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In emission computed tomography one wants to determine the location and intensity of radiation emitted by sources in the presence of an attenuating medium. If the attenuation is known everywhere and equals a constant α in a convex neighborhood of the support of f, then the problem reduces to that of inverting the exponential X-ray transform P_α.

The exponential X-ray transform P_μ with the attenuation μ variable is of interest mathematically. For the exponential X-ray transform in two dimensions we show that for a large class of approximate δ-functions E, convolution kernels K exist for use in the convolution backprojection algorithm. $P_\mu f$, K and E are related by the formula $f * E = P^*_{-\mu}(P_\mu f * K)$ where $P^*_{-\mu}$ is the adjoint of $P_{-\mu}$. This approximate inversion formula leads to methods for inverting the exponential X-ray transform.

For the case where the attenuation is constant we derive exact formulas for calculating the convolution kernels from radial point spread functions in $\mathbb{R}^n$. From these an exact inversion formula for the constantly attenuated transform is obtained. Examples of radial point spread functions and calculations of the corresponding convolution kernels are given. Also explicit formulas for calculating errors in the convolution-backprojection algorithm are derived.

The Exponential X-Ray Transform

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I. NOTATION AND BACKGROUND FROM COMPUTED TOMOGRAPHY

The set of all real numbers is denoted by $\mathbb{R}$, the set of all complex numbers by $\mathbb{C}$. The m-dimensional real (complex) Euclidean space is $\mathbb{R}^m(\mathbb{C}^m)$, i.e., if $x \in \mathbb{R}^m(\mathbb{C}^m)$ then $x = (x_1,\ldots,x_m)$ where $x_1,\ldots,x_m \in \mathbb{R}(\mathbb{C})$. The Euclidean inner product of $x,y \in \mathbb{R}^m(\mathbb{C}^m)$ where $x = (x_1,\ldots,x_m)$ and $y = (y_1,\ldots,y_m)$ is $\langle x,y\rangle = \Sigma_{i=1}^m x_i\bar{y}_i$; $\bar{y}_i$ denotes the complex conjugate of y_i. The Euclidean norm of $x \in \mathbb{R}^m(\mathbb{C}^m)$ is $|x| = \sqrt{\langle x,x\rangle}$. The differential operator $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_m}^{\alpha_m}$ where $\alpha = (\alpha_1,\ldots,\alpha_m)$ is a multi-index with $\alpha_1,\ldots,\alpha_m$ non-negative integers and $\partial_{x_i} = \partial/\partial x_i$. The order of the operator ∂_x^α is $|\alpha| = \Sigma_{j=1}^m \alpha_j$. If Ω is a subset of $\mathbb{R}^m$ then $C(\Omega)$ denotes the space of all continuous functions on Ω. For $\ell = 0,1,2,\ldots$, $C^\ell(\Omega)$ is the space of all functions f on Ω such that $\partial_x^\alpha f \in C(\Omega)$ for all $|\alpha| \leq \ell$. Note that $C^0(\Omega) = C(\Omega)$. The subspace of $C^\ell(\Omega)$ consisting of functions with compact support in Ω is denoted by $C_0^\ell(\Omega)$. For $1 \leq p < \infty$, $L^p(\mathbb{R}^m)$ denotes the space of all Lebesgue measurable functions on $\mathbb{R}^m$ such that

$$||f||_p = (\int |f(x)|^p dx)^{1/p} < \infty$$

For $p = \infty$, $L^\infty(\mathbb{R}^m)$ is the space of essentially bounded functions on $\mathbb{R}^m$. The subspace of $L^p(\mathbb{R}^m)$ consisting of all functions that vanish outside Ω is denoted by $L_0^p(\Omega)$. The space of all functions f so that

$$\left(\int_K |f(x)|^p dx\right)^{1/p} < \infty$$

for all compact subsets K of $\mathbb{R}^m$ is $L^p_{loc}(\mathbb{R}^m)$. The unit sphere in $\mathbb{R}^m$ is $S^{m-1} = \{x \in \mathbb{R}^m : |x| = 1\}$. For $\theta \in S^{m-1}$, $\theta^\perp = \{x \in \mathbb{R}^m : \langle x,\theta\rangle = 0\}$ is the subspace of $\mathbb{R}^m$ orthogonal to θ. If X is a vector space, X' will denote its dual space. For $x' \in X'$ and $x \in X$ we will write $x'(x)$ as $\langle x',x\rangle$.

In 1917, J. Radon [17], proposed and solved the problem of determining a function from its integrals over all hyperplanes in $\mathbb{R}^n$. This is now known as Radon's problem. There were other independent discoveries of Radon's problem prior to and after 1917 in such fields as astronomy, probability and radiology. For more information and references to these discoveries see [6]. Much research has been done on the Radon problem as well as on applications of it in various fields of science and technology. Applications to radiology will be the standard example used in this work.

In classical radiology, one is interested in determining the intensity loss of the X-ray beam when passing through an inhomogeneous medium. Let $\mu(x)$ be the X-ray attenuation coefficient of the tissue at a point x with respect to photons at energy E. Let I_0 be the initial intensity of the X-ray beam, I its intensity leaving the body. Assuming that the photons are at a constant energy E, it has been experimentally verified that

$$I = I_0\, e^{-\int_L \mu(x)dx} \tag{1.1}$$

where L is the straight line along which the X-ray beam, assumed to be infinitely thin, travels.

In 1963 A. Cormack [4, 5] proposed the problem of determining the X-ray attenuation coefficient $\mu(x)$ in inhomogeneous media, a problem that had received little attention till then. From (1.1) we have

$$\ln(I_0/I) = \int_L \mu(x)dx \tag{1.2}$$

where $\ln(I_0/I)$ can be measured experimentally. So the problem proposed by Cormack is determining μ knowing the line integrals intersecting the region of interest. In the two-dimensional case this is the Radon problem. In practice a cross section of the human body is scanned by a very thin X-ray beam, the intensity loss is recorded and a mathematical alogrithm is applied to the data to produce a two-dimensional image of the cross section. This in fact is sufficient because the body can be viewed as a succession of two-dimensional layers.

This method of reconstructing two-dimensional cross sections is known as Computed Tomography (CT), where "tomos" in Greek means slice. The above is an example of Transmission Computed Tomography (TCT). Another form of CT is Emission Computed Tomography (ECT). Here radionuclides are injected into the body and upon decay emit photons, whose intensity after leaving the body are recorded by a detector. These photons emitted by sources inside the body are attenuated by the amount of material present along the line joining the source to the detector. For more detail on ECT see [2]. In ECT both the location

and the intensity of the emitted radiation are unknown. We will be interested in one kind of ECT namely single photon ECT, which can be described mathematically as follows. Let $f(x)$ be the intensity of the photons emitted from a point x inside the body along a straight line L, then assuming (1.1) still holds, the intensity of the ray leaving the body is

$$f(x)\, e^{-\int_{L(x)} \mu(y)dy}$$

where $L(x)$ is the segment of L between the point x and the detector. The cumulative intensity I of all sources on L will then be

$$I = \int_L f(x)\, e^{-\int_{L(x)} \mu(y)dy}\, dx. \tag{1.3}$$

In single photon ECT one seeks to find f knowing the values of the integrals in (1.3) for a finite set of lines intersecting the region of interest. The attenuation coefficient μ can be estimated by TCT methods; see for instance [21] and [15]. Finding f in (1.3) with μ known is a generalized form of Radon's problem. We shall be interested in solving integral equations of the kind in (1.3).

In practice the important dimensions are two and three. However, it is of equal interest from a mathematical point of view to study such integral equations in higher dimensions. Let $n \geq 2$. A straight line L in $\mathbb{R}^n$ can be parameterized by the pair (θ,x) where $\theta \in S^{n-1}$ and $x \in \theta^{\perp}$ so that

$$L = L(\theta,x) = \{x + t\theta : t \in \mathbb{R}\}.$$

Using this parameterization, (1.2) becomes

$$\ln(I_0/I) = \int_{-\infty}^{\infty} \mu(x+t\theta)dt \tag{1.4}$$

and (1.3) becomes

$$I = \int_{-\infty}^{\infty} f(x+t\theta)\, e^{-\int_t^{\infty} \mu(x+s\theta)ds}\, dt. \tag{1.5}$$

The expression in (1.4) is called the X-ray transform of μ and the expression in (1.5) the attenuated X-ray transform of f and it will be denoted by $T_\mu f$.

Suppose in (1.5) f has support in a bounded set. Suppose also that the attenuation μ is known everywhere and is constant in a convex set D containing the support of f, i.e., the emitter. The attenuation can otherwise be arbitrary except that it be bounded, measurable and have compact support. Let α be the constant value that μ takes on D and for $\theta \in S^{n-1}$, $x \in \theta^{\perp}$ let $d = d(\theta,x)$ be chosen so that $x + d\theta \in D$. Define a real valued function ε^{μ} on $\{(\theta,x) | \theta \in S^{n-1}, x \in \theta^{\perp}\}$ by

$$\varepsilon^{\mu}(\theta,x) = \begin{cases} e^{\int_d^{\infty} \mu(x+s\theta)ds + \alpha d} & \text{if } L(\theta,x) \cap D \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Then it is not hard to see that

$$e^{\alpha t} = \varepsilon^{\mu}(\theta,x)\, e^{-\int_t^{\infty} \mu(x+s\theta)ds} \tag{1.6}$$

for all $x+s\theta \in D$. An important consequence of (1.6) is that

$$\varepsilon^{\mu}(\theta,x)T_{\mu}f(\theta,x) = \int_{-\infty}^{\infty} f(x+t\theta)\, e^{\alpha t}\, dt. \tag{1.7}$$

The right hand side of (1.7) is called the exponential X-ray transform of f and is denoted by $\mathcal{P}_{\alpha}f$. All the above has been shown in Markoe [14]. Because of (1.7) many results on the attenuated X-ray transform can be deduced from results on $\mathcal{P}_{\alpha}$, when μ satisfies the conditions above. In [14], Markoe used (1.7) and analytic continuation to develop an inversion method for T_{μ}.

The purpose of this work is to find inversion and approximate inversion formulas for the operator $\mathcal{P}_{\alpha}$, where α is a constant, and a generalization of it; namely the exponential X-ray transform with variable attenuation μ, also denoted by $\mathcal{P}_{\mu}$ and defined for functions f on $\mathbb{R}^n$ by

$$\mathcal{P}_{\mu}f(\theta,x) = \int_{-\infty}^{\infty} f(x+t\theta)\, e^{\mu(\theta)t}\, dt$$

where $\theta \in S^{n-1}$, $x \in \theta^{\perp}$ and the attenuation $\mu : S^{n-1} \to \mathbb{R}$.

We wish to determine f, or an approximation of f from the data $\mathcal{P}_{\mu}f$. In TCT there are methods used to solve integral equations of the kind in (1.4) known as "convolution-backprojection" algorithms; see Smith [21] for general reference. Following along the same lines, we choose E to be an approximate δ-function; typically these are integrable functions with $\int_{\mathbb{R}^n} E(x)dx = 1$; and for $\rho > 0$ we set

$E_\rho(x) = \rho^{-n}E(x/\rho)$. E_ρ is called a point spread function. We look for functions K_ρ, called convolution kernels, such that

$$f * E_\rho = P^*_{-\mu}(P_\mu f * K_\rho) \tag{1.8}$$

where P^*_μ denotes the formal adjoint of P_μ and will be defined in the next chapter. The right hand side of (1.8) resembles the convolution-backprojection algorithm of TCT, where $(P_\mu f * K_\rho)$ is the convolution step and the action of $P^*_{-\mu}$ is the backprojection step. With suitable conditions on E and f one can show that $E_\rho * f \to f$ uniformly, in L^p norm, or pointwise almost everywhere; see for instance Theorem 2, page 62 of Stein [22]. In practice, the data $P_\mu f$ are incomplete and noisy and so one considers determining a smoothed approximation of f from the data. Formula (1.8) provides a method to achieve that.

In Chapter III, the transform P_μ is introduced formally and some of its basic properties developed. Of these the most important is the relationship

$$f * P^*_{-\mu}K = P^*_{-\mu}(P_\mu f * K) \tag{1.9}$$

Comparison of this formula with (1.8) suggests that we set $P^*_{-\mu}K = E$ and solve for K. In Chapter IV we show that in two dimensions, $P^*_{-\mu}K = E$ can be solved for a large class of approximate δ-functions E. Unfortunately we do not have an explicit expression for K in terms of E in the general case. But in the case where the attenuation is constant we do solve explicitly for K in n dimensions when E is radial. This is done in Chapter V.

Using the results of Chapter V, in Chapter VII we establish an inversion formula for P_μ, when μ is constant, for a large class of functions f, by actually evaluating the limit

$$\lim_{\rho\to 0} P^*_{-\mu}(P_\mu f * K_\rho)$$

with K_ρ as in (1.8).

Relationship (1.9) has also been observed by Tretiak and Metz [24] for the constantly attenuated transform in two dimensions and has been used to invert the transform. The approach used in [24] for inverting the transform is different from the one used here. In [24] they choose a convolution kernel K and then show the corresponding E is an approximate δ-function. Our approach follows that of Madych and Nelson [12] for the Radon transform.

In the course of the work an operator denoted by Λ_μ arises naturally. Chapter VI studies some properties of this operator. In Chapter VIII examples of point spread functions and computations of corresponding convolution kernels are given. In Chapter IX we derive formulas that can be used to check the extent to which the convolution-backprojection algorithm accurately reconstructs f from the data $P_\mu f$, when f is the characteristic function of a disc in the plane. In the next chapter some known results from analysis are presented.

II. BACKGROUND FROM ANALYSIS

Some known results on the Fourier transform and Sobolev spaces are presented here. The material on Sobolev spaces is taken mainly from Triebel [25], although the notation used here is different.

For $f \in L^1(\mathbb{R}^m)$, the Fourier transform of f, denoted by $\hat{f}$, is defined by

$$\hat{f}(\xi) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} f(x)\, e^{-i\langle x,\xi\rangle}\, dx.$$

The Fourier transform can be extended to functions in $L^p(\mathbb{R}^m)$ for $1 \leq p \leq 2$. In fact, the Fourier transform maps $L^p(\mathbb{R}^m)$ continuously into $L^q(\mathbb{R}^m)$ for $1 \leq p \leq 2$ and $1/p + 1/q = 1$. In particular for $p = 2$ the Fourier transform is an isometry.

Let $\mathcal{S}$ denote the space of all functions $\phi \in C^\infty(\mathbb{R}^m)$ such that for all ℓ, k non-negative integers

$$|\phi|_{\ell,k} = \max_{|\alpha|\leq \ell} \sup_x (1+|x|^2)^k |\partial_x^\alpha \phi(x)| < \infty,$$

i.e., the Schwartz space of rapidly decreasing functions. The dual space of $\mathcal{S}$, denoted by $\mathcal{S}'$ is the space of temperate distributions on $\mathbb{R}^m$. Since $\mathcal{S} \subseteq L^1(\mathbb{R}^m)$ the Fourier transform is defined on $\mathcal{S}$. In fact the Fourier transform is a homeomorphism of $\mathcal{S}$ onto $\mathcal{S}$ and thus can be extended to a homeomorphism of $\mathcal{S}'$ onto $\mathcal{S}'$ by duality, i.e., for $f \in \mathcal{S}'$, its Fourier transform $\hat{f}$ is given by

$$\langle \hat{f},\phi\rangle = \langle f,\hat{\phi}\rangle \quad \text{for all} \quad \phi \in \mathcal{S}.$$

The inverse Fourier transform of f denoted by $\tilde{f}$ is

$$\tilde{f}(x) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} f(\xi)\, e^{i<x,\xi>}\, d\xi.$$

Some properties of the Fourier transform that will be used are

a) $(\hat{f})^{\sim} = f$ (2.1)

b) $(\partial_{x_j} f)^{\wedge}(\xi) = i\xi_j \hat{f}(\xi)$ (2.2)

c) $(f*g)^{\wedge} = (2\pi)^{m/2} \hat{f}(\xi)\hat{g}(\xi)$ (2.3)

where $f * g$ denotes the convolution of f and g, i.e.,

$$f * g(x) = \int_{\mathbb{R}^m} f(x-y)g(y)dy.$$

d) If F is radial and f is a function of one variable so that $F(x) = f(|x|)$, $x \in \mathbb{R}^m$, then $\hat{F}$ is radial and

$$\hat{F}(\xi) = |\xi|^{(2-m)/2} \int_0^\infty f(s) J_{(m-2)/2}(s|\xi|) s^{m/2}\, ds \tag{2.4}$$

where J_ν denotes the Bessel function of order ν.

