) = \frac{x^{m+1}}{[m+1]}\left\{\frac{x^{S_{m-1}+1}(1-x^{m-2})}{[m][m-1]\ldots[2][1]^2} + \frac{x^{S_m}}{[m][m-1]\ldots[1]}\right.$$

$$\left. + \frac{x^{S_{m-1}}}{[m-1][m-2]\ldots[1]}\right\}$$

$$= \frac{x^{m+1}}{[m+1]}\left\{\frac{x^{S_{m-1}}(1-x^{m-2})+x^{S_m}(1-x)+x^{S_{m-1}}(1-x^m)(1-x)}{[m][m-1]\ldots[2][1]^2}\right\}$$

$$= \frac{x^{S_{m-1}+m+1}(x(1-x^{m-2})+x^m(1-x)+(1-x^m)(1-x))}{[m+1][m]\ldots[2][1]^2}$$

$$= \frac{x^{S_m+1}(x-x^{m-1}+x^m-x^{m+1}+1-x^m-x+x^{m+1})}{[m+1][m]\ldots[2][1]^2}$$

$$= \frac{x^{S_m+1}(1-x^{m-1})}{[m+1][m]\ldots[2][1]^2}$$

and (a) is proved.

By LEMMA 4.5,

$$A(x;m,2) = x^{m+2}\{A(x;m,2)+A(x;m-1,2)+A(x;m,1)+A(x;m-1,1)\}$$

$$= \frac{x^{m+2}}{[m+2]}\{A(x;m-1,2)+A(x;m,1)+A(x;m-1,1)\}$$

By part (a) and assuming (b) for $m_1 = m-1$, we have

$$\frac{x^{m+2}}{[m+2]}\left\{\frac{x^{S_{m-1}+3}(1-x^{m-2}+x^{m-1}-x^{m+1})(1-x^{m-3})}{[m+1][m]\ldots[3][2]^2[1]^2} + \frac{x^{S_m+1}(1-x^{m-1})}{[m+1][m]\ldots[2][1]^2}\right.$$

$$\left. + \frac{x^{S_{m-1}+1}(1-x^{m-2})}{[m][m-1]\ldots[2][1]^2}\right\}$$

$$= \frac{x^{S_{m-1}+1+m+2}}{D_m^2}\{x^2(1-x^{m-2}+x^{m-1}-x^{m+1})(1-x^{m-3})+x^m(1-x^{m-1})(1-x^2)$$

$$+ (1-x^{m-2})(1-x^{m+1})(1-x^2)\}$$

where

$$D_m^2 = [m+2][m+1]\ldots[3][2]^2[1]^2$$

Hence, we obtain

$$\frac{x^{S_m+3}}{D_m^2}(x^2-x^m+x^{m+1}-x^{m+3}-x^{m-1}+x^{2m-3}-x^{2m-2}+x^{2m}+x^m$$

$$-x^{2m-1}-x^{m+2}+x^{2m+1}+1-x^{m-2}-x^{m+1}+x^{2m-1}-x^2$$

$$+x^m+x^{m+3}-x^{2m+1})$$

$$=\frac{x^{S_m+3}(-x^{m-1}+x^{2m-3}-x^{2m-2}+x^{2m}+x^m-x^{m+2}+1-x^{m-2})}{[m+2][m+1]\ldots[3][2]^2[1]^2}$$

$$=\frac{x^{S_m+3}(1-x^{m-1}+x^m-x^{m+2})(1-x^{m-2})}{[m+2][m+1]\ldots[3][2]^2[1]^2}$$

and (b) is proved.

By LEMMA 4.5,

$$A(x;m,3) = x^{m+3}\{A(x;m,3)+A(x;m-1,3)+A(x;m,2)+A(x;m-1,2)\}$$

$$=\frac{x^{m+3}}{[m+3]}\{A(x;m-1,3)+A(x;m-2)+A(x;m-1,2)\}$$

By part (b) and assuming (c) for $m_1 = m-1$, we have

$$\frac{x^{m+3}}{[m+3]}\left\{\frac{x^{S_{m-1}+6}}{D_{m-1}^3}\{(1-x^{m-3}+x^{m-1}-x^{m+1})(1-x^{m+2})+x^{2m-5}(1-x)(1-x^2)\}\right.$$

$$\times(1-x^{m-4})+\frac{x^{S_m+3}(1-x^{m-1}+x^m-x^{m+2})(1-x^{m-2})}{[m+2][m+1]\ldots[3][2]^2[1]^2}$$

$$+ \left. \frac{x^{S_{m-1}+3}(1-x^{m-2}+x^{m-1}-x^{m+1})(1-x^{m-3})}{[m+1][m]\ldots[3][2]^2[1]^2} \right\}$$

where

$$D^3_{m-1} = [m+2][m+1]\ldots[4][3]^2[2]^2[1]^2$$

$$= \frac{x^{S_{m-1}+3+m+3}}{D^3_m} \{x^3((1-x^{m-3}+x^{m-1}-x^{m+1})(1-x^{m+2})+x^{2m-5}(1-x)$$

