ORTHOGONAL POLYNOMIALS

by

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A THESIS

submitted to the

OREGON STATE COLLEGE

in partial fulfillment of
the requirements for the
degree of

MASTER OF ARTS

October 1944

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ACKNOWLEDGEMENT

I wish to express my great appreciation of the help and guidance I received from Dr. Milne at every stage of my work on this thesis.

TABLE OF CONTENTS

ORTHOGONAL POLYNOMIALS

INTRODUCTION

The subject of orthogonal polynomials finds a place in diverse branches of study in mathematics, pure and applied. Their properties make their use in typically modern problems in quantum mechanics, for example, fairly prevalent, while in the field of analysis they have proved to be of inestimable value in the study of differential equations with boundary conditions.

By definition, if we have a system

$$P_n(x) = \sum_{n=0}^{n} a_n x^n$$

with the property

$$\int_a^b P_m(x)\, P_n(x)\, dx = 0, \; m \neq n, \; (m,n = 0, 1, 2 - - -),$$

then any two of the polynomials of different degrees are said to be mutually orthogonal over the interval (a,b).

If we are granted the further hypotheses:

(i) a weight function

$$w(x) \geqslant 0 \qquad \text{in } (a,b).$$

(ii) all moments

$$m_j = \int_a^b w(x)\, x^j dx \text{ exist; } (j = 0, 1, 2 - - -)$$

(iii) $m_0 > 0$,

then

$$\int_a^b w(x)\, P_m(x)\, P_n(x)\, dx = 0 \quad (m \neq n)$$

defines a unique system of polynomials which are mutually orthogonal with respect to w(x).[9]

If, further, the system be normalized,[4] we have

Eq. 1 $$\int_a^b w(x)\, P_m(x)\, P_n(x)\, dx = \delta_{m,n}$$

where δ is the Kronecker delta;

i.e. $$\delta_{m,n} \begin{array}{ll} = 0 & (m \neq n \\ = 1 & (m = n. \end{array}$$

By analogy with the expansion of an arbitrary function f(x) in a Fourier Series, f(x) may be expanded in the form

$$\sum_{n=0}^{\infty} f_n P_n(x)$$

where

$$f_n = \int_a^b w(x)\, f(x)\, P_n(x)\, dx.$$

It is clear that the type of polynomials set up will depend on the choice of w(x) and on whether (a,b) is finite or infinite. The reader should bear in mind, too, that we have here assumed for the independent variable the property of continuity. This is by no means necessary, and in many important applications is not even true. In problems of statistics, or probability, involving distribution aggregates, for instance, the independent variable assumes a set of discrete values, so that the defining equation, instead of being an integral proper, is a sum of a number of discrete terms.

While this thesis is the outcome of an examination of orthogonal polynomials with certain chosen weight functions, no attempt has been made to unify the treatment of x as a continuous variable or as a set of discrete points by utilizing the convenient properties of Stieltjes integrals. For such a treatment the reader is referred to Szegö's "Orthogonal Polynomials."

CHAPTER I

CLASSICAL ORTHOGONAL POLYNOMIALS

1. Fourier Series. The paper by Fourier, "Théorie analytique de chaleur" (1822) wherein the assertion was first put forward that an arbitrary function, given in a fixed interval, could be expressed in a certain trigonometric series, may be taken as the starting point of the present study.

Let

$$I_{m,n} = \int_{-\pi}^{\pi} \operatorname{Sin} mx \operatorname{Sin} nx \, dx$$

and let

$$I'_{m,n} , I''_{m,n}$$

denote the corresponding integrals with Cos mx Sin nx and Cos mx Cos nx respectively. We find

$$I_{m,n} = 0; \; m \neq n.$$

Similarly,

$$I'_{m,n} = 0, \; I''_{m,n} = 0,$$

the former equation holding also for

$$m = n.$$

These equations express the fact that the system of trigonometric functions 1, Cos x, Sin x, Cos 2x, Sin 2x - - - is orthogonal over the interval $(-\pi, \pi)$.

An arbitrary function $f(x)$ may be expanded in the form

$$a_0/2 + \sum_{k=1}^{\infty} a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx$$

where $a_k = \quad f(x) \operatorname{Cos} kx\, dx$,

$b_k = \quad f(x) \operatorname{Sin} kx\, dx$

are the Fourier coefficients.

2. Tchebichef polynomials. In an analogous manner, if we substitute

$$x = \operatorname{Cos} \theta,$$

then

$$T_n(x) = \operatorname{Cos} n\theta$$

$$U_n(x) = \frac{1}{n+1} T'_{n+1}(x) = \operatorname{Sin}(n+1)\theta / \operatorname{Sin}\theta,$$

$$(n = 0, 1, 2 - - -)$$

are polynomials of degree n in x. (Tchebichef polynomials[8])

e.g.

$$\operatorname{Cos} n\theta = 2^{n-1} \operatorname{Cos}^n \theta - \frac{n}{1!} 2^{n-3} \operatorname{Cos}^{n-2} \theta$$

$$+ \frac{n(n-3)}{2!} 2^{n-5} \operatorname{Cos}^{n-4} \theta \; - - -$$

$$\operatorname{Sin}(n+1)\theta / \operatorname{Sin}\theta = 2^n \operatorname{Cos}^n \theta - \frac{n-1}{1!} 2^{n-2} \operatorname{Cos}^{n-2} \theta$$

$+$ [3]

3. Properties of Orthogonal Polynomials. Properties of $T_n(x)$ and $U_n(x)$, of heuristic value when sets of orthogonal

polynomials are being developed, are

(a) $$\left.\begin{aligned} \int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} \quad T_m(x)\, T_n(x)\, dx &= 0 \\ \int_{-1}^{1} (1 - x^2)^{\frac{1}{2}} \quad U_m(x)\, U_n(x)\, dx &= 0 \end{aligned}\right\} m \neq n$$

ie. $T_m(x)$, $T_n(x)$ are mutually orthogonal (-1, 1) with respect to the weight function $(1 - x^2)^{-\frac{1}{2}}$. The same is true for $U_m(x)$, $U_n(x)$ if we substitute $(1 - x^2)^{\frac{1}{2}}$ for $(1 - x^2)^{-\frac{1}{2}}$ for the weight function.

(b) The zeros of $T_n(x)$ and $U_n(x)$ are all real, distinct, and lie within the interval (-1, 1).

(c) Between any three successive polynomials a relation of recurrence exists.

$$T_{n+1}(x) = xT_n(x) - (1 - x^2)\; U_{n-1}(x).$$

$$U_n(x) = xU_{n-1}(x) + T_n(x). \quad n = 1, 2, 3 - - -.$$

(d) The polynomials satisfy a second order linear differential equation.

$$(1 - x^2)\, T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0$$

$$(1 - x^2)\, U_n''(x) - 3xU_n'(x) + n\,(n + 2)\, U_n(x) = 0.$$

If, in equation 1, we put (a,b) = (-1, 1),

$$w(x) = (1 - x)^{\alpha}\, (1 + x)^{\beta}, \quad (\alpha, \beta > -1),$$

we have the Jacobi polynomials, of which the Tchebichef polynomials form a subclass;

$$(\alpha = \beta = -\tfrac{1}{2})$$

while, if

$$\alpha = \beta = 0,$$

the Legendre polynomials are formed. If the interval be infinite at one end, $(0, \infty)$, $w(x) = x^{\alpha}e^{-x}$, $(\alpha > -1)$, we have the Laguerre polynomials; if the interval be infinite at both ends,

$(-\infty, \infty)$, $w(x) = e^{-x^2}$, the Hermite polynomials result.