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^m)$ is

$$H^s(\mathbb{R}^m) = \{f \in \mathcal{S}' : \hat{f} \text{ is a function and } \|f\|_{H^s} < \infty\}$$

where

$$\|f\|^2_{H^s} = \int_{\mathbb{R}^m} (1+|\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi$$

If k is a non-negative integer then there exist positive constants c_k and C_k such that

$$c_k \|f\|^2_{H^k} \leq \sum_{|\alpha|\leq k} \|\partial^\alpha_x f\|^2_2 \leq C_k \|f\|^2_{H^k} . \tag{2.5}$$

For the proof see Theorem 2.3.3 of Triebel [25]. A Schwarz inequality holds for this norm, namely

$$\int |fg| dx \leq \|f\|_{H^s} \|g\|_{H^{-s}} \tag{2.6}$$

for all $s \in \mathbb{R}$.

Let $\Omega = \{x \in \mathbb{R}^m : |x| < r, r > 0\}$, i.e., Ω is a ball of radius r in $\mathbb{R}^m$. For $s \in \mathbb{R}$, the Sobolev space $H^s_0(\Omega)$ is

$$H^s_0(\Omega) = \{f \in H^s(\mathbb{R}^m) : \text{support of } f \subseteq \overline{\Omega}\}$$

Here $\overline{\Omega}$ denotes the closure of Ω in $\mathbb{R}^m$. The norm on $H^s_0(\Omega)$ denoted by $\| \ \|_{H^s_0}$ is defined by

$$\|f\|_{H^s_0} = \|f\|_{H^s} .$$

<u>Remark 2.1</u>. For $s \in \mathbb{R}$ and Ω an open ball in $\mathbb{R}^m$, $C^\infty_0(\Omega)$ is dense in $H^s_0(\Omega)$.

The next remark is known as Rellich's Lemma.

<u>Remark 2.2</u>. For $s < t$ the inclusion map $I : H^t_0(\Omega) \to H^s(\mathbb{R}^m)$ is compact.

The Sobolev space $H^s(\Omega)$ is defined by

$$H^s(\Omega) = \{f \in \mathcal{S}' : f = g|_\Omega \text{ for some } g \in H^s(\mathbb{R}^m)\}$$

The norm $\| \ \|_{H^s(\Omega)}$ on $H^s(\Omega)$ is defined by

$$\|f\|_{H^s(\Omega)} = \inf\{\|g\|_{H^s} : g|_\Omega = f\}.$$

<u>Remark 2.3</u>. The dual space of $H_0^s(\Omega)$ is $H^{-s}(\Omega)$ for all $s \in \mathbb{R}$.

The above remark is Theorem 4.8.1 of Triebel [25].

Let $Z = S^1 \times \mathbb{R}$. Define the norm $\| \ \|_{H^s(Z)}$ by

$$\|f\|_{H^s(Z)} = \int_{S^1} \|g(\theta,\cdot)\|^2_{H^s} \, d\theta \tag{2.7}$$

The Sobolev space associated with this norm is denoted by $H^s(Z)$.

<u>Remark 2.4</u>. The dual space of $H^s(Z)$ is $H^{-s}(Z)$ for all $s \in \mathbb{R}$.

The Sobolev spaces $H^s(\mathbb{R}^n)$, $H_0^s(\Omega)$ and $H^s(Z)$ are Banach spaces and hence so are their dual spaces.

III. PROPERTIES OF THE EXPONENTIAL X-RAY TRANSFORM

Definition 3.1. The exponential X-ray transform of a function f in $\mathbb{R}^n$, denoted by $P_\mu f$, is

$$P_\mu f(\theta,x) = \int_{-\infty}^{\infty} f(x+t\theta)\, e^{\mu(\theta)t}\, dt$$

where $\theta \in S^{n-1}$ and $\mu \in C(S^{n-1})$.

In this chapter some basic properties of P_μ will be developed. First some notation is introduced. Let $T = \{(\theta,x) : \theta \in S^{n-1},\ x \in \theta^\perp\}$; T is known as the tangent bundle to S^{n-1}. For $1 \le p < \infty$, $L^p(T)$ denotes the space of all measurable functions g on T such that

$$||g||_p^p = \int_{S^{n-1}} \int_{\theta^\perp} |g(\theta,x)|^p\, dx d\theta < \infty.$$

The inner integral will be denoted by $||g(\theta,\cdot)||^p_{L^p(\theta^\perp)}$. For $p = \infty$, $L^\infty(T)$ is the space of all essentially bounded functions on T.

Proposition 3.2. Let $M = \max|\mu(\theta)|$. If $e^{M|x|}f \in L^1(\mathbb{R}^n)$ then for each $\theta \in S^{n-1}$

$$||P_\mu f(\theta,\cdot)||_{L^1(\theta^\perp)} \le \int_{\mathbb{R}^n} |f(y)|e^{M|y|}\, dy < \infty.$$

Proof. Setting $y = x + t\theta$ and using Fubini's theorem gives

$$\int_{\theta^\perp} |P_\mu f(\theta,x)|dx \le \int_{\theta^\perp}\int_{-\infty}^{\infty} |f(x+t\theta)|e^{\mu(\theta)t}\, dtdx$$

$$\le \int_{\mathbb{R}^n} |f(y)|e^{M|y|}\, dy.$$

<u>Proposition 3.3</u>. The transform P_μ maps $L_0^p(\mathbb{R}^n)$ into $L^p(T)$. Moreover if Ω is a bounded open subset of $\mathbb{R}^n$ then for $f \in L_0^p(\Omega)$

$$||P_\mu(\theta,\cdot)||_{L^p(\theta^\perp)} \le c\,||f||_p$$

where c is a constant depending on Ω, μ and on p. Hence $P_\mu : L_0^p(\Omega) \to L^p(T)$ is continuous.

<u>Proof</u>. Let $f \in L_0^p(\Omega)$ and let χ_Ω be the characteristic function of Ω. Then for $1 \le p < \infty$ and $1/p + 1/q = 1$, Hölder's inequality gives

$$\int_{\theta^\perp} |P_\mu f(\theta,x)|^p\, dx$$

$$= \int_{\theta^\perp} \left| \int_{-\infty}^{\infty} f(x+t\theta)e^{\mu(\theta)t}\, dt\right|^p dx$$

$$= \int_{\theta^\perp} \left| \int_{-\infty}^{\infty} \chi_\Omega(x+t\theta)f(x+t\theta)e^{\mu(\theta)t}\, dt\right|^p dx$$

$$\le \int_{\theta^\perp}\int_{-\infty}^{\infty} |f(x+t\theta)|^p\, dt \left(\int_{-\infty}^{\infty} |\chi_\Omega(x+t\theta)e^{\mu(\theta)t}|^q\, dt\right)^{p/q} dx$$

$$\le c_1\,||f||_p^p$$

where c_1 is a constant depending on Ω, μ and p. So

$$||P_\mu f(\theta,\cdot)||^p_{L^p(\theta^\perp)} \le c_1 ||f||^p_p$$

Integration over S^{n-1} gives

$$||P_\mu f||^p_p \le c ||f||^p_p$$

and the continuity follows. The case $p = \infty$ is similar but simpler and is omitted.

Let E_θ be the orthogonal projection in $\mathbb{R}^n$ on the subspace $\theta^\perp$. The operator P^*_μ is defined for functions g on T by

$$P^*_\mu g(x) = \int_{S^{n-1}} g(\theta,E_\theta x)e^{\mu(\theta)<x,\theta>}\, d\theta \tag{3.1}$$

<u>Proposition 3.4</u>.

$$\int_{S^{n-1}} \int_{\theta^\perp} P_\mu f(\theta,x)g(\theta,x)\, dxd\theta = \int_{\mathbb{R}^n} f(x)P^*_\mu g(x)\, dx.$$

The equality holds when f and g are non-negative or when either side of the equation is finite when f is replaced by $|f|$ and g by $|g|$.

<u>Proof</u>. In either case Fubini's theorem holds. Making the change of variable $y = x + t\theta$ with $x = E_\theta y$ and using (3.1) gives

$$\int_{S^{n-1}} \int_{\theta^\perp} \mathcal{P}_\mu f(\theta,x) g(\theta,x)\, dx d\theta$$

$$= \int_{S^{n-1}} \int_{\theta^\perp} \left(\int_{-\infty}^{\infty} f(x+t\theta) e^{\mu(\theta)t}\, dt \right) g(\theta,x)\, dx d\theta$$

$$= \int_{S^{n-1}} \int_{\mathbb{R}^n} f(y) g(\theta, E_\theta y) e^{\mu(\theta)\langle y,\theta\rangle}\, dy d\theta$$

$$= \int_{\mathbb{R}^n} f(y) \mathcal{P}_\mu^* g(y)\, dy.$$

<u>Corollary 3.5</u>. $\mathcal{P}_\mu^*$ is the formal adjoint of $\mathcal{P}_\mu$. In particular for $p > 1$ if $g \in L^p(T)$ then $\mathcal{P}_\mu^* g \in L^p_{loc}(\mathbb{R}^n)$.

The next result is the main motivation for all the work in this thesis.

<u>Theorem 3.6</u>.

$$(f * \mathcal{P}_{-\mu}^* K)(x) = \mathcal{P}_{-\mu}^* (\mathcal{P}_\mu f * K)(x). \tag{3.2}$$

The equality holds when f and K are non-negative or when either side of the equation is finite when f is replaced by $|f|$ and K by $|K|$.

<u>Proof</u>. The hypotheses allow the use of Fubini's theorem. With $y = y' + t\theta$ where $y' = E_\theta y$ and (3.1) we have

$$(f*P^*_{-\mu}K)(x) = \int_{\mathbb{R}^n} f(y)P^*_{-\mu}K(x-y)\,dy$$

$$= \int_{\mathbb{R}^n} f(y) \int_{S^{n-1}} K(\theta,E_\theta(x-y))e^{-\mu(\theta)<x-y,\theta>}\,d\theta dy$$

$$= \int_{S^{n-1}} \int_{\theta^\perp} K(\theta,E_\theta x-y')e^{-\mu(\theta)<x,\theta>} \int_{-\infty}^{\infty} f(y'+t\theta)e^{\mu(\theta)t}\,dtdy'd\theta$$

$$= \int_{S^{n-1}} \int_{\theta^\perp} K(\theta,E_\theta x-y')e^{-\mu(\theta)<x,\theta>}P_\mu f(\theta,y')\,dy'd\theta$$

$$= \int_{S^{n-1}} (P_\mu f*K)(\theta,E_\theta x)e^{-\mu(\theta)<x,\theta>}\,d\theta$$

$$= P^*_{-\mu}(P_\mu f*K)(x).$$

Relationship (3.2) will be used to show that the transforms can be inverted. In the case of constant attenuation we use it to get an exact inversion formula as described briefly in Chapter I.

The next theorem has been observed by Natterer [15] in two dimensions for the constantly attenuated transform. It can easily be extended to the exponential X-ray transform in higher dimensions.

<u>Theorem 3.7</u>. If $\theta \in S^{n-1}$, $\xi \in \theta^\perp$ then

$$(P_\mu f)\hat{}\,(\theta,\xi) = (2\pi)^{1/2}\hat{f}(\xi+i\mu(\theta)\theta)$$

where $(P_\mu f)\hat{}$ is the Fourier transform of $P_\mu f$ with respect to the second variable.

<u>Proof</u>. With $y = x + t\theta$, $\theta \in S^{n-1}$ and $x \in \theta^\perp$

$$(P_\mu f)^\wedge(\theta,\xi) = (2\pi)^{-(n-1)/2} \int_{\theta^\perp} P_\mu f(\theta,x) e^{-i\langle x,\xi\rangle}\, dx$$

$$= (2\pi)^{-(n-1)/2} \int_{\theta^\perp} \int_{-\infty}^{\infty} f(x+t\theta) e^{\mu(\theta)t}\, e^{-i\langle x,\xi\rangle}\, dt dx$$

$$= (2\pi)^{-(n-1)/2} \int_{\mathbb{R}^n} f(y) e^{-i\langle y,\xi+i\mu(\theta)\theta\rangle}\, dy$$

$$= (2\pi)^{1/2} \hat{f}(\xi+i\mu(\theta)\theta).$$

An immediate consequence of this theorem and (2.3) is the following.

<u>Corollary 3.8</u>. $P_\mu(f*g) = P_\mu f * P_\mu g$.

Theorem 3.7 can be used to extend the operator P_μ to distributions with compact support. Let u be such a distribution. Then $\hat{u}$ extends to an entire function on $\mathbb{C}^n$, and $P_\mu u$ is defined by setting

$$(P_\mu u)^\wedge(\theta,\xi) = \hat{u}(\xi+i\mu(\theta)\theta) \tag{3.3}$$

<u>Lemma 3.9</u>. Let V be an infinite subset of S^{n-1}. If g is holomorphic on $\mathbb{C}^n$ and $g(\xi+i\mu(\theta)\theta) = 0$ for $\theta \in V$ and $\xi \in \theta^\perp$ then $g = 0$ on $\mathbb{C}^n$.

The proof is almost identical to that given by Markoe in [14] for the case μ constant and is omitted.

<u>Theorem 3.10</u>. $P_\mu : L_0^1(\mathbb{R}^n) \to L^1(T)$ is one to one.

Proof. If $f \in L_0^1(\mathbb{R}^n)$ then $\hat{f}$ extends to a holomorphic function on $\mathbb{C}^n$. By the previous lemma, Theorem 3.8 and the uniqueness of the Fourier transform we conclude that $P_\mu : L_0^1(\mathbb{R}^n) \to L^1(T)$ is one to one.

Similarly since the Fourier transform of a distribution with compact support extends to a holomorphic function on $\mathbb{C}^n$ we have

Corollary 3.11. P_μ is one to one on the space of distributions with compact support.

In [14], Markoe has proved Theorem 3.10 for the constantly attenuated X-ray transform in n dimensions. In [7], Finch and Hertle used similar techniques to prove a uniqueness result for the variably attenuated exponential Radon transform. (Apart from notation, the exponential Radon transform and the exponential X-ray transform are the same in two dimensions.)

IV. AN EXISTENCE RESULT FOR THE EXPONENTIAL X-RAY TRANSFORM IN TWO DIMENSIONS

Let E be an approximate δ-function. With relationship (3.2) in mind, we would like to find a solution K for

$$P^*_{-\mu} K = E. \tag{4.1}$$

While we have not been able to find an explicit solution for K (unless μ is constant), we have been able to establish the existence of and regularity results for solutions of (4.1) in two dimensions.

Throughout this chapter we let

$$Z = S^1 \times \mathbb{R},$$

$$B = \{x \in \mathbb{R}^2 : |x| < 2r,\ r > 0\},$$

$$\Omega = \{x \in \mathbb{R}^2 : |x| < r,\ r > 0\}.$$

Before proceeding a short comment on notation is needed. For $\phi \in [0,2\pi)$, let $\theta = (\cos\phi, \sin\phi) \in S^1$. The subspace $\theta^\perp$ of $\mathbb{R}^2$ is spanned by the vector $(-\sin\phi, \cos\phi)$ which we will also denote by $\theta^\perp$. If $x = s\theta$, $\theta \in S^1$, $s \in \mathbb{R}$, we let $x = s\theta^\perp$. In such a case we will write $P_\mu f(\theta,s)$ for $P_\mu f(\theta,x^\perp)$.

Most of the work in this chapter is devoted to proving the following theorem.

<u>Theorem 4.1</u>. Let $f \in H^s_0(\Omega)$, $\mu \in C^\infty(S^1)$. Then for each $s \in \mathbb{R}$ there exists constants $c_s, C_s > 0$ depending only on s such that

$$c_s \|f\|_{H^s} \leq \|P_\mu f\|_{H^{s+1/2}(Z)} \leq C_s \|f\|_{H^s} .$$

A consequence of this theorem is the following result which establishes the existence and regularity of solutions to (4.1).

Theorem 4.2. For all $s \in \mathbb{R}$, the map $P_\mu^* : H^s(Z) \to H^{s+1/2}(\Omega)$ is onto.