$$\times(1-x^2))(1-x^{m-4})+x^m(1-x^{m-1}+x^m-x^{m+2})$$

$$\times(1-x^{m-2})(1-x^3)+(1-x^{m-2}+x^{m-1}-x^{m+1})(1-x^{m-3})$$

$$\times(1-x^{m+2})(1-x^3)\}$$

where

$$D^3_m = [m+3][m+2]\ldots[4][3]^2[2]^2[1]^2$$

$$= \frac{x^{S_m+6}}{D^3_m} (x^3-x^m+x^{m+2}-x^{m+4}-x^{m+5}+x^{2m+2}-x^{2m+4}+x^{2m+6}+x^{2m-2}$$

$$-x^{2m-1}-x^{2m}+x^{2m+1}-x^{m-1}+x^{2m-4}-x^{2m-2}+x^{2m}+x^{2m+1}$$

$$-x^{3m-2}+x^{3m}-x^{2m+2}-x^{3m-6}+x^{3m-5}+x^{3m-4}-x^{3m-3}+x^m$$

$$-x^{2m-1}+x^{2m}-x^{2m+2}-x^{2m-2}+x^{3m-3}-x^{3m-2}+x^{3m}-x^{m+3}$$

$$+x^{2m+2}-x^{2m+3}+x^{2m+5}+x^{2m+1}-x^{3m}+x^{3m+1}-x^{3m+3}$$

$$+1-x^{m-2}+x^{m-1}-x^{m+1}-x^{m-3}+x^{2m-5}-x^{2m-4}+x^{2m-2}$$

$$-x^{m+2}+x^{2m}-x^{2m+1}+x^{2m+3}+x^{2m-1}-x^{3m-3}+x^{3m-2}-x^{3m}$$

$$-x^3+x^{m+1}-x^{m+2}+x^{m+4}+x^m-x^{2m-2}+x^{2m-1}-x^{2m+1}+x^{m+5}$$

$$-x^{2m+3}+x^{2m+4}-x^{2m+6}-x^{2m+2}+x^{3m}-x^{3m+1}+x^{3m+3})$$

Cancelling like terms and rearranging the remaining terms, we obtain

$$\frac{x^{S_m+6}}{D_m^3}(1-x^{m-2}+x^m-x^{m+2}-x^{m+3}+x^{2m+1}-x^{2m+3}+x^{2m+5}+x^{2m-3}-x^{2m-2}$$

$$-x^{2m-1}+x^{2m}-x^{m-3}+x^{2m-5}-x^{2m-3}+x^{2m-1}+x^{2m}-x^{3m-2}+x^{3m}$$

$$-x^{3m-2}-x^{3m-6}+x^{3m-5}+x^{3m-4}-x^{3m-3})$$

$$=\frac{x^{S_m+6}}{D_m^3}\{(1-x^{m-2}+x^m-x^{m+2})(1-x^{m+3})+x^{2m-3}(1-x)(1-x^2)\}(1-x^{m-3})$$

where again $D_m^3 = [m+3][m+2]\ldots[4][3]^2[2]^2[1]^2$

and (c) is proved.

By LEMMA 4.5,

$$A(x;m,4) = x^{m+4}\{A(x;m,4)+A(x;m-1,4)+A(x;m,3)+A(x;m-1,3)\}$$

$$= x^{m+4}\{A(x;m-1,4)+A(x;m,3)+A(x;m-1,3)\}$$

By part (c) and assuming (d) for $m_1 = m-1$, we have

$$\frac{x^{m+4}}{[m+4]}\left\{\frac{x^{S_{m-1}+10}}{D_{m-1}^4}\{(1-x^{m-4}+x^{m-1}-x^{m+1})(1-x^{m+2})(1-x^{m+3}) + x^{2m-7}(1-x^2)(1-x^3)(1-x^{m-2}+x^{m-1}-x^{m+4})\}(1-x^{m-5})\right.$$

$$+\frac{x^{S_m+6}}{D_m^3}\{(1-x^{m-2}+x^m-x^{m+2})(1-x^{m+3}) + x^{2m-3}(1-x)(1-x^2)\}(1-x^{m-3})$$

$$+\ \frac{x^{S_{m-1}+6}}{D^3_{m-1}}\{(1-x^{m-3}+x^{m-1}-x^{m+1})(1-x^{m+2}) + x^{2m-5}(1-x)(1-x^2)\}(1-x^{m-4})\ \Bigg\}$$

where

$$D^4_{m-1} = [m+3][m+2]\ldots[5][4]^2[3]^2[2]^2[1]^2$$

$$= \frac{x^{S_{m-1}+6+m+4}}{D^4_m}\{x^4((1-x^{m-4}+x^{m-1}-x^{m+1})(1-x^{m+2})(1-x^{m+3}) + x^{2m-7}(1-x^2)(1-x^3)(1-x^{m-2}+x^{m-1}-x^{m+4}))(1-x^{m-5})$$

$$+ x^m((1-x^{m-2}+x^m-x^{m+2})(1-x^{m+3})$$

$$+x^{2m-3}(1-x)(1-x^2))(1-x^{m-3})(1-x^4)$$

$$+((1-x^{m-3}+x^{m-1}-x^{m+1})(1-x^{m+2})$$

$$+x^{2m-5}(1-x)(1-x^2))(1-x^{m-4})(1-x^{m+3})(1-x^4)\ \}$$

where

$$D^4_m = [m+4][m+3]\ldots[5][4]^2[3]^2[2]^2[1]^2$$

$$= \frac{x^{S_m+10}}{D^4_m}(x^4-x^m+x^{m+3}-x^{m+5}-x^{m+6}+x^{2m+2}-x^{2m+5}+x^{2m+7}$$