These four kinds of orthogonal polynomials constitute the Classical Orthogonal Polynomials, and a study of them should precede any investigation in the construction of a set of orthogonal polynomials.

CHAPTER II

CONTINUOUS VARIABLE

1. Construction of Orthogonal Polynomials. We have defined the J^{th} moment:

$$m_j = \int_a^b w(x)\, x^j\, dx \qquad (j = 0, 1, 2 - - -)$$

If we form

$$D_n = \begin{vmatrix} m_0 & m_1 & \cdots & m_{n-1} \\ m_1 & m_2 & \cdots & m_n \\ \vdots & & & \\ m_{n-1} & m_n & \cdots & m_{2n-2} \end{vmatrix}$$

and define

$$P_n(x) = \begin{vmatrix} 1 & x & x^2 \cdots & x^n \\ m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & & & \\ m_{n-1} & m_n & \cdots & m_{2n-1} \end{vmatrix}$$

then

$$\text{Eq. 2 (a)} \quad \frac{1}{D_{n+1}} \int_a^b w(x)\, x^m\, P_n(x)\, dx = \delta_{m,n}$$

$$\text{(b)} \quad \frac{1}{D_n D_{n+1}} \int_a^b w(x)\, P_m(x)\, P_n(x)\, dx = \delta_{m,n}.$$

For a detailed proof, the reader is referred to "Numerical Calculus" by W. E. Milne, p. 60, where the subject is treated in a theory of least squares, or to notes by the same author on Orthogonal Polynomials. The properties of the polynomials so constructed will be analogous with those of the Tchebichef polynomials discussed in the first chapter.

2. Example. We will now consider

$$m_n = \int_{-1}^{1} w(x)\, x^n dx$$

with the given weight function defined as

$$\begin{aligned} w(x) &= 1 + x & \quad x &< 0 \\ &= 1 - x & \quad x &> 0. \end{aligned}$$

Then $m_n = \int_{-1}^{0} (1 + x)\, x^n dx + \int_{0}^{1} (1 - x)\, x^n dx.$

Whence

$$\text{Eq. 3} \qquad \begin{aligned} m_n &= 0 & \quad & n \text{ odd.} \\ &= \frac{2}{(n+1)(n+2)} & \quad & n \text{ even.} \end{aligned}$$

It is apparent that the interval of orthogonality is symmetric with respect to the origin, as is $w(x)$.

It follows that $(-1)^n P_n(-x) = P_n(x)$;

i.e., $P_n(x)$ contains only even (odd) powers of x for n even (odd).

By definition,

$$P_n(x) = \begin{vmatrix} 1 & x & x^2 & - & - & - & x^n \\ m_0 & m_1 & - & - & - & - & m_n \\ \vdots & & & & & & \\ m_{n-1} & - & - & - & - & & m_{2n-1} \end{vmatrix}$$

and any polynomial of this group, *e.g.* $P_4(x)$ would, with the aid of Eq. 3, have the form

$$P_4(x) = \begin{vmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1 & 0 & 1/6 & 0 & 1/15 \\ 0 & 1/6 & 0 & 1/15 & 0 \\ 1/6 & 0 & 1/15 & 0 & 1/28 \\ 0 & 1/15 & 0 & 1/28 & 0 \end{vmatrix}$$

3. Evaluating Determinants by Pivotal Condensation. The evaluation of each determinant becomes increasingly laborious. Much time can be saved, however, by use of the theorem.[10],[6],[1]

$$\left| a_{i,j} \right| = \frac{1}{(a_{1,1})^{n-2}} \left| \begin{matrix} \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} & \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} & - - - & \begin{vmatrix} a_{1,1} & a_{1,n} \\ a_{2,1} & a_{2,n} \end{vmatrix} \\ \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} & \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} & - - - & \begin{vmatrix} a_{1,1} & a_{1,n} \\ a_{3,1} & a_{3,n} \end{vmatrix} \\ - & - & - & - \\ \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{n,1} & a_{n,2} \end{vmatrix} & \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{n,1} & a_{n,3} \end{vmatrix} & - - - & \begin{vmatrix} a_{1,1} & a_{1,n} \\ a_{n,1} & a_{n,n} \end{vmatrix} \end{matrix} \right|$$

Each operation of this kind reduces the order of the determinant by one.

Work can be simplified further by moving columns or rows, if necessary, so as to bring a simpler term (e.g. 1) to the leading element position.

By these methods we find

$$P_0(x) = 1$$

$$P_1(x) = -x$$

$$P_2(x) = 1/36(6x^2 - 1)$$

$$P_3(x) = - 7/5400\ (5x^3 - 2x)$$

$$P_4(x) = \frac{19}{4(6300)^2}\ (490x^4 - 310x^2 + 19)$$

$$P_5(x) = \frac{-683}{735(180 \times 420)^2}\ (798x^5 - 700x^3 + 109x)$$

In working this problem, it had been hoped that, at least, a recursion formula might be found so that values of $P_n(x)$ could be tabulated therefrom. Unlike the Legendre polynomials, whose coefficients are convenient to handle, those of the present problem became unmanageable. Nor was any luck experienced in finding a function which would generate the polynomials.