Proof. For $s \in \mathbb{R}$, $P_\mu : H_0^{-s-1/2}(\Omega) \to H^{-s}(Z)$ by Theorem 4.1. Then by Remarks 2.3 and 2.4, $P_\mu^* : H^s(Z) \to H^{s+1/2}(\Omega)$. From Theorem 4.1 we conclude that P_μ and P_μ^{-1} are continuous. By the closed graph theorem P_μ has closed range. Since P_μ is one to one P_μ^* has dense range in $H^{s+1/2}(\Omega)$. And since P_μ has closed range so does P_μ^*; see [18] or [19]. Thus P_μ^* is onto.

Having established the existence of convolution kernels K for a large class of point spread functions E we get the following inversion result for P_μ.

Corollary 4.3. For $s \geq 1/2$ let $E \in H_0^s(\Omega)$ with $E(x) \geq 0$ and $\int E(x)dx = 1$. For $\rho > 0$ let $E_\rho(x) = \rho^{-n}E(x/\rho)$. Let $M = \max|\mu(\theta)|$ and $f(\cdot)e^{M|\cdot|} \in L^1(\mathbb{R}^n)$. If $K_\rho \in H^{s-1/2}(Z)$ is such that $P_{-\mu}^* K_\rho = E_\rho$ then for almost every x

$$\lim_{\rho\to 0} P_{-\mu}^*(P_\mu f * K_\rho)(x) = f(x)$$

Proof. By Theorem 3.6,

$$P_{-\mu}^*(P_\mu f * K_\rho)(x) = E_\rho * f(x),$$

and $E_\rho * f(x) \to f(x)$ for almost ever x by Theorem 2, page 62 of Stein [22].

To prove Theorem 4.1 we introduce next a slightly modified operator which preserves data from $\mathcal{P}_\mu$ on certain Sobolev spaces, but allows us to use results on pseudo-differential operators.

<u>Definition 4.4</u>. The transform $\mathcal{P}_{\mu,\gamma}$ of a function f on $\mathbb{R}^2$ is defined by

$$\mathcal{P}_{\mu,\gamma} f(\theta,s) = \int_{-\infty}^{\infty} f(t\theta+s\theta^\perp)M(t\theta+s\theta^\perp,\theta)\, dt$$

for $\theta \in S^1$ and $s \in \mathbb{R}$ and where for $x \in \mathbb{R}^2$

$$M(x,\theta) = e^{\mu(\theta)\langle x,\theta\rangle}\, \gamma(x) \tag{4.2}$$

with $\mu \in C(S^1)$, $\gamma \in C_0^\infty(B)$ and $\gamma = 1$ in a neighborhood of $\overline{\Omega}$.

Clearly $\mathcal{P}_\mu$ and $\mathcal{P}_{\mu,\gamma}$ agree for functions with support in $\overline{\Omega}$. Our aim now is to show that inequalities similar to those in Theorem 4.1 are satisfied by $\mathcal{P}_{\mu,\gamma}$ and consequently by $\mathcal{P}_\mu$ for $f \in H_0^s(\Omega)$. The method used follows closely that of Heike [10] for the attenuated X-ray transform.

As in [10], we start by defining two operators A_+ and A_- for functions f on $\mathbb{R}^2$ by

$$(A_+ f)\hat{\ }(\xi^\perp) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\langle x,\xi^\perp\rangle} M(x,\xi/|\xi|)f(x)\, dx \tag{4.3}$$

$$(A_-f)\hat{}(\xi^\perp) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i<x,\xi^\perp>} M(x,-\xi/|\xi|)f(x)\,dx \tag{4.4}$$

where M is as in (4.2). Many properties for $\mathcal{P}_{\mu,\gamma}$ will be deduced from those for A_+ and A_-. The next theorem gives a Fourier transform relationship between $\mathcal{P}_{\mu,\gamma}$ and A_+, A_-.

Theorem 4.5.

$$(A_+f)\hat{}(\sigma\theta^\perp) = (2\pi)^{-1/2}(\mathcal{P}_{\mu,\gamma}f)\hat{}(\theta,\sigma) \quad \text{if} \quad \sigma \geq 0$$

$$(A_-f)\hat{}(\sigma\theta^\perp) = (2\pi)^{-1/2}(\mathcal{P}_{\mu,\gamma}f)\hat{}(-\theta,-\sigma) \quad \text{if} \quad \sigma \geq 0.$$

where $(\mathcal{P}_{\mu,\gamma}f)\hat{}$ is the Fourier transform of $\mathcal{P}_{\mu,\gamma}f$ with respect to the second variable.

Proof. We prove the first equality. The proof of the second is similar. Let $\xi^\perp = \sigma\theta^\perp$ with $\sigma \geq 0$ and $\theta \in S^1$. Setting $x = t\theta + s\theta^\perp$ and using (4.3) we get

$$\begin{aligned}
(A_+f)\hat{}(\sigma\theta^\perp) &= (2\pi)^{-1}\int_{\mathbb{R}^2} e^{-i\sigma<x,\theta^\perp>} M(x,\theta)f(x)\,dx \\
&= (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-is\sigma}\int_{-\infty}^{\infty} M(t\theta+s\theta^\perp,\theta)f(t\theta+s\theta^\perp)\,dtds \\
&= (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-is\sigma}\mathcal{P}_{\mu,\gamma}f(\theta,s)\,ds \\
&= (2\pi)^{-1/2}(\mathcal{P}_{\mu,\gamma}f)\hat{}(\theta,\sigma).
\end{aligned}$$

<u>Lemma 4.6</u>. For all $s \in \mathbb{R}$, define $S : \mathbb{R}^2\backslash\{0\} \to H^s(\mathbb{R}^2)$ by

$$S(\xi)(x) = e^{-i\langle x,\xi^{\perp}\rangle} M(x, \xi/|\xi|)$$

where M is as in (4.2). Then S is continuous and is bounded on bounded subsets of $\mathbb{R}^2\backslash\{0\}$.

<u>Proof</u>. Let $s = m$ be a non-negative integer. Then by (2.5), if $\xi \in \mathbb{R}^2\backslash\{0\}$

$$||S(\xi)||_{H^m} \leq c_m \sum_{|\alpha|\leq m} ||\partial_x^\alpha S(\xi)(\cdot)||_2^2 .$$

From (4.2) we have that for each α, the function $G_\alpha(\xi,x) = \partial_x^\alpha S(\xi)(x)$ is continuous in x and ξ and is bounded on bounded subsets of $\mathbb{R}^2\backslash\{0\}$. Moreover for each ξ, G has support in B. From the above and the dominated convergence theorem the continuity and boundedness of S follow. For any $s \in \mathbb{R}$ if m is a non-negative integer such that $s < m$ the inclusion map $H^m(\mathbb{R}^2) \to H^s(\mathbb{R}^2)$ is continuous. Hence for $s \in \mathbb{R}$, $S : \mathbb{R}^2\backslash\{0\} \to H^s(\mathbb{R}^2)$ is continuous as it is the composition of two continuous functions. The boundedness also follows.

<u>Lemma 4.7</u>. If $f \in C_0^\infty(\mathbb{R}^2)$ then for each $s \in \mathbb{R}$ there exists constants $c_s, C_s > 0$ such that

$$\begin{aligned} c_s ||A_\pm f||^2_{H^{s-1/2}} &\leq ||P_{\mu,\gamma} f||^2_{H^s(Z)} \\ &\leq C_s(||A_+ f||^2_{H^{s-1/2}} + ||A_- f||^2_{H^{s-1/2}} + ||f||^2_{H^{s-1/2}}) \end{aligned} \tag{4.5}$$

<u>Proof</u>. Let $f \in C_0^\infty(\mathbb{R}^n)$ and $\theta = (\cos\phi, \sin\phi)$. Then Formula (2.7), the polar coordinates formula in $\mathbb{R}^2$ and Theorem 4.5 give

$$\begin{aligned}
\|P_{\mu,\gamma}f\|^2_{H^s(Z)} &= \int_0^{2\pi}\int_{-\infty}^{\infty} (1+\sigma^2)^s |(P_{\mu,\gamma}f)\hat{}(\theta,\sigma)|^2 \, d\sigma d\phi \\
&\geq \int_0^{2\pi}\int_0^{\infty} (1+\sigma^2)^s |(P_{\mu,\gamma}f)\hat{}(\theta,\sigma)|^2 \, d\sigma d\phi \\
&= \int_{\mathbb{R}^2} \frac{(1+|\xi|^2)^s}{|\xi|} |(P_{\mu,\gamma}f)\hat{}(\xi/|\xi|,|\xi|)|^2 \, d\xi \\
&\geq \int_{\mathbb{R}^2} (1+|\xi|^2)^{s-1/2} |(P_{\mu,\gamma}f)\hat{}(\xi/|\xi|,|\xi|)|^2 \, d\xi \\
&= 2\pi \int_{\mathbb{R}^2} (1+|\xi|^2)^{s-1/2} |(A_+f)\hat{}(\xi)|^2 \, d\xi \\
&= 2\pi \|A_+f\|^2_{H^{s-1/2}} .
\end{aligned}$$

Similarly

$$\|P_{\mu,\gamma}f\|_{H^s(Z)} \geq 2\pi \|A_-f\|^2_{H^{s-1/2}}$$

which gives the first inequality.

Now using Theorem 4.5 we get

$$\begin{aligned}
\|P_{\mu,\gamma}f\|^2_{H^s(Z)} &= \int_{S^1}\int_{-\infty}^{\infty} (1+\sigma^2)^s |(P_{\mu,\gamma}f)\hat{}(\theta,\sigma)|^2 \, d\sigma d\theta \\
&= \int_0^{2\pi}\int_0^{\infty} (1+\sigma^2)^s |(P_{\mu,\gamma}f)\hat{}(\theta,\sigma)|^2 \, d\sigma d\phi \\
&\quad + \int_0^{2\pi}\int_{-\infty}^{0} (1+\sigma^2)^s |(P_{\mu,\gamma}f)\hat{}(\theta,\sigma)|^2 \, d\sigma d\phi \quad =
\end{aligned}$$

$$= \int_0^{2\pi} \int_0^{\infty} (1+\sigma^2)^s \, |(P_{\mu,\gamma} f)\hat{}(\theta,\sigma)|^2 \, d\sigma d\phi$$

$$+ \int_0^{2\pi} \int_0^{\infty} (1+\sigma^2)^s |(P_{\mu,\gamma} f)\hat{}(\theta,-\sigma)|^2 \, d\sigma d\phi$$

$$= 2\pi \int_{\mathbb{R}^2} \frac{(1+|\xi|^2)^s}{|\xi|} \, |(A_+ f)\hat{}(\xi)| \, d\xi$$

$$+ 2\pi \int_{\mathbb{R}^2} \frac{(1+|\xi|^2)^s}{|\xi|} \, |(A_- f)\hat{}(\xi)| \, d\xi. \qquad (4.6)$$

Now

$$\int_{\mathbb{R}^2} \frac{(1+|\xi|^2)^s}{|\xi|} \, |(A_+ f)\hat{}(\xi)|^2 \, d\xi = \int_{|\xi|\le 1} \frac{(1+|\xi|^2)^s}{|\xi|} \, |(A_+ f)\hat{}(\xi)|^2 \, d\xi$$

$$+ \int_{|\xi|>1} \frac{(1+|\xi|^2)^s}{|\xi|} \, |(A_+ f)\hat{}(\xi)|^2 \, d\xi \qquad (4.7)$$

and

$$\int_{|\xi|>1} \frac{(1+|\xi|^2)^s}{|\xi|} |(A_+ f)\hat{}(\xi)|^2 \, d\xi$$

$$= \int_{|\xi|>1} |(A_+ f)\hat{}(\xi)|^2 (1+|\xi|^2)^{s-1/2} \frac{(1+|\xi|^2)^{1/2}}{|\xi|} \, d\xi$$

$$\le c \int_{|\xi|>1} |(A_+ f)\hat{}(\xi)|^2 (1+|\xi|^2)^{s-1/2} \, d\xi$$

$$\le c \, ||A_+ f||^2_{H^{s-1/2}}$$

where c is a constant. Throughout this proof, c_s will be used to denote a constant depending only on s. However it may take different values in different places. Next we estimate the first integral on the

right hand side of (4.7). Using (2.6) with M as in (4.2) gives

$$|(A_+f)\hat{}(\xi^\perp)|^2 = |\int_{\mathbb{R}^2} e^{-i\langle x,\xi^\perp\rangle} M(x, \xi/|\xi|)f(x)dx|^2$$

$$\leq \|f\|_{H^{s-1/2}} \|e^{-i\langle\cdot,\xi^\perp\rangle} M(\cdot, \xi/|\xi|)\|^2_{H^{1/2-s}} .$$

For $|\xi| \leq 1$, $\|e^{-i\langle\cdot,\xi^\perp\rangle} M(\cdot, \xi/|\xi|)\|^2_{H^{1/2-s}} \leq c_s$ by Lemma 4.6, and so $|(A_+f)\hat{}(\xi^\perp)|^2 \leq c_s \|f\|_{H^{s-1/2}}$. Hence

$$\int_{|\xi|\leq 1} \frac{(1+|\xi|^2)^s}{|\xi|} |(A_+f)\hat{}(\xi)|^2 \, d\xi$$

$$\leq \sup_{|\xi|\leq 1} |(A_+f)\hat{}(\xi)|^2 \int_{|\xi|\leq 1} \frac{(1+|\xi|^2)^s}{|\xi|} d\xi$$

$$\leq c_s \|f\|^2_{H^{s-1/2}} .$$

We have shown that the first integral in (4.6)

$$\int_{\mathbb{R}^2} \frac{(1+|\xi|^2)^s}{|\xi|} |(A_+f)\hat{}(\xi)| d\xi \leq c_s(\|A_+f\|^2_{H^{s-1/2}} + \|f\|^2_{H^{s-1/2}})$$

Similarly we have an estimate for the second integral

$$\int_{\mathbb{R}^2} \frac{(1+|\xi|^2)^s}{|\xi|} |(A_-f)\hat{}(\xi)| d\xi \leq c_s(\|A_-f\|^2_{H^{s-1/2}} + \|f\|^2_{H^{s-1/2}}) .$$

Thus establishing the second inequality.

For $f \in C_0^\infty(\Omega)$ the inequalities in (4.5) continue to hold when $P_{\mu,\gamma}$ is replaced by P_μ. The inequalities in (4.5) resemble those of Theorem 4.1. In fact if for $s \in \mathbb{R}$, A_+ and A_- extend to continuous operators on $H_0^s(\Omega)$, the second inequality in Theorem 4.1 follows immediately. While if A_+ or A_- has a continuous inverse on $H_0^s(\Omega)$, $s \in \mathbb{R}$, the first inequality follows.

In the remaining part of this chapter we show that A_+ and A_- are continuous by showing they are pseudo-differential operators of order 0 on $C_0^\infty(\Omega)$, and that A_+^{-1} and A_-^{-1} are continuous by showing that A_+ and A_- are semi-Fredholm operators.

Before proceeding some results on pseudo-differential operators and semi-Fredholm operators will be presented mostly without proof. The material on pseudo-differential operators is taken from Kohn and Nirenberg [11]. The material on semi-Fredholm operators is from Schechter [19].

<u>Definition 4.8</u>. A function $a : \mathbb{R}^2 \times \mathbb{R}^2\backslash\{0\} \to \mathbb{R}$ is called a symbol if

1. $a \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2\backslash\{0\})$;
2. a is positive homogeneous of degree 0 in the second variable;
3. for any integer p and any α, β positive integers

$$(1+|x|^p)\partial_x^\alpha \partial_\xi^\beta a(x,\xi) \to 0$$

as $|x| \to \infty$ uniformly in ξ for $|\xi| = 1$.

To a is assigned an operator A called a pseudo-differential operator defined by

$$(Af)^\wedge(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\langle x,\xi\rangle}\, a(x,\xi)f(x)\, dx \tag{4.8}$$

and a is said to be the symbol of A.

<u>Definition 4.9</u>. A linear operator $T : \mathcal{S} \to \mathcal{S}$ is said to have order r if for all $s \in \mathbb{R}$ there exists a constant $c_s > 0$ such that

$$\|Tu\|_{H^s} \le c_s \|u\|_{H^{s+r}} .$$

The infimum of all orders r of T is called the true order of T.

<u>Remark 4.10</u>. The composition of two operators has order equal to the sum of their orders.

The following is a statement of Theorem 1 in [11].