$$-x^{m+7}+x^{2m+3}-x^{2m+6}+x^{2m+8}+x^{2m+9}-x^{3m+5}+x^{3m+8}$$

$$-x^{3m+10}+x^{2m-3}-x^{2m-1}-x^{2m}+x^{2m+2}-x^{3m-5}+x^{3m-3}$$

$$+x^{3m-2}-x^{3m}+x^{3m-4}-x^{3m-2}-x^{3m-1}+x^{3m+1}-x^{3m}$$

$$+x^{3m+2}+x^{3m+3}-x^{3m+5}-x^{m-1}+x^{2m-5}-x^{2m-2}+x^{2m}$$

$$+x^{2m+1}-x^{3m-3}+x^{3m}-x^{3m+2}+x^{2m+2}-x^{3m-2}+x^{3m+1}$$

$$-x^{3m+3}-x^{3m+4}+x^{4m}-x^{4m+3}+x^{4m+5}-x^{3m-8}+x^{3m-6}$$

$$+x^{3m-5}-x^{3m-3}+x^{4m-10}-x^{4m-8}-x^{4m-7}+x^{4m-5}-x^{4m-9}$$

$$+x^{4m-7}+x^{4m-6}-x^{4m-4}+x^{4m-5}-x^{4m-3}-x^{4m-2}+x^{4m}$$

$$+x^{m}-x^{2m-2}+x^{2m}-x^{2m+2}-x^{2m+3}+x^{3m+1}-x^{3m+3}+x^{3m+5}$$

$$+x^{3m-3}-x^{3m-2}-x^{3m-1}+x^{3m}-x^{2m-3}+x^{3m-5}-x^{3m-3}$$

$$+x^{3m-1}+x^{3m}-x^{4m-2}+x^{4m}-x^{4m+2}-x^{4m-6}+x^{2m+2}$$

$$-x^{2m+4}+x^{2m+6}+x^{2m+7}-x^{3m+5}+x^{3m+7}-x^{3m+9}-x^{3m+1}$$

$$+x^{3m+2}+x^{3m+3}-x^{3m+4}+x^{2m+1}-x^{3m-1}+x^{3m+1}-x^{3m+3}$$

$$-x^{3m+4}+x^{4m+2}-x^{4m+4}+x^{4m+6}+x^{4m-2}-x^{4m-1}$$

$$-x^{4m}+x^{4m+1}+1-x^{m-3}+x^{m-1}-x^{m+1}-x^{m+2}+x^{2m-1}$$

$$-x^{2m+1}+x^{2m+3}+x^{2m-5}-x^{2m-4}-x^{2m-3}+x^{2m-2}-x^{m-4}$$

$$+x^{2m-7}-x^{2m-5}+x^{2m-3}+x^{2m-2}-x^{3m-5}+x^{3m-3}-x^{3m-1}$$

$$-x^{3m-9}+x^{3m-8}+x^{3m-7}-x^{3m-6}-x^{m+3}+x^{2m}-x^{2m+2}$$

$$+x^{2m+4}+x^{2m+5}-x^{3m+2}+x^{3m+4}-x^{3m+6}-x^{3m-2}+x^{3m-1}$$

$$+x^{3m}-x^{3m+1}+x^{2m-1}-x^{3m-4}+x^{3m-2}-x^{3m}-x^{3m+1}+x^{4m-2}$$

$$-x^{4m}+x^{4m+2}+x^{4m-6}-x^{4m-5}-x^{4m-4}+x^{4m-3}-x^{4}+x^{m+1}$$

$$-x^{m+3}+x^{m+5}+x^{m+6}-x^{2m+3}+x^{2m+5}-x^{2m+7}-x^{2m-1}+x^{2m}$$

$$+x^{2m+1}-x^{2m+4}+x^{m}-x^{2m-3}+x^{2m-1}-x^{2m+1}-x^{2m+2}$$

$$+x^{3m-1}-x^{3m+1}+x^{3m+3}+x^{3m-5}-x^{3m-4}-x^{3m-3}+x^{3m-2}$$

$$+x^{m+7}-x^{2m+4}+x^{2m+6}-x^{2m+8}-x^{2m+9}+x^{3m+6}-x^{3m+8}$$

$$+x^{3m+10}+x^{3m+2}-x^{3m+3}-x^{3m+4}+x^{3m+5}-x^{2m+3}+x^{3m}$$

$$-x^{3m+2}+x^{3m+4}+x^{3m+5}-x^{4m+2}+x^{4m+4}-x^{4m+6}-x^{4m-2}$$

$$+x^{4m-1}+x^{4m}-x^{4m+1}+x^{4m-5}+x^{4m-4}-x^{4m-3}+x^{m+4})$$

Cancelling like terms and rearranging the remaining terms, we obtain

$$\frac{x^{S_m+10}}{D_m^4}(1-x^{m-3}+x^{m}-x^{m+2}-x^{m+3}+x^{2m}-x^{2m+3}+x^{2m+5}-x^{m+4}+x^{2m+1}$$