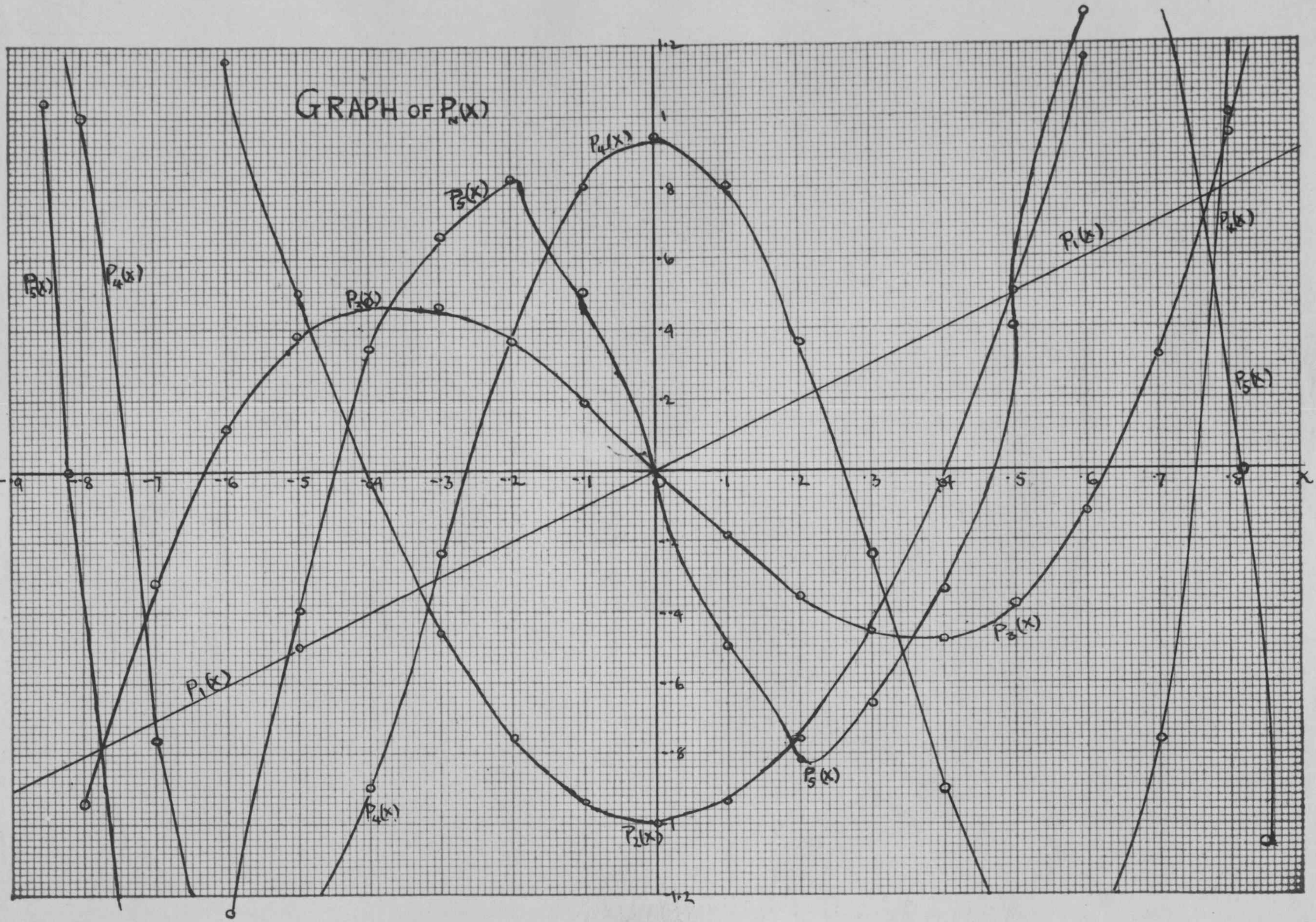
12. Roots of $P_n(x)$. The graph of the polynomials given up to $P_5(x)$ should be compared with that of the Legendre polynomials. It will be observed that between every two zeros of $P_n(x)$ a zero of $P_{n+1}(x)$ occurs. This is a property shared by all orthogonal polynomials in addition to those cited in the first chapter.

Roots of $P_n(x)$ up to $P_5(x)$ obtained algebraically are listed below.

$P_n(x)$	Roots
$P_1(x)$	0
$P_2(x)$	$\pm$ .408
$P_3(x)$	0, $\pm$ .632
$P_4(x)$	$\pm$ .262, $\pm$ .751
$P_5(x)$	0, $\pm$ .820, $\pm$ .451

4. Tables of $P_n(x)$.

x	$P_1(x)$	$36\,P_2(x)$	$-\frac{5400}{7}\,P_3(x)$	$\frac{4}{19}(6300)^2\,P_4(x)\frac{1}{20}$	$-\frac{(180.420)^2.735}{683}\,P_5(x)\frac{1}{20}$
-1.0	1.0000	5.0000	-3.0000	9.9500	10.3500
- .9	.9000	3.8600	-1.8450	4.8435	2.9600
- .8	.8000	2.8400	- .9600	1.0655	- .4855
- .7	.7000	1.9400	- .3150	- .7625	- 1.4800
- .6	.6000	1.1600	.1200	-1.4548	- 1.2874
- .5	.5000	.5000	.3750	-1.4062	- .4032
- .4	.4000	- .0400	.4800	- .9028	.3485
- .3	.3000	- .4600	.4650	- .2466	.6650
- .2	.2000	- .7600	.3600	.3692	.8228
- .1	.1000	- .9400	.1950	.7974	.5104
0.0	0	-1	0	.9500	0
.1	- .1000	- .9400	- .1950	.7974	- .5104
.2	- .2000	- .7600	- .3600	.3692	- .8228
.3	- .3000	- .4600	- .4650	- .2466	- .6650
.4	- .4000	- .0400	- .4800	- .9028	- .3485
.5	- .5000	.5000	- .3750	-1.4062	.4032
.6	- .6000	1.1600	- .1200	-1.4548	1.2874
.7	- .7000	1.9400	.3150	- .7625	1.4800
.8	- .8000	2.8400	.9600	1.0655	.4855
.9	- .9000	3.8600	1.8450	4.8435	- 2.9600
1.0	-1.0000	5.0000	3.0000	9.9500	-10.3500



GRAPH OF $P_n(x)$
$P_1(x)$
$P_2(x)$
$P_3(x)$
$P_4(x)$
$P_5(x)$
x

CHAPTER III

ORTHOGONAL POLYNOMIALS FOR DISCRETE POINTS

1. Factorial Polynomials. Mention was made in the introduction to this thesis that there is no need to treat the two cases separately, provided that we use Stieltjes integrals, for the underlying principles and the resulting formulae are identical. Since we are not using that method, we propose instead to develop the case for discrete points.

It will be convenient to recall some properties of factorial polynomials and some equations from the finite calculus.

Eq. 4

(a) $x^{(n)} = x(x-1) - - - (x-n+1) = n!\binom{x}{n}$

(b) $\Delta x^{(n)} = (x+1)^{(n)} - x^{(n)} = nx^{(n-1)}$

cf. $\dfrac{dx^n}{dx} = nx^{n-1}$

(c) $\displaystyle\sum_{s=0}^{x} s^{(n)} = \frac{(x+1)^{(n+1)}}{n+1}$

cf. $\displaystyle\int_0^x s^n\,ds = \frac{x^{n+1}}{n+1}$

Eq. 5

$$\Delta_x \Delta_y f(x,y) = \Delta_y f(x+1, y) - \Delta_y f(x,y)$$

$$= f(x+1, y+1) - f(x+1, y) - f(x, y+1) + f(x,y).$$

The analogy of factorial polynomials and the binomial coefficients with permutations and combinations respectively, is noteworthy;

<u>e</u>.<u>g</u>. from

3(a) $\quad {}^{x}C_{n} = \binom{x}{n}\ ; \quad {}^{x}P_{n} = x^{(n)}$

3(b) $\quad {}^{x+1}P_{n} - {}^{x}P_{n} = n\ {}^{x}P_{n-1}.$

2. <u>Example</u>: <u>Representation</u> <u>of</u> <u>Given</u> <u>Function</u>. Consider

$$\sum_{x=0}^{\infty} x^{(n)} 2^{-x}.$$

The series

$$\sum_{x=0}^{\infty} Z^{x}$$

is generated by the function $(1 - Z)^{-1}$. We obtain, after n successive differentiations,

$$n!\ (1 - Z)^{-n-1} = \sum_{x=0}^{\infty} x^{(n)}\ Z^{x-n}$$

whence

$$n!\,Z^{n}\ (1 - Z)^{-n-1} = \sum_{x=0}^{\infty} x^{(n)} Z^{x},$$

a series which is convergent for all Z in the unit circle.