<u>Theorem 4.11</u>. The pseudo-differential operator A, defined by (4.8), has order 0.

Let $\zeta \in C^\infty(\mathbb{R}^2)$ with $0 \le \zeta(\xi) \le 1$, $\zeta(\xi) = 1$ if $|\xi| > 1$ and vanishes for $|\xi| < 1/2$. Let C be the operator defined on by

$$(Cf)^\wedge(\xi) = \zeta(\xi)\hat{f}(\xi)$$

then C has order 0 and $I - C$ has true order $-\infty$. Here I

denotes the identity operator. In what follows let $a(x,\xi)$ by a symbol and A its corresponding pseudo-differential operator as in Definition 4.8. Let A' be the operator defined on $\mathcal{S}$ by $(A'f)^\wedge(\xi) = \zeta(\xi)(Af)^\wedge(\xi)$.

Remark 4.12. $A - A'$ has true order $-\infty$.

Proof. From the definition of A' we see that $A' = C \circ A$ and $A - A' = (I-C) \circ A$. Since $I - C$ has true order $-\infty$ and A has order 0, $A - A'$ has true order $-\infty$ by Remark 4.10.

Let $b(x,\xi)$ be a symbol and B the corresponding pseudo-differential operator. Let B' be the operator defined on $\mathcal{S}$ by $(B'f)^\wedge(\xi) = \zeta(\xi)(Bf)^\wedge(\xi)$. Let P be the pseudo-differential operator with symbol $a(x,\xi)b(x,\xi)$ and P' the operator defined by $(P'f)^\wedge(\xi) = \zeta(\xi)(Pf)^\wedge(\xi)$. The first lemma below is a special case of Lemma 5.1 of [11], the second follows from the first and Remarks 4.10 and 4.12.

Lemma 4.13. $A' \circ B' - P'$ has order -1.

Lemma 4.14. $A \circ B - P$ has order -1.

Next some results on semi-Fredholm operators are presented.

Definition 4.15. Let X, Y be Banach spaces. The continuous linear operator $A : X \to Y$ is called a semi-Fredholm operator is

a) the range of A is closed in Y, and

b) the dimension of the nullspace of A is finite.

Definition 4.16. Let X be a Banach space with norm $\| \ \|_X$ a seminorm $|\cdot|$ is said to be compact relative to $\| \ \|_X$ if whenever

$\{x_n\}$ is a sequence of elements of X such that $\|x_n\|_X \leq c$, a constant, then it has a subsequence which is Cauchy in $|\cdot|$.

The next result is Theorem 6.2, p. 127 of Schechter [19].

Theorem 4.17. Let X and Y be Banach spaces with norms $\| \ \|_X$ and $\| \ \|_Y$ respectively. Then $A : X \to Y$ is a semi-Fredholm operator if and only if there is a seminorm $|\cdot|$ compact relative to $\| \ \|_X$ such that for all $x \in X$

$$\|x\|_X \leq c\,\|Ax\|_Y + |x|.$$

The following is a corollary to Definition 4.8 and Theorem 4.11.

Corollary 4.18. Let $\mu \in C^\infty(S^1)$. If $V_+, V_- : \mathbb{R}^2 \times \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ are defined by

$$V_\pm(x,\xi^\perp) = M(x, \pm\xi/|\xi|) \tag{4.9}$$

where M is as in (4.2), then V_+ and V_- are symbols with A_+ and A_- in (4.3) and (4.4) the corresponding pseudo-differential operators. Moreover A_+ and A_- have order 0.

Corollary 4.19. For all $s \in \mathbb{R}$, $A_\pm$ extend to continuous maps $A_\pm : H^s(\mathbb{R}^2) \to H^s(\mathbb{R}^2)$. Hence $A_\pm : H^s_0(\Omega) \to H^s(\mathbb{R}^2)$ are continuous.

Proof. This is an immediate consequence of Definition 4.9 and the fact that $\mathcal{S}$ is dense in $H^s(\mathbb{R}^n)$ for all s.

By the above corollary the second inequality of Theorem 4.1 is established.

Since $C_0^\infty(\Omega)$ is dense in $H_0^s(\Omega)$, the previous corollary and Lemma 4.7 imply that for each $s \in \mathbb{R}$, $P_{\mu,\gamma}$ extends to a continuous map from $H_0^s(\Omega)$ to $H^{s+1/2}(Z)$. Since P_μ and $P_{\mu,\gamma}$ agree on $C_0^\infty(\Omega)$, $P_\mu = P_{\mu,\gamma}$ on $H_0^s(\Omega)$. Consequently we have

<u>Corollary 4.20</u>. For all $s \in \mathbb{R}$, the maps

$$A_\pm : H_0^s(\Omega) \to H^s(\mathbb{R}^n)$$

are one to one.

<u>Proof</u>. Since $P_\mu = P_{\mu,\gamma}$ on $H_0^s(\Omega)$, by Corollary 3.11 $P_{\mu,\gamma}$ cannot vanish on an infinite subset of S^1. Hence by Theorem 4.5, $A_\pm$ are one to one.

We will next show that A_+ and A_- have closed ranges by showing that they are semi-Fredholm operators. This together with the above implies that A_+^{-1} and A_-^{-1} are continuous, thus establishing the first inequality of Theorem 4.1.

We define the functions $W_+, W_- : \mathbb{R}^2 \times \mathbb{R}^2\backslash\{0\} \to \mathbb{R}$ by

$$W_\pm(x,\xi^\perp) = e^{\mu(\pm\xi/|\xi|)<x,\mp\xi/|\xi|>}\,\gamma(x) \tag{4.10}$$

Clearly W_+, W_- are symbols. Let B_+ and B_- be the corresponding pseudo-differential operators respectively. Then B_+ and B_- have order 0. Note that for $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^2\backslash\{0\}$

$$W_\pm(x,\xi) = V_\pm^{-1}(x,\xi) \tag{4.11}$$

with V_+ and V_- as in (4.9).

Lemma 4.21. For $f \in \mathcal{S}$,

$$(B_\pm \circ A_\pm)f = P_\pm f + K_\pm f$$

where P_+, P_- = identity map on $C_0^\infty(\Omega)$ and $K_\pm$ have order -1.

Proof. Only one case will be shown. The proof of the second is similar. Let $p : \mathbb{R}^2 \times \mathbb{R}^2 \backslash \{0\}$ be defined by

$$p(x,\xi^\perp) = W_+(x,\xi^\perp)V_+(x,\xi^\perp) = \gamma^2(x)$$

then p is a symbol. Let P_+ be the corresponding pseudo-differential operator. With $f \in C_0^\infty(\Omega)$, Formulas (4.11) and(4.8) give

$$(P_+f)\hat{}(\xi) = \frac{1}{2\pi}\int_\Omega e^{-i<x,\xi>} f(x)\, dx = \hat{f}(\xi).$$

So $P_+f = f$. Let $K_+ = B_+ \circ A_+ - P_+$. By Lemma 4.14 K_+ has order -1.

Theorem 4.22. $A_\pm : H_0^s(\Omega) \to H^s(\mathbb{R}^n)$ are semi-Fredholm operators.

Proof. We prove it for A_+. The proof for A_- is similar. Let $f \in C_0^\infty(\Omega)$ then by Lemma 4.21

$$f = (B_+ \circ A_+)f - K_+f.$$

B_+ has order 0, so

$$||f||_{H^s} \leq C_{B_+} ||A_+f||_{H^s} + ||K_+f||_{H^s} . \tag{4.12}$$

Since K_+ has order -1, it extends to a continuous map from $H_0^s(\Omega)$ to $H^s(\mathbb{R}^n)$. This and the continuity of A_+ on $H_0^s(\Omega)$ implies that (4.12) holds for all $f \in H_0^s(\Omega)$.

For $f \in H_0^s(\Omega)$ define

$$|f| = ||K_+ f||_{H^s} .$$

Then $|\cdot|$ is a seminorm. Let $\{f_n\}$ be a sequence in $H_0^s(\Omega)$ such that $||f_n||_{H^s} \leq c$. We want to show that $\{f_n\}$ has a subsequence Cauchy in $|\cdot|$. By Remark 2.2 the inclusion map $H_0^s(\Omega) \to H^{s-1}(\mathbb{R}^2)$ is compact, hence there is a subsequence $\{f_{n_i}\} \subseteq \{f_n\}$ that is Cauchy in $|| \ ||_{H^{s-1}}$. Since K_+ is of order -1

$$|f_{n_i} - f_{n_j}| = ||K_+ f_{n_i} - K_+ f_{n_j}||_{H^s} \leq c_{K_+} ||f_{n_i} - f_{n_j}||_{H^{s-1}} .$$

So $\{f_{n_i}\}$ is Cauchy in $|\cdot|$. By Definition 4.16, $|\cdot|$ is compact with respect to $|| \ ||_{H^s}$. Hence by Theorem 4.17 A_+ is a semi-Fredholm operator.

We have thus completed the proof of Theorem 4.1.

V. POINT SPREAD FUNCTIONS AND CONVOLUTION KERNELS FOR THE EXPONENTIAL X-RAY TRANSFORM WITH CONSTANT ATTENUATION

To the end of this thesis we will be dealing with the exponential X-ray transform with constant attenuation μ. In this chapter an explicit solution for

$$\mathcal{P}^*_{-\mu} K = E \tag{5.1}$$

will be given. Recall that E is a point spread function and K the corresponding convolution kernel. Some of the results developed in this chapter appeared in Hazou and Solmon [9].

Let $\rho > 0$ and set $E_\rho(x) = \rho^{-n} E(x/\rho)$. Let K_ρ be the convolution kernel corresponding to E_ρ. Suppose that there is an even function k of one variable so that $K(\theta,x) = k(|x|,\mu)$, for $\theta \in S^{n-1}$, $x \in \theta^\perp$. (In solving $\mathcal{P}^*_{-\mu} K = E$, K will depend on μ. However, this dependence will be suppressed in the notation for K but emphasized in that of k.) For such a kernel K, $E = \mathcal{P}^*_{-\mu} K$ is also radial. For let U be an orthogonal transformation on $\mathbb{R}^n$ and let U^t be its transpose, then

$$|E_\theta(Ux)| = |E_{U^t\theta}\, x|$$

and a change of variable $\omega = U^t\theta$ gives,

$$E(Ux) = P^*_{-\mu}K(Ux)$$

$$= \int_{S^{n-1}} k(|E_\theta(Ux)|,\mu)e^{-\mu<Ux,\theta>}\, d\theta$$

$$= \int_{S^{n-1}} k(|E_\omega x|,\mu)e^{-\mu<x,\omega>}\, d\omega$$

$$= E(x).$$

With the above as motivation, we choose E to be a radial function and look for solutions K of (5.1) of the form $K(\theta,x) = k(|x|,\mu)$. To this end, let e be an even function of one variable and let $E(x) = e(|x|)$. Setting $x = r\phi$ for $r \geq 0$, $\phi \in S^{n-1}$ (5.1) becomes

$$e(r) = \int_{S^{n-1}} k(|E_\theta(r\phi)|,\mu)e^{-\mu r<\phi,\theta>}\, d\theta. \tag{5.2}$$

Throughout this section and to the end of the thesis we let the constant $\gamma_{n-2} = 2\pi^{(n-1)/2}/\Gamma((n-1)/2)$, the $(n-2)$-dimensional surface area measure of S^{n-2}.

<u>Theorem 5.1</u>. Let $n \geq 2$. If $e \in C^1(\mathbb{R}\backslash\{\pm a\})$, $a > 0$, and if its first derivative e' have right and left hand limits at $\pm a$, then (5.2) has a unique solution $k \in C(\mathbb{R}\backslash\{\pm a\})$ given by

$$k(s,\mu) = \frac{s^{2-n}}{\pi\gamma_{n-2}} \frac{d}{ds} \int_0^s t^{n-1}e(t) \frac{\cos(\mu\sqrt{s^2-t^2})}{\sqrt{s^2-t^2}}\, dt. \tag{5.3}$$

<u>Proof</u>. Since $E_\theta(r\phi) = r\phi - \langle r\phi,\theta\rangle\theta$, (5.2) becomes

$$e(r) = \int_{S^{n-1}} k(r|\phi-\langle\phi,\theta\rangle\theta|,\mu)e^{-\mu r\langle\phi,\theta\rangle}\, d\theta$$

$$= \int_{S^{n-1}} k(r\sqrt{1-\langle\phi,\theta\rangle^2},\mu)e^{-\mu r\langle\phi,\theta\rangle}\, d\theta.$$

Letting $t = \langle\phi,\theta\rangle$ and then $s = r^2(1-t^2)$ gives

$$e(r) = \gamma_{n-2}\int_{-1}^{1} k(r\sqrt{1-t^2},\mu)e^{-\mu rt}(1-t^2)^{(n-3)/2}\, dt$$

$$= 2\gamma_{n-2}\int_{0}^{1} k(r\sqrt{1-t^2},\mu)\cosh(\mu rt)(1-t^2)^{(n-3)/2}\, dt$$

$$= \gamma_{n-2}r^{2-n}\int_{0}^{r^2} (\sqrt{s})^{n-3}k(\sqrt{s},\mu)\,\frac{\cosh(\mu\sqrt{r^2-s})}{\sqrt{r^2-s}}\, ds.$$

Now let

$$k_1(s,\mu) = (\sqrt{s})^{n-3}k(\sqrt{s},\mu)$$

and

$$e_1(r) = (\sqrt{r})^{n-2}e(\sqrt{r})$$

to obtain

$$e_1(r) = \gamma_{n-2}\int_{0}^{r} k_1(s,\mu)\,\frac{\cosh(\mu\sqrt{r-s})}{\sqrt{r-s}}\, ds. \tag{5.4}$$

This is a generalized Abel integral equation. Assume for now that e is bounded by a function of exponential growth. Then (5.4) can be solved using Laplace transforms. Let $\mathcal{L}$ denote the Laplace transform.

We get

$$L(e_1)(t) = \gamma_{n-2} L(k_1)(t,\mu) L\left(\frac{\cosh \mu\sqrt{\cdot}}{\sqrt{\cdot}}\right)(t).$$

Using formula 29.3.77 in [1] we have

$$L(k_1)(t,\mu) = \frac{\sqrt{t}}{\sqrt{\pi}\ \gamma_{n-2}} e^{-\mu^2/4t} L(e_1)(t)$$

Let

$$e_1(a^2+) = \lim_{\substack{r\to a^2 \\ r>a^2}} e_1(r) \quad \text{and} \quad e_1(a^2-) = \lim_{\substack{r\to a^2 \\ r<a^2}} e_1(r).$$

Since e_1 has a jump discontinuity at a^2,

$$L(e_1')(t) = tL(e_1)(t) - e_1(0) - (e_1(a^2+) - e_1(a^2-))e^{-a^2 t}$$

where e_1' is the derivative of e_1 on $[0,\infty)\backslash\{a^2\}$. Taking inverse Laplace transforms and using formula 29.3.76 of [1] we get

$$k_1(s,\mu) = \frac{1}{\pi\gamma_{n-2}} \left[\int_0^s e_1'(r) \frac{\cos(\mu\sqrt{s-r})}{\sqrt{s-r}} dr + \frac{e_1(0)\cos \mu\sqrt{s}}{\sqrt{s}} + h(s,\mu) \right]$$

where

$$h(s,\mu) = \begin{cases} 0 & \text{if} \quad s < a^2 \\ (e_1(a^2+) - e_1(a^2-)) \dfrac{\cos(\mu\sqrt{s-a^2})}{\sqrt{s-a^2}} & \text{if} \quad s > a^2 \end{cases}$$

Substituting back for k_1 and e_1 and then setting $t = \sqrt{r}$ gives

$$k(s,\mu) = \frac{s^{3-n}}{\pi\gamma_{n-2}}\left[\int_0^s ((n-2)t^{n-3}e(t) + t^{n-2}e'(t))\,\frac{\cos(\mu\sqrt{s^2-t^2})}{\sqrt{s^2-t^2}}\,dt\right.$$

$$\left. + e_1(0)\,\frac{\cos\mu s}{s} + h(s^2,\mu)\right] \tag{5.5}$$

Now if $s < a$

$$k(s,\mu) = \frac{s^{2-n}}{\mu\pi\gamma_{n-2}}\left[\int_0^s \frac{d}{dt}\,(t^{n-2}e(t))\,\frac{\mu s\cos(\mu\sqrt{s^2-t^2})}{\sqrt{s^2-t^2}}\,dt + \mu e_1(0)\cos\mu s\right]$$

$$= \frac{s^{2-n}}{\mu\pi\gamma_{n-2}}\left[\frac{d}{ds}\int_0^s \frac{d}{dt}\,(t^{n-2}e(t))\sin(\mu\sqrt{s^2-t^2})\,dt + \mu e_1(0)\cos\mu s\right]$$

Integration by parts gives (5.3) for $s < a$.

