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Title: 3D Cone Beam Reconstruction Formulas for the Transverse-ray Transform with Source Points on a Curve

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David V. Finch

3D vector tomography has been explored and results have been achieved in the last few decades. Among these was a reconstruction formula for the solenoidal part of a vector field from its Doppler transform with sources on a curve. The Doppler transform of a vector field is the line integral of the component parallel to the line. In this work, we shall study the transverse ray transform of a vector field, which instead integrates over lines the component of the vector field perpendicular to the line. We provide a reconstruction procedure for the transverse ray transform of a vector field with sources on a curve fulfilling Tuy's condition of order 3. We shall recover both the potential and solenoidal parts. We present two steps for the reconstruction. The first one is to reconstruct the solenoidal part and the techniques we use are inspired by work of Katsevich and Schuster. A procedure for recovering the potential part will be the second step. The main ingredient is the difference between the measured data and the reprojection of the solenoidal part. We also provide a variation of the Radon inversion formula for the vector part of a quaternionic-valued function (or vector field) and an inversion formula in cone-beam setting with sources on the sphere.

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3D Cone Beam Reconstructions Formulas for the Transverse-ray Transform with Source Points on a Curve

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I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Patcharee Wongsason, Author

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1 INTRODUCTION

To see the interior of objects or patients without seeing inside or opening it up, we need some experiment or certain test. During a CAT (Computerized Axial Tomography) scan the patient will lie on the table that is attached to the scanner, which is a large doughnut-shaped machine. The CT scanner sends x-rays through the body area being studied. Each rotation of the scanner provides a picture of a thin slice of an organ or area which can be saved and reassembled on a computer. The process of creating the image of the patient's interior from such x-ray measurements is modeled mathematically as the problem of recovering a function from its line integrals.

In 2 dimensional space, the fan-beam tomography is introduced for examples in [17, 24] which is a scanning method to reconstruct density functions. The process begin by emitting x-rays from an x-ray source and the intensities are measured on a single row of detectors. To continue, the x-ray sources and detectors are rotated to a new angle and the x-rays are emitted and measured again. This process is repeated until the x-rays have been measured over a sufficient number of angles. The object is supported in the plane of rotation. Cone-beam tomography, basically uses the same methods as in the fan beam case but is in dimensions 3 and the number of detector rows increases from the fan-beam case.

Cone beam scalar tomography: The scalar cone beam transform has been thoroughly studied over the last three decades. Inversion formulas have been achieved by Tuy [32], Finch [4], Smith [27], Grangeat [7], Katsevich [13, 11, 12, 10] for helical source trajectory [12], and by Louis [15, 16] for general source orbits satisfying Tuy's condition. Tuy's condition says that any plane intersecting the support of the object function f must hit the trajectory $a(\lambda)$ transversally at least once. This condition arose for the first time in Tuy's article [32] and was fundamental for his inversion formula.

Vector tomography considers the reconstruction of vector fields from line integrals of some components of the vector fields. Doppler tomography is one example of vector tomography which integrates the component parallel to the line. Thus the Doppler transform $D\mathbf{f}$ of a vector field $\mathbf{f}$ can be defined by

$$D\mathbf{f}(\underline{x}, \underline{\theta}) = \int_0^\infty \underline{\theta} \cdot \mathbf{f}(\underline{x} + t\underline{\theta})dt. \tag{1.1}$$

The Helmholtz decomposition states that a vector field can be written as a sum of an irrotational (or curl-free) vector field and a solenoidal (or divergence-free) vector field. In 2D vector tomography, Norton [19] used the Helmholtz decomposition to show that the irrotational part of a vector field cannot be imaged in acoustical time-of flight flow imaging. He proposed that boundary measurements be made in order to recover the irrotational component. Braun and Hauck called the standard acoustical time-of flight measurements the *longitudinal measurements*, and proposed a new set of measurements, which they called the t*ransverse measurements*, which would allow recovery of both the solenoidal and irrotational components of a vector field.

Several authors have proposed extensions of vector tomography to 3D. Most of them consider the Doppler tomography. Juhlin [9] proved that complete data are sufficient to recover the solenoidal part of a vector field $\mathbf{f}$, see also Sharafutdinov [26], Prince [21] and Denisjuk [3]. Several stable solvers of filtered-back projection type can be found in Sparr et al [31] and for a parallel slice-by slice scanning in Schuster [24]. An approximate inversion approach for the cone beam setting that does not lead to an exact inversion has been described and implemented in Schuster et al [25, 23]. It is known (Sharafutdinov [26], Schuster [22] Denisjuk [3]) that only the solenoidal part of the vector field can be reconstructed from the Doppler transform.

Cone beam vector Doppler tomography was presented recently by Katsevich and Schuster in [13] describing the reconstruction of a smooth solenoidal vector field from its Doppler transform cone beam data. They present an exact inversion formula for smooth solenoidal vector fields from the Doppler transform with cone beam data for a general trajectory Γ fulfilling Tuy's condition of order 3.

In this work, we study the transverse ray transform of a vector field, which integrates over lines the component of the vector field perpendicular to the line. An object with compact support is surrounded by a general curve satisfying Tuy's condition of order 3. Our reconstruction procedure has two parts. The first one is to recover the solenoidal part and the techniques we use are inspired by the work of Katsevich and Schuster. The solenoidal part will be considered in two pieces according to the decomposition of the Radon inversion formula into tangential and normal parts. The tangential part of the Radon transform is obtained from the measured data. The second piece of the solenoidal part can be recovered by using data from the first part. A procedure for recovering the potential part uses the difference of the measured data and the reprojection of the solenoidal part.

The organization of this dissertation is as follows:

Chapter 2 will mainly present mathematical background by first recalling the definitions of x-ray and Radon transforms for real and vector-valued functions. The background of quaternions also will be presented in this chapter including Clifford algebra which is a generalization to higher dimension of quaternions. The Dirac operator $\partial_{\underline{x}}$ which plays an important role in quaternionic analysis and is closely related to the Cauchy-Riemann operator in the complex analysis and the Laplace operator shall be described. Background on quaternions and Clifford algebra will be used only in sections 4.1 and 4.2. However, the essential part of this dissertation is section 4.3.

Furthermore, we shall present the definition of plane wave decomposition, definitions of Clifford x-ray and Radon transforms established by F. Sommen in [1]. The Helmholtz-Hodge decomposition of a vector field will be reviewed and at the end of this chapter we shall talk about the surface gradient $\nabla_{\underline{\eta}}$.

In chapter 3 we present the x-ray and Radon decompositions of the Cauchy kernel which again were introduced by Sommen [28, 30]. These decompositions use the plane wave decomposition in Chapter 2 as the main ingredient. We include a proof that clarifies the x-ray decomposition. The definition of new transforms such as the quaternionic Doppler transform and cone beam transform will be included as well. At the end of this chapter we shall recall some facts about the spherical harmonics and Gegenbauer polynomials. Also, some orthogonal expansions for vector fields in $L_2(B^3)$ which were introduced and explained in [14] in detail will be briefly reviewed.

Chapter 4 which presents our main results has 3 parts as follows:

Part 1 : We present a variation on the Radon inversion formula of the vector part of a quaternionic-valued function in term of the Dirac operator as the following form:

$$\underline{\mathbf{f}}(\underline{x}) = -\frac{1}{8\pi^2}\partial_{\underline{x}} \int_{S^2} \underline{\theta}\mathcal{R}'\underline{\mathbf{f}}(\underline{\theta}\cdot\underline{x},\underline{\theta})d\underline{\theta}. \tag{1.2}$$

We provide two derivations to verify the formula. The first derivation bases on the Laplacian of the back-projection of the Radon transform. Using the fact that the Laplcian is negative of the square of the Dirac operator gives (1.2). The second derivation uses arguments involving the boundary values of the Cauchy transform and the plane wave decomposition of the fundamental solution of the Dirac operator (Cauchy Kernel). The second derivation is more complicated than the first one. We can see that, however, the second one shows how the quaternions can be used for recovering functions. We remark at the end of this section that the formula (1.2) is equivalent to well known Radon inversion formulas for scalar case, for example see [17].

Part 2 : In this part we shall discuss a reconstruction formula for the cone beam transform with sources on the sphere. This is one of the simplest cases for the cone beam tomography.

Part 3 : This part contains the most substantial results of this thesis. We consider the problem of reconstructing a smooth vector field $\underline{\mathbf{f}}$, supported in the open unit ball

$B^3 = \{\underline{x} \in \mathbb{R}^3 \ : \ |\underline{x}| < 1\}$ from the transverse ray transform in cone beam geometry with sources on a curve. We may occasionally use the notation for vector field $\mathbf{f}$ or $\underline{\mathbf{f}}$ depending on their contexts. We define the transverse ray transform with source $\underline{a}$ in direction $\underline{\theta}$ is given by

$$T\underline{\mathbf{f}}(\underline{a}, \underline{\theta}) = \int_0^\infty (\underline{\theta} \times \underline{\mathbf{f}})(\underline{a} + t\underline{\theta})dt. \tag{1.3}$$

This transform has the property:

$$T\underline{\mathbf{f}}(\underline{a}, \underline{\theta}) \times \underline{\theta} = E_{\underline{\theta}} \int_0^\infty \underline{\mathbf{f}}(\underline{a} + t\underline{\theta})dt \tag{1.4}$$

where $E_{\underline{\theta}}$ is the orthogonal projection on $\underline{\theta}^\perp$. So we are able to call both (1.3) and (1.4) transverse ray transforms. Moreover, knowing the transforms (1.3) and (1.4) is equivalent to knowing the quantity, for example for fixed $\underline{\theta} \in S^2$

$$\underline{\eta} \cdot T\underline{\mathbf{f}}(a, \underline{\theta}), \quad \text{for all } \underline{\eta} \in \underline{\theta}^\perp.$$

We assume that we have sources lying on a regular curve Γ. Here, $\underline{a} = \underline{a}(\lambda), \lambda \in \Lambda \subset \mathbb{R}$ denote the parametrization of a source trajectory $\Gamma \subset (\mathbb{R}^3 \backslash \overline{B^3})$ fulfilling a certain property called Tuy's condition of order 3.

To our knowledge, this work is the first to investigate the transverse ray transform for sources on a curve. The Doppler transform (1.1) is an established mathematical model of vector tomography. In this work, we hope that the transform (1.3-1.4) can also be a useful mathematical model in vector tomography. The main purpose of this work is to present mathematical framework leading to an inversion procedure of $\underline{\mathbf{f}}$.

We will first pay attention to the reconstruction of the solenoidal part $\underline{\mathbf{f}}^d$ and then we shall discuss the reconstruction for the potential part using the data from the solenoidal part. To recover $\underline{\mathbf{f}}^d$, we follow the outlines in [14, 13] as we shall clearly point out along the way. The 3 dimensional Radon inversion of the solenoidal part $\underline{\mathbf{f}}^d$ of $\underline{\mathbf{f}}$ will be considered as the starting point. In other words, we shall consider the inversion formula

$$\underline{\mathbf{f}}^d(\underline{x}) = -\frac{1}{8\pi^2} \int_{S^2} \partial_s^2 \mathcal{R}\underline{\mathbf{f}}^d(s, \underline{\theta})|_{s=\underline{x}\cdot\underline{\theta}} d\underline{\theta}. \tag{1.5}$$

In [13] Katsevich and Schuster decomposed the solenoidal part $\underline{\mathbf{f}}^d$ of a vector field $\underline{\mathbf{f}}$ into two terms:

$$\underline{\mathbf{f}}^d = \underline{\mathbf{f}}_1^d + \underline{\mathbf{f}}_2^d \tag{1.6}$$

according to the tangential and normal parts of the Radon transform of $\underline{\mathbf{f}}$

$$\begin{aligned}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\eta}) &= \frac{\partial}{\partial s}\mathcal{R}\underline{\mathbf{f}}(s,\underline{\eta}) - \mathcal{R}^{\mathrm{nor}}\underline{\mathbf{f}}(s,\underline{\eta}) \\ \mathcal{R}^{\mathrm{nor}}\underline{\mathbf{f}}(s,\underline{\eta}) &= \underline{\eta}(\underline{\eta}\cdot\frac{\partial}{\partial s}\mathcal{R}\underline{\mathbf{f}}(s,\underline{\eta})), \quad s\in[-1,1], \quad \underline{\eta}\in S^2.\end{aligned} \tag{1.7}$$

This means that equations (1.5-1.7) lead to

$$\underline{\mathbf{f}}_1^d(\underline{x}) = -\frac{1}{8\pi^2}\int_{S^2}\partial_s^2\mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s,\underline{\theta})|_{s=\underline{x}\cdot\underline{\theta}}d\underline{\theta} \text{ and } \underline{\mathbf{f}}_2^d(\underline{x}) = -\frac{1}{8\pi^2}\int_{S^2}\partial_s^2\mathcal{R}^{\mathrm{nor}}\underline{\mathbf{f}}^d(s,\underline{\theta})|_{s=\underline{x}\cdot\underline{\theta}}d\underline{\theta}. \tag{1.8}$$

The first was the determination of $\underline{\mathbf{f}}_1^d$. They showed that the integrand in the first integral of (1.8) can be determined from the measured data. In other words, they obtained (1.8) from the measured data. This step used an important relation in [13], theorem 5.1 describing the connection between the Doppler transform and the Radon transform called a Grangeat type formula. The second part was the reconstruction of $\underline{\mathbf{f}}_2^d$. They showed that the integrand of the second integral in (1.8) which is $\partial_s^2\mathcal{R}^{\mathrm{nor}}\underline{\mathbf{f}}^d$ could be recovered from $\partial_s^2\mathcal{R}^{\tan}\underline{\mathbf{f}}^d$.

In this thesis, we shall recover $\underline{\mathbf{f}}^d$ by using the same decomposition as in (1.6) but the integrands in relation (1.8) will be replaced by $\underline{\eta}\times\mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s,\underline{\theta})$ and $\underline{\eta}\times\mathcal{R}^{\mathrm{nor}}\underline{\mathbf{f}}^d(s,\underline{\theta})$, respectively. Therefore, as in the work of Katsevich and Schuster in [13] we have two parts to reconstruct. Firstly, we shall modify theorem 5.1 in [14] to obtain $\underline{\eta}\times\frac{\partial}{\partial s}\mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s,\underline{\theta})$. Using a technique in [13], $\underline{\eta}\times\frac{\partial}{\partial s}\mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s,\underline{\theta})$ can be written in the sum of products of particular functions depending on source points and the quantities of the form

$$\int_{S^2}\delta''(\underline{\theta}\cdot\underline{\eta})\int_0^\infty\underline{\theta}\cdot(\underline{\eta}\times\underline{\mathbf{f}})(\underline{x}+t\underline{\theta})dtd\underline{\theta}. \tag{1.9}$$

We perhaps call it here a Grangeat-type formula. We shall work out the details in chapter 4.

Secondly, we shall recover $\underline{\mathbf{f}}_2^d$ and of course we consider the procedures to obtain $\underline{\eta} \times \frac{\partial^2}{\partial s^2}\mathcal{R}^{\text{nor}}\underline{\mathbf{f}}^d$. We follow techniques in the second part of [13] to recover $\frac{\partial^2}{\partial s^2}\mathcal{R}^{\text{nor}}\underline{\mathbf{f}}^d$ from $\frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}^d$ by the following formula

$$\frac{\partial^2}{\partial s^2}\mathcal{R}^{\text{nor}}\underline{\mathbf{f}}^d(s,\underline{\eta}) = \underline{\eta}\int_{S^2}\frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\theta})\cdot\nabla_{\underline{\theta}}K(\alpha\cdot\underline{\theta})d\underline{\theta}. \tag{1.10}$$

Here K is a kernel whose explicit expression is given below

$$K(\underline{\alpha}\cdot\underline{\eta}) = \sum_{n\geq 0}\frac{2n+3}{4\pi(n+2)}P_{n+1}(\underline{\alpha}\cdot\underline{\eta})$$

and

$$P_n(\underline{\alpha}\cdot\underline{\eta}) = \sum_{|l|\leq n+1} Y_{n,l}(\underline{\eta})\overline{Y}_{n,l}(\underline{\alpha}).$$

To obtain (1.10), we shall follow framework in [14] where they heavily used the orthogonal series expansions of vector fields. Using the fact that the product of two quaternions is -1 together with (1.10) give the second part :

$$\begin{aligned}\underline{\mathbf{f}}_2^d(\underline{x}) &= -\frac{1}{8\pi^2}\int_{S^2}\frac{\partial^2}{\partial s^2}\mathcal{R}^{\text{nor}}\underline{\mathbf{f}}^d(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}}d\underline{\eta}\\ &= -\int_{S^2}\underline{\eta}\int_{S^2}\underline{\theta}\left(\underline{\theta}\times\frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\theta})|_{s=\underline{x}\cdot\underline{\theta}}\right)\cdot\nabla_{\underline{\theta}}K(\alpha\cdot\underline{\eta})d\underline{\theta}d\underline{\eta}\end{aligned}$$

which we have known $\underline{\eta} \times \frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}^d$ from the first part. This is done in section (4.3). We note that computing the normal part of the Radon transform $\mathcal{R}\underline{\mathbf{f}}$ from the tangential part ignores the potential part of $\underline{\mathbf{f}}$ see [14].

According to the Helmholtz-Hodge decomposition of $\underline{\mathbf{f}}$

$$\underline{\mathbf{f}} = \nabla p + \underline{\mathbf{f}}_0^d + \nabla h$$

where $\underline{\mathbf{f}}_0^d + \nabla h$ is the solenoidal part and $p = 0$ on the boundary, we shall show that the tangential part of the Radon transform of ∇p vanishes and so only its solenoidal part can be reconstructed from such transverse ray transform. Another important thing left to be presented is the reconstruction formula of the potential part ∇p of a vector field $\underline{\mathbf{f}}$. To do so, we begin with the Radon inversion formula of p and then use the Grangeat formula on $T\underline{\mathbf{f}} - T\underline{\mathbf{f}}^d$ where T is the transverse ray transform.

In this last part of the introduction we shall present imaging models and methods for 2 and 3 dimensional flow fields as already explained by several authors, [24, 21, 2, 9, 31]. The 2 dimensional case was introduced by Norton [19] in 1988 and summarized by Schuster [24]. We shall follow their presentation.

First we present physical motivation for the Doppler transform. It arises when fluid velocity fields are investigated by ultrasonic time-of-flight measurements. Let $\Omega \subset \mathbb{R}^2$ be a convex, open, bounded domain of 2 dimension fluid flow with piecewise smooth boundary $\partial\Omega$. The velocity flow is given by $\mathbf{f}(x) \in \mathbb{R}^2$ having magnitude $|\mathbf{f}(x)|$. The aim of 2 dimensional imaging is to recover $\mathbf{f}$ from the time-of-flight measurements which are taken between two positions a and b located on $\partial\Omega$. The local speed of sound is denoted by $c(x)$. We assume that the ultrasound beam is traveling from a to b along a straight line L which is justified provided that the variations in c are small and the path length $|L|$ is short. The signal is the travel time $t(a, b)$ needed for traveling from a to b and is given by

$$t(a, b) = \int_{L(a,b)} \frac{1}{c(x) + \mathbf{f}(x) \cdot \eta} dl(x) \tag{1.11}$$

where $L(a, b)$ is the line connecting a and b, η is the unit vector of direction of the line and dl is the 1-dimensional Lebesgue measure along L. Writing

$$\frac{1}{c(x) + \mathbf{f}(x) \cdot \eta} = \frac{1}{c(x)(1 + (\mathbf{f}(x)/c(x)) \cdot \eta)} \tag{1.12}$$

and considering the right-hand side as function of $\mathbf{f}/c$, the first order Taylor-approximation gives

$$t(a, b) \simeq \int_{L(a,b)} \left(\frac{1}{c(x)} - \frac{\mathbf{f}(x) \cdot \eta}{c(x)^2} \right) dl(x). \tag{1.13}$$

Transmitting the ultrasound signal in the opposite direction from b to a we get (1.10) with η replaced by $-\eta$ and compute

$$\begin{aligned} t(a,b)+t(b,a) &= 2\int_{L(a,b)} \frac{1}{c(x)} dl(x) \\ t(a,b)-t(b,a) &= -2\int_{L(a,b)} \frac{\mathbf{f}(x)\cdot\eta}{c(x)^2} dl(x). \end{aligned}$$

From (1.11) we obtain $c(x)$ applying any inversion scheme for scalar 2D computerized tomography. Assuming $c(x)$ to be known in the interior of Ω we can define

$$\mathbf{g}(x) = -\frac{2\mathbf{f}(x)}{c(x)^2}.$$

Extending $\mathbf{g}$ by zero outside Ω gives the measured data the Doppler transform. More precisely, for every $a, b \in \partial\Omega$,

$$t(a,b)-t(b,a) = D\mathbf{g}(a,\underline{\eta})$$

where $D\mathbf{g}$ is defined in (1.1).

Schlieren tomography is a method to measure temperature in gases using the change of index of refraction due to heating. Braun and Hauck [2] show that the deflection of a laser beam shining through the gas is proportional to

$$\int_0^l (\nabla\eta \times \mathbf{t}_0) ds \tag{1.14}$$

where $\mathbf{t}_0$ is the direction of the undeflected beam and η is the index of refraction. This is the transverse ray transform of $\nabla\eta$.

2 MATHEMATICAL BACKGROUND

A complex number $a + bi$ can be viewed as an element (a,b) $\in \mathbb{R}^2$ with a new algebraic structure together with an imaginary number i with $i = -1$. More complicated algebraic structure was introduced to $\mathbb{R}^4$ with imaginary units $e_1^2 = e_2^2 = e_3^2 = -1$ where $e_i,\ i = 1, 2, 3$ are standard bases of $\mathbb{R}^3$ called Quaternions. In this sense, it can be considered as a higher dimensional generalization of complex numbers.

We begin this chapter by recalling the definition of x- ray and Radon transforms for scalar case and also a vector-valued function.

Section 2 shall present the definition and elementary properties of quaternions and the Dirac operator which play important role in quaternion analysis. The Dirac operator will be considered as the generalization of the Cauchy-Riemann operator in complex analysis and also its fundamental solution or the Cauchy kernel will be introduced. The null space of the Dirac operator are called the monogenic functions. The analogue of quaternions in higher dimension is Clifford algebra which is briefly presented in Section 3.

In Section 4 we introduce the Cauchy transform of a quaternionic-valued function. Its non-tangential boundary values provide the so called Plemelj-Sokhotski formula (Theorem 2.2). Subsequently, the quaternion-(Clifford) Hilbert transform will be introduced. The difference between the non-tangential limits of the Cauchy transform leads to the function we are considering. Section 5 presents four families of distributions in quaternions. Section 6 shall introduce the plane wave decomposition of the Cauchy kernel. This decomposition is in terms of the integral of the scalar product. The Clifford-Radon and x-ray transforms of a quaternionic or Clifford-valued function already worked out in details in [28, 30] will be described in section 7 and section 8, respectively. We also mentioned that the classical Radon transform is a boundary value of the quaternions or Clifford one. The x-ray transform, moreover, in the classical sense has closely related structure to the quaternionic one.

The last part of this chapter will recall the Helmholtz-Hodge decomposition of a vector field which is in section 9. The surface gradient is in section 10.

Throughout this work, we denote a real-valued function by a symbolic f and a vector-valued function by a bold $\mathbf{f}$. Furthermore, the quaternionic-valued function will be denoted by $\mathbf{f} = f_0 e_0 + \underline{\mathbf{f}}$ where f_0 is a scalar-valued function, $e_0 = (1, \ldots, 0)$ and $\underline{\mathbf{f}}$ is the vector part of $\mathbf{f}$. Here, we are interested in functions on the Euclidean space $\mathbb{R}^3$ and $\mathbb{R}^4$.

2.1 Real-Valued and Vector-Valued Radon and X-ray Transforms

The x-ray transform of a real-valued function f is defined by integrating over lines. In detail, if f is a compactly supported on $\mathbb{R}^n$, then the x-ray transform of f is the function Xf defined on the set of all lines in $\mathbb{R}^n$ by

$$Xf(L) = \int_L f = \int_{\mathbb{R}} f(x + t\theta)dt$$

where x is point on the line and θ is a unit vector giving the direction of the line L.

The Radon transform is an operator R defined on $L_1(\mathbb{R}^n)$ whereby for any integrable function f on $\mathbb{R}^n$, the function Rf is defined for $\theta \in S^{n-1}$ and $s \in \mathbb{R}$ by

$$Rf(s, \theta) = \int_{<\theta, x>=s} f(x)dx = \int_{y\cdot\theta=0} f(y + s\theta)dy$$

whenever the integrals exist. The left hand side is read "the value of the Radon transform of a function f" on the hyperplane $< \theta, x >= s$. Alternative expressions are

$$Rf(s, \theta) = \int_{y\in\theta^\perp} f(y + s\theta)dy \quad \text{or} \quad Rf(s, \theta) = \int_{\mathbb{R}^m} \delta(< x, \theta > -s)f(x)dx. \qquad (2.1)$$

In two dimensions, the x-ray and the Radon transform coincide up to notation because a line is the same as a hyperplane.

For a vector-valued function $\mathbf{f}$, for example, $\mathbf{f}$ is a vector field, in $\mathbb{R}^n$ we define the x-ray and Radon transforms of $\mathbf{f}$ by

$$X\mathbf{f}(L) = \int_L \mathbf{f} = \int_{\mathbb{R}} \mathbf{f}(x + t\theta)dt \tag{2.2}$$

and

$$R\mathbf{f}(s,\theta) = \int_{<\theta,y>=0} \mathbf{f}(y + s\theta)dy = \int_{\mathbb{R}^m} \delta(\underline{x}\cdot\underline{\theta} - s)\mathbf{f}(\underline{x})d\underline{x} \tag{2.3}$$

where the integrals are understood componentwise. We would like to remark that by a vector-valued function $\mathbf{f}$ we mean either a vector field or a Clifford (quaternion)-valued function depending on the context. There are some crucial properties of both transforms that we shall use for the inversion formulas, for example see [17].

During the last fifty years, Quaternion analysis has gradually developed into a comprehensive function theory offering a higher dimensional generalization of the theory of holomorphic functions of one complex variable. We shall introduce the quaternions in the following section.

2.2 Quaternions

In this section, we introduce the definition and elementary properties of quaternions together with all the algebraic properties which are used throughout this work. We can observe afterward that quaternions are generalization of complex numbers in $\mathbb{R}^2$ to higher dimension which is exactly $\mathbb{R}^4$. This means that quaternions can be viewed as $\mathbb{R}^4$ with certain structures similar to the case of complex numbers and $\mathbb{R}^2$.

Let $\mathbb{R}^4$ be the 4-dimensional Euclidean vector space and let

$$e_0 = (1,0,0,0),\ e_1 = (0,1,0,0),\ e_2 = (0,0,1,0) \text{ and } e_3 = (0,0,0,1)$$

be the standard orthonormal basis of $\mathbb{R}^4$. Hence, a vector $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ can be written as

$$x = x_0 e_0 + \sum_{j=1}^{3} x_j e_j$$

where $x_j's$ are scalars. Here scalars mean $\mathbb{R}$ or $\mathbb{C}$ and we will restrict our attention to $\mathbb{R}$. With the notation $\underline{x} = \sum_{j=1}^{3} x_j e_j$ we obtain that for each $x \in \mathbb{R}^4$,

$$x = x_0 e_0 + \underline{x}.$$

We call x_0 and $\underline{x}$ the real and the vector parts of x, denoted by Re$\underline{x}$ and Vec$\underline{x}$ respectively. The product of any two elements $\underline{x} = x_0 e_0 + \underline{x}$ and $\underline{y} = y_0 e_0 + \underline{y}$ in $\mathbb{R}^4$ is given by the multiplication law as:

$$\begin{aligned} xy &= (x_0 y_0 - \underline{x} \cdot \underline{y}) e_0 + \underline{x} \times \underline{y} + x_0 \underline{y} + y_0 \underline{x} \\ &= x_0 y_0 e_0 + x_0 \underline{y} + y_0 \underline{x} + \underline{xy} \end{aligned} \tag{2.4}$$

where $\underline{x} \cdot \underline{y}$ and $\underline{x} \times \underline{y}$ are the scalar(inner) and the vector(cross) products in $\mathbb{R}^3$, respectively. It is obvious to see that the product is not commutative. In this way the vector space $\mathbb{R}^4$ is furnished with the algebraic structure of a ring. It is denoted by $\mathbb{H}$ and will be named *The Quaternions.*

We shall next present some significant properties of quaternions closely related to properties of complex variables. Define the conjugate of $x = x_0 e_0 + \underline{x} \in \mathbb{H}$ by

$$\bar{x} = x_0 e_0 - \underline{x}.$$

From (2.4) it follows that

$$x \bar{x} = \bar{x} x = \sum_{j=0}^{3} x_j^2 = |x|^2.$$

and the product of two vectors $\underline{x}$ and $\underline{y}$ is given by

$$\underline{xy} = -\underline{x} \cdot \underline{y} + \underline{x} \times \underline{y}. \tag{2.5}$$

By (2.4) and (2.5), it is straightforward to see that

$$Re(x) = x_0 e_0 = \frac{1}{2}(x + \bar{x}), \; Vec(x) = \frac{1}{2}(x - \bar{x}).$$

The inverse of the quaternion $x \neq 0$ is obtained as $x^{-1} = \bar{x}|\bar{x}|^{-2}$. By using the multiplication rule (2.4), the basis quaternions fulfill the following relations:

1. $e_0^2 = 1 = e_0, e_i^2 = -1 = -e_0, \; i = 1, 2, 3,$

2. $e_i e_j + e_j e_i = 0, \; i, j = 1, 2, 3, \quad i \neq j$

3. e_0 commutes with other basis vectors.

Therefore, we can put $e_0 = 1$ which is the identity of the space. A straightforward computation leads to the identities stating that for any quaternions x and y,

1. $\overline{xy} = \overline{y}\;\overline{x},$

2. $|xy| = |yx|,$

3. $\mathrm{Re}(xy) = \mathrm{Re}(yx),$

4. -2$\underline{x} \cdot \underline{y} = \underline{x}\underline{y} + \underline{y}\underline{x}, \quad 2\underline{x} \times \underline{y} = \underline{x}\underline{y} - \underline{y}\underline{x}.$

The Euclidean space $\mathbb{R}^3$ is embedded in the Quaternions $\mathbb{H}$ by identifying a point $(x_1, x_2, x_3) \in \mathbb{R}^3$ with the vector $\underline{x} = (0, x_1, x_2, x_3)$ in $\mathbb{H}$. In other word, $\mathbb{R}^3$ is embedded in $\mathbb{R}^4$ or $\mathbb{H}$ by $0 \times \mathbb{R}^3$.

At the heart of Quaternion analysis is the *Dirac operator*, which is a direct and elegant generalization to higher dimension of the Cauchy-Riemann operator in the complex plane. This Dirac operator in $\mathbb{R}^3$ is the elliptic, rotation invariant, vector-valued differential operator of first order defined by

$$\partial_{\underline{x}} = \sum_{j=1}^{3} e_j \partial_{x_j}. \tag{2.6}$$

Its null space contains functions called the *monogenic* functions which are generalization of the holomorphic functions in complex analysis. Also, its fundamental solution is given by

$$E(\underline{x}) = -\frac{1}{a_3}\frac{\underline{x}}{|\underline{x}|^3}, \quad \underline{x} \neq 0 \tag{2.7}$$

with $a_3 = \dfrac{2\pi^{3/2}}{\Gamma(3/2)} = 4\pi$ the area of the unit sphere S^2 in $\mathbb{R}^3$. This means that

1. E is vector valued and belongs to $L_1^{loc}(\mathbb{R}^3)$
2. $\lim_{|\underline{x}|\to\infty} E(\underline{x}) = 0$
3. $\partial_{\underline{x}} E(\underline{x}) = \delta(\underline{x})$ in distributional sense, δ being the classical δ-distribution in $\mathbb{R}^3$, i.e, for each test function ϕ defined on $\mathbb{R}^3$ and with values in $\mathbb{H}$, one has $<\delta, \phi>= \phi(0)$. We see that $E(\underline{x})$ is not differentiable everywhere due to the singularity. So it makes sense to define $\partial_{\underline{x}} E(\underline{x})$ in distributional sense as the following, for any test function φ on $\mathbb{R}^3$,

$$< \partial_{\underline{x}} E(\underline{x}), \varphi >= - < E(\underline{x}), \partial_{\underline{x}}\varphi >= -E(\underline{x}) * \partial_{\underline{x}}\varphi = \varphi(0)$$

where $E(\underline{x}) * \partial_{\underline{x}}\varphi$ is defined by

$$E(\underline{x}) * \partial_{\underline{x}}\varphi = -\partial_{\underline{x}} E(\underline{x}) * \varphi = -\delta * \varphi = -\varphi(0).$$

We shall see next that what can we say about the Dirac operator of the product of two quaternionic-valued functions. Recall that in complex variables, the Cauchy-Riemann operator is defined by

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and for any complex-valued functions u and v, the operator satisfies the natural multiplication law

$$\frac{\partial}{\partial \bar{z}}(uv) = (\frac{\partial}{\partial \bar{z}}u)v + u(\frac{\partial}{\partial \bar{z}}v).$$

In particular, it follows that the product of two analytic functions is again an analytic function. Due to the non-commutativity of quaternions, the assertion cannot be true for the $\mathbb{H}$-valued functions. The generalized Leibniz rule for $\mathbb{H}$-valued functions is the following theorem:

Theorem 2.1. *For any two differentiable $\mathbb{H}$-valued functions u and v,*

$$\partial_{\underline{x}}(uv) = (\partial_{\underline{x}}u)v + \bar{u}(\partial_{\underline{x}}v) + 2Re[(u)\partial_{\underline{x}})]v. \tag{2.8}$$

For the proof see [23, 8].