Hence, put $Z = \frac{1}{2}$ and obtain

$$n!\,(\tfrac{1}{2})^{n}\,(\tfrac{1}{2})^{-n-1} = \sum_{x=0}^{\infty} x^{(n)}\,2^{-x}$$

or

$$\text{Eq. 6} \qquad \sum_{x=0}^{\infty} x^{(n)}\,2^{-x} = 2n!$$

We now construct $P_k(x) = \begin{vmatrix} 1 & x^{(1)} & x^{(2)} & \cdots & x^{(k)} \\ m_0 & m_1 & & \cdots & m_k \\ m_1 & m_2 & & & m_{k+1} \\ \vdots & & & & \\ m_{k-1} & m_k & & \cdots & m_{2k-1} \end{vmatrix}$.

where

$$m_j = 2j! = \sum_{x=0}^{\infty} x^{(j)}\,2^{-x}$$

(from Eq. 6)

Thus, $P_k(x) = \begin{vmatrix} 1 & x^{(1)} & x^{(2)} & --- & x^{(k)} \\ 2 & 2.1! & 2.2! & --- & 2.k! \\ 2.1! & 2.2! & 2.3! & --- & 2.(k+1)! \\ \vdots & & & & \\ 2.(k-1)! & 2.k! & 2.(k+1)! & --- & 2.(2k-1)! \end{vmatrix}.$

If we factor out all the elements of the first row and column and, in addition, divide each column by the factorial of the exponent of n heading that column, we will have

$$P_k(x) = \prod_{j=0}^{n-1} 2j!(j+1)! \begin{vmatrix} 1 & \binom{x}{1} & \binom{x}{2} & --- & \binom{x}{k} \\ 1 & 1 & 1 & --- & 1 \\ 1 & 2 & 3 & --- & \binom{k+1}{1} \\ 1 & 3 & 6 & --- & \binom{k+2}{2} \\ \vdots & & & & \\ 1 & - & - & --- & \binom{2k-1}{k} \end{vmatrix}.$$

For example,

$$P_4(x) = \begin{vmatrix} 1 & x^{(1)} & x^{(2)} & x^{(3)} & x^{(4)} \\ 2 & 2.1! & 2.2! & 2.3! & 2.4! \\ 2.1! & 2.2! & 2.3! & 2.4! & 2.5! \\ 2.2! & 2.3! & 2.4! & 2.5! & 2.6! \\ 2.3! & 2.4! & 2.5! & 2.6! & 2.7! \end{vmatrix}.$$

$$= \prod_{0}^{3} 2j!(j+1)! \begin{vmatrix} 1 & \binom{x}{1} & \binom{x}{2} & \binom{x}{3} & \binom{x}{4} \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \end{vmatrix}$$

We proceed to show that

$$P_n(x) = \prod_{j=0}^{n-1} 2j!\,(j+1)! \sum (-1)^j \binom{n}{j}\binom{x}{j}$$

or

Eq. 7

$$P_n(x) = \frac{2^n}{n!} L_n(x) \prod_{0}^{n} (n!)^2$$

where $L_n(x)$ is the "normalized"

Eq. 8

$$L_n(x) = \sum_{j=0}^{n} (-1)^n \binom{n}{j}\binom{x}{j}$$

Before doing so, we discuss some properties of a special type of determinant.

3. Persymmetric Determinants.[2] If we write

$$D_{n+1} = \begin{vmatrix} m_0 & - & - & - & m_n \\ \vdots & & & & \\ \vdots & & & & \\ \vdots & & & & \\ m_n & - & - & - & m_{2n} \end{vmatrix}$$

Where the m_j are given by Eq. 3, and factor out the terms of the first row and column, we may write

$$D_{n+1} = 2^{n+1} \prod_{n=1}^{n} (n!)^2 \begin{vmatrix} \binom{0}{0} & \binom{1}{0} & - & - & - & \binom{n}{0} \\ \vdots & & & & & \\ \vdots & & & & & \\ \vdots & & & & & \\ \binom{n}{n} & \binom{n+1}{n} & - & - & - & \binom{2n}{n} \end{vmatrix}$$

The "normalized" determinant is persymmetric; i.e., all elements in any diagonal at right angles to a_{ii} are alike. If we adopt the notation

$$\begin{vmatrix} a_{0,0} & - & - & - & a_{n,0} \\ \vdots & & & & \\ \vdots & & & & \\ \vdots & & & & \\ a_{0,n} & - & - & - & a_{n,n} \end{vmatrix}, \qquad \text{then } a_{i,j} = a_{i+j}$$

Persymmetric determinants in general have the property of lending themselves to rapid evaluation by the successive operations

$$\Delta \text{ row } 1 \;=\; \text{row } 2 \;-\; \text{row } 1$$

$$\Delta \text{ row } 1 \;=\; \text{row } 3 \;-\; \text{row } 2 \;+\; \text{row } 1, \text{ and so on.}$$

In practice, we subtract row 1 from row 2, row 2 from row 3, row 3 from row 4 - - -, row (n - 1) from row n, then repeat the operation omitting row 1, repeat again omitting row 2, then omitting row 3, and so on down the line.

We see that since the first row (column) is a polynomial in n of degree 0, $\binom{n}{1}$ is of degree 1, $\binom{n+1}{2}$ is of degree 2, or in general, $\binom{n+k-1}{k}$ is of degree k in n, it follows, from the equations

$$\Delta \binom{n+k-1}{k} = \binom{n+k}{k} - \binom{n+k-1}{k}$$

$$\binom{n+k}{k} - \binom{n+k-1}{k-1} = \binom{n+k-1}{k}$$

etc., that all the elements below the leading diagonal vanish, while

$$a_{ii} = 1;$$

i.e.,

$$a_{i,j} = \delta_{i,j} \quad ; \quad i \geqslant j,$$

so that the determinant has the value 1.