For $s > a$ we write the integral in (5.5) as

$$\frac{1}{\mu s}\frac{d}{ds}\left[\int_0^a \frac{d}{dt}\,((t^{n-2}e(t))\sin(\mu\sqrt{s^2-t^2})\,dt\right.$$

$$\left. + \int_a^s \frac{d}{dt}\,(t^{n-2}e(t))\sin(\mu\sqrt{s^2-t^2})\,dt\right]$$

Integration by parts and the fact that

$$g(s) = \int_0^s t^{n-1}e(t)\,\frac{\cos(\mu\sqrt{s^2-t^2})}{\sqrt{s^2-t^2}}\,dt$$

is continuous on $[0,\infty)$ and belongs to $C^1([0,\infty)\backslash\{a\})$ give us (5.3) for $s > a$. Thus (5.3) provides a solution to (5.2) when e is bounded by a function of exponential growth. For the general case, one can check a posteriori that (5.3) provides a continuous solution and the uniqueness can be deduced from results on generalized Abel

integral equations, §41 of Yosida [26].

As a consequence of the proof one can easily see that the theorem continues to hold if e is piecewise continuously differentiable.

With E and K as before, the following proposition gives a relationship between their Fourier transforms. The Fourier transform of $K(\theta,x)$ is taken with respect to the second variable, i.e., it is the (n-1)-dimensional Fourier transform on $\theta^{\perp}$.

<u>Proposition 5.2</u>. If in addition E is integrable on $\mathbb{R}^n$ then

$$\hat{K}(\theta,\xi) = \begin{cases} \dfrac{|\xi|^{3-n}(\sqrt{|\xi|^2-\mu^2})^{n-2}}{\sqrt{2\pi}\,\gamma_{n-2}}\,\hat{E}(\sqrt{|\xi|^2-\mu^2}\,\xi/|\xi|) & \text{when } 0 < |\mu| < |\xi| \\ 0 & \text{when } |\mu| > |\xi| \end{cases} \tag{5.6}$$

<u>Proof</u>. Since K is radial, (2.4) and (5.3) give

$$\hat{K}(\theta,\xi) = \frac{1}{\pi\gamma_{n-2}}\int_0^\infty (|\xi|s)^{-(n-3)/2}J_{(n-3)/2}(|\xi|s)\,\frac{d}{ds}$$

$$\times \int_0^s t^{n-1}e(t)\,\frac{\cos(\mu\sqrt{s^2-t^2})}{\sqrt{s^2-t^2}}\,dtds.$$

Integration by parts and formula 3.1.1, page 67 of [13] give

$$\hat{K}(\theta,\xi) = \frac{1}{\pi\gamma_{n-2}} \int_0^\infty \int_0^s \left(t^{n-1}e(t) \frac{\cos(\mu\sqrt{s^2-t^2})}{\sqrt{s^2-t^2}} dt\right) \frac{d}{ds}$$

$$\times \left((|\xi|s)^{-(n-3)/2} J_{(n-3)/2}(|\xi|s)\right) ds$$

$$= \frac{1}{\pi\gamma_{n-2}} \int_0^\infty t^{n-1}e(t)$$

$$\times \int_t^\infty \frac{\cos(\mu\sqrt{s^2-t^2})}{\sqrt{s^2-t^2}} |\xi|^2 s(|\xi|s)^{-(n-1)/2} J_{(n-1)/2}(|\xi|s)\, dsdt.$$

Letting $r^2 = s^2 - t^2$, using Fubini's theorem and then applying formula 6.726, no. 2, p. 756 of [8] gives

$$\hat{K}(\xi,\mu) = \frac{|\xi|^2}{\pi\gamma_{n-2}} \int_0^\infty t^{n-1}e(t)$$

$$\times \int_0^\infty (\cos \mu r)(\sqrt{r^2+t^2})^{-(n-1)/2} |\xi|^{-(n-1)/2} J_{(n-1)/2}(|\xi|\sqrt{r^2+t^2}) drdt$$

$$= \begin{cases} \dfrac{|\xi|^{3-n}(\sqrt{|\xi|^2-\mu^2})^{n-2}}{\sqrt{2\pi}\,\gamma_{n-2}} \\ \times \displaystyle\int_0^\infty t^{n-1}e(t)(t\sqrt{|\xi|^2-\mu^2})^{-(n-2)/2} J_{(n-2)/2}(t\sqrt{|\xi|^2-\mu^2})dt & \text{when } 0 < |\mu| < |\xi| \\ 0 & \text{when } |\mu| > |\xi| \end{cases}$$

Now using (2.4) again for E gives (5.6) and the proof is complete.

In particular when $n = 2$

$$\hat{K}(\theta,\xi) = \begin{cases} \dfrac{|\xi|}{2\sqrt{2\pi}}\,\hat{E}(\sqrt{|\xi|^2-\mu^2}\;\xi/|\xi|) & \text{when } 0 < |\mu| < |\xi| \\ 0 & \text{when } |\mu| > |\xi| \end{cases} \tag{5.7}$$

and taking inverse Fourier transforms

$$k(s,\mu) = \frac{1}{2\pi}\int_{\mu}^{\infty} \sigma\hat{E}(\sqrt{\sigma^2-\mu^2}\;\theta)\cos s\sigma d\sigma \tag{5.8}$$

where $\theta \in S^1$.

For $\rho > 0$ and $E_\rho(x) = \rho^{-n}E(x/\rho)$ let e_ρ be the even function of one variable so that $E_\rho(x) = e_\rho(|x|)$ then for $r \in \mathbb{R}$, $e_\rho(r) = \rho^{-n}e(r/\rho)$. If K_ρ is the convolution kernel corresponding to E_ρ and k_ρ the even function of one variable so that $K_\rho(\theta,x) = k_\rho(|x|,\mu)$ then

$$k_\rho(r,\mu) = \rho^{-n}k(r/\rho,\rho\mu) \tag{5.9}$$

For by (5.3)

$$k_\rho(r,\mu) = \frac{r^{n-2}}{\pi\gamma_{n-2}}\frac{d}{dr}\int_0^r t^{n-1}e_\rho(t)\,\frac{\cos(\mu\sqrt{r^2-t^2})}{\sqrt{r^2-t^2}}\,dt$$

Substituting $\rho^{-n}e(t/\rho)$ for $e_\rho(t)$ and making the change of variable $s = t/\rho$ gives

$$k_\rho(r,\mu) = \frac{\rho^{-n}(r/\rho)^{2-n}}{\pi\gamma_{n-2}}\frac{d}{d(r/\rho)}\int_0^{r/\rho} s^{n-1}e(s)\,\frac{\cos(\mu\rho\sqrt{(r/\rho)^2-s^2})}{\sqrt{(r/\rho)^2-s^2}}\,ds$$

$$= \rho^{-n}k(r/\rho,\rho\mu).$$

In practice with a suitable choice of E and ρ, K and hence K_ρ can be calculated say by (5.3). Then one applies the algorithm $\mathcal{P}^*_{-\mu}(\mathcal{P}_\mu f * K_\rho)$ to the data $\mathcal{P}_\mu f$ to reconstruct an approximation of f. See Corollary 4.3.

In Chapter VIII, we will give examples of point spread functions and corresponding convolution kernels that may be used in recovering the function f from the data $\mathcal{P}_\mu f$. Formula (5.3) will also be used to obtain an exact inversion formula for $\mathcal{P}_\mu$ when μ is constant. In doing so we encounter an operator which will be denoted by Λ_μ and the purpose of the next chapter is to introduce this operator and study some of its properties.

VI. THE OPERATOR Λ_μ

In finding an exact inversion formula for P_μ, μ constant, the operator Λ_μ plays an important role. In the next chapter we establish an exact inversion formula for P_μ when $P_\mu f$ belongs to certain potential spaces. This chapter is devoted to introducing these potential spaces and the operator Λ_μ as well as studying some properties of Λ_μ on these spaces. The material on potential spaces is taken from Stein [22].

<u>Definition 6.1</u>. For $1 \leq p \leq \infty$, the potential spaces $\mathcal{L}_1^p(\mathbb{R}^m)$ are defined by

$$\mathcal{L}_1^p(\mathbb{R}^m) = \{f \in L^p(\mathbb{R}^m) : f = G * g, \text{ for some } g \in L^p(\mathbb{R}^m)\}$$

where G is given by

$$\hat{G}(\xi) = (1+|\xi|^2)^{-1/2}$$

The norm $\| \ \|_{p,1}$ of $f \in \mathcal{L}_1^p(\mathbb{R}^m)$ is defined by

$$\|f\|_{p,1} = \|g\|_p$$

where $g \in L^p(\mathbb{R}^m)$ and $f = G * g$.

<u>Definition 6.2</u>. The space $L_1^p(\mathbb{R}^m)$ consists of all $f \in L^p(\mathbb{R}^m)$ such that for each j, $j = 1,\ldots,m$ there is $g \in L^p(\mathbb{R}^m)$ for which

$$\langle f, \partial_{x_j} \phi\rangle = -\langle g, \phi\rangle$$

for all $\phi \in C_0^\infty(\mathbb{R}^m)$. This defines g almost everywhere and one writes $\partial_{x_j} f = g$. The norm $\| \ \|_{L_1^p}$ of $f \in L_1^p(\mathbb{R}^m)$ is defined by

$$\|f\|_{L_1^p} = \|f\|_p + \sum_{j=1}^{m} \|\partial_{x_j} f\|_p .$$

The spaces $\mathcal{L}_1^p$ and L_1^p are complete in the given norms. An important result relating the spaces in the above definitions is

<u>Theorem 6.3</u>. For $1 < p < \infty$,

$$\mathcal{L}_1^p(\mathbb{R}^m) = L_1^p(\mathbb{R}^m)$$

and the corresponding norms are equivalent. Moreover, if $f \in L_1^p(\mathbb{R}^m)$ then for $j = 1,2,\ldots,m$, $\partial_{x_j} f(x)$ exists for almost every x in the usual sense and defines a function in $L^p(\mathbb{R}^m)$.

A consequence of the above theorem which will be of importance in inverting $\mathcal{P}_\mu$ is the fact that for $1 < p < \infty$ the spaces $L_1^p(\mathbb{R}^m)$ are independent of the choice of orthogonal coordinates on $\mathbb{R}^m$.

With this as background we now turn to introducing the operator Λ_μ. For $x \in \mathbb{R}^m$, let $H_{j,\mu}$ be defined for $j = 1,\ldots,m$ as follows

$$H_{j,\mu}(x) = \frac{\Gamma((m+1)/2)}{\pi^{(m+1)/2}} \frac{x_j}{|x|^{m+1}} \cos \mu|x| \tag{6.1}$$

where x_j is the j-th coordinate function. The Cauchy principle value distribution v.p. $H_{j,\mu}$ is defined by

$$\langle \text{v.p. } H_{j,\mu}, \phi \rangle = \lim_{\rho \to 0} \int_{|x| \geq \rho} H_{j,\mu}(x)\phi(x)\, dx$$

for all $\phi \in \mathcal{S}$. Note that for $\mu = 0$, $H_{j,\mu}$ is just the Riesz kernel. For a function f on $\mathbb{R}^m$ the operator Λ_μ is defined by

$$\Lambda_\mu f = \sum_{j=1}^{m} \partial_{x_j} (\text{v.p. } H_{j,\mu} * f) \tag{6.2}$$

where the derivative is a distribution derivative and the convolution is a Cauchy principle value convolution, i.e.,

$$\text{v.p. } H_{j,\mu} * f(x) = \lim_{\rho \to 0} \int_{|y| \geq \rho} H_{j,\mu}(y) f(x-y)\, dy.$$

<u>Lemma 6.4</u>. For $f \in C_0^1(\mathbb{R}^m)$ and $1 < p < \infty$

$$\text{v.p. } H_{j,\mu} * f(x) = \lim_{\rho \to 0} \int_{|y| \geq \rho} \frac{y_j}{|y|^{m+1}} \cos(\mu|y|) f(x-y)\, dy \tag{6.3}$$

exists for every x, in L^p norm, and is continuous.

<u>Proof</u>. The proof is standard but is included here for completeness; see §3.4, page 37 of Stein [22]. Let

$$K(y) = \frac{y_j}{|y|^{m+1}} \cos \mu|y|.$$

Then

a) $|K(y)| \leq |y|^{-m}$ for $|y| > 0$

b) $\int_{R_1 < |y| < R_2} K(y)dy = 0$ for $0 < R_1 < R_2 < \infty$ since K is odd.

Using b) we can write

$$\int_{|y|\geq\rho} K(y)f(x-y)dy = \int_{|y|\geq 1} K(y)f(x-y)dy$$

$$+ \int_{1\geq|y|\geq\rho} K(y)[f(x-y)-f(x)]\, dy.$$

For $1 < p < \infty$, K is an L^p function for $|y| \geq 1$ as a result of a) and f is an L^1 function. So the first integral on the right hand side of the equation exists for every x and is a continuous L^p function since it is the convolution of continuous L^1 and L^p functions.

Since $f \in C_0^1(\mathbb{R}^n)$, the second integral will have compact support in x and $|f(x-y)-f(x)| \leq A|y|$ by the mean value theorem. So the second integral will converge uniformly in x as $\rho \to 0$. Hence

$$\lim_{\rho\to 0} \int_{|y|\geq\rho} K(y)f(x-y)\, dy$$

exists for every x, converges in L^p and is continuous.

In fact we can show that for $f \in L^p(\mathbb{R}^m)$, $1 < p < \infty$ the limit in (6.3) exists in L^p norm and pointwise almost everywhere. To do so we need a result of Chen [3], that will be stated here without proof.

<u>Theorem 6.5</u>. For $m \geq 2$, let $\Omega \in L^q(S^{m-1})$ for $q > 1$ satisfy the following

a) Ω is homogeneous of degree 0

b) $\int_{S^{m-1}} \Omega d\theta = 0$

Let h be any radial bounded function on $\mathbb{R}^m$. For $1 < p < \infty$ and $f \in L^p(\mathbb{R}^m)$, let

$$T_\rho f(x) = \int_{|y|>\rho} h(y) \frac{\Omega(y)}{|y|^m} f(x-y)\, dy$$

then

$$\| \sup_{0<\rho<\infty} |T_\rho f(x)| \|_p \leq A_p \|h\|_\infty \|f\|_p$$

where A_p is a constant depending only on p and m.

<u>Theorem 6.6</u>. For $m \geq 2$, $1 < p < \infty$, if $f \in L^p(\mathbb{R}^m)$, then the limit

$$\text{v.p. } H_{j,\mu} * f(x) = \lim_{\rho\to 0} \int_{|y|\geq\rho} \frac{y_j}{|y|^{m+1}} \cos(\mu|y|) f(x-y)\, dy \quad (6.4)$$

exists a) in L^p norm, b) for almost every x.

<u>Proof</u>. The proof is standard and is included here for completeness; see §3.4, page 37 and §4.6.3, page 42 of [22]. Let

$$T_\rho f(x) = \int_{|y|\geq\rho} \frac{y_j}{|y|^{m+1}} \cos(\mu|y|) f(x-y)\, dy \quad (6.5)$$

With $h(y) = \cos \mu|y|$, $\Omega(y) = y_j/|y|$, the hypotheses of the above theorem are satisfied, so there is a constant A_p, depending only on p and the dimension, such that

$$||T_\rho f||_p \leq A_p ||f||_p, \quad 1 < p < \infty \tag{6.6}$$

Now let $f \in L^p(\mathbb{R}^m)$, and let $\varepsilon > 0$, then $f = f_1 + f_2$ where $f_1 \in C_0^1(\mathbb{R}^m)$ and $||f_2||_p < \varepsilon/3A_p$. By the previous lemma $T_\rho f_1$ converges in L^p norm as $\rho \to 0$, so there exists $\rho_0 > 0$ such that if $\rho_1, \rho_2 < \rho_0$, $||T_{\rho_1} f_1 - T_{\rho_2} f_1||_p < \varepsilon/3$, but

$$\begin{aligned} ||T_{\rho_1} f - T_{\rho_2} f||_p &= ||T_{\rho_1} f_1 + T_{\rho_1} f_2 - T_{\rho_2} f_1 - T_{\rho_2} f_2||_p \\ &\leq ||T_{\rho_1} f_1 - T_{\rho_2} f_1||_p + ||T_{\rho_1} f_2||_p + ||T_{\rho_2} f_2||_p < \varepsilon \end{aligned}$$

by the above and (6.6). So $\{T_\rho f\}$ is Cauchy in L^p, hence converges in the L^p norm.