$$-x^{2m+4}+x^{2m+6}+x^{2m+7}-x^{3m+4}+x^{3m+7}-x^{3m+9}+x^{2m-5}-x^{2m-3}$$

$$-x^{2m-2}+x^{2m}-x^{3m-6}+x^{3m-4}+x^{3m-3}-x^{3m-1}+x^{3m-5}-x^{3m-3}$$

$$-x^{3m-2}+x^{3m}-x^{3m-1}+x^{3m+1}+x^{3m+2}-x^{3m+4}-x^{m-4}+x^{2m-7}$$

$$-x^{2m-4}+x^{2m-2}+x^{2m-1}-x^{3m-4}+x^{3m-1}-x^{3m+1}+x^{2m}-x^{3m-3}$$

$$+x^{3m}-x^{3m+2}-x^{3m+3}+x^{4m}-x^{4m+3}+x^{4m+5}-x^{3m-9}+x^{3m-7}$$

$$+x^{3m-6}-x^{3m-4}+x^{4m-10}-x^{4m-8}-x^{4m-7}+x^{4m-5}-x^{4m-9}$$

$$+x^{4m-7}+x^{4m-6}-x^{4m-4}+x^{4m-5}-x^{4m-3}-x^{4m-2}+x^{4m})$$

$$= \frac{x^{S_m+10}}{D_m^4}\{(1-x^{m-3}+x^m-x^{m+2})(1-x^{m+3})(1-x^{m+4})$$
$$+x^{2m-5}(1-x^2)(1-x^3)(1-x^{m-1}+x^m-x^{m+4})\}(1-x^{m-4})$$

where again

$$D_m^4 = [m+4][m+3]\ldots[5][4]^2[3]^2[2]^2[1]^2$$

and (d) is proved.

COROLLARY 4.8.

$$A(x;5,4) = \frac{x^{25}}{D_5^4}\{(1-x^2)(1+x^5)(1-x^8)(1-x^9)$$
$$+x^5(1-x^4)(1+x^5)(1-x^2)(1-x^3)\}(1-x)$$

where

$$D_5^4 = [9][8][7][6][5][4]^2[3]^2[2]^2[1]^2$$

$$A(x;6,4) = \frac{x^{31}}{D_6^4}\{(1-x^3+x^6-x^8)(1-x^9)(1-x^{10})$$
$$+x^7(1-x^5+x^6-x^{10})(1-x^2)(1-x^3)\}(1-x^2)$$

where

$$D_6^4 = [10][9][8][7][6][5][4]^2[3]^2[2]^2[1]^2$$

Proof: Let $m = 5, 6$ in part (d) of THEOREM 4.7.

THEOREM 4.9.

$$A(x;6,5) = \frac{x^{36}}{R_6^5}\{(1-x^2)(1+x^6)(1-x^9)(1-x^{10})(1-x^{11})$$
$$+x^5(1-x^4)(1+x^6)(1-x^3)(1-x^4)(1-x^{11})$$
$$+x^{14}(1-x)(1-x^2)(1-x^3)(1-x^4)\}$$

where

$$R_6^5 = [11][10][9][8][7][6][5]^2[4]^2[3]^2[2]^2[1]$$

Proof: By LEMMA 3.5,

$$A(x;6,5) = x^{11}\{A(x;6,5)+A(x;5,5)+A(x;6,4)+A(x;5,4)\}$$
$$= \frac{x^{11}}{[11]}\{A(x;6,4)+A(x;5,4)\}$$

since $A(x;5,5) = 0$.

By COROLLARY 4.8, we have

$$A(x;6,5) = \frac{x^{11}}{[11]}\left\{\frac{x^{31}}{R_6^4}\{(1-x^3+x^6-x^8)(1-x^9)(1-x^{10}) +x^7(1-x^5+x^6-x^{10})(1-x^2)(1-x^3)\}\right.$$
$$\left. + \frac{x^{25}}{R_5^4}\{(1-x^2)(1+x^5)(1-x^8)(1-x^9) +x^5(1-x^4)(1+x^5)(1-x^2)(1-x^3)\}\right\}$$

where

$$R_6^4 = [10][9][8][7][6][5][4]^2[3]^2[2][1]^2$$

and

$$
\begin{aligned}
R_5^4 &= [9][8][7][6][5][4]^2[3]^2[2]^2[1] \\
&= \frac{x^{36}}{R_6^5}\{x^6(1-x^3+x^6-x^8)(1-x^9)(1-x^{10})(1+x) \\
&\qquad +x^{13}(1-x^5+x^6-x^{10})(1-x^2)(1-x^3)(1+x) \\
&\qquad +(1-x^2)(1+x^5)(1-x^8)(1-x^9)(1-x^{10}) \\
&\qquad +x^5(1-x^4)(1+x^5)(1-x^2)(1-x^3)(1-x^{10})\}(1-x^5)
\end{aligned}
$$