As mentioned earlier we consider a quaternionic-valued function as a vector-valued function so one can denote the quaternionic-valued function with domain in $\mathbb{R}^n$ by $\mathbf{f} = f_0e_0 + \underline{\mathbf{f}}$ where $\underline{\mathbf{f}} = f_1e_1 + f_2e_2 + f_3e_3, \quad f_i(\underline{x}) \in \mathbb{R}, \; i = 1, 2, 3.$

2.3 Clifford Algebra

The Clifford algebra is a generalization of the quaternions to higher dimensions $\mathbb{R}^m$. For $m \in \mathbb{N}$ and $p, q \in \mathbb{N}_0$ such that $p + q = m$, let $\mathbb{R}^{p,q}$ be the real vector space $\mathbb{R}^m$, endowed with a non-degenerate quadratic form of signature (p, q), and let $\{e_1, e_2, \dots, e_m\}$ be an orthonormal basis for $\mathbb{R}^{p,q}$. Then the linear, real and associative *univeral Clifford algebra* $\mathbb{R}_{p,q}$ constructed over $\mathbb{R}^{p,q}$ has a non-commutative product governed by the rules:

$$\begin{aligned} e_j^2 &= 1, \quad j = 1, \dots, p \\ e_{p+j}^2 &= -1, \; j = 1, \dots, q \\ e_je_k &= -e_ke_j, \; j \neq k, \; j, k = 1, \dots, m. \end{aligned}$$

For a set $A = \{j_1, \cdots, j_h\} \subset \{1, 2, \cdots, m\} = M$ with $1 \leq j_1 < j_2 < \dots < j_h \leq m$, let $e_A = e_{j_1}e_{j_2}\dots e_{j_h}$. We put $e_0 = 1$, the latter being the identity element of the algebra. Then $(e_A : A \subseteq M)$ is a canonical basis for the 2^m-dimensional Clifford algebra $\mathbb{R}_{p,q}$. Any Clifford number $v \in \mathbb{R}_{p,q}$ may thus be written as $v = \sum_{A \subseteq M} e_Av_A$ with $v_A \in \mathbb{R}$ or as $v = \sum_{k=0}^{m}[v]_k$ where $[v]_k = \sum_{|A|=k} e_Av_A$ is the so-called *k-vector part* of v $(k = 0, 1, \cdots m)$.

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebra $\mathbb{R}_{0,m}$ by identifying the point $(x_1, x_2, \cdots, x_m)$ with the Clifford-vector variable $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} e_j x_j.$$

In particular, when $m = 3$, $\mathbb{R}_{0,3}$ has $2^3 = 8$ dimensions with the basis $\{e_0, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}$ where other notations for $e_1e_2, e_2e_3, e_1e_3, e_1e_2e_3$ are e_{12}, e_{23}, e_{23} and e_{123}, respectively. $\mathbb{H}$ coincides the 4 dimensional subspace $\{e_0, e_{12}, e_{13}, e_{23}\}$ of $\mathbb{R}_{0,3}$. The Clifford product of any two vectors $\underline{x}$ and $\underline{y}$ is given by

$$\underline{xy} = -\underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y}$$

with the first term of the right-hand side is the usual inner product in $\mathbb{R}^m$ and

$$\underline{x} \wedge \underline{y} = \sum_{j=1}^{m} \sum_{k=j+1}^{m} e_j e_k (x_j y_k - x_k y_j)$$

being the 2-vector (also called *bivector*). Similar to the case of quaternions, the Dirac operator in $\mathbb{R}^m$ is defined as

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}.$$

Its fundamental solution is given by

$$E(\underline{x}) = -\frac{1}{a_m} \frac{\underline{x}}{|\underline{x}|^m}, \quad \underline{x} \neq 0$$

where $a_m = \dfrac{2\pi^{m/2}}{\Gamma(m/2)}$ is the area of the unit sphere S^{m-1} in $\mathbb{R}^m$. Properties of $E(\underline{x})$ in $\mathbb{R}^m$ are similar to the ones in $\mathbb{R}^3$ as in (2.7). We shall focus on a Clifford valued function $\mathbf{f} : L_2(\mathbb{R}^m) \to \mathbb{R}_{0,m}$.

2.4 The Distributions in Clifford analysis

A *distribution* on $\mathbb{R}^m$ is an element of continuous dual of $C_0^\infty(\mathbb{R}^m)$. More precisely, a scalar-valued distribution u is a linear operator on $C_0^\infty(\mathbb{R}^m)$ defined by

$$u(\varphi) = \int_{\mathbb{R}^3} \varphi(x)u(x)dx, \quad \varphi \in C_0^\infty(\mathbb{R}^m).$$

We shall follow [1] to discuss about definitions, elementary properties of distributions in Clifford analysis which have scalar and vector values. These distributions will be used in a small portion in section 4.2.

2.4.1 The general spherical means operators $\Sigma^{(0)}$ and $\Sigma^{(1)}$.

Let ϕ be a scalar-valued test function defined on $\mathbb{R}^m$; putting $\underline{x} = r\underline{\omega}, r = |x|,\ \underline{\omega} \in S^{m-1}$, the authors in [1] defined the generalized spherical means

$$\Sigma^{(0)}[\phi] = \frac{1}{a_m}\int_{S^{m-1}} \phi(r\underline{\omega})dS(\underline{\omega})$$

and

$$\Sigma^{(1)}[\phi] = \sum{}^{(0)}[\underline{\omega}\phi] = \frac{1}{a_m}\int_{S^{m-1}} \underline{\omega}\phi(r\underline{\omega})dS(\underline{\omega}).$$

To discuss the behaviors of the derivatives of the spherical means at the origin $r = 0$, we introduce the constants

$$C(k) = \frac{2^{2k}k!}{(2k)!}\left(\frac{m}{2}+k-1\right)\ldots\left(\frac{m}{2}\right), \quad k \in \mathbb{N}_0$$

in order to make the formula compact. The following 2 propositions proposed in [1] are important properties of $\sum^{(0)}$ and $\sum^{(1)}$, respectively.

2.4.2 The distributions T_λ^* and U_λ^*

Let λ be a complex parameter and let ϕ be a scalar-valued test function defined on $\mathbb{R}^m$. The authors in [1] defined the scalar-valued distribution T_λ and the vector-valued distribution U_λ by:

$$\langle T_\lambda, \phi\rangle = a_m < \mathrm{Fp} r_+^z, \Sigma\ ^{(0)}[\phi] > \tag{2.9}$$

and

$$\langle U_\lambda, \phi\rangle = a_m < \mathrm{Fp} r_+^z, \Sigma\ ^{(1)}[\phi] > \tag{2.10}$$

where they have put $z = \lambda + m - 1$ and $\mathrm{Fp} r_+^z$ is a distribution finite part on the real line see [1]. Both families of distributions inherit an infinite sequence of singular points from $\mathrm{Fp} r_+^z$, namely, $z = -n, n \in \mathbb{N}$. However, by the propositions 2.2 and 2.3 we notice that half of those singularities disappear. More precisely, for $k \in \mathbb{N}_0$ the residue for $\lambda = -m - 2k - 1$ of (2.9) is

$$a_m \left\langle \mathrm{Res}_{z=-2k-2} \mathrm{Fp} r_+^z, \Sigma\ ^{(0)}[\phi] \right\rangle = a_m \left\langle \frac{(-1)^{2k+1}}{(2k+1)!} \delta^{(2k+1)}(r), \Sigma\ ^{(0)}[\phi] \right\rangle = 0$$

while the residue for $\lambda = -m - 2k$ of the second one yields

$$a_m \left\langle \mathrm{Res}_{\mu=-2k-1} \mathrm{Fp} r_+^z, \Sigma\ ^{(1)}[\phi] \right\rangle = a_m \left\langle \frac{(-1)^{2k}}{(2k)!} \delta^{(2k)}(r), \Sigma\ ^{(1)}[\phi] \right\rangle = 0.$$

In [1] use the well-known technique which is the method of dividing an appropriate Gamma-function to show the distributions can be normalized as the following formulas:

$$\begin{aligned} T_\lambda^* &= \pi^{\frac{\lambda+m}{2}} \frac{T_\lambda}{\Gamma(\frac{\lambda+m}{2})}, && \lambda \neq -m - 2k \\ T_{-m-2k}^* &= \frac{\pi^{\frac{m}{2}-k}}{2^{2k}\Gamma(\frac{m}{2}+k)} (-\triangle)^k \delta(\underline{x}), && k \in \mathbb{N}_0 \end{aligned}$$

and

$$\begin{aligned} U_\lambda^* &= \pi^{\frac{\lambda+m+1}{2}} \frac{U_\lambda}{\Gamma(\frac{\lambda+m+1}{2})}, && \lambda \neq -m - 2k \\ U_{-m-2k-1}^* &= -\frac{\pi^{\frac{m}{2}-k}}{2^{2k+1}\Gamma(\frac{m}{2}+k+1)} \partial_{\underline{x}}^{2k+1} \delta(\underline{x}), && k \in \mathbb{N}_0 \end{aligned}$$

For the last part of this section we shall briefly talk about that the Riesz potentials I^α introduced in [17] can be written as one of these distribution. For a complex parameter

α and scalar valued rapidly decreasing function f defined in $\mathbb{R}^m$, they are defined by

$$\begin{aligned} I^\alpha[f](\underline{y}) &= \frac{1}{2^\alpha \pi^{m/2}} \frac{\Gamma((m-\alpha)/2)}{\Gamma(\alpha/2)} \mathrm{Fp} \int_{\mathbb{R}^m} |\underline{x}-\underline{y}|^{\alpha-m} f(\underline{x}) d\underline{x} \\ &= \frac{1}{2^\alpha \pi^{m/2}} \frac{\Gamma((m-\alpha)/2)}{\Gamma(\alpha/2)} (\mathrm{Fp}|\underline{x}|^{\alpha-m} * f(\underline{x}))(\underline{y}). \end{aligned}$$

2.5 The Clifford-Hilbert Transform on $\mathbb{R}^3$

In this section, we shall present a vector-valued Hilbert transform on $\mathbb{R}^3$ using Clifford algebra which is a generalization of a one-dimensional Hilbert transform called Clifford-Hilbert transform. The first part of this section shall give its definition and some properties. The second part of this section shall introduce the Cauchy kernel and the Cauchy transform of a function $\mathbf{f} = f_0 + \underline{\mathbf{f}} \in L_2(\mathbb{R}^3)$. A significant property of the Cauchy transform, for example the differences of the non-tangential boundary limits, leads to the reconstruction formula of $\underline{\mathbf{f}}$ by using the quaternionic x-ray and Radon transforms which we shall discuss in Chapter 3.

2.5.1 Definition and properties

We introduce two open subspaces of $\mathbb{R}^4$: the upper and lower half spaces $\mathbb{R}^4_\pm$, respectively given by

$$\begin{aligned} \mathbb{R}^4_+ &= \{x = (x_0, \underline{x}) \in \mathbb{R}^4 : x_0 > 0\} \\ \mathbb{R}^4_- &= \{x = (x_0, \underline{x}) \in \mathbb{R}^4 : x_0 < 0\}. \end{aligned}$$

Identifying the Euclidean space $\mathbb{R}^3$ with the hyperplane $x_0 = 0$ in $\mathbb{R}^4$, we obtain that the boundaries of the spaces $\mathbb{R}^4_\pm$ are given by $\partial\mathbb{R}^4_+ = \partial\mathbb{R}^4_- = \mathbb{R}^3$.

Now, let $\mathbf{f} \in L_2(\mathbb{R}^3)$; F. Sommen [1] provided the definition of *the quaternionic-*

Hilbert transform $\mathcal{H}[\mathbf{f}]$ of $\mathbf{f}$ on $\mathbb{R}^3$ which was defined by

$$\mathcal{H}[\mathbf{f}](\underline{x}) = \frac{2}{a_4} P.V \int_{\mathbb{R}^3} \frac{\bar{\underline{x}} - \bar{\underline{y}}}{|\underline{x} - \underline{y}|^4} \mathbf{f}(\underline{y}) d\underline{y}$$

$$= \frac{2}{a_4} \lim_{\epsilon \to 0+} \int_{\mathbb{R}^3 \setminus B(x,\epsilon)} \frac{\bar{\underline{x}} - \bar{\underline{y}}}{|\underline{x} - \underline{y}|^4} \mathbf{f}(\underline{y}) d\underline{y} \qquad (2.11)$$

or for an appropriate distribution $\mathbf{f}$, by means of the convolution

$$\mathcal{H}[\mathbf{f}](\underline{x}) = (H * \mathbf{f})(\underline{x})$$

with H the convolution kernel given by the distribution

$$H(\underline{x}) = \frac{2}{a_4} P.V \frac{\bar{\underline{w}}}{r^3} = -\frac{2}{a_4} U^*_{-3,0}.$$

Similar to the Hilbert transform in the real line, the Hilbert transform (2.11) then satisfies the following properties. Again by a_m we denote the surface area of the unit sphere S^{m-1}.

Property 1 *The Hilbert transform is a convolution operator, which is equivalent with saying that the Hilbert transform commutes with translations, i.e,*

$$\tau_{\underline{a}}[\mathcal{H}\mathbf{f}] = \mathcal{H}[\tau_{\underline{a}}[\mathbf{f}]]$$

with $\tau_{\underline{a}}[\mathbf{f}]] = \mathbf{f}(\underline{x} - \underline{a})$, $\underline{a} \in \mathbb{R}^3$.

Property 2 *The Hilbert kernel H is a homogeneous distribution of degree -3, which, for a convolution operator, is equivalent saying that the Hilbert transform commutes with dilations,i.e.,*

$$d_a[\mathcal{H}[\mathbf{f}]] = \mathcal{H}[d_a[\mathbf{f}]]$$

with $d_a[\mathbf{f}](\underline{x}) = \frac{1}{a^{3/2}} \mathbf{f}\left(\frac{\underline{x}}{a}\right), \quad a > 0.$

Property 3 *The Hilbert transform is a bounded linear operator on $L_2(\mathbb{R}^3)$ and it is a norm preserving,i.e.,*

$$\|\mathcal{H}[\mathbf{f}]\|_{L_2} = \|\mathbf{f}\|_{L_2}.$$

More generally, it also preserves the inner product

$$< \mathcal{H}[\mathbf{f}], \mathcal{H}[\underline{\mathbf{g}}] >=< \mathbf{f}, \underline{\mathbf{g}} > .$$

Property 4 *The Hilbert transform* $\mathcal{H} : L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)$ *is an involution, i.e., it is invertible with* $\mathcal{H}^{-1} = \mathcal{H}$.

Property 5 *The Hilbert transform* $\mathcal{H} : L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)$ *is unitary, its adjoint being given by* $\mathcal{H}^* = \mathcal{H}$, *i.e.,*

$$< \mathcal{H}[\mathbf{f}], \mathbf{g} >=< \mathbf{f}, \mathcal{H}[\mathbf{g}] > \quad \mathbf{f}, \mathbf{g} \in L_2(\mathbb{R}^3).$$

Property 6 *The Hilbert transform anti-commutes with the Dirac operator, i.e., if* $\mathbf{f}$. *And* $\partial_{\underline{x}}\mathbf{f}$ *are in* $L_2(\mathbb{R}^3)$ or if $\mathbf{f}$ *is an appropriate distribution, then*

$$\mathcal{H}[\partial_{\underline{x}}\mathbf{f}(\underline{x})](\underline{y}) = -\partial_{\underline{y}}[\mathcal{H}[\mathbf{f}](\underline{y})].$$

2.5.2 The Cauchy Kernel and the Cauchy Transform

F. Sommen [1] defined *The Cauchy Kernel*

$$C(x) = C(x_0, \underline{x}) = \frac{1}{a_4}\frac{\bar{x}}{|x|^4} = \frac{1}{a_4}\frac{x_0 - \underline{x}}{|x_0 - \underline{x}|^4}, x \neq 0. \tag{2.12}$$

to be the fundamental solution of the Dirac operator ∂_x in $\mathbb{R}^4$ being given by

$$\partial_x = \partial_{x_0} + \partial_{\underline{x}}.$$

This means that

1. $C(x)$ is ∂_x-monogenic in $\mathbb{R}^4 \setminus \{0\}$

2. $\lim_{|x| \to +\infty} C(x) = 0$

3. $\partial_x C(x) = \delta(x)$ in distributional sense.

Let $\mathbf{f} \in L_2(\mathbb{R}^3)$, the *Cauchy integral* $\mathcal{C}[\mathbf{f}]$ is defined in $\mathbb{R}^4 \setminus \mathbb{R}^3$ by

$$\mathcal{C}[\mathbf{f}](x_0, \underline{x}) = (C(x_0, \cdot) * \mathbf{f}(\cdot))(\underline{x}) = \int_{\mathbb{R}^3} C(x_0, \underline{x} - \underline{y})\mathbf{f}(\underline{y})d\underline{y}$$

where $C(x)$ is the Cauchy Kernel (2.12).

On account of those properties, it may be clear that the Cauchy integral is ∂_x - monogenic in both the upper half space $\mathbb{R}^4_+$ and the lower half space $\mathbb{R}^4_-$. Further, for a function $\mathbf{f} \in L_2(\mathbb{R}^3)$, taking the supremun either in $\mathbb{R}^4_+$ or $\mathbb{R}^4_-$ we also have that

$$\sup_{x_0 \geq 0, x_0 \leq 0} \int_{\mathbb{R}^3} |\mathcal{C}[\mathbf{f}](x_0, \underline{x})|^2 d\underline{x} < +\infty.$$

The recovering of $\underline{\mathbf{f}}$ proposed in [28, 30] made use of the non-tangential boundary limits of the Cauchy integral $\mathcal{C}[\mathbf{f}]$. Then we have the so-called Plemelj-Sokhotzki formulae.

Theorem 2.2 (Plemelj-Sokhotzki formulae)**.** *Let* $\mathbf{f} \in L_2(\mathbb{R}^3)$*, then the non-tangential boundary limits of the Cauchy integral* $\mathcal{C}[\mathbf{f}]$ *are given by*

$$\mathcal{C}^+[\mathbf{f}](\underline{x}) = \lim_{x_0 \to 0+} \mathcal{C}[\mathbf{f}](x_0, \underline{x}) = \frac{1}{2}\mathbf{f}(\underline{x}) + \frac{1}{2}\mathcal{H}[\mathbf{f}](\underline{x})$$

$$\mathcal{C}^-[\mathbf{f}](\underline{x}) = \lim_{x_0 \to 0-} \mathcal{C}[\mathbf{f}](x_0, \underline{x}) = -\frac{1}{2}\mathbf{f}(\underline{x}) + \frac{1}{2}\mathcal{H}[\mathbf{f}](\underline{x})$$

For a function $\mathbf{f} \in L_2(\mathbb{R}^3)$, we then call $\mathcal{C}^+[\mathbf{f}]$ and $\mathcal{C}^-[\mathbf{f}]$ its Cauchy transforms and they satisfy the following properties.

1. $\mathcal{C}^+$ and $\mathcal{C}^-$ are bounded linear operators on $L_2(\mathbb{R}^3)$

2. $\mathbf{f} = \mathcal{C}^+[\mathbf{f}] - \mathcal{C}^-[\mathbf{f}]$ and $\mathcal{H}[\mathbf{f}] = \mathcal{C}^+[\mathbf{f}] + \mathcal{C}^-[\mathbf{f}]$

3. $\mathcal{C}^+$ and $\mathcal{C}^-$ are orthogonal , i.e., $< \mathcal{C}^+[\mathbf{f}], \mathcal{C}^-[\mathbf{f}] >= 0$.

2.6 Plane wave decomposition

A plane wave is a function of scalar product $< \underline{x}, \underline{\theta} >$, $\underline{x}, \underline{\theta} \in \mathbb{R}^m$. In [28] F. Sommen has shown that the fundamental solution of the Dirac operator $\partial_x = \partial_{x_0} + \partial_{\underline{x}}$ in $\mathbb{R}^{m+1}$ admits the following plane wave decompositions :

for m even, we have that in $\mathbb{R}^{m+1}_\pm$

$$\frac{1}{a_{m+1}} \frac{x_0 - \underline{x}}{|x_0 - \underline{x}|^{m+1}} = \pm \frac{(-1)^{m/2}(m-1)!}{2(2\pi)^m} \int_{S^{m-1}} (< \underline{x}, \underline{\theta} > -x_0 \underline{\theta})^{-m} d\underline{\theta} \ ,$$

whereas for m odd,

$$\frac{1}{a_{m+1}}\frac{x_0-\underline{x}}{|x_0-\underline{x}|^{m+1}} = \pm\frac{(-1)^{(m+1)/2}(m-1)!}{2(2\pi)^m}\int_{S^{m-1}}(<\underline{x},\underline{\theta}> -x_0\underline{\theta})^{-m}\underline{\theta}d\underline{\theta}. \qquad (2.13)$$

In particular for $m=3$, the decomposition reduces to

$$\frac{x_0-\underline{x}}{|x_0-\underline{x}|^4} = \pm\frac{1}{4\pi}\int_{S^2}(<\underline{x},\underline{\theta}> -x_0\underline{\theta})^{-3}\underline{\theta}d\underline{\theta}. \qquad (2.14)$$

In Chapter 3, we shall see that the decomposition (2.13) will lead to the x-ray and the Radon decompositions giving rise to the reconstruction of a quaternionic-valued function. For the proof of decomposition (2.13) and (2.14) we will refer to [28, 30, 1] by F. Sommen.

Let $\alpha \in \mathbb{N}$. The distributional boundary values of the plane wave powers, $(<\underline{x},\underline{\theta}> -0^+\underline{\theta})^\alpha$ and $(<\underline{x},\underline{\theta}> -0^-\underline{\theta})^\alpha$, defined in $\mathbb{R}^m$, are the limits of $(<\underline{x},\underline{\theta}> -x_0\underline{\theta})^\alpha$ when $x_0 \to \pm 0$ in the distributional sense as

$$\lim_{x_0\to\pm 0}\int <(<\underline{x},\underline{\theta}> -x_0\underline{\theta})^\alpha, \phi > d\underline{x} =<(<\underline{x},\underline{\theta}> -0^\pm\underline{\theta})^\alpha, \phi >$$

where ϕ is a Clifford-valued compactly supported function on $\mathbb{R}^n$. We will read $(<\underline{x},\underline{t}> -0^\pm\underline{t})^\alpha$ the limits of the plane wave generalized powers from the upper and the lower half spaces, respectively. The following lemma is one of important properties that we shall use for the reconstruction formula of a function $\mathbf{f}$.

Lemma 2.3. *For any positive integer m, we have, for $\underline{x},\underline{\theta}\in\mathbb{R}^m$,*

$$(<\underline{x},\underline{\theta}> -0^+\underline{\theta})^{-(m-1)} - (<\underline{x},\underline{\theta}> -0^-\underline{\theta})^{-(m-1)} = 2\pi\underline{\theta}\frac{(-1)^{m-2}}{(m-2)!}\delta^{m-2}(<\underline{x},\underline{\theta}>). \quad (2.15)$$

For the proof we refer to [29].

2.7 The Quaternionic Radon Transform

As seen in section 2.1, we have defined the Radon transform of a vector-valued function $\mathbf{f}$, with values in $\mathbb{R}^3$. We shall next focus on a Clifford-valued function, with values in $\mathbb{R}_{0,3}$ and introduce a new type of Radon transform as seen in [28, 30], called the Clifford Radon transform.

Definition 2.4. *Let* $\underline{\theta} \in S^2$ *and* $x_0 \neq 0$. *Then the Clifford Radon transform of* $\mathbf{f} \in \mathcal{S}(\mathbb{R}^3)$ *is given by*

$$\mathcal{R}(\mathbf{f})(s, \underline{\theta}, x_0) = \frac{1}{2\pi} \int_{\mathbb{R}^3} (< \underline{u}, \underline{\theta} > -s + x_0 \underline{\theta})^{-1} \underline{\theta} \mathbf{f}(\underline{u}) d\underline{u}. \tag{2.16}$$

The Radon transform is a monogenic function as the following proposition

Proposition 2.5. *If* $x_0 \neq 0$, $R(\varphi)(s, \underline{\theta}, x_0)$ *satisfies the Cauchy Riemann system*

$$(\frac{\partial}{\partial x_0} + \partial_{\underline{x}})(\mathcal{R}(\varphi)(s, \underline{\theta}, x_0)) = 0.$$

Here we use notation $\mathcal{R}$ for the quaternionic-valued Radon transform to distinguish from the scalar-valued one denoted by R. The integral make sense because we assume that x_0 is non zero making the integrand is an integrable function.

In [28, 30], F. Sommen showed that the *classical Radon transform* $\mathcal{R}\mathbf{f}$ defined in $\mathbb{R}^3$ which was the transform

$$\mathcal{R}(\mathbf{f})(s, \underline{\theta}) = \int_{\mathbb{R}^3} \delta(< \underline{u}, \underline{\theta} > - < \underline{x}, \underline{\theta} >) \mathbf{f}(\underline{u}) d\underline{u}, \tag{2.17}$$

was the boundary value of the quaternion-valued Radon transform defined in $\mathbb{R}^4$. More precisely, by taking the non-tangential boundary limits and applying lemma (2.3) we then have

$$\begin{aligned}
\mathcal{R}(\mathbf{f})(s, \underline{\theta}) &= \mathcal{R}(\mathbf{f})(s, \underline{\theta}, 0^+) - \mathcal{R}(\mathbf{f})(s, \underline{\theta}, 0^-) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^3} \left((< \underline{x} - \underline{u}, \underline{\theta} > -0^+ \underline{\theta})^{-1} - (< \underline{x} - \underline{u}, \underline{\theta} > -0^- \underline{\theta})^{-1} \right) \underline{\theta} \mathbf{f}(\underline{u}) d\underline{u} \\
&= \frac{1}{2\pi} (2\pi) \int_{\mathbb{R}^3} \delta(< \underline{u}, \underline{\theta} > - < \underline{x}, \underline{\theta} >) \mathbf{f}(\underline{u}) d\underline{u}.
\end{aligned}$$

In particular, the vector-valued Radon transform will be

$$\mathcal{R}(\mathbf{f})(\underline{x} \cdot \underline{\theta}, \underline{\theta}) = \int_{\mathbb{R}^3} \delta(< \underline{u}, \underline{\theta} > - < \underline{x}, \underline{\theta} >) \mathbf{f}(\underline{u}) d\underline{u}. \tag{2.18}$$

In section 4.1 we shall provide a different proof that the Cauchy transform of **f** can be written as the following

$$\mathcal{C}[\mathbf{f}](x_0+x) = \frac{-1}{8\pi}\partial_{\underline{x}}^2 \int_{S^2} (<\underline{x},\underline{\theta}> - x_0\underline{\theta})^{-1}\underline{\theta}d\underline{\theta} \tag{2.19}$$

$$\mathcal{C}[\mathbf{f}](x_0+x) = -\frac{1}{8\pi} \int_{S^2} \partial_{\underline{x}}^2 (<\underline{x},\underline{\theta}> - x_0\underline{t})^{-1}\underline{\theta}d\underline{\theta}. \tag{2.20}$$

The reconstruction of **f** by using the Radon transform is contained by taking the non-tangential boundary of its Cauchy transform as follows:

$$\begin{aligned}\mathbf{f}(\underline{x}) &= \mathcal{C}[\mathbf{f}](\underline{x}+0) - \mathcal{C}(\mathbf{f})(\underline{x}-0)\\ &= \frac{-1}{8\pi}\partial_{\underline{x}}^2 \int_{S^2} (\mathcal{R}\mathbf{f}(0^+,\underline{x},\underline{\theta}) - \mathcal{R}\mathbf{f}(0^-,\underline{x},\underline{\theta}))d\theta\\ &= -\frac{1}{8\pi}\partial_{\underline{x}}^2 \int_{S^2} \mathcal{R}\mathbf{f}(\underline{\theta}\cdot\underline{x},\underline{\theta})d\underline{\theta}. \end{aligned} \tag{2.21}$$

2.8 The Quaternionic X-Ray transform

Definition 2.6. *Let $\underline{\theta} \in S^2$ and $x_0 \neq 0$. Then the quaternionic x-ray transform or the x-ray type transform of $\mathbf{f} \in \mathcal{S}(\mathbb{R}^3)$ is given by*

$$\mathcal{X}(\mathbf{f})(x_0,\underline{x},\underline{\theta}) = \frac{-1}{a_3}\int_{\mathbb{R}^3} \frac{x_0\underline{\theta} - (\underline{x}-\underline{u})\times\underline{\theta}}{|x_0\underline{\theta} - (\underline{x}-\underline{u})\times\underline{\theta}|^3}\underline{\theta}\mathbf{f}(\underline{u})d\underline{u}. \tag{2.22}$$

Similar to the case of the quaternionic-valued Radon transform we shall use the notation $\mathcal{X}$ to distinguish between the quaternionic-valued x-ray transform and the scalar-valued one denoted by X. With the integral (2.2), Sommen [28] shows that

$$X\mathbf{f}(\underline{x},\underline{\theta}) = \mathcal{X}\mathbf{f}(0^+,\underline{x},\underline{\theta}) - \mathcal{X}\mathbf{f}(0^-,\underline{x},\underline{\theta}).$$

and that

$$\begin{aligned}X\mathbf{f}(\underline{x},\underline{\theta}) &= \mathcal{X}\mathbf{f}(0^+,\underline{x},\underline{\theta}) - \mathcal{X}\mathbf{f}(0^-,\underline{x},\underline{\theta})\\ &= -\frac{1}{a_3}\int_{\mathbb{R}^3} \frac{(\underline{u}-\underline{x})\times\underline{\theta}}{|(\underline{u}-\underline{x})\times\underline{\theta}|^3}\underline{\theta}\mathbf{f}(\underline{u})d\underline{u}\\ &= -\frac{1}{a_3}\int_{\underline{\theta}^\perp} \frac{-(E_{\underline{\theta}}\underline{x}-u')}{|E_{\underline{\theta}}x-u'|^3}\mathcal{X}\mathbf{f}(u',\underline{\theta})du', \end{aligned} \tag{2.23}$$

where $E_{\underline{\theta}}\underline{x}$ is the orthogonal projection of $\underline{x}$ on the plane $\underline{\theta}^{\perp}$.

The next proposition tells that the Cauchy transform of a Clifford-valued function can be written in term of back projection of the x-ray transform. This proposition and the proof were proposed in [28, 30] and we just add more details in the existing proof.

Proposition 2.7. *Let* $\mathbf{f} \in \mathcal{S}(\mathbb{R}^3)$*. Then we have* $x \in \mathbb{R}^4_{\pm}$ *,*

$$\mathcal{C}(\mathbf{f})(x_0 + \underline{x}) = \frac{1}{2(2\pi)^3}\partial_{\underline{x}} \int_{S^2} \mathcal{X}\mathbf{f}(x_0, \underline{x}, \underline{\theta})d\underline{\theta}. \tag{2.24}$$

Proof. Using the following x-ray decomposition of the Cauchy kernel introduced by F. Sommen [30], for which we shall provide details in chapter 3 section 3.1,

$$\frac{x_0 - \underline{x}}{|x_0 - \underline{x}|^4} = \frac{1}{8\pi}(\text{sgn}x_0)\partial_{\underline{x}} \int_{S^2} \frac{x_0\underline{\theta} - \underline{x} \times \underline{\theta}}{|x_0\underline{\theta} - \underline{x} \times \underline{\theta}|^3}\underline{\theta}d\underline{\theta}$$

we have

$$\begin{aligned}
\mathcal{C}(\mathbf{f})(x_0 + \underline{x}) &= \frac{1}{a_4}\int_{\mathbb{R}^3} \frac{x_0 - \underline{x} + \underline{u}}{|x_0 - \underline{x} + \underline{u}|^4}\mathbf{f}(\underline{u})d\underline{u} \\
&= \int_{\mathbb{R}^3} \frac{(\text{sgn}x_0)\partial_{\underline{x}}}{2\pi a_3 a_2} \int_{S^2} \frac{x_0\underline{\theta} - (\underline{x} - \underline{u}) \times \underline{\theta}}{|x_0\underline{\theta} - (\underline{x} - \underline{u}) \times \underline{\theta}|^3}\underline{\theta}d\underline{\theta}\mathbf{f}(\underline{u})d\underline{u} \\
&= -\frac{1}{2(2\pi)^3}(\text{sgn}x_0)\partial_{\underline{x}} \int_{S^2}\int_{\mathbb{R}^3} \frac{x_0\underline{\theta} - (\underline{x} - \underline{u}) \times \underline{\theta}}{|x_0\underline{\theta} - (\underline{x} - \underline{u}) \times \underline{\theta}|^3}\underline{\theta}\mathbf{f}(\underline{u})d\underline{u} \\
&= \frac{1}{2(2\pi)^3}(\text{sgn}x_0)\partial_{\underline{x}} \int_{S^2} \mathcal{X}\mathbf{f}(x_0, \underline{x}, \underline{\theta})d\underline{\theta}.
\end{aligned}$$

□

Then $\mathbf{f}$ is obtained from the non-tangential boundary values of the Cauchy transform as follows:

$$\begin{aligned}
\mathbf{f}(\underline{x}) &= \mathcal{C}[\mathbf{f}](\underline{x} + 0) - \mathcal{C}[\mathbf{f}](\underline{x} - 0) \\
&= \frac{1}{2(2\pi)^3}\partial_{\underline{x}} \int_{S^2} ((\text{sgn}x_0)\mathcal{X}\mathbf{f}(0^+, \underline{x}, \underline{\theta}) - (\text{sgn}x_0)\mathcal{X}\mathbf{f}(0^-, \underline{x}, \underline{\theta}))d\underline{\theta} \\
&= \frac{1}{2(2\pi)^3}\partial_{\underline{x}} \int_{S^2}\int_{\mathbb{R}^3} \frac{(\underline{u} - \underline{x}) \times \underline{\theta}}{|(\underline{u} - \underline{x}) \times \underline{\theta}|^3}\underline{\theta}\mathbf{f}(\underline{u})d\underline{u}d\underline{\theta} \\
&= \frac{1}{2(2\pi)^3}\partial_{\underline{x}} \int_{S^2}\int_{\underline{\theta}^{\perp}} \frac{-(E_{\theta}x - u')}{|E_{\theta}x - u'|^3}\mathcal{X}\mathbf{f}(u', \theta)du'd\underline{\theta}.
\end{aligned}$$

2.9 Helmholtz-Hodge decomposition of a vector field f

We shall remind the reader of the well-known Helmholtz-Hodge decomposition of a vector field by following [14, 13]. Denote B^3 by the unit ball in $\mathbb{R}^3$. First of all, we will recall the following subspaces of $L^2(B^3)$: let

$$H_0^1(B^3) = \{p : H^1(B^3) \to \mathbb{C} \mid p = 0 \text{ on } S^2\}$$

be the subspace of $H^1(B^3)$ with elements vanish on the boundary of the unit ball. The norm of a scalar-valued function f in H^1 is defined by

$$\|f\|_{H^1} = \left(\int_{B^3} (|f|^2 + |\nabla f|^2)\right)^{1/2}.$$

Denote

$$\nabla H_0^1(B^3) = \{\nabla p \mid p \in H_0^1(B^3)\}$$

the space of potential fields with potentials that vanish on the boundary of the unit ball,

$$H(\text{div}; B^3) = \{\mathbf{f} \in L^2(B^3) \mid \ <\mathbf{f}, \nabla p>= 0 \text{ for all } p \in C_0^\infty(\mathbb{B}^3)\}$$

the space of divergence-free vector fields with the subspace

$$H_0(\text{div}; B^3) = \{\mathbf{f} \in H(\text{div}; B^3) \mid \xi \cdot \mathbf{f}(\xi) = 0, \xi \in \partial(B^3)\}$$

consisting of those vector fields in $H(\text{div}; B^3)$ that are tangential to the boundary and finally the space of harmonic fields $\nabla \mathrm{H}(B^3)$ where

$$\mathrm{H}(B^3) = \{h \in H^1(B^3) : \int_{B^3} \nabla h(x) \cdot \nabla p(x) dx = 0\} \text{ for all } p \in C_0^\infty(B^3)$$

is the space of harmonic functions. The Helmholtz-Hodge decomposition of a vector field is the following theorem:

Theorem 2.8. *Every vector field* $\mathbf{f} \in L^2(B^3)$ *can be uniquely decomposed into the form*

$$\mathbf{f} = \nabla p + \mathbf{f}_0^d + \mathbf{f}^h, \tag{2.25}$$

where the first part ∇p is called the irrotational part of $\mathbf{f}$*, the second and the third parts* $\mathbf{f}^d = \mathbf{f}_0^d + \mathbf{f}^h$ *are called solenoidal part with* $\mathbf{f}_0^d$ *has a tangential flow at the boundary* $\partial B^3 = S^2$ *and a harmonic field* $\mathbf{f}^h = \nabla h$ *with a harmonic function* h*. In direct sum notations, the space of square integrable vector fields* $L_2(B^3)$ *is an orthogonal sum of three subspaces*

$$L_2(B^3) = \nabla H_0^1(B^3) \oplus H_0(div; B^3) \oplus \nabla H(B^3).$$

Here ∇ denote the gradient in $\mathbb{R}^3$, respectively.

