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We generalize overpartition rank and crank generating functions to obtain k-fold variants, and give a combinatorial interpretation for each. The k-fold crank generating function is interpreted by extending the first and second residual cranks to a natural infinite family. The k-fold rank generating functions generate two families of *buffered Frobenius representations*, which generalize the first and second Frobenius representations studied by Lovejoy.

Overpartition Ranks, Cranks, and Frobenius Representations

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Dean of the Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Thomas Morrill, Author

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OVERPARTITION RANKS, CRANKS, AND FROBENIUS REPRESENTATIONS

1 INTRODUCTION

Given a set $A \subseteq \mathbb{Z}$, how many ways may any integer n be represented as a sum of integers in A? This is the formative question of additive number theory. Even Fermat's Last Theorem may be cast in this light: *If $k \geq 3$ and $C \geq 0$, how many ways may C^k be written as the sum of two perfect kth powers?* The answer: *Exactly one, $C^k = C^k + 0^k$.*

This question leads to the formulation of partitions. A *partition* λ of n is a non-increasing sequence of positive integers $(\ell_1, \ell_2, \ldots, \ell_k)$ which sum to n. We call each ℓ_i a *part* of λ. For this dissertation, we will use the generalized notion of *overpartitions*. An overpartition of n is a non-increasing sequence of positive integers which sum to n, in which the first occurrence of any part may be overlined. Here, an overlined integer is distinguished from a non-overlined integer.

For example, the overpartitions of 4 are

$$\begin{array}{ccccc} (4), & (\overline{4}), & (3,1), & (3,\overline{1}), & (\overline{3},1), \\ (\overline{3},\overline{1}), & (2,2), & (\overline{2},2), & (2,1,1), & (2,\overline{1},1), \\ (\overline{2},1,1), & (\overline{2},\overline{1},1), & (1,1,1,1), & (\overline{1},1,1,1), & \end{array}$$

and of these, only

$$(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad (1,1,1,1),$$

are ordinary partitions. If we let $p(n)$ and $\overline{p}(n)$ denote the number of partitions and overpartitions of n, respectively, then we see that $p(4) = 5$ and $\overline{p}(n) = 14$. We write $|\lambda|$ to denote the sum of parts of a partition or overpartition. For all of the above, $|\lambda| = 4$.

A powerful tool in combinatorics is the *generating series*, which is a power series

$$\sum_{n \geq 0} a(n) q^n$$

in which the coefficient function $a(n)$ is of combinatorial interest. In this setting[1], $a(n)$ will relate to partitions or overpartitions with $|\lambda| = n$. For example, Euler proved [2] that the generating series for $p(n)$ may be expressed as an infinite product,

$$\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n}. \tag{1.0.1}$$

Additionally, Euler observed [2] that both sides of the equation

$$\prod_{i=1}^{\infty}(1+q^i) = \prod_{i=1}^{\infty} \frac{(1-q^{2i})}{(1-q^i)} = \prod_{i=1}^{\infty} \frac{1}{(1-q^{2i-1})} \tag{1.0.2}$$

can be interpreted as the generating series for certain restricted partitions. The combinatoric interpretation of (1.0.2) implies that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Manipulating a generating series may be performed either in a ring of formal power series or in an analytic setting. For example, the infinite products in (1.0.1) and (1.0.2) converge absolutely when $|q| < 1$. If we take $q = e^{2\pi i\tau}$, then (1.0.1) and (1.0.2) define analytic functions for $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. Most of the q-series presented in this dissertation will converge to analytic functions of τ under these conditions. In this light, the combinatorics of partitions and overpartitions have a deep connection to analytic number theory, particularly q-hypergeometric series, elliptic functions, eta-quotients, and modular forms. However, we will work in terms of formal power series.

In this dissertation, we are focused on generating series for the *rank* and *crank* functions of overpartitions. Dyson [10] developed the rank function for ordinary partitions, in order to give a combinatorial proof of the Ramanujan congruences,

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.0.3}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{1.0.4}$$

[1]In the study of partitions, it is conventional to use q as the variable of a power series. Such series are loosely referred to as *q-series*.

$$p(11n+6) \equiv 0 \pmod{11}, \tag{1.0.5}$$

which were originally proven using modular forms and elliptic functions [18]. By analyzing the generating series of the rank function, Atkin and Swinnerton-Dyer [5] gave a combinatoric explanation for (1.0.3) and (1.0.4).

Dyson [10] was aware that the rank does not provide a combinatoric explanation for (1.0.5), and conjectured the existence of another function which would fill the gap. He stated necessary conditions on the unknown function and named it the *crank* of a partition. Andrews and Garvan [4] later developed the crank function according to Dyson's specifications, and showed that the crank gives a combinatoric explanation for all three Ramanujan congruences.

Because overpartitions generalize partitions, it is natural to ask if overpartition ranks and cranks can be defined in a meaningful way. Jeremy Lovejoy [15] [16] developed the Dyson rank and M_2-rank for overpartitions by adapting Dyson's rank of partitions and Berkovich and Garvan's [6] M_2-rank on partitions whose odd parts may not repeat. We use $\overline{N}(m,n)$ and $\overline{N2}(m,n)$ to denote the number of overpartitions of n with Dyson rank m, or M_2-rank m, respectively.

Bringmann, Lovejoy, and Osburn [7] developed the *first residual crank* and *second residual crank* of overpartitions by adapting Andrews and Garvan's crank of partitions. We use $\overline{M}(m,n)$ and $\overline{M2}(m,n)$ to denote the number of overpartitions of n with first residual crank m, or second residual crank m, respectively.

Our motivating question comes from studying the positive moments of these rank and crank functions. For $\ell \geq 1$, the *ℓth positive moments* of the rank functions are defined to be

$$\overline{N}_\ell^+(n) := \sum_{m\geq 0} m^\ell \overline{N}(m,n) \tag{1.0.6}$$

$$\overline{N2}_\ell^+(n) := \sum_{m\geq 0} m^\ell \overline{N2}(m,n), \tag{1.0.7}$$

and the ℓth positive moments of the crank functions are defined to be

$$\overline{M}_\ell^+(n) := \sum_{m\geq 0} m^\ell \overline{M}(m,n) \tag{1.0.8}$$

$$\overline{M2}_\ell^+(n) := \sum_{m\geq 0} m^\ell \overline{M2}(m,n). \tag{1.0.9}$$

By manipulating the generating series for (1.0.6), (1.0.7), (1.0.8), (1.0.9), Larsen, Rust, and Swisher [14] proved that the ℓth positive moments fall into a nesting inequality,

$$\overline{N2}_\ell^+(n) < \overline{M2}_\ell^+(n) < \overline{N}_\ell^+(n).$$

Larsen, Rust, and Swisher also gave closed form expansions for the generating series for the positive rank moments $\overline{N}_\ell^+(n)$ and $\overline{N2}_\ell^+(n)$

$$\sum_{n\geq 0} \overline{N}_\ell^+(n) q^n = 2\left(\prod_{i\geq 1} \frac{1+q^i}{1-q^i}\right) \sum_{n\geq 1} (-1)^{n+1} \frac{q^{n^2+n} A_\ell(q^n)}{(1+q^n)(1-q^n)^\ell} \tag{1.0.10}$$

$$\sum_{n\geq 0} \overline{N2}_\ell^+(n) q^n = 2\left(\prod_{i\geq 1} \frac{1+q^i}{1-q^i}\right) \sum_{n\geq 1} (-1)^{n+1} \frac{q^{n^2+2n} A_\ell(q^{2n})}{(1+q^{2n})(1-q^{2n})^\ell}, \tag{1.0.11}$$

and closed form expansions for the generating series for the positive crank moments $\overline{M}_\ell^+(n)$ and $\overline{M2}_\ell^+(n)$

$$\sum_{n\geq 0} \overline{M}_\ell^+(n) q^n = \left(\prod_{i\geq 1} \frac{1+q^i}{1-q^i}\right) \sum_{n\geq 1} (-1)^{n+1} \frac{q^{\frac{n^2+n}{2}} A_\ell(q^n)}{(1-q^n)^\ell} \tag{1.0.12}$$

$$\sum_{n\geq 0} \overline{M2}_\ell^+(n) q^n = \left(\prod_{i\geq 1} \frac{1+q^i}{1-q^i}\right) \sum_{n\geq 1} (-1)^{n+1} \frac{q^{2\frac{n^2+n}{2}} A_\ell(q^{2n})}{(1-q^{2n})^\ell}. \tag{1.0.13}$$

Here, $A_\ell(q)$ denotes the ℓth Eulerian polynomial.

During a presentation by Swisher, Dr. Long of Louisiana State University observed that the generating series for $\overline{N}_\ell^+(n)$ and $\overline{N2}_\ell^+(n)$ closely resemble one another, as do the generating series for $\overline{M2}_\ell^+(n)$ and $\overline{M2}_\ell^+(n)$. She then raised the question of whether there exist an overpartition k-rank and a k-crank for $k \geq 3$ so that

$$2\left(\prod_{i\geq 1}\frac{1+q^i}{1-q^i}\right)\sum_{n\geq 1}(-1)^{n+1}\frac{q^{n^2+kn}A_\ell(q^{kn})}{(1+q^{kn})(1-q^{kn})^\ell} \tag{1.0.14}$$

is the generating series for the ℓth positive moments of the k-rank and

$$\left(\prod_{i\geq 1}\frac{1+q^i}{1-q^i}\right)\sum_{n\geq 1}(-1)^{n+1}\frac{q^{k\frac{n^2+n}{2}}A_\ell(q^{kn})}{(1-q^{kn})^\ell} \tag{1.0.15}$$

is the generating series for the ℓth positive moments of the k-crank. In this dissertation, we answer Long's question *yes*.

In the rank case, $\overline{R[k]}_\ell^+(q)$ is the generating series for the ℓth positive moments of the *full ranks* of *buffered Frobenius representations* of overpartitions, our generalization of two Frobenius representations of overpartitions defined by Lovejoy [15] [16]. Briefly, a buffered Frobenius representation is an array

$$\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_k\\ \beta_1 & \beta_2 & \dots & \beta_k\end{pmatrix},$$

where the entries α_i and β_i are partitions or overpartitions. These new objects naturally lie over the set of overpartitions, although the correspondence is not one-to-one. We dedicate Chapter 5 to profuse details regarding the nature of buffered Frobenius representations.

Let $\mathcal{B}_1^k$ denote the set of buffered Frobenius representations of the first kind with at most k columns. We define

$$\overline{N[k]}(m,n) := \sum_{\substack{\nu\in\mathcal{B}_1^k\\ |\nu|=n\\ \rho_1(\nu)=km}} (-1)^{h(\nu)}\prod_{i=1}^{k}\zeta_k^{(i-1)\rho_1^i(\nu)}, \tag{1.0.16}$$

where ζ_k denotes a primitive kth root of unity and $\rho_1(\nu)$ denotes the full rank of ν. The functions $h(\nu)$, $\rho_1^i(\nu)$, and the full rank are presented in Chapter 5. Despite the roots of unity in (1.0.16), the function $\overline{N[k]}(m, n)$ takes integer values for all m and n. In fact, if $k = 1$, then $\overline{N}(m, n) = \overline{N[1]}(m, n)$.

We also generalize $\overline{N2}(m, n)$. Let $\mathcal{B}_2^k$ denote the set of buffered Frobenius representations of the second kind with at most k columns, and define

$$\overline{N2[k]}(m, n) = \sum_{\substack{\nu \in \mathcal{B}_2^k \\ |\nu| = n \\ \rho_2(\nu) = km}} (-1)^{h(\nu)} \prod_{i=1}^{k} \zeta_k^{(i-1)\rho_2^i(\nu)},$$

where the functions ρ_2^i are again defined in Chapter 5. We will see that $\overline{N2[k]}(m, n)$ takes integer values and $\overline{N2}(m, n) = \overline{N2[1]}(m, n)$. We state our results for these families as a single theorem

Theorem 1.0.1 (Morrill [17])**.** *For all $k \geq 1$ we have*

$$\sum_{n \geq 0} \left(\sum_{m \geq 1} m^\ell \overline{N[k]}(m, n) \right) q^n = 2 \left(\prod_{i \geq 1} \frac{1 + q^i}{1 - q^i} \right) \sum_{n \geq 1} (-1)^{n+1} \frac{q^{n^2 + kn} A_\ell(q^{kn})}{(1 + q^{kn})(1 - q^{kn})^\ell}$$
$$\sum_{n \geq 0} \left(\sum_{m \geq 1} m^\ell \overline{N2[k]}(m, n) \right) q^n = 2 \left(\prod_{i \geq 1} \frac{1 + q^i}{1 - q^i} \right) \sum_{n \geq 1} (-1)^{n+1} \frac{q^{n^2 + 2kn} A_\ell(q^{2kn})}{(1 + q^{2kn})(1 - q^{2kn})^\ell}.$$

That is, the generating series for the ℓth positive moments of $\overline{N[k]}(m, n)$ and $\overline{N2[k]}(m, n)$ generalize (1.0.6) and (1.0.7) as desired. In order to generalize (1.0.8) and (1.0.9), we introduce the *kth residual crank* in Chapter 3. This produces the following theorem.

Theorem 1.0.2 (Morrill)**.** *Let $\overline{M[k]}(m, n)$ denote the number of overpartitions with kth residual crank m and weight n. Then,*

$$\sum_{n \geq 0} \left(\sum_{m \geq 1} m^\ell \overline{M[k]}(m, n) \right) q^n = \left(\prod_{i \geq 1} \frac{1 + q^i}{1 - q^i} \right) \sum_{n \geq 1} (-1)^{n+1} \frac{q^{k\frac{n^2+n}{2}} A_\ell(q^{kn})}{(1 - q^{kn})^\ell}.$$

The remainder of this dissertation is as follows. In Chapter 2, we give an overview of ordinary partitions and their Frobenius representations. In Chapter 3, we give a similar overview of overpartitions and their Frobenius representations, develop the kth residual crank, and prove Theorem 1.0.2. In Chapter 4, we introduce q-hypergeometric series and prove two k-fold q-series transformations. In Chapter 5, we develop our major contribution, the buffered Frobenius representations of overpartitions, and prove Theorem 1.0.1. The proof rests on results from each of the preceding chapters. In Chapter 6, we conclude by discussing where our main result may be improved and give areas for future investigation.

2 PARTITIONS

Let n be a nonnegative integer. A *partition* of n is a non-increasing sequence of integers $\lambda = (\ell_1, \ell_2, \ldots, \ell_k)$ such that

$$\sum_{i=1}^{k} \ell_i = n. \tag{2.0.1}$$

The empty partition $\lambda = \emptyset$ is defined to be the unique partition of 0. We abbreviate (2.0.1) by writing $|\lambda| = n$, and we call n the *weight* of λ. Each ℓ_i is called a *part* of the partition λ. For example, the partitions of 4 are given by

$$(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad (1,1,1,1).$$

Sometimes it is useful to weaken the definition of partitions. A partition of n *into nonnegative parts* is a partition λ in which 0 may occur as a part. This allows us flexibility in counting the number of parts of a partition. For example, both of $(86, 75, 30, 9)$ and $(182, 36, 9, 0)$ are partitions into four nonnegative parts. We will use partitions into nonnegative parts in and throughout Chapters 2, 3, and 5.

Weakening the definition of partitions comes at a cost. The number of partitions of n into nonnegative parts always infinite, as

$$n = n + 0 = n + 0 + 0 = \ldots.$$

Thus, the reader may be concerned that our combinatoric proofs are invalid. We will always fix the number of parts in a partition before counting partitions which have nonnegative parts. For example, there are only two partitions of $n = 2$ into exactly four nonnegative parts, $(2, 0, 0, 0)$ and $(1, 1, 0, 0)$. Any references we make to partitions that are not clear in context should be taken to mean partitions into positive parts.

We draw heavily from Andrews' book on partitions [2]. Any uncited results may be found there.

2.1 Young Tableaux and Conjugation of Partitions

Although we defined a partition λ to be a sequence $(\ell_1, \ell_2, \dots, \ell_k)$, we may also represent λ graphically as an array of boxes. The *Young tableau* of a partition $\lambda = (\ell_1, \ell_2, \dots, \ell_n)$ is a left aligned array in which the kth row of the array consists of ℓ_k boxes. If λ is a partition into nonnegative parts, we use a vertical line rather than a box to represent a row of 0 boxes in the Young tableau of λ. For example, the Young tableau of $(4, 4, 2, 1)$ is given in Figure 2.1.

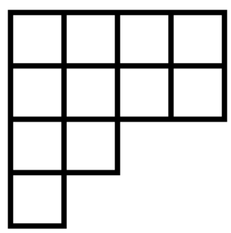

FIGURE 2.1: The Young tableau of $(4, 4, 2, 1)$.

Certain transformations of partitions are more easily described in terms of their Young tableaux than their sequence form. The *conjugate* of a partition λ is obtained by mirroring the Young tableaux of λ across its NW-SE diagonal. For example, the conjugate of $\lambda = (4, 4, 2, 1)$ is shown in Figure 2.2. If we denote the conjugate of λ by λ', then we see that λ' is a partition such that $|\lambda'| = |\lambda|$. It easy to verify that the conjugate of a partition is unique and that conjugating a second time yields $(\lambda')' = \lambda$.

Defining conjugation in terms of the parts of λ is less obvious. If $\lambda = (\ell_1, \ell_2, \dots, \ell_k)$, then the parts of $\lambda' = (m_1, m_2, \dots, m_{\ell_1})$ are given by

$$m_i = \#\{1 \leq j \leq k : \ell_j \geq i\}. \tag{2.1.1}$$

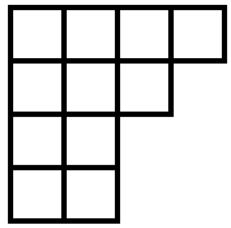

FIGURE 2.2: The Young tableaux of $(4, 3, 2, 2)$.