where

$$
\begin{aligned}
R_6^5 &= [11][10][9][8][7][6][5]^2[4]^2[3]^2[2]^2[1] \\
&= \frac{x^{36}}{R_6^5}\{(x^6-x^9+x^{12}-x^{14}+x^7-x^{10}+x^{13}-x^{15}+1-x^2+x^5-x^7-x^8+x^{10} \\
&\qquad -x^{13}+x^{15})(1-x^9)(1-x^{10}) \\
&\qquad +(x^{13}-x^{18}+x^{19}-x^{23}+x^{14}-x^{19}+x^{20}-x^{24}+x^5-x^9+x^{10}-x^{14} \\
&\qquad -x^{15}+x^{19}-x^{20}+x^{24})(1-x^2)(1-x^3)\}(1-x^5) \\
&= \frac{x^{36}}{R_6^5}\{(1-x^2+x^5+x^6-x^8-x^9+x^{12}-x^{14})(1-x^9)(1-x^{10}) \\
&\qquad +(x^5-x^9+x^{10}+x^{13}-x^{15}-x^{18}+x^{19}-x^{23})(1-x^2)(1-x^3)\}(1-x^5) \\
&= \frac{x^{36}}{R_6^5}(1-x^2+x^5+x^6-x^8-x^9+x^{12}-x^{14}-x^9+x^{11}-x^{14}-x^{15}+x^{17}+x^{18}-x^{21} \\
&\qquad +x^{23}-x^{10}+x^{12}-x^{15}-x^{16}+x^{18}+x^{19}-x^{22}+x^{24}+x^{19}-x^{21}+x^{24}+x^{25} \\
&\qquad -x^{27}-x^{28}+x^{31}-x^{33}+x^5-x^9+x^{10}+x^{13}-x^{15}-x^{18}+x^{19}-x^{23}-x^7+x^{11} \\
&\qquad -x^{12}-x^{15}+x^{17}+x^{20}-x^{21}+x^{25}-x^8+x^{12}-x^{13}-x^{16}+x^{18}+x^{21}-x^{22}
\end{aligned}
$$

$$+x^{26}+x^{10}-x^{14}+x^{15}+x^{18}-x^{20}-x^{23}+x^{24}-x^{28}-x^{5}+x^{7}-x^{10}-x^{11}$$

$$+x^{13}+x^{14}-x^{17}+x^{19}+x^{14}-x^{16}+x^{19}+x^{20}-x^{22}-x^{23}+x^{26}-x^{28}+x^{15}$$

$$-x^{17}+x^{20}+x^{21}-x^{23}-x^{24}+x^{27}-x^{29}-x^{24}+x^{26}-x^{29}-x^{30}+x^{32}+x^{33}$$

$$-x^{36}+x^{38}-x^{10}+x^{14}-x^{15}-x^{18}+x^{20}+x^{23}-x^{24}+x^{28}+x^{12}-x^{16}+x^{17}$$

$$+x^{20}-x^{22}-x^{25}+x^{26}-x^{30}+x^{13}-x^{17}+x^{18}+x^{21}-x^{23}-x^{26}+x^{27}-x^{31}$$

$$-x^{15}+x^{19}-x^{20}-x^{23}+x^{25}+x^{28}-x^{29}+x^{33})$$

Cancelling like terms and rearranging the remaining terms, we have

$$\frac{x^{36}}{R_6^5}(1-x^{2}+x^{6}-x^{8}-x^{9}+x^{11}-x^{15}+x^{17}-x^{10}+x^{12}-x^{16}+x^{18}+x^{19}-x^{21}+x^{25}$$

$$-x^{27}-x^{11}+x^{13}-x^{17}+x^{19}+x^{20}-x^{22}+x^{26}-x^{28}+x^{21}-x^{23}+x^{27}-x^{29}$$

$$-x^{30}+x^{32}-x^{36}+x^{38}+x^{5}-x^{9}+x^{11}-x^{15}-x^{8}+x^{12}-x^{14}+x^{18}-x^{9}+x^{13}$$

$$-x^{15}+x^{19}+x^{12}-x^{16}+x^{18}-x^{22}-x^{16}+x^{20}-x^{22}+x^{26}+x^{19}-x^{23}+x^{25}$$

$$-x^{29}+x^{20}-x^{24}+x^{26}-x^{30}-x^{23}+x^{27}-x^{29}+x^{33}+x^{14}-x^{15}-x^{16}-x^{22}$$

$$-x^{23}+x^{24})$$

$$= \frac{x^{36}}{R_6^5}\{(1-x^{2})(1+x^{6})(1-x^{9})(1-x^{10})(1-x^{11})$$

$$+x^{5}(1-x^{4})(1+x^{6})(1-x^{3})(1-x^{4})(1-x^{11})$$

$$+x^{14}(1-x)(1-x^{2})(1-x^{3})(1-x^{4})\}$$

where

$$R_6^5[11][10][9][8][7][6][5]^2[4]^2[3]^2[2]^2[1]$$

Hence the theorem is proved.