2.10 Surface Differential Operators

In this section we shall present the definition of the surface divergence operator $\text{div}_{\underline{\eta}}$ as a dual of a surface gradient operator $\nabla_{\underline{\eta}}$ where $\underline{\eta} \in S^2$. This definition is global meaning that it is independent of local coordinates. We shall provide some calculations to show how this global definition works for spherical coordinates on S^2,

Definition 2.9. *The surface divergence on* S^2 *is defined by*

$$\int_{S^2} f\, div_{\underline{\eta}} v dA = -\int_{S^2} \nabla_{\underline{\eta}} f \cdot v dA \ , f \in C_0^1(S^2) \tag{2.26}$$

where v *is tangential vector field. That is, for each* $\underline{\eta} \in S^2$*,* $v(\underline{\eta}) \in \underline{\eta}^{\perp}$*.*

We recall that the spherical coordinate for $\underline{x} \in \mathbb{R}^3$ is defined by

$$\underline{x} = r\underline{\eta},\ r > 0,\ \underline{\eta} = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix},\ \varphi \in [0, 2\pi], \theta \in (0, \pi) \tag{2.27}$$

We claim that $\text{div}_{\underline{\eta}}$ has the following expression:

$$\text{div}_{\underline{\eta}} v = \frac{1}{\sin\theta} \left(\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{\partial}{\partial\varphi} v_\varphi \right) \tag{2.28}$$

where $\underline{\eta} = \underline{\eta}(\varphi, \theta)$ is defined in (2.27) and

$$v = v_\varphi e_1(\underline{\eta}) + v_\theta e_2(\underline{\eta}) \tag{2.29}$$

is a tangential vector on S^2, supported away from poles. Here $e_1(\underline{\eta})$ and $e_2(\underline{\eta})$ are two orthogonal vectors spanning the tangent plane $\underline{\eta}^\perp$,

$$e_1(\underline{\eta}) = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\underline{\eta} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \quad e_2(\underline{\eta}) = \frac{\partial}{\partial\theta}\underline{\eta} = \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ -\sin\theta \end{pmatrix}$$

The definition of *surface gradient* of a scalar function $f(\underline{\eta})$ on the sphere S^2 was reviewed in [14] by Schuster and Kazantsev denoted by $\nabla_{\underline{\eta}} f(\underline{\eta})$ and defined by

$$\nabla_{\underline{\eta}} f = \frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}e_1(\underline{\eta}) + \frac{\partial f}{\partial\theta}e_2(\underline{\eta}). \tag{2.30}$$

By using (2.26) and (2.29) and the formula for the integration on the sphere,

$$\int_{S^2} f(\alpha)dS(\alpha) = \int_0^\pi \int_0^{2\pi} f(\varphi,\theta)\sin\theta d\varphi d\theta, \tag{2.31}$$

the right hand side of (2.26) in these coordinates becomes

$$-\int_0^\pi \int_0^{2\pi} \left(\frac{\partial f}{\partial\varphi}\frac{1}{\sin\theta}v_\varphi + \frac{\partial f}{\partial\theta}v_\theta\right)\sin\theta d\varphi d\theta. \tag{2.32}$$

Integrating by parts in (2.32), the integral becomes

$$\int_0^\pi \int_0^{2\pi} f\frac{\partial v_\varphi}{\partial\varphi}d\varphi d\theta + \int_0^\pi \int_0^{2\pi} f\frac{\partial}{\partial\theta}(v_\theta\sin\theta)d\theta d\varphi \tag{2.33}$$

where we have used the periodicity of f. Simplifying (2.33) gives

$$\int_0^\pi \int_0^{2\pi} f\frac{1}{\sin\theta}\left(\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right)\sin\theta d\varphi d\theta. \tag{2.34}$$

Consequently, by (2.26) and (2.34) we have verified the claim (2.28).

3 METHODS AND CONSTRUCTIONS

We shall begin this chapter by introducing x-ray and Radon decompositions of the Cauchy kernel where the Cauchy kernel means the fundamental solution of the Dirac operator. These decompositions were first introduced by Sommen [28, 30]. As in section 2.7 and 2.8 these give reconstruction formulas for a quaternionic-valued function. We briefly describe and follow notations in [28, 30]. Furthermore, in this section we shall give a proof of important expression leading to the x-ray decomposition of a Cauchy kernel. To do so, we shall define the Dirac operator in a plane, say $\partial_{x_{\parallel}}$ where $\underline{x} = x_{\parallel} + \underline{\theta} < \underline{x}, \underline{\theta} >$, $x_{\parallel}$ is on the plane and $\underline{\theta}$ is a unit vector normal to the plane.

In section 2 we define the quaternionic-Doppler transform which has real and vector parts. The real part corresponds to the scalar Doppler transform which already exists and widely used for many authors for reconstruction of vector fields [11, 12, 10, 4]. One of the reasons why we are interested in this transform is to see how it contributes the reconstruction formula for the vector part $\underline{\mathbf{f}}$ of a quaternionic-valued function $\mathbf{f}$. We shall investigate this question in chapter 4.

Section 3 shall present cone-beam transform of a quaternionic-valued function $\mathbf{f}$ which is defined componentwise of the scalar one. The Grangeat formula will be mentioned again in componentwise sense. We furthermore, shall present notations, properties and conditions for a curve trajectory already described in [13] which we will use in Chapter 4. The vector part is a new term that we shall introduce in this work.

In section 4 we shall follow outline in [14] to describe the derivations of series expansions of a vector field in $L_2(B^3)$. The first part shall give its orthogonal expansions and the second part will present the orthogonal series expansion of its solenoidal part. We also mention that a vector field can be viewed as a vector part $\underline{\mathbf{f}}$ of a quaternionic-function $\mathbf{f}$ and discuss that these expansions work for $\underline{\mathbf{f}}$ as well.

3.1 Radon and X-ray decompositions of the Cauchy Kernel

As in chapter 2 section 2.6, we see that the fundamental solution of the Dirac operator (Cauchy kernel) can be decomposed as a plane wave integral and this leads to the Radon and x-ray decompositions. The Radon decomposition was introduced and explained in [28, 30] and the x-ray decomposition in [30]. With the Radon decomposition, the reconstruction of a quaternion-valued function coincides with the classical one via the filter back-projection of the Radon transform. The x-ray decomposition gives an inversion formula for the vector part of a quaternionic-valued function with the Dirac operator form instead of the lambda operator. Making use of the generalized Leibniz rule (2.8), we obtain that, for $\underline{x} \in \mathbb{R}^3, \underline{\theta} \in S^2$,
for m odd

$$\partial_{\underline{x}}^{m-1}(< \underline{x}, \underline{\theta} > -x_0\underline{\theta})^{-1} = (m-1)!(-1)^{(m-1)/2}(< \underline{x}, \underline{\theta} > -x_0\underline{\theta})^{-m}, \tag{3.1}$$

and for m even

$$\partial_{\underline{x}}^{m-1}(< \underline{x}, \underline{\theta} > -x_0\underline{\theta})^{-1} = (m-1)!(-1)^{m/2}\underline{\theta}(< \underline{x}, \underline{\theta} > -x_0\underline{\theta})^{-m}.$$

For dimension 3, by using the plane wave decomposition (2.14) and the equation (3.1), one obtains the *Radon decomposition* of the Cauchy Kernel given by the following theorem:

Theorem 3.1. *For $x = x_0 + \underline{x}$, $x_0 \neq 0$,*

$$\frac{x_0 - \underline{x}}{|x_0 - \underline{x}|^4} = -\frac{\partial_{\underline{x}}^2}{8\pi}\int_{S^2} \frac{\underline{\theta}}{< \underline{x}, \underline{\theta} > -x_0\underline{\theta}} d\underline{\theta}. \tag{3.2}$$

See [28, 30] for the proof.

In particular for the x-ray decomposition of the Cauchy kernel, we introduce the following two useful lemmas. They were first presented in [30]. The proof of the first one is clear. The second one has been stated in [30]. However, the definition of the Dirac operator in a plane and the proof have not been treated yet. So here we shall introduce the definition of the Dirac operator on a plane and provide the proof of the lemma.

Lemma 3.2. *For any smooth function φ on S^2 we have that*

$$\begin{aligned}\int_{S^2}\varphi(\underline{w})d\underline{w} &= \frac{1}{2\pi}\int_{S^2}\int_{S^2}\delta(<\underline{\omega},\underline{t}>)\varphi(\underline{\omega})d\underline{\omega}d\underline{t}\\ &= \frac{1}{2\pi}\int_{S^2}\int_{S^2}\delta(<\underline{\omega},\underline{t}>)d\underline{t}\varphi(\underline{\omega})d\underline{\omega}. \end{aligned} \tag{3.3}$$

Lemma 3.3. *Let $\underline{x}\in\mathbb{R}^3$, $\underline{t}\in S^2$. We decompose $\underline{x}$ in tangential and normal parts as $\underline{x}=x_{\|}+\underline{t}<\underline{t},\underline{x}>$. Then*

$$\partial_{x_{\|}}\int_{S^2}\frac{\delta(<\underline{\omega},\underline{t}>)}{<\underline{x},\underline{\omega}>-x_0\underline{\omega}}\underline{\omega}d\underline{\omega} = 2\pi\frac{x_0\underline{t}-\underline{x}\times\underline{t}}{|x_0\underline{t}-\underline{x}\times\underline{t}|^3}\underline{t} \tag{3.4}$$

where $\partial_{x_{\|}}$ is the Dirac operator on the plane $\underline{\theta}^{\perp}$.

Proof. We write $x_{\|}=a_1\omega_1+a_2\omega_2$ where $a_1=r\cos\theta, a_2=r\sin\theta, \omega_1,\omega_2$ is an orthonormal basis and $0\le\theta<2\pi, r>0$. We define the Dirac operator $\partial_{x_{\|}}$ on the plane $\underline{t}^{\perp}$ by

$$\partial_{x_{\|}}=\omega_1\frac{\partial}{\partial a_1}+\omega_2\frac{\partial}{\partial a_2}.$$

This operator appears in [30] but was given no precise definition. So with the definition just given we use it to prove the identity (3.4). Let

$$I=\int_{S^2}\frac{\delta(<\underline{\omega},\underline{t}>)}{<\underline{x},\underline{\omega}>-x_0\underline{\omega}}\underline{\omega}d\underline{\omega}. \tag{3.5}$$

Firstly, compute I.

$$I=\int_0^{2\pi}\frac{\cos\varphi\omega_1+\sin\varphi\omega_2}{(r\cos\theta\cos\varphi+r\sin\varphi\sin\theta)-x_0(\cos\varphi\omega_1+\sin\varphi\omega_2)}d\varphi$$

$$=\int_0^{2\pi}\frac{(\cos\varphi\omega_1+\sin\varphi\omega_2)(r\cos(\theta-\varphi)+x_0(\cos\varphi\omega_1+\sin\varphi\omega_2))}{[(r\cos(\theta-\varphi))-x_0(\cos\varphi\omega_1+\sin\varphi\omega_2)][(r\cos(\theta-\varphi))+x_0(\cos\varphi\omega_1+\sin\varphi\omega_2)]}d\varphi$$

$$=\int_0^{2\pi}\frac{r\cos(\theta-\varphi)\cos\varphi\omega_1-x_0\cos^2\varphi+r\cos(\theta-\varphi)\sin\varphi\omega_2-x_0\sin^2\varphi}{r^2\cos^2(\theta-\varphi)-x_0^2(-\cos^2\varphi-\sin^2\varphi)}d\varphi$$

$$=\int_0^{2\pi}\frac{r\cos(\theta-\varphi)\cos\varphi\omega_1+r\cos(\theta-\varphi)\sin\varphi\omega_2-x_0}{r^2\cos^2(\theta-\varphi)+x_0^2}d\varphi.$$

$$= \int_0^{2\pi} \frac{-x_0 + r\cos(\theta - \varphi)(\cos\varphi\omega_1 + \sin\varphi\omega_2)}{r^2\cos^2(\theta - \varphi) + x_0^2} d\varphi.$$

Making the change of variable $\psi = \varphi - \theta$ gives

$$I = \int_0^{2\pi} \frac{-x_0 + r\cos\psi[\cos(\psi + \theta)\omega_1 + \sin(\psi + \theta)\omega_2]}{r^2\cos^2(\psi) + x_0^2} d\psi. \tag{3.6}$$

We claim next that the integrand in I is independent of the choice of bases in $\underline{\theta}^\perp$ by changing the orthogonal basis ω_1, ω_2 to $\tilde{\omega}_1, \tilde{\omega}_2$ as follows:

$$\tilde{\omega}_1 = \cos\beta\omega_1 + \sin\beta\omega_2$$

$$\tilde{\omega}_2 = -\sin\beta\omega_1 + \cos\beta\omega_2$$

$$\omega_1 = \cos\beta\tilde{\omega}_1 - \sin\beta\tilde{\omega}_2$$

$$\omega_2 = \sin\beta\tilde{\omega}_1 + \cos\beta\tilde{\omega}_2.$$

So $x_\parallel$ can be written in new coordinates as

$$x_\parallel = r\cos\theta\omega_1 + r\sin\theta\omega_2$$

$$\begin{aligned} x_\parallel &= r\cos\theta(\cos\beta\tilde{\omega}_1 - \sin\beta\tilde{\omega}_2) + r\sin\theta(\sin\beta\tilde{\omega}_1 + \cos\beta\tilde{\omega}_2) \\ &= r\cos\theta\cos\beta\tilde{\omega}_1 - r\cos\theta\sin\beta\tilde{\omega}_2 + r\sin\theta\sin\beta\tilde{\omega}_1 + r\sin\theta\cos\beta\tilde{\omega}_2 \\ &= r\cos(\theta - \beta)\tilde{\omega}_1 + r\sin(\theta - \beta)\tilde{\omega}_2. \end{aligned}$$

This means that if we change ω_1, ω_2 to $\tilde{\omega}_1, \tilde{\omega}_2$ by β, $x_\parallel$ can be written in the new basis by β. So

$$I = \int_0^{2\pi} \frac{-x_0 + r\cos\psi[\cos(\psi + \theta - \beta)\tilde{\omega}_1 + \sin(\psi + \theta - \beta)\tilde{\omega}_2]}{r^2\cos^2(\psi) + x_0^2} d\psi.$$

The numerator of the integrand of I is

$$\begin{aligned} \text{N of Integrand} &= -x_0 + r\cos\psi[\cos(\psi + \theta)\cos\beta\tilde{\omega}_1 + \sin(\psi + \theta)\sin\beta\tilde{\omega}_1 \\ &\quad + \sin(\psi + \theta)\cos\beta\tilde{\omega}_2 - \cos(\psi + \theta)\sin\beta\tilde{\omega}_2] \\ &= -x_0 + r\cos\psi[\cos(\psi + \theta)(\cos\beta\tilde{\omega}_1 - \sin\beta\tilde{\omega}_2) \\ &\quad + \sin(\psi + \theta)(\sin\beta\tilde{\omega}_1 + \cos\beta\tilde{\omega}_2)] \\ &= -x_0 + r\cos\psi[\cos(\psi + \theta)\omega_1 + \sin(\psi + \theta)\omega_2.] \end{aligned}$$

Therefore, the claim has proved.

Next we shall evaluate I by considering term by term I_1, I_2 and I_3 where they are defined as the following :

$$I_1 = \int_0^{2\pi} \frac{-x_0}{r^2\cos^2(\psi) + x_0^2} d\psi,$$
$$I_2 = \int_0^{2\pi} \frac{r\cos\psi\cos(\psi+\theta)}{r^2\cos^2(\psi) + x_0^2} d\psi,$$
$$I_3 = \int_0^{2\pi} \frac{r\cos\psi\sin(\psi+\theta)}{r^2\cos^2(\psi) + x_0^2} d\psi.$$

Consider I_1.

$$\begin{aligned} I_1 &= -x_0 \int_0^{2\pi} \frac{1}{r^2\cos^2\psi + x_0^2} d\psi \\ &= -x_0 \int_0^{2\pi} \frac{1}{r^2(\frac{\cos 2\psi + 1}{2}) + x_0^2} d\psi \\ &= -x_0 \int_0^{2\pi} \frac{2}{r^2(\cos 2\psi + 1) + 2x_0^2} d\psi. \end{aligned}$$

Making change of variable $\omega = 2\psi$ gives

$$I_1 = -\frac{2x_0}{r^2} \int_0^{2\pi} \frac{1}{\cos\omega + 1 + \frac{2x_0^2}{r^2}} d\omega.$$

Let $z = e^{i\omega}$ and so $\cos\omega = z + z^{-1}$ and then

$$\begin{aligned} I_1 &= -\frac{2x_0}{r^2} \int_{S^1} \frac{1}{\frac{z+z^{-1}}{2} + A} \frac{d\Omega}{iz}, \qquad A = 1 + \frac{2x_0^2}{r^2} > 1 \\ &= -\frac{2x_0}{ir^2} \int_{S^1} \frac{1}{\frac{z^2+1}{2z} + A} \frac{d\Omega}{z}, \qquad A = 1 + \frac{2x_0^2}{r^2} > 1 \\ &= -\frac{4x_0}{ir^2} \int_{S^1} \frac{1}{z^2 + 1 + 2Az} d\Omega. \end{aligned}$$

$z^2+1+2Az=0$ gives $z=\dfrac{-2A\pm\sqrt{(2A)^2-4}}{2}=-A\pm\sqrt{A^2-1}$. Let $z_+=-A+\sqrt{A^2-1}$, $z_-=-A-\sqrt{A^2-1}$. We note that z_+ lies in the unit circle and by using the Residue theorem we get

$$\begin{aligned}
I_1 &= -\frac{4x_0}{ir^2}\int_{S^1}\frac{1}{(z-z_+)(z-z_-)}d\Omega \\
&= -\frac{4x_0}{ir^2}(2\pi i)Res(f,z_+) \\
&= -\frac{8x_0\pi}{r^2}\frac{1}{(z_+-z_-)} \\
&= -\frac{2\pi}{\sqrt{x_0^2+r^2}} \\
&= f_1(r,\theta,x_0).
\end{aligned}$$

Next we will be considering

$$\begin{aligned}
I_2 &= \int_0^{2\pi}\frac{r\cos\psi\cos(\psi+\theta)}{r^2\cos^2(\psi)+x_0^2}d\psi \\
&= \frac{1}{r}\int_0^{2\pi}\frac{\cos\psi(\cos\psi\cos\theta-\sin\psi\sin\theta)}{\cos^2\psi+B}d\psi, \quad B=x_0^2/r^2 \\
&= \frac{1}{r}\int_0^{2\pi}\frac{\cos^2\psi\cos\theta-\cos\psi\sin\psi\sin\theta)}{\cos^2\psi+B}d\psi, \\
&= \frac{1}{r}\cos\theta I_{21}-\frac{1}{r}\sin\theta I_{22}.
\end{aligned}$$

The breaking factor I_{21} is given by

$$\begin{aligned}
I_{21} &= \int_0^{2\pi} \frac{\cos^2\psi}{\cos^2\psi + B} d\psi \\
&= \int_0^{2\pi} \frac{\cos 2\psi + 1}{\cos 2\psi + 1 + 2B} d\psi \\
&= \frac{1}{2}\int_0^{4\pi} \frac{\cos\omega + 1}{\cos\omega + A} d\omega, \quad ,\omega = 2\psi, \quad A = 1 + 2B \\
&= \int_0^{2\pi} \frac{\cos\omega + 1}{\cos\omega + A} d\omega \\
&= \int_{S^1} \frac{z^2 + 2z + 1}{z^2 + 2Az + 1}\frac{d\Omega}{iz}, \quad z = e^{i\omega} \\
&= \frac{1}{i}\int_{S^1} \frac{z^2 + 2z + 1}{z(z^2 + 2Az + 1)} d\Omega \\
&= \frac{1}{i}\int_{S^1} \frac{z^2 + 2z + 1}{z(z - z_+)(z - z_-)} dz
\end{aligned}$$

where $z_+ = -A + \sqrt{A^2 - 1}$, $z_- = -A - \sqrt{A^2 - 1}$. Notice that z_+ and $z = 0$ lie in the unit circle and by the residue theorem we obtain

$$\begin{aligned}
I_{21} &= \frac{1}{i}(2\pi i)\left(\frac{z^2 + 2z + 1}{(z - z_+)(z - z_-)}|_{z=0} + \frac{z^2 + 2z + 1}{z(z - z_-)}|_{z=z_+}\right) \\
&= 2\pi\left(\frac{1}{(z_+)(z_-)} + \frac{z_+^2 + 2z_+ + 1}{z_+(z_+ - z_-)}\right) \\
&= 2\pi\left(1 + \frac{(A - \sqrt{A^2 - 1})(A - 1)}{(-A + \sqrt{A^2 - 1})\sqrt{A^2 - 1}}\right) \\
&= 2\pi\left(1 - \frac{x_0}{\sqrt{x_0^2 + r^2}}\right) = f_2(r, x_0, \theta).
\end{aligned}$$

The second integral I_{22} is given by

$$\begin{aligned}
I_{22} &= \int_0^{2\pi} \frac{\cos\psi\sin\psi}{\cos^2\psi + B} d\psi \\
&= -1\int \frac{u}{u^2 + B} du, \quad u = \cos\psi \\
&= \frac{-1}{2}\int \frac{1}{v} dv \quad v = u^2 + B \\
&= \frac{-1}{2}\ln(\cos^2\psi + B)|_0^{2\pi} = 0.
\end{aligned}$$

Hence,

$$I_2 = \frac{1}{r}\cos\theta f_2(r, x_0, \theta).$$

Next we will be considering

$$\begin{aligned} I_3 &= \int_0^{2\pi} \frac{r\cos\psi\sin(\psi+\theta)}{r^2\cos^2(\psi)+x_0^2}d\psi \\ &= \frac{1}{r}\int_0^{2\pi} \frac{\cos\psi(\sin\psi\cos\theta+\cos\psi\sin\theta)}{\cos^2\psi+B}d\psi, \quad B = x_0^2/r^2 \\ &= \frac{1}{r}\int_0^{2\pi} \frac{\cos\psi\sin\psi\cos\theta+\cos^2\psi\sin\theta)}{\cos^2\psi+B}d\psi \\ &= \frac{1}{r}\sin\theta I_{31} + \frac{1}{r}\cos\theta I_{32}, \end{aligned}$$

where

$$I_{31} = \int_0^{2\pi} \frac{\cos^2\psi}{\cos^2\psi+B}d\psi, \quad I_{32} = \int_0^{2\pi} \frac{\cos\psi\sin\psi}{\cos^2\psi+B}d\psi.$$

Thus

$$I_3 = \frac{1}{r}\sin\theta f_2(r, x_0, \theta),$$

and so

$$\begin{aligned} I &= f_1(r, x_0, \theta) + \frac{1}{r}\cos\theta f_2(r, x_0, \theta)\omega_1 + \frac{1}{r}\sin\theta f_2(r, x_0, \theta)\omega_2 \\ &= -\frac{2\pi}{\sqrt{x_0^2+r^2}} + \frac{1}{r}f_2(r, x_0, \theta)(\cos\theta\omega_1+\sin\theta\omega_2) \\ &= -\frac{2\pi}{\sqrt{x_0^2+r^2}} + \frac{2\pi}{r}\left(1-\frac{x_0}{\sqrt{x_0^2+r^2}}\right)(\cos\theta\omega_1+\sin\theta\omega_2). \end{aligned}$$

Now, let's compute

$$J = \int_{S^1} \frac{1}{(<x,t> - x_0 t)^2}dt = \int_{S^1} \frac{(<x,t> + x_0 t)^2}{|<x,t>^2 + x_0^2|^2}dt = \int_{S^1} \frac{<x,t>^2 + 2x_0 <x,t> t - x_0^2}{|<x,t>^2 + x_0^2|^2}dt.$$

Let $x = r(\cos\theta, \sin\theta)$, $\quad t = (\cos\omega, \sin\omega)$. Then

$$\begin{aligned} J &= \int \frac{<x,t>^2 - x_0^2}{|<x,t>^2 + x_0^2|^2}dt + 2x_0 \int \frac{<x,t> t}{|<x,t>^2 + x_0^2|^2}dt \\ &= J_1 + 2x_0 J_2. \end{aligned}$$

$$
\begin{aligned}
J_1 &= \int \frac{r^2\cos^2(\theta-\omega)-x_0^2}{(r^2\cos^2(\theta-\omega)+x_0^2)^2}d\omega \\
&= 2\int_0^{\pi} \frac{r^2\cos^2(\omega)-x_0^2}{(r^2\cos^2(\omega)+x_0^2)^2}d\omega \\
&= 2\int_0^{\pi} \frac{r^2\frac{(\cos 2\omega+1)}{2}-x_0^2}{(r^2\frac{(\cos 2\omega+1)}{2}+x_0^2)^2}d\omega \\
&= \int_0^{2\pi} \frac{r^2\frac{(\cos\varphi+1)}{2}-x_0^2}{(r^2\frac{(\cos\varphi+1)}{2}+x_0^2)^2}d\varphi, \quad \varphi = 2\omega \\
&= 2\int_0^{2\pi} \frac{r^2\cos\varphi+r^2-2x_0^2}{(r^2\cos\varphi+r^2+2x_0^2)^2}d\varphi \\
&= \frac{2}{r^2}\int_0^{2\pi} \frac{\cos\varphi+\alpha}{(\cos\varphi+\beta)^2}d\varphi, \quad \alpha = 1-\frac{2x_0^2}{r^2}, \quad \beta = 1+\frac{2x_0^2}{r^2} \\
&= \frac{2}{r^2}\int_{S^1} \frac{\frac{z^2+1}{2z}+\alpha}{(\frac{z^2+1}{2z}+\beta)^2}\frac{d\Omega}{iz}, \quad z = e^{i\omega} \\
&= \frac{4}{ir^2}\int_{S^1} \frac{z^2+1+2\alpha z}{(z^2+1+2\beta z)^2}d\Omega \\
&= \frac{4}{ir^2}\int_{S^1} \frac{z^2+1+2\alpha z}{((z-z_+)(z-z_-))^2}d\Omega
\end{aligned}
$$

where

$$
z_\pm = \frac{-2\beta \pm \sqrt{4\beta^2-4}}{2} = -\beta \pm \sqrt{\beta^2-1}, \quad \beta = 1+\frac{2x_0^2}{r^2} > 1.
$$

Also, notice that z_+ lies in the unit circle. By the residue theorem

$$\begin{aligned}
J_1 &= \frac{4}{ir^2}(2\pi i)\frac{d}{dz}\left(\frac{z^2+1+2\alpha z}{(z-z_-)^2}\right)|_{z=z_+} \\
&= \frac{8\pi}{r^2}\frac{(z-z_-)^2(2z+2\alpha)-(z^2+1+2\alpha z)2(z-z_-)}{(z-z_-)^4}|_{z=z_+} \\
&= \frac{16\pi}{r^2}\frac{(z-z_-)(z+\alpha)-(z^2+1+2\alpha z)}{(z-z_-)^3}|_{z=z_+} \\
&= \frac{16\pi}{r^2}\frac{-\alpha z_+ - 1 - \alpha z_- - z_+ z_-}{(z_+-z_-)^3} \\
&= \frac{16\pi}{r^2}\left(\frac{-\alpha(z_++z_-)-2}{(z_+-z_-)^3}\right) \\
&= \frac{16\pi}{r^2}\left(\frac{-\alpha(-2\beta)-2}{(2\sqrt{\beta^2-1})^3}\right) \\
&= \frac{16\pi}{r^2}\frac{2}{8}\frac{\alpha\beta-1}{(\beta^2-1)\sqrt{\beta^2-1}} \\
&= \frac{4\pi}{r^2}\frac{(-4x_0^4/r^4)}{(\beta^2-1)\sqrt{\beta^2-1}} \\
&= \frac{-2x_0\pi}{(x_0^2+r^2)^{3/2}}, \qquad \beta^2-1 = \frac{4x_0^2}{r^2}\left(\frac{x_0^2}{r^2}+1\right)
\end{aligned}$$

$$\begin{aligned}
J_2 &= \int_0^{2\pi} \frac{r\cos(\theta-\omega)(\cos\omega\omega_1+\sin\omega\omega_2)}{(r^2\cos^2(\theta-\omega)+x_0^2)^2}d\omega \\
&= \int_0^{2\pi}\frac{r\cos\omega\cos(\theta-\omega)}{(r^2\cos^2(\theta-\omega)+x_0^2)^2}d\omega\omega_1 + \int_0^{2\pi}\frac{r\sin\omega\cos(\theta-\omega)}{(r^2\cos^2(\theta-\omega)+x_0^2)^2}d\omega\omega_2 \\
&= J_{21}\omega_1 + J_{22}\omega_2 \\
J_{21} &= \int_0^{2\pi}\frac{r\cos(\varphi+\theta)\cos\varphi}{(r^2\cos^2\varphi+x_0^2)^2}d\varphi, \qquad \varphi=\omega-\theta \\
&= \frac{\cos\theta}{r^3}\int_0^{2\pi}\frac{\cos^2\varphi}{(\cos^2\varphi+\frac{x_0^2}{r^2})^2}d\varphi - \frac{\sin\theta}{r^3}\int_0^{2\pi}\frac{\cos\varphi\sin\varphi}{(\cos^2\varphi+\frac{x_0^2}{r^2})^2}d\varphi \\
&= \frac{\cos\theta}{r^3}A_1 - \frac{\sin\theta}{r^3}A_2
\end{aligned}$$

.

$$
\begin{aligned}
A_1 &= \int_0^{2\pi} \frac{\cos^2\varphi}{(\cos^2\varphi + B)^2} d\varphi, \quad B = \frac{x_0^2}{r^2} \\
&= \int_0^{2\pi} \frac{\frac{\cos 2\varphi+1}{2}}{(\frac{\cos 2\varphi+1}{2} + B)^2} d\varphi = 2\int_0^{2\pi} \frac{\cos 2\varphi + 1}{(\cos 2\varphi + 1 + 2B)^2} d\varphi \\
&= \int_0^{4\pi} \frac{\cos\omega}{(\cos\omega + C)^2} d\omega, \qquad C = 1 + 2B \\
&= 2\int_0^{2\pi} \frac{\cos\omega + 1}{(\cos\omega + C)^2} d\omega = \int_{S^1} \frac{\frac{z^2+1}{2z} + 1}{(\frac{z^2+1}{2z} + C)^2} \frac{d\Omega}{iz} \\
&= \frac{4}{i}\int_{S^1} \frac{z^2+1+2z}{(z^2+1+2Cz)^2} d\Omega = \frac{4}{i}\int_{S^1} \frac{z^2+1+2z}{(z-z_{c-})^2(z-z_{c+})^2} d\Omega \\
&= 8\pi \frac{d}{dz}\left(\frac{z^2+1+2z}{(z-z_{c-})^2}\right)_{z=z_{c+}} = 8\pi(2)\left(\frac{-(z_+ + z_-) - z_+ z_- - 1}{(z_+ - z_-)^3}\right) \\
&= \frac{16\pi(C-1)}{(C^2-1)^{3/2}} = \frac{r^4}{2x_0(x_0^2+r^2)^{3/2}}.
\end{aligned}
$$

To finish this proof, we need to compute J_{22}.

$$
\begin{aligned}
J_{22} &= \int_0^{2\pi} \frac{r\sin\omega\cos(\theta-\omega)}{(r^2\cos^2(\theta-\omega) + x_0^2)^2} d\omega \\
&= \int \frac{r\sin(\varphi+\theta)\cos\varphi}{(r^2\cos\varphi + x_0^2)^2} d\varphi \\
&= \frac{\cos\theta}{r^3}\int \frac{\sin\varphi\cos\varphi}{(r^2\cos\varphi + B)^2} d\varphi + \frac{\sin\theta}{r^3}\int \frac{\cos^2\varphi}{(r^2\cos\varphi + B)^2} d\varphi, \qquad B = \frac{x_0^2}{r^2} \\
&= \frac{\cos\theta}{r^3} A_2 + \frac{\sin\theta}{r^3} A_1, \qquad A_2 = 0.
\end{aligned}
$$

So,

$$
J_2 = \frac{A_1}{r^3}(\cos\theta\omega_1 + \sin\theta\omega_2)
$$

and so

$$
\begin{aligned}
J = J_1 + 2x_0 J_2 &= \frac{-2x_0\pi}{(x_0^2+r^2)^{3/2}} + 2x_0\frac{A_1}{r^3}(\cos\theta\omega_1 + \sin\theta\omega_2) \\
&= \frac{-2x_0\pi}{(x_0^2+r^2)^{3/2}} + \frac{2x_0}{r^3}\frac{r^4}{2x_0(x_0^2+r^2)^{3/2}}(\cos\theta\omega_1 + \sin\theta\omega_2) \\
&= \frac{-2x_0\pi}{(x_0^2+r^2)^{3/2}} + \frac{r}{(x_0^2+r^2)^{3/2}}(\cos\theta\omega_1 + \sin\theta\omega_2) \\
&= \frac{-2\pi x_0}{|x_0 - \underline{x}|^3} + \frac{\underline{x}}{|x_0 - \underline{x}|^3}.
\end{aligned}
$$