Next we want to show the existence of the limit almost everywhere. With $f = f_1 + f_2$ as before, let

$$Lf(x) = |\limsup_{\rho \to 0} T_\rho f(x) - \liminf_{\rho \to 0} T_\rho f(x)|.$$

Then clearly

$$Lf(x) \leq 2(\sup|T_\rho f(x)|)$$

and (6.7)

$$Lf(x) = L(f_1 + f_2)(x) \leq Lf_1(x) + Lf_2(x).$$

By the previous lemma $Lf_1(x) = 0$. By (6.7) and (6.6)

$$||Lf_2||_p < \frac{2}{3}\varepsilon,$$

so $Lf_2 = 0$ almost everywhere, thus $Lf = 0$ almost everywhere, which

gives the pointwise convergence almost everywhere of $\lim_{\rho\to 0} T_\rho f$.

Remark 6.7. We are also interested in the existence of the limit in (6.4) for the case $m = 1$. In this case (6.4) is

$$\begin{aligned}
\text{v.p. } H_{j,\mu} * f(x) &= \lim_{\rho\to 0} \int_{|y|\geq\rho} \frac{\cos \mu y}{y} f(x-y)\, dy \\
&= \lim_{\rho\to 0} \int_{|x-y|\geq\rho} \frac{\cos \mu(x-y)}{x-y} f(y)\, dy \\
&= (\cos \mu x) \lim_{\rho\to 0} \int_{|x-y|\geq\rho} \frac{\cos \mu y}{x-y} f(y)\, dy \\
&\quad + (\sin \mu x) \lim_{\rho\to 0} \int_{|x-y|\geq\rho} \frac{\sin \mu y}{x-y} f(y)\, dy
\end{aligned}$$

where the last two integrals are just the Hilbert transforms of $\cos \mu(\cdot)f(\cdot)$ and $\sin \mu(\cdot)f(\cdot)$ respectively and which for $f \in L^p(\mathbb{R}^m)$, $1 < p < \infty$, exist in L^p norm and pointwise almost everywhere [22].

Remark 6.8. Let $Tf = \lim_{\rho\to 0} T_\rho f$ with T_ρ as in (6.5) and $m \geq 1$, then it is easy to see that (6.6) holds for T too. So T maps $L^p(\mathbb{R}^m)$ continuously into $L^p(\mathbb{R}^m)$. Note that

$$Tf = \text{v.p. } H_{j,\mu} * f. \tag{6.8}$$

With T as above, we have

Remark 6.9. For $1 < p < \infty$, $T : L_1^p(\mathbb{R}^m) \to L_1^p(\mathbb{R}^m)$ is continuous.

Proof. Using Theorem 6.3 we replace $L_1^p(\mathbb{R}^m)$ by $\mathcal{L}_1^p(\mathbb{R}^m)$. If $f \in \mathcal{S}$ then $f = G * g$ for some $g \in L^p(\mathbb{R}^m)$ and

$$Tf = T(G*g) = G * Tg.$$

The continuity of T on L^p and the definition of $\| \ \|_{p,1}$ give

$$\|Tf\|_{p,1} = \|G*Tg\|_{p,1} = \|Tg\|_p \leq c\|g\|_p = c\|f\|_{p,1}.$$

The result now follows from the fact that $\mathcal{S}$ is dense in $L_1^p(\mathbb{R}^m)$.

Theorem 6.10.

a) $$(\text{v.p. } H_{j,\mu})\hat{}(\xi) = \begin{cases} -i(2\pi)^{-m/2}\xi_j|\xi|^{-m}(|\xi|^2-\mu^2)^{(m-1)/2} & \text{if } |\xi| > |\mu| \\ 0 & \text{if } |\xi| < |\mu| \end{cases}$$

b) For $1 < p < \infty$, $\Lambda_\mu : L_1^p(\mathbb{R}^m) \to L^p(\mathbb{R}^m)$ is continuous, and if $p \leq 2$

$$(\Lambda_\mu f)\hat{}(\xi) = \begin{cases} |\xi|^{2-m}(|\xi|^2-\mu^2)^{(m-1)/2}\hat{f}(\xi) & \text{if } |\xi| > |\mu| \\ 0 & \text{if } |\xi| < |\mu| \end{cases}$$

c) For $g \in L^1(\mathbb{R}^m)$, $f \in L_1^p(\mathbb{R}^m)$, $1 < p < \infty$ then $f * g \in L_1^p(\mathbb{R}^m)$ and

$$\Lambda_\mu f * g = \Lambda_\mu(f*g).$$

Before proceeding with the proof we note that if $\chi_{\varepsilon,M}$ denotes the characteristic function of $\{x \in \mathbb{R}^m : \varepsilon \leq |x| \leq M,\ \varepsilon, M > 0\}$, then for all $\phi \in \mathcal{S}$

$$\langle \text{v.p. } H_{j,\mu},\phi\rangle = \lim_{\substack{\varepsilon\downarrow 0\\ M\to\infty}} \int_{\varepsilon\le|x|\le M} H_{j,\mu}(x)\phi(x)\,dx$$

$$= \lim_{\substack{\varepsilon\downarrow 0\\ M\to\infty}} \int_{\mathbb{R}^m} \chi_{\varepsilon,M}(x)H_{j,\mu}(x)\phi(x)\,dx$$

$$= \lim_{\substack{\varepsilon\downarrow 0\\ M\to\infty}} \langle \chi_{\varepsilon,M}H_{j,\mu},\phi\rangle.$$

<u>Proof of Theorem 6.10</u>. a) By the above we need only evaluate

$$\lim_{\substack{\varepsilon\downarrow 0\\ M\to\infty}} (\chi_{\varepsilon,M}H_{j,\mu})\hat{}\,.$$

We will denote it by $(H_{j,\mu})\hat{}\,(\xi)$ to simplify notation. For each ε and each M, $\chi_{\varepsilon,M}H_{j,\mu}$ is integrable so using polar coordinates in $\mathbb{R}^m$ with $x = r\theta$, $\xi = |\xi|\phi$, $\theta,\phi \in S^{m-1}$ and $r > 0$ we get

$$(H_{j,\mu})\hat{}\,(\xi) = (2\pi)^{-m/2} \lim_{\substack{\varepsilon\downarrow 0\\ M\to\infty}} \int_{\mathbb{R}^m} \chi_{\varepsilon,M}(x)H_{j,\mu}(x)e^{-i\langle x,\xi\rangle}\,dx$$

$$= (2\pi)^{-m/2} \frac{\Gamma((m+1)/2)}{\pi^{(m+1)/2}} \lim_{\substack{\varepsilon\downarrow 0\\ M\to\infty}} \int_{\varepsilon}^{M}$$

$$\times \int_{S^{m-1}} \theta_j \frac{\cos \mu r}{r} e^{-ir|\xi|\langle\theta,\phi\rangle}\,d\theta dr$$

$$= -i(2\pi)^{-m/2} \frac{\Gamma((m+1)/2)}{\pi^{(m+1)/2}} \lim_{\substack{\varepsilon\downarrow 0\\ M\to\infty}} \int_{\varepsilon}^{M} \frac{\cos \mu r}{r}$$

$$\times \int_{S^{m-1}} \theta_j \sin(r|\xi|\langle\theta,\phi\rangle)\,d\theta dr \tag{6.9}$$

since θ_j is odd on S^{m-1}. If we first set $t = \langle\theta,\phi\rangle$ and then let $\theta = t\phi + \sqrt{1-t^2}\,\omega$, where $\omega \in S^{m-2}$ and $\langle\omega,\phi\rangle = 0$, the inner integral becomes

$$\int_{S^{m-1}} \theta_j \sin(r|\xi|\langle\theta,\phi\rangle)\, d\theta$$

$$= \int_{-1}^{1} \int_{\langle\theta,\phi\rangle=t} \theta_j \sin(r|\xi|t)(1-t^2)^{-1/2}\, d\theta dt$$

$$= \int_{-1}^{1} \int_{\phi^\perp\cap S^{m-1}} (t\phi_j+\sqrt{1-t^2}\,\omega_j)d\omega \sin(r|\xi|t)(1-t^2)^{(m-3)/2}\, dt$$

$$= \gamma_{m-2}\phi_j \int_{-1}^{1} t \sin(r|\xi|t)(1-t^2)^{(m-3)/2}\, dt,$$

since $\int_{\phi^\perp\cap S^{m-1}} \omega_j\, d\omega = 0$. For the last integral we use formula 3.771, No. 10 of [8] and noting that $t \sin(r|\xi|t)$ is even in t, we get

$$\int_{S^{m-1}} \theta_j \sin(r|\xi|\langle\theta,\phi\rangle)\, d\theta$$

$$= 2^{(m-2)/2} \sqrt{\pi}\, \Gamma(\tfrac{m-1}{2})\gamma_{m-2}\phi_j (r|\xi|)^{-(m-2)/2} J_{m/2}(r|\xi|).$$

Substituting in (6.9) we have

$$(H_{j,\mu})^\wedge(\xi) = -i(2\pi)^{-m/2}2^{m/2} \frac{\Gamma((m+1)/2)}{\sqrt{\pi}} \varepsilon_j \lim_{\substack{\varepsilon\downarrow 0\\ M\to\infty}}$$

$$\times \int_{\varepsilon}^{M} (r|\xi|)^{-m/2} \cos(\mu r) J_{m/2}(r|\xi|)\, dr.$$

The integral in the last expression can be evaluated by using the third formula, p. 62 of [16] to get a).

The continuity in b) is clear. If $p \leq 2$ we get from a)

$$(\Lambda_\mu f)\hat{}(\xi) = \sum_{j=1}^{m} \partial_{x_j} (\text{v.p. } H_{j,\mu} * f)\hat{}(\xi)$$

$$= \sum_{j=1}^{m} i(2\pi)^{m/2} \xi_j (H_{j,\mu})\hat{}(\xi) \hat{f}(\xi)$$

$$= \begin{cases} \sum_{j=1}^{m} \xi_j^2 |\xi|^{-m} (|\xi|^2 - \mu^2)^{(m-1)/2} \hat{f}(\xi) & \text{if } |\xi| > |\mu| \\ 0 & \text{if } |\xi| < |\mu| \end{cases}$$

$$= \begin{cases} |\xi|^{2-m} (|\xi|^2 - \mu^2)^{(m-1)/2} \hat{f}(\xi) & \text{if } |\xi| > |\mu| \\ 0 & \text{if } |\xi| < |\mu| \end{cases}$$

By continuity and denseness it suffices to show c) for $f \in \mathcal{S}$. Then

$$(\Lambda_\mu (f*g))\hat{}(\xi) = \begin{cases} |\xi|^{2-m} (|\xi|^2 - \mu^2)^{(m-1)/2} (f*g)\hat{}(\xi) & \text{if } |\xi| > |\mu| \\ 0 & \text{if } |\xi| < |\mu| \end{cases}$$

$$= \begin{cases} (2\pi)^{m/2} |\xi|^{2-m} (|\xi|^2 - \mu^2)^{(m-1)/2} \hat{f}(\xi) \hat{g}(\xi) & \text{if } |\xi| > |\mu| \\ 0 & \text{if } |\xi| < |\mu| \end{cases}$$

$$= \Lambda_\mu f * g.$$

<u>Remark 6.11</u>. Λ_μ is formally self adjoint.

<u>Proof</u>. Let $f \in L_1^p(\mathbb{R}^m)$, $1 < p < \infty$ then for all $g \in \mathcal{S}$

$$\langle \Lambda_\mu f, g\rangle = \sum_{j=1}^{m} \langle \partial_{x_j} (\text{v.p. } H_{j,\mu} * f), g\rangle$$

$$= \sum_{j=1}^{m} \langle \text{v.p. } H_{j,\mu} * f, -\partial_{x_j} g\rangle$$

$$= \sum_{j=1}^{m} \langle f, \text{v.p. } H_{j,\mu} * \partial_{x_j} g\rangle$$

$$= \langle f, \sum_{j=1}^{n} \partial_{x_j} (\text{v.p. } H_{j,\mu} * g)$$

since $H_{j,\mu}$ is odd, and we are done.

VII. INVERSION OF THE EXPONENTIAL X-RAY TRANSFORM WITH CONSTANT ATTENUATION

In this chapter an inversion formula for the exponential X-ray transform is derived in the case where the attenuation μ is constant. Using this formula we deduce a way for evaluating convolution kernels from point spread functions that are not necessarily radial. Also another approximate inversion formula for P_μ is given.

First we need to extend the notion of potential spaces to functions defined on $T = \{(\theta,x) : \theta \in S^{n-1}, x \in \theta^\perp\}$. For $1 < p < \infty$, the space $L_1^p(T)$ consists of all functions $g \in L^p(T)$ such that

$$\|g\|_{L_1^p(T)} = \int_{S^{n-1}} \|g(\theta,\cdot)\|_{L_1^p} \, d\theta < \infty.$$

Here $\| \ \|_{L_1^p}$ is as in Definition 6.1 with $\mathbb{R}^m$ replaced by $\theta^\perp$. Thus for almost every θ, $g(\theta,\cdot) \in L_1^p(\theta^\perp)$ and by Definition 6.2 and Theorem 6.3 (with $\mathbb{R}^m$ replaced by $\theta^\perp$), the first order derivatives of $g(\theta,\cdot)$ in the directions $v_1,\ldots,v_{n-1}$ exist almost everywhere and belong to $L^p(\theta^\perp)$, for any choice of orthonormal basis $v_1,\ldots,v_{n-1}$ of $\theta^\perp$. In what follows, if $g \in L_1^p(T)$ then for each θ, Λ_μ acts on g as a function on $\theta^\perp$.

Throughout this chapter the constant $c_n = 1/(2\pi\gamma_{n-2})$. Recall that γ_{n-2} is the (n-2)-dimensional surface area measure of S^{n-2}.

For the next theorem it is assumed that E, the point spread function, is non-negative, continuous, radial and integrable on $\mathbb{R}^n$

and $\int_{\mathbb{R}^n} E(x)dx = 1$. For $\rho > 0$, $E_\rho(x) = \rho^{-n}E(x/\rho)$ and K_ρ is the corresponding convolution kernel. For the main result of this chapter we have

Theorem 7.1.

$$f(x) = c_n P^*_{-\mu} \Lambda_\mu P_\mu f(x). \tag{7.1}$$

Equality holds pointwise if $f \in C_0^2(\mathbb{R}^n)$ and pointwise almost everywhere if $f \in L_0^1(\mathbb{R}^n)$ and $P_\mu f \in L_1^p(T)$ for some $p > 1$.

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^n)$, then for each $x \in \mathbb{R}^n$

$$\begin{aligned}\phi(x) &= \lim_{\rho\to 0} \phi * E_\rho(x) \\ &= \lim_{\rho\to 0} P^*_{-\mu}(P_\mu\phi * K_\rho)(x)\end{aligned}$$

by Theorem 3.6. Now for each fixed θ and $x \in \theta^\perp$

$$(P_\mu\phi * K_\rho)(\theta,x) = (2\pi)^{(n-1)/2}[(P_\mu\phi)\hat{}\,\hat{K}_\rho]^\sim(\theta,x)$$

where $\sim$ denotes the inverse Fourier transform on $\theta^\perp$. So using (5.6)

$$\begin{aligned}(P_\mu\phi * K_\rho)(\theta,x) = \sqrt{2\pi}\, c_n \int_{|\xi|>|\mu|} & e^{i\langle x,\xi\rangle} (P_\mu\phi)\hat{}\,(\theta,\xi) \\ & \times |\xi|^{3-n}(|\xi|^2-\mu^2)^{(n-2)/2}\hat{E}_\rho(\sqrt{|\xi|^2-\mu^2}\,\xi/|\xi|)d\xi.\end{aligned}$$

The integrand is bounded by

$$c\,\|E\|_1|\xi|\,|(P_\mu\phi)\hat{}\,(\theta,\xi)|.$$

Since $\phi \in C_0^\infty(\mathbb{R}^n)$, $P_\mu\phi \in C_0^\infty(T)$ and $|(P_\mu\phi)\hat{}(\theta,\xi)|$ decays faster than any power of $|\xi|$, $|\xi||(P_\mu\phi)\hat{}(\theta,\xi)|$ is integrable on $\theta^\perp$ and also on T.