In view of the formulas for $A(x;m,1)$, $A(x;m,2)$, $A(x;m,3)$, $A(x;m,4)$ and $A(x;6,5)$, we conjecture that for $m > 5$,

$$\begin{aligned} A(x;m,5) = \frac{x^{S_m+15}}{D_m^5}\{&(1-x^{m-4}+x^m-x^{m+2})(1-x^{m+3})(1-x^{m+4})(1-x^{m+5}) \\ &+x^{2m-7}(1-x^{m-2}+x^m-x^{m+4})(1-x^3)(1-x^4)(1-x^{m+5}) \\ &+x^{3m-4}(1-x)(1-x^2)(1-x^3)(1-x^4)\}(1-x^{m-5}) \end{aligned}$$

where

$$D_m^5 = [m+5][m+4]\ldots[6][5]^2[4]^2[3]^2[2]^2[1]^2$$

and for $m > 6$,

$$\begin{aligned} A(x;m,6) = \frac{x^{S_m+21}}{D_m^6}\{&(1-x^{m-5}+x^m-x^{m+2})(1-x^{m+3})(1-x^{m+4})(1-x^{m+5})(1-x^{m+6}) \\ &+x^{2m-9}(1-x^{m-3}+x^m-x^{m+4})(1-x^4)(1-x^5)(1-x^{m+5}) \\ &\times(1-x^{m+6}) \\ &+x^{3m-7}(1-x^{m-1}+x^m-x^{m+6})(1-x^2)(1-x^3)(1-x^4)(1-x^5)\} \\ &\times(1-x^{m-6}) \end{aligned}$$

where

$$D_m^6[m+6][m+5]\ldots[7][6]^2[5]^2[4]^2[3]^2[2]^2[1]^2$$

In general, for $m_1 > m_2 \geq 0$, we conjecture that

$$
\begin{aligned}
&A(x; m_1, m_2) \\
&= \frac{x^{S_{m_1}+S_{m_2}}}{D_{m_1}^{m_2}} \Big\{ (1-x^{m_1-(m_2-1)}+x^{m_1}-x^{m_1+2})(1-x^{m_1+3})\ldots(1-x^{m_1+m_2}) \\
&\qquad + x^{2m_1-(2m_2-3)}(1-x^{m_1-(m_2-3)}+x^{m_1}-x^{m_1+4})(1-x^{m_2-2}) \\
&\qquad\qquad \times (1-x^{m_2-1})(1-x^{m_1+5})\ldots(1-x^{m_1+m_2}) \\
&\qquad + x^{3m_1-(3m_2-11)}(1-x^{m_1-(m_2-5)}+x^{m_1}-x^{m_1+6})(1-x^{m_2-4}) \\
&\qquad\qquad \times (1-x^{m_2-3})(1-x^{m_2-2})(1-x^{m_2-1})(1-x^{m_1+7})\ldots(1-x^{m_1+m_2}) \\
&\qquad + x^{4m_1-(4m_2-23)}(1-x^{m_1-(m_2-7)}+x^{m_1}-x^{M_1+8})(1-x^{m_2-6}) \\
&\qquad\qquad \times \ldots(1-x^{m_2-1})(1-x^{m_1+9})\ldots(1-x^{m_1+m_2}) \\
&\qquad + x^{5m_1-(5m_2-39)}(1-x^{m_1-(m_2-9)}+x^{m_1}-x^{m_1+10})(1-x^{m_2-8}) \\
&\qquad\qquad \times \ldots(1-x^{m_2-1})(1-x^{m_1+11})\ldots(1-x^{m_1+m_2}) \\
&\qquad + \ldots + E_{m_1, m_2} \Big\} (1-x^{m_1-m_2})
\end{aligned}
$$

where

$$D_{m_1}^{m_2} = [m_1+m_2][m_1+m_2-1]\ldots[m_2+1][m_2]^2[m_2-1]^2,,,[1]^2$$

and

$$E_{m_1, m_2}$$

$$= \begin{cases} x^{\frac{m_2 m_1}{2} - (\frac{m_2 m_2}{2} - m_2(\frac{m_2}{2} - 1) + 1)} (1 - x^{m_1 - (m_2 - m_2 + 1)} + x^{m_1} - x^{m_1 + m_2}) \\ \quad \times (1 - x^{m_2 - (m_2 - 2)}) \ldots (1 - x^{m_2 - 1}) \text{ if } m_2 \text{ is even} \\ x^{\frac{(m_2+1)m_1}{2} - (\frac{(m_2+1)m_2}{2} - (m_2+1)(\frac{m_2+1}{2} - 1) + 1)} (1 - x^{m_2 - (m_2 - 1)}) \\ \quad \times \ldots (1 - x^{m_2 - 1}) \text{ if } m_2 \text{ is odd} \end{cases}$$

$$= \begin{cases} x^{\frac{m_2 m_1}{2} - m_2 - 1} (1 - x^{m_1 - 1} + x^{m_1} - x^{m_1 + m_2})(1 - x^2) \ldots (1 - x^{m_2 - 1}) \\ \quad \text{if } m_2 \text{ is even} \\ x^{\frac{(m_2+1)m_1}{2} - \frac{m_2+3}{2}} (1 - x) \ldots (1 - x^{m_2 - 1}) \text{ if } m_2 \text{ is odd} \end{cases}$$