Let $x_\| = a_1\omega_1 + a_2\omega_2 = r\cos\theta\omega_1 + r\sin\theta\omega_2 . \partial_{x_\|} = \omega_1 \dfrac{\partial}{\partial a_1} + \omega_2 \dfrac{\partial}{\partial a_2}, \quad A = x_0^2 + a_1^2 + a_2^2 = x_0^2 + r^2$. Recall that

$$I = -\frac{2\pi}{\sqrt{x_0^2 + r^2}} + \frac{2\pi}{r}\left(1 - \frac{x_0}{\sqrt{x_0^2 + r^2}}\right)(\cos\theta\omega_1 + \sin\theta\omega_2)$$

$$\begin{aligned}
\frac{\partial}{\partial a_1} I &= \frac{a_1}{A^{3/2}} + \frac{1}{a_1^2 + a_2^2}\left(1 - \frac{x_0}{A^{1/2}}\right)\omega_1 \\
&\quad (a_1\omega_1 + a_2\omega_2)\left[\frac{1}{a_1^2 + a_2^2}\left(\frac{(-x_0)(-a_1)}{A^{3/2}}\right) + \left(1 - \frac{x_0}{A^{1/2}}\right)\frac{(-2a_1)}{(a_1^2 + a_2^2)^2}\right] \\
&= \frac{a_1}{A^{3/2}} + \frac{\omega_1}{a_1^2 + a_2^2} - \frac{x_0}{(a_1^2 + a_2^2)A^{1/2}}\omega_1 \\
&\quad + \frac{a_1^2 x_0}{(a_1^2 + a_2^2)A^{3/2}}\omega_1 - \frac{2a_1^2}{(a_1^2 + a_2^2)^2}\left(1 - \frac{x_0}{A^{1/2}}\right)\omega_1 \\
&\quad + \frac{x_0 a_1 a_2}{(a_1^2 + a_2^2)A^{3/2}}\omega_2 - \frac{2a_1 a_2}{(a_1^2 + a_2^2)^2}\left(1 - \frac{x_0}{A^{1/2}}\right)\omega_2, \\
\omega_1 \frac{\partial}{\partial a_1} I &= \frac{a_1\omega_1}{A^{3/2}} - \frac{1}{r^2} + \frac{x_0}{r^2 A^{1/2}} - \frac{a_1^2 x_0}{r^2 A^{3/2}} + \frac{2a_1^2}{r^4}\left(1 - \frac{x_0}{A^{1/2}}\right) \\
&\quad + \frac{x_0 a_1 a_2}{r^2 A^{3/2}}(\omega_1 \times \omega_2) - \frac{2a_1 a_2}{r^4}\left(1 - \frac{x_0}{A^{1/2}}\right)(\omega_1 \times \omega_2) \\
\frac{\partial}{\partial a_2} I &= \frac{a_2}{A^{3/2}} + \frac{1}{a_1^2 + a_2^2}\left(1 - \frac{x_0}{A^{1/2}}\right)\omega_2 \\
&\quad (a_1\omega_1 + a_2\omega_2)\left[\frac{1}{a_1^2 + a_2^2}\left(\frac{(-x_0)(-a_2)}{A^{3/2}}\right) + \left(1 - \frac{x_0}{A^{1/2}}\right)\frac{(-2a_2)}{(a_1^2 + a_2^2)^2}\right] \\
&= \frac{a_2}{A^{3/2}} + \frac{\omega_2}{a_1^2 + a_2^2} - \frac{x_0}{(a_1^2 + a_2^2)A^{1/2}}\omega_2 \\
&\quad + \frac{a_2^2 x_0}{(a_1^2 + a_2^2)A^{3/2}}\omega_2 - \frac{2a_2^2}{(a_1^2 + a_2^2)^2}\left(1 - \frac{x_0}{A^{1/2}}\right)\omega_2 \\
&\quad + \frac{x_0 a_1 a_2}{(a_1^2 + a_2^2)A^{3/2}}\omega_1 - \frac{2a_1 a_2}{(a_1^2 + a_2^2)^2}\left(1 - \frac{x_0}{A^{1/2}}\right)\omega_1 \\
\omega_2 \frac{\partial}{\partial a_2} I &= \frac{a_2\omega_2}{A^{3/2}} - \frac{1}{r^2} + \frac{x_0}{r^2 A^{1/2}} - \frac{a_2^2 x_0}{r^2 A^{3/2}} + \frac{2a_2^2}{r^4}\left(1 - \frac{x_0}{A^{1/2}}\right) \\
&\quad + \frac{x_0 a_1 a_2}{r^2 A^{3/2}}(\omega_2 \times \omega_1) - \frac{2a_1 a_2}{r^4}\left(1 - \frac{x_0}{A^{1/2}}\right)(\omega_2 \times \omega_1)
\end{aligned}$$

$$\left(\omega_1\frac{\partial}{\partial a_1}+\omega_2\frac{\partial}{\partial a_2}\right)I = \frac{a_1\omega_1+a_2\omega_2}{A^{3/2}}-\frac{2}{r^2}+\frac{2x_0}{r^2A^{1/2}}$$
$$-\frac{x_0(a_1^2+t_2^2)}{r^2A^{3/2}}+\frac{2(a_1^2+a_2^2)}{r^4}\left(1-\frac{x_0}{A^{1/2}}\right)$$
$$=\frac{x_\parallel}{|x_0-x|^3}-\frac{2}{r^2}+\frac{2x_0}{r^2A^{1/2}}-\frac{x_0}{|x_0-x|^3}+\frac{2}{r^2}-\frac{2x_0}{r^2A^{1/2}}$$
$$=-\frac{x_0-x_\parallel}{|x_0-x|^3}.$$

□

Proposition 3.4. *For $x = x_0+\underline{x}\in\mathbb{R}^4$, $x_0\neq 0$. We have the X-ray decomposition of the Cauchy Kernel in dimensions 3 as*

$$\frac{x_0-\underline{x}}{|x_0-\underline{x}|^4} = \pm\frac{-(sgnx_0)\partial_{\underline{x}}}{8\pi}\int_{S^2}\frac{x_0\underline{\theta}-\underline{x}\times\underline{\theta}}{|x_0\underline{\theta}-\underline{x}\times\underline{\theta}|^3}\underline{\theta}d\underline{\theta}. \tag{3.7}$$

Proof. We start from the Radon decomposition (3.3) and then use lemmas 3.2 and 3.3 as follows

$$\begin{aligned}
\frac{x_0-\underline{x}}{|x_0-\underline{x}|^4} &= -\frac{\partial^2_{\underline{x}}}{8\pi}\int_{S^2}(<\underline{x},\underline{\omega}> -x_0\underline{\omega})^{-1}\underline{\omega}d\underline{\omega}\\
&= -\frac{\partial^2_{\underline{x}}}{16\pi^2}\int_{S^2}\int_{S^2}\frac{\delta(<\underline{\omega},\underline{\theta}>)}{<\underline{x},\underline{\omega}> -x_0\underline{\omega}}\underline{\omega}d\underline{\theta}d\underline{\omega}.\\
&= -\frac{\partial^2_{\underline{x}}}{16\pi^2}\partial^2_x\int_{S^2}\int_{S^2}\frac{\delta(<\underline{\omega},\underline{\theta}>)}{<\underline{x},\underline{\omega}> -x_0\underline{\omega}}\underline{\omega}d\underline{\omega}d\underline{\theta}.\\
&= -\frac{\partial^2_{\underline{x}}}{16\pi}\partial_{\underline{x}}\int_{S^2}2\pi\partial_{x_\parallel}\int_{S^2}\frac{\delta(<\underline{\omega},\underline{\theta}>)}{<\underline{x},\underline{\omega}> -x_0\underline{\omega}}\underline{\omega}d\underline{\omega}d\underline{\theta}, \quad \text{for each } \underline{\theta}\\
&= \frac{-(\text{sgn}x_0)\partial_{\underline{x}}}{8\pi}\int_{S^2}\frac{x_0\underline{\theta}-\underline{x}\times\underline{\theta}}{|x_0\underline{\theta}-\underline{x}\times\underline{\theta}|^3}\underline{\theta}d\underline{\theta}.
\end{aligned}$$

□

The integrals make sense because $|x_0\underline{\theta}-\underline{x}\times\underline{\theta}|$ does not vanish. Even though both $x_0\underline{\theta}$ and $\underline{x}\times\underline{\theta}$ are vector valued functions, the first one is a multiple of $\underline{\theta}$ and the latter is orthogonal to both $\underline{x}$ and $\underline{\theta}$.

3.2 The quaternionic-Doppler transform

The 3 dimensional Doppler transform has been introduced in, for example [14, 13, 25, 24, 21] representing a mathematical model of *vector tomography*, this means the problem of reconstructing velocity fields using the Doppler effect.

In concrete definition, the Doppler transform D maps a vector field $\mathbf{f}$ to the line integral of the field component parallel to the line L, more precisely,

$$D\mathbf{f}(L) = \int_L \theta \cdot \mathbf{f}(x) ds \tag{3.8}$$

where $\theta \in S^2$ denotes the direction of the line L. It is well-known that D has a non-trivial null space and that only the solenoidal part of $\mathbf{f}$ can be recovered by $D\mathbf{f}$. We refer to Sharafutdinov [26] for proofs. Here any vector field $\mathbf{f}$ can be viewed as a quaternionic-valued function $\underline{\mathbf{f}} = f_1e_1 + f_2e_2 + f_3e_3, f_i$ are scalar-valued functions, by identifying $\underline{\mathbf{f}} = f_1e_1 + f_2e_2 + f_3e_3$. One of the differences between them is, for example, the product of an element in $\mathbb{R}^3$ with $\underline{\mathbf{f}}$ is defined according to the properties of quaternions while this is undefined for a vector field.

Here we define a new transform in quaternions as the following:

Definition 3.5. *Let L be an oriented line in $\mathbb{R}^3$, $\underline{\theta} \in S^2$ and $\underline{\mathbf{f}}$ be the vector part of a quaternionic-valued function. The quaternionic-Doppler transform maps $\underline{\mathbf{f}}$ to its line integral in the following way:*

$$D^q\underline{\mathbf{f}}(L) = \int_L \underline{\theta}\ \underline{\mathbf{f}} ds = -\int_L \underline{\theta} \cdot \underline{\mathbf{f}} ds + \int_L \underline{\theta} \times \underline{\mathbf{f}} ds \tag{3.9}$$

where $\underline{\theta}$ is the direction of a line L and the integration is to be understood componentwise in the vector part.

Alternatively, for $\underline{a} \in \mathbb{R}^3$ and $\underline{\theta}$ a unit vector in S^2 ,

$$D^q\underline{\mathbf{f}}(\underline{a}, \underline{\theta}) = \int_{\mathbb{R}} \underline{\theta}\ \underline{\mathbf{f}}(\underline{a} + \underline{\theta} t) dt.$$

We think of $\underline{a}$ as the source of a ray with direction $\underline{\theta}$. We shall occasionally use the notation $D_{\underline{a}}\underline{\mathbf{f}}(\underline{\theta})$ instead of $D\underline{\mathbf{f}}(\underline{a},\underline{\theta})$.

The following proposition is an inversion formula for the quaternionic-Doppler transform.

Proposition 3.6. *The quaternionic-Doppler transform in (3.9) contribute a reconstruction formula of* $\underline{\mathbf{f}}$ *in parallel beam setting by using the X-ray transform:*

$$\underline{\mathbf{f}}(\underline{x}) = -\frac{1}{4\pi}\partial_{\underline{x}}\int_{S^2}\int_{\theta^\perp}\frac{(E_{\underline{\theta}}\underline{x}-u')\times\theta}{|(E_{\underline{\theta}}\underline{x}-u')\times\underline{\theta}|^3}D(\underline{\mathbf{f}})(u',\theta)du'd\underline{\theta}. \tag{3.10}$$

Proof. Applying Fubini's theorem to

$$\underline{\mathbf{f}}(\underline{x}) = -\frac{1}{4\pi}\partial_{\underline{x}}\int_{S^2}\int_{\mathbb{R}^3}\frac{(\underline{u}-\underline{x})\times\theta}{|(\underline{u}-\underline{x})\times\underline{\theta}|^3}\underline{\theta}\underline{\mathbf{f}}(\underline{u})d\underline{u}d\underline{\theta}$$

in section 2.8 gives,

$$\begin{aligned}
\underline{\mathbf{f}}(\underline{x}) &= -\frac{1}{4\pi}\partial_{\underline{x}}\int_{S^2}\int_{\theta^\perp}\int_{\mathbb{R}}\frac{(\underline{x}-(u'+t\underline{\theta}))\times\theta}{|(\underline{x}-\underline{u})\times\underline{\theta}|^3}\underline{\theta}\underline{\mathbf{f}}(\underline{u})dtdu'd\underline{\theta}\\
&= -\frac{1}{4\pi}\partial_{\underline{x}}\int_{S^2}\int_{\theta^\perp}\int_{\mathbb{R}}\frac{(\underline{x}-u')\times\theta}{|(\underline{x}-u')\times\underline{\theta}|^3}\underline{\theta}\underline{\mathbf{f}}(u'+t\underline{\theta})dtdu'd\underline{\theta}\\
&= -\frac{1}{4\pi}\partial_{\underline{x}}\int_{S^2}\int_{\theta^\perp}\frac{(\underline{x}-u')\times\theta}{|(\underline{x}-u')\times\underline{\theta}|^3}\underline{\theta}\int_{\mathbb{R}}\underline{\mathbf{f}}(u'+t\underline{\theta})dtdu'd\underline{\theta}\\
&= -\frac{1}{4\pi}\partial_{\underline{x}}\int_{S^2}\int_{\theta^\perp}\frac{(E_{\underline{\theta}}\underline{x}-u')\times\theta}{|(E_{\underline{\theta}}\underline{x}-u')\times\underline{\theta}|^3}D(\underline{\mathbf{f}})(u',\theta)du'd\underline{\theta}.
\end{aligned} \tag{3.11}$$

Since we are recovering the vector part of $\mathbf{f}$, its real part is zero and from the properties of multiplication of the following quaternion terms

$$\begin{aligned}
\frac{(\underline{x}-\underline{u})\times\theta}{|(\underline{x}-\underline{u})\times\underline{\theta}|^3}\underline{\theta}\underline{\mathbf{f}}(\underline{u}) &= \frac{(\underline{x}-\underline{u})\times\theta}{|(\underline{x}-\underline{u})\times\underline{\theta}|^3}(-\underline{\theta}\cdot\underline{\mathbf{f}}(\underline{u})+(\underline{\theta}\times\underline{\mathbf{f}}(\underline{u})))\\
&= -\frac{(\underline{x}-\underline{u})\times\theta}{|(\underline{x}-\underline{u})\times\underline{\theta}|^3}(\underline{\theta}\cdot\underline{\mathbf{f}}(\underline{u})) - \frac{(\underline{x}-\underline{u})\times\theta}{|(\underline{x}-\underline{u})\times\underline{\theta}|^3}\cdot(\underline{\theta}\times\underline{\mathbf{f}}(\underline{u}))\\
&\quad + \frac{(\underline{x}-\underline{u})\times\theta}{|(\underline{x}-\underline{u})\times\underline{\theta}|^3}\times(\underline{\theta}\times\underline{\mathbf{f}}(\underline{u}))
\end{aligned}$$

gives that the equation (3.10) becomes

$$\begin{aligned}\underline{\mathbf{f}}(\underline{x}) = &-\frac{1}{4\pi}\partial_{\underline{x}} \int_{S^2}\int_{\mathbb{R}^3} -\frac{(\underline{x}-\underline{u})\times\underline{\theta}}{|(\underline{x}-\underline{u})\times\underline{\theta}|^3}\cdot(\underline{\theta}\times\underline{\mathbf{f}}(\underline{u}))d\underline{u}d\underline{\theta} \\ &-\frac{1}{4\pi}\partial_{\underline{x}}\times\int_{S^2}\int_{\mathbb{R}^3}\frac{(\underline{x}-\underline{u})\times\underline{\theta}}{|(\underline{x}-\underline{u})\times\underline{\theta}|^3}(\underline{\theta}\cdot\underline{\mathbf{f}}(\underline{u}))d\underline{u}d\underline{\theta} \\ &+\frac{1}{4\pi}\partial_{\underline{x}}\times\int_{S^2}\int_{\mathbb{R}^3}\frac{(\underline{x}-\underline{u})\times\underline{\theta}}{|(\underline{x}-\underline{u})\times\underline{\theta}|^3}\times(\underline{\theta}\times\underline{\mathbf{f}}(\underline{u}))d\underline{u}d\underline{\theta}.\end{aligned}$$

□

3.3 Cone (divergent) beam transform of a quaternionic-valued function

The cone beam transform of a scalar-valued function f at source point $a \in \mathbb{R}^3$ and direction $\theta \in S^2$, is defined by

$$Df(a,\theta) = \int_0^\infty f(a+t\theta)dt. \tag{3.12}$$

For this scalar cone beam transform, the recovery of a function by using (3.12) as a data has been achieved by Tuy [32], Gel'fand and Goncharov [6], Sparr [31] Grangeat [7], Katsevich[11, 12] and Louis [15, 16]. In the case of the cone beam transform of a vector-valued function $\mathbf{f}$ or vector field, we define the transform as

$$D^c\mathbf{f}(a,\theta) = \int_0^\infty \mathbf{f}(a+t\theta)dt. \tag{3.13}$$

where the definition is to be understood componentwise. Katsevich and Schuster [13] have an excellent paper describing an exact inversion formula for a vector field by using a Doppler transform (3.8) where a cone beam transform (3.13) has been used with sources on a particular curve.

One of the important ingredients for vector tomography is Grangeat's formula presenting a connection between the Radon transform and the cone beam transform. In the scalar case, Grangeat formula is the following:

Theorem 3.7. *If f is compactly supported function defined in $\mathbb{R}^3$ and $a \in \mathbb{R}^3$ satisfies $a \cdot \theta = s$, then*

$$\frac{\partial}{\partial s} Rf(s,\theta) = \int_{S^2 \cap \theta^\perp} \nabla_\theta Df(a,\omega) d\omega. \tag{3.14}$$

An alternative formulation of (3.14)

$$\frac{\partial}{\partial s} Rf(s,\theta)|_{s=a\cdot\theta} = -\int_{S^2} \delta'(\theta \cdot \eta) Df(a,\eta) d\eta. \tag{3.15}$$

In a typical 3D tomographic set up, the object f of compact support would be surrounded by a source curve Γ, and $Df(a,\theta)$ would be measured for $a \in \Gamma$ and $\theta \in S^2$. It is for this situation that we try to invert D. For this purpose, we give some relations between D and Radon transform R, which in turn permit the inversion of D, provided the source curve Γ satisfies certain conditions. We restrict the discussion to the 3D case.

Schuster [23] proved an extension of the formula of Grangeat for vector and tensor fields for arbitrary rank. However, he pointed out that this formula could not be used to derive an exact inversion formula similar to how it was done for scalar-valued functions in [11, 12, 10]. Schuster et al [25] presented an inversion algorithm for D which was inexact but demonstrated good performance for vector field with a particular structure.

A breakthrough was marked by Kazantsev and Schuster [14], where the authors found another type Grangeat- type formula for a vector field,

$$\frac{\partial^2}{\partial s^2} \mathcal{R}^{\tan}\mathbf{f}(s,\theta)|_{s=\theta\cdot a} = \sum_a \phi(a,\eta) \int_{S^2} \delta''(\theta \cdot \eta)\eta \cdot D\mathbf{f}(a,\eta) d\eta \tag{3.16}$$

where ϕ is a function depending on a source point a and a unit vector η, which, in turn allowed Kazantsev and Schuster to obtain an asymtotic inversion formula for D.

For the rest of this section we shall present the properties, notations of a curve established by Katsevich [11] which we will mainly use in chapter 4 for our main result (section 4.3).

Let Γ be a finite union of C^∞-curves in $\mathbb{R}^3$:

$$I := \bigcup_{l=1}^{L_\Gamma} [a_l, b_l] \to \mathbb{R}^3, \quad I \ni \lambda \to a(\lambda) \in \mathbb{R}^3, \quad |a'(\lambda)| \neq 0 \text{ on } \ I,$$

where $-\infty < a_l < b_l < \infty$ and $a'(\lambda) = da/d\lambda$. Also, let

$$\beta(\lambda, x) := \frac{x - a(\lambda)}{|x - a(\lambda)|}, \quad x \in \mathbb{R}^3 \setminus \Gamma, \lambda \in I;$$

be a unit vector in the direction from $a(\lambda)$ to a point x and let

$$\Pi(x, \xi) := \{z \in \mathbb{R}^3 : (z - x) \cdot \xi = 0\}$$

be the plane passing through x with normal unit vector ξ. Katsevich begins by assuming that $f \in C_0^\infty(\mathbb{R}^3)$ and $\text{dist}(\Gamma, \text{supp} f) > 0$. Given $x \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3 \setminus 0$, let $a(\lambda_j)$, where $\lambda_j = \lambda_j(\xi, \xi \cdot x)$, $j = 1, 2, \cdots$ denote points of intersection of $\Pi(x, \xi)$ with Γ. For $\beta \in S^2$, $\beta^\perp$ denotes the great circle $\{\eta \in S^2 : \eta \cdot \beta = 0\}$. Introduce the sets

$$\begin{aligned}
\text{Crit}(\lambda, x) &:= \{\eta \in \beta^\perp(\lambda, x) : \Pi(x, \eta) \text{ is tangent to } \Gamma \\
&\qquad \text{or } \Pi(x, \eta) \text{ contains an endpoint of } \Gamma\}, \\
I_{\text{reg}}(x) &:= \{\lambda \in I : \text{Crit}(\lambda, x) \subsetneq \beta^\perp(\lambda, x)\}, \\
\text{Crit}(x) &:= \bigcup_{\lambda \in I} \text{Crit}(\lambda, x).
\end{aligned}$$

If $\beta(\lambda, x)$ is parallel to $a'(\lambda)$, or the line through $a(\lambda) \in \Gamma$ and x contains an endpoint of Γ, then $\text{Crit}(\lambda, x)$ coincides with $\beta^\perp(\lambda, x)$. In [6,7], it is shown that the set $\text{Crit}(x)$ has Lebesgue measure zero and the set I_{reg} is open.

The main assumptions for the curve trajectory curve Γ are the following properties, for fixed any $x \in \mathbb{R}^3$ where f need to be computed,

Property 1 (completeness condition). Any plane through x intersects Γ in at least at one point.

Property 2 The number of directions in $\text{Crit}(\lambda, x)$ is uniformly bounded on $I_{\text{reg}}(x)$.

Property 3 The number of points in $\Pi(x, \eta) \cap \Gamma$ is uniformly bounded on $S^2 \setminus \text{Crit}(x)$.

Property 1 is the most important from the practical point of view.

Remark 1 An important ingredient in the reconstruction formula is the weight function $w_0(\lambda, x, \eta)$, $\lambda \in I_{\text{reg}}(x)$ and $\eta \in \beta^\perp(\lambda, x) \setminus \text{Crit}(\lambda, x)$

Remark 2 The weight function w_0 can be described as follows: x and η determine the plane $\Pi(x,\eta)$, and the weight w_0 assigned to $a(\lambda) \in \Pi(x,\eta) \cap \Gamma$ depends on the location of x. In view of this interpretation, $w_0(\lambda, x, \eta) = w_0(\lambda, x, -\eta)$.

We denote $\sum_j$ the sum over all λ_j such that $a(\lambda_j) \in \Gamma \cap \Pi(x,\eta)$. Define

$$w_\Sigma(x,\eta) := \sum_j w_0(\lambda_j, x, \eta), \quad \lambda_j = \lambda_j(\eta, \eta \cdot x), \quad \eta \in S^2 \setminus \mathrm{Crit}(x) \tag{3.17}$$

$$w(\lambda, x, \eta) := \frac{w_0(\lambda, x, \eta)}{w_\Sigma(x,\eta)}. \tag{3.18}$$

The main assumptions about w_0 are the following properties:

Property 4 $w_\Sigma(x,\eta) \geq c$ a.e. on S^2 for some $c > 0$;

Property 5 There exists finitely many C^1- functions $\eta_k(\lambda, x) \in \beta^\perp(\lambda, x), \lambda \in I_{\mathrm{reg}}(x)$, such that $w(\lambda, x, \eta)$ is locally constant in a neighborhood of any (λ, η) where $\lambda \in I_{\mathrm{reg}}(x)$ and $\eta \in \beta^\perp(\lambda, x), \eta \notin (\bigcup_k \eta_k(\lambda, x)) \cup \mathrm{Crit}(\lambda, x)$.

We also recall the following definitions:

Definition 3.8. *A collection of subsets $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ of a manifold M is called locally finite, if for all $m \in M$ there is an neighborhood O of M with $U_\alpha \cap O \neq \emptyset$ for only a finite subset of A.*

Definition 3.9. *A partition of unity on a manifold M is a collection of a smooth functions $\{\phi_i : M \to R : \ i \in I\}$ such that*

1. *$\{$the support of ϕ_i $i \in I\}$ is locally finite*
2. *$\phi_i(p) \geq 0$ for all p in M, $i \in I$ and,*
3. *$\sum_{i \in I} \phi_i(p) = 1$ for all $p \in M$*

Definition 3.10. *A function $f : A \to B$ is bounded away from zero on the set $C \subseteq A$ if there exists $\epsilon > 0$ such that*

$$|f(c)| \geq \epsilon \quad \text{for all } c \in C.$$

3.4 Series expansions in subspaces of $L^2(B^3)$

In this section we shall talk about the fact that any vector field in $L_2(B^3)$ can be represented by the series of orthogonal functions where B^3 is the unit ball in $\mathbb{R}^3$. We will describe it in two parts by following outline in [14].

3.4.1 Orthogonal expansions for vector fields in $L_2(B^3)$

We define the inner product of two vector fields $\mathbf{f}$ and $\mathbf{g}$ by

$$< \mathbf{f}, \mathbf{g} >= \int_{B^3} \mathbf{f}(x) \cdot \mathbf{g}(x) dx.$$

Let $\prod_n^3$ be the space of polynomials on $\mathbb{R}^3$ of degree at most n. We denote by V_n^3 the orthocomplement in $\prod_n^3$ of $\prod_{n-1}^3$

$$V_n^3 = \{P \in \prod_n^3 \mid \; < P, Q >_{L^2(B^3)} = 0, \; \forall \; Q \in \prod_{n-1}^3\}.$$

Then

$$\dim \prod_n^3 = \begin{pmatrix} n+3 \\ n \end{pmatrix}, \; \dim V_n^3 = \begin{pmatrix} n+2 \\ n \end{pmatrix}.$$

The Zernike polynomials

$$\{Z_{n-2k,l}^{(n)}(x) \; , \; k = 1, 2, \ldots, [n/2], |l| \leq n-2k\}, \qquad x \in B^3,$$

form a basis in V_n^3. By [14], the *Zernike polynomials* can be written in term of Gegenbauer polynomials as

$$Z_{n-2k,l}^{(n)}(x) = \frac{1}{4\pi} \int_{S^2} C_n^{(3/2)}(x \cdot \xi) Y_{n-2k,l}(\xi) d\xi, \tag{3.19}$$

where $Y_{n-2k,l}$ are spherical harmonics of degree $n-2k, k = 0, 1, \ldots, [n/2], |l| \leq n-2k, n = 0, 1, 2 \ldots.$

The *summation formulas for spherical harmonics* are given by

$$C_n^{(3/2)}(\eta\cdot\xi) = 4\pi \sum_{k=0}^{[n/2]}\sum_l Y_{n-2k,l}(\eta)\overline{Y_{n-2k,l}(\xi)}, \tag{3.20}$$

$$C_n^{(3/2)}(x\cdot\xi) = 4\pi \sum_{k=0}^{[n/2]}\sum_l Z_{n-2k,l}^{(n)}(x)\overline{Y_{n-2k,l}(\xi)}. \tag{3.21}$$

Furthermore two relationships for *Gegenbauer polynomials* are needed:

$$\int_{S^2} C_{n-2k}^{(3/2)}(x\cdot\xi)C_n^{3/2}(\xi\cdot\eta)d\xi = 4\pi C_{n-2k}^{3/2}(x\cdot\eta), \quad x\in B^3, \eta\in S^2, \tag{3.22}$$

$$k = 0,1,\ldots,[n/2]$$

$$\int_{B^3} C_n^{(3/2)}(x\cdot\xi)C_n^{3/2}(x\cdot\eta)d\xi = \frac{4\pi}{2n+3}C_n^{3/2}(\xi\cdot\eta), \quad \xi,\eta\in S^2. \tag{3.23}$$

The family $\{Z_{n-2k,l}^{(n)}\}$ forms an orthogonal basis of $L_2(B^3)$. The norm is given by

$$\|Z_{n-2k,l}^{(n)}\|_{L^2(B^3)} = \frac{1}{\sqrt{2n+3}}.$$

Any $\mathbf{f}\in L_2(B^3)$ has a unique representation in Zernike polynomials

$$\mathbf{f}(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\sum_l \mathbf{f}_{n-2k,l}^{(n)} Z_{n-2k,l}^{(n)}(x). \tag{3.24}$$

By applying (3.20) to (3.23) we then have,

$$\begin{aligned}\mathbf{f}(x) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\sum_l \mathbf{f}_{n-2k,l}^{(n)}\frac{1}{4\pi}\int_{S^2} Y_{n-2k,l}(\eta)C_n^{3/2}(x\cdot\eta)d\eta \\ &= \sum_{n=0}^{\infty}\frac{1}{4\pi}\int_{S^2}\mathbf{f}_n(\eta)C_n^{(3/2)}(x\cdot\eta)d\eta, \end{aligned}\tag{3.25}$$

where

$$\mathbf{f}_n(\eta) = \sum_{k=0}^{[n/2]}\sum_l \mathbf{f}_{n-2k,l}^{(n)} Y_{n-2k,l}(\eta). \tag{3.26}$$

By the virtue of (2.24), (2.26) and (2.27) we then have

$$\begin{aligned}\int_{B^3}\mathbf{f}(x)C_n^{3/2}(x\cdot\eta_1)dx &= \frac{1}{4\pi}\int_{B^3}\int_{S^2}\mathbf{f}_n(\eta)C_n^{3/2}(x\cdot\eta_1)C_n^{3/2}(x\cdot\eta)dxd\eta \\ &= \frac{1}{2n+3}\int_{S^2}\mathbf{f}_n(\eta)C_n^{3/2}(\eta\cdot\eta_1)d\eta \\ &= \frac{4\pi}{2n+3}\mathbf{f}_n(\eta_1).\end{aligned}$$

This implies

$$\mathbf{f}_n(\eta) = \frac{2n+3}{4\pi} \int_{B^3} \mathbf{f}(x) C_n^{3/2}(x \cdot \eta) dx. \tag{3.27}$$

By using (3.26) we have

$$\begin{aligned}\mathbf{f}(x) &= \sum_{n=0}^{\infty} \int_{S^2} \frac{2n+3}{16\pi^2} \int_{B^3} \mathbf{f}(y) C_n^{(3/2)}(x \cdot \eta) C_n^{3/2}(y \cdot \eta) dy d\eta \\ &= \sum_{n=0}^{\infty} \int_{B^3} \mathbf{f}(y) P_n(x, y) dy,\end{aligned}$$

with

$$P_n(x, y) = \frac{2n+3}{16\pi^2} \int_{S^2} C_n^{(3/2)}(x \cdot \eta) C_n^{3/2}(y \cdot \eta) dy d\eta. \tag{3.28}$$

If one of the variables of P_n, say $\xi \in S^2$, is a unit vector, then by using (3.23) we have,

$$P_n(x, \xi) = \frac{2n+3}{16\pi^2} \int_{S^2} C_n^{(3/2)}(x \cdot \eta) C_n^{3/2}(\xi \cdot \eta) d\eta = \frac{2n+3}{4\pi} C_n^{3/2}(x \cdot \xi). \tag{3.29}$$

Combining equation (3.21) and definition (3.30), we then have

$$P_n(x, y) = (2n+3) \sum_{k=0}^{[n/2]} \sum_{l} Z_{n-2k,l}^{(n)}(x) \overline{Z_{n-2k,l}^{(n)}(y)}. \tag{3.30}$$

$P_n(x, .)$ is called the reproducing kernel of V_n^3.

Definition 3.11 (Reproducing kernel of V_n^3). *A function $q(x, .)$ is called the reproducing kernel of V_n^3 if it satisfies*

$$p(x) = \int_{B^3} q(x, y) p(y) dy \quad \textit{for} \quad \textit{all}\ p \in V_n^3. \tag{3.31}$$

The orthogonal expansion can be stated as

$$\mathbf{f}(x) = \sum_{n=0}^{\infty} [\mathrm{Proj}_n \mathbf{f}](x) = \sum_{n=0}^{\infty} \mathbf{f}_n(x), \tag{3.32}$$

where

$$[\mathrm{Proj}_n \mathbf{f}](x) = \int_{B^3} \mathbf{f}(y) P_n(x, y) dy = \mathbf{f}_n(x).$$

Proposition 3.12. *Each vector field* $\mathbf{f} \in L^2(B^3)$ *has a unique representation as a series of orthogonal functions*

$$\begin{aligned}\mathbf{f}(x) &= \sum_{n=0}^{\infty} \mathbf{f}_n(x) = \sum_{n=0}^{\infty} \frac{1}{4\pi} \int_{S^2} \mathbf{f}_n(\eta) C_n^{3/2}(x\cdot\eta)d\eta \\ &= \sum_{n=0}^{\infty} \int_{B^3} \mathbf{f}(y) P_n(x,y)dy. \end{aligned} \tag{3.33}$$

The coefficient functions are defined on the unit sphere as

$$\mathbf{f}_n(\eta) = \frac{2n+3}{4\pi} \int_{B^3} \mathbf{f}(x) C_n^{3/2}(x\cdot\eta)dx \tag{3.34}$$

and have the representation

$$\mathbf{f}_n(\eta) = \sum_{k=0}^{[n/2]} \sum_{l} \mathbf{f}_{n-2k,l}^{(n)} Y_{n-2k,l}(\eta) = \frac{1}{4\pi} \int_{S^2} \mathbf{f}_n(\xi) C_n^{(3/2)}(\xi\cdot\eta)d\xi. \tag{3.35}$$

We also have the Parseval equality

$$\| \mathbf{f} \|_{L^2(B^3)}^2 = \sum_{n=0}^{\infty} \frac{1}{2n+3} \|\mathbf{f}_n\|_{L^2(S^2)}^2.$$

3.4.2 Orthogonal expansion of the solenoidal part of a vector field

This section will present a unique representation of any vector field in term of series of orthogonal bases of $\nabla H_0^1(B^3), \nabla \mathrm{H}(B^3)$ and $H_0(\mathrm{div}; B^3)$. Once again, we will follow the outline in [14] for notations and details to describe the procedures of the derivations.