2.2 Frobenius Representation

According to Andrews [3], Frobenius was interested in a way to represent partitions which would illuminate when two partitions are conjugates. In terms of Young tableaux, conjugating a partition λ fixes the boxes along the NW-SE diagonal of the corresponding array. Thus, conjugation exchanges the rows to the right of the NW-SE diagonal with the columns below the NW-SE diagonal.

Let $\lambda = (\ell_1, \ell_2, \ldots, \ell_k)$ be a partition, and let n be the length NW-SE diagonal in the Young tableau of λ. The *Frobenius representation of* λ is an array

$$\nu = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}$$

in which a_i is the number of boxes in the ith row of the tableau to the right of the NW-SE diagonal and b_i is the number of boxes in the ith column of the tableau below the NW-SE diagonal for all $1 \leq i \leq n$.

For example, let $\lambda = (4, 4, 2, 1)$, as in Figure 2.1. The Frobenius representation of λ is given by

$$\nu = \begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix}.$$

A graphical representation of the Frobenius representation is shown in Figure 2.3.

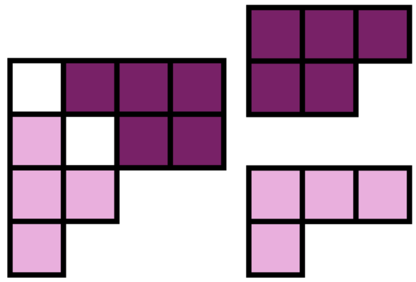

FIGURE 2.3: The Frobenius representation of $(4,4,2,1)$ visualized in terms of its Young tableau.

Let λ be a partition with Frobenius representation

$$\nu = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}.$$

It is easy to verify that the conjugate of λ has Frobenius representation

$$\nu' = \begin{pmatrix} b_1 & b_2 & \dots & b_n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}.$$

In terms of its Young tableau ν, the weight of λ is given by $|\lambda| = n + \sum a_i + b_i$.

Note that $\{a_i\}$ and $\{b_i\}$ must be non-increasing sequences of nonnegative integers. Thus, we may define a generic *Frobenius representation* to be an array

$$\nu = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$$

where $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ are partitions into distinct nonnegative parts. We also define the weight ν to be $|\nu| = n + \sum a_i + b_i$. It is easy to verify that these Frobenius representations ν are in one-to-one correspondence with partitions λ, with $|\nu| = |\lambda|$. This concept is the foundation of the notion of generalized Frobenius representations in Chapter 3 and buffered Frobenius partitions in Chapter 5.

2.3 q-Pochhammer Symbols

The *q-Pochhammer symbol* is a convenient shorthand which frequently appears in the theory of partitions. For $a \in \mathbb{C}$, we define

$$(a;q)_n := \prod_{i=0}^{n-1}(1-aq^i), \tag{2.3.1}$$

$$(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i), \tag{2.3.2}$$

$$(a_1,a_2,\ldots,a_k;q)_n := (a_1;q)_n(a_2;q)_n\ldots(a_k;q)_n, \tag{2.3.3}$$

$$(a_1,a_2,\ldots,a_k;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\ldots(a_k;q)_\infty. \tag{2.3.4}$$

We will use the q-Pochhammer symbols extensively throughout this dissertation. Even the generating series for $p(n)$ can be written in terms of q-Pochhammer symbols, which we see below.

Proposition 2.3.1 (Euler [2])**.** *Let $p(n)$ denote the number of partitions of n. Then,*

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{2.3.5}$$

Proof. We seek to count all a partitions λ with weight n. Expanding each term of the product (2.3.5) using a geometric series yields

$$\begin{aligned}\frac{1}{(q;q)_\infty} = \prod_{i\geq 1}\frac{1}{1-q^i} &= \prod_{i\geq 1}(1+q^i+q^{i+i}+\ldots)\\ &= q^0+q^1+(q^2+q^{1+1})+(q^3+q^{2+1}+q^{1+1+1})+\ldots.\end{aligned}$$

Recall that the empty partition $\emptyset$ is defined to be the unique partition of $n=0$. We see that the occurrences of q^n correspond to each of the partitions of n, which proves the claim. □

Unlike Andrews [2], we will exclusively work in formal power series, rather than give convergence conditions. For example, the generating series in (2.3.5) is defined in $\mathbb{Z}[[q]]$, the ring of formal power series in q with integer coefficients. In Chapter 2 and Chapter 3, we will work in $\mathbb{Z}((z))[[q]]$, the ring of formal power series in q with coefficients taken from the ring of formal Laurent series in z. This allows the coefficient of $z^m q^n$ to count the number of partitions $|\lambda| = n$ with $r(\lambda) = m$, which we will see in Proposition 2.5.1. In Chapter 4 and Chapter 5 we will also work in $\mathbb{Z}((x_1, x_2, \ldots, x_k))[[q]]$.

In the proof of Proposition 2.3.1, it is common to say that

$$\frac{1}{1-q^i} = 1 + q^i + q^{i+i} + \ldots \tag{2.3.6}$$

generates the parts of size i in partitions λ. This means that the coefficient of q^{ki} is equal to the number of ways to select allow the part i appear exactly k times in a generic partition λ (here, the number of ways is 1). When we multiply (2.3.6) by another q-series, calculating the resulting coefficients is equivalent to enumerating pairs of partitions in the Cartesian product of two or more sets of partitions.

We could also say that (2.3.6) generates the rows of size i in the Young tableau of a partition. Equivalently, we could have structured the proof so that (2.3.6) generates the columns of the Young tableau of a partition instead. The row and column interpretations show us that the coefficient of q^n in

$$\frac{1}{(q;q)_k} = \prod_{i=1}^{k} (1 + q^i + q^{i+i} + \ldots) \tag{2.3.7}$$

is equal to the number partitions whose parts are k or less. The same coefficient of q^n in (2.3.7) is also equal to the number of partitions with k or fewer parts. Therefore, the number of number partitions whose parts are at most k is equal to the number of partitions with k or fewer parts[2].

[2]This fact may also be proven bijectively by considering the conjugates of such partitions.

We will always specify whether a q-Pochhammer symbol should be interpreted as counting the rows or columns of a Young tableau. Although counting rows is more natural, we will find that counting columns is more useful, particularly in finding generating series of overpartitions and buffered Frobenius representations in Chapters 3 and 5.

Manipulating the q-Pochhammer symbols often entails expanding the individual terms of the product and canceling, as seen in the following proposition.

Proposition 2.3.2. *For all positive integers m and n,*

$$\frac{(a;q)_m}{(aq;q)_{m+n}} = \frac{(1-a)}{(aq^m;q)_{n+1}}.$$

By expanding the q-Pochhammer symbols and canceling like terms, we have

$$\begin{aligned}\frac{(a;q)_m}{(aq;q)_{m+n}} &= \frac{(1-a)(1-aq)\dots(1-aq^{m-1})}{(1-aq)(1-aq^2)\dots(1-aq^{m-1})(1-aq^m)\dots(1-aq^{m+n})}\\ &= \frac{(1-a)}{(1-aq^m)\dots(1-aq^{m+n})} = \frac{(1-a)}{(aq^m;q)_{n+1}}.\end{aligned}$$

We will use Proposition 2.3.2 extensively throughout this dissertation.

2.4 Partition Statistics

Recall that P is the set of partitions. We use the term *partition statistic* loosely to refer to any integer valued function $f : P \to \mathbb{Z}$. We have already seen the weight function, $|\cdot| : P \to \mathbb{Z}$, which sends each partition λ to the sum of its parts, $|\lambda|$. Some other simple examples of partition statistics are the largest part function and the length function. If $\lambda = (\ell_1, \ell_2, \dots, \ell_k)$ is a partition then we define $\ell(\lambda) := \ell_1$ and $\#(\lambda) := k$.

Our main interests, the *rank* and *crank* functions, are defined in terms of other partition statistics. In 1944, Dyson [10] introduced the theory of partition ranks to give a combinatorial explanation for the famous Ramanujan congruences for $p(n)$.

Theorem 2.4.1 (Ramanujan [18]). *For all* $n \geq 0$*,*

$$p(5n+4) \equiv 0 \pmod 5, \tag{2.4.1}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{2.4.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{2.4.3}$$

Ramanujan's proof [18] of these congruences is based on transformations of the generating series of $p(n)$ using q-geometric series and elliptic functions. However, this method does not provide a combinatorial explanation for the congruences. The search to find such a combinatorial explanation drove the theory of partition statistics until the discovery of the crank function by Andrews and Garvan [4] in 1988.

The goal is an equivalence relation on the set of partitions such that the partitions of n fall into 5 equally sized equivalence classes whenever $n \equiv 4 \pmod 5$, and similarly for 7 and 11. Atkin and Swinnerton-Dyer [5] gave a partial solution by considering Dyson's rank of a partition modulo 5 and 7.

2.5 Rank of a Partition

Given a partition $\lambda = (\ell_1, \ell_2, \ldots, \ell_n)$, Dyson [10] defined the *rank* of λ to be the largest part of λ minus the number of parts in λ, which we write as

$$r(\lambda) := \ell(\lambda) - \#(\lambda).$$

As an example, the partitions of $n = 4$ are given with their ranks in Table 2.1. A generating series for the ranks of partitions is given in Proposition 2.5.1 below.

Note that each equivalence class of $\mathbb{Z}/5\mathbb{Z}$ occurs once as $r(\lambda)$ in Table 2.1. If we let $N(m, n, k)$ denote number of partitions of n with rank m modulo k, then Atkin and

TABLE 2.1: The ranks of partitions of 4.

λ	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
$r(\lambda)$	3	1	0	-1	-3

Swinnerton-Dyer [5] proved that

$$N(i, 5n+4, 5) = N(j, 5n+4, 5) \tag{2.5.1}$$

for all $n \geq 0$ and all $i,\ j \in \mathbb{Z}$. Consequently, the set of partitions of $5n+4$ can be separated into five classes of equal size, by taking their ranks modulo 5. With Atkin and Swinnderton-Dyer's result in place, the combinatorial explanation of (2.4.1) is that

$$p(5n+4) = \sum_{i=0}^{4} N(i, 5n+4, 5) = 5 \times N(0, 5n+4, 5).$$

Atkin and Swinnerton-Dyer [5] also proved that

$$N(i, 7n+5, 7) = N(j, 7n+5, 7)$$

for all i, j, which explains (2.4.2) in a similar manner.

However, counting ranks modulo 11 fails to divide the partitions of $11n+6$ into equal classes. We see Table 2.2 that $N(\pm 1, 6, 11) = 2$ but that $N(\pm 4, 6, 11) = 0$ and $N(i, 6, 11) = 1$ for all other residue classes $[i] \in \mathbb{Z}/11\mathbb{Z}$.

TABLE 2.2: The ranks of partitions of 6.

λ	(6)	$(5,1)$	$(4,2)$	$(4,1,1)$	$(3,3)$	$(3,2,1)$
$r(\lambda)$	5	3	2	1	1	0
λ	$(1,1,1,1,1,1)$	$(2,1,1,1,1)$	$(2,2,1,1)$	$(3,1,1,1)$	$(2,2,2)$	
$r(\lambda)$	-5	-3	-2	-1	-1	

Proposition 2.5.1. *The generating series for the ranks of partitions is given by*

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} N(m,n)z^m q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(zq, z^{-1}q;q)_n}.$$

The proof of Proposition 2.5.1 demonstrates how to treat Young tableaux as separate pieces, which fit together like a jigsaw puzzle. This method is crucial to our later proofs in Chapters 2, 3, and 5.

For example, let $\lambda = (\ell_1, \ell_2, \ldots, \ell_k)$ be a partition, and let n be the side length of the largest square that fits in top left corner of the Young tableau of λ. We call this square the *Durfee square* of λ. As seen in Figure 2.4, we can decompose the Young tableau of λ into its Durfee square (pink), and the Young tableaux for two other partitions (white).

The rows to the right of the Durfee square of λ form a partition α into exactly n nonnegative parts. In Figure 2.4, we have$\alpha = (2,2)$. We see that $\ell(\lambda) = \ell(\alpha) + n$. Similarly, the columns below the Durfee square of λ form a partition β. The parts of β must be read vertically, with the largest part on the left, unlike our usual treatment of Young tableaux. In Figure 2.4, we have $\beta = (2,0)$. Like α, we see that β can have at most n parts. we also note that $\#(\lambda) = \ell(\beta) + n$.

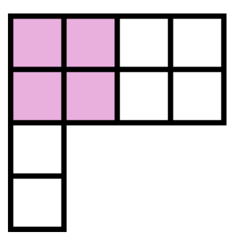

FIGURE 2.4: The Young tableau of $(4,4,1,1)$ with the Durfee square highlighted.

Proof of Proposition 2.5.1. When $n = 0$, the q-Pochhammer symbols take their trivial value, and the summand

$$\frac{q^0}{(zq, z^{-1}q;q)_0}$$

reduces to 1. This corresponds to the empty partition $\lambda = \emptyset$, for which both $|\lambda| = 0$ and $r(\lambda) = 0$. For $n \geq 1$, we seek to demonstrate that the coefficient of $z^m q^{n'}$

$$\frac{q^{n^2}}{(zq, z^{-1}q; q)_n} \tag{2.5.2}$$

is equal to the number of partitions $\lambda = (\ell_1, \ell_2, \ldots, \ell_k)$ with $|\lambda| = n'$ and $r(\lambda) = m$, such that the Durfee square of λ has side length of length n.

The term q^{n^2} generates the Durfee square. The term

$$\frac{1}{(zq; q)_n} = \prod_{i=1}^{n} \frac{1}{1 - zq^i}$$

generates partitions α into at most n parts as follows. Each factor

$$\frac{1}{1 - zq^i} = 1 + zq^i + z^2 q^{i+i} + \ldots$$

generates columns, rather than the rows, of the Young tableau of α, where i is the number of boxes in the column. Since $1 \leq i \leq n$, we see that α can have at most n parts. The exponent of z is increased by one for each column generated, regardless of size. Thus the exponent of z is equal to $\ell(\alpha)$.

By symmetry, the term

$$\frac{1}{(z^{-1}q; q)_n}$$

generates partitions β into at most n parts, where the coefficient of z is equal to $-\ell(\beta)$. When we multiply these terms to produce (2.5.2), we must add the exponents of z. Since $r(\lambda) = (\ell(\alpha) + n) - (\ell(\beta) + n) = \ell(\alpha) - \ell(\beta)$, the exponent of z is the rank of λ. By summing over all $n \geq 0$, we generate all partitions and their ranks. □

Corollary 2.5.1 (Corollary to Proposition 2.5.1)**.** *For all* $n \geq 0$*,*

$$N(m, n) = N(-m, n).$$

Corollary 2.5.1 follows by observing that mapping $z \mapsto z^{-1}$ reverse the sign of the ranks of partitions, but the generating series is unchanged. This fact reflected in Tables 2.1 and 2.2.

2.6 Crank of a Partition

Knowing that the rank does not give a combinatorial explanation for Congruence (2.4.3), Dyson [10] conjectured that another partition statistic would fill the gap. He named this statistic the *crank* of a partition ahead of its discovery, and gave necessary conditions on its distribution.

The crank was later established by Andrews and Garvan [4] and shown to satisfy Dyson's criteria. Given a partition $\lambda = (\ell_1, \ell_2, \ldots, \ell_k)$, let $w(\lambda)$ denote the number of occurrences of 1 in λ. Further, let λ_w be the subpartition of λ consisting of all parts of λ greater than $w(\lambda)$. Then the crank of λ is defined to be

$$cr(\lambda) := \begin{cases} \ell(\lambda), & \text{if } w(\lambda) = 0 \\ \#(\lambda_w) - w(\lambda), & \text{if } w(\lambda) > 0. \end{cases}$$

For example, the crank of $(4,4,1,1)$ is 0, because $w(\lambda) = 2$ and $\lambda_w = (4,4)$. Notably, the crank function gives a combinatorial interpretation not only for Congruence (2.4.3), but to all three Ramanujan congruences as in Equation 2.5.1.

Proposition 2.6.1 (Andrews-Garvan [4]). *Let $M(m,n,k)$ denote the number of partitions λ of n with $cr(\lambda) = m$. Then for all $i, j \in \mathbb{Z}$,*

$$M(i, 5n+5, 5) = M(j, 5n+5, 5),$$

$$M(i, 7n+5, 7) = M(j, 7n+5, 7),$$

$$M(i, 11n+5, 11) = M(j, 11n+5, 11).$$

The proof of Proposition 2.6.1 rests on Garvan's earlier work [11] in vector partitions[3]. We see a generating series for the crank of partitions below.

Proposition 2.6.2 (Andrews, Garvan [4]). *Let $M(m,n)$ denote the number of partitions of n with crank m. Then*

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} M(m,n)z^m q^n = \frac{(q;q)_n}{(zq,z^{-1}q;q)_n}. \tag{2.6.1}$$

By observing the symmetry between z and z^{-1} in Proposition 2.6.2, we see that for all $n \geq 0$ and all $m \in \mathbb{Z}$ we have $M(m,n) = M(-m,n)$ as was the case with the rank counting function $N(m,n)$.