For $m_1 = \mu$ and $m_2 = \mu - 1$, $S_\mu + S_{\mu-1} = \mu^2$ and

$$A(x; \mu, \mu - 1)$$

$$= \frac{x^{\mu^2}}{R_\mu^{\mu-1}} \{(1 - x^2 + x^\mu - x^{\mu+2})(1 - x^{\mu+3}) \ldots (1 - x^{2\mu-1})$$
$$+ x^5 (1 - x^4 + x^\mu - x^{\mu+4})(1 - x^{\mu-3})(1 - x^{\mu-2})(1 - x^{\mu+5}) \ldots (1 - x^{2\mu-1})$$
$$+ x^{14} (1 - x^6 + x^\mu - x^{\mu+6})(1 - x^{\mu-5}) \ldots (1 - x^{\mu-2})(1 - x^{\mu+7}) \ldots (1 - x^{2\mu-1})$$
$$+ \ldots + E_{\mu, \mu-1}\}$$

where

$$R_{\mu}^{\mu-1} = [2\mu-1][2\mu-2]\ldots[\mu][\mu-1]^2[\mu-2]^2\ldots[2]^2[1]$$

and

$$E_{\mu,\mu-1} = \begin{cases} x^{\frac{\mu(\mu-3)}{2}}(1-x^{\mu-1}+x^{\mu}-x^{2\mu-1})(1-x^2)\ldots(1-x^{\mu-2}) & \text{if } \mu \text{ is odd} \\ x^{\frac{(\mu-2)(\mu+1)}{2}}(1-x)\ldots(1-x^{\mu-2}) & \text{if } \mu \text{ is even} \end{cases}$$

since

$$\frac{(\mu-1)\mu}{2} - (\mu-1) - 1 = \frac{(\mu-1)\mu}{2} - \frac{2\mu}{2}$$

$$= \frac{\mu^2-3\mu}{2}$$

$$= \frac{\mu(\mu-3)}{2}$$

and

$$\frac{\mu(\mu)}{2} - \frac{\mu+2}{2} = \frac{\mu^2-\mu-2}{2}$$

$$= \frac{\mu(\mu-2)(\mu+1)}{2}$$

Hence,

$$A(x;\mu,\mu-1) = x^{\mu^2}\left\{\frac{(1-x^2)(1+x^{\mu})}{[\mu+2][\mu+1]\ldots[\mu][\mu-1]^2\ldots[2]^2[1]}\right.$$

$$+ \frac{x^5(1-x^4)(1+x^{\mu})(1-x^{\mu-3})(1-x^{\mu-2})}{[\mu+4][\mu+3]\ldots[\mu][\mu-1]^2\ldots[2]^2[1]} +$$

$$+\frac{x^{14}(1-x^6)(1+x^\mu)(1-x^{\mu-5})\dots(1-x^{\mu-2})}{[\mu+6][\mu+5]\dots[\mu][\mu-1]^2\dots[2]^2[1]}$$

$$\left.+\dots+\frac{E_{\mu,\mu-1}}{[2\mu-1][2\mu-2]\dots[\mu][\mu-1]^2\dots[2]^2[1]}\right\}$$

where

$$E_{\mu,\mu-1}=\begin{cases} x^{\frac{\mu(\mu-3)}{2}}(1-x^{\mu-1})(1+x^\mu)(1-x^2)\dots(1-x^{\mu-2}) & \text{if } \mu \text{ is odd} \\ x^{\frac{(\mu-2)(\mu+1)}{2}}(1-x)\dots(1-x^{\mu-2}) & \text{if } \mu \text{ is even} \end{cases}$$

From Equation (3), we have

$$\sum_{n=0}^{\infty} c_2(n)x^n = \lim_{\mu\to\infty}\frac{A(x;\mu,\mu-1)}{x^{\mu^2}}$$

$$=\lim_{\mu\to\infty}\left\{\frac{(1-x^2)(1+x^\mu)}{[\mu+2]\dots[\mu][\mu-1]^2\dots[2]^2[1]}\right.$$

$$+\frac{x^5(1-x^4)(1+x^\mu)(1-x^{\mu-3})(1-x^{\mu-2})}{[\mu+4]\dots[\mu][\mu-1]^2\dots[2]^2[1]}$$

$$+\frac{x^{14}(1-x^6)(1+x^\mu)(1-x^{\mu-5})\dots(1-x^{\mu-2})}{[\mu+6]\dots[\mu][\mu-1]^2\dots[2]^2[1]}$$

$$\left.+\dots+\frac{E_{\mu,\mu-1}}{[\mu-1]\dots[\mu][\mu-1]^2\dots[2]^2[1]}\right\}$$

Consider each ith term $(i \geq 1)$,

(4)

$$\frac{x^{S_{2i-1}-1}(1-x^{2i})(1+x^{\mu})(1-x^{\mu-2i+1})\ldots(1-x^{\mu-2})}{[\mu+2i]\ldots[\mu][\mu-1]^2\ldots[2]^2[1]^2}$$

$$= \frac{x^{S_{2i-1}-1}(1-x^{2i})(1+x^{\mu})}{[\mu+2i]\ldots[\mu][\mu-1]^2[\mu-2]\ldots[\mu-2i+1][\mu-2i]^2\ldots[2]^2[1]}$$