We begin with spherical harmonics $Y_{n,l} \in L^2(S^2)$ and define vector spherical harmonics

$$y_{n,l}^{(1)}(\theta), \quad y_{n,l}^{(2)}(\theta), \quad y_{n,l}^{(3)}(\theta)$$

for $\theta \in S^2$ by

$$y_{n,l}^{(1)}(\theta) = \theta Y_{n,l}(\theta), \quad y_{n,l}^{(2)}(\theta) = \nabla_\theta Y_{n,l}(\theta), \quad y_{n,l}^{(3)}(\theta) = \theta \times y_{n,l}^{(2)}(\theta)$$

where ∇_θ denotes the surface gradient on S^2 and $\times$ the cross product on $\mathbb{R}^3$. From the definition of the three vectors, $y_{n,l}^{(i)}, i = 1,2,3$, it is clear that

$$\theta \times y_{n,l}^{(1)}(\theta) = 0, \quad \theta \cdot y_{n,l}^{(2)}(\xi) = 0, \quad \theta \cdot y_{n,l}^{(3)}(\theta) = 0.$$

Then the system

$$\left\{y_{0,0}^{(1)},\ y_{n,l}^{(j)}\ :\ n\in\mathbb{N},\ |l|\le n,\ j=1,2,3\right\}$$

forms a complete orthogonal system in $L_2(S^2)$. The vector spherical harmonics satisfy a Funk-Hecke theorem which gives the following formulas :

$$\int_{S^2} y_{n-1-2k,l}^{(1)}(\theta)C_n^{3/2}(\eta\cdot\theta)d\theta = 4\pi y_{n-1-2k,l}^{(1)}(\eta),\quad k=0,\ldots,[(n-1)/2], \tag{3.36}$$

$$\int_{S^2} y_{n+1,l}^{(2)}(\theta)C_n^{3/2}(\eta\cdot\theta)d\theta = \frac{4\pi(n+2)}{2n+3}\left((n+1)y_{n+1,l}^{(1)}(\eta)+y_{n+1,l}^{(2)}(\eta)\right) \tag{3.37}$$

$$\int_{S^2} y_{n+1-2k,l}^{(2)}(\theta)C_n^{3/2}(\eta\cdot\theta)d\theta = 4\pi y_{n+1-2k,l}^{(2)}(\eta),\quad k=1,\ldots,[n/2], \tag{3.38}$$

$$\int_{S^2} y_{n-2k,l}^{(3)}(\theta)C_n^{3/2}(\eta\cdot\theta)d\theta = 4\pi y_{n-2k,l}^{(3)}(\eta),\quad k=0,\ldots,[(n-1)/2]. \tag{3.39}$$

With the help of the vector valued functions $y_{n,l}^{(i)}$ Kazantsev and Schuster report that it is possible to generate orthogonal bases for $\nabla H_0^1(B^3), \nabla \mathrm{H}(B^3)$ and $H_0(\mathrm{div};B^3)$. Setting

$$A_{n-1-2k,l}^{(n)}(x) := \int_{S^2} y_{n-1-2k,l}^{(1)}(\theta)C_n^{3/2}(x\cdot\theta)d\theta,\quad n\ge 1 \tag{3.40}$$

$$B_{1,l}^{(0)}(x) := \int_{S^2} y_{1,l}^{(2)}(\theta)d\theta, \tag{3.41}$$

$$B_{n+1-2k,l}^{(n)}(x) := \int_{S^2} y_{n+1-2k,l}^{(2)}(\theta)C_n^{3/2}(x\cdot\theta)d\theta,\quad n\ge 1 \tag{3.42}$$

$$C_{n-2k,l}^{(n)}(x) := \int_{S^2} y_{n-2k,l}^{(3)}(\theta)C_n^{3/2}(x\cdot\theta)d\theta,\quad n\ge 1, \tag{3.43}$$

then

$$\left\{A_{n-1-2k,l}^{(n)}\ :\ n\in\mathbb{N},\ k=0,\ldots,[(n-1)/2],\ |l|\le n-1-2k\right\}$$

is an orthogonal basis of $\nabla H_0^1(B^3)$,

$$\left\{B_{n+1,l}^{(n)}\ :\ n\in\mathbb{N}_0,\ |l|\le n+1\right\}$$

is an orthonormal basis of $\nabla\mathrm{H}(B^3)$ and

$$\left\{B_{n+1-2k,l}^{(n)}\ :\ n\in\mathbb{N}, k=1,\ldots,[n/2],\ |l|\le n-1-2k\right\}$$

$$\cup \left\{ C^{(n)}_{n-2k,l} \ : \ n \in \mathbb{N}, k = 0, \ldots, [(n-1)/2], \ |l| \le n - 2k \right\}$$

is an orthogonal basis of $H_0(\mathrm{div}; B^3)$.

Let $\mathbf{f} \in L_2(B^3)$ be a solenoidal field. Then $\mathbf{f}$ has a unique representation

$$\begin{aligned} \mathbf{f}(x) = & \sum_{n=0}^{\infty} \sum_{l} b^{(n)}_{n+1,l} B^{(n)}_{n+1,l}(x) \\ & + \sum_{n=2}^{\infty} \sum_{k=1}^{[n/2]} \sum_{l} b^{(n)}_{n+1-2k,l} B^{(n)}_{n+1-2k,l}(x) + \sum_{n=1}^{\infty} \sum_{k=1}^{[(n-1)/2]} \sum_{l} c^{(n)}_{n-2k,l} C^{(n)}_{n-2k,l}(x). \end{aligned} \tag{3.44}$$

4 RESULTS

Any scalar-valued function is well known as the Laplacian or of the back-projection of its Radon transform ; that is :

$$f(x) = -\frac{1}{8\pi^2}\triangle\int_{S^2} Rf(x\cdot\theta,\theta)d\theta \qquad (*).$$

Many authors have proposed derivations of the proof, for example in [17, 5, 20, 18]. The first part of this chapter will provide an extension to vector-valued or quaternionic-valued functions. The form (4.7) is an alternative formulation of (*) in term of quaternion notation. We provide two proofs for (4.7). The first proof use elementary calculations. The second proof use the Cauchy kernel in term of the Dirac operator acting on a function in $\mathbb{R}^3$. In other words, the second proof uses the Radon decomposition mentioned in chapter 3. The first proof is much simpler. This is one of many variations of Radon inversion formulas and has both scalar and vector parts.

The second section (section 4.2) describes the cone beam reconstruction of the vector part of a quaternionic-valued function with sources on the sphere considered as a complete data type reconstruction. It requires the knowledge of the cone beam data for all sources on the sphere of certain radius and for all directions. We have used the symmetry property of the cone beam transform

$$D_a f(\frac{a-x}{|a-x|}) + D_a f(\frac{x-a}{|x-a|}) = D_x f(\frac{a-x}{|a-x|}) + D_x f(\frac{x-a}{|x-a|}) \qquad (4.1)$$

defined in [17] for a scalar case.

In the last part, which is section 4.3, we shall provide the reconstruction procedures for a whole vector field $\underline{\mathbf{f}}$ by using the transverse ray transform. We begin with the procedure for its solenoidal part. Then use the reprojection of the solenoidal part to recover the potential part according to the Helmholtz-Hodge decomposition (2.25). One assume under the condition that sources will be a curve and the curve satisfies certain condition called Tuy's condition of order three. Here we shall follow the outline in [14, 13]. The solenoidal part $\underline{\mathbf{f}}^d$ of $\underline{\mathbf{f}}$ is given by the back projection of the second derivative of its Radon transform. We split the Radon transform into the tangential and the normal parts. Thus $\underline{\mathbf{f}}^d$ has two parts, say $\mathbf{f}_1^d$ and $\mathbf{f}_2^d$. Both parts need the relation, for fixed $\underline{\eta} \in S^2$,

$$-\int_{S^2}\int_0^\infty \underline{\eta}\cdot(\underline{\theta}\times\underline{\mathbf{f}}(\underline{x}+t\underline{\theta}))dt\delta''(\underline{\eta}\cdot\underline{\theta})d\underline{\theta} = \mathrm{div}_{\underline{\eta}}\left(\underline{\eta}\times\frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\eta})\right)|_{s=\underline{\eta}\cdot\underline{x}} + \underline{x}\cdot\left(\underline{\eta}\times\frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\eta})\right)|_{s=\underline{\eta}\cdot\underline{x}} \quad (4.2)$$

to obtain $\underline{\eta}\times\partial_s\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\eta})$. We consider and define the inner integral on the left hand side of (4.2):

$$\int_0^\infty \underline{\eta}\cdot(\underline{\theta}\times\underline{\mathbf{f}}(\underline{x}+t\underline{\theta}))dt$$

as the transverse ray transform.

To reach the reconstruction procedure, we shall use the framework in [13] by splitting the solenoidal part $\underline{\mathbf{f}}^d$ into two parts, say $\underline{\mathbf{f}}_1^d$ and $\underline{\mathbf{f}}_2^d$ with

$$\underline{\mathbf{f}}_1^d(\underline{x}) = \int_{S^2}\partial_s^2\mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s,\underline{\eta})d\underline{\eta}, \quad \underline{\mathbf{f}}_2^d(\underline{x}) = \int_{S^2}\partial_s^2\mathcal{R}^{\mathrm{nor}}\underline{\mathbf{f}}^d(s,\underline{\eta})d\underline{\eta}. \quad (4.3)$$

By using simple geometric knowledge or using an elementary property of quaternions, both integrands in (4.3) can be replaced by $\underline{\eta}\times\partial_s^2\mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s,\underline{\eta})$ and $\underline{\eta}\times\partial_s^2\mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s,\underline{\eta})$, respectively. Then we apply the measurements (4.2) to get the reconstruction procedure.

4.1 A variation on Radon inversion formula

Recall that, in section 2.6 equation (2.14), the plane wave decomposition of the fundamental solution of the Dirac operator $\partial_{x_0+\underline{x}}$ in $\mathbb{R}^4$ reads

$$\frac{1}{a_4}\frac{x_0-\underline{x}}{|x_0-\underline{x}|^4} = \frac{1}{(2\pi)^3}\int_{S^2}(<\underline{x},\underline{t}> -x_0\underline{t})^{-3}\underline{t}d\underline{t} \quad (4.4)$$

where a_4 is the surface area of the unit sphere S^3 in $\mathbb{R}^4$. Also, by straightforward computation we have, for each $i = 1, 2, 3$,

$$e_i \frac{\partial}{\partial x_i}(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-1} = -e_i(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-2} t_i.$$

Therefore, for $\partial_{\underline{x}} = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}$,

$$\begin{aligned} \partial_{\underline{x}}(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-1} &= (e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3})(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-1} \\ &= -\underline{t}(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-2}. \end{aligned} \tag{4.5}$$

To get $\partial_{\underline{x}}^2(< \underline{x}, \underline{t} - x_0 \underline{t} >)^{-1}$ we apply $\partial_{\underline{x}}$ to both sides of (4.5) and use the generalized Leibniz rule which leads to

$$\begin{aligned} -\partial_{\underline{x}}^2(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-1} &= \partial_{\underline{x}}[\underline{t}(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-2}] \\ &= (\partial_{\underline{x}} \underline{t})(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-2} - \underline{t} \partial_{\underline{x}}(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-2} \\ &+ 2(Re\ \underline{t} \partial_{\underline{x}})(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-2} \\ &= -2(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-3} \\ &- 2(t_1 \frac{\partial}{\partial x_1} + t_2 \frac{\partial}{\partial x_2} + t_3 \frac{\partial}{\partial x_3})(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-2} \\ &= -2(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-3} + 4(t_1^2 + t_2^2 + t_3^2)(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-3} \\ &= 2(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-3}. \end{aligned}$$

Hence we obtain the following lemma:

Lemma 4.1. *For* $x = x_0 + \underline{x} \in \mathbb{R}^4/\mathbb{R}^3 = \mathbb{R}^4_{\pm}$,

$$\frac{1}{a_4} \frac{x_0 - \underline{x}}{|x_0 - \underline{x}|^4} = -\frac{1}{2(2\pi)^3} \partial_x \int_{S^2} \underline{t}(< \underline{x}, \underline{t} > -x_0 \underline{t})^{-2} \underline{t} d\underline{t}. \tag{4.6}$$

Proof.

$$\begin{aligned}
\frac{1}{a_4}\frac{x_0-\underline{x}}{|x_0-\underline{x}|^4} &= \frac{1}{(2\pi)^3}\int_{S^2}(<\underline{x},\underline{t}>-x_0\underline{t})^{-3}\underline{t}d\underline{t}\\
&= -\frac{1}{(2\pi)^3}\int_{S^2}\frac{1}{2}\partial_{\underline{x}}^2(<\underline{x},\underline{t}>-x_0\underline{t})^{-1}\underline{t}d\underline{t}\\
&= -\frac{1}{2(2\pi)^3}\int_{S^2}\partial_{\underline{x}}^2(<\underline{x},\underline{t}>-x_0\underline{t})^{-1}\underline{t}d\underline{t}\\
&= -\frac{1}{2(2\pi)^3}\partial_{\underline{x}}\int_{S^2}\partial_{\underline{x}}(<\underline{x},\underline{t}>-x_0\underline{t})^{-1}\underline{t}d\underline{t}\\
&= -\frac{1}{2(2\pi)^3}\partial_{\underline{x}}\int_{S^2}\underline{t}(<\underline{x},\underline{t}>-x_0\underline{t})^{-2}\underline{t}d\underline{t}.
\end{aligned}$$

We can interchange the integral and the Dirac operator because the integrand is bounded away from zero. □

In Chapter 3, we have seen that a quaternionic-valued function $\underline{\mathbf{f}}$ is the differences of the non-tangential boundary value of the Cauchy transform $\mathcal{C}[\underline{\mathbf{f}}]$. By applying Lemma 4.1 to $\mathcal{C}[\mathbf{f}]$ and interchanging the integrals we then have for $\underline{\mathbf{f}} \in \mathcal{S}(\mathbb{R}^3)$ and $x \in \mathbb{R}^4_\pm$,

$$\begin{aligned}
\mathcal{C}[\mathbf{f}](x_0+\underline{x}) &= \int_{\mathbb{R}^3}\frac{x_0-\underline{x}+\underline{u}}{|x_0-\underline{x}+\underline{u}|^4}\underline{\mathbf{f}}(\underline{u})d\underline{u}\\
&= \int_{\mathbb{R}^3}\frac{-2\pi^2}{2(2\pi)^3}\partial_{\underline{x}}\int_{S^2}\underline{t}(<\underline{x}-\underline{u},\underline{t}>-x_0\underline{t})^{-2}\underline{t}d\underline{t}\underline{\mathbf{f}}(\underline{u})d\underline{u}\\
&= -\frac{1}{8\pi}\partial_{\underline{x}}\int_{S^2}\int_{\mathbb{R}^3}\underline{t}(<\underline{x}-\underline{u},\underline{t}>-x_0\underline{t})^{-2}\underline{t}\underline{\mathbf{f}}(\underline{u})d\underline{u}d\underline{t}.
\end{aligned}$$

The following theorem is the inversion formula in term of Dirac operator.

Theorem 4.2. *For a quaternionic function* $\underline{\mathbf{f}} \in \mathcal{S}(\mathbb{R}^3)$,

$$\underline{\mathbf{f}}(\underline{x}) = -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\ \mathcal{R}\underline{\mathbf{f}}\ '(\underline{\theta}\cdot\underline{x},\underline{\theta})d\underline{\theta}. \tag{4.7}$$

We shall provide two proofs for this theorem. The first one has a much simpler proof by using fact that $\partial_{\underline{x}}^2 = -\Delta$ where Δ is the Laplacian in a Euclidian space as follows

$$\underline{\mathbf{f}}(\underline{x}) = \frac{1}{8\pi^2}\Delta\int_{S^2}\mathcal{R}\underline{\mathbf{f}}(\underline{x}\cdot\underline{\theta},\underline{\theta})d\underline{\theta} = -\frac{1}{8\pi^2}\partial_{\underline{x}}^2\int_{S^2}\mathcal{R}\underline{\mathbf{f}}(\underline{x}\cdot\underline{\theta},\underline{\theta})d\underline{\theta} = -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\partial_{\underline{x}}\mathcal{R}\underline{\mathbf{f}}(\underline{x}\cdot\underline{\theta},\underline{\theta})d\underline{\theta} \tag{4.8}$$

$$= \frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\partial_{\underline{x}}\cdot\mathcal{R}\underline{\mathbf{f}}(\underline{x}\cdot\underline{\theta},\underline{\theta})d\underline{\theta} - \frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\partial_{\underline{x}}\times\mathcal{R}f(\underline{x}\cdot\underline{\theta},\underline{\theta})d\underline{\theta}$$

$$= \frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\cdot\mathcal{R}'\underline{\mathbf{f}}(\underline{x}\cdot\underline{\theta},\underline{\theta})d\underline{\theta} - \frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\times\mathcal{R}'\underline{\mathbf{f}}(\underline{x}\cdot\underline{\theta},\underline{\theta})d\underline{\theta} \tag{4.9}$$

$$= -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\mathcal{R}'\underline{\mathbf{f}}(\underline{x}\cdot\underline{\theta},\underline{\theta})d\underline{\theta}. \tag{4.10}$$

The second proof shall begin from the non-tangential boundary value of the Cauchy transform of a function defined by the integration of convolution of Cauchy kernel and the function. Then the equation (4.4) and Lemma 4.1 will be used, respectively.

Proof. The non-tangential boundary value of the Cauchy transform $\mathcal{C}[\mathbf{f}]$ gives $\underline{\mathbf{f}}$ as follows:

$$\begin{aligned}\underline{\mathbf{f}}(\underline{x}) &= \mathcal{C}[\mathbf{f}](\underline{x}+0^+) - \mathcal{C}[\mathbf{f}](\underline{x}-0^-)\\ &= -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\int_{\mathbb{R}^3}\underline{\theta}\ \underline{\theta}\ \delta'(<\underline{x}-\underline{u},\underline{\theta}>)\underline{\theta}\underline{\mathbf{f}}(\underline{u})d\underline{u}d\underline{\theta}\\ &= \frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\int_{\mathbb{R}^3}\delta'(<\underline{x}-\underline{u},\underline{\theta}>)\underline{\theta}\underline{\mathbf{f}}(\underline{u})d\underline{u}d\underline{\theta}\end{aligned}$$

where we have applied lemma 3.2 in the second equation. Let $\underline{y} = \underline{x} - \underline{u}$ so $\underline{u} = \underline{x} - \underline{y}$ and

$$\begin{aligned}\underline{\mathbf{f}}(\underline{x}) &= -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\int_{\mathbb{R}^3}\delta'(<\underline{y},\underline{\theta}>)\underline{\theta}\underline{\mathbf{f}}(\underline{x}-\underline{y})d\underline{y}d\underline{\theta}\\ &= -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\int_{\mathbb{R}^3}\delta'(<|\underline{y}|\frac{\underline{y}}{|\underline{y}|},\underline{\theta}>)\underline{\theta}\underline{\mathbf{f}}(\underline{x}-\underline{y})d\underline{y}d\underline{\theta}.\end{aligned}$$

Apply polar coordinate in the inner integral by letting $y = \rho\underline{w}$ where $\rho > 0$ and $\underline{\omega} \in S^2$. It follows that

$$\underline{\mathbf{f}}(\underline{x}) = -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\int_{S^2}\int_0^\infty\delta'(<\rho\underline{\omega},\underline{\theta}>)\underline{\theta}\underline{\mathbf{f}}(\underline{x}-\rho\underline{\omega})\rho^2 d\rho d\underline{\omega}d\underline{\theta}.$$

Since δ' is a homogeneous function of degree -2, we then reach to the reconstruction formula as:

$$\begin{aligned}
\underline{\mathbf{f}}(\underline{x}) &= -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\int_{S^2}\int_0^\infty \rho^{-2}\delta'(<\underline{\omega},\underline{\theta}>)\underline{\theta}\underline{\mathbf{f}}(\underline{x}-\rho\underline{\omega})\rho^2 d\rho d\underline{\omega} d\underline{\theta} \\
&= -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\int_{S^2}\delta'(<\underline{\omega},\underline{\theta}>)\int_0^\infty \underline{\mathbf{f}}(\underline{x}-\rho\underline{\omega})d\rho d\underline{\omega} d\underline{\theta} \\
&= -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\int_{S^2}\delta'(<\underline{\omega},\underline{\theta}>)D\underline{\mathbf{f}}(x,-\underline{\omega})d\underline{\omega} d\underline{\theta} \qquad (4.11)\\
&= -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\ \mathcal{R}\underline{\mathbf{f}}'(<\underline{\theta},\underline{\theta}\cdot x>)d\underline{\theta}
\end{aligned}$$

where we have applied the Grangeat's formula to the third equation. □

The theorem has an interesting special case as follows: if $\underline{\mathbf{f}} = f_1 e_1$, then

$$\begin{aligned}
\underline{\mathbf{f}}(\underline{x}) &= -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}(\mathcal{R}f_1)'e_1(s,\underline{\theta})|_{s=\underline{x}\cdot\underline{\theta}}d\underline{\theta} \\
&= -\frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}(\mathcal{R}f_1)'(s,\underline{\theta})|_{s=\underline{x}\cdot\underline{\theta}}d\underline{\theta}e_1 \\
&= -\frac{1}{8\pi^2}\left[\nabla\cdot\int_{S^2}\underline{\theta}(\mathcal{R}f_1)'(s,\underline{\theta})|_{s=\underline{x}\cdot\underline{\theta}}d\underline{\theta}\right]e_1 \\
&= -\frac{1}{8\pi^2}\int_{S^2}(\mathcal{R}f_1)''(<\underline{\theta},\underline{x}\cdot\underline{\theta}>)d\underline{\theta}e_1
\end{aligned}$$

where we used the associative rule for quaternions and that f_1 is real-valued.

In particular, if f is scalar, then Rf is real as well and the scalar part of $f(x)$ is

$$\begin{aligned}
f(\underline{x}) &= -\frac{1}{8\pi^2}\nabla\cdot\int_{S^2}\underline{\theta}(\mathcal{R}f)'(\underline{\theta},\underline{x}\cdot\underline{\theta})d\theta \\
&= -\frac{1}{8\pi^2}\int_{S^2}\sum\theta_j\frac{\partial}{\partial_{x_j}}(\mathcal{R}f)'(\underline{\theta},\underline{x}\cdot\underline{\theta})d\underline{\theta} \\
&= -\frac{1}{8\pi^2}\int_{S^2}(\mathcal{R}f)''(\underline{\theta},\underline{x}\cdot\underline{\theta})\sum\theta_j^2 d\underline{\theta} \\
&= -\frac{1}{8\pi^2}\int_{S^2}(\mathcal{R}f)''(\underline{\theta},\underline{x}\cdot\underline{\theta})d\underline{\theta}.
\end{aligned}$$

We furthermore point out that $\underline{\mathbf{f}}$ can be decomposed as from (4.9), one can see that any $\underline{\mathbf{f}}$ can be decomposed similarly to the Helmholtz decomposition in the case of vector fields as

$$\underline{\mathbf{f}}(\underline{x}) = -\frac{1}{8\pi 2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}(\mathcal{R}\underline{\mathbf{f}})'(s,\underline{\theta})|_{s=\underline{x}\cdot\underline{\theta}}d\underline{\theta}$$
$$= \frac{1}{8\pi^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\cdot(\mathcal{R}\underline{\mathbf{f}})'(s,\underline{\theta})|_{s=\underline{x}\cdot\underline{\theta}}d\underline{\theta} - \frac{1}{8\pi^2}\partial_{\underline{x}}\times\int_{S^2}\underline{\theta}\times(\mathcal{R}\underline{\mathbf{f}})'(s,\underline{\theta})|_{s=\underline{x}\cdot\underline{\theta}}d\underline{\theta}. \quad (4.12)$$

Since $\underline{\mathbf{f}}$ is a vector-valued function, the real part

$$\partial_{\underline{x}}\cdot\int_{S^2}\underline{\theta}\times(\mathcal{R}\underline{\mathbf{f}})'(s,\underline{\theta})|_{s=\underline{x}\cdot\underline{\theta}}d\underline{\theta}$$

vanishes.

4.2 The Cone Beam Reconstruction With Sources on The Sphere

The simplest situation for the divergent beam transform occurs when we restrict our attention to functions with support inside a sphere in $\mathbb{R}^3$. Similar to the reconstruction in the case of scalar-valued function, the following lemma is crucial.

Lemma 4.3. *Let S_r be an 2 sphere of radius r, let $\underline{x}$ be a point in $\mathbb{R}^3$ which is inside S_r and let $\underline{\mathbf{f}}$ be an integrable function on S^2. Then*

$$\int_{S^2}\underline{\mathbf{f}}(\underline{\theta})d\underline{\theta} = \frac{1}{r}\int_{S_r}\underline{\mathbf{f}}\left(\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|}\right)|\underline{a}-\underline{x}|^{-2}|<\underline{a},\underline{a}-\underline{x}>|d\underline{a} \quad (4.13)$$

where $d\underline{\theta}$ is the Lebesgue measure on the unit sphere S^2, $d\underline{a}$ is the measure on the sphere S_r and the integration is to be understood componentwise. The scalar case of this lemma can be found in [17] and the proof of this lemma can be done easily by the componentwise definition.

Theorem 4.4. *For $\underline{x}\in\mathbb{R}^3$, then*

$$\underline{\mathbf{f}}(\underline{x}) = -\frac{1}{16\pi}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\frac{1}{r}\int_{S_r}\delta'(<\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|},\underline{\theta}>)\left(D_{\underline{a}}\underline{\mathbf{f}}(\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|}) + D_{\underline{a}}\underline{\mathbf{f}}(\frac{\underline{x}-\underline{a}}{|\underline{x}-\underline{a}|})\right)$$
$$\times|\underline{a}-\underline{x}|^{-2}|<\underline{a},\underline{a}-\underline{x}>|d\underline{a}d\theta \quad (4.14)$$

Proof. We start from theorem 4.2. Using the fact that δ' is homogeneous of degree -2 and for any (scalar-valued) quaternionic-valued function function $\underline{\mathbf{g}}$ on S^2,

$$\int_{S^2} \underline{\mathbf{g}}(\underline{\theta})d\underline{\theta} = \int_{S^2} \underline{\mathbf{g}}(-\underline{\theta})d\underline{\theta} \tag{4.15}$$

we have the following equations:

$$\begin{aligned}\underline{\mathbf{f}}(\underline{x}) &= -\frac{1}{8\pi}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\int_{S^2}\delta'(<\underline{\omega},\underline{\theta}>)D_{\underline{x}}\underline{\mathbf{f}}(\underline{\omega})d\underline{\omega}d\theta \\ &= -\frac{1}{8\pi}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\int_{S^2}\delta'(<-\underline{\omega},\underline{\theta}>)D_{\underline{x}}\underline{\mathbf{f}}(-\underline{\omega})d\underline{\omega}d\theta \\ &= -\frac{1}{8\pi}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\int_{S^2}\delta'(<\underline{\omega},\underline{\theta}>)D_{\underline{x}}\underline{\mathbf{f}}(-\underline{\omega})d\underline{\omega}d\theta\end{aligned}$$

since $\delta'(<\underline{\omega},\underline{\theta}>) = \delta'(-<\underline{\omega},\underline{\theta}>)$. Therefore,

$$\begin{aligned}\underline{\mathbf{f}}(\underline{x}) &= -\frac{1}{16\pi}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\int_{S^2}\delta'(<\underline{\omega},\underline{\theta}>)(D_{\underline{x}}\underline{\mathbf{f}}(\underline{\omega}) + D_{\underline{x}}\underline{\mathbf{f}}(-\underline{\omega}))d\underline{\omega}d\theta \\ &= -\frac{1}{16\pi}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\frac{1}{r}\int_{S_r}\delta'(<\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|},\underline{\theta}>)\left(D_{\underline{x}}\underline{\mathbf{f}}(\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|}) + D_{\underline{x}}\underline{\mathbf{f}}(\frac{\underline{x}-\underline{a}}{|\underline{x}-\underline{a}|})\right) \\ &\quad \times |\underline{a}-\underline{x}|^{-2}|<\underline{a},\underline{a}-\underline{x}>|d\underline{a}d\theta \\ &= -\frac{1}{16\pi}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\frac{1}{r}\int_{S_r}\delta'(<\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|},\underline{\theta}>)\left(D_{\underline{a}}\underline{\mathbf{f}}(\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|}) + D_{\underline{a}}\underline{\mathbf{f}}(\frac{\underline{x}-\underline{a}}{|\underline{x}-\underline{a}|})\right) \\ &\quad \times |\underline{a}-\underline{x}|^{-2}|<\underline{a},\underline{a}-\underline{x}>|d\underline{a}d\theta\end{aligned} \tag{4.16}$$

by the symmetry property of the line integral transform

$$D_{\underline{a}}\underline{\mathbf{f}}(\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|}) + D_{\underline{a}}\underline{\mathbf{f}}(\frac{\underline{x}-\underline{a}}{|\underline{x}-\underline{a}|}) = D_{\underline{x}}\underline{\mathbf{f}}(\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|}) + D_{\underline{x}}\underline{\mathbf{f}}(\frac{\underline{x}-\underline{a}}{|\underline{x}-\underline{a}|}). \tag{4.17}$$

□

Furthermore, we have the reconstruction formula of $\underline{\mathbf{f}}$ with sources on the sphere in term of the Doppler transform as the following theorem:

Theorem 4.5. *Assume that $\underline{\mathbf{f}}$ has compact support in the unit ball B^3. Then*

$$\begin{aligned}\underline{\mathbf{f}}(\underline{x}) = \frac{1}{12(2\pi)^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\frac{1}{r}\int_{S_r}\delta'(<\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|},\underline{\theta}>)\underline{\omega}\left(D_{\underline{a}}\underline{\omega}\underline{\mathbf{f}}(\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|}) + D_{\underline{a}}\underline{\omega}\underline{\mathbf{f}}(\frac{\underline{x}-\underline{a}}{|\underline{x}-\underline{a}|})\right) \\ \times |\underline{a}-\underline{x}|^{-2}|<\underline{a},\underline{a}-\underline{x}>|d\underline{a}d\theta.\end{aligned}$$

Proof. Recall that from (4.10) we obtain

$$\begin{aligned}
\underline{\mathbf{f}}(\underline{x}) &= -\frac{1}{2(2\pi)^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\int_{S^2}\delta'(<\underline{\omega},\underline{\theta}>)D_{\underline{x}}\underline{\mathbf{f}}(-\underline{\omega})d\underline{\omega}d\underline{\theta} \\
&= \frac{1}{2(2\pi)^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\int_{S^2}\delta'(<\underline{\omega},\underline{\theta}>)\underline{\omega}D_{\underline{x}}\underline{\omega}\underline{\mathbf{f}}(-\underline{\omega})d\underline{\omega}d\underline{\theta}. \qquad (4.18) \\
&= \frac{1}{2(2\pi)^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\int_{S^2}\delta'(<\underline{\omega},\underline{\theta}>)\underline{\omega}D_{\underline{x}}\underline{\omega}\underline{\mathbf{f}}(\underline{\omega})d\underline{\omega}d\underline{\theta} \qquad (4.19)
\end{aligned}$$

By combining (4.18) and (4.19) we have

$$\begin{aligned}
\underline{\mathbf{f}}(x) &= \frac{1}{12(2\pi)^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\int_{S^2}\delta'(<\underline{\omega},\underline{\theta}>)\underline{\omega}(D_{\underline{x}}\underline{\omega}f(\underline{\omega})+D_{\underline{x}}\underline{\omega}\underline{\mathbf{f}}(-\underline{\omega}))d\underline{\omega}d\theta \\
&= \frac{1}{12(2\pi)^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\frac{1}{r}\int_{S_r}\delta'(<\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|},\underline{\theta}>)\underline{\omega}\left(D_{\underline{x}}\underline{\omega}\underline{\mathbf{f}}(\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|})+D_{\underline{x}}\underline{\omega}\underline{\mathbf{f}}(\frac{\underline{x}-\underline{a}}{|\underline{x}-\underline{a}|})\right) \\
&\quad\times|\underline{a}-\underline{x}|^{-2}|<\underline{a},\underline{a}-\underline{x}>|d\underline{a}d\theta \\
&= \frac{1}{12(2\pi)^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}\frac{1}{r}\int_{S_r}\delta'(<\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|},\underline{\theta}>)\underline{\omega}\left(D_{\underline{a}}\underline{\omega}\underline{\mathbf{f}}(\frac{\underline{a}-\underline{x}}{|\underline{a}-\underline{x}|})+D_{\underline{a}}\underline{\omega}\underline{\mathbf{f}}(\frac{\underline{x}-\underline{a}}{|\underline{x}-\underline{a}|})\right) \\
&\quad\times|\underline{a}-\underline{x}|^{-2}|<\underline{a},\underline{a}-\underline{x}>|d\underline{a}d\theta. \qquad (4.20)
\end{aligned}$$

□

We can also mention that the lambda operator can be viewed as the operator $\mathcal{H}\partial_{\underline{x}}$ in the quaternion notation. Recall that for the scalar-valued function f,

$$f = \Lambda(R_1 * f) \quad \text{where } R_1 \text{ is a Riesz potential} \qquad (4.21)$$

and the Riesz potential is defined by

$$R_\alpha(x) = \rho_{\alpha,n}|x|^{\alpha-n}, \qquad \rho_{\alpha,n} = \frac{\Gamma((n-\alpha)/2)}{2^\alpha\pi^{n/2}\Gamma(\alpha/2)}.$$

With the help of quaternion distributions one can see that

$$(R_1 * \underline{\mathbf{f}}) = \rho_{\alpha,n}(T^*_{1-m}\underline{\mathbf{f}}).$$

Then,

$$\begin{aligned}\mathcal{H}\partial_{\underline{x}}(R_1 * \underline{\mathbf{f}}) &= \mathcal{H}\partial_{\underline{x}}(T^*_{1-m} * \underline{\mathbf{f}}) = U^*_{-3} * (\partial_{\underline{x}}(T^*_{1-m} * \underline{\mathbf{f}})) \\ &= (U^*_{-m}\partial_{\underline{x}}) * (R^*_{1-m} * \underline{\mathbf{f}}) = T^*_{-m-1} * (T^*_{1-m} * \underline{\mathbf{f}}) \\ &= (T^*_{-m-1} * T^*_{1-m}) * \underline{\mathbf{f}}) = \delta(\underline{x}) * \underline{\mathbf{f}} = \underline{\mathbf{f}}. \qquad (4.22)\end{aligned}$$

4.3 The Cone Beam Reconstruction for the Transverse ray Transform with Sources on a Curves Satisfying Tuy's Condition of order 3

In this section we present the reconstruction formula of a vector field supported in the unit ball with sources on any curve located outside an object, provided the curve satisfies geometric conditions which state that any plane that passes through the unit ball intersects the curve in at least three points that are not located on a line. We begin by introducing the normal and the tangential parts of the first derivative of the Radon transform as in [14, 13].