In fact, persymmetric determinants whose corresponding elements in each row (column) reduce to binomial coefficients of successive differences, may be shown to have the value 1 by premultiplying with a determinant whose elements are the binomial coefficients of successive order. For example, if we premultiply

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} \quad \text{with} \quad \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{vmatrix},$$

we obtain

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

By either method, we find

Eq. 9

$$D_{n+1} = 2^{n+1} \prod_{1}^{n} (n!)^2$$

We are now ready to prove that

4. $\underline{L_n(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{x}{j}}$. If we subject the determinant $L_n(x)$ to the operation, column 1 $-\binom{n}{1}$ column 2 $+\binom{n}{2}$ column 3 - - -

$$= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \text{ column } (1+j),$$

we obtain, for example, $n = 5$.

$$\begin{vmatrix} 1-\binom{5}{1}\binom{x}{1}+\binom{5}{2}\binom{x}{2}-\binom{5}{3}\binom{x}{3}+\binom{5}{4}\binom{x}{4}-\binom{5}{5}\binom{x}{5} & \binom{x}{1} & \binom{x}{2} & \binom{x}{3} & \binom{x}{4} & \binom{x}{5} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 & 5 & 6 \\ 0 & 3 & 6 & 10 & 15 & 21 \\ 0 & 4 & 10 & 20 & 35 & 56 \\ 0 & 5 & 15 & 35 & 70 & 126 \end{vmatrix}$$

Consistent with the notation which we have adopted for D_{n+1}, if we write

$$L_n(x) = \begin{vmatrix} a_{0,\,-1} & a_{1,\,-1} & - - - & a_{n,\,-1} \\ a_{0,\,0} & & - - - & a_{n,\,0} \\ \vdots & & & \\ a_{0,\,n-1} & & - - - & a_{n,\,n-1} \end{vmatrix}$$

we observe that the element $a_{i,\,j} = \binom{i+j}{i}$. The cofactor of the leading element is composed of rows of differences of successive order.

$$\begin{array}{r} a_{i,\,0} \\ \Delta^2 a_{i,\,0} \\ \Delta^3 a_{i,\,0} \\ \Delta^4 a_{i,\,0} \\ \Delta^5 a_{i,\,0} \\ \Delta^6 a_{i,\,0} \end{array} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

Thus, the value of $L_5(x)$ is

$$a_{0,\,-1}\,A_{0,\,-1} = \sum_{j=0}^{5} (-1)^j \binom{5}{j}\binom{x}{j}$$

and, in general,

$$L_n(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{x}{j}$$

if we can show that our operation

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \text{ column } (1 + j)$$

gives

$$a_{0,\,k} = 0 \qquad (k = 0, 1, 2, - - - n - 1).$$

Since

$$a_{i,\,j} = a_{i+j} = \binom{i+j}{i}$$

it suffices to show that

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{i+j}{i} = 0.$$

Let the generating function

$$\mu(t) = (1 - t)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} t^j .$$

Differentiating i times,

$$\frac{d^i\left[t^i u(t)\right]}{dt^i} = i! \sum (-1)^j \binom{n}{j}\binom{i+j}{i} t^i$$

Apply Leibniz rule for the i^{th} derivative of a product and obtain

$$t^i_{(i)} u + \binom{i}{1} t^i_{(i-1)} u_{(1)} + \binom{i}{2} t^i_{(i-2)} u_{(2)}$$

$$+ - - - \binom{i}{i-1} t^i_1 u_{i-1} + u_i,$$

where the suffixes indicate the order of the derivative.

Putting t = 1 in the above expression, after differentiating, we get

$$u = (1-1)^n = 0 = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{i+j}{i}.$$

5. Laguerre Polynomials for Finite Differences. It should be observed that since

$$\Delta_x^n \; 2^x = 2^x,$$

the weight function 2^{-x} chosen is analogous to the weight function e^{-x} of the Laguerre polynomials (defined on page 7 with $\alpha = 0$).

We accordingly define the set of polynomials $L_n(x)$ as Laguerre Polynomials for finite differences.

6. Partial Difference Equation. From the equations

$$\Delta_x L_n(x) = L_n(x + 1) - L_n(x)$$

$$= \sum_{j=0}^{n+1} (-1)^j \binom{n}{j}\binom{x}{j-1},$$

we get

$$\Delta_n \Delta_x L_n(x) = \Delta_n \left[L_n(x + 1) - L_n(x) \right]$$

$$= \sum_{j=0}^{n+1} (-1)^j \binom{n}{j-1}\binom{x}{j-1}$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k}\binom{x}{k} = - L_n(x).$$

Thus

Eq. 10

$$\Delta_n \Delta_x L_n(x) + L_n(x) = 0.$$

This equation affords a rapid means of tabulating the values L(n, x). Thus,

$$\Delta_n \Delta_x L_n(x) + L_n(x) = L_{n+1}(x + 1)$$

$$- L_{n+1}(x) + 2 L_n(x) - L_n(x + 1) = 0$$

or

Eq. 11

$$L_{n+1}(x) + L_n(x + 1) - 2 L_n(x) = L_{n+1}(x + 1).$$

Since

$$L_0(x) = L_n(0) = 1,$$

we tabulate values

x \ n	0	1	2	
0	1	1	1	-
1	1	0	-1	-
2	1	-1	-2	-
'	-	-	-	
'				

in accordance with Eq. 11; i.e., in any block

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} \qquad b + c - 2a = d,$$

a relationship which must hold for every block of 4 cells.

Eq. 12

$$n\left[L_n(x) - L_{n-1}(x)\right] = x\left[L_n(x) - L_n(x-1)\right]$$

follows from the property of symmetry in x and n.

7. Laguerre Polynomials for Finite Differences. Weight function 2^{-x}.

x	$L_0(x)$	$L_1(x)$	$L_2(x)$	$L_3(x)$	$L_4(x)$	$L_5(x)$	$L_6(x)$	$L_7(x)$	$L_8(x)$	$L_9(x)$	$L_{10}(x)$	2^x
0	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	2
2	1	-1	-2	-2	-1	1	4	8	13	19	26	4
3	1	-2	-2	0	3	6	8	8	5	-2	-14	8
4	1	-3	-1	3	6	6	2	-6	-17	-29	-39	16
5	1	-4	1	6	6	0	-10	-20	-25	-20	-1	32
6	1	-5	4	8	2	-10	-20	-20	-5	25	64	64
7	1	-6	8	8	-6	-20	-20	0	35	70	84	128
8	1	-7	13	5	-17	-25	-5	35	70	70	14	256
9	1	-8	19	-2	-29	-20	25	70	70	0	-126	512
10	1	-9	26	-14	-39	-1	64	84	14	-126	-252	1024
11	1	-10	34	-32	-43	34	100	56	-98	-252	-252	2048
12	1	-11	43	-57	-36	84	116	-28	-238	-294	-42	4096
13	1	-12	53	-90	-12	144	92	-168	-350	-168	378	8192
14	1	-13	64	-132	36	204	8	-344	-358	174	888	16384
15	1	-14	76	-184	116	248	-152	-512	-182	708	1248	32768
16	1	-15	89	-247	237	253	-395	-603	239	1311	1142	65536
17	1	-16	103	-322	409	188	-713	-526	919	1752	273	131072
18	1	-17	118	-410	643	13	-1076	-176	1795	1709	-1522	262144
19	1	-18	134	-512	951	-322	-1424	552	2699	818	-4122	524288
20	1	-19	151	-629	1346	-878	-1658	1742	3337	-1243	-7001	1048576