Now as $\rho \to 0$, $\hat{E}_\rho(\xi) \to (2\pi)^{-n/2}$. By the dominated convergence theorem and Theorem 6.10 b) with $m = n - 1$

$$\lim_{\rho\to 0} (P_\mu\phi * K_\rho)(\theta,x)$$

$$= (2\pi)^{-(n-1)/2} c_n \int_{|\xi|>|\mu|} e^{i\langle x,\xi\rangle} (P_\mu\phi)\hat{}(\theta,\xi)|\xi|^{3-n}(|\xi|^2-\mu^2)^{(n-2)/2} d\xi$$

$$= c_n \Lambda_\mu P_\mu \phi(\theta,x).$$

Hence

$$\phi(x) = c_n P^*_{-\mu} \Lambda_\mu P_\mu \phi(x)$$

which proves the theorem for functions $\phi \in C_0^\infty(\mathbb{R}^n)$.

Let

$$h(x) = c_n P^*_{-\mu} \Lambda_\mu P_\mu f(x).$$

If $f \in C_0^2(\mathbb{R}^n)$, then, by the definition of Λ_μ and Lemma 6.4, h is continuous. If $f \in L_0^1(\mathbb{R}^n)$ and $P_\mu f \in L_1^p(T)$, for some $p > 1$, then $\Lambda_\mu P_\mu f \in L^p(T)$ and $h \in L^p_{loc}(\mathbb{R}^n)$. In either case to prove the theorem it suffices to show that $h = f$ almost everywhere. Let $\phi \in C_0^\infty(\mathbb{R}^n)$. Then

$$\langle h,\phi\rangle = \langle c_n P^*_{-\mu} \Lambda_\mu P_\mu f,\phi\rangle$$

$$= \langle f, c_n P^*_\mu \Lambda_\mu P_{-\mu}\phi\rangle$$

$$= \langle f,\phi\rangle,$$

since the theorem holds for $\phi \in C_0^\infty(\mathbb{R}^n)$. The proof is complete.

Remark 7.2. If $E \in L_0^1(\mathbb{R}^n)$ and $\mathcal{P}_\mu E \in L_1^p(T)$, $p > 1$, then

$$E(x) = \mathcal{P}_{-\mu}^*(c_n \Lambda_\mu \mathcal{P}_\mu E)(x).$$

So we define the corresponding convolution kernel by

$$K = c_n \Lambda_\mu \mathcal{P}_\mu E.$$

We thus have a formula for evaluating the convolution kernel, that does not require that E be radial. In the case where E is radial we have another formula for the unique solution k of (5.2).

Due to practical limitations such as finite sampling of and noise in the data $\mathcal{P}_\mu f$, the exact inversion formula (7.1) is not very useful in practice. A smoothed approximation of f is what one usually seeks to determine. We have already seen approximate inversion formulas in Chapter III. Theorem 7.1 leads to another approximate inversion formula for $\mathcal{P}_\mu$, which is given in the following theorem.

Theorem 7.3. Suppose $f \in L_0^1(\mathbb{R}^n)$ then for $E \in L_0^1(\mathbb{R}^n)$ and $\mathcal{P}_\mu E \in L_1^p(\mathbb{R}^n)$, $p > 1$,

$$f * E = c_n \mathcal{P}_{-\mu}^*(\mathcal{P}_\mu f * \Lambda_\mu \mathcal{P}_\mu E). \tag{7.2}$$

Proof. The previous theorem, Corollary 3.8 and Theorem 6.10 c) give

$$f * E = c_n P^*_{-\mu} \Lambda_\mu P_\mu (f*E)$$

$$= c_n P^*_{-\mu} \Lambda_\mu (P_\mu f * P_\mu E)$$

$$= c_n P^*_{-\mu} (P_\mu f * \Lambda_\mu P_\mu E).$$

For E an approximate δ-function, (7.2) gives an approximation for f. An advantage that (7.2) has over (7.1) is that Λ_μ, which involves differentiation and singular integration acts on $P_\mu E$ rather than on the data $P_\mu f$ and E can be chosen a priori to suit the needs of the problem at hand.

VIII. EXAMPLES

In this chapter, we will give examples of point spread functions E and calculations of convolution kernels K using formulas derived in Chapter V. We do not claim that these kernels are the best suited for use in the convolution-backprojection algorithm but they are reasonable ones to use and demonstrate the use of the formulas. The examples given are in two dimensions, as this is the space of interest in practice. Here e, k, are as defined in Chapter V.

<u>Example 8.1</u>. We start here with a simple example where E is a constant multiple of χ_D, the characteristic function of the unit disc $D = \{x \in \mathbb{R}^2 : |x| \leq 1\}$, i.e.,

$$E(x) = \frac{1}{\pi}\chi_D(x) = \begin{cases} \frac{1}{\pi} & \text{if} \quad |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and so

$$e(r) = \begin{cases} \frac{1}{\pi} & \text{if} \quad |r| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Using (5.3) with $n = 2$ and setting $t = \sqrt{s^2-r^2}$ gives

$$k(s,\mu) = \begin{cases} \frac{1}{2\pi^2}\frac{d}{ds}\int_0^s \cos \mu t \, dt & \text{if} \quad |s| \leq 1 \\ \frac{1}{2\pi^2}\frac{d}{ds}\int_{\sqrt{s^2-1}}^s \cos \mu t \, dt & \text{if} \quad |s| > 1 \end{cases}$$

and hence

$$k(s,\mu) = \begin{cases} \dfrac{1}{2\pi^2} \cos \mu s & \text{if} \quad |s| \le 1 \\[2ex] \dfrac{1}{2\pi^2} \left[\cos \mu s - \dfrac{|s|\cos(\mu\sqrt{s^2-1})}{\sqrt{s^2-1}}\right] & \text{if} \quad |s| > 1. \end{cases}$$

<u>Example 8.2</u>. Let

$$E(x) = \frac{1}{2\pi} e^{-(|x|^2/2)}$$

so the corresponding e is

$$e(r) = \frac{1}{2\pi} e^{-r^2/2} .$$

By (5.3) with $n = 2$ we have after setting $u = t^2$ and $\sigma = s^2$

$$k(\sqrt{\sigma},\mu) = \frac{\sqrt{\sigma}}{(2\pi)^2} \frac{d}{d\sigma} \int_0^\sigma e^{-u/2} \frac{\cos(\mu\sqrt{\sigma-u})}{\sqrt{\sigma-u}} du.$$

Integrating by parts and then setting $t^2 = \sigma - u$ gives

$$k(\sqrt{\sigma},\mu) = \frac{\sqrt{\sigma}}{\mu(2\pi)^2} \frac{d}{d\sigma} \left(2 \sin \mu\sqrt{\sigma} - \int_0^\sigma e^{-u/2} \sin(\mu\sqrt{\sigma-u}) du\right)$$

$$= \frac{1}{(2\pi)^2} \cos \mu\sqrt{\sigma} - \frac{\sqrt{\sigma}}{2(2\pi)^2} \int_0^\sigma e^{-u/2} \frac{\cos(\mu\sqrt{\sigma-u})}{\sqrt{\sigma-u}} du$$

$$= \frac{1}{(2\pi)^2} \cos \mu\sqrt{\sigma} - \frac{\sqrt{\sigma}}{(2\pi)^2} e^{-\sigma/2} \int_0^{\sqrt{\sigma}} e^{t^2/2} \cos \mu t \, dt.$$

Hence

$$k(s,\mu) = \frac{1}{(2\pi)^2}\left(\cos \mu s - s e^{-s^2/2} \int_0^s e^{t^2/2} \cos \mu t \, dt\right),$$

which is the kernel of Tretiak and Metz [24]. Since $\hat{E}(\xi) = e^{-|\xi|^2/2}$ the Fourier transform formula (5.7) gives

$$\hat{K}(\theta,\xi) = \begin{cases} \dfrac{1}{2(2\pi)^{3/2}} |\xi| e^{-(|\xi|^2-\mu^2)/2} & \text{if} \quad |\sigma| > |\mu| \\ 0 & \text{if} \quad |\sigma| < |\mu|. \end{cases}$$

<u>Example 8.3</u>. For $\nu > 0$ let

$$E(x) = \begin{cases} \dfrac{\nu+1}{\pi} (1-|x|^2)^\nu & \text{if} \quad |x| \le 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$

and so the corresponding e is

$$e(r) = \begin{cases} \dfrac{\nu+1}{\pi} (1-r^2)^\nu & \text{if} \quad |r| \le 1 \\ 0 & \text{if} \quad |r| > 1. \end{cases}$$

Using (5.5) with $n = 2$ and s replaced by $\sqrt{s}$ we get

$$k(\sqrt{s},\mu) = \begin{cases} \dfrac{\nu+1}{2\pi^2} \left[-\nu\sqrt{s} \displaystyle\int_0^s (1-r)^{\nu-1} \frac{\cos(\mu\sqrt{s-r})}{\sqrt{s-r}} dr + \cos \mu\sqrt{s}\right] & \text{if} \quad 0 < s < 1 \\ \dfrac{\nu+1}{2\pi^2} \left[-\nu\sqrt{s} \displaystyle\int_0^1 (1-r)^{\nu-1} \frac{\cos(\mu\sqrt{s-r})}{\sqrt{s-r}} dr + \cos \mu\sqrt{s}\right] & \text{if} \quad s \ge 1. \end{cases} \tag{8.1}$$

Assuming ν an integer we have

$$\int_0^s (1-r)^{\nu-1} \frac{\cos(\mu\sqrt{s-r})}{\sqrt{s-r}}\,dr = \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} \int_0^s r^j \frac{\cos(\mu\sqrt{s-r})}{\sqrt{s-r}}\,dr.$$

The integral in the left hand side is evaluated by Laplace transform. Applying formulas 29.3.7, 29.3.76 and 29.3.80 of [1] we get

$$\int_0^s r^j \frac{\cos(\mu\sqrt{s-r})}{\sqrt{s-r}}\,dr = \sqrt{\pi}\,\Gamma(j+1)\left(\frac{4s}{\mu^2}\right)^{(j+1/2)/2} J_{j+1/2}(\mu\sqrt{s}).$$

Hence

$$\int_0^s (1-r)^{\nu-1} \frac{\cos(\mu\sqrt{s-r})}{\sqrt{s-r}}\,dr$$

$$= \sum_{j=0}^{\nu-1} (-1)^j \frac{(\nu-1)!}{(\nu-1-j)!} \left(\frac{2}{\mu}\right)^{j+1/2} (\sqrt{s})^{j+1/2} J_{j+1/2}(\mu\sqrt{s}), \qquad (8.2)$$

which, for $0 < s < 1$, evaluates the first integral appearing in (8.1). Now

$$\int_0^1 (1-r)^{\nu-1} \frac{\cos(\mu\sqrt{s-r})}{\sqrt{s-r}}\,dr = \int_0^s (1-r)^{\nu-1} \frac{\cos(\mu\sqrt{s-r})}{\sqrt{s-r}}\,dr$$

$$- \int_1^s (1-r)^{\nu-1} \frac{\cos(\mu\sqrt{s-r})}{\sqrt{s-r}}\,dr. \qquad (8.3)$$

The first integral in the right hand side is evaluated by (8.2). For the second we first let $x = r - 1$ and then use Laplace transforms and formulas 29.3.7, 29.3.76 and 29.3.80 of [1] we have

$$\int_1^s (1-r)^{\nu-1} \frac{\cos(\mu\sqrt{s-r})}{\sqrt{s-r}}\, dr$$

$$= (-1)^{\nu-1} \int_0^{s-1} x^{\nu-1} \frac{\cos(\mu\sqrt{s-1-x})}{\sqrt{s-1-x}}\, dx$$

$$= (-1)^{\nu-1}\sqrt{\pi}(\nu-1)!\,(\tfrac{2}{\mu})^{\nu-1/2}(\sqrt{s-1})^{\nu-1/2}J_{\nu-1/2}(\mu\sqrt{s-1})$$

Substituting back in (8.2) and using (8.3) we have for (8.1) with s instead of $\sqrt{s}$

$$k(s,\mu) = \frac{(\nu+1)s}{2\pi\sqrt{\pi}} \sum_{j=0}^{\nu-1} (-1)^{j+1} \frac{\nu!}{(\nu-1-j)!} (\tfrac{2s}{\mu})^{j+1/2} J_{j+1/2}(\mu s) + \frac{\nu+1}{2\pi^2} \cos \mu s$$

$$+ \begin{cases} 0 & \text{if } 0 < s < 1 \\ \dfrac{(-1)^{\nu-1}(\nu+1)\nu!}{2\pi\sqrt{\pi}} (\tfrac{2}{\mu})^{\nu-1/2} s(\sqrt{s^2-1})^{\nu-1/2} J_{\nu-1/2}(\mu\sqrt{s^2-1}) & \text{if } s > 1. \end{cases}$$

<u>Example 8.4</u>. For $\nu > 0$, let $E(x) = \frac{\Gamma((\nu+2)/2)}{2\pi\nu} (\frac{|x|}{2})^{-\nu/2} J_{\nu/2}(|x|)$, then

$$\hat{E}(\xi) = (1/2\pi)\ (1-|\xi|^2)^{(\nu-2)/2} \quad \text{if} \quad |\xi| < 1 \quad \text{and} \quad 0 \quad \text{otherwise.}$$

And by (5.8)

$$k(s,\mu) = \frac{1}{4\pi^2} \int_\mu^{\sqrt{1+\mu^2}} \sigma(1+\mu^2-\sigma^2)^{(\nu-2)/2} \cos(s\sigma)\, d\sigma \tag{8.4}$$

Assume ν is an even integer, $\nu \geq 2$. Near the origin expanding the cosine in its Taylor series and integrating term by term we get

$$k(s,\mu) = \frac{1}{4\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} s^{2k} \int_\mu^{\sqrt{1+\mu^2}} \sigma^{2k+1}(1+\mu^2-\sigma^2)^{(\nu-2)/2}\, d\sigma$$

If $\nu = 2\gamma$, γ an integer, then integration by parts gives for s near the origin

$$k(s,\mu) = \frac{1}{4\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k s^{2k}}{(2k)!} \frac{1}{\nu} \sum_{j=0}^{k} \prod_{\ell=1}^{j} \left(\frac{k+1-\ell}{\gamma+\ell}\right) \mu^{2(k-j)} .$$

For larger values of s, let $m = (\nu-2)/2$, $a^2 = 1 + \mu^2$ and $P_{2m+1}(\sigma) = \sigma(a^2-\mu^2)^m$. Then using formula 2.634 of [8] we get for (8.4)

$$\int_{\mu}^{\sqrt{1+\mu^2}} P_{2m+1}(\sigma)\cos(s\sigma)d\sigma = g(\sigma) \frac{\sin(s\sigma)}{s} + h(\sigma) \frac{\cos(s\sigma)}{s}\Big]_{\sigma=\mu}^{\sigma=\sqrt{1+\mu^2}}$$

where

$$g(\sigma) = \sum_{k=0}^{m} \frac{(-1)^k}{s^{2k}} \frac{d^{2k}}{d\sigma^{2k}} [\sigma(a^2-\sigma^2)^m]$$

and

$$h(\sigma) = \frac{1}{s} \frac{dg}{d\sigma} .$$

It is not hard to see that g satisfies the differential equation

$$[1 + \frac{1}{s^2} \frac{d^2}{d\sigma^2}]g(\sigma) = \sigma(a^2-\sigma^2)^m$$

whose solution is

$$g(\sigma) = \sigma \sum_{k=0}^{m} a_k(a^2-\sigma^2)^k$$

where

$$a_m = 1$$

$$a_{m-1} = \frac{2m(2m+1)}{s^2}$$

and for $k = 0,1,\ldots,m-2$

$$a_k = \frac{2(k+1)}{s^2}\,[(2k+3)(a_{k+1} - a^2 a_{k+2}) - a^2 a_{k+2}] \;.$$

And hence

$$h(\sigma) = \sum_{k=0}^{m} c_k (a^2-\sigma^2)^k$$

where

$$c_m = (2m+1)a_m$$

and for $k = 0,1,\ldots,m-1$

$$c_k = (2k+1)a_k - 2a^2(k+1)a_{k+1} \;.$$

IX. ESTIMATION OF ERRORS IN THE CONVOLUTION-BACKPROJECTION ALGORITHM

To check to what extent algorithms in TCT accurately reconstruct the desired attentuation coefficient of a cross section of the body, Shepp and Logan [20] suggested reconstructing phantoms, i.e., simulations of the object of interest. The data, in this case given by (1.4), can be exactly computed. For instance, Shepp and Logan [20] suggested as a simulation of the cross section of the human head a superposition of ellipses with the attenuation coefficient μ constant on each ellipse. The same simulation of a cross section of the head can be used in our case where the intensity of the radiation f is assumed constant on each ellipse. Here we will be satisfied with finding ways to check the convolution-backprojection algorithm for the case f is the characteristic function of a disc in the plane.