Define the normal part of the first derivative of the Radon transform of a vector field $\underline{\mathbf{f}}$, denoted by $\mathcal{R}^{\text{nor}}[\underline{\mathbf{f}}](s,\underline{\eta})$ as

$$\mathcal{R}^{\text{nor}}[\underline{\mathbf{f}}](s,\underline{\eta}) = \underline{\eta}\left(\underline{\eta}\cdot\frac{\partial}{\partial s}\mathcal{R}\underline{\mathbf{f}}(s,\underline{\eta})\right) \qquad (4.23)$$

and its tangential part as

$$\begin{aligned}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\eta}) &= \frac{\partial}{\partial s}\mathcal{R}\underline{\mathbf{f}}(s,\underline{\eta}) - \mathcal{R}^{\text{nor}}[\underline{\mathbf{f}}](s,\underline{\eta}) \\ &= \frac{\partial}{\partial s}\mathcal{R}\underline{\mathbf{f}}(s,\underline{\eta}) - \underline{\eta}\left(\underline{\eta}\cdot\frac{\partial}{\partial s}\mathcal{R}\underline{\mathbf{f}}(s,\underline{\eta})\right) \\ &= \left(\underline{\eta}\times\frac{\partial}{\partial s}\mathcal{R}\underline{\mathbf{f}}(s,\underline{\eta})\right)\times\underline{\eta}.\end{aligned}$$

To reach reconstruction procedures, the following theorem is the main ingredient which has not been introduced in any work and is new here.

Theorem 4.6. *Let* $\underline{\eta} \in S^2$, $\underline{x} \in \mathbb{R}^3$, *and let* $\underline{\mathbf{f}}$ *be a vector field. Then*

$$div_{\underline{\eta}}(\underline{\eta} \times \mathcal{R}^{(\tan)}\underline{\mathbf{f}})(\underline{x} \cdot \underline{\eta}, \underline{\eta}) = (div_{\underline{\eta}}(\underline{\eta} \times \mathcal{R}^{(\tan)}\underline{\mathbf{f}})(s, \underline{\eta}))|_{s=\underline{x}\cdot\underline{\eta}} + \underline{x} \cdot \left(\underline{\eta} \times \frac{\partial}{\partial s}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta})\right)|_{s=\underline{x}\cdot\underline{\eta}} \tag{4.24}$$

where $div_{\underline{\eta}}$ *is the surface divergence defined in (2.28).*

We begin the proof of the theorem by observing that $\mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta}) \in \underline{\eta}^{\perp}$ and by considering its expression with respect to an orthonormal basis of $\underline{\eta}^{\perp}$ as

$$\mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta}) = X_{\varphi}(s, \underline{\eta})e_1(\underline{\eta}) + X_{\theta}(s, \underline{\eta})e_2(\eta) \tag{4.25}$$

$$\mathcal{R}^{\tan}\underline{\mathbf{f}}(\underline{\eta} \cdot \underline{x}, \underline{\eta}) = X_{\varphi}(\underline{\eta} \cdot x, \underline{\eta})e_1(\underline{\eta}) + X_{\theta}(\underline{\eta} \cdot \underline{x}, \underline{\eta})e_2(\underline{\eta}). \tag{4.26}$$

Here $e_1(\underline{\eta}), e_2(\underline{\eta})$ are two orthogonal vectors spanning the tangent plane $\underline{\eta}^{\perp}$ and these notations are defined to be different from the notations on orthonormal basis of the embedded $\mathbb{R}^3$ in $\mathbb{H}$ in chapter 2. $X_{\varphi}(s, \underline{\eta})$ and $X_{\theta}(s, \underline{\eta})$ are real coefficients corresponding the expression. Here

$$\underline{\eta} = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix} \quad \text{where} \quad \varphi \in [0, 2\pi], \theta \in (0, \pi),$$

$$e_1(\underline{\eta}) = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\underline{\eta}, \quad e_2(\underline{\eta}) = \frac{\partial}{\partial\theta}\underline{\eta}.$$

The following two lemmas will be useful in the proof of the theorem 4.6. The first one which is lemma 4.7 has been originally established in [14].

Lemma 4.7. *For* $\mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta})$ *defined as above,*

$$div_{\underline{\eta}}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta}) = \frac{\partial}{\partial\theta}X_{\theta}(s, \underline{\eta}) + \cot\theta X_{\theta}(s, \underline{\eta}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}X_{\varphi}(s, \underline{\eta}). \tag{4.27}$$

Proof. Applying the definition of surface divergence from (2.28) to $\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\eta})$ gives

$$\begin{aligned}\mathrm{div}_{\underline{\eta}}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\eta}) &= \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta X_\theta(s,\underline{\eta}))\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}X_\varphi(s,\underline{\eta})\\ &= \frac{1}{\sin\theta}\left(\sin\theta\frac{\partial}{\partial\theta}X_\theta(s,\underline{\eta})+X_\theta(s,\underline{\eta})\cos\theta\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}X_\varphi(s,\underline{\eta})\\ &= \frac{\partial}{\partial\theta}X_\theta(s,\underline{\eta})+\cot\theta X_\theta(s,\underline{\eta})+\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}X_\varphi(s,\underline{\eta}).\end{aligned}\tag{4.28}$$

□

Similarly,

$$\mathrm{div}_{\underline{\eta}}\mathcal{R}^{\tan}\underline{\mathbf{f}}(\underline{x}\cdot\underline{\eta},\underline{\eta}) = \frac{\partial}{\partial\theta}X_\theta(\underline{x}\cdot\underline{\eta},\underline{\eta})+\cot\theta X_\theta(\underline{x}\cdot\underline{\eta},\underline{\eta})+\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}X_\varphi(\underline{x}\cdot\underline{\eta},\underline{\eta}).\tag{4.29}$$

We also note that

$$\underline{x}\cdot\frac{\partial}{\partial s}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\eta})\mid_{s=\underline{x}\cdot\underline{\eta}}=\frac{\partial}{\partial s}X_\varphi(s,\underline{\eta})\mid_{s=\underline{x}\cdot\underline{\eta}}(\underline{x}\cdot e_1(\underline{\eta}))+\frac{\partial}{\partial s}X_\theta(s,\underline{\eta})\mid_{s=\underline{x}\cdot\underline{\eta}}(\underline{x}\cdot e_2(\underline{\eta})).\tag{4.30}$$

By the chain rule

$$\frac{\partial}{\partial\theta}X_\theta(\underline{x}\cdot\underline{\eta},\underline{\eta}) = \frac{\partial}{\partial\theta}X_\theta(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}}+\frac{\partial}{\partial s}X_\theta(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}}(\underline{x}\cdot e_2(\underline{\eta}))\tag{4.31}$$

and

$$\frac{\partial}{\partial\varphi}X_\varphi(\underline{x}\cdot\underline{\eta},\underline{\eta}) = \frac{\partial}{\partial\varphi}X_\varphi(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}}+\frac{\partial}{\partial s}X_\varphi(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}}\sin\theta(\underline{x}\cdot e_1(\underline{\eta}))\tag{4.32}$$

where we have used that

$$\frac{\partial}{\partial\theta}(\underline{x}\cdot\underline{\eta}) = \underline{x}\cdot e_2(\underline{\eta})\quad\text{and}\quad\frac{\partial}{\partial\varphi}(\underline{x}\cdot\underline{\eta}) = \sin\theta(\underline{x}\cdot e_1(\underline{\eta})).\tag{4.33}$$

By the virtue of the equations (4.28-4.33), the equation (4.27) can be recast as

$$\mathrm{div}_{\underline{\eta}}\mathcal{R}^{(\tan)}\underline{\mathbf{f}}(\underline{x}\cdot\underline{\eta},\underline{\eta}) = (\mathrm{div}_{\underline{\eta}}\mathcal{R}^{(\tan)}\underline{\mathbf{f}}(s,\underline{\eta}))\mid_{s=\underline{x}\cdot\underline{\eta}}+\underline{x}\cdot\frac{\partial}{\partial s}\mathcal{R}^{tan}\underline{\mathbf{f}}(s,\underline{\eta})\mid_{s=\underline{x}\cdot\underline{\eta}}\tag{4.34}$$

where we have used the fact that $\mathcal{R}^{(\tan)}\underline{\mathbf{f}}(\underline{x}\cdot\underline{\eta},\underline{\eta})$ is a tangential vector field. Lemma 4.8 will be presented here for the first time which we slightly modify form lemma 4.7.

Lemma 4.8.

$$div_{\underline{\eta}}(\underline{\eta} \times \mathcal{R}^{(\tan)}\underline{\mathbf{f}})(s, \underline{\eta}) = -\cot\theta X_{\varphi}(s, \underline{\eta}) - \frac{\partial}{\partial\theta} X_{\varphi}(s, \underline{\eta}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} X_{\theta}(s, \underline{\eta}) \tag{4.35}$$

and

$$div_{\underline{\eta}}(\underline{\eta} \times \mathcal{R}^{(\tan)}\underline{\mathbf{f}})(\underline{x}\cdot\underline{\eta}, \underline{\eta}) = -\cot\theta X_{\varphi}(\underline{x}\cdot\underline{\eta}, \underline{\eta}) - \frac{\partial}{\partial\theta} X_{\varphi}(\underline{x}\cdot\underline{\eta}, \underline{\eta}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} X_{\theta}(\underline{x}\cdot\underline{\eta}, \underline{\eta}). \tag{4.36}$$

Proof. From the equation (4.25) and for each $\underline{\eta} \in S^2$, we get

$$\begin{aligned}\underline{\eta} \times \mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta}) &= X_{\varphi}(s, \underline{\eta})(\underline{\eta} \times e_1(\underline{\eta})) + X_{\theta}(s, \underline{\eta})(\eta \times e_2(\underline{\eta})) \\ &= X_{\theta}(s, \underline{\eta})e_1(\underline{\eta}) - X_{\varphi}(s, \underline{\eta})e_2(\underline{\eta})\end{aligned}$$

where we have used the fact

$$\eta \times e_1(\underline{\eta}) = -e_2(\underline{\eta}) \quad \text{and} \quad \eta \times e_2(\underline{\eta}) = e_1(\underline{\eta}). \tag{4.37}$$

Use the definition of the surface divergence, we then have

$$\mathrm{div}_{\underline{\eta}}(\underline{\eta} \times \mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta})) = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(-\sin\theta X_{\varphi}(s, \underline{\eta}))\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} X_{\theta}(s, \underline{\eta})$$

Simplifying the above equation we get (4.36). □

Proof of the theorem 4.6

In the same manner as (4.31) and (4.32) we obtain

$$\frac{\partial}{\partial\theta} X_{\varphi}(\underline{x}\cdot\underline{\eta}, \underline{\eta}) = \frac{\partial}{\partial\theta} X_{\varphi}(s, \underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}} + \frac{\partial}{\partial s} X_{\varphi}(s, \underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}}(\underline{x}\cdot e_2(\underline{\eta})) \tag{4.38}$$

and

$$\frac{\partial}{\partial\varphi} X_{\theta}(\underline{x}\cdot\underline{\eta}, \underline{\eta}) = \frac{\partial}{\partial\varphi} X_{\theta}(s, \underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}} + \frac{\partial}{\partial s} X_{\theta}(s, \underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}} \sin\theta(\underline{x}\cdot e_1(\underline{\eta})). \tag{4.39}$$

Thus

$$\begin{aligned}
\mathrm{div}_{\underline{\eta}}(\underline{\eta} \times \mathcal{R}^{(\tan)}\underline{\mathbf{f}})(\underline{x}\cdot\underline{\eta},\underline{\eta}) &= -\cot\theta X_{\varphi}(\underline{x}\cdot\underline{\eta},\underline{\eta}) - \frac{\partial}{\partial\theta}X_{\varphi}(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}} \\
&\quad - \frac{\partial}{\partial s}X_{\varphi}(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}}(\underline{x}\cdot e_2(\underline{\eta})) \\
&\quad + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}X_{\theta}(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}} + \frac{\partial}{\partial s}X_{\theta}(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}}(\underline{x}\cdot e_1(\underline{\eta})) \\
&= (div_{\underline{\eta}}(\underline{\eta}\times\mathcal{R}^{(\tan)}\underline{\mathbf{f}})(s,\underline{\eta}))|_{s=\underline{x}\cdot\underline{\eta}} \\
&\quad - \frac{\partial}{\partial s}X_{\varphi}(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}}(\underline{x}\cdot e_2(\underline{\eta})) \\
&\quad + \frac{\partial}{\partial s}X_{\theta}(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}}(\underline{x}\cdot e_1(\underline{\eta})).
\end{aligned}$$

By using (4.30), we have that

$$\begin{aligned}
\mathrm{div}_{\underline{\eta}}(\underline{\eta} \times \mathcal{R}^{(\tan)}\underline{\mathbf{f}})(\underline{x}\cdot\underline{\eta},\underline{\eta}) &= (\mathrm{div}_{\underline{\eta}}(\underline{\eta}\times\mathcal{R}^{(\tan)}\underline{\mathbf{f}})(s,\underline{\eta}))|_{s=\underline{x}\cdot\underline{\eta}} \\
&\quad + \frac{\partial}{\partial s}X_{\varphi}(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}}(\underline{x}\cdot(\eta\times e_1(\underline{\eta}))) \\
&\quad + \frac{\partial}{\partial s}X_{\theta}(s,\underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}}(\underline{x}\cdot(\underline{\eta}\times e_2(\underline{\eta}))) \\
&= (\mathrm{div}_{\underline{\eta}}(\underline{\eta}\times\mathcal{R}^{(\tan)}\underline{\mathbf{f}})(s,\underline{\eta}))|_{s=\underline{x}\cdot\underline{\eta}} \\
&\quad + \underline{x}\cdot\left(\underline{\eta}\times\frac{\partial}{\partial s}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\eta})\right)|_{s=\underline{x}\cdot\underline{\eta}}.
\end{aligned}$$

□

Next, we shall show that $\mathrm{div}_{\underline{\eta}}(\underline{\eta}\times\mathcal{R}^{(\tan)}\underline{\mathbf{f}})(\underline{x}\cdot\underline{\eta},\underline{\eta})$ can be written as a function of the divergent beam transform of $\underline{\mathbf{f}}$,

$$\underline{\mathbf{g}}(\underline{x},\underline{\xi}) = D\underline{\mathbf{f}}(\underline{x},\underline{\xi}) = \int_0^{\infty}\underline{\mathbf{f}}(\underline{x}+t\underline{\xi})d\underline{\xi}.$$

where $\underline{x}$ is located on a source trajectory Γ that surrounds the object B^3 and the direction $\underline{\theta}$ are contained in a cone $\mathcal{C}\subset\mathbb{R}^3$ and $\underline{\mathbf{g}}(\underline{x}) = g_1(\underline{x})e_1+g_2(\underline{x})e_2+g_3(\underline{x})e_3$, $g_i(\underline{x})$ are scalar-valued functions.

Theorem 4.9. *Assume that* $\underline{\mathbf{f}}\in C^2(B^3)$. *Then*

$$div_{\underline{\eta}}(\underline{\eta}\times\mathcal{R}^{(\tan)}\underline{\mathbf{f}})(\underline{x}\cdot\underline{\eta},\underline{\eta}) = \int_{S^2}\delta''(\underline{\eta}\cdot\underline{\xi})(\underline{\eta}\cdot(\underline{\xi}\times\underline{\mathbf{g}}))d\underline{\xi}. \tag{4.40}$$

Proof. Recall that $\nabla_{\underline{\eta}}$ is the surface gradient introduced in (2.30). In [13] Katsevich and Schuster provide one of the following useful identities

$$\operatorname{div}_{\underline{\eta}}(\mathcal{R}^{(\tan)}\underline{\mathbf{f}})(\underline{x}\cdot\underline{\eta},\underline{\eta}) = -\int_{S^2}\operatorname{div}_{\underline{\eta}}([\nabla_{\underline{\eta}}(\underline{\eta}\cdot\underline{\mathbf{g}}(\underline{x},\underline{\xi})]\delta'(\underline{\eta}\cdot\underline{\xi}))d\underline{\xi}.$$

By the definition of surface gradient we obtain

$$\begin{aligned}\nabla_{\underline{\eta}}[\underline{\eta}\cdot\underline{\mathbf{g}}(\underline{x},\underline{\xi})] &= \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}(\underline{\eta}\cdot\underline{\mathbf{g}}(\underline{x},\underline{\xi}))e_1(\underline{\eta}) + \frac{\partial}{\partial\theta}(\underline{\eta}\cdot\underline{\mathbf{g}}(\underline{x},\underline{\xi}))e_2(\underline{\eta})\\ &= (-\sin\varphi g_1(\underline{x},\underline{\xi}) + \cos\varphi g_2(\underline{x},\underline{\xi}))e_1(\underline{\eta})\\ &\quad + (\cos\varphi\cos\theta g_1(\underline{x},\underline{\xi}) + \sin\varphi\cos\theta g_2(\underline{x},\underline{\xi}) - \sin\theta g_3(\underline{x},\underline{\xi}))e_2(\underline{\eta})\end{aligned}$$

and hence

$$([\nabla_\eta(\eta\cdot\underline{\mathbf{g}}(x,\xi)]\delta'(\eta\cdot\xi)) = A_\varphi(\underline{x},\underline{\xi})e_1(\eta) + A_\theta(\underline{x},\underline{\xi})e_2(\eta)$$

where

$$\begin{aligned}A_\varphi(\underline{x},\underline{\xi},\underline{\eta}) &= (-\sin\varphi g_1(\underline{x},\underline{\xi}) + \cos\varphi g_2(\underline{x},\underline{\xi}))\delta'(\underline{\eta}\cdot\underline{\xi}),\\ A_\theta(\underline{x},\underline{\xi},\underline{\eta}) &= (\cos\varphi\cos\theta g_1(\underline{x},\underline{\xi}) + \sin\varphi\cos\theta g_2(\underline{x},\underline{\xi}) - \sin\theta g_3(\underline{x},\underline{\xi}))\delta'(\underline{\eta}\cdot\underline{\xi}).\end{aligned}$$

So

$$\operatorname{div}_{\underline{\eta}}([\nabla_{\underline{\eta}}(\underline{\eta}\cdot g(\underline{x},\underline{\xi})]\delta'(\underline{\eta},\underline{\xi})) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta(\underline{x},\underline{\xi},\underline{\eta})) + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}(A_\varphi(\underline{x},\underline{\xi},\underline{\eta}))$$

and

$$\begin{aligned}\operatorname{div}_{\underline{\eta}}(R^{(\tan)}\underline{\mathbf{f}})(\underline{x}\cdot\underline{\eta},\underline{\eta}) &= -\int_{S^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta(\underline{x},\underline{\xi},\underline{\eta})) + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}(A_\varphi(\underline{x},\underline{\xi},\underline{\eta}))\right)d\underline{\xi}\\ &= -\frac{\partial}{\partial\theta}\int_{S^2}A_\theta(\underline{x},\underline{\xi},\underline{\eta})d\underline{\xi} - \cot\theta\int_{S^2}A_\theta(\underline{x},\underline{\xi},\underline{\eta})d\underline{\xi}\\ &\quad - \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\int_{S^2}(A_\varphi(\underline{x},\underline{\xi},\underline{\eta}))d\underline{\xi}.\end{aligned} \tag{4.41}$$

Equating (4.29) and (4.41), we obtain that

$$X_\varphi(\underline{x}\cdot\underline{\eta},\underline{\eta}) = -\int_{S^2}A_\varphi(\underline{x},\underline{\xi},\underline{\eta})d\underline{\xi} \quad\text{and}\quad X_\theta(\underline{x}\cdot\underline{\eta},\underline{\eta}) = -\int_{S^2}A_\theta(\underline{x},\underline{\xi},\underline{\eta})d\underline{\xi}. \tag{4.42}$$

To get the conclusion, we will compute 3 terms in (4.36) where $X_\varphi(s,\underline{\eta})$ and $X_\theta(s,\underline{\eta})$ are defined as in (4.42):

$$
\begin{aligned}
\frac{\partial}{\partial\varphi}(X_\theta(\underline{x}\cdot\underline{\eta},\underline{\eta})) = &\int_{S^2}(\big[\cos\varphi\cos\theta g_1(\underline{x},\underline{\xi})+\sin\varphi\cos\theta g_2(\underline{x},\underline{\xi})-\sin\theta g_3\big]\,\delta''(\underline{\eta}\cdot\underline{\xi})\\
&\times[-\sin\varphi\sin\theta\xi_1+\cos\varphi\sin\theta\xi_2]\\
&+\delta'(\underline{\eta}\cdot\underline{\xi})(-\sin\varphi\cos\theta g_1(\underline{x},\underline{\xi})+\cos\varphi\cos\theta g_2(\underline{x},\underline{\xi})))d\underline{\xi}\\
&=\int_{S^2}\delta''(\underline{\eta}\cdot\underline{\xi})[-\sin\varphi\cos\varphi\sin\theta\cos\theta g_1\xi_1+\cos^2\varphi\cos\theta\sin\theta g_1\xi_2\\
&-\sin^2\varphi\sin\theta\cos\theta g_2\xi_1+\sin\theta\cos\theta\sin\varphi\cos\varphi g_2\xi_2]\\
&+\sin^2\theta\sin\varphi g_3\xi_1-\sin^2\theta\cos\varphi g_3\xi_2]d\underline{\xi}\\
&+\int_{S^2}\delta'(\underline{\eta}\cdot\underline{\xi})(-\sin\varphi\cos\theta g_1+\cos\varphi\cos\theta g_2)d\underline{\xi}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}(X_\theta(\underline{x}\cdot\underline{\eta},\underline{\eta})) = &\int_{S^2}\delta''(\underline{\eta}\cdot\underline{\xi})[-\sin\varphi\cos\varphi\cos\theta g_1\xi_1+\cos^2\varphi\cos\theta g_1\xi_2\\
&-\sin^2\varphi\cos\theta g_2\xi_1+\cos\theta\sin\varphi\cos\varphi g_2\xi_2]\\
&+\sin\theta\sin\varphi g_3\xi_1-\sin\theta\cos\varphi g_3\xi_2]d\underline{\xi}\\
&+\int_{S^2}\delta'(\underline{\eta}\cdot\underline{\xi})(-\sin\varphi\cot\theta g_1+\cos\varphi\cot\theta g_2)d\underline{\xi}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\theta}(X_\varphi(\underline{x}\cdot\underline{\eta},\underline{\eta})) = &\int_{S^2}(-\sin\varphi g_1(\underline{x},\underline{\xi})+\cos\varphi g_2(\underline{x},\underline{\xi}))\delta''(\underline{\eta}\cdot\underline{\xi})\\
&\times\left[\cos\varphi\cos\theta\xi_1+\sin\varphi\cos\theta\xi_2-\sin\theta\underline{\xi}_3\right]d\underline{\xi}\\
&=\int_{S^2}\delta''(\underline{\eta}\cdot\underline{\xi})[-\sin\varphi\cos\varphi\cos\theta g_1\xi_1-\sin^2\varphi\cos\theta g_1\xi_2+\sin\varphi\sin\theta g_1\xi_3\\
&+\cos^2\varphi\cos\theta g_2\xi_1+\cos\varphi\sin\varphi\cos\theta g_2\xi_2-\sin\theta\cos\varphi g_2\xi_3]d\underline{\xi}\\
\cot\theta X_\varphi(\underline{x}\cdot\underline{\eta},\underline{\eta}) = &\int_{S^2}(-\cot\theta\sin\varphi g_1+\cot\theta\cos\varphi g_2)\delta'(\underline{\eta}\cdot\underline{\xi})d\underline{\xi}.
\end{aligned}
$$

By the virtue of the equations we just computed and (4.36), we have that

$$\begin{aligned}\mathrm{div}_{\underline{\eta}}(\underline{\eta} \times \mathcal{R}^{(\tan)}\underline{\mathbf{f}})(\underline{x} \cdot \underline{\eta}, \underline{\eta}) &= \int_{S^2} \delta''(\underline{\eta} \cdot \underline{\xi})[(g_2\xi_1 - g_1\xi_2)\cos\theta \\ &\quad - (g_3\xi_1 - g_1\xi_3)\sin\varphi\sin\theta \\ &\quad + (g_3\xi_2 - g_2\xi_3)\sin\theta\cos\varphi]d\underline{\xi} \\ &= \int_{S^2} \delta''(\underline{\eta} \cdot \underline{\xi})(\underline{\eta} \cdot (\underline{\xi} \times \underline{\mathbf{g}}))d\underline{\xi}\end{aligned}$$

since

$$\underline{\eta} \cdot e_1 = \cos\varphi\sin\theta,\ \underline{\eta} \cdot e_2 = \sin\varphi\sin\theta, \underline{\eta} \cdot e_3 = \cos\theta \tag{4.43}$$

where e_1, e_2 and e_3 are defined in section 2.2. □

The main purpose of this section is to reconstruct a function $\underline{\mathbf{f}}$ from the transverse-ray transform in $\mathbb{R}^3$ with source points on a curve. We shall begin in the case of the solenoidal part $\underline{\mathbf{f}}^d$ of $\underline{\mathbf{f}}^d$ by using the ideas in [13]. The authors in [13] proposed the two decompositions of $\underline{\mathbf{f}}^d$ by using the Radon inversion formula:

$$\begin{aligned}\underline{\mathbf{f}}^d(\underline{x}) &= -\frac{1}{8\pi^2}\int_{S^2} \frac{\partial^2}{\partial s^2}\mathcal{R}\underline{\mathbf{f}}^d(s, \underline{\eta})|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta} \\ &= -\frac{1}{8\pi^2}\int_{S^2} \frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s, \underline{\eta})|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta} - \frac{1}{8\pi^2}\int_{S^2} \frac{\partial^2}{\partial s^2}\mathcal{R}^{\mathrm{nor}}\underline{\mathbf{f}}^d(s, \underline{\eta})|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta} \\ &= -\frac{1}{8\pi^2}\int_{S^2} \frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta})|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta} - \frac{1}{8\pi^2}\int_{S^2} \frac{\partial^2}{\partial s^2}\mathcal{R}^{\mathrm{nor}}\underline{\mathbf{f}}^d(s, \underline{\eta})|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta}\end{aligned}$$

where we have used the fact that $\mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta}) = \mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s, \underline{\eta})$. Subsequently, using the fact that the product of unit quaternions is -1 or simple geometriy, we obtain

$$\begin{aligned}\underline{\mathbf{f}}^d &= -\frac{1}{8\pi^2}\int_{S^2} \underline{\eta}\underline{\eta}\frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta})|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta} + \frac{1}{8\pi^2}\int_{S^2} \frac{\partial^2}{\partial s^2}\mathcal{R}^{\mathrm{nor}}\underline{\mathbf{f}}^d(p, \underline{\eta})|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta} \\ &= -\frac{1}{8\pi^2}\int_{S^2} \underline{\eta}(\underline{\eta} \times \frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta}))|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta} + \frac{1}{8\pi^2}\int_{S^2} \frac{\partial^2}{\partial s^2}\mathcal{R}^{\mathrm{nor}}\underline{\mathbf{f}}^d(s, \underline{\eta})|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta} \\ &= -\frac{1}{8\pi^2}\int_{S^2} \underline{\eta}(\underline{\eta} \times \frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta})|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta} + \frac{1}{8\pi^2}\int_{S^2} \frac{\partial^2}{\partial s^2}\mathcal{R}^{\mathrm{nor}}\underline{\mathbf{f}}^d(p, \underline{\eta})|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta}.\end{aligned} \tag{4.44}$$

We denote the first and the second term in (4.44) by $\underline{\mathbf{f}}_1^d(\underline{x})$ and $\underline{\mathbf{f}}_2^d(\underline{x})$, respectively.

Throughout this section, we shall assume that a curve Γ which we are considering has a function $\underline{a}(\lambda) : \Lambda \subset \mathbb{R} \to \Gamma$ as its parametrization and it satisfies Tuy's condition of order three. For simplicity we assume furthermore that

$$B^3 \subset \underline{a}(\lambda) + \mathcal{C} \quad \text{for all} \quad \lambda \in \Lambda$$

and that the support of $\underline{\mathbf{f}}$ is contained in B^3. Here $\mathcal{C}$ is a cone and $\underline{\eta} \in \mathcal{C}$. The statement for the Tuy's condition of order 3 is stated as follow:

Definition 4.10 (Tuy's condition for vector tomography)**.** *A source trajectory* $\Gamma \subset \mathbb{R}^3 \backslash \overline{B^3}$ *satisfies a Tuy condition of order 3 if any plane that passes through* B^3 *intersects the trajectory* Γ *in at least 3 points that are not located on a line. That means to any* $s \in (-1, 1)$ *and* $\underline{\eta} \in S^2$ *there exist at least 3 parameters* $\lambda_i \in \Lambda, i = 1, 2, 3,$ *with*

$$a_1 \cdot \underline{\eta} = a_2 \cdot \underline{\eta} = a_3 \cdot \underline{\eta} = s = \underline{x} \cdot \underline{\eta}, \quad a_i = a(\lambda_i(s, \underline{\eta})), i = 1, 2, 3$$

and $a_1 - a_3$ *and* $a_2 - a_3$ *are not collinear.*

Definition 4.11. *For* $\underline{x} \in \mathbb{R}^3$ *and* $\underline{\eta} \in S^2$*, the transverse-ray transform* T *of* $\underline{\mathbf{f}}$ *is defined by*

$$T\underline{\mathbf{f}}(\underline{x}, \underline{\theta}) = \int_0^\infty (\underline{\theta} \times \underline{\mathbf{f}})(\underline{x} + t\underline{\theta}) dt, \quad \textit{where} \ , \ \ \underline{\theta} \in S^2. \tag{4.45}$$

Furthermore, we define a function G*, an integration of the transverse- ray transform, as*

$$G(\underline{x}, \underline{\eta}) = -\int_{S^2} \delta''(\underline{\eta} \cdot \underline{\theta})[\underline{\eta} \cdot (\underline{\theta} \times \underline{\mathbf{g}}(\underline{x}, \underline{\theta}))] d\underline{\theta} \tag{4.46}$$

$$= -\int_{S^2} \delta''(\underline{\eta} \cdot \underline{\theta})[(\underline{\eta} \cdot T\underline{\mathbf{f}}(\underline{x}))(\underline{x}, \underline{\theta}) d\underline{\theta} \tag{4.47}$$

where $\underline{\mathbf{g}}$ *is defined by*

$$\underline{\mathbf{g}}(x, \underline{\theta}) = D^c \underline{\mathbf{f}}(\underline{x}, \underline{\theta}) = \int_0^\infty \mathbf{f}(\underline{x} + t\underline{\theta}) dt, \quad \mathbf{f} \textit{ is any vector-valued function.} \tag{4.48}$$

We also define another type on the transverse ray transform $\tilde{T}\underline{\mathbf{f}}(\underline{x}, \underline{\theta})$ *as*

$$\tilde{T}\underline{\mathbf{f}}(\underline{x}, \underline{\theta}) = \underline{\eta} \cdot D^c \underline{\mathbf{f}}(\underline{x}, \underline{\theta}), \ \ \underline{\eta} \in \underline{\theta}^{\perp}.$$

Proposition 4.12. *The distributional senses of $t^2\delta'$ are as follows : for any test function ψ,*

$$< t^2\delta', \psi > = < \delta', t^2\psi > = -\frac{d}{dt}(t^2\psi)|_{t=0} = 0.$$

By straightforward computation or induction, we obtain

$$< t^{2n}\delta', \psi > = 0, \quad \textit{where } n \in \mathbb{N}.$$

The following lemma is a general result for the operator appearing in the definition of G. It is also useful for the reconstruction procedure.