Since $\frac{1}{1+\frac{1}{2^s}} < 1 < \frac{1}{1-\frac{1}{2^s}}$ for a positive integer s, for $|x| < \frac{1}{2}$,

each ith term (4) is less than

$$\frac{\frac{1}{2^{S_{2i-1}-1}}(1-\frac{1}{2^{2i}})(1+\frac{1}{2^{\mu}})}{(1-\frac{1}{2})(1-\frac{1}{2^2})^2(1-\frac{1}{2^3})^2(1-\frac{1}{2^4})^2\ldots} \leq \frac{\frac{1}{2^{S_{2i-1}-1}}(\frac{3}{2})}{(\frac{1}{2})(\frac{3}{4})^2(\frac{7}{8})^2(\frac{15}{16})^2\ldots},$$

since $\mu \geq 1$ and $1-\frac{1}{2^{2i}} < 1$. Hence we have

$$\frac{3}{2^{S_{2i-1}-1}}(\frac{4}{3})^2(\frac{8}{7})^2(\frac{16}{15})^2\ldots$$

$$= \frac{3}{2^{S_{2i-1}-1}}\prod_{j=2}^{\infty}(\frac{2^j}{2^j-1})^2$$

$$= \frac{3}{2^{S_{2i-1}-1}}V^2$$

where

$$V = \prod_{j=2}^{\infty}(\frac{2^j}{2^j-1}) = \prod_{j=2}^{\infty}(1+\frac{1}{2^j-1})$$

Hence,

$$\log V = \sum_{j=2}^{\infty} \log(1+\frac{1}{2^j-1})$$

$$= \sum_{j=2}^{\infty} \left\{ \frac{1}{2^j-1} - (\frac{1}{2(2^j-1)} - \frac{1}{3(2^j-1)}) - (\frac{1}{4(2^j-1)} - \frac{1}{5(2^j-1)}) - \cdots \right\}$$

$$< \sum_{j=2}^{\infty} \frac{1}{2^j-1}$$

$$< \sum_{j=1}^{\infty} \frac{1}{2^j} = 2$$

Hence $V < e^2$ and therefore each ith term (4) is less than

$$\frac{3}{2^{S_{2i-1}-1}} e^4$$

Since $i \geq 1$, the sum over all i is less than $6e^4$. Thus,

$$\sum_{n=0}^{\infty} c_2(n)x^n = \sum_{i=1}^{\infty} \lim_{\mu\to\infty} \frac{x^{S_{2i-1}-1}(1-x^{2i})(1+x^{\mu})}{[\mu+2i]\ldots[\mu][\mu-1]^2[\mu-2]\ldots[\mu-2i+1][\mu-2i]^2\ldots[2]^2[1]}$$

$$= \frac{(1-x^2)}{D} + \frac{x^5(1-x^4)}{D} + \frac{x^{14}(1-x^6)}{D} + \ldots$$

where

$$D = [1][2]^2[3]^2[4]^2 \ldots$$

$$\sum_{h=0}^{\infty} c_2(n)x^n = \frac{x-x^3+x^6-x^{10}+x^{15}-x^{30}+\ldots}{x[1][2]^2[3]^2\ \ldots}$$

$$= \frac{1-(1-x+x^3-x^6+x^{10}-x^{15}+\ldots)}{x[1][2]^2[3]^2\ldots}$$

$$= \frac{x^{-1}(1-x)(1-\Sigma_{i=0}^{\infty}(-1)^i S_i)}{[1]^2[2]^2[3]^2\ldots}$$

$$= x^{-1}(1-x)(1-\sum_{i=0}^{\infty}(-1)^i S_i)P(x)^2 \tag{5}$$

which agrees with Gordon's result (2, p. 98), for $k = 2$.

AN IDENTITY: Consider any 2-line slant partition of a positive integer N.

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1h_1} \\ & a_{22} & a_{23} & \cdots & a_{2h_2} \end{array}$$

Let $a_{11} = \mu$. From DEFINITION 3.2, $a_{ij} \leq \mu$ for all $i \geq 2$ and $j \geq 1$. Removing a_{11} from this partition, yields a 2-line partition of $N - \mu$ where each element is less than or equal to μ. Transposing the graph of such a partition gives rise to a 2-line partition of $N - \mu$ whose number of non-zero parts in each line is less than or equal to μ. The correspondence is one-to-one. Let $t_2^{\mu}(N-\mu)$ denote the number of 2-line partitions of $N-\mu$ into at most μ non-zero parts in each row. From formula (3) of the introduction, $t_2^{\mu}(N-\mu)$ is the coefficient of $x^{N-\mu}$ of the generating function

$$\frac{(1-x)}{(1-x)^2(1-x^2)^2\ldots(1-x^{\mu})^2(1-x^{\mu+1})} = \frac{1}{[1][2]^2\ldots[\mu]^2[\mu+1]} .$$

That is, $t_2^{\mu}(N-\mu)$ is the coefficient of x^N of the generating function

$$\frac{x^{\mu}}{[1][2]^2\ldots[\mu]^2[\mu+1]} .$$

Letting $\mu = 1, 2, 3, \ldots,$ we see that the number of 2-line slant partitions of N is the coefficient of x^N of the series

$$\frac{x}{[1][2]^2[3]} + \frac{x^2}{[1][2]^2[3]^2[4]} + \frac{x^3}{[1][2]^2[3]^2[4]^2[5]} + \ldots .$$

Defining $t_2^0(0) = 1,$ we have

$$\sum_{\mu=0}^{1} \frac{x^{\mu}}{[1][2]^2\ldots[\mu]^2[\mu+1]} = x^{-1}(1-x)(1-\sum_{i=0}^{\infty}(-1)^i S_i)P(x)^2$$

from Equation (5).

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