Comparing the convolution-backprojection algorithm with the exact inversion formula (7.1) we expect that for a suitable choice of point spread function and ρ, the convolution step $\mathcal{P}_\mu f * K_\rho$ is a smoothed approximation of $c_n \Lambda_\mu \mathcal{P}_\mu f$. For the case where f is the characteristic function of a disc in the plane we give explicit formulas to evaluate $(1/4\pi)\Lambda_\mu \mathcal{P}_\mu f$, $\mathcal{P}_\mu f * K_\rho$ and the difference between them. Also a formula that can be used to check the backprojection step is given.

Let τ_a denote translation in $\mathbb{R}^m$ defined by $\tau_a f(x) = f(x-a)$. Then direct consequences of the definition of $\mathcal{P}_\mu$ and the fact that convolution and differentiation commute with translation are

$$P_\mu(\tau_a f)(\theta,s) = e^{\mu<a,\theta>}(\tau_{<a,\theta^\perp>} P_\mu f)(\theta,s) \tag{9.1}$$

and

$$[\Lambda_\mu P_\mu(\tau_a f)](\theta,s) = e^{\mu<a,\theta>}(\tau_{<a,\theta^\perp>} \Lambda_\mu P_\mu f)(\theta,s). \tag{9.2}$$

<u>Lemma 9.1</u>. Let χ_a be the characteristic function of a disc in the plane with center a and radius $\delta > 0$. Then

$$\frac{1}{4\pi}\Lambda_\mu P_\mu \chi_a(\theta,s) = \frac{e^{\mu<a,\theta>}}{2\pi}[\cos\mu(s-<a,\theta^\perp>)-h(s-<a,\theta^\perp>)] \tag{9.3}$$

where

$$h(t) = \begin{cases} 0 & \text{if } |t| < \delta \\ \dfrac{|t|\cos(\mu\sqrt{t^2-\delta^2})}{\sqrt{t^2-\delta^2}} & \text{if } |t| > \delta. \end{cases} \tag{9.4}$$

<u>Proof</u>. If $a = 0$ then

$$P_\mu\chi_0(\theta,s) = \begin{cases} \dfrac{2\sinh(\mu\sqrt{\delta^2-s^2})}{\mu} & \text{if } |s| \le \delta \\ 0 & \text{if } |s| > \delta \end{cases} \tag{9.5}$$

and is clearly in $L_1^p(\mathbb{R}^1)$ if $p < 2$. Hence, by Remark 7.2, $(1/4\pi)\Lambda_\mu P_\mu \chi_0$ is just the convolution kernel corresponding to the point spread function χ_0. A calculation identical to that done in Example 8.1 gives (9.3) when $a = 0$. In general $\chi_a = \tau_a\chi_0$, and by (9.2) we get the desired result.

There is another way for getting (9.3) which involves complex variable methods and the Hilbert transform. This way is sketched briefly in the additional remark at the end of this chapter.

Let χ_a be as in Lemma 9.1. The result that relates the convolution step $P_\mu\chi_a * K$ with $(1/4\pi)\Lambda_\mu P_\mu\chi_a$ is a direct consequence of the following theorem, and is given in Remark 9.3.

<u>Theorem 9.2</u>. Let e be an even function which is piecewise C^1 on $\mathbb{R}$ and is continuous at 0. For $x \in \mathbb{R}^2$, let $E(x) = e(|x|)$ and let k be given by (5.3) with $n = 2$. If $M \geq \max\{|s-\delta-|a||, s+\delta+|a|\}$ then

$$(k*P_\mu\chi_a)(\theta,s)$$

$$= e^{\mu\langle a,\theta\rangle} \int_0^M [\cos \mu(s-\langle a,\theta^\perp\rangle) - L(r,s-\langle a,\theta^\perp\rangle)]re(r)\, dr \qquad (9.6)$$

where

$$L(r,s) = (2\pi)^{-1} \int_{S^1} h(s-r\langle\phi,\theta\rangle)e^{\mu r\langle\phi,\theta\rangle}\, d\phi$$

and h is as in (9.4).

<u>Proof</u>. It suffices to show (9.6) for $a = 0$, as the general case follows from (9.1). So let $a = 0$. For fixed s, the convolution $(k*P_\mu\chi_0)(\theta,s)$ depends only on the values of k on $[s-\delta,s+\delta]$ and hence on its values on $[-M,M]$. Let

$$E_M(x) = \begin{cases} E(x) & \text{if} \quad |x| \leq M \\ 0 & \text{if} \quad |x| > M \end{cases}$$

and let k_M be the corresponding convolution kernel given by Remark (7.2). Clearly $k_M(t) = k(t)$ if $|t| < M$. Since e is piecewise C^1 then $P_\mu E_M(\theta,\cdot) \in L_1^p(\mathbb{R})$ if $p < 2$. Hence

$$(k*P_\mu\chi_0)(\theta,s) = (k_M*P_\mu\chi_0)(\theta,s)$$

$$= \frac{1}{4\pi}(\Lambda_\mu P_\mu E_M*P_\mu\chi_0)(\theta,s)$$

$$= \frac{1}{4\pi}(P_\mu E_M*\Lambda_\mu P_\mu\chi_0)(\theta,s)$$

Fubini's theorem and the polar coordinates formula in $\mathbb{R}^2$ give for any g

$$(P_\mu E_M*g)(\theta,s) = \int_0^M re(r)\int_{S^1} g(\theta,s-r\langle\phi,\theta^\perp\rangle)e^{\mu r\langle\phi,\theta\rangle}\,d\phi dr. \tag{9.7}$$

With $g = (1/4\pi)\Lambda_\mu P_\mu\chi_0$, and $t = \langle\phi,\theta\rangle$ we have from Theorem 9.1

$$\int_{S^1}\cos\mu(s-r\langle\phi,\theta^\perp\rangle)e^{\mu r\langle\phi,\theta\rangle}\,d\phi$$

$$= \cos\mu s\int_{S^1}\cos(\mu r\langle\phi,\theta^\perp\rangle)e^{\mu r\langle\phi,\theta\rangle}\,d\phi$$

$$= 4\cos\mu s\int_0^1\cos(\mu rt)\,\frac{\cosh(\mu r\sqrt{1-t^2})}{\sqrt{1-t^2}}\,dt$$

$$= 2\pi\cos\mu s,$$

where the last equality follows from p. 38 of [8]. Substituting back in (9.7) and using (9.3) we get (9.6) for $a = 0$, and so we are done.

Remark 9.3. If $M \leq \infty$ is chosen in (9.6) so that

$$2\pi \int_0^M re(r)dr = \int_{|x|\leq M} E(x)dx = 1$$

then using (9.3) we can write (9.6) as

$$(k*P_\mu\chi_a)(\theta,s) = \frac{1}{4\pi}\Lambda_\mu P_\mu\chi_a(\theta,s)$$

$$+ \frac{e^{\mu<a,\theta>}}{2\pi}\int_0^M [h(s-<a,\theta^\perp>) - L(r,s-<a,\theta^\perp>)]re(r)\,dr.$$

where the second term on the right hand side of the equality can be used to estimate the difference between $k * P_\mu\chi_a$ and $(1/4\pi)\Lambda_\mu P_\mu\chi_a$.

Remark 9.4. In the special case when $s - <a,\theta^\perp> = 0$, $L(r,0) = 0$ if $|r| < \delta$ and $L(r,0) = 2\pi$ if $|r| > \delta$. With M chosen as in Remark 9.3

$$\frac{1}{4\pi}\Lambda_\mu P_\mu\chi_a(\theta,s) - (k*P_\mu\chi_a)(\theta,s) = e^{\mu<a,\theta>}\int_\delta^M re(r)\,dr$$

$$= \frac{e^{\mu<a,\theta>}}{2\pi}\int_{\delta\leq|x|\leq M} E(x)\,dx$$

$$= \frac{e^{\mu<a,\theta>}}{2\pi}[1 - \int_{|x|\leq\delta} E(x)\,ds].$$

From the above results we see that, for a good choice of E, $k * P_\mu\chi_a$ should be a good approximation of $(1/4\pi)\Lambda_\mu P_\mu\chi_a$ and hence of the right hand side of (9.3). Remark 9.3 gives an exact formula

for the difference and allows one to check for errors in numerical calculations of the convolution kernel and estimate errors in the convolution computations.

As an immediate corollary to Lemma 4.1 and the inversion formula (7.2) we get the following which can be used to check the backprojection step in the algorithm.

Corollary 9.5. Let $g(\theta,s)$ be given by the right hand side of (9.3). Then

$$P^*_{-\mu} g(x) = \chi_a(x).$$

Additional Remark. As mentioned earlier Lemma 9.1 can be proved using complex variable methods and the Hilbert transform. We give a brief description of this method here. Again it suffices to consider the case $a = 0$. For simplicity let $\delta = 1$. By (9.5) we have

$$P_\mu \chi_0(\theta,s) = \begin{cases} \dfrac{2 \sinh(\mu\sqrt{1-s^2})}{\mu} & \text{if} \quad |s| \le 1 \\ 0 & \text{if} \quad |s| > 1 \end{cases}$$

Proceeding as in Remark 6.7 one has

$$\Lambda_\mu P_\mu \chi_0(\theta,s) = \frac{2}{\mu} \frac{d}{ds} [\cos(\mu s)(Hg_1)(s) + \sin(\mu s)(Hg_2)(s)]$$

where

$$g_1(t) = \begin{cases} \cos(\mu t)\sinh(\mu\sqrt{1-t^2}) & \text{if} \quad |t| < 1 \\ 0 & \text{if} \quad |t| > 1, \end{cases}$$

$$g_2(t) = \begin{cases} \sin(\mu t)\sinh(\mu\sqrt{1-t^2}) & \text{if} \quad |t| < 1 \\ 0 & \text{if} \quad |t| > 1 \end{cases}$$

and

$$Hg = H_{1,0} * g$$

is the Hilbert transform of g with $H_{1,0}$ as in (6.1).

A theorem in complex variables gives a method to evaluate the Hilbert transform. The following is a statement of this theorem; see for instance Theorem 93, p. 125 of [23].

<u>Theorem</u>. Let F be analytic on $\operatorname{Im} z > 0$ and

$$||F_y||_2 \leq M$$

where $F_y(x) = F(x+iy)$, $x,y \in \mathbb{R}$. Then if g is the real boundary function of F on the real axis then Hg is the imaginary boundary function of F.

To evaluate Hg_1 we pick F_1 such that

$$F_1(z) = \sinh \mu(\sqrt{1-z^2} + iz)$$

and to evaluate Hg_2 we pick F_2 such that

$$F_2(z) = -i[\cosh(\mu\sqrt{1-z^2} + i\mu z)].$$

F_1 and F_2 satisfy the hypothesis of the above theorem. The real boundary function of F_1 on the real axis is g_1 and the imaginary boundary function of F_1 is

$$h_1(t) = \begin{cases} \sin(\mu t + \mu\sqrt{t^2-1}) & \text{if} \quad t < -1 \\ \sin(\mu t)\cosh(\mu\sqrt{1-t^2}) & \text{if} \quad |t| < 1 \\ \sin(\mu t - \mu\sqrt{t^2-1}) & \text{if} \quad t > 1 \end{cases}$$

The real boundary function of F_2 on the real axis is g_2 and the imaginary boundary function is

$$h_2(t) = \begin{cases} 1 - \cos(\mu t + \mu\sqrt{t^2-1}) & \text{if} \quad t < -1 \\ 1 - \cos(\mu t)\cosh(\mu\sqrt{1-t^2}) & \text{if} \quad |t| < 1 \\ 1 - \cos(\mu t - \mu\sqrt{t^2-1}) & \text{if} \quad t > 1. \end{cases}$$

It is worth noting here that in the definitions of F_1 and F_2 we use the analytic branch of the square root defined off of the negative real axis. The verifications of the estimates and indicated boundary functions are omitted.

BIBLIOGRAPHY

1. M. Abramowitz, I. Stegun, eds., Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, National Bureau of Standards, 1964.

2. T.F. Budinger, G.T. Gullberg and R.H. Huesmann, Emission Computed Tomography. In G.T. Herman (ed.): Image Reconstruction from Projections:Implementation and Applications. Topics in Applied Physics 32, Springer, New York 1979, 147-246.

3. L.K. Chen, On a Singular Integral, Studia Mathematica, to appear.

4. A. Cormack, Representation of a Function by its Line Integrals, with Some Radiological Applications I, J. Appl. Physics 34 (1963), 2722-2727.

5. A. Cormack, Representation of a Function by its Line Integrals, with Some Radiological Applications II, J. Appl. Physics 35 (1965), 2908-2913.

6. A. Cormack, Computed Tomography: Some History and Recent Developments in Computed Tomography. In: L.A. Shepp (ed.), Computed Tomography, Proc. Symp. Appl. Math. 27, AMS, Providence, R.I. 1982. 35-42.

7. D. Finch, A. Hertle, The Exponential X-ray Transform, Proceedings Summer Research Conference on Integral Geometry (1984) to appear.

8. I. Gradshteyn, I. Ryzhik, Table of Integrals, Series and Products, Academic Press, New York, 1980.

9. I. Hazou, D. Solmon, Approximate Inversion Formulas and Convolution Kernels for the Exponential X-ray Transform, Zeit. Ang. Math and Mech. 66 (1986), 370-372.

10. U. Heike, Die Inversion der Gedämpften Radontransformation, ein Rekonstruktionsverfahren der Emissiontomographie, Ph.D. Dissertation, University of Münster (1984).

11. J. Kohn, L. Nirenberg, An Algebra of Pseudo-differential Operators, Communications on Pure and Applied Mathematics 18, (1965), 269-305.

12. W.R. Madych, S.A. Nelson, Stable Inversion Formulas in Computed Tomography. SIAM J. Math. Anal. to appear.

13. W. Magnus, F. Obberhettinger, R. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, 3rd enl. ed., Springer-Verlag, New York, 1966.

14. A. Markoe, Fourier Inversion of the Attenuated X-ray Transform, SIAM J. Math. Anal. 15-4 (1984), 718-722.

15. F. Natterer, Computerized Tomography with Unknown Sources. SIAM J. Appl. Math. 43-5 (1983), 1201-1210.

16. F. Oberhettinger, Tabellen zur Fouriertransformation, Springer-Verlag, Berlin, 1957.

17. J. Radon, Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewissen Mannigfaltigkeiten, Ber. Verh. Sachs, Akad. Wiss., Leipzig, Math.-Nat. Kl. 69 (1917), 262-277.

18. W. Rudin, Functional Analysis, McGraw-Hill, New York, 1973.

19. M. Schechter, Principles of Functional Analysis, Academic Press, New York, 1971.

20. L.A. Shepp, B.F. Logan, The Fourier Reconstruction of a Head Section, IEEE Trans. Nucl. Sci. NS-21 (1974), 21-43.

21. K.T. Smith, Reconstruction Formulas in Computed Tomography, In: L.A. Shepp (ed.), Computed Tomography, Proc. Symp. Appl. Math. 27, AMS, Providence, R.I. (1983), 7-23.

22. E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970.

23. E.C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford Press, London, 1937.

24. O. Tretiak, C. Metz, The Exponential Radon Transform, SIAM J. Appl. Math. 39-2 (1980), 341-354.

25. H. Triebel, Interpolation Theory, Function Spaces, Differential Operators. North-Holland Pub. Co., Amsterdam, New York, 1978.

26. K. Yosida, Lectures on Differential and Integral Equations, Interscience Publishers, New York, 1960.