Lemma 4.13. *For any smooth scalar-valued function w on the unit sphere, we have the identity for $\underline{\eta} \in S^2$*

$$\int_{S^2} w(\underline{\theta})\delta''(\underline{\theta}\cdot\underline{\eta})d\underline{\theta} = \int_{S^2}[-(\underline{\eta}\cdot\nabla_{\underline{\theta}}w(\underline{\theta}))\delta'(\underline{\eta}\cdot\underline{\theta})]d\underline{\theta} \tag{4.49}$$

$$= \int_{S^2}[\underline{\eta}\cdot\nabla_{\underline{\theta}}(\underline{\eta}\cdot\nabla_{\underline{\theta}}w(\underline{\theta}))\delta(\underline{\eta}\cdot\underline{\theta})]d\underline{\theta}. \tag{4.50}$$

Proof. Straightforward computation of the surface gradient ∇_θ in (2.30) applied to a scalar-valued function f gives the following formulas:

$$\nabla_{\underline{\theta}} f(\underline{\theta}\cdot\underline{\eta}) = (\underline{\eta} - (\underline{\eta}\cdot\underline{\theta})\underline{\theta})f'(\underline{\theta}\cdot\underline{\eta}), \tag{4.51}$$

$$\nabla_{\underline{\theta}}\delta'(\underline{\theta}\cdot\underline{\eta})) = (\underline{\eta} - (\underline{\eta}\cdot\underline{\theta})\underline{\theta})\delta''(\underline{\theta}\cdot\underline{\eta}), \tag{4.52}$$

$$\underline{\eta}\cdot\nabla_{\underline{\theta}}\delta'(\underline{\theta}\cdot\underline{\eta})) = (1 - (\underline{\eta}\cdot\underline{\theta})^2)\delta''(\underline{\theta}\cdot\underline{\eta}). \tag{4.53}$$

Therefore,

$$\begin{aligned}\int_{S^2} w(\underline{\theta})\delta''(\underline{\theta}\cdot\underline{\eta})d\underline{\theta} &= \int_{S^2} w(\underline{\theta})\frac{1}{(1-(\underline{\eta}\cdot\underline{\theta})^2)}(\underline{\eta}\cdot\nabla_{\underline{\theta}}\delta'(\underline{\theta}\cdot\underline{\eta}))d\underline{\theta} \\ &= -\int_{S^2}\underline{\eta}\cdot\nabla_{\underline{\theta}}\left(w(\underline{\theta})\frac{1}{(1-(\underline{\eta}\cdot\underline{\theta})^2)}\right)\delta'(\underline{\eta}\cdot\underline{\theta})d\underline{\theta} \\ &= -\int_{S^2}(\underline{\eta}\cdot\nabla_{\underline{\theta}}w(\underline{\theta}))\frac{1}{(1-(\underline{\eta}\cdot\underline{\theta})^2)}\delta'(\underline{\eta}\cdot\underline{\theta})d\underline{\theta}. \end{aligned} \tag{4.54}$$

The first line use the definition of G. The second line follows from (4.49). The third one use the definition of distributions and the last one is the product rule of the operator $\nabla_{\underline{\theta}}$. Using the geometric series, we have

$$\frac{1}{1-(\underline{\eta}\cdot\underline{\theta})^2}\delta'(\underline{\eta}\cdot\underline{\theta}) = [1+(\underline{\eta}\cdot\underline{\theta})^2 + O((\underline{\eta}\cdot\underline{\theta})^4)]\delta'(\underline{\eta}\cdot\underline{\theta}) \tag{4.55}$$

$$= [1+(\underline{\eta}\cdot\underline{\theta})^2 + (\underline{\eta}\cdot\underline{\theta})^4 a(\underline{\eta}\cdot\underline{\theta})]\delta'(\underline{\eta}\cdot\underline{\theta})$$

$$= \delta'(\underline{\eta}\cdot\underline{\theta}) \tag{4.56}$$

for some constant $a(\underline{\eta}\cdot\underline{\theta})$ and we have applied proposition 4.12 to the second equation. So substituting (4.55) in (4.53) gives

$$\int_{S^2} w(\underline{\theta})\delta''(\underline{\theta}\cdot\underline{\eta})d\underline{\theta} = -\int_{S^2}(\underline{\eta}\cdot\nabla_{\underline{\theta}}w(\underline{\theta}))\delta'(\underline{\eta}\cdot\underline{\theta})d\underline{\theta}. \tag{4.57}$$

which proved (4.48). To reach (4.49), we need to simplify (4.48) by considering, for any smooth scalar-valued function v,

$$\begin{aligned}
\int_{S^2} v(\underline{\theta})\delta'(\underline{\eta}\cdot\underline{\theta})d\underline{\theta} &= \int \delta'(\tau)\int_{\underline{\eta}^\perp} v(\tau\underline{\eta}+\sqrt{1-\tau^2}\varphi)d\varphi\sqrt{1-\tau^2}d\tau \\
&= -\frac{d}{d\tau}\left(\int_{\underline{\eta}^\perp} v(\tau\underline{\eta}+\sqrt{1-\tau^2}\varphi)d\varphi\sqrt{1-\tau^2}\right)|_{\tau=0} \\
&= -\frac{d}{d\tau}\left(\int_{\underline{\eta}^\perp} v(\tau\underline{\eta}+\sqrt{1-\tau^2}\varphi)d\varphi\sqrt{1-\tau^2}\right)|_{\tau=0}\cdot\sqrt{1-\tau^2}_{\tau=0} \\
&- \left(\int_{\underline{\eta}^\perp} v(\varphi)d\varphi\right)\frac{(-\tau)}{\sqrt{1-\tau^2}}_{\tau=0} \\
&= -\int_{S^2}\underline{\eta}\cdot\nabla_{\underline{\theta}}v(\underline{\theta})\delta(\underline{\theta}\cdot\underline{\eta})d\underline{\theta}.
\end{aligned} \tag{4.58}$$

Consequently, (4.49) is proved. □

Theorem 4.14. *Assume that* $\underline{\mathbf{f}}\in C^2(B^3)$ *and it has the Helmholtz-Hodge decomposition (2.25),* $\underline{\mathbf{f}}$ *and* $\underline{\mathbf{f}}_0^d$ *have compact support in* B^3*. We assume furthermore that the curve surrounding* $\underline{\mathbf{f}}$ *satisfies Tuy's condition of order 3 and* I *is the interval that parametrized* *the curve. Then its first part* $\underline{\mathbf{f}}_1^d(\underline{x})$ *according to the decomposition (4.44) reads*

$$\underline{\mathbf{f}}_1^d(\underline{x}) = \int_I \frac{1}{|x - a(\lambda)|}[\int_0^{2\pi} [\partial_\gamma^2(\underline{\eta}(\gamma)(\Phi(\underline{\eta}(\gamma)))) + \underline{\eta}(\gamma)(\Phi(\underline{\eta}(\gamma)))] \tag{4.59}$$

$$\times \int_0^{2\pi} \frac{1}{\cos^2\alpha} H(\alpha, \gamma, a)] d\alpha d\gamma d\lambda \tag{4.60}$$

where $\eta(\gamma) = (\cos\gamma, \sin\gamma, 0),\ \beta = (0, 0, 1),\ \underline{\theta} = \cos\alpha\underline{\eta}^\perp(\gamma) + \sin\alpha\beta$ *and*

$$H(\alpha, \gamma, a(\lambda)) = \underline{\eta}(\gamma) \cdot \left(\underline{\theta} \times \int_0^\infty \underline{\mathbf{f}}(a(\lambda) + t\underline{\theta})dt\right).$$

Proof. Suppose that Γ satisfies Tuy's condition of order 3. For $\underline{x} \in \mathbb{B}^3$ and $\underline{\eta} \in S^2$, consider the plane

$$\Pi(\underline{x}, \underline{\eta}) := \{\underline{z} \in \mathbb{R}^3 : \underline{z} \cdot \underline{\eta} = \underline{x} \cdot \underline{\eta}\}$$

which passes through x and is perpendicular to $\underline{\eta}$.

We shall use framework in [14, 13], theorem 4.6 and 4.9 to find $\underline{\eta} \times \frac{\partial}{\partial s}\mathcal{R}\underline{\mathbf{f}}^d$ as a function of the transverse-ray transform (4.49). If there are three intersection points, $a_i = a(\lambda_i),\ i = 1, 2, 3$, i.e., then by taking account into equation (4.28) we obtain that for s and $\underline{\eta}$ fixed,

$$div_{\underline{\eta}}[\underline{\eta} \times \mathcal{R}^{\tan}\underline{\mathbf{f}}](s, \underline{\eta}) + a_1 \cdot \left(\underline{\eta} \times \frac{\partial}{\partial s}[\mathcal{R}^{\tan}\underline{\mathbf{f}}](s, \underline{\eta})\right) = G(a_1, \underline{\eta}) \tag{4.61}$$

$$div_{\underline{\eta}}[\eta \times \mathcal{R}^{\tan}\underline{\mathbf{f}}](s, \underline{\eta}) + a_2 \cdot \left(\underline{\eta} \times \frac{\partial}{\partial s}[R^{\tan}\underline{\mathbf{f}}](s, \underline{\eta})\right) = G(a_2, \underline{\eta}) \tag{4.62}$$

$$div_{\underline{\eta}}[\underline{\eta} \times \mathcal{R}^{\tan}\underline{\mathbf{f}}](s, \underline{\eta}) + a_3 \cdot \left(\underline{\eta} \times \frac{\partial}{\partial s}[\mathcal{R}^{\tan}\underline{\mathbf{f}}](s, \underline{\eta})\right) = G(a_3, \underline{\eta}). \tag{4.63}$$

Then by subtracting (4.62) from (4.60) and (4.61) respectively, we get

$$(a_1 - a_3) \cdot \left(\underline{\eta} \times \frac{\partial}{\partial s}[\mathcal{R}^{\tan}\underline{\mathbf{f}}](s, \underline{\eta})\right) = G(a_1, \underline{\eta}) - G(a_3, \underline{\eta}) \tag{4.64}$$

$$(a_2 - a_3) \cdot \left(\underline{\eta} \times \frac{\partial}{\partial s}[\mathcal{R}^{\tan}\underline{\mathbf{f}}](s, \underline{\eta})\right) = G(a_2, \underline{\eta}) - G(a_3, \underline{\eta}). \tag{4.65}$$

Since $\underline{\eta} \times \partial_s\mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta}) \in \underline{\eta}^\perp$, there exist scalar $A(s, \underline{\eta}), B(s, \underline{\eta}) \in \mathbb{R}$ such that

$$\underline{\eta} \times \frac{\partial}{\partial s}[\mathcal{R}^{\tan}\underline{\mathbf{f}}](s, \underline{\eta}) = A(s, \underline{\eta})e_1(\underline{\eta}) + B(s, \underline{\eta})e_2(\underline{\eta}).$$

Substituting the above expression into the systems (4.63) and (4.64) gives

$$A(s,\underline{\eta})(a_1-a_3)\cdot e_1(\underline{\eta})+B(s,\underline{\eta})(a_1-a_3)\cdot e_2(\underline{\eta})=G(a_1,\underline{\eta})-G(a_3,\underline{\eta})$$

$$A(s,\underline{\eta})(a_2-a_3)\cdot e_1(\underline{\eta})+B(s,\underline{\eta})(a_2-a_3)\cdot e_2(\underline{\eta})=G(a_2,\underline{\eta})-G(a_3,\underline{\eta}).$$

By taking into account that a_1-a_3 and a_2-a_3 are not collinear, we have that the determinant of the system matrix is non zero and this leads to the solution of the above system is unique. By solving the system we have

$$A(s,\underline{\eta})=\frac{[(a_2-a_3)\cdot e_2]G(a_1,\underline{\eta})-[(a_1-a_3)\cdot e_2]G(a_2,\underline{\eta})-[(a_2-a_1)\cdot e_2]G(a_3,\underline{\eta})}{[(a_2-a_3)\cdot e_2][(a_1-a_3)\cdot e_1]-[(a_1-a_3)\cdot e_2][(a_2-a_3)\cdot e_1]}$$

$$B(s,\underline{\eta})=-\frac{[(a_2-a_3)\cdot e_1]G(a_1,\underline{\eta})-[(a_1-a_3)\cdot e_1]G(a_2,\underline{\eta})-[(a_2-a_1)\cdot e_1]G(a_3,\underline{\eta})}{[(a_2-a_3)\cdot e_2][(a_1-a_3)\cdot e_1]-[(a_1-a_3)\cdot e_2][(a_2-a_3)\cdot e_1]}.$$

So

$$\begin{aligned}\underline{\eta}\times\frac{\partial}{\partial s}[\mathcal{R}^{\tan}f](s,\underline{\eta})|_{s=\underline{\eta}\cdot x}=&\frac{([(a_2-a_3)\cdot e_2]e_1-[(a_2-a_3)\cdot e_1]e_2)G(a_1,\underline{\eta})}{[(a_2-a_3)\cdot e_2][(a_1-a_3)\cdot e_1]-[(a_1-a_3)\cdot e_2][(a_2-a_3)\cdot e_1]}\\&-\frac{([(a_1-a_3)\cdot e_2]e_1+[(a_1-a_3)\cdot e_1]e_2)G(a_2,\eta)}{[(a_2-a_3)\cdot e_2][(a_1-a_3)\cdot e_1]-[(a_1-a_3)\cdot e_2][(a_2-a_3)\cdot e_1]}\\&-\frac{([(a_2-a_1)\cdot e_2]e_1+[(a_2-a_1)\cdot e_1]e_2)G(a_3,\underline{\eta})}{[(a_2-a_3)\cdot e_2][(a_1-a_3)\cdot e_1]-[(a_1-a_3)\cdot e_2][(a_2-a_3)\cdot e_1]}.\end{aligned}$$

Let $\lambda_j=\lambda_j(x,\underline{\eta})$ be the parameters of intersection of the source of trajectory Γ and $\Pi(\underline{x},\underline{\eta})$. The authors in [13] also assume that there can be more than three intersection points so in general, one can form multiple triples from these points. Denote those triples by $\mathcal{L}_m=\mathcal{L}_m(x,\eta)$, where the subscript m denotes the index of a triple. Therefore, we can write

$$\underline{\eta}\times\frac{\partial}{\partial s}[\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\eta})]|_{s=\underline{\eta}\cdot\underline{x}}=\sum_{\lambda_i\in\mathcal{L}_m}\phi(a(\lambda_i),\mathcal{L}_m)G(a(\lambda_i),\underline{\eta})$$

where

$$\phi(a(\lambda_1),\mathcal{L}_m)=\frac{([(a_2-a_3)\cdot e_2]e_1-[(a_2-a_3)\cdot e_1]e_2)}{[(a_2-a_3)\cdot e_2][(a_1-a_3)\cdot e_1]-[(a_1-a_3)\cdot e_2][(a_2-a_3)\cdot e_1]}$$

$$\phi(a(\lambda_2),\mathcal{L}_m)=-\frac{([(a_1-a_3)\cdot e_2]e_1+[(a_1-a_3)\cdot e_1]e_2)}{[(a_2-a_3)\cdot e_2][(a_1-a_3)\cdot e_1]-[(a_1-a_3)\cdot e_2][(a_2-a_3)\cdot e_1]}$$

$$\phi(a(\lambda_3),\mathcal{L}_m)=-\frac{([(a_2-a_1)\cdot e_2]e_1+[(a_2-a_1)\cdot e_1]e_2)}{[(a_2-a_3)\cdot e_2][(a_1-a_3)\cdot e_1]-[(a_1-a_3)\cdot e_2][(a_2-a_3)\cdot e_1]}.$$

Hence, we obtain the formula

$$\underline{\mathbf{f}}_1^d(\underline{x}) = -\frac{1}{8\pi^2}\int_{S^2}\underline{\eta}(\underline{\eta}\times\frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}f(s,\underline{\eta}))|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta}$$
$$= -\frac{1}{8\pi^2}\int_{S^2}\underline{\eta}\left[\sum_{\text{all triple m}}\left(\sum_{\lambda_i\in\mathcal{L}_m}\phi(\lambda_i,\mathcal{L}_m)G(a(\lambda_i),\underline{\eta})\right)w_m(\underline{x},\underline{\eta})\right]d\underline{\eta} \qquad (4.66)$$

where $w_m(\underline{x},\underline{\eta})$ is the weight assigned to each triple and is precisely defined in (3.18). $\underline{\eta}\in S^2\setminus\mathrm{Crit}(\underline{x})$. Using the technique in [13], the summation in the integral (4.65) can be written in the form of the integral on the curve as the following

$$\underline{\mathbf{f}}_1^d(\underline{x}) = -\frac{1}{8\pi^2}\int_{S^2}\underline{\eta}\left(\int_I|\underline{\eta}\cdot a'(\lambda)|\delta(\underline{\eta}\cdot(\underline{x}-a(\lambda)))\left[\sum_{m,\lambda\in\mathcal{L}_m}\phi(\lambda,\mathcal{L}_m)w_m(\underline{x},\underline{\eta})\right]G(\lambda,\underline{\eta})d\lambda\right)d\underline{\eta}. \qquad (4.67)$$

Let

$$\beta(\lambda,\underline{x}) = \frac{\underline{x}-a(\lambda)}{|\underline{x}-a(\lambda)|},\qquad \Phi = |\underline{\eta}\cdot a'(\lambda)|\sum_{m,\lambda\in\mathcal{L}_m}\phi(\lambda,\mathcal{L}_m)w_m(\underline{x},\underline{\eta}).$$

By changing the order of integral and using the fact that the homogeneity of δ function is -1, one can rewrite the equation (4.66) as

$$\underline{\mathbf{f}}_1^d(\underline{x}) = -\frac{1}{8\pi^2}\int_{S^2}\underline{\eta}\left(\int_I\frac{1}{|\underline{x}-a(\lambda)|}\delta(\underline{\eta}\cdot\beta(\lambda,\underline{x}))\Phi(\lambda,\underline{\eta})G(\lambda,\underline{\eta})d\lambda\right)d\underline{\eta}$$
$$= -\frac{1}{8\pi^2}\int_I\frac{1}{|\underline{x}-a(\lambda)|}\left[\int_{S^2}\underline{\eta}(\Phi(\lambda,\underline{\eta})G(\lambda,\underline{\eta})\delta(\underline{\eta}\cdot\beta(\lambda,\underline{x}))d\underline{\eta})\right]d\lambda. \qquad (4.68)$$

Let us consider the inner integral of (4.67), where we have omitted λ for convenience,

$$h_1^d(\beta) = \int_{S^2}\underline{\eta}(\Phi(\underline{\eta}))G(\underline{\eta})\delta(\underline{\eta}\cdot\beta)d\underline{\eta}$$

The integral is over the plane $\beta^\perp$ so we introduce γ to be an polar angle that describes $\underline{\eta}(\gamma)$. Then

$$h_1^d(\beta) = \int_0^{2\pi}\underline{\eta}(\gamma)(\Phi(\underline{\eta}(\gamma))G(\underline{\eta}(\gamma))\delta(\underline{\eta}(\gamma)\cdot\beta)d\gamma. \qquad (4.69)$$

We shall choose a coordinate system in which $\beta = (0,0,1)$. It follows that

$$\underline{\eta}(\gamma) = (\cos\gamma, \sin\gamma, 0) \quad \text{and} \quad \underline{\eta}^{\perp}(\gamma) = (-\sin\gamma, \cos\gamma, 0). \tag{4.70}$$

To see $h_1^d(\beta)$ it suffices to compute $G(\underline{\eta})$. One can see that by lemma 4.13 when $w(\underline{\theta})$ is replaced by $\underline{\eta} \cdot (\underline{\theta} \times \underline{\mathbf{g}}))$ we then have,

$$G(\underline{\eta}) = \int_{S^2} \underline{\eta} \cdot \nabla_{\underline{\theta}}(\underline{\eta} \cdot \nabla_{\underline{\theta}}(\underline{\eta} \cdot (\underline{\theta} \times \underline{\mathbf{g}})))\delta(\underline{\eta} \cdot \underline{\theta})d\underline{\theta}. \tag{4.71}$$

To reach the concrete formula for $G(\underline{\eta})$, we shall see the precise form of $\nabla_{\underline{\theta}}$. We begin by parametrizing $\underline{\theta}$ in (4.70) which is a vector on $\underline{\eta}^{\perp}(\gamma)$ as:

$$\underline{\theta} = \cos\alpha\underline{\eta}^{\perp}(\gamma) + \sin\alpha\beta = \begin{pmatrix} -\cos\alpha\sin\gamma \\ \cos\alpha\cos\gamma \\ \sin\alpha \end{pmatrix}$$

where $\alpha \in [0, 2\pi)$ and we have used $\underline{\eta}(\gamma)$, $\underline{\eta}^{\perp}(\gamma)$ in (4.69). Using the same process in [4], the following two unit vectors are basis of $\underline{\theta}^{\perp}$:

$$e_1(\underline{\theta}) = \frac{1}{\cos\alpha}\frac{\partial}{\partial\gamma}\underline{\theta} = \begin{pmatrix} -\cos\gamma \\ -\sin\gamma \\ 0 \end{pmatrix}, \quad e_2(\underline{\theta}) = \frac{\partial}{\partial\alpha}\underline{\theta} = \begin{pmatrix} \sin\alpha\sin\gamma \\ -\sin\alpha\cos\gamma \\ \cos\alpha \end{pmatrix}.$$

In this case, we shall see the form of the surface gradient of a scalar function $u(\underline{\theta})$ on the sphere S^2 in the same manner of (2.30). Denote $\nabla_{\underline{\theta}} u(\underline{\theta})$ by such surface gradient which is defined as

$$\nabla_{\underline{\theta}} u = \frac{1}{\cos\alpha}\frac{\partial u}{\partial\gamma}e_1(\underline{\theta}) + \frac{\partial u}{\partial\alpha}e_2(\underline{\theta}).$$

Applying the dot product with $\underline{\eta}$ gives

$$\underline{\eta} \cdot \nabla_{\underline{\theta}} u(\underline{\theta}) = \frac{1}{\cos\alpha}\frac{\partial u}{\partial\gamma}(\underline{\eta} \cdot e_1(\underline{\theta})) + \frac{\partial u}{\partial\alpha}(\underline{\eta} \cdot e_2(\underline{\theta})). \tag{4.72}$$

By applying the surface gradient operator to (4.71) we obtain

$$\begin{aligned}
\nabla_{\underline{\theta}}(\underline{\eta}\cdot\nabla_{\underline{\theta}}u(\underline{\theta})) &= \frac{1}{\cos\alpha}\frac{\partial}{\partial\gamma}\left[\frac{1}{\cos\alpha}\frac{\partial u}{\partial\gamma}(\underline{\eta}\cdot e_1)+\frac{\partial u}{\partial\alpha}(\underline{\eta}\cdot e_2)\right]e_1 \\
&\quad+\frac{\partial}{\partial\alpha}\left[\frac{1}{\cos\alpha}\frac{\partial u}{\partial\gamma}(\underline{\eta}\cdot e_1)+\frac{\partial u}{\partial\alpha}(\underline{\eta}\cdot e_2)\right]e_2 \\
&= \frac{1}{\cos^2\alpha}\left[\partial_\gamma u(\underline{\eta}\cdot\partial_\gamma e_1)+(\underline{\eta}\cdot e_1)\partial_\gamma^2 u\right]e_1 \\
&\quad+\frac{1}{\cos\alpha}\left[\partial_\alpha u(\underline{\eta}\cdot\partial_\gamma e_2)+(\underline{\eta}\cdot e_2)\partial_\gamma\partial_\alpha u\right]e_1 \\
&\quad+\frac{1}{\cos\alpha}\left[\partial_\gamma u(\underline{\eta}\cdot\partial_\alpha e_1)+(\underline{\eta}\cdot e_1)\partial_\gamma\partial_\alpha u\right]e_2+\frac{\sin\alpha}{\cos^2\alpha}\partial_\gamma u(\underline{\eta}\cdot e_1)e_2 \\
&\quad+\left[\partial_\alpha u(\underline{\eta}\cdot\partial_\alpha e_2)+(\underline{\eta}\cdot e_2)\partial_\alpha^2 u\right]e_2.
\end{aligned}$$

Thus

$$\begin{aligned}
\underline{\eta}\cdot\nabla_{\underline{\theta}}(\underline{\eta}\cdot\nabla_{\underline{\theta}}u(\underline{\theta})) &= (\underline{\eta}\cdot e_1)\left[\frac{1}{\cos^2\alpha}\partial_\gamma u(\underline{\eta}\cdot\partial_\gamma e_1)+\frac{1}{\cos\alpha}\partial_\alpha u(\underline{\eta}\cdot\partial_\gamma e_2)\right] \\
&\quad+(\underline{\eta}\cdot e_1)^2\frac{1}{\cos^2\alpha}\partial_\gamma^2 u+(\underline{\eta}\cdot e_2)^2\partial_\alpha^2 u \\
&\quad+(\underline{\eta}\cdot e_1)(\underline{\eta}\cdot e_2)\left[\frac{2}{\cos\alpha}\partial_\gamma\partial_\alpha u+\frac{\sin\alpha}{\cos^2\alpha}\partial_\gamma u\right] \\
&\quad+\quad(\underline{\eta}\cdot e_2)\left[\frac{1}{\cos\alpha}\partial_\gamma u(\underline{\eta}\cdot\partial_\alpha e_1)+\partial_\alpha u(\underline{\eta}\cdot\partial_\gamma e_2)\right].
\end{aligned}$$

Using the following identities

$$\partial_\gamma e_1=\begin{pmatrix}\sin\gamma\\ -\cos\gamma\\ 0\end{pmatrix}=e_1^\perp,\quad \partial_\alpha e_1=0,\quad \partial_\gamma e_2=-\sin\alpha e_1,\quad \partial_\alpha e_2=-\underline{\theta}$$

gives

$$\begin{aligned}
\underline{\eta}\cdot\nabla_{\underline{\theta}}(\underline{\eta}\cdot\nabla_{\underline{\theta}}u(\underline{\theta})) &= (\underline{\eta}\cdot e_1)(\underline{\eta}\cdot e_1^\perp)\frac{1}{\cos^2\alpha}\partial_\gamma u+(\underline{\eta}\cdot e_1)^2\left[\frac{1}{\cos^2\alpha}\partial_\gamma^2 u-\tan\alpha\partial_\alpha u\right] \\
&\quad+(\underline{\eta}\cdot e_1)(\underline{\eta}\cdot e_2)\left[\frac{2}{\cos\alpha}\partial_\gamma\partial_\alpha u+\frac{\sin\alpha}{\cos^2\alpha}\partial_\gamma u\right] \\
&\quad-(\underline{\eta}\cdot e_2)(\underline{\eta}\cdot\underline{\theta})\partial_\alpha u+(\underline{\eta}\cdot e_2)^2\partial_\alpha^2 u.
\end{aligned}\tag{4.73}$$

Plugging in

$$\underline{\eta}\cdot e_1=1,\quad \underline{\eta}\cdot e_1^\perp=\underline{\eta}\cdot e_2=0$$

to (4.72) gives

$$\underline{\eta} \cdot \nabla_{\underline{\theta}}(\underline{\eta} \cdot \nabla_{\underline{\theta}} u(\underline{\theta})) = \left[\frac{1}{\cos^2 \alpha}\partial_\gamma^2 u - \tan\alpha\partial_\alpha u\right]. \tag{4.74}$$

Substituting (4.73) into the equation (4.70) we have

$$G(\underline{\eta}(\gamma)) = \int_0^{2\pi} \left[\frac{1}{\cos^2\alpha}\partial_\gamma^2 - \tan\alpha\partial_\alpha\right] H(\alpha,\gamma,a(\lambda))d\alpha \tag{4.75}$$

where

$$H(\alpha,\gamma,a(\lambda)) = \underline{\eta}\cdot(\underline{\theta}\times\underline{\mathbf{g}}) = (\cos\gamma,\sin\gamma,0)\cdot((-\cos\alpha\sin\gamma,\cos\alpha\cos\gamma,\sin\alpha)\times\underline{\mathbf{g}}).$$

Substituting (4.74) into (4.68) and integrating by part gives,

$$\begin{aligned} h_1^d(\beta) &= \int_0^{2\pi} \underline{\eta}(\gamma)\Phi(\underline{\eta}(\gamma)) \int_0^{2\pi} \left[\frac{1}{\cos^2\alpha}\partial_\gamma^2 - \tan\alpha\partial_\alpha\right] H(\alpha,\gamma,a(\lambda))d\alpha d\gamma \\ &= \int_0^{2\pi} \left[\partial_\gamma^2(\underline{\eta}(\gamma)(\Phi(\underline{\eta}(\gamma)))) + \underline{\eta}(\gamma)(\Phi(\underline{\eta}(\gamma)))\right] \int_0^{2\pi} \frac{H(\alpha,\gamma,a)}{\cos^2\alpha}d\alpha d\gamma. \end{aligned} \tag{4.76}$$

To finish this we shall put (4.75) into (4.67) which gives the reconstruction formula

$$\begin{aligned} \underline{\mathbf{f}}_1^d(\underline{x})(\underline{x}) = \int_I \frac{1}{|x-a(\lambda)|}[\int_0^{2\pi} &\left[\partial_\gamma^2(\underline{\eta}(\gamma)(\Phi(\underline{\eta}(\gamma)))) + \underline{\eta}(\gamma)(\Phi(\underline{\eta}(\gamma)))\right] \\ \times &\int_0^{2\pi} \frac{H(\alpha,\gamma,a)}{\cos^2\alpha} d\alpha d\gamma d\lambda. \end{aligned}$$

□

The following theorem is the reconstruction formula for the second part of the solenoidal part of $\underline{\mathbf{f}}$ according to the decomposition (4.44).

Theorem 4.15. *Assume that the assumptions in theorem 4.14 is true. Then second part $\underline{\mathbf{f}}_2^d$ of the solenoidal part reads*

$$\underline{\mathbf{f}}_2^d(\underline{x}) = \frac{1}{8\pi^2}\int_{S^2}\underline{\eta}\int_I \frac{1}{|x-a(\lambda)|} h_2^d(\beta,\lambda)d\lambda d\underline{\eta} \tag{4.77}$$

where

$$h_2^d(\beta,\lambda) = \int_{S^2} \underline{\theta}\Phi(\lambda,\underline{\theta})G(\lambda,\underline{\theta})\delta(\underline{\theta}\cdot\beta(\lambda(\underline{x})))\cdot\nabla_{\underline{\theta}}K(\underline{\eta}\cdot\underline{\theta})d\underline{\theta}$$

where $\Phi(\lambda,\underline{\theta})$, $G(\lambda,\underline{\theta})$ are defined in theorem 4.14.

To reconstruct the second part $\underline{\mathbf{f}}_2^d$ of the solenoidal part $\underline{\mathbf{f}}^d$ of $\underline{\mathbf{f}}$, we shall use the formula in [13] to recover $\partial_s^2\mathcal{R}^{\text{nor}}\underline{\mathbf{f}}^d$ from $\partial_s^2\mathcal{R}^{\tan}\underline{\mathbf{f}}$ as in the following form:

$$\partial_s^2\mathcal{R}^{\text{nor}}\underline{\mathbf{f}}^d(s,\underline{\eta}) = \underline{\eta}\int_{S^2}[\partial_s^2\mathcal{R}^{\tan}\underline{\mathbf{f}}](s,\theta)\cdot\nabla_{\underline{\theta}}K(\underline{\eta}\cdot\underline{\theta})d\underline{\theta} \tag{4.78}$$

where

$$K(\underline{\eta}\cdot\underline{\theta}) = \sum_{n\geq 0}\frac{2n+3}{4\pi(n+2)}P_{n+1}(\underline{\eta}\cdot\underline{\theta}), \tag{4.79}$$

$$P_{n+1}(\underline{\eta}\cdot\underline{\theta}) = \sum_{|l|\leq n+1}\frac{4\pi}{(2n+3)}Y_{n+1,l}(\underline{\eta})\overline{Y}_{n+1,l}(\underline{\theta}), \tag{4.80}$$

see appendix B and $Y_{n,l}$ is a spherical harmonic as introduced in Chapter 3. We shall briefly verify the formula (4.77) by following framework in [14, 13] but here we shall provide details at some points. The authors in [14] proposed a unique representation of the solenoidal part $\underline{\mathbf{f}}^d$ of $\mathbf{f}$ as

$$\begin{aligned}\underline{\mathbf{f}}^d(\underline{x}) &= \sum_{n=0}^{\infty}\sum_{|l|\leq n+1}b_{n+1,l}^{(n)}B_{n+1,l}^{(n)}(\underline{x})\\ &+\sum_{n=2}^{\infty}\sum_{k=1}^{[n/2]}\sum_{|l|\leq n+1}b_{n+1-2k,l}^{(n)}B_{n+1-2k,l}^{(n)}(\underline{x}) + \sum_{n=1}^{\infty}\sum_{k=0}^{[(n-1)/2]}\sum_{|l|\leq n+1}c_{n-2k,l}^{(n)}C_{n-2k,l}^{(n)}(\underline{x})\end{aligned} \tag{4.81}$$

where $B_{n,l}^{(n)}(\underline{x})$ and $C_{n,l}^{(n)}(\underline{x})$ are defined in (3.37-3.40) and $b_{k,l}^{(n)},\ c_{k,l}^{(n)},\ k\in\mathbb{N}$ are constants. Then the second derivative of its Radon transform takes the form

$$\begin{aligned}\frac{\partial^2}{\partial s^2}\mathcal{R}\underline{\mathbf{f}}^d(s,\underline{\eta}) &= \frac{\partial^2}{\partial s^2}\int_{B^3}\underline{\mathbf{f}}^d(\underline{x})\delta(s-\underline{x}\cdot\underline{\eta})d\underline{x}\\ &= \sum_{n=0}^{\infty}\sum_{|l|\leq n+1}b_{n+1,l}^{(n)}\frac{\partial^2}{\partial s^2}\int_{B^3}\int_{S^2}y_{n+1,l}^{(2)}(\underline{\xi})C_n^{3/2}(\underline{x}\cdot\underline{\xi})d\underline{\xi}\delta(s-\underline{x}\cdot\underline{\eta})d\underline{x}\\ &+\sum_{n=2}^{\infty}\sum_{k=1}^{[n/2]}\sum_{|l|\leq n+1}b_{n+1-2k,l}^{(n)}\frac{\partial^2}{\partial s^2}\int_{B^3}\int_{S^2}y_{n+1-2k,l}^{(2)}(\underline{\xi})C_n^{3/2}(\underline{x}\cdot\underline{\xi})d\underline{\xi}\delta(s-\underline{x}\cdot\underline{\eta})d\underline{x}\\ &+\sum_{n=1}^{\infty}\sum_{k=0}^{[(n-1)/2]}\sum_{|l|\leq n+1}c_{n-2k,l}^{(n)}\frac{\partial^2}{\partial s^2}\int_{B^3}\int_{S^2}y_{n-2k,l}^{(3)}(\underline{\xi})C_n^{3/2}(\underline{x}\cdot\underline{\xi})d\underline{\xi}\delta(s-\underline{x}\cdot\underline{\eta})d\underline{x}.\end{aligned} \tag{4.82}$$

By using (3.38), (3.39) and (3.40) stating that

$$\int_{S^2} y^{(2)}_{n+1,l}(\underline{\xi}) C_n^{3/2}(\underline{\eta}\cdot\underline{\xi}) d\underline{\xi} = \frac{4\pi(n+2)}{2n+3}\left((n+1)y^{(1)}_{n+1,l}(\underline{\eta}) + y^{(2)}_{n+1,l}(\underline{\eta})\right)$$

$$\int_{S^2} y^{(2)}_{n-2k+1,l}(\underline{\xi}) C_n^{3/2}(\underline{\eta}\cdot\underline{\xi}) d\underline{\xi} = 4\pi y^{(2)}_{n-2k+1,l}(\underline{\eta}), k = 1, \ldots, [n/2]$$

$$\int_{S^2} y^{(3)}_{n-2k,l}(\underline{\xi}) C_n^{3/2}(\underline{\eta}\cdot\underline{\xi}) d\underline{\xi} = 4\pi y^{(3)}_{n-2k,l}(\underline{\eta}) \quad , k = 0, \ldots, [(n-1)/2]$$

one claim that

$$\int_{B^3} \delta(s - \underline{x}\cdot\underline{\eta}) C_n^{3/2}(\underline{x}\cdot\underline{\xi}) d\underline{x} = \frac{2\pi(1-s^2)C_n^{3/2}(s)C_n^{3/2}(\underline{\eta}\cdot\underline{\xi})}{(n+1)(n+2)} \tag{4.83}$$

see appendix D. Substituting (4.82) in to (4.81) which is the second derivative of the Radon transform of $\underline{\mathbf{f}}^d$ yields

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}\mathcal{R}\underline{\mathbf{f}}^d(s,\underline{\eta}) &= \sum_{n=0}^{\infty} \frac{d^2}{ds^2}\left(\frac{8\pi^2(1-s^2)C_n^{3/2}(s)}{(n+1)(2n+3)}\right) \sum_{|l|\le n+1} b^{(n)}_{n+1,l}\left((n+1)y^{(1)}_{n+1,l}(\underline{\eta}) + y^{(2)}_{n+1,l}(\underline{\eta})\right) \\
&+ \sum_{n=2}^{\infty} \frac{d^2}{ds^2}\left(\frac{8\pi^2(1-s^2)C_n^{3/2}(s)}{(n+1)(n+2)}\right) \sum_{k=1}^{[n/2]} \sum_{|l|\le n+1} b^{(n)}_{n+1-2k,l} y^{(2)}_{n+1-2k,l}(\underline{\eta}) \\
&+ \sum_{n=1}^{\infty} \frac{d^2}{ds^2}\left(\frac{8\pi^2(1-s^2)C_n^{3/2}(s)}{(n+1)(n+2)}\right) \sum_{k=0}^{[(n-1)/2]} \sum_{|l|\le n+1} c^{(n)}_{n-2k,l} 4\pi y^{(3)}_{n-2k,l}(\underline{\eta}).
\end{aligned} \tag{4.84}$$

By using the decomposition together with the fact that for $j \in \mathbb{N}$, $y^{(1)}_{j,l}(\underline{\eta})$ is parallel to $\underline{\eta}$ and $y^{(2)}_{j,l}(\underline{\eta})$, $y^{(3)}_{j,l}(\underline{\eta})$ are perpendicular to $\underline{\eta}$, we then obtain

$$\frac{\partial^2}{\partial s^2}\mathcal{R}^{\text{nor}}\underline{\mathbf{f}}^d(s,\underline{\eta}) = \sum_{n=0}^{\infty} \frac{d^2}{ds^2}\left(\frac{8\pi^2(1-s^2)C_n^{3/2}(s)}{(2n+3)}\right) \sum_{|l|\le n+1} b^{(n)}_{n+1,l} y^{(1)}_{n+1,l}(\underline{\eta}) \tag{4.85}$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s,\underline{\eta}) &= \sum_{n=0}^{\infty} \frac{d^2}{ds^2}\left(\frac{8\pi^2(1-s^2)C_n^{3/2}(s)}{(n+1)(2n+3)}\right) \sum_{|l|\le n+1} b^{(n)}_{n+1,l} y^{(2)}_{n+1,l}(\underline{\eta}) \\
&+ \sum_{n=2}^{\infty} \frac{d^2}{ds^2}\left(\frac{8\pi^2(1-s^2)C_n^{3/2}(s)}{(n+1)(n+2)}\right) \sum_{k=1}^{[n/2]} \sum_{l} b^{(n)}_{n+1-2k,l} y^{(2)}_{n+1-2k,l}(\underline{\eta}) \\
&+ \sum_{n=1}^{\infty} \frac{d^2}{ds^2}\left(\frac{8\pi^2(1-p^2)C_n^{3/2}(s)}{(n+1)(n+2)}\right) \sum_{k=0}^{[(n-1)/2]} \sum_{l} c^{(n)}_{n-2k,l} y^{(3)}_{n-2k,l}(\underline{\eta}).
\end{aligned} \tag{4.86}$$

Applying dot product to (4.83) by $\overline{y}^{(2)}_{n+1}(\underline{\eta})$, integrating over S^2 and using the fact that $y^{(j)}_{n,l}$ are orthogonal system gives

$$\int_{S^2} \frac{\partial^2}{\partial s^2}\mathcal{R}\underline{\mathbf{f}}^d(s,\underline{\eta})\cdot \overline{y}^{(2)}_{n+1,l}(\underline{\eta})d\underline{\eta} = \frac{d^2}{ds^2}\left(\frac{8\pi^2(1-s^2)C_n^{3/2}(s)}{(n+1)(2n+3)}\right) b^{(n)}_{n+1,l}\|y^{(2)}_{n+1,l}\|^2 \tag{4.87}$$

see appendix D for the verification.