8. Difference Equation. During the discussion of the series $T_n(x)$ and $U_n(x)$, (also known as Tchebichef polynomials of the first and second kinds respectively) we found that a relation of recurrence existed between any three successive polynomials; also that the polynomials satisfied a 2nd order linear differential equation. With this in mind, if we set up a table for j = 0, 1, 2 from the relations already established, viz.,

$$L_n(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{x}{j}$$

$$\Delta_x L_n(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{x}{j-1}$$

$$\Delta_x^2 L_n(x-1) = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{x}{j-2},$$

we will have

j	$L_n(x)$	$\Delta_x L_n(x)$	$\Delta_x^2 L_n(x-1)$
0	1	0	0
1	$-nx$	$-n$	0
2	$\binom{n}{2}\binom{x}{2}$	$\binom{n}{2}\binom{x}{1}$	$\binom{n}{2}$

from which we find that the coefficients of $L_n(x)$, $\Delta_x L_n(x)$ and $\Delta_x^2 L_n(x-1)$ which will make the sum vanish are n, $(1-x)$ and $2x$ respectively. That is, we have established the difference equation

Eq. 13

$$2x \Delta_x^2 L_n(x - 1) + (1 - x) \Delta_x L_n(x) + n L_n(x) = 0.$$

9. Recurrence Formula. The relation of symmetry,

$$L(x, n) = L(n, x)$$

conveniently establishes the recurrence formula as well. Thus, from equation 12,

$$2x \left[L_n(x + 1) - 2 L_n(x) + L_n(x - 1)\right] + (1 - x) \left[L_n(x + 1) - L_n(x)\right] + n L_n(x) = 0.$$

Simplifying, we get

$$(x + 1) L_n(x + 1) + (n - 3x - 1) L_n(x) + 2x L_n(x - 1) = 0,$$

and using the symmetry relation mentioned, we have

Eq. 14

$$(n + 1) L_{n+1}(x) + (x - 3n - 1) L_n(x) + 2n L_{n-1}(x) = 0.$$

10. $P_n(x)$. Modification of Formula. An essential part of the proof of equations 2(a) and 2(b) is the fact that when the first row of the determinant $P_n(x)$ is multiplied by $x^m\,dx$ and each element integrated from a to b, two rows become identical for $m < n$. For the $P_n(x)$ which we are integrating, however, while no difficulty is experienced on account of a summation replacing the integration, we are

dealing by the nature of the case with a factorial polynomial. When, therefore, we multiply $P_n(x)$ by $2^{-x}\, x^{(n)}$ and sum from 0 to ∞, we do not obtain a row which corresponds to the last one of D_{n+1}, and our equations 2 no longer hold.

We make use of the relation

$$(x + m)^m\, x^{(n)} = (x + m)^{(m + n)}$$

and replace

$$\sum_{x = 0}^{\infty} x^{(k)}\, 2^{-x} = m_k$$

by

$$\sum_{x = 0}^{\infty} (x + m)^{(k)}\, 2^{-x} = C_k^m .$$

If we let $x + m = s$, then we obtain

$$\sum_{s = m}^{\infty} s^{(k)}\, 2^{-s + m} = 2^m \sum_{s = m}^{\infty} s^{(k)}\, 2^{-s}$$

$$= 2^m \left[\sum_{s = 0}^{\infty} s^{(k)}\, 2^{-s} - \sum_{s = 0}^{n - 1} s^{(k)}\, 2^{-s} \right].$$

But the last expression vanishes for

$$S = 0, 1, 2 - - - k - 1,$$

whence

Eq. 15

$$\sum_{x=0}^{\infty} (x + m)^{(k)} 2^{-x} = 2^m \ 2k! = 2^{m+1} k!$$

For $k \geqslant m$.

We accordingly redefine

$$P_n(x) = \begin{vmatrix} 1 & x^{(1)} & x^{(2)} & - - - & x^{(m)} \\ C^0_0 & C^0_1 & - & - - - & C^0_n \\ C^1_1 & C^1_2 & - & - - - & C^1_{n+1} \\ \vdots & & & & \\ C^{n-1}_{n-1} & C^{n-1}_n & - & - - - & C^{n-1}_{2n-1} \end{vmatrix}$$

where

$$C^m_k = 2^{m+1} k! = \sum_{x=0}^{\infty} (x + m)^{(k)} 2^{-x}$$

This relation, then, supersedes the one established by equation 5.

By factoring out elements in the first row and column, proceeding as for equation 6, that equation is now superseded by

Eq. 16

$$P_n(x) = \frac{2^{n!}}{n!} L_n(x) \left(\prod_{0}^{n} (n!)^2 \right.$$

11. <u>Evaluation</u> <u>of</u> $\sum_{x=0}^{\infty} 2^{-x} L_n^2(x)$. We proceed to obtain the formal equivalent of equation 2(b).

If we multiply

$$L_n(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{x}{j}$$

by $2^{-x} L_n(x)$, and sum on x from 0 to ∞, we get

$$\sum_{x=0}^{\infty} 2^{-x} L_n^2(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{x=0}^{\infty} \binom{x}{j} 2^{-x} L_n(x),$$

and this, by virtue of the orthogonal property, vanishes for $j \neq n$.

We obtain

$$\sum_{x=0}^{\infty} 2^{-x} L_n^2(x) = (-1)^n \sum_{x=0}^{\infty} \binom{x}{n} 2^{-x} L_n(x),$$

which for convenience we express as

$$(-1)^n \sum_{x=0}^{\infty} \binom{x+n}{n} 2^{-x} L_n(x),$$

which again we can do, because of the orthogonality of the $L_n(x)$ polynomials. This permits us to write

$$\sum_{x=0}^{\infty} 2^{-x} L_n^2(x) = (-1)^n \sum_{x=0}^{\infty} \binom{x+n}{n} 2^{-x} \sum_{j=0}^{n} (-1)^n \binom{n}{j}\binom{x}{j}$$

$$= (-1)^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{x=0}^{\infty} \binom{x+n}{n}\binom{x}{j} 2^{-x}$$

$$= (-1)^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{x=0}^{\infty} \frac{(x+n)^{(n+j)}}{n!\quad j!} 2^{-x}$$

With the aid of equation 14, we write

$$\sum_{x=0}^{\infty} 2^{-x} L_n^2(x) = (-1)^n \sum_{j=0}^{n} (-1)^j \, {n \atop j} \, \frac{2^{n+1} \quad (n+j)!}{n!j!}$$

$$= (-1)^n 2^{n+1} \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{n+j}{j}$$

From the formula(8)

$$F(x) = \sum_{j=0}^{n} \binom{x+j}{j} \Delta^j F(-j-1).$$

If we suppose

$$F(x) = \binom{x}{n},$$

we have

$$\binom{x}{n} = \sum_{j=0}^{n} \binom{x+j}{j}\binom{-j-1}{n-j} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j}\binom{x+j}{j},$$

by virtue of the relations[9]

$$\binom{n}{k} = \binom{n}{n-k}; \quad \binom{-s}{k} = (-1)^k \binom{s+k-1}{k}.$$