Note that the generating series in (2.6.1) implicitly defines $M(-1,1) = 1$, which may be verified by hand, but also $M(1,1) = 1$ and $M(0,1) = -1$. The standard interpretation is to count the partition $\lambda = (1)$ multiple times; once as $\lambda_+ = (1)$ with $cr(\lambda_+) = 1$, once as $\lambda_- = (1)$ with $cr(\lambda_-) = -1$, and once as the "negative" partition $\lambda_0 = (1)$ with weight $|\lambda_0| = 1$ and crank $cr(\lambda_0) = 0$. Then there are two "positive" partitions of 1 and one "negative" partition of 1, for a total of one partition of 1. This agrees with the values of $M(m,1)$, and preserves the relation

$$\sum_{m\in\mathbb{Z}} M(m,n) = p(n), \tag{2.6.2}$$

Despite this technicality, Equation (2.6.1) has been accepted as the correct generating series to work with. It is the basis for the first and second residual cranks of overpartitions developed by [7] and for our kth residual cranks of overstrains in Section 3.7. We will see in Chapter 3 how the crank of $\lambda = (1)$ leads to a more complicated technicality in the overpartition case.

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[3] A *vector partition* of n is a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ in which each λ_i is a partition, and $|\lambda_1| + |\lambda_2| + \cdots + |\lambda_k| = n$. See Garvan [?] for more details.

2.7 Bracket of a Partition

Our work in Chapter 5 requires another partition statistic, which measures the right edge of a Young tableau. This statistic appears in Franklin's proof of Euler's Pentagonal Number Theorem [2], but is not given a name there. It is also similar to the initial run of a partition, which appears in work of Choi, Kang, and Lovejoy [8]. We take the opportunity to name it the *bracket* of a partition. Two variations of the bracket appear below, and two more are introduced in Chapter 3.

Definition 2.7.1 (Franklin [2])**.** Given a partition $\lambda = (\ell_1, \ell_2, \ldots, \ell_n)$, the bracket of λ is equal to the length of the longest substring of the form $(\ell_1, \ell_2, \ldots, \ell_k)$ such that for all $1 \leq i < k$, we have $\ell_i = \ell_{i+1} + 1$. We retain Andrews' notation of $\sigma(\lambda)$ to denote the bracket of λ.

For example, if $\lambda = (7, 6, 5, 3, 2)$, then we consider the sequences

$$(7), \quad (7,6), \quad (7,6,5),$$

the longest of which has length three. Therefore, $\sigma(\lambda) = 3$. The Young tableau of λ is shown in Figure 2.5 with the boxes corresponding to $\sigma(\lambda)$ highlighted. We see a a relation between the rank and bracket of partitions in the following lemma.

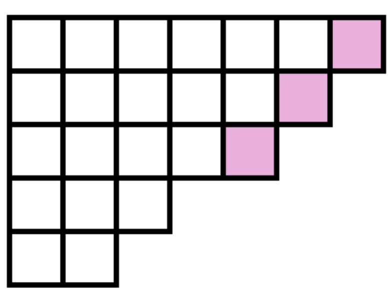

FIGURE 2.5: The Young tableau of $\lambda = (7, 6, 5, 3, 2)$. The boxes counted by $\sigma(\lambda)$ have been highlighted.

Lemma 2.7.1 (Morrill [17]). *Fix integers $0 \leq s \leq t$. The coefficient of $z^m q^n$ in the series*

$$\frac{q^{\frac{t^2+t}{2}}}{(zq^s;q)_{t-s+1}} \tag{2.7.1}$$

is equal to the number of partitions λ of n into t distinct parts with bracket $\sigma(\lambda) \geq s$ and rank m.

This lemma is revisited in our generating series work in Chapter 5. The proof of Lemma 2.7.1 demonstrates another way to manipulate partitions by their Young tableaux. Given a partition $\lambda = (\ell_1, \ell_2, \ldots, \ell_k)$ into k nonnegative parts, we construct a partition

$$\lambda' = (\ell_1 + k, \ell_2 + (k-1), \ldots, \ell_k + 1),$$

with weight $|\lambda'| = |\lambda| + \frac{k^2+k}{2}$. Note that λ' has strictly positive parts. Because the parts of λ are non-increasing, λ' must have distinct parts. This manipulation is commonly called *adding a staircase* to λ, as seen in work of Lovejoy [15] [16].

For example, let $\lambda = (2,0,0)$. Adding a staircase to λ produces the partition $\lambda' = (5,2,1)$, which is shown in Figure 2.6 by its Young tableaux. Recall, we use a vertical line to denote a row of zero boxes in the Young tableau. The notion of staircases will continue to appear in Chapters 3 and 5.

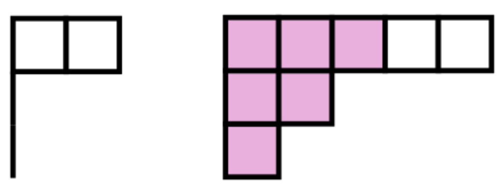

FIGURE 2.6: Adding a staircase to $\lambda = (2,0,0)$.

Proof of Lemma 2.7.1. Fix integers $0 \leq s \leq t$. Similarly to the proof of Proposition 2.5.1, the product

$$\frac{1}{(zq^s;q)_{t-s+1}} = \frac{1}{(1-zq^s)} \frac{1}{(1-zq^{s+1})} \cdots \frac{1}{(1-zq^t)} \tag{2.7.2}$$

$$= \prod_{i=s}^{t}(1 + zq^i + z^2q^{i+i} + \dots). \qquad (2.7.3)$$

corresponds to the columns in the Young tableaux of partitions λ. The length of these columns is bounded between s and t. Thus, the coefficient of z^mq^n is equal to the number of partitions of n into exactly t nonnegative parts, with at least s occurrences of its largest part, and $\ell(\lambda) = m$. Any such λ begins with a substring of the form

$$(\ell_1, \ell_2, \dots, \ell_s) = (\ell_1, \ell_1, \dots, \ell_1)$$

This includes the possibility that λ consists only of zeroes.

Given such a λ, we produce a partition λ' by adding a staircase to λ. Then λ' is a partition into t positive parts. Then $\#(\lambda') = \#(\lambda) + \frac{t^2+t}{2}$. This accounts for the contribution of $q^{\frac{t^2+t}{2}}$ in (2.7.1). By construction, λ' contains the sequence

$$(\ell'_1, \ell'_2, \dots, \ell'_s) = (\ell_1 + t, \ell_1 + t - 1, \dots, \ell_1 + t - s + 1).$$

By the definition of the bracket of a partition, $\sigma(\lambda) \geq s$.

It remains to show that the exponent of z is equal to $r(\lambda')$. Because $\#(\lambda') = t$ and $\ell(\lambda') = \ell(\lambda) + t$, we see that $r(\lambda') = \ell(\lambda)$, which is equal to the exponent of z. Therefore, the coefficient of z^mq^n in (2.7.1) is equal to the number of partitions of n into exactly t parts, with $\sigma(\lambda') \geq s$ and $r(\lambda') = m$. □

2.8 Second Rank of a Partition

Berkovich and Garvan [6] originally defined the M_2-rank on partitions whose odd parts do not repeat. In order to extend the M_2-rank to overpartitions, Lovejoy [16] considered another partition statistic which resembles the rank function. The second rank appears unnamed in Definition (3.1) of Lovejoy's work [16] on the second Frobenius representation of overpartitions.

Definition 2.8.1 (Lovejoy [16])**.** Let λ be a partition into nonnegative parts whose odd parts do not repeat. The *second rank* of λ is defined to be

$$r_2(\lambda) := \left\lfloor \frac{\ell(\lambda)}{2} \right\rfloor - \#(\lambda_<),$$

where $\lambda_<$ denotes the subpartition of λ whose parts are the odd parts of λ less than $\ell(\lambda)$.

For example, if $\lambda = (6,5)$, we first compute that $\lfloor \frac{6}{2} \rfloor = 3$. Next, $\lambda_< = (5)$, so we have $\#(\lambda_<) = 1$. Therefore, $r_2(\lambda) = 2$.

2.9 Second Bracket of a Partition

We also require a variation of the bracket function for our work in Chapter 5.

Definition 2.9.1 (Morrill [17])**.** Let $\lambda = (\ell_1, \ell_2, \ldots, \ell_n)$ be a partition into nonnegative parts. The *second bracket* of λ is the length of the longest substring of λ of the form $(\ell_1, \ell_2, \ldots, \ell_k)$, in which for all $1 \leq i < k$, one of the following holds.

- $\ell_i = \ell_{i+1} + 1$,
- $\ell_i = \ell_{i+1}$ and ℓ_i is even,
- $\ell_i = \ell_{i+1} + 2$ and ℓ_i is odd.

We denote the second bracket of λ by $\sigma_2(\lambda)$.

For example, if $\lambda = (6,6,5,4,2,2,0)$, then we consider the substrings

$$(6), \quad (6,6), \quad (6,6,5), \quad (6,6,5,4).$$

Then we see that $\sigma_2(\lambda) = 4$. The Young tableau of λ is shown in Figure 2.7 with the boxes corresponding to $\sigma_2(\lambda)$ highlighted. We see a relation between the M_2-rank and second bracket of partitions in the following lemma.

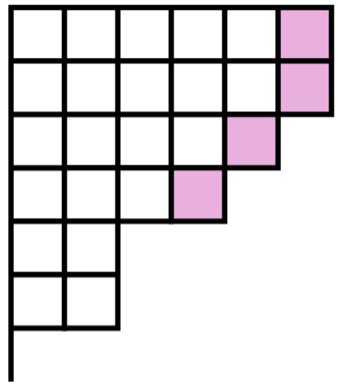

FIGURE 2.7: The Young tableau of $\lambda = (6, 6, 5, 4, 2, 2, 0)$. The boxes counted by $\sigma_2(\lambda)$ have been highlighted.

Lemma 2.9.1 (Morrill [17]). *Fix integers $s \leq t$. The coefficient of $z^m q^n$ in the series*

$$\frac{(-q; q^2)_t}{(zq^{2s}; q^2)_{t-s+1}}$$

is equal to the number of partitions λ of n into exactly t nonnegative parts where odd parts may not repeat, such that $r_2(\lambda) = m$ and $\sigma_2(\lambda) \geq s$.

Note that we have changed the base variable of the q-Pochhammer symbol in Lemma 2.9.1. Here,

$$\frac{(-q; q^2)_t}{(zq^{2s}; q^2)_{t-s+1}} = \prod_{i=0}^{t}(1 + q^{2i+1}) \times \prod_{j=0}^{t-s} \frac{1}{1 - zq^{2j+2s}}.$$

In Chapters 4 and 5, we will work with q-Pochhammer symbols of the form $(a; q^k)_n$, where k is some positive integer.

The proof of Lemma 2.9.1 requires an algorithm which combines the Young tableaux of partitions λ and μ without changing $\#(\lambda)$. Algorithm 2.9.1, below, is Lovejoy's modification [16] of the Joichi-Stanton Map [13]. The original Joichi-Stanton Map appears as Algorithm 3.5.1 in Chapter 3. Algorithmic manipulation of Young tableaux and is required for the generating series proofs in Chapters 3 and 5. In particular, Lemma 2.9.1 is required for the proof of Theorem 5.3.2.

Algorithm 2.9.1 (Lovejoy [16]).

Input: A partition $\lambda = (\ell_1, \ell_2, \ldots, \ell_n)$ into n nonnegative even parts, and a partition $\mu = (m_1, m_2, \ldots, m_k)$ into k distinct odd parts, where $m_i < 2n$ for all i.

Output: A partition $\lambda' = (\ell'_1, \ell'_2, \ldots, \ell'_n)$ into n nonnegative parts with k distinct odd parts.

(1) Delete the largest part of μ, which we write as $m_1 = 2s + 1$.

(2) Add 2 to the first s parts of λ, then add 1 to ℓ_{s+1}, which makes ℓ_{s+1} odd. If $s = 0$, then we add 1 to λ_1. Then this operation is well defined, as λ has exactly n parts and $s + 1 \leq n$.

(3) Relabel the parts of μ, if any exist, so that the largest part of μ is m_1. We now repeat Steps (1) and (2) until the parts of μ are exhausted.

When the process terminates, we label the resulting partition λ' .

An example of Algorithm 2.9.1 is shown in Table 2.3. Because the parts of μ are distinct, the odd parts of λ' must also be distinct. Hence, λ is a partition into n parts having k distinct odd parts. We now prove Lemma 2.9.1.

Iteration	λ	μ
0	$(4, 4, 2, 0)$	(7, 3, 1)
1	$(6, 6, 4, 1)$	(3, 1)
2	$(8, 7, 4, 1)$	(1)
3	$(9, 7, 4, 1)$	$\emptyset$

TABLE 2.3: Example of Algorithm 2.9.1.

Proof of Lemma 2.9.1. Fix integers $s \leq t$. By substituting $q \mapsto q^2$ in (2.7.2) in the proof of Lemma 2.7.1, we see that the coefficient of $z^m q^n$ in

$$\frac{1}{(zq^{2s};q^2)_{t-s+1}} = \frac{1}{(1-zq^{2s})}\frac{1}{(1-zq^{2s+2})}\cdots\frac{1}{(1-zq^{2t})} \tag{2.9.1}$$

is equal to the number of partitions λ of n into t nonnegative even parts with at least s occurrences of their largest part, where $\ell(\lambda) = 2m$.

The coefficient of q^n in

$$(-q;q^2)_t = (1+q)(1+q^3)\dots(1+q^{2t-1}) \tag{2.9.2}$$

is equal to the number of partitions μ of n into distinct odd parts, all of which are less than $2t$.

Given such a λ and μ, we apply Algorithm 2.9.1 to produce a partition λ'. We see that λ' is a partition into t nonnegative parts in which odd parts may not repeat. It remains to determine $\sigma(\lambda')$ and $r_2(\lambda')$.

Let $(\lambda^{(i)}, \mu^{(i)})$ denote the result of iterating the steps of Algorithm 3.5.1 i times on the pair (λ, μ). Then $\lambda^{(0)} = \lambda$ and $\lambda^{(\#(\mu))} = \lambda'$. We claim that $\sigma_2(\lambda^{(i)}) = \sigma(\lambda^{(i+1)})$ and $r_2(\lambda^{(i)}) = r_2(\lambda^{(i+1)})$ for all i. That is, both the second bracket and the second rank are invariant under the steps of Algorithm 2.9.1.

Write $\lambda^{(i)} = (\ell_1^{(i)}, \ell_2^{(i)}, \dots, \ell_t^{(i)})$ and $2m+1 = \ell(\mu^{(i)})$. Let $\sigma = \sigma_2(\lambda^{(i)})$. Consider the substring $(\ell_1^{(i)}, \ell_2^{(i)}, \dots, \ell_\sigma^{(i)})$ of $\lambda^{(i)}$ under the iteration of Algorithm 2.9.1. We seek to prove that the corresponding substring $(\ell_1^{(i+1)}, \ell_2^{(i+1)}, \dots, \ell_\sigma^{(i+1)})$ of $\lambda^{(i+1)}$ is the substring measured by $\sigma_2(\lambda^{(i+1)})$.

We break into cases. If $m \geq \sigma$, then for all $1 \leq j \leq \sigma$ we have that $\ell_j^{(i+1)} = \ell_j^{(i)} + 2$. Because Algorithm 2.9.1 treats the parts of μ in decreasing order, none of the parts $\ell_j^{(i)}$ can be odd. Then we must have $\ell_j^{(i)} = \ell_1^{(i)}$ for all $1 \leq j \leq \sigma$, and $\lambda^{(I+1)}$ begins with the substring

$$(\ell_1^{(i+1)}, \ell_2^{(i+1)}, \ldots, \ell_\sigma^{(i+1)}) = ((\overbrace{\ell_1^{(i)} + 2, \ldots, \ell_1^{(i)} + 2}^{\sigma}).$$

This meets the definition of second bracket, which implies that $\sigma_2(\lambda^{(i+1)}) \geq \sigma$.

Otherwise, $m < \sigma$. As before, none of the parts $\ell_j^{(i)}$ can be odd for $1 \leq j \leq m+1$. This implies that $\ell_j^{(i)} = \ell_1^{(i)}$ for all $1 \leq j \leq m+1$.

After iterating Algorithm 2.9.1 on $(\lambda^{(i)}, \mu^{(i)})$ we now have $\ell_j^{(i+1)} = \ell_j^i + 2$ for all $1 \leq j \leq m$, and $\ell_{m+1}^{(i+1)} = \ell_{m+1}^{(i)} + 1$. Then, $\lambda^{(i+1)}$ begins with the substring

$$(\ell_1^{(i+1)}, \ell_2^{(i+1)} \ldots, \ell_\sigma^{(i+1)}) = (\overbrace{\ell_1^{(i)} + 2, \ldots, \ell_1^{(i)} + 2}^{m}, \ell_1^{(i)} + 1, \ell_{m+2}^{(i)}, \ldots, \ell_\sigma^{(i)}).$$

Because $\ell_{m+2}^{(i)}$ is even and $\sigma_2(\lambda^{(i)}) = \sigma$, we have two subcases. Either $\ell_{m+1}^{(i)} = \ell_{m+2}^{(i)}$ or $\ell_{m+1}^{(i)} = \ell_{m+2}^{(i)} + 1$. These correspond to $\ell_{m+1}^{(i+1)} = \ell_{m+2}^{(i)} + 1$ or $\ell_{m+1}^{(i)} = \ell_{m+2}^{(i)} + 2$, respectively. Because $\ell_{m+2}^{(i+1)}$ is odd, the substring $(\ell_1^{(i+1)}, \ell_2^{(i+1)}, \ldots, \ell_{m+1}^{(i+1)})$ meets the definition of second bracket in both subcases. The substring $(\overline{\ell_{m+1}^{(i)}}, \ell_{m+2}^{(i)}, \ldots, \ell_\sigma^{(i)})$ also meets the definition of second bracket, as none of the parts $m_j^{(i+1)}$ have changed for $j \geq m+1$. Therefore, $\sigma_2(\lambda^{(i+1)}) \geq \sigma$.