Since $y^{(2)}_{j,l}$ is perpendicular to $\underline{\eta}$,

$$y^{(2)}_{n+1,l}\cdot \frac{\partial^2}{\partial s^2}\mathcal{R}^{\text{nor}}\mathbf{f}^d(s,\underline{\eta}) = \nabla_{\underline{\eta}}Y_{n,l}(\underline{\eta})\cdot\frac{\partial^2}{\partial s^2}\mathcal{R}^{\text{nor}}\mathbf{f}^d(s,\underline{\eta}) = 0.$$

Consequently, the equation (4.86) can be replaced by

$$\int_{S^2} \frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s,\underline{\eta})\cdot \overline{y}^{(2)}_{n+1}(\underline{\eta})d\underline{\eta} = \frac{d^2}{ds^2}\left(\frac{8\pi^2(1-s^2)C_n^{3/2}(s)}{(n+1)(2n+3)}\right) b^{(n)}_{n+1,l}\|y^{(2)}_{n+1,l}\|^2. \tag{4.88}$$

Multiplying (4.87) by $\dfrac{(n+1)}{\|y^{(2)}_{n+1,l}\|^2}y^{(1)}_{n+1,l}(\underline{\theta})$ and summing over n,l provides

$$\sum_{n=0}^{\infty}\sum_{|l|\le n+1}\int_{S^2}\frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\mathbf{f}(s,\underline{\eta})\cdot\overline{y}^{(2)}_{n+1,l}(\underline{\eta})d\underline{\eta}\frac{(n+1)}{\|y^{(2)}_{n+1,l}\|^2}y^{(1)}_{n+1,l}(\underline{\theta}) \tag{4.89}$$

$$=\sum_{n=0}^{\infty}\sum_{|l|\le n+1}\frac{d^2}{ds^2}\left(\frac{8\pi^2(1-s^2)C_n^{3/2}(s)}{(2n+3)}\right) b^{(n)}_{n+1,l}y^{(1)}_{n+1,l}(\underline{\theta}) \tag{4.90}$$

$$=\frac{\partial^2}{\partial s^2}\mathcal{R}^{\text{nor}}\underline{\mathbf{f}}^d(s,\underline{\theta}). \tag{4.91}$$

The authors in [13], furthermore, provide the matrix form of the sum in (4.89) by using the fact that $y^{(1)}_{n,l}(\underline{\theta}) = \underline{\theta}Y_{n,l}(\underline{\theta})$, $y^{(2)}_{n,l}(\underline{\eta}) = \nabla_{\underline{\eta}}Y_{n,l}(\underline{\eta})$ and $\|y^{(2)}_{n+1,l}\|^2 = (n+1)(n+2)$:

$$\underline{\theta}\nabla_{\underline{\eta}}\sum_{|l|\le n}Y_{n+1}(\underline{\eta})\overline{Y}_{n+1,l}(\underline{\theta}) = \frac{2n+3}{4\pi}\underline{\theta}\nabla_{\underline{\eta}}P_{n+1}(\underline{\theta}\cdot\underline{\eta}). \tag{4.92}$$

By some calculations according to (4.90) and the operator $\nabla_{\underline{\eta}}$, one can have

$$\frac{\partial^2}{\partial s^2}\mathcal{R}^{\text{nor}}\underline{\mathbf{f}}^d(s,\underline{\theta}) = \underline{\theta}\int_{S^2}\frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s,\underline{\eta})\cdot\nabla_{\underline{\eta}}\left(\sum_{n\ge 0}\frac{2n+3}{4\pi(n+2)}P_{n+1}(\underline{\theta}\cdot\underline{\eta})\right)d\underline{\eta}. \tag{4.93}$$

In the same manner as the case of $\mathcal{R}^{\tan}\underline{\mathbf{f}}_1^d$, we obtain

$$\begin{aligned}
\underline{\mathbf{f}}_2^d(\underline{x}) &= -\frac{1}{8\pi^2}\int_{S^2}\frac{\partial^2}{\partial s^2}\mathcal{R}^{\text{nor}}\underline{\mathbf{f}}^d(s,\underline{\eta})|_{s=\underline{\eta}\cdot\underline{x}}d\underline{\eta} \\
&= -\frac{1}{8\pi^2}\int_{S^2}\eta\int_{S^2}\frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\theta})|_{s=\underline{\theta}\cdot\underline{x}}\cdot\nabla_{\underline{\theta}}K(\underline{\eta}\cdot\underline{\theta})d\underline{\theta}d\underline{\eta} \\
&= -\frac{1}{8\pi^2}\int_{S^2}\underline{\eta}\int_{S^2}\left(\underline{\theta}[\underline{\theta}\times\frac{\partial^2}{\partial s^2}\mathcal{R}^{\tan}\underline{\mathbf{f}}(s,\underline{\theta})]\right)|_{s=\underline{\theta}\cdot\underline{x}}\cdot\nabla_{\underline{\theta}}K(\underline{\eta}\cdot\underline{\theta})d\underline{\theta}d\underline{\eta} \\
&= -\frac{1}{8\pi^2}\int_{S^2}\underline{\eta}\int_{S^2}\left(\underline{\theta}\sum_{\lambda_i\in\mathcal{L}}\phi(a(\lambda_i))G(a(\lambda_i),\underline{\theta})\right)\cdot\nabla_{\underline{\theta}}K(\underline{\eta}\cdot\underline{\theta})d\underline{\theta}d\underline{\eta} \\
&= \frac{1}{8\pi^2}\int_{S^2}\underline{\eta}\int_I\frac{1}{|x-a(\lambda)|}h_2^d(\beta,\lambda)d\lambda d\underline{\eta}.
\end{aligned}$$

Combining theorem 4.14 and 4.15 gives the reconstruction procedure for the solenoidal part of $\underline{\mathbf{f}}$ by using the 3 dimensional transverse-ray transform with source points on a curve. We would like to discuss an important remark:

Remark Only the solenoidal part of $\underline{\mathbf{f}}$ can be recovered from such transverse-ray transform. As mentioned in chapter 3, we consider the Helmholtz-Hodge decomposition (2.25) of a vector field of the form

$$\underline{\mathbf{f}} = \nabla p + \underline{\mathbf{f}}^d, \quad \underline{\mathbf{f}}^d = \underline{\mathbf{f}}_{\mathbf{0}}^{\mathbf{d}} + \nabla h$$

where $\underline{\mathbf{f}}^d$ is the solenoidal part and ∇p is the potential part with $p = 0$ on S^2. For fixed $\underline{\eta}\in S^2$ and $e_1(\underline{\eta})$, $e_2(\underline{\eta})$ are defined in section 2.10, one can write ∇p in the form

$$\nabla p = < e_1(\underline{\eta}),\nabla > pe_1(\underline{\eta}) + < e_2(\underline{\eta}),\nabla > pe_2(\underline{\eta}) + < \underline{\eta},\nabla > p\underline{\eta}. \tag{4.94}$$

Consider the Radon transform

$$\begin{aligned}
R(< e_1(\underline{\eta}),\nabla p >)(s,\underline{\eta}) &= \int_{\mathbb{R}^3}(e_1\cdot\nabla p)(\underline{x})\delta(s-\underline{x}\cdot\underline{\eta})d\underline{x} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}(e_1\cdot\nabla p)(re_1+te_2+s\underline{\eta})dtdr \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\partial p}{\partial r}(re_1+te_2+s\underline{\eta})drdt = 0
\end{aligned}$$

where we have used the fact that $p = 0$ on S^2. Similarly,

$$R(< e_1(\underline{\eta}), \nabla p >)(s, \underline{\eta}) = 0.$$

Therefore,

$$R(\nabla p)(s, \underline{\eta}) = R(< \underline{\eta}, \nabla p >)(s, \underline{\eta})\underline{\eta} = \underline{\eta}\left(\underline{\eta} \cdot R\nabla p(s, \underline{\eta})\right).$$

Substituting the previous equation in the decomposition of ∇p

$$\begin{aligned} R^{\tan}\nabla p &= \frac{\partial}{\partial s} R\nabla p(s, \underline{\eta}) - \underline{\eta}(\underline{\eta} \cdot \frac{\partial}{\partial s} R\nabla p) \\ &= \underline{\eta}(\underline{\eta} \cdot \frac{\partial}{\partial s} R\nabla p(s, \underline{\eta})) - \underline{\eta}(\underline{\eta} \cdot \frac{\partial}{\partial s} R\nabla p)(s, \underline{\eta}) \end{aligned}$$

gives

$$R^{\tan}\nabla p = 0.$$

Consequently,

$$\underline{\eta} \times \mathcal{R}^{\tan}\underline{\mathbf{f}}(s, \underline{\eta}) = \underline{\eta} \times \mathcal{R}^{\tan}\underline{\mathbf{f}}^d(s, \underline{\eta})$$

and so the remark is clarified.

The last part of this section shall discuss about the recovering the potential part of $\underline{\mathbf{f}}$ by using a different type of transverse-ray transform.

Proposition 4.16. *For a given measured data $\tilde{T}(\underline{\mathbf{f}})$. Under the same assumptions as in theorem 4.14, one can recover the potential from $\tilde{T}(\underline{\mathbf{f}}) - \tilde{T}(\underline{\mathbf{f}}^d)$.*

Proof. By assumptions on the curve, for each $\underline{\eta}$, there exist at least 3 points $a(\lambda_i),\ i =$

$1, 2, 3$ on it such that $a(\lambda_i) \cdot \underline{\eta} = \underline{x} \cdot \underline{\eta}$. The potential field can be viewed as:

$$\begin{aligned} p(\underline{x}) &= -\frac{1}{8\pi^2} \int_{S^2} \frac{\partial^2}{\partial s^2} \mathcal{R}p(s, \underline{\eta})|_{s=\underline{x}\cdot\underline{\eta}} d\underline{\eta} \\ &= -\frac{1}{8\pi^2} \int_{S^2} \underline{\eta} \cdot \frac{\partial}{\partial s} \mathcal{R}(\nabla p)|_{s=\underline{x}\cdot\underline{\eta}} d\underline{\eta} \\ &= -\frac{1}{24\pi^2} \int_{S^2} \int_{S^2} \delta'(\underline{\theta} \cdot \underline{\eta}) \underline{\eta} \cdot \sum_{i=1}^{3} D_{a(\lambda_i)}(\nabla p)(\underline{\theta}) d\underline{\theta} d\underline{\eta} \\ &= -\frac{1}{24\pi} \int_{S^2} \int_{S^2} \delta'(\underline{\theta} \cdot \underline{\eta}) \sum_{i=1}^{3} \tilde{T}_{a(\lambda_i)}(\nabla p)(\underline{\theta}) d\underline{\theta} d\underline{\eta} \\ &= -\frac{1}{24\pi} \int_{S^2} \int_{S^2} \delta'(\underline{\theta} \cdot \underline{\eta}) \sum_{i=1}^{3} (\tilde{T}_{a(\lambda_i)} \underline{\mathbf{f}} - \tilde{T}_{a(\lambda_i)} \underline{\mathbf{f}}^d)(\underline{\theta}) d\underline{\theta} d\underline{\eta} \end{aligned}$$

Using the technique in [13], we obtain that p can be written in the form of integral on the curve as

$$p(\underline{x}) = \frac{1}{24\pi} \int_{S^2} \int_I \frac{1}{|\underline{x} - a(\lambda)|} \int_{S^2} \delta'(\underline{\theta} \cdot \underline{\eta}) \phi(\lambda) (\tilde{T}_{a(\lambda)} \underline{\mathbf{f}} - \tilde{T}_{a(\lambda)} \underline{\mathbf{f}}^d)(\underline{\theta}) d\lambda d\underline{\eta}$$

where $\phi(\lambda)$ is defined in theorem 4.14. $\square$

5 DISCUSSION

Many physical phenomena can be described by quaternions rather than vector fields. This is one of the reasons inspiring us to consider the reconstruction of the quaternionic-valued function. We have paid attention to the vector part of a quaternionic-valued function which is similar to the vector field in some sense. We presented the reconstructions both in the parallel beam setting and cone beam setting. The formula for the former one is,

$$\underline{\mathbf{f}}(\underline{x}) = -\frac{1}{2(2\pi)^2}\partial_{\underline{x}}\int_{S^2}\underline{\theta}R\underline{\mathbf{f}}'(<\underline{\theta},\underline{\theta}\cdot\underline{x}>)d\underline{\theta}. \tag{5.1}$$

The existence of such a formula might not be surprising since this formula is in the scalar case, but we have seen that the quaternionic version gives new insight. The formula (5.1), however, involved the quaternion arguments which is the non-tangential boundary values of the Cauchy transform of $\underline{\mathbf{f}}$. This boundary value is analogue of the Riemann-Hilbert problem in the complex analysis. We would like to remark that the Dirac operator is related to the Laplacian or the lambda operator in Euclidean space. Moreover, this formula can ensure that the vector part of a quaternionic-valued function also can be decomposed as the divergence and the curl free parts in the Helmholtz decomposition for the case of vector fields.

In the latter case, we presented the reconstruction formula both with sources on the sphere and on a curve satisfying the Tuy's condition of order 3. For the sources on the sphere, we begin by using (5.1) and then use (4.18) which already proposed in the scalar case in [17]. To get this reconstruction, we may begin with another form rather than (5.1) but here we consider the equation (5.1) because it provides the orthogonality or the transform $\underline{\theta} \to -\underline{\theta}$ in its arguments.

The certain property of a curve called Tuy's condition of order 3 provides the data where sources are on the curve. More precisely, its properties make sense for the relation

$$-\int_{S^2} D\underline{\mathbf{f}}(\underline{x},\underline{\theta})\delta''(\underline{\theta}\cdot\underline{\eta})d\underline{\theta} = (\mathrm{div}_{\underline{\eta}}(\underline{\eta}\times R^{\tan}\underline{\mathbf{f}})(s,\underline{\eta}))|_{s=\underline{\eta}\cdot\underline{x}} + \underline{x}\cdot\left(\underline{\eta}\times\frac{\partial}{\partial s}(R^{\tan}\underline{\mathbf{f}})(s,\underline{\eta})\right)|_{s=\underline{\eta}\cdot\underline{x}} \tag{5.2}$$

where

$$D\underline{\mathbf{f}}(\underline{x},\underline{\theta}) = \int_0^\infty \underline{\eta}\cdot(\underline{\theta}\times\underline{\mathbf{f}})(\underline{x}+t\underline{\theta})dt. \tag{5.3}$$

We followed the procedures in the paper of Katsevich and Schuster [13]. Their inversion formula is exact. The obtained formula in this work, however, is indirect for the mathematical point of view. The transform (5.3) is the transverse-ray transform. Only the divergence free part of a vector field can be reconstructed by the transform (5.3) together with (5.2). However, the potential part can be recovered by using the measured data and the reprojection of the solenoidal part.

6 CONCLUSIONS

As in scalar 3D computerized tomography, the inversion of the cone beam transform is a special interest from a practical point of view. The cone beam transform for vector fields have been achieved recently Katsevich and Schuster. A number of quantities in physics behave as some parts of quaternions rather than vector fields. We thus further pay attention to the reconstructions of the quaternionic-valued functions. To see the procedures of the reconstructions, we shall follow some work done in the vector field case since vector fields and the vector parts of quaternions have some properties in common.

The reconstructions for the parallel beam setting has been discussed. This formula is the Dirac operator acting on the back-projection of the first derivative of the Radon transform weight with unit vectors on the sphere. This formula is closely related to a well-known back-projection formula established in scalar case where the Laplacian and the lambda operators have been used. In one of procedures, we have used the Grangeat's formula which we define in vector-valued case in componentwise sense.

For the simplest cone beam case where sources are on the sphere, we begin with the previous result. Since this result contains the first of the Radon transform, we can use Grangeat 's formula to connect cone beam transform with sources on the sphere and the data. For the last part which is the cone beam setting with sources on a curve fulfilling Tuy's conditions, we have obtained the indirect data by using the transverse-ray transform. The whole vector field can be recovered by using this transverse-ray transform.

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APPENDICES

A APPENDIX Gegenbauer polynomials

Gegenbauer polynomials $C_n^{(\nu)}(s)$ are orthogonal polynomials on the interval [-1,1] with respect to the weight function $(1-s^2)^{\nu-1/2}$. Some of the characterizations of the Gegenbauer polynomials are the following:

- They can be defined in terms of their generating function

$$\frac{1}{(1-2st+t^2)^\nu} = \sum_{n=0}^{\infty} C_n^\nu(s)t^n;$$

- They are given as Gaussian hypergeometric series

$$C_n^\nu(s) \sum_{k=0}^{[n/2]} (-1)^k \frac{\Gamma(n-k+\nu)}{\Gamma(\nu)k!(n-2k)!}(2s)^{n-2k}.$$

They furthermore have the orthogonality and normalization : For fixed ν, the polynomials are orthogonal on [-1,1] with respect to the weight function $w(s) = (1-s^2)^{\nu-1/2}$. More precisely, , for $m \neq n$,

$$\int_{-1}^{1} C_n^\nu(s)C_m^\nu(s)(1-s^2)^{\nu-1/2}ds = 0. \tag{A.1}$$

They are normalized by

$$\int_{-1}^{1} [C_n^\nu(s)]^2(1-s^2)^{\nu-1/2}ds = \frac{\pi 2^{1-2\nu}\Gamma(n+2\nu)}{n!(n+\nu)[\Gamma(\nu)]^2} \tag{A.2}$$

where

$$\Gamma(t+1) = t\Gamma(t), \quad t > 0, \quad \Gamma(1/2) = \sqrt{\pi}.$$

and for $n \in \mathbb{N}$,

$$\Gamma(n+1) = n!.$$

B APPENDIX Spherical harmonics

Recall that spherical coordinate for $x \in \mathbb{R}^3$ is $x = r\eta$ where

$$\eta = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, -\sin\theta) \in S^2, \quad \varphi \in [0, 2\pi], \; \theta \in (0, \pi).$$

In obtaining the solutions to Laplaces equation in spherical coordinates, it is traditional to introduce the spherical harmonics, $Y_{n,l}(\theta,\varphi)$

$$
\begin{aligned}
Y_{n,l}(\theta,\varphi) &= (-1)^l\sqrt{\frac{(2n+1)(n-l)!}{4\pi(n+l)!}}P_{n,l}(\cos\theta)e^{il\varphi} \\
&= a_{n,l}P_{n,l}(\cos\theta)e^{il\varphi}
\end{aligned}
$$

for

$$
\begin{cases}
n = 0, 1, 2, 3, \ldots, \\
l = -n, -n+1, \ldots, n-1, n.
\end{cases}
$$

Here $P_{n,l}(x)$ are Legendre polynomials with the property

$$P_{n,-l}(\cos\underline{\theta}) = (-1)^l\frac{(n-l)!}{(n+l)!}P_{n,l}(\cos\underline{\theta}).$$

This leads to

$$Y_{n,-l}(\underline{\theta},\underline{\varphi}) = (-1)^l\overline{Y_{n,l}(\underline{\theta},\underline{\varphi})}.$$

A well-behaved function f of θ and φ can be written as

$$f(\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{l=-n}^{n}a_{nl}Y_{n,l}(\theta,\varphi).$$

For example, we list a few low order spherical harmonics :

$$
\begin{aligned}
&n = 0, \quad Y_{0,0}(\theta,\varphi) = \frac{1}{4\pi} \\
&n = 1, \begin{cases}
Y_{1,1}(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi} \\
Y_{1,0}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta
\end{cases} \\
&n = 2, \begin{cases}
Y_{2,1}(\theta,\varphi) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\varphi} \\
Y_{2,0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{5}{4\pi}}(3\cos^2\theta - 1) \\
Y_{2,2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\varphi}.
\end{cases}
\end{aligned}
$$

We next establish some relations involving $y_{n,l}$ from section 3.4.2. We claim that

$$\sum_{|l|\le n+1} \frac{\overline{y}^{(2)}_{n+1,l}(\eta)\overline{y}^{(1)}_{n+1,l}(\underline{\alpha})}{(n+2)} = \sum_{|l|\le n+1} \frac{\overline{\nabla_\eta Y_{n+1,l}(\eta)}\underline{\alpha} Y_{n+1,l}(\underline{\alpha})}{(n+2)}. \tag{B.1}$$

Consider

$$\begin{aligned}
\nabla_\eta Y_{n,l}(\eta) &= \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}(Y_{n,l})e_1(\eta) + \frac{\partial}{\partial\theta}(Y_{n,l})e_2(\eta)\\
&= a_{n,l}P_{n,l}(\cos\theta)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}e^{il\varphi} + a_{n,l}e^{il\varphi}\frac{\partial}{\partial\theta}(P_{n,l}(\cos\theta))\\
&= a_{n,l}P_{n,l}(\cos\theta)\frac{1}{\sin\theta}(il)e^{il\varphi} + a_{n,l}e^{il\varphi}\frac{\partial}{\partial\theta}(P_{n,l}(\cos\theta)).
\end{aligned}$$

Thus,

$$\overline{\nabla_\eta Y_{n,l}(\eta)} = -a_{n,l}P_{n,l}(\cos\theta)\frac{1}{\sin\theta}(il)e^{-il\varphi} + a_{n,l}e^{-il\varphi}\frac{\partial}{\partial\theta}(P_{n,l}(\cos\theta)).$$

On the other hand,

$$\begin{aligned}
\nabla_\eta \overline{Y_{n,l}(\eta)} &= \nabla_\eta(a_{n,l}P_{n,l}(\cos\theta)e^{-il\varphi})\\
&= a_{n,l}P_{n,l}(\cos\theta)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}e^{-il\varphi} + a_{n,l}e^{-il\varphi}\frac{\partial}{\partial\theta}(P_{n,l}(\cos\theta))\\
&= -a_{n,l}P_{n,l}(\cos\theta)\frac{1}{\sin\theta}(il)e^{-il\varphi} + a_{n,l}e^{-il\varphi}\frac{\partial}{\partial\theta}(P_{n,l}(\cos\theta)).
\end{aligned}$$

Thus,

$$\nabla_\eta\overline{Y_{n,l}(\eta)} = \overline{\nabla_\eta Y_{n,l}(\eta)}.$$

So (B.1) is equal to

$$\sum_{|l|\le n+1} \frac{\overline{y}^{(2)}_{n+1,l}(\eta)\overline{y}^{(1)}_{n+1,l}(\underline{\alpha})}{(n+2)} = \underline{\alpha}\nabla_\eta \sum_{|l|\le n+1} \frac{\overline{Y_{n+1,l}(\eta)}Y_{n+1,l}(\underline{\alpha})}{(n+2)}. \tag{B.2}$$

Equation (3.25) in [13] and (B.2)

$$\frac{2n+3}{4\pi}P_{n+1}(\alpha\cdot\eta) = \sum_{|l|\le n+1} \frac{\overline{Y_{n+1,l}(\eta)}Y_{n+1,l}(\underline{\alpha})}{(n+2)}.$$

C APPENDIX Reconstruction formula of scalar and vector fields

In an image processing context the original image f can be recovered from the "sinogram" data Rf by applying a ramp filter (in s variable) and then back-projecting. As the filtering step can be performed efficiently (for example usinf digital signal processing techniques) and the back projection step is simply an accumulation of values in the pixel of the image, this results in a highly efficient, and hence widely used, algorithm.
Explicitly

Definition 0.1 (Lambda operator)**.** *For any real number α and for any tempered distribution u for which $|\xi|^{\alpha}\hat{u}(\xi)$ is also tempered distribution, define*

$$\Lambda^{\alpha} u = \mathcal{F}^{-1}(|\xi|^{\alpha}\hat{u}(\xi)) \tag{C.1}$$

Theorem 0.2. *Let P be the k-plane transform, $0 < k < n$. If f is an L^1 function such that $|\xi|^{-\alpha}\hat{f}(\xi) \in L^1$, then*

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^{\alpha} P^{\sharp} \Lambda^{k-\alpha} P f = f \tag{C.2}$$

D APPENDIX Formula for second derivative in term of series

Before we provide the proof of the Radon transform of a Gegenbauer polynomial we shall recall lemma 2.3 in [15]

Lemma 0.3. *Let* $Rf \in L_2(\mathbb{R} \times S^2, w_\nu^{-1}), w_\nu = (1-s^2)^{\nu-1/2}$ *and* $\phi_{mkl} = w_\nu C_m^\nu Y_{lk}$*. Then*

$$Rf(s,w) = w_\nu(s) \sum_{m=0}^{\infty} C_m^\nu(s) q_m(w) \tag{D.1}$$

with

$$q_m(w) = \sum_{l=0 m+l \ even}^{m} \sum_{k=1}^{M(N,l)} d_{mkl} Y_{lk}(w) \tag{D.2}$$

$$= h_{m\nu}^{-1} \int_{-1}^{1} C_m^\nu(s) Rf(s,w) ds, \tag{D.3}$$

$$d_{mlk} = h_{m\nu}^{-1} \int_{S^{N-1}} Y_{lk}(\omega) \int_{-1}^{1} Rf(s,\omega) C_m^\nu(s) ds d\omega \tag{D.4}$$

and

$$h_{m\nu} = \int_{-1}^{1} w_\nu(s) [C_m^\nu(s)]^2 ds. \tag{D.5}$$

Then we shall derive the following:

Lemma 0.4. *The Radon transform of the Gegenbeaur polynomial of order 3/2, for* $\underline{\xi}, \underline{\eta}$ *are fixed: reads*

$$\int_{B^3} \delta(s - \underline{x} \cdot \underline{\eta}) C_n^{3/2}(\underline{x} \cdot \underline{\xi}) d\underline{x} = \frac{2\pi(1-s^2) C_n^{3/2}(s) C_n^{3/2}(\underline{\xi} \cdot \underline{\eta})}{(n+1)(n+2)} \tag{D.6}$$

Proof. Since $C_n^{3/2}(\cdot)$ is real, taking conjugate in both sides of (3.22), we obtain

$$C_n^{3/2}(\underline{x} \cdot \underline{\xi}) = 4\pi \sum_{k=0}^{[n/2]} \sum_{|l| \le n-2k} Z_{n-2k,l}^{(n)}(\underline{x}) \overline{Y_{n-2k,l}(\underline{\xi})}. \tag{D.7}$$

Then

$$RC_n^{3/2}(< \cdot, \underline{\xi} >)(s, \underline{\eta}) = 4\pi \sum_{k=0}^{[n/2]} \sum_{l} \overline{Y_{n-2k,l}(\underline{\xi})} R Z_{n-2k,l}^{(n)}(\cdot)(s, \underline{\eta}). \tag{D.8}$$

From the lemma (0.3), n is fixed

$$RZ^{(n)}_{n-2k,l}(\cdot)(s,\underline{\eta}) = (1-s^2)\sum_{m=0}^{\infty} C_m^{3/2}(s)q_m(\underline{\eta}) \tag{D.9}$$

where

$$q_m(\underline{\eta}) = h_{m\nu}^{-1}\int_{-1}^{1} RZ^{(n)}_{n-2k,l}(s,\underline{\eta})C_m^{3/2}(s)ds. \tag{D.10}$$

By using the adjoint of the Radon transform

$$\int_{-1}^{1} Rf(s,\underline{\eta})h(s)ds = \int_{B^3} f(x)h(\underline{x}\cdot\underline{\eta})d\underline{x} \tag{D.11}$$

we then have

$$\begin{aligned}
q_m(\underline{\eta}) &= h_{m\nu}^{-1}\int_{B^3} Z^{(n)}_{n-2k,l}(x)C_m^{3/2}(\underline{x}\cdot\underline{\eta})d\underline{x} \\
&= h_{m\nu}^{-1}\int_{B^3} Z^{(n)}_{n-2k,l}(x)4\pi\sum_{j=0}^{[m/2]}\sum_{|d|\le m-2j}\overline{Z^{(m)}_{m-2j,d}(\underline{x})Y_{m-2j,d}(\underline{\eta})d\underline{x}} \\
&= h_{m\nu}^{-1}4\pi\sum_{j=0}^{[m/2]}\sum_{|d|\le m-2j} Y_{m-2j,d}(\underline{\eta})\int_{B^3} Z^{(n)}_{n-2k,l}(\underline{x})\overline{Z^{(m)}_{m-2j,d}(\underline{x})d\underline{x}} \\
&= h_{n\nu}^{-1}4\pi Y_{n-2k,l}(\underline{\eta})\frac{1}{2n+3}
\end{aligned} \tag{D.12}$$

where we have used the the orthogonality of the Zernike polynomials and that

$$\|Z^{(n)}_{n-2k,l}\|^2_{L^2(B^3)} = \frac{1}{2n+3}.$$

To verify the last equation of (D.2) , we will consider in 2 cases, for $m > n$ and $m < n$. Thus by substituting D.12 into D.9 and D.9 into D.8 we then have,

$$RC_n^{3/2}(<\cdot,\underline{\xi}>)(s,\underline{\eta}) = 4\pi\sum_{k=0}^{[n/2]}\sum_{l}\overline{Y_{n-2k,l}(\underline{\xi})}(1-s^2)C_n^{3/2}(s)h_{n\nu}^{-1}4\pi Y_{n-2k,l}(\underline{\eta})\frac{1}{2n+3}.$$

$$\begin{aligned}
h_{n\nu} &= \int_{-1}^{1}(C_n^{\nu})^2(1-s^2)^{\nu-1/2}ds \\
&= \frac{\pi 2^{-2}\Gamma(n+3)}{n!(n+3/2)(\pi/4)} = \frac{2(n+1)(n+2)}{(2n+3)}.
\end{aligned}$$

Hence,

$$RC_n^{3/2}(<\cdot,\underline{\xi}>)(s,\underline{\eta}) = \frac{2\pi(1-s^2)C_n^{3/2}(s)}{(n+1)(n+2)}4\pi\sum_{k=0}^{[n/2]}\sum_{l}\overline{Y_{n-2k,l}(\underline{\xi})}Y_{n-2k,l}(\underline{\eta}) \tag{D.13}$$

$$= \frac{2\pi(1-s^2)C_n^{3/2}(s)C_n^{3/2}(\underline{\eta}\cdot\underline{\xi})}{(n+1)(n+2)}. \tag{D.14}$$

□

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