Let $x = n$, then we will have

$$1 = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j}\binom{n+j}{j},$$

whence

$$(-1)^n = \sum_{j=0}^{n} (-1)^n \binom{n}{j}\binom{n+j}{j};$$

so that

$$\sum_{x=0}^{\infty} 2^{-x} L_n^2(x) = (-1)^n 2^{n+1} (-1)^n,$$

and we finally emerge with

Eq. 17

$$\sum_{x=0}^{\infty} 2^{-x} L_n^2(x) = 2^{n+1}$$

As will presently be seen, this equation is of great importance when we seek to expand an arbitrary function in the form

$$\sum_{k=0}^{\infty} a_k L_k(x).$$

12. <u>Expansion of an Arbitrary Function in Series of $L_n(x)$.</u> If we can assume that we may write the function

$$f(x) = a_0 L_0(x) + a, L, (x) + - - -,$$

then, if we multiply by $2^{-x} L_k(x)$ and sum on x from 0 to ∞, we will have

$$\sum_{x=0}^{\infty} f(x)\, 2^{-x} L_k(x) = A_k \sum_{x=0}^{\infty} 2^{-x} L_k^2(x).$$

This is true because each sum on the right is zero except the one containing $L_k^2(x)$, by virtue of the orthogonality property. Thus

Eq. 18

$$A_k = \frac{\sum_{x=0}^{\infty} f(x)\, 2^{-x} L_k(x)}{2^{k+1}}$$

We first give a few simple examples to illustrate and confirm the results arrived at.

Ex. 1 $f(x) = x$

From eq. 17. $a_0 = \frac{\sum 2^{-x} x}{2} = 1$

$$a_1 = \frac{\sum 2^{-x}\, x(1 - x)}{4}$$

$$= -\frac{\sum 2^{-x}\, x^{(2)}}{4} = -1;$$

$$f(x) = x = L_0(x) - L_1(x);$$

a result which is readily verified.

Ex. 2 $\quad f(x) = x^2$

We have

$$a_0 = \sum 2^{-x} \; x^2/_2 = \sum 2^{-x} \left[x^{(2)} + x^{(1)}\right]/_2 = 3$$

$$a_1 = \sum 2^{-x} \; x^2(1 - x)/_4 = -\sum 2^{-x} \left[2x^{(2)} + x^{(3)}\right]/_4 = -5$$

$$a_2 = \frac{1}{8}\left[\sum 2^{-x} \; x^2 \; \left[1 - 2x + \frac{x(x - 1)}{2}\right)\right]$$

$$= \frac{1}{8} \sum 2^{-x} \left(x^2 - \frac{5}{2} x^3 + \frac{x^4}{2}\right).$$

To convert the expression in parentheses on the right to factorial form, we make use of

Eq. 19[10][7]

$$f(x) = f(0) + \binom{x}{1} \Delta f(0) + \binom{x}{2} \Delta^2 f(0) + - - - + \binom{x}{m} \Delta^m f(0);$$

and set up the table of differences

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	0				
		-1			
1	-1		-6		
		-7		3	
2	-8		-3		12
		-10		15	
3	-18		12		
		2			
4	-16				

From equation 18 we now have

$$f(x) = -x^{(1)} - 3x^{(2)} + \frac{1}{2}x^{(3)} + \frac{1}{2}x^{(4)}$$

i.e.

$$a_2 = \frac{1}{8} \sum 2^{-x} \left(-x^{(1)} - 3x^{(2)} + \frac{1}{2}x^{(3)} + \frac{1}{2}x^{(4)}\right)$$

$$= \frac{1}{8} \quad (-2 - 12 + 6 + 24) = 2,$$

and

$$f(x) = x^2 = 3\,L_0(x) - 5\,L_1(x) + 2\,L_2(x).$$

As before, this result is easily substantiated.

13. Biorthogonal Functions. The problem of expanding an arbitrary function is much more complex when, instead of a polynomial, we have to deal with a nonterminating series. We do not propose to enter into the problem of examining the series for convergence, and we leave open the question of the validity of the series as a representation of the function, since it would take us outside the scope of this thesis. What we do have to take into account is the concept of biorthogonal functions, since we obtain thereby a new set of coefficients for our $L_n(x)$ which give us a more rapidly convergent series for large values of x.

By definition, if two sets of functions

$$u_i(x),\ v_i(x),\ (i = 0, 1, 2 - - -)$$

bear the relation

$$\int_a^b u_i\, v_j\, dx = 0,\ i \neq j,$$

then the two systems of functions are said to be biorthogonal over the interval (a, b). In the case of the functions

$$L_0(x) \quad , \quad L_1(x) \quad - - - L_n(x)$$

$$2^{-x} L_0(x) \, , \, 2^{-x} L_1(x) - - - 2^{-x} L_n(x),$$

since we may write

$$\sum_{x = 0}^{\infty} 2^{-x} L_m(x) L_n(x) = 0 \quad m \neq n,$$

the $L_i(x)$, $2^{-x} L_i(x)$ are two sets of mutually biorthogonal functions, for the case of discrete points.

The a_j given by equation 17 is now replaced by

$$\sum 2^{-x} b_i L_i(x),$$

where the b_k are given by

Eq. 20

$$b_k = \frac{\sum f(x) L_k(x)}{2^{k+1}} \quad .$$

Example. $f(x) = \text{Sech } x$.