Whether $m \geq \sigma$ or $m < \sigma$, any longer substring of $\lambda^{(i+1)}$ cannot meet the second bracket criteria. Otherwise, we could recover a corresponding string in $\lambda^{(i)}$ by reversing the steps of Algorithm 2.9.1. This would imply that $\sigma_2(\lambda^{(i+1)}) > s'$, a contradiction. Therefore, $\overline{\sigma}(\lambda^{(i)}) = \overline{\sigma}(\lambda^{(i+1)})$.

By considering all i, this implies that $\overline{\sigma}(\lambda') = \overline{\sigma}(\lambda)$. Because λ is a partition into even nonnegative parts, λ does not have any odd parts. By the definition of second bracket, $\sigma_2(\lambda)$ is equal to the number of occurrences of the largest part of λ. Because λ has at least s occurrences of its largest part, then $\overline{\sigma}(\lambda') \geq s$.

It remains to show that $r_2(\lambda^{(i)}) = r_2(\lambda^{(i+1)})$. Recall that $\ell(m^{(i)}) = 2m + 1$.

We divide the effect of Algorithm 3.5.1 into two cases depending on m. If $m = 0$, then $\lambda^{(i+1)} = \lambda^{(i)} + 1$, and no other parts are changed. Then

$$\left\lceil \frac{\ell(\lambda^{(i+1)})}{2} \right\rceil = \left\lceil \frac{\ell(\lambda^{(i)})}{2} \right\rceil .$$

Further, the number of odd parts in $\lambda^{(i+1)}$ less than $\ell(\lambda^{(i+1)})$ is equal to the number of odd parts in $\lambda^{(i)}$ less than $\ell(\lambda^{(i)})$. This implies that $r_2(\lambda^{(i)}) = r_2(\lambda^{(i+1)})$.

Otherwise, if $m > 0$, then $\lambda^{(i+1)}$ is obtained by adding 2 to the first m parts of $\lambda^{(i)}$, and adding 1 to $\ell_{m+1}^{(i)}$, which must be smaller than $\ell(\lambda^{(i+1)})$. Then

$$\left\lceil \frac{\ell(\lambda^{(i+1)})}{2} \right\rceil = \left\lceil \frac{\ell(\lambda^{(i)})}{2} \right\rceil + 1,$$

and $\lambda^{(i+1)}$ has one more odd part less than $\ell(\lambda^{(i+1)})$ than $\lambda^{(i)}$ has less than $\ell(\lambda^{(i)})$. Again, this implies that $r_2(\lambda^{(i)}) = r_2(\lambda^{(i+1)})$.

Because $\lambda^{(0)} = \lambda$ and $\lambda^{(\#(\mu))} = \lambda'$, this implies that $r_2(\lambda') = r_2(\lambda)$. By definition, the second rank of a partition λ into even nonnegative parts is equal to $\frac{\ell(\lambda)}{2} = m$. Since m is the exponent of z is equal to $\ell(\lambda)$, then exponent of z is equal to $r_2(\lambda')$.

Therefore, λ' is a partition into t nonnegative parts with $\overline{\sigma}(\lambda) \geq s$, where the exponent of z is $\overline{r}_{CL}(\lambda') + 1$. Because Algorithm 2.9.1 is invertible, then the coefficient of $z^m q^n$ in the series

$$\frac{(-q;q^2)_t}{(zq^{2s};q^2)_{t-s+1}}$$

is equal to the number of partitions λ of n into exactly t nonnegative parts where odd parts may not repeat, such that $r_2(\lambda) = m$ and $\sigma_2(\lambda) \geq s$. □

This concludes our overview of partition theory.

3 OVERPARTITIONS

Let n be a nonnegative integer. An *overpartition* of n is a non-increasing sequence of positive integers $\lambda = (\ell_1, \ell_2, \ldots, \ell_k)$ such that

$$\sum_{i=1}^{k} \ell_i = n,$$

in which the first occurrence of each integer may be overlined. The only significance of the overline is to increase the count of these objects, which gives overpartitions a combinatoric relation to q-hypergeometric series [9].

For example, the overpartitions of $n = 4$ are given by

$$\begin{array}{ccccc} (4) & (\overline{4}) & (3,1) & (3,\overline{1}) & (\overline{3},1) \\ (\overline{3},\overline{1}) & (2,2) & (\overline{2},2) & (2,1,1) & (2,\overline{1},1) \\ (\overline{2},1,1) & (\overline{2},\overline{1},1) & (1,1,1,1) & (\overline{1},1,1,1). & \end{array}$$

Note that every partition is an overpartition. The empty overpartition $\lambda = \emptyset$ is the unique overpartition of $n = 0$, as it has no parts to be overlined. If λ is an overpartition, we retain the use of $\ell(\lambda)$, $|\lambda|$, and $\#(\lambda)$ to denote the largest part, weight, and number of parts of λ, respectively. We denote the number of overpartitions of n by $\overline{p}(n)$, and the set of overpartitions by $\overline{P}$. We will use the phrase "partitions or overpartitions" to remind the reader that $P \subset \overline{P}$.

As in Chapter 2, it is useful sometimes to weaken the definition of overpartitions. An overpartition of n *into nonnegative parts* is a partition in which 0 may occur as a part. For example, the overpartitions of 1 into three nonnegative parts are given by

$$(1,0,0) \quad (1,\overline{0},0) \quad (\overline{1},0,0) \quad (\overline{1},\overline{0},0).$$

We will always fix the number of parts in an overpartition before counting overpartitions which have nonnegative parts, so that the count will be finite. Any references to overpartitions that are not clear should be taken to mean overpartitions into positive parts.

Generating series for overpartitions typically take the form

$$\frac{(-aq;q)_n}{(bq;q)_n}$$

For example, we see the generating series for the number of overpartitions of n in the following proposition.

Proposition 3.0.1 (Corteel, Lovejoy [9]). *Let $\overline{p}(n)$ denote the number of overpartitions of n. Then,*

$$\sum_{n\geq 0}\overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

Proof. Consider the coefficient of q^n. Let M and N be nonnegative integers such that $M+N=n$. The coefficient of q^M in $(-q;q)_\infty$ is equal to the number of partitions λ_1 which have distinct parts and weight M. By Proposition 2.3.1, the coefficient of q^N in

$$\frac{1}{(q;q)_\infty}$$

is equal to the number of partitions λ_2 with weight N. Given such a λ_1 and λ_2, we create an overpartition λ with weight n by overlining the parts of λ_1 and inserting them into λ_2 in non-increasing order. Because every overpartition can be decomposed in this way, we have enumerated all overpartitions of n. $\square$

3.1 Young Tableaux

We represent overpartitions graphically by modifying the definition of Young tableaux from Section 2.1. The Young tableau of an overpartition $\lambda = (\ell_1, \ell_2, \ldots, \ell_n)$ is a Young tableau in which whenever the first occurrence of a part i is overlined in λ we mark the last row of i boxes with a dot. This convention ensures that conjugating the tableau across its NW-SE diagonal will produce another Young tableau of an overpartition. An example is given in Figure 3.1.

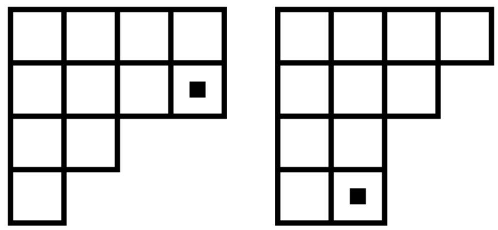

FIGURE 3.1: Young tableaux of $(\overline{4},4,2,1)$ and its conjugate, $(4,3,\overline{2},2)$.

3.2 Dyson Rank

Because conjugating partitions was useful for studying the rank of partitions in Section 2.5, it is natural to ask if Dyson's formulation of partition rank is meaningful for overpartitions. The *Dyson rank*, or *D-rank* of an overpartition λ is defined to be

$$\overline{r}(\lambda) := \ell(\lambda) - \#(\lambda).$$

For example, if $\lambda = (\overline{4},4,2,1)$, then the Dyson rank of λ is 0.

We use Lovejoy's terminology [15] *Dyson rank* in order to distinguish between several notions of the rank of overpartitions which appear in the following sections. We see the generating series for Dyson ranks of overpartitions in the following proposition.

Proposition 3.2.1 (Lovejoy [15])**.** *Let* $\overline{N}(m,n)$ *denote the number of overpartitions* λ *with* $|\lambda| = n$ *and* $\overline{r}(\lambda) = m$*. Then*

$$\sum\sum \overline{N1}(m,n)z^m q^n = \sum_{n\geq 0} \frac{(-1;q)_n q^{(n^2+n)/2}}{(zq,z^{-1}q;q)_n}. \tag{3.2.1}$$

Note the similarity of (3.2.1) to the generating series in Proposition 2.5.1. The proof of Proposition 3.2.1 may be seen as the case $k=1$ in the proof of Corollary 5.2.1.

3.21 The CL-Rank

Given an overpartition λ, we define the *CL-rank* of λ to be $\ell(\lambda) - 1$ minus the number of overlined parts less than the largest part of λ. We denote the overpartition rank of λ

by $\overline{r}_{CL}(\lambda)$. For example, if $\lambda = (\overline{5}, \overline{3}, 3, \overline{1})$, then $\overline{r}_{CL}(\lambda) = (5-1) - 2 = 2$.

Corteel and Lovejoy [9] originally called this statistic the rank of an overpartition. We will see in Proposition 3.4.1 and Lemma 3.5.1 how the CL-rank of overpartitions occurs naturally in combinatorial interpretations of q-Pochhammer symbols. Corteel and Lovejoy [9] motivated their terminology with the following observation. Suppose that λ is an overpartition in which all parts are overlined[4]. Then $\overline{r}_{CL}(\lambda) = \ell\lambda - 1 - (\#(\lambda) - 1) = \ell\lambda - \#(\lambda)$, which coincides with the definition of rank of a partition. Since we are interested in many different ranks of overpartitions, we use the term CL-rank instead for clarity.

3.22 The Second Rank of an Overpartition

We also use a variation of the overpartition rank implied by the work of Lovejoy [16]. Given an overpartition λ into odd parts, the *second overpartition rank* of λ is equal to $\frac{\ell(\lambda)-1}{2}$ minus the number of parts of λ less than $\ell(\lambda)$ which are overlined. For example, if $\lambda = (3, \overline{1})$, then the second overpartition rank of λ is given by $1 - 1 = 0$. We denote the second overpartition rank of λ by $\overline{r}_2(\lambda)$.

3.3 M_2-rank

Lovejoy developed [16] an M_2-rank for overpartitions, which expands Berkovich and Garvan's M_2-rank [6] for ordinary partitions where odd parts may not repeat. Given an overpartition $\lambda = (\ell_1, \ell_2, \ldots, \ell_k)$, the *$M_2$-rank* of an overpartition λ is defined to be

$$\overline{r}_{M_2}(\lambda) := \left\lceil \frac{\ell(\lambda)}{2} \right\rceil - \#(\lambda) + n(\lambda_o) - \chi(\lambda), \tag{3.3.1}$$

[4] Note that λ has distinct parts.

where λ_o is the subpartition of λ consisting of all non-overlined odd parts of λ, and the indicator function χ is defined to be

$$\chi(\lambda) := \begin{cases} 1, & \text{if the largest part of } \lambda \text{ is both odd and non-overlined} \\ 0, & \text{otherwise.} \end{cases}$$

For example, let $\lambda = (\overline{2}, 1, 1)$. Then $\lambda_o = (1, 1)$ and $\chi(\lambda) = 0$. Then the M_2-rank of λ is equal to $1 - 3 + 2 - 0 = 0$. We see the generating function for M_2-ranks of overpartitions in the following proposition.

Proposition 3.3.1 (Lovejoy [16]). *Let $\overline{N2}(m, n)$ denote the number of overpartitions λ with $|\lambda| = n$ and $\overline{r}_{M_2}(\lambda) = m$. Then*

$$\sum_{m\in\mathbb{Z}} \sum_{n\geq 0} \overline{N2}(m, n) z^m q^n = \sum_{n\geq 0} \frac{(-1; q)_{2n} q^n}{(zq^2, z^{-1}q^2; q^2)_n}. \tag{3.3.2}$$

The proof of Proposition 3.3.1 may be seen as the case $k = 1$ in the proof of Corollary 5.3.1.

3.4 The First and Second Frobenius Representations of Overpartitions

Recall from Section 2.2 that a Frobenius representation is an array

$$\nu = \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ b_1 & b_2 & \dots & b_k \end{pmatrix},$$

where both of $(a_1, a_2, \dots, a_k)$ and $(b_1, b_2, \dots, b_k)$ are partitions into distinct parts. Andrews [3] developed the study of generalized Frobenius partitions, in which $(a_1, a_2, \dots, a_k)$ and $(b_1, b_2, \dots, b_k)$ may be any partitions. Lovejoy [16] takes a similar approach in his definition of generalized Frobenius representations.

Definition 3.4.1 (Andrews, Lovejoy [3] [16])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be sets of partitions or overpartitions into possibly nonnegative parts. A generalized Frobenius representation, or F-partition, is a two rowed array

$$\nu = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix},$$

where $(a_1, a_2, \dots, a_n) \in \mathcal{A}$, and $(b_1, b_2, \dots, b_n) \in \mathcal{B}$.

We have chosen to retain Andrew's abbreviation of F-partition. We know from Section 2.2 that ordinary partitions are in one-to-one correspondence with their Frobenius representations. Because $\overline{p}(n) > p(n)$ for $n \geq 1$, we require more choices for $(a_1, a_2, \dots, a_k)$ and $(b_1, b_2, \dots, b_k)$ than just partitions into distinct parts in order to construct Frobenius representations of overpartitions.

Unlike Section 2.2, we define the *weight* of an F-partition ν to be the sum of its entries[5],

$$|\nu| = \sum_{i=1}^{k} a_i + b_i. \tag{3.4.1}$$

For example, if $\mathcal{A}$ is the set of partitions and $\mathcal{B}$ is the set of overpartitions into non-negative parts, then

$$\begin{pmatrix} 6 & 5 & 5 & 2 \\ 6 & 4 & \overline{0} & 0 \end{pmatrix}$$

is a generalized Frobenius representation of $n = 28$. It remains to establish an appropriate choice of sets $\mathcal{A}$ and $\mathcal{B}$ in order for the corresponding F-partitions to be in one-to-one correspondence with overpartitions.

[5] Lovejoy uses two different definitions for the weight of a Frobenius representation in [15] and [16]. We have chosen to follow the definition from [16]. Definitions and results quoted from [15] have been adjusted for consistency.

Proposition 3.4.1 (Lovejoy [15])**.** *There is a one-to-one correspondence between overpartitions λ and F-partitions $\nu = (\alpha, \beta)^T$ where α is a partition into distinct parts and β is an overpartition into nonnegative parts such that $|\lambda| = |\nu|$ and*

$$\overline{r}(\lambda) = (\ell(\alpha) - 1) - \overline{r}_{CL}(\beta).$$

Lovejoy later developed a second Frobenius representation of overpartitions, which corresponds to the M_2-rank.

Proposition 3.4.2 (Lovejoy [16])**.** *There is a one-to-one correspondence between overpartitions λ and generalized Frobenius partitions $\nu = (\alpha, \beta)^T$ where α is an overpartition into odd parts and β is a partition into nonnegative parts where odd parts may not repeat such that $|\lambda| = |\nu|$ and*

$$\overline{r}_{M_2}(\lambda) = r_2(\beta) - \overline{r}_2(\alpha).$$

Note the relations between the Dyson ranks and M_2-ranks of overpartitions and the statistics of their Frobenius representations.

3.5 The Overpartition Bracket

We require two variants of the partition bracket function from Section 2.7 for our work in Chapter 5.

Definition 3.5.1 (Morrill [17])**.** Let $\lambda = (\ell_1, \ell_2, \ldots, \ell_n)$ be an overpartition. The overpartition bracket of λ is the length of the longest substring of the form $(\ell_1, \ell_2, \ldots, \ell_k)$ in which for all $1 \leq i < k$, we either have $\ell_i = \ell_{i+1}$, or $\ell_i = \ell_{i+1} + 1$ and at least one of ℓ_i and ℓ_{i+1} are overlined. We denote the overpartition bracket of λ by $\overline{\sigma}(\lambda)$.

For example, if $\lambda = (7, 7, \overline{6}, 5, 4)$, then we consider the sequences

$$(7), \quad (7,7), \quad (7,7,\overline{6}), \quad (7,7,\overline{6},5),$$

the longest of which has length four. Therefore, $\overline{\sigma}(\lambda) = 4$.

Unlike the partition bracket, we allow successive parts to be equal in the definition of the overpartition bracket. Note that if all parts of λ are overlined, then λ must have distinct parts. In this case, $\overline{\sigma}(\lambda)$ is equal to longest substring of the form $(\ell_1, \ell_1{+}1, \ell_1{+}2, \ldots, \ell_1{+}k)$, which coincides with the definition of the bracket of a partition. This suggests a relation between overpartition rank and overpartition bracket, which we see in the following lemma.

Lemma 3.5.1 (Morrill [17])**.** *Fix positive integers $1 \leq s \leq t$. The coefficient of $z^m q^n$ in*

$$\frac{(-1;q)_t}{(zq^s;q)_{t-s+1}} \tag{3.5.1}$$

is equal to the number of overpartitions λ of n into t nonnegative parts with overpartition bracket $\overline{\sigma}(\lambda) \geq s$ and CL-rank $\overline{r}_{CL}(\lambda) = m + 1$.

We see that the coefficient of $z^m q^n$ in (3.5.1) is equal to the number of certain pairs of partitions (λ, mu). In order to prove Lemma 3.5.1, we use the Joichi-Stanton Map [13] to combine such a λ and μ into a single overpartition.

Algorithm 3.5.1 (Joichi, Stanton [13])**.**

Input: a partition $\lambda = (\ell_1, \ell_2, \ldots, \ell_n)$ into n nonnegative parts, and a partition $\mu = (m_1, m_2, \ldots, m_k)$ into k distinct nonnegative parts each less than n.

Output: An overpartition $\lambda' = (\ell'_1, \ell'_2, \ldots, \ell'_n)$ into n nonnegative parts.

(1) Delete m_1 from μ, and add 1 to the first m_1 parts of λ. This operation is well defined, as all parts of μ are strictly less than the number of parts of λ. Because μ is a partition into nonnegative parts, 0 may occur as a part of μ. In this step, if $m_1 = 0$, then the parts of λ are unchanged.

(2) Overline the $(m_1 + 1)$*st part of* λ*. Here, if* $m_1 = 0$*, then we overline* ℓ_1*.*

(3) If $\mu = \emptyset$*, terminate the algorithm, and take* λ' *to be the resulting overpartition. Otherwise, relabel the parts of* μ*, so that* m_1 *is the largest part of* μ *and return to Step (1).*

An example of Algorithm 3.5.1 shown in Table 3.1. Because the parts of μ are distinct, only the first occurrence of any part in λ may be overlined. Thus, we see that λ' is an overpartition into n nonnegative parts. Note that if $\mu = (t, t-1, \ldots, 1)$, then λ' consists of t distinct parts, which are all overlined. This coincides with the notion of adding a staircase to an ordinary partition (Chapter 2).

It is easy to verify that Algorithm 3.5.1 is invertible. The reverse algorithm also appears in work of Lovejoy [15]. We may now prove Lemma 3.5.1.

TABLE 3.1: Example of Algorithm 3.5.1.

Iteration	λ	μ
0	$(4, 3, 2, 2)$	$(3, 1, 0)$
1	$(5, 4, 3, \overline{2})$	$(1, 0)$
2	$(6, \overline{4}, 3, \overline{2})$	(0)
3	$(\overline{6}, \overline{4}, 3, \overline{2})$	$\emptyset$

Proof of Lemma 3.5.1. Let $1 \leq s \leq t$ be integers. As in the proof of Lemma 2.7.1, the coefficient of $z^m q^n$ in

$$\frac{1}{(zq^s; q)_{t-s+1}} \tag{3.5.2}$$

is equal to the number of partitions λ of n into t nonnegative parts with at least s occurrences of their largest part, where $\ell(\lambda) = m$.

The coefficient of q^n in

$$(-1;q)_t = (1+q)(1+q^2)\dots(1+q^{t-1}) \tag{3.5.3}$$

is equal to the number of partitions μ of n into distinct parts, all of which are less than t.

Given such a λ and μ, we apply Algorithm 3.5.1 to produce an overpartition λ'. We see that λ' is an overpartition into t nonnegative parts. It remains to determine $\overline{\sigma}(\lambda')$ and $\overline{r}_{CL}(\lambda')$.

Let $(\lambda^{(i)}, \mu^{(i)})$ denote the result of iterating the steps of Algorithm 3.5.1 i times on the pair (λ, μ). Then $\lambda^{(0)} = \lambda$ and $\lambda^{(\#(\mu))} = \lambda'$. We claim that $\overline{\sigma}(\lambda^{(i)}) = \overline{\sigma}(\lambda^{(i+1)})$ and $\overline{r}_{CL}(\lambda^{(i)}) = \overline{r}_{CL}(\lambda^{(i+1)})$ for all i. That is, both the overpartition bracket and the CL-rank are invariant under the steps of Algorithm 3.5.1.

Write $\lambda^{(i)} = (\ell_1^{(i)}, \ell_2^{(i)}, \dots, \ell_t^{(i)})$. Let $m = \ell(\mu^{(i)})$ and $\sigma = \overline{\sigma}(\lambda^{(i)})$. Consider the substring $(\ell_1^{(i)}, \ell_2^{(i)}, \dots, \ell_\sigma^{(i)})$ of $\lambda^{(i)}$ under the iteration of Algorithm 3.5.1. We seek to prove that the corresponding substring $(\ell_1^{(i+1)}, \ell_2^{(i+1)}, \dots, \ell_\sigma^{(i+1)})$ of $\lambda^{(i+1)}$ is the substring measured by $\overline{\sigma}(\lambda^{(i+1)})$.

We break into cases. If $m \geq \sigma$, then for all $1 \leq j \leq \sigma$ we have that $\ell_j^{(i+1)} = \ell_j^{(i)} + 1$. Because Algorithm 3.5.1 treats the parts of μ in decreasing order, none of the parts $\ell_j^{(i)}$ can appear overlined for $1 \leq j \leq m+1$. Then we must have $\ell_j^{(i)} = \ell_1^{(i)}$ for all $1 \leq j \leq \sigma$, and $\lambda^{(I+1)}$ begins with the substring

$$(\ell_1^{(i+1)}, \ell_2^{(i+1)}, \dots, \ell_\sigma^{(i+1)}) = ((\ell_1^{(i)} + 1, \dots, \ell_1^{(i)} + 1).$$

This meets the definition of overpartition bracket, which implies that $\overline{\sigma}(\lambda^{(i+1)}) \geq \sigma$.

Otherwise, $m < \sigma$. As before, none of the parts $\ell_j^{(i)}$ can appear overlined for $1 \leq j \leq m+1$. This implies that $\ell_j^{(i)} = \ell_1^{(i)}$ for all $1 \leq j \leq m+1$.

After iterating Algorithm 3.5.1 on $(\lambda^{(i)}, \mu^{(i)})$ we now have $\ell_j^{(i+1)} = \ell_j^i + 1$ for all $1 \leq j \leq$

m, and $\ell_{m+1}^{(i+1)}$ is now overlined. Then, $\lambda^{(i+1)}$ begins with the substring

$$(\ell_1^{(i+1)}, \ell_2^{(i+1)} \ldots, \ell_\sigma^{(i+1)}) = (\overbrace{\ell_1^{(i)} + 1, \ldots, \ell_1^{(i)} + 1}^{m}, \overline{\ell_1^{(i)}}, \ell_{m+2}^{(i)}, \ldots, \ell_\sigma^{(i)}).$$

Since $\ell_m^{(i+1)} = \ell_{m+1}^{(i+1)} + 1$ and $\ell_{m+1}^{(i+1)}$ is overlined, the substring $(\ell_1^{(i+1)}, \ell_2^{(i+1)}, \ldots, \ell_{m+1}^{(i+1)})$ meets the definition of overpartition bracket. Because $\overline{\sigma}(\lambda^{(i)}) = \sigma$, the substring $(\overline{\ell_{m+1}^{(i)}}, \ell_{m+2}^{(i)}, \ldots, \ell_\sigma^{(i)})$ also meets the definition of overpartition bracket. Therefore, $\overline{\sigma}(\lambda^{(i+1)}) \geq \sigma$.

In either of these cases, any longer substring of $\lambda^{(i+1)}$ cannot meet the overpartition bracket criteria. Otherwise, we could recover a corresponding string in $\lambda^{(i)}$ by reversing the steps of Algorithm 3.5.1. This would imply that $\overline{\sigma}(\lambda^{(i+1)}) > s'$, a contradiction. Therefore, $\overline{\sigma}(\lambda^{(i)}) = \overline{\sigma}(\lambda^{(i+1)})$.

By considering all i, this implies that $\overline{\sigma}(\lambda') = \overline{\sigma}(\lambda)$. Because λ is a partition into nonnegative parts, λ does not have any overlined parts. By the definition of overpartition bracket, $\overline{\sigma}(\lambda)$ is equal to the number of occurrences of the largest part of λ. Because λ has at least s occurrences of its largest part, then $\overline{\sigma}(\lambda') \geq s$.

It remains to show that $\overline{r}_{CL}(\lambda^{(i)}) = \overline{r}_{CL}(\lambda^{(i+1)})$. Recall that $m = \ell(m^{(i)})$. We divide the effect of Algorithm 3.5.1 into two cases depending on m. If $m = 0$, then $\lambda^{(i+1)}$ is obtained by overlining the largest part of $\lambda^{(i)}$. Since the CL-rank does not detect whether the largest part of an overpartition is overlined, $\overline{r}_{CL}(\lambda^{(i)}) = \overline{r}_{CL}(\lambda^{(i+1)})$.

Otherwise, if $m > 0$, then $\lambda^{(i+1)}$ is obtained by adding 1 to the first m parts of $\lambda^{(i)}$, and overlining $\ell_{m+1}^{(i)}$, which must be smaller than $\ell(\lambda^{(i+1)})$. Then $\ell(\lambda^{(i+1)}) = \ell(\lambda^{(i+1)}) + 1$, and $\lambda^{(i+1)}$ has one more overlined part which is counted by the CL-rank than $\lambda^{(i)}$ does. By the definition of CL-rank, $\overline{r}_{CL}(\lambda^{(i)}) = \overline{r}_{CL}(\lambda^{(i+1)})$.

Because $\lambda^{(0)} = \lambda$ and $\lambda^{(\#(\mu))} = \lambda'$, this implies that $\overline{r}_{CL}(\lambda') = \overline{r}_{CL}(\lambda)$. By definition, the CL-rank of a partition λ into nonnegative parts is equal to $\ell(\lambda) - 1$. Since the exponent of z is equal to $\ell(\lambda)$, then exponent of z is equal to $\overline{r}_{CL}(\lambda') + 1$.

Therefore, λ' is an overpartition into t nonnegative parts with $\overline{\sigma}(\lambda) \geq s$, where the exponent of z is $\overline{r}_{CL}(\lambda') + 1$. Because Algorithm 3.5.1 is invertible, then the coefficient of $z^m q^n$ in the series

$$\frac{(-1;q)_t}{(zq^s;q)_{t-s+1}}$$

is equal to the number of overpartitions λ of n into t nonnegative parts with overpartition bracket $\overline{\sigma}(\lambda) \geq s$ and CL-rank $\overline{r}_{CL}(\lambda) = m + 1$. □

In Chapter 5, we will consider what happens to the product

$$\frac{1}{(zq^s;q)_{t-s+1}}$$

when $s = 0$. This case is excluded from Lemma 3.5.1 because it contains the geometric series

$$\frac{1}{1-z}$$

as a factor. Instead, modifying the proof of lemma Lemma 3.5.1 gives the following proposition.

Proposition 3.5.1 (Morrill [17]). *Fix $t \geq 0$. The coefficient of $z^m q^n$ in*

$$\frac{(-1;q)_t}{(zq;q)_t}$$

is equal to the number of overpartitions λ of n into t nonnegative parts in which the largest part cannot be overlined, with CL-rank m.

3.6 Second Bracket of an Overpartition

The next statistic extends the notion of overpartition bracket.

Definition 3.6.1 (Morrill [17])**.** Let $\lambda = (\ell_1, \ell_2, \ldots, \ell_k)$ be an overpartition. The *second overpartition bracket* of λ is the length of the longest substring of λ of the form $(n_1, n_2, \ldots, n_j)$, in which for all $1 \leq i < j$, one of the following holds.

- $\ell_i = \ell_{i+1}$,
- $\ell_i = \ell_{i+1} + 2$ and at least one of n_i or n_{i+1} is overlined.

We denote the second overpartition bracket of λ by $\overline{\sigma}_2(\lambda)$.

For example, if $\lambda = (5, \overline{3}, 3, 1)$, the substrings we consider are

$$(5),$$
$$(5, \overline{3}),$$
$$(5, \overline{3}, 3),$$

the longest of which has length 3. Therefore, $\overline{\sigma}_2(\lambda) = 3$.

We find a generating series for the second ranks of overpartitions with bounded second overpartition brackets at a corollary to Lemma 3.5.1.

Corollary 3.6.1 (Morrill [17])**.** *Fix nonnegative integers $s \leq t$. The coefficient of $z^m q^n$ in*

$$\frac{(-1; q^2)_t q^t}{(zq^{2s}; q^2)_{t-s+1}} \tag{3.6.1}$$

is equal to the number of overpartitions λ of n into t odd parts with $\overline{r_2}(\lambda) = m$ and $\overline{\sigma}_2(\lambda) \geq s$.

Proof of Corollary 3.6.1. By substituting $q \mapsto q^2$ in Lemma 3.5.1, we see that the coefficient of $z^m q^n$ in

$$\frac{(-1; q^2)_t}{(zq^{2s}; q^2)_{t-s+1}}$$

is equal to the number of overpartitions λ of n into t nonnegative even parts which satisfy the following. First, m is equal to $\frac{\ell(\lambda)}{2}$ minus the number of overlined parts of λ which are less than $\ell(\lambda)$. Second, λ contains a sequence $(\ell_1, \ell_2, \ldots, \ell_s)$ in which either $\ell_i = \ell_{i+1}$, or $\ell_i = \ell_{i+1} + 2$ and at least one of n_i or n_{i+1} is overlined.

Given such a λ, we create an overpartition λ' by adding 1 to each part of λ. Note that λ' is an overpartition into t odd parts with $|\lambda'| = |\lambda| + t$. This accounts for the contribution of q^t in (3.6.1).

Then m is equal to $\frac{\ell(\lambda')-1}{2}$ minus the number of overlined parts of λ which are less than $\ell(\lambda)$, which is the definition of $\overline{r}_2(\lambda')$. Also, λ contains a sequence

$$(\ell_1', \ell_2', \ldots, \ell_s') = (\ell_1 + 1, \ell_2 + 1, \ldots, \ell_s + 1),$$

in which either $\ell_i' = \ell_{i+1}'$, or $\ell_i' = \ell_{i+1}' + 2$ and at least one of n_i or n_{i+1} is overlined. Then by the definition of second bracket, $\overline{\sigma}_2(\lambda') \geq s$. $\square$

3.7 The kth Residual Cranks

Bringmann, Lovejoy, and Osburn [7] developed two analogs of the partition crank function. The *first residual crank* of an overpartition λ is the partition crank applied to the partition λ' whose parts consist of the non-overlined parts of λ.

For example, if $\lambda = (\overline{4}, 3, 2)$, then the first residual crank of λ is equal to 3, because no 1's appear in the partition $(3, 2)$. We see the generating function for first residual cranks[6] of overpartitions in the following theorem of Bringmann, Lovejoy, and Osburn [7].

Theorem 3.7.1. *[7] The coefficient of $z^m q^n$ in the series*

$$\overline{C}(z; q) = \frac{(-q; q)_\infty (q; q)_\infty^2}{(q; q)_\infty (zq, z^{-1}q; q)_\infty},$$

[6]The generating series is subject to modifications when dealing with the crank function and its extensions; see Section **??** for details.

is equal to the number of overpartitions of n with first residual crank m.

The *second residual crank* of an overpartition λ is the partition crank applied to the partition λ' whose parts consist of one half of the non-overlined even parts of λ.

For example, let $\lambda = (\overline{4}, 4, 2)$. Because the partition crank of $(2, 1)$ is equal to 0, the second residual crank of λ is equal to 0. We see the generating function for second residual cranks of overpartitions in the following theorem of Bringmann, Lovejoy, and Osburn [7].

Proposition 3.7.2 (Bringmann [7]). *The coefficient of $z^m q^n$ in the series*

$$\overline{C2}(z;q) = \frac{(-q;q)_\infty (q^2;q^2)_\infty^2}{(q;q)_\infty (zq^2, z^{-1}q^2; q^2)_\infty},$$

is equal to the number of overpartitions of n with second residual crank m.

We take these functions to their logical conclusion.

Definition 3.7.1 (Morrill). Let λ be an overpartition. The *k-th residual crank* of an overpartition λ is defined to be

$$cr_k(\lambda) = cr(\lambda_k),$$

where λ_k is an ordinary partition whose parts are $\frac{1}{k}$th the nonoverlined parts of λ that are divisible by k, and cr denotes the crank of a partition.

For example, if $\lambda = (\overline{4}, 3, 3)$, the third residual crank of λ is equal to the partition crank of $(1, 1)$, which is -2. Our third main result gives the generating series for k-th residual cranks of overpartitions.

Theorem 3.7.3 (Morrill). *The coefficient of $z^m q^n$ in*

$$\overline{C[k]}(z;q) := \frac{(-q;q)_\infty (q^k;q^k)_\infty^2}{(q;q)_\infty (zq^k, z^{-1}q^k; q^k)_\infty}. \tag{3.7.1}$$

is equal to the number of overpartitions of n with k-th residual crank m.

Proof. By Proposition 3.7.2, we see that the coefficient of $z^m q^{kn}$ in the factor

$$\frac{(q^k;q^k)_\infty}{(zq^k,z^{-1}q^k;q^k)_\infty}$$

is equal to the number of partitions λ_1 with $|\lambda_1| = n$ and $cr(\lambda_1) = m$.

As in the proof of Proposition 2.3.1, we see that the coefficient of q^n in

$$\frac{(q^k;q^k)_\infty}{(q;q)_\infty} = \prod_{\substack{i\geq 1\\ k\nmid i}} \frac{1}{1-q^i}$$

is equal to the number of partitions λ_2 into parts which are not divisible by k, where $|\lambda_2| = n$. Finally, as in the proof of Proposition 3.0.1, the coefficient of q^n in

$$(-q;q)_\infty$$

is equal to the number of partitions μ into distinct parts, where $|\mu| = n$

Therefore, the coefficient of $z^m q^n$ in

$$\frac{(-q;q)_\infty (q^k;q^k)^2_\infty}{(q;q)_\infty (zq^k,z^{-1}q^k;q^k)_\infty} \tag{3.7.2}$$

is equal to the number of triples $(\lambda_1, \lambda_2, \mu)$ with $m = cr(\lambda_1)$ and $n = k|\lambda_1| + |\lambda_2| + |\mu|$, where λ_1, λ_2, and μ are partitions as described above.

Given such a triple $(\lambda_1, \lambda_2, \mu)$, we create an overpartition λ' as follows. First, replace each occurrence of the integer part n in λ_1 with the integer kn. That is,

$$(\ell_1, \ell_2, \ldots, \ell_s) \mapsto (k\ell_1, k\ell_2, \ldots, k\ell_s)$$

We then overline the parts of μ, and append the parts of μ and λ_2 to λ_1 and arrange the parts in non-increasing order to produce λ'.

Then λ' is an overpartition with $|\lambda'| = n$. By definition, we see that $cr_k(\lambda') = m$. Moreover, any overpartition may be generated this way. therefore, the coefficient of $z^m q^n$ in (3.7.2) is equal to the number of overpartitions λ' with $|\lambda'| = n$ and $cr_k(\lambda') = m$ $\square$

3.8 The ℓth Positive Moments of $\overline{M[k]}(m,n)$

In order to prove Theorem 1.0.2, we require results of Bringmann, Lovejoy, and Osburn [7], and Larsen, Rust, and Swisher [14].

Proposition 3.8.1 (Larsen, Rust, Swisher [14])**.** *Let $a(m,n)$ be a sequence defined for $n \geq 0$, $m \in Z$ such that $a(m,0) = 0$ for all $m \geq 1$. Consider the generating series*

$$G(z,q) = \sum_{m\in\mathbb{Z}} \sum_{n\geq 0} a(m,n) z^m q^n,$$

and write

$$\frac{\partial}{\partial z} G(z,q) = \sum_{m\in\mathbb{Z}} \sum_{n\geq 0} b(m,n) z^m q^n.$$

Then the generating series for the ℓth positive moments of $a(m,n)$ is given by

$$\sum_{n\geq 0} \left(\sum_{m>1} m^\ell a(m,n) \right) q^n = \left[\left(z \frac{\partial}{\partial z} \right)^{\ell-1} \sum_{m>1} \sum_{n\geq 0} b(m,n) z^m q^n \right]_{z=1}.$$

Proposition 3.8.2 (Larsen, Rust, Swisher [14])**.** *The generating series for the ℓth positive moment of $\overline{M}(m,n)$ is given by*

$$\sum_{n\geq 0} \left(\sum_{m\geq 0} \overline{M}(m,n) \right) q^n = \frac{(q;q)_{-\infty}}{(q;q)_\infty} \sum_{n\geq 1} (-1)^{n+1} \frac{q^{\frac{n^2+n}{2}} A_\ell(q^n)}{(1-q^n)^\ell},$$

where $A_\ell(q)$ denotes the ℓth Eulerian polynomial.

We now restate and prove Theorem 1.0.2

Theorem 3.8.3 (Morrill)**.** *Let $\overline{M[k]}(m,n)$ denote the number of overpartitions with kth residual crank m and weight n. The generating series for the ℓth positive moment of $\overline{M[k]}(m,n)$ is given by*

$$\sum_{n\geq 0} \left(\sum_{m\geq 1} m^\ell \overline{M[k]}(m,n) \right) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 1} (-1)^{n+1} \frac{q^{k\frac{n^2+n}{2}} A_\ell(q^{kn})}{(1-q^{kn})^\ell}$$

Proof. Using Proposition 3.8.1, we first seek to calculate

$$\frac{\partial}{\partial z}\frac{(-q;q)_\infty(q^k;q^k)_\infty^2}{(q;q)_\infty(zq^k,z^{-1}q^k;q^k)_\infty} = \frac{(-q;q)_\infty(q^k;q^k)_\infty}{(q;q)_\infty(-q^k;q^k)_\infty}\frac{\partial}{\partial z}\frac{(-q^k;q^k)_\infty}{(zq^k,z^{-1}q^k;q^k)_\infty}$$
$$=: \frac{(-q;q)_\infty(q^k;q^k)_\infty}{(q;q)_\infty(-q^k;q^k)_\infty}\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} b(m,n)z^m q^n.$$

If we write

$$\frac{\partial}{\partial z}\frac{(-q;q)_\infty}{(zq,z^{-1}q;q)_\infty} = \sum_{m\in\mathbb{Z}}\sum_{n\geq 0} b'(m,n)z^m q^n$$

Then we have

$$\sum_{m\geq 0}\sum_{n\geq 0} b(m,n)z^m q^n = \frac{(-q,-q^2,\ldots,-q^{k-1},q^k,q^k;q^k)_\infty}{(q;q)_\infty}\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} b'(m,n)z^m q^{kn}. \quad (3.8.1)$$

We now apply Proposition 3.8.1 to (3.8.1). Note that

$$\left[\left(z\frac{\partial}{\partial z}\right)^{\ell-1}\sum_{m>1}\sum_{n\geq 0} b(m,n)z^m q^n\right]_{z=1}$$
$$= \frac{(-q;q)_\infty(q^k;q^k)_\infty}{(q;q)_\infty(-q^k;q^k)_\infty}\left[\left(z\frac{\partial}{\partial z}\right)^{\ell-1}\sum_{m>1}\sum_{n\geq 0} b(m,n)z^m q^{kn}\right]_{z=1} \quad (3.8.2)$$

By Proposition 3.8.1 and Proposition 3.8.2, we see that

$$\left[\left(z\frac{\partial}{\partial z}\right)^{\ell-1}\sum_{m>1}\sum_{n\geq 0} b(m,n)z^m q^{kn}\right]_{z=1} = \frac{(-q^k;q^k)_\infty}{(q^k;q^k)_\infty}\sum_{n\geq 1}(-1)^{n+1}\frac{(q^k)^{\frac{n^2+n}{2}}A_\ell(q^{kn})}{(1-q^{kn})^\ell}.$$

It follows from (3.8.2) that

$$\sum_{n\geq 0}\left(\sum_{m\geq 1} m^\ell\overline{M[k]}(m,n)\right)q^{kn} = \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n+1}\frac{q^{k\frac{n^2+n}{2}}A_\ell(q^{kn})}{(1-q^{kn})^\ell}$$

as desired. ∎

4 q-HYPERGEOMETRIC SERIES

Gauss was interested in the *hypergeometric function*

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix} ; z\right] = 1 + \sum_{n\geq 1} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \tag{4.0.1} $$

where $(a)_n$ denotes the rising factorial

$$ (a)_n := a(a+1)\dots(a+n-1). $$

Gauss' hypergeometric function (4.0.1) is a special case of the formula

$$ {}_rF_{r-1}\left[\begin{matrix} a_1 & a_2 & \dots & a_{r-1} & a_r \\ & b_1 & b_2 & \dots & b_{r-1} \end{matrix} ; z\right] = 1 + \sum_{n\geq 1} \frac{(a_1)_n(a_2)_n\dots(a_r)_n}{(b_1)_n(b_2)_n\dots(b_{r-1})_n n!} z^n. $$

Slater [19] attributes the introduction of q-analogs to the hypergeometric series to Heine. For any $a \in \mathbb{C}$ and $n \geq 0$,

$$ \lim_{q\to 1} \frac{1-q^a}{1-q^n} = \frac{a}{n}. $$

It follows that

$$ \lim_{q\to 1} \frac{(q^a;q)_n}{(q;q)_n} = \frac{(a)_n}{n!}. \tag{4.0.2} $$

The q-hypergeometric series is then defined to be

$$ {}_r\Phi_{r-1}\left[\begin{matrix} a_1 & a_2 & \dots & a_{r-1} & a_r \\ & b_1 & b_2 & \dots & b_{r-1} \end{matrix} ; q;\ z\right] := \sum_{n\geq 0} \frac{(a_1, a_2, \dots, a_r; q)_n}{(b_1, b_2, \dots, b_{r-1}, q; q)_n} z^n. \tag{4.0.3} $$

We may substitute $a_i \mapsto q^{a_i}$ and use (4.0.2) to recover the ordinary hypergeometric series. In practice, however, we will take the parameters a_i to be complex numbers or simple expressions of the variables q, x, and z.

The inquisitive reader may find more examples of hypergeometric series and their q analogs in Slater's book [19]. We caution reader that the convention for the generic case ${}_p\Phi_q$ has changed since the time of Slater, but that the conventions agree when $p = q + 1$. See Gasper and Rahman for a contemporary treatment [12].

We focus on Andrews' k-fold generalization of the Watson-Whipple transformation.

Theorem 4.0.1 (Andrews [1])**.** *Let $a, b_1, c_1, b_2, c_2, \dots, b_k, c_k$ be complex numbers, and let $k \geq 1$ and $N \geq 0$. Then,*

$$
\begin{aligned}
&{}_{2k+4}\Phi_{2k+3}\left[\begin{matrix} a & a^{\frac{1}{2}}q & -a^{\frac{1}{2}}q & b_1 & c_1 & \dots & b_k & c_k & q^{-N} \\ & a^{\frac{1}{2}} & -a^{\frac{1}{2}} & \frac{aq}{b_1} & \frac{aq}{c_1} & \dots & \frac{aq}{b_k} & \frac{aq}{c_k} & aq^{N+1} \end{matrix} ; q; \frac{a^k q^{k+N}}{b_1 c_1 \dots b_k c_k}\right] \\
&= \frac{(aq, \frac{aq}{b_k c_k}; q)_N}{(\frac{aq}{b_k}, \frac{aq}{c_k}; q)_N} \sum_{\substack{n_1 \geq 0 \\ \vdots \\ n_{k-1} \geq 0}} \frac{(\frac{aq}{b_1 c_1}; q)_{n_1}}{(q;q)_{n_1}} \cdots \frac{(\frac{aq}{b_{k-1} c_{k-1}}; q)_{n_{k-1}}}{(q;q)_{n_{k-1}}} \\
&\times \frac{(b_2, c_2; q)_{N_1}}{(\frac{aq}{b_1}, \frac{aq}{c_1}; q)_{N_1}} \frac{(b_3, c_3; q)_{N_2}}{(\frac{aq}{b_2}, \frac{aq}{c_2}; q)_{N_2}} \cdots \frac{(b_k, c_k; q)_{N_{k-1}}}{(\frac{aq}{b_{k-1}}, \frac{aq}{c_{k-1}}; q)_{N_{k-1}}} \\
&\times \frac{(q^{-N}; q)_{N_{k-1}}}{(\frac{b_k c_k q^{-N}}{a}; q)_{N_{k-1}}} \frac{(aq)^{N_1 + N_2 + \cdots + N_{k-2}} q^{N_{k-1}}}{(b_2 c_2)^{N_1} (b_3 c_3)^{N_2} \dots (b_{k-1} c_{k-1})^{N_{k-2}}},
\end{aligned}
\tag{4.0.4}
$$

where we write $N_0 = 0$ and , $N_i = n_1 + n_2 + \cdots + n_i$ for all $i \geq 1$.

Corollary 4.0.1. *Let $a, b_1, c_1, b_2, c_2, \dots, b_k, c_k$ be complex numbers, and let $k \geq 1$ and $N \geq 0$. Then,*

$$
\begin{aligned}
&{}_{2k+4}\Phi_{2k+3}\left[\begin{matrix} a & a^{\frac{1}{2}}q & -a^{\frac{1}{2}}q & b_1 & c_1 & \dots & b_k & c_k & q^{-N} \\ & a^{\frac{1}{2}} & -a^{\frac{1}{2}} & \frac{aq}{b_1} & \frac{aq}{c_1} & \dots & \frac{aq}{b_k} & \frac{aq}{c_k} & aq^{N+1} \end{matrix} ; q; \frac{a^k q^{k+N}}{b_1 c_1 \dots b_k c_k}\right] \\
&= \frac{(aq, \frac{aq}{b_k c_k}; q)_N}{(\frac{aq}{b_k}, \frac{aq}{c_k}; q)_N} \sum_{n_1, n_2, \dots, n_{k-1} \geq 0} \frac{(\frac{aq}{b_{k-1} c_{k-1}}; q)_{n_1}}{(q;q)_{n_1}} \cdots \frac{(\frac{aq}{b_1 c_1}; q)_{n_{k-1}}}{(q;q)_{n_{k-1}}} \\
&\times \frac{(b_{k-2}, c_{k-2}; q)_{N_1}}{(\frac{aq}{b_{k-1}}, \frac{aq}{c_{k-1}}; q)_{N_1}} \frac{(b_{k-3}, c_{k-3}; q)_{N_2}}{(\frac{aq}{b_{k-2}}, \frac{aq}{c_{k-2}}; q)_{N_2}} \cdots \frac{(b_1, c_1; q)_{N_{k-2}}}{(\frac{aq}{b_2}, \frac{aq}{c_2}; q)_{N_{k-2}}} \frac{(b_k, c_k; q)_{N_{k-1}}}{(\frac{aq}{b_1}, \frac{aq}{c_1}; q)_{N_{k-1}}} \\
&\times \frac{(q^{-N}; q)_{N_{k-1}}}{(\frac{b_k c_k q^{-N}}{a}; q)_{N_{k-1}}} \frac{(aq)^{N_1 + N_2 + \cdots + N_{k-2}} q^{N_{k-1}}}{(b_{k-2} c_{k-2})^{N_1} (b_{k-3} c_{k-3})^{N_2} \dots (b_1 c_1)^{N_{k-2}}},
\end{aligned}
$$

where we write $N_0 = 0$ *and ,* $N_i = n_1 + n_2 + \cdots + n_i$ *for all* $i \geq 1$.

Proof. The left hand side of (4.0.4) is a symmetric function in the variables b_i and c_i. Thus, the hypergeometric series is invariant under permutation of the b_i and c_i on the right hand side. We map

$$1 \mapsto (k-1), \quad 2 \mapsto (k-2), \quad \ldots, \quad (k-1) \mapsto 1, \quad k \mapsto k\cdot$$

This permutes the parameters in the hypergeometric to give that

$$\begin{aligned}
&{}_r\Phi_{r-1}\left[\begin{matrix} a & a^{\frac{1}{2}}q & -a^{\frac{1}{2}}q & b_1 & c_1 & \ldots & b_k & c_k & q^{-N} \\ & a^{\frac{1}{2}} & -a^{\frac{1}{2}} & \frac{a}{b_1} & \frac{a}{c_1} & \ldots & \frac{a}{b_k} & \frac{a}{c_k} & aq^{N+1} \end{matrix} ; q; \frac{a^k q^{k+N}}{b_1c_1\ldots b_kc_k}\right] \\
&= {}_r\Phi_{r-1}\left[\begin{matrix} a & a^{\frac{1}{2}}q & -a^{\frac{1}{2}}q & b_{k-1} & c_{k-1} & \ldots & c_1 & b_k & c_k & q^{-N} \\ & a^{\frac{1}{2}} & -a^{\frac{1}{2}} & \frac{aq}{b_{k-1}} & \frac{aq}{c_{k-1}} & \ldots & \frac{aq}{c_1} & \frac{aq}{b_k} & \frac{aq}{c_k} & aq^{N+1} \end{matrix} ; q; \frac{a^k q^{k+N}}{b_1c_1\ldots b_kc_k}\right].
\end{aligned}$$

□

Theorem 4.0.2 (Morrill [17])**.** *Let* k *be a positive integer, and let* $x_1, x_2, \ldots, x_k$ *be any nonzero complex numbers. Then,*

$$\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2+kn}\prod_{i=1}^{k}\frac{(1-x_i)(1-x_i^{-1})}{(1-x_iq^n)(1-x_i^{-1}q^n)}\right) \\
&\qquad = \sum_{\substack{n_1\geq 0 \\ \vdots \\ n_k \geq 0}} (-1;q)_{N_k} q^{\frac{N_k^2-N_k}{2}} \prod_{i=1}^{k}\frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})q^{N_i}}{(x_{k-i+1}q^{N_{i-1}}, x_{k-i+1}^{-1}q^{N_{i-1}};q)_{n_i+1}}, \quad (4.0.5)
\end{aligned}$$

where we write $N_0 = 0$ *and* $N_i = n_1 + n_2 + \cdots + n_i$ *for all* $i \geq 1$.

Proof. Recall the definition of the q-hypergeometric series. We begin by substituting $k \mapsto k+1$ in Corollary 4.0.1. This yields

$${}_{2k+6}\Phi_{2k+5}\left[\begin{matrix} a & a^{\frac{1}{2}}q & -a^{\frac{1}{2}}q & b_1 & c_1 & \ldots & b_{k+1} & c_{k+1} & q^{-N} \\ & a^{\frac{1}{2}} & -a^{\frac{1}{2}} & \frac{aq}{b_1} & \frac{aq}{c_1} & \ldots & \frac{aq}{b_{k+1}} & \frac{aq}{c_{k+1}} & aq^{N+1} \end{matrix} ; q; \frac{a^{k+1}q^{k+1+N}}{b_1c_1\ldots b_{k+1}c_{k+1}}\right]$$

$$= \frac{(aq, \frac{aq}{b_{k+1}c_{k+1}}; q)_N}{(\frac{aq}{b_{k+1}}, \frac{aq}{c_{k+1}}; q)_N} \sum_{\substack{n_1 \geq 0 \\ \vdots \\ n_k \geq 0}} \frac{(\frac{aq}{b_k c_k}; q)_{n_1}}{(q;q)_{n_1}} \cdots \frac{(\frac{aq}{b_1 c_1}; q)_{n_k}}{(q;q)_{n_k}}$$

$$\times \frac{(b_{k-1}, c_{k-1}; q)_{N_1}}{(\frac{aq}{b_k}, \frac{aq}{c_k}; q)_{N_1}} \frac{(b_{k-2}, c_{k-2}; q)_{N_2}}{(\frac{aq}{b_{k-1}}, \frac{aq}{c_{k-1}}; q)_{N_2}} \cdots \frac{(b_1, c_1; q)_{N_{k-1}}}{(\frac{aq}{b_2}, \frac{aq}{c_2}; q)_{N_{k-1}}} \frac{(b_{k+1}, c_{k+1}; q)_{N_k}}{(\frac{aq}{b_1}, \frac{aq}{c_1}; q)_{N_k}}$$

$$\times \frac{(q^{-N}; q)_{N_k}}{(\frac{b_{k+1}c_{k+1}q^{-N}}{a}; q)_{N_k}} \frac{(aq)^{N_1+N_2+\cdots+N_{k-1}} q^{N_k}}{(b_{k-1}c_{k-1})^{N_1}(b_{k-2}c_{k-2})^{N_2} \ldots (b_1c_1)^{N_{k-1}}}. \tag{4.0.6}$$

We next take the limit $N \to \infty$. Because we are working in formal power series in the variable q, then for arbitrary d we have

$$\lim_{N\to\infty} (d;q)_N = (d;q)_\infty \tag{4.0.7}$$

$$\lim_{N\to\infty} (dq^N;q)_n = \lim_{N\to\infty} \prod_{i=0}^{n} (1 - dq^{N+i}) = 1 \tag{4.0.8}$$

$$\lim_{N\to\infty} (dq^{-N};q)_n q^{Nn} = \lim_{N\to\infty} \prod_{i=0}^{n} (q^N - dq^i) = (-1)^n (dq)^{\frac{n^2-n}{2}}, \tag{4.0.9}$$

Then the left side of (4.0.6) becomes

$$\sum_{n=0}^{\infty} \frac{(a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b_1, c_1, \ldots, b_{k+1}, c_{k+1}; q)_n (-1)^n q^{\frac{n^2-n}{2}}}{(q, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, \frac{aq}{b_1}, \frac{aq}{c_1}, \ldots, \frac{aq}{b_{k+1}}, \frac{aq}{c_{k+1}}; q)_n} \left(\frac{a^{k+1}q^{k+1}}{b_1c_1 \ldots b_{k+1}c_{k+1}} \right)^n.$$

When $n = 0$, the q-Pochhammer symbols take their trivial value, and the summand is equal to 1. For $n > 0$, we may simplify the summand using the relation

$$\frac{(a, a^{\frac{1}{2}}q, -a^{\frac{1}{2}}q; q)_n}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}; q)_n} = \prod_{i=0}^{n-1} \frac{(1-aq^i)(1-a^{\frac{1}{2}}q^{i+1})(1+a^{\frac{1}{2}}q^{i+1})}{(1-a^{\frac{1}{2}}q^i)(1+a^{\frac{1}{2}}q^i)} \tag{4.0.10}$$

$$= \prod_{i=0}^{n-1} \frac{(1-aq^i)(1-aq^{2i+2})}{(1-aq^{2i})} = (1-aq^{2n})(aq;q)_{n-1}. \tag{4.0.11}$$

Thus the left hand side is equal to

$$1 + \sum_{n=1}^{\infty} (1-aq^{2n})(aq;q)_{n-1} \frac{(b_1, c_1, \ldots, b_{k+1}, c_{k+1}; q)_n (-1)^n q^{\frac{n^2-n}{2}}}{(q, \frac{aq}{b_1}, \frac{aq}{c_1}, \ldots, \frac{aq}{b_{k+1}}, \frac{aq}{c_{k+1}}; q)_n} \left(\frac{a^{k+1}q^{k+1}}{b_1c_1 \ldots b_{k+1}c_{k+1}} \right)^n.$$

In order to take the limit on the right side of (4.0.6), we use the relation

$$\lim_{N\to\infty} \frac{(q^{-N};q)_{N_k}}{(a^{-1}b_{k+1}c_{k+1}q^{-N};q)_{N_k}} = \lim_{N\to\infty} \prod_{i=0}^{N_k-1} \frac{(q^N - q^i)}{(q^N - a^{-1}b_{k+1}c_{k+1}q^i)} \tag{4.0.12}$$

$$= \prod_{i=0}^{N_k-1} \frac{-q^i}{-a^{-1}b_{k+1}c_{k+1}q^i} = \left(\frac{a}{b_{k+1}c_{k+1}}\right)^{N_k} \tag{4.0.13}$$

to obtain

$$\begin{aligned}
1+\sum_{n=1}^{\infty}&(1-aq^{2n})(aq;q)_{n-1}\frac{(b_1,c_1,\dots,b_{k+1},c_{k+1};q)_n(-1)^nq^{\frac{n^2-n}{2}}}{(q,\frac{aq}{b_1},\frac{aq}{c_1},\dots,\frac{aq}{b_{k+1}},\frac{aq}{c_{k+1}};q)_n}\left(\frac{a^{k+1}q^{k+1}}{b_1c_1\dots b_{k+1}c_{k+1}}\right)^n \\
&= \frac{(aq,\frac{aq}{b_{k+1}c_{k+1}};q)_\infty}{(\frac{aq}{b_{k+1}},\frac{aq}{c_{k+1}};q)_\infty}\sum_{\substack{n_1\ge 0\\ \vdots \\ n_k\ge 0}}\frac{(\frac{aq}{b_kc_k};q)_{n_1}}{(q;q)_{n_1}}\cdots\frac{(\frac{aq}{b_1c_1};q)_{n_k}}{(q;q)_{n_k}} \\
&\times \frac{(b_{k-1},c_{k-1};q)_{N_1}}{(\frac{aq}{b_k},\frac{aq}{c_k};q)_{N_1}}\frac{(b_{k-2},c_{k-2};q)_{N_2}}{(\frac{aq}{b_{k-1}},\frac{aq}{c_{k-1}};q)_{N_2}}\cdots\frac{(b_1,c_1;q)_{N_{k-1}}}{(\frac{aq}{b_2},\frac{aq}{c_2};q)_{N_{k-1}}}\frac{(b_{k+1},c_{k+1};q)_{N_k}}{(\frac{aq}{b_1},\frac{aq}{c_1};q)_{N_k}} \\
&\times \frac{(aq)^{N_1+N_2+\cdots+N_{k-1}}(aq)^{N_k}}{(b_{k-1}c_{k-1})^{N_1}\dots(b_1c_1)^{N_{k-1}}(b_{k+1}c_{k+1})^{N_k}}.
\end{aligned} \tag{4.0.14}$$

Next, set $a=1$. On the left hand side of (4.0.14), we have

$$(1-q^{2n})\frac{(q;q)_{n-1}}{(q;q)_n} = \frac{1-q^{2n}}{1-q^n}\frac{(q;q)_{n-1}}{(q;q)_{n-1}} = 1+q^n.$$

Then, (4.0.14) becomes

$$\begin{aligned}
1+\sum_{n=1}^{\infty}&(1+q^n)\frac{(b_1,c_1,\dots,b_{k+1},c_{k+1};q)_n(-1)^nq^{\frac{n^2-n}{2}}}{(\frac{q}{b_1},\frac{q}{c_1},\dots,\frac{q}{b_{k+1}},\frac{q}{c_{k+1}};q)_n}\left(\frac{q^{k+1}}{b_1c_1\dots b_{k+1}c_{k+1}}\right)^n \\
&= \frac{(q,\frac{q}{b_{k+1}c_{k+1}};q)_\infty}{(\frac{q}{b_{k+1}},\frac{q}{c_{k+1}};q)_\infty}\sum_{n_1,n_2,\dots,n_k\ge 0}\frac{(\frac{q}{b_kc_k};q)_{n_1}}{(q;q)_{n_1}}\cdots\frac{(\frac{q}{b_1c_1};q)_{n_k}}{(q;q)_{n_k}} \\
&\times \frac{(b_{k-1},c_{k-1};q)_{N_1}}{(\frac{q}{b_k},\frac{q}{c_k};q)_{N_1}}\frac{(b_{k-2},c_{k-2};q)_{N_2}}{(\frac{q}{b_{k-1}},\frac{q}{c_{k-1}};q)_{N_2}}\cdots\frac{(b_1,c_1;q)_{N_{k-1}}}{(\frac{q}{b_2},\frac{q}{c_2};q)_{N_{k-1}}}\frac{(b_{k+1},c_{k+1};q)_{N_k}}{(\frac{q}{b_1},\frac{q}{c_1};q)_{N_k}} \\
&\times \frac{q^{N_1+N_2+\cdots+N_k}}{(b_{k+1}c_{k+1})^{N_1}\dots(b_1c_1)^{N_{k-1}}(b_{k+1}c_{k+1})^{N_k}}.
\end{aligned} \tag{4.0.15}$$

For $1\le i\le k$, we set $b_i=x_i$, $c_i=x_i^{-1}$, and $b_{k+1}=-1$. This cancels the terms

$$\frac{(-1)^n}{b_{k+1}^n} \text{ and } \frac{(\frac{q}{b_{k-i+1}c_{k-i+1}};q)_{n_i}}{(q;q)_{n_i}}.$$

We use the identity

$$(1+q^n)\frac{(-1;q)_n}{(-q;q)_n} = (1+q^0)\frac{(-q;q)_n}{(-q;q)_n} = 2,$$

on the left hand side of (4.0.15) and obtain

$$
\begin{aligned}
1+2\sum_{n=1}^{\infty}&\frac{(x_1,x_1^{-1},\dots x_k,x_k^{-1},c_{k+1};q)_n}{(x_1q,x_1^{-1}q,\dots x_kq,x_k^{-1}q,qc_{k+1}^{-1};q)_n}\frac{q^{\frac{n^2-n}{2}+(k+1)n}}{c_{k+1}^n}\\
=\frac{(q,\frac{-q}{c_{k+1}};q)_\infty}{(-q,\frac{q}{c_{k+1}};q)_\infty}&\sum_{n_1,n_2,\dots,n_k\geq 0}\frac{(x_{k-1},x_{k-1}^{-1};q)_{N_1}}{(x_kq,x_k^{-1}q;q)_{N_1}}\frac{(x_{k-2},x_{k-2}^{-1};q)_{N_2}}{(x_{k-1}q,x_{k-1}^{-1}q;q)_{N_2}}\cdots\frac{(x_1,x_1^{-1};q)_{N_{k-1}}}{(x_2q,x_2^{-1}q;q)_{N_{k-1}}}\\
&\times\frac{(-1,c_{k+1};q)_{N_k}}{(x_1q,x_1^{-1}q;q)_{N_k}}\frac{q^{N_1+N_2+\cdots+N_k}}{(-c_{k+1})^{N_k}}.
\end{aligned}
$$

Let $c_{k+1}\to\infty$. We use the fact that

$$\lim_{c_{k+1}\to\infty}\frac{(c_{k+1};q)_n}{c_{k+1}^n}=\lim_{c_{k+1}\to\infty}\prod_{i=0}^{n-1}\left(\frac{1}{c_{k+1}}-q^i\right)=(-1)^nq^{\frac{n^2-n}{2}} \tag{4.0.16}$$

$$\lim_{c_{k+1}\to\infty}(c_{k+1}^{-1}q;q)_n=\lim_{c_{k+1}\to\infty}\prod_{i=0}^{n-1}\left(1-\frac{q^i}{c_{k+1}}\right)=1 \tag{4.0.17}$$

to obtain

$$
\begin{aligned}
1+2\sum_{n=1}^{\infty}&\frac{(x_1,x_1^{-1},\dots,x_k,x_k^{-1};q)_n(-1)^nq^{n^2+kn}}{(x_1q,x_1^{-1}q,\dots,x_kq,x_k^{-1}q;q)_n}\\
&=\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{\substack{n_1\geq 0\\ \vdots\\ n_k\geq 0}}\frac{(x_{k-1},x_{k-1}^{-1};q)_{N_1}}{(x_kq,x_k^{-1}q;q)_{N_1}}\frac{(x_{k-2},x_{k-2}^{-1};q)_{N_2}}{(x_{k-1}q,x_{k-1}^{-1}q;q)_{N_2}}\cdots\frac{(x_1,x_1^{-1};q)_{N_{k-1}}}{(x_2q,x_2^{-1}q;q)_{N_{k-1}}}\\
&\qquad\times\frac{(-1;q)_{N_k}}{(x_1q,x_1^{-1}q;q)_{N_k}}q^{N_1+N_2+\cdots+N_k+\frac{N_k^2-N_k}{2}}.
\end{aligned}
\tag{4.0.18}
$$

Recall that $N_i=N_{i-1}+n_i$ and $N_0=0$. We may use Lemma 2.3.2 to write

$$\frac{(x_i,x_i^{-1};q)_n}{(x_iq,x_i^{-1}q;q)_n}=\frac{(1-x_i)(1-x_i^{-1})}{(1-x_iq^n)(1-x_i^{-1}q^n)} \tag{4.0.19}$$

$$\frac{(x_{k-i+1},{x_{k-i+1}}^{-1};q)_{N_{i-1}}}{(x_{k-i+1}q,{x_{k-i+1}}^{-1}q;q)_{N_i}}=\frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})}{(x_{k-i+1}q^{N_{i-1}}x_{k-i+1}^{-1}q^{N_{i-1}};q)_{n_i}} \tag{4.0.20}$$

$$\frac{1}{(x_kq,x_k^{-1}q;q)_{n_1}}=\frac{(1-x_k)(1-x_k^{-1})}{(x_kq^{N_0},x_k^{-1}q^{N_0};q)_{n_1+1}}. \tag{4.0.21}$$

Then (4.0.18) becomes

$$\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2+kn}\prod_{i=1}^{k}\frac{(1-x_i)(1-x_i^{-1})}{(1-x_iq^n)(1-q^nx_i^{-1})}\right)$$
$$=\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{\substack{n_1\geq 0\\ \vdots\\ n_k\geq 0}}(-1;q)_{N_k}q^{\frac{N_k^2-N_k}{2}}\prod_{i=1}^{k}\frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})q^{N_i}}{(x_{k-i+1}q^{N_{i-1}},x_{k-i+1}^{-1}q^{N_{i-1}};q)_{n_i+1}}. \quad (4.0.22)$$

Multiplying (4.0.22) by $\frac{(-q;q)_\infty}{(q;q)_\infty}$ gives us the desired equation,

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2+kn}\prod_{i=1}^{k}\frac{(1-x_i)(1-x_i^{-1})}{(1-x_iq^n)(1-q^nx_i^{-1})}\right)$$
$$=\sum_{\substack{n_1\geq 0\\ \vdots\\ n_k\geq 0}}(-1;q)_{N_k}q^{\frac{N_k^2-N_k}{2}}\prod_{i=1}^{k}\frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})q^{N_i}}{(x_{k-i+1}q^{N_{i-1}},x_{k-i+1}^{-1}q^{N_{i-1}};q)_{n_i+1}}. \quad (4.0.23)$$

□

Theorem 4.0.3. *Fix $k \geq 1$. For $|q| < 1$ and $x_1, x_2, \ldots, x_k$ nonzero complex numbers,*

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}q^{n^2+2kn}\prod_{i=1}^{k}\frac{(1-x_i)(1-x_i^{-1})}{(1-x_iq^{2n})(1-x_i^{-1}q^{2n})}\right)$$
$$=\sum_{\substack{n_1\geq 0\\ \vdots\\ n_k\geq 0}}\frac{(-1;q)_{2N_k}}{q^{N_k}}\prod_{i=1}^{k}\frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})q^{2N_i}}{(x_{k-i+1}q^{2N_{i-1}},x_{k-i+1}^{-1}q^{2N_{i-1}};q^2)_{n_i+1}}, \quad (4.0.24)$$

where we write $N_0 = 0$ and $N_i = n_1 + n_2 + \cdots + n_i$ for all $i \geq 1$.

The proof of Theorem 4.0.3 draws from the techniques used in the proof of Theorem 4.0.2

Proof of Theorem 4.0.3. Let $k \geq 1$. We begin from (4.0.15) in the proof of Theorem 4.0.2,

$$1+\sum_{n=1}^{\infty}(1+q^n)\frac{(b_1,c_1,\ldots,b_{k+1},c_{k+1};q)_n(-1)^nq^{\frac{n^2-n}{2}}}{(\frac{q}{b_1},\frac{q}{c_1},\ldots,\frac{q}{b_{k+1}},\frac{q}{c_{k+1}};q)_n}\left(\frac{q^{k+1}}{b_1c_1\ldots b_{k+1}c_{k+1}}\right)^n$$

$$= \frac{(q, \frac{q}{b_{k+1}c_{k+1}}; q)_\infty}{(\frac{q}{b_{k+1}}, \frac{q}{c_{k+1}}; q)_\infty} \sum_{n_1, n_2, \ldots, n_k \geq 0} \frac{(\frac{q}{b_k c_k}; q)_{n_1}}{(q;q)_{n_1}} \cdots \frac{(\frac{q}{b_1 c_1}; q)_{n_k}}{(q;q)_{n_k}}$$

$$\times \frac{(b_{k-1}, c_{k-1}; q)_{N_1}}{(\frac{q}{b_k}, \frac{q}{c_k}; q)_{N_1}} \frac{(b_{k-2}, c_{k-2}; q)_{N_2}}{(\frac{q}{b_{k-1}}, \frac{q}{c_{k-1}}; q)_{N_2}} \cdots \frac{(b_1, c_1; q)_{N_{k-1}}}{(\frac{q}{b_2}, \frac{q}{c_2}; q)_{N_{k-1}}} \frac{(b_{k+1}, c_{k+1}; q)_{N_k}}{(\frac{q}{b_1}, \frac{q}{c_1}; q)_{N_k}}$$

$$\times \frac{q^{N_1+N_2+\cdots+N_k}}{(b_{k+1}c_{k+1})^{N_1} \ldots (b_1 c_1)^{N_{k-1}} (b_{k+1}c_{k+1})^{N_k}}. \quad (4.0.25)$$

Next, we substitute $q \mapsto q^2$. We see that (4.0.25) becomes

$$1 + \sum_{n=1}^{\infty} (1+q^{2n}) \frac{(b_1, c_1, b_2, c_2, \ldots, b_{k+1}, c_{k+1}; q^2)_n (-1)^n q^{n^2-n}}{(\frac{q^2}{b_1}, \frac{q^2}{c_1}, \ldots, \frac{q^2}{b_{k+1}}, \frac{q^2}{c_{k+1}}; q^2)_n} \left(\frac{q^{2k+2}}{b_1 c_1 \ldots b_{k+1} c_{k+1}} \right)^n$$

$$= \frac{(q^2, \frac{q^2}{b_{k+1}c_{k+1}}; q^2)_\infty}{(\frac{q^2}{b_{k+1}}, \frac{q^2}{c_{k+1}}; q^2)_\infty} \sum_{\substack{n_1 \geq 0 \\ \vdots \\ n_k \geq 0}} \frac{(\frac{q^2}{b_k c_k}; q^2)_{n_1}}{(q^2;q^2)_{n_1}} \frac{(\frac{q^2}{b_{k-1}c_{k-1}}; q^2)_{n_2}}{(q^2;q^2)_{n_2}} \cdots \frac{(\frac{q^2}{b_1 c_1}; q^2)_{n_k}}{(q^2;q^2)_{n_k}}$$

$$\times \frac{(b_{k-1}, c_{k-1}; q^2)_{N_1}}{(\frac{q^2}{b_k}, \frac{q^2}{c_k}; q^2)_{N_1}} \frac{(b_{k-2}, c_{k-2}; q^2)_{N_2}}{(\frac{q^2}{b_{k-1}}, \frac{q^2}{c_{k-1}}; q^2)_{N_2}} \cdots \frac{(b_1, c_1; q^2)_{N_{k-1}}}{(\frac{q^2}{b_2}, \frac{q^2}{c_2}; q^2)_{N_{k-1}}} \frac{(b_{k+1}, c_{k+1}; q^2)_{N_k}}{(\frac{q^2}{b_1}, \frac{q^2}{c_1}; q^2)_{N_k}}$$

$$\times \frac{q^{2N_1+2N_2+\cdots+2N_k}}{(b_{k+1}c_{k+1})^{N_k} (b_{k-1}c_{k-1})^{N_1} (b_{k-2}c_{k-2})^{N_2} \ldots (b_1 c_1)^{N_{k-1}}}. \quad (4.0.26)$$

Next substitute $b_{k+1} \mapsto -1$, $c_{k+1} \mapsto -q$, and $b_i \mapsto x_i$ and $c_i \mapsto x_i^{-1}$ for $1 \leq i \leq k$. Then the term

$$\frac{(-1)^n}{b_{k+1}^n} \frac{(c_{k+1}; q^2)_n}{(\frac{q^2}{c_{k+1}}; q^2)_n} \mapsto 1,$$

and (4.0.26) becomes

$$1 + \sum_{n=1}^{\infty} (1+q^{2n}) \frac{(x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_k, x_k^{-1}, -1; q^2)_n q^{n^2+2kn}}{(x_1 q^2, x_1^{-1} q^2, x_2 q^2, x_2^{-1} q^2, \ldots, x_k q^2, x_k^{-1} q^2, -q^2; q^2)_n}$$

$$= \frac{(q, q^2; q^2)_\infty}{(-q, -q^2; q^2)_\infty} \sum_{\substack{n_1 \geq 0 \\ \vdots \\ n_k \geq 0}} \frac{(x_{k-1}, x_{k-1}^{-1}; q^2)_{N_1}}{(x_k q^2, x_k^{-1} q^2; q^2)_{N_1}} \frac{(x_{k-2}, x_{k-2}^{-1}; q^2)_{N_2}}{(x_{k-1} q^2, x_{k-1}^{-1} q^2; q^2)_{N_2}} \cdots \frac{(x_1, x_1^{-1}; q^2)_{N_{k-1}}}{(x_2 q^2, x_2^{-1} q^2; q^2)_{N_{k-1}}}$$

$$\times \frac{(-1, -q; q^2)_{N_k} q^{2N_1+2N_2+\cdots+2N_{k-1}+N_k}}{(x_1 q^2, x_1^{-1} q^2; q^2)_{N_k}}. \quad (4.0.27)$$

Note that

$$\frac{(q, q^2; q^2)_\infty}{(-q, -q^2; q^2)_\infty} = \prod_{i=0}^{\infty} \frac{(1-q^{2i+1})(1-q^{2i+2})}{(1+q^{2i+1})(1+q^{2i+2})} = \prod_{i=0}^{\infty} \frac{(1-q^i)}{(1+q^i)} = \frac{(q;q)_\infty}{(-q;q)_\infty},$$

$$(-1,-q;q^2)_n = \prod_{i=0}^{n-1}(1+q^{2i})(1+q^{2i+1}) = \prod_{i=0}^{2n-1}(1+q^i) = (-1;q)_{2n}.$$

By applying (4.0.19), (4.0.20), and (4.0.21) from the proof of Theorem 4.0.2, we can reduce (4.0.27) to

$$\left(1+2\sum_{n=1}^{\infty} q^{n^2+2kn}\prod_{i=1}^{k}\frac{(1-x_i)(1-x_i^{-1})}{(1-x_iq^{2n})(1-x_i^{-1}q^{2n})}\right)$$
$$= \frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n_1,n_2,\ldots,n_k\geq 0}\frac{(-1;q)_{2N_k}}{q^{N_k}}\prod_{i=1}^{k}\frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})q^{2N_i}}{(x_{k-i+1}q^{2N_{i-1}},x_{k-i+1}^{-1}q^{2N_{i-1}};q^2)_{n_i+1}}. \quad (4.0.28)$$

Here we write q^{N_k} as q^{2N_k}/q^{N_k} in order to simplify the product notation. Multiplying 4.0.28 by $\frac{(-q;q)_\infty}{(q;q)_\infty}$ gives us the desired equation,

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty} q^{n^2+2kn}\prod_{i=1}^{k}\frac{(1-x_i)(1-x_i^{-1})}{(1-x_iq^{2n})(1-x_i^{-1}q^{2n})}\right)$$
$$= \sum_{\substack{n_1\geq 0\\ \vdots \\ n_k\geq 0}}\frac{(-1;q)_{2N_k}}{q^{N_k}}\prod_{i=1}^{k}\frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})q^{2N_i}}{(x_{k-i+1}q^{2N_{i-1}},x_{k-i+1}^{-1}q^{2N_{i-1}};q^2)_{n_i+1}}. \quad (4.0.29)$$

□

With the series identities established, we may now work towards a combinatorial interpretation.

5 BUFFERED FROBENIUS REPRESENTATIONS

5.1 Buffered Frobenius Partitions

We now introduce a combinatorial object to extend the notion of F-partitions. Recall from Chapter 3 that an F-partition is an array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_k \\ b_1 & b_2 & \dots & b_k \end{pmatrix},$$

where each $(a_1, a_2, \dots, a_k)$ and $(b_1, b_2, \dots, b_k)$ is either a partition or overpartition. Buffered Frobenius partitions, or B-partitions, provide a connection between the hypergeometric series (4.0.5) and (4.0.24), and F-partitions.

The definition of B-partitions is simplified by using the following notation for arrays. Let A_1, A_2, …, A_k and B_1, B_2, …, B_k be sets. We write

$$\nu \in \begin{pmatrix} A_1 & A_2 & \dots & A_k \\ B_1 & B_2 & \dots & B_k \end{pmatrix}$$

to denote that ν is an array

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix}$$

where for all i, we have $\alpha_i \in A_i$ and $\beta_i \in B_i$.

Definition 5.1.1 (Morrill)**.** Let $\overline{P}$ denote the set of overpartitions, and let P_0 denote the set of partitions into nonnegative parts. A buffered Frobenius partition, or B-partition, is a two rowed array

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix} \in \begin{pmatrix} \overline{P} & P_0 & \dots & P_0 \\ \overline{P} & P_0 & \dots & P_0 \end{pmatrix},$$

where for all i, we have $\#(\alpha_i) \geq \#(\alpha_{i+1})$ and $\#(\beta_i) = \#(\alpha_i)$. Additionally, we may mark either of α_i or β_i with a hat for any $i < k$.

Like the function of overlining parts of overpartitions, the hat notation serves to increase the count of these objects and to enrich their combinatorics. We define the *weight* of B-partition to be

$$|\nu| := \sum_{1 \leq i \leq k} |\alpha_i| + |\beta_i|.$$

For example, the empty array $\nu = \emptyset$ is a B-partition with $|\nu| = 0$. As a nontrivial example, consider

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} (4, \widehat{3, 2,} 1) & (1,1,1) \\ (\overline{3}, \overline{2}, 2, 2) & (1,1,1) \end{pmatrix}. \tag{5.1.1}$$

Note that the entires of each row appear by decreasing order of their number of parts. This requirement is analogous to the definition of F-partitions, where integers must appear in decreasing order. Requiring $\#(\beta_i) = \#(\alpha_i)$ is not very restrictive for partitions into nonnegative parts. For example,

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} (3, \widehat{3, 2,} 1) & (1,0,0,0) \\ (\overline{3}, \overline{2}, 2, 2) & (4,1,1,0) \end{pmatrix} \tag{5.1.2}$$

is also a B-partition. Note that $\ell(\beta_2) > \ell(\beta_1)$, which is allowed in the definition of B-partitions.

5.11 Buffered Young Tableaux

We have seen in Chapters 2 and 3 how to represent partitions and overpartitions geometrically by their Young tableaux. We construct the *buffered Young tableaux*[7] of a B-partition using k colors as follows.

[7] Plural, as we will construct one tableau for each row of the B-partition ν.

Given a B-partition

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix},$$

we first draw the Young tableau for α_1 in the first color. Next, we draw the Young tableau for α_2 in the second color. However, we left justify the boxes for α_2 to the right edge of the tableau for α_1. If an entry α_i is marked with a hat, we shift the tableau for α_{i+1} to the right by one unit and leave a buffer between α_i and α_{i+1}. For example, if $\alpha_1 = (\widehat{3, 2, 1})$ and $\alpha_2 = (2, 2, 1)$, then we produce the tableaux in figure 5.1.

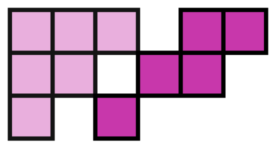

FIGURE 5.1: The buffered Young tableaux of $((\widehat{3, 2, 1}), (2, 2, 1))$

We then continue by drawing the tableau for each α_i in the ith color, justified to the right edge of the preceding tableau. We draw the tableaux for β_1, β_2, $\dots$, β_k in the same manner.

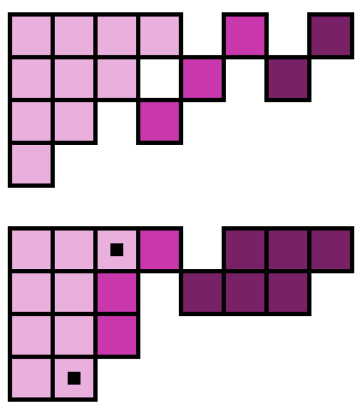

FIGURE 5.2: The buffered Young Tableaux of (5.1.3).

For example, Figure 5.2 shows the buffered Young tableaux of

$$\nu = \begin{pmatrix} (4, \widehat{3, 2, 1}) & (\widehat{1, 1, 1}) & (1, 1) \\ (\overline{3}, \overline{2}, 2, 2) & (\widehat{1, 1, 1}) & (3, 3) \end{pmatrix}. \tag{5.1.3}$$

Note that entries marked with a hat increase the total width of the tableaux without increasing the number of boxes. Forbidding α_k and β_k from appearing with a hat reflects the fact that there are no tableaux which could indicate a buffer to the right of α_k or β_k.

5.12 The Jigsaw Map

Visualizing B-partitions with their buffered Young tableaux suggests that we may interpret B-partitions as exploded Young tableaux of F-partitions. To reassemble the F-partition, we introduce the *jigsaw map*. Let ν be a B-partition

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix},$$

where for all i,

$$\alpha_i = (a_{(i,1)}, a_{(i,2)}, \dots, a_{(i,k_1)})$$
$$\beta_i = (b_{(i,1)}, b_{(i,2)}, \dots, b_{(i,k_1)}).$$

We seek to construct an F-partition

$$j(\nu) = \begin{pmatrix} a_1 & a_2 & \dots & a_{k_1} \\ b_1 & b_2 & \dots & b_{k_1} \end{pmatrix},$$

where $(a_1, a_2, \dots, a_{k_1})$ and $(b_1, b_2, \dots, b_{k_1})$ are overpartitions into nonnegative parts.

First, we remove any hats from the entries of ν. Given a B-partition ν without hats, we rewrite each α_i and β_i as a partition into k_1 nonnegative parts,

$$\alpha_i = (\overbrace{a_{(i,1)}, a_{(i,2)}, \dots, a_{(i,k_i)}, 0, \dots, 0}^{k_1})$$
$$\beta_i = (\overbrace{b_{(i,1)}, b_{(i,2)}, \dots, b_{(i,k_i)}, 0, \dots, 0}^{k_1}).$$

For all $1 \leq j \leq k_1$, we define the integers a_j to be

$$a_j = \sum_{i=1}^{k} a_{(i,j)}$$

$$b_j = \sum_{i=1}^{k} b_{(i,j)}.$$

Finally, we overline a_i or b_i if and only if the ith part of α_1 or β_1 is overlined, respectively[8].

For example, taking ν as in (5.1.3),

$$\nu = \begin{pmatrix} (4,\widehat{3,2,}1) & (\widehat{1,1,1}) & (1,1) \\ (\overline{3},\overline{2},2,2) & (\widehat{1,1,1}) & (3,3) \end{pmatrix},$$

then we first remove all hats from the entries of ν and rewrite each entry so that $\#(\alpha_i) = \#(\beta_i) = 4$. This produces

$$\nu = \begin{pmatrix} (4,3,2,1) & (1,1,1,0) & (1,1,0,0) \\ (\overline{3},\overline{2},2,2) & (1,1,1,0) & (3,3,0,0) \end{pmatrix}.$$

By summing the parts of α_i and β_i, we see that

$$j(\nu) = \begin{pmatrix} 6 & 5 & 3 & 1 \\ \overline{7} & \overline{6} & 3 & 2 \end{pmatrix}.$$

A visualization of the jigsaw map acting on ν is shown in Figure 5.3.

To see that the jigsaw map produces an F-partition, we need only check that the integers a_j and b_j are non-increasing,

$$a_j = \sum_{i=1}^{k} a_{(i,j)} \geq \sum_{i=1}^{k} a_{(i,j+1)} = a_{j+1}$$
$$b_j = \sum_{i=1}^{k} b_{(i,j)} \geq \sum_{i=1}^{k} b_{(i,j+1)} = b_{j+1},$$

which holds since each α_i and β_i is an overpartition or a partition into nonnegative parts. Then for any B-partition ν, we see that $j(\nu)$ is an F-partition as defined in Chapter 3. The Young tableau for $(a_1, a_2, \ldots, a_{\#(\alpha_1)})$ and $(b_1, b_2, \ldots, b_{\#(\alpha_1)})$ may be obtained by decoloring the buffered Young tableaux for ν and left-aligning all the boxes.

[8]This is why only α_1 and β_1 are allowed to be overpartitions.

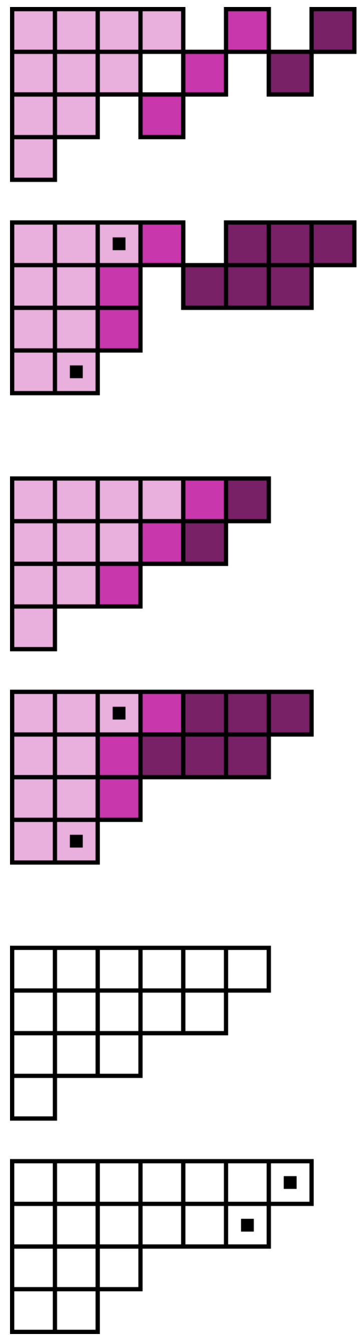

FIGURE 5.3: Demonstration of the jigsaw map acting on (5.1.3).

Recall from Chapter 3 that the Young tableau of an overpartition represents overlined parts ℓ by marking the last row of ℓ boxes. Because we use the convention that the first part of an overpartition may be overlined when written as a sequence, we must track the placement of any overlined parts under the jigsaw map separately from the tableaux.

5.2 Buffered Frobenius Representations of the First Kind

We are now able to work towards the proof of the first half of Theorem 1.