The expansion is of the form

$$\text{Sech } x = 2^{-x} \sum b_k L_k(x).$$

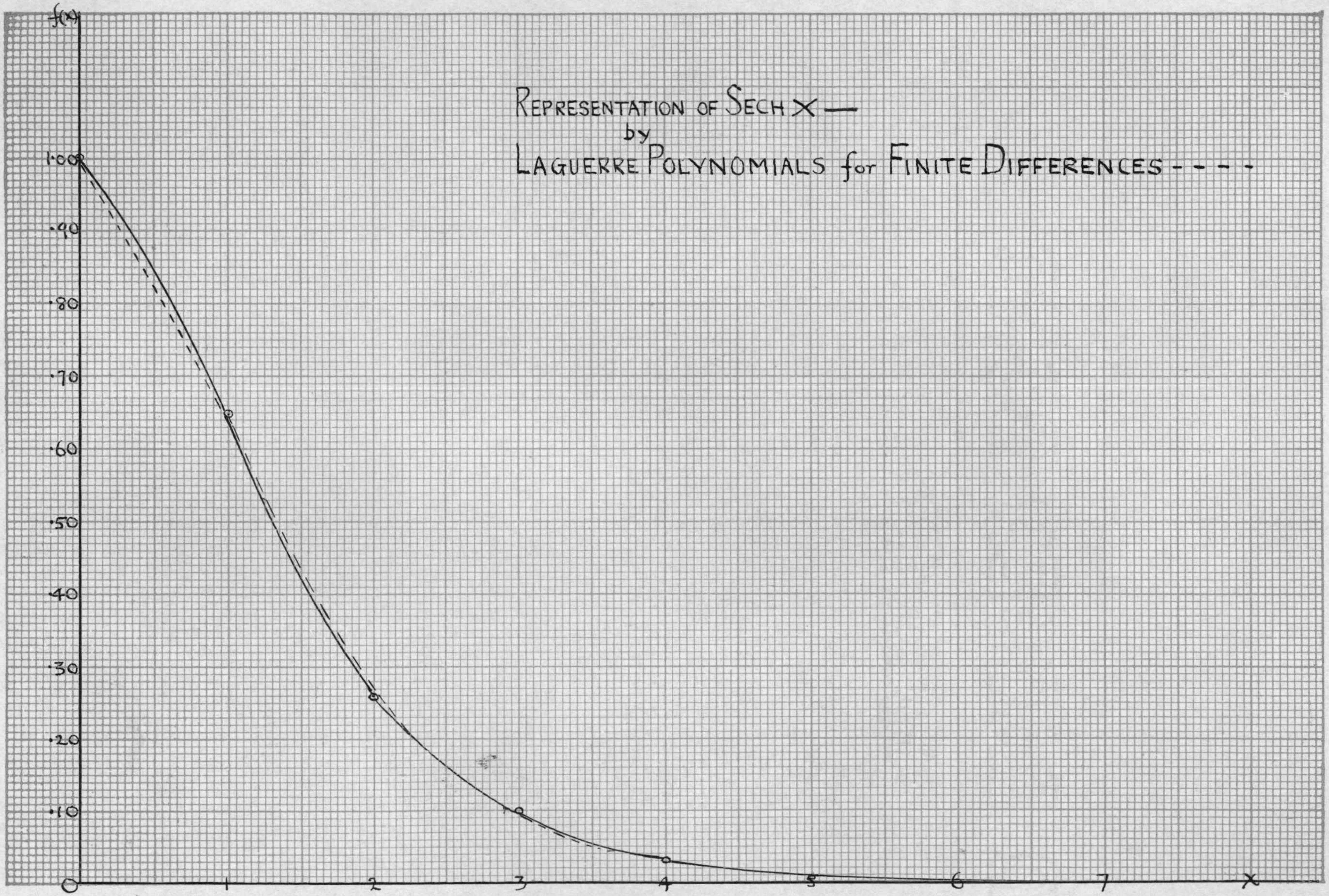
To calculate the coefficients

$$b_k = \frac{\sum \text{Sech } x \, L_k(x)}{2^{k+1}}$$

We find it sufficient to take values for x up to 10. (Sech 10 = .0001.) We then compute

k	$\sum L_k$ Sech x	b_k	Sech x	$2^{-x} \sum b_k L_k(x)$
0	2.0711	1.0355	1.000	1.002
1	.3286	.0822	.648	.640
2	-.3515	-.0438	.266	.273
3	-.5809	-.0363	.099	.099
4	-.6340	-.0198	.0366	.0360
5	-.6188	-.0097	.0135	.0100
6	-.4728	-.0037	.0050	.0051
7	-.5105	-.0020	.0018	.0030

The accompanying graph shows how close an approximation to the given function we have obtained.


REPRESENTATION OF SECH X —
by
LAGUERRE POLYNOMIALS for FINITE DIFFERENCES - - - -
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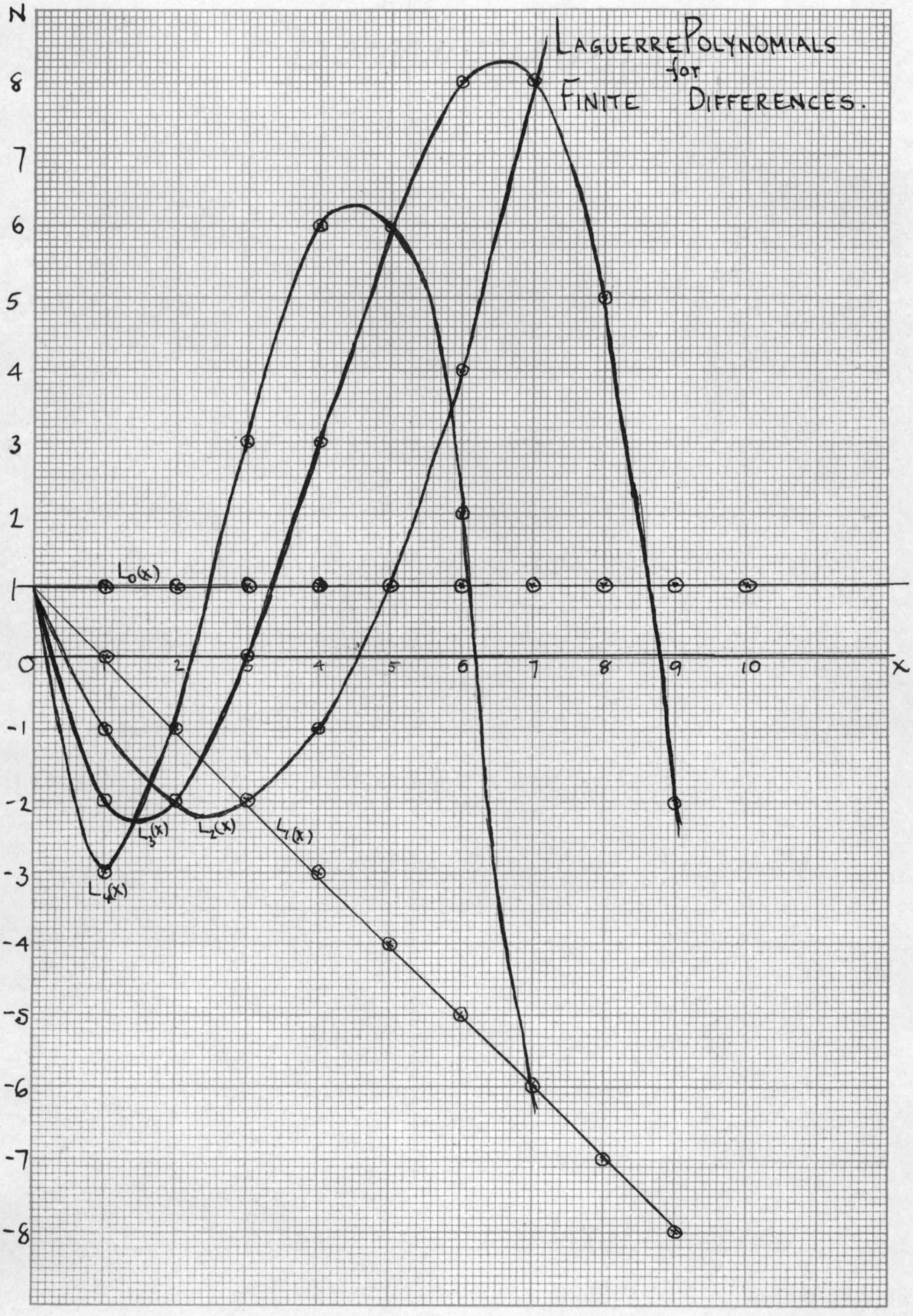
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X
$L_0(x)$
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$L_2(x)$
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$L_4(x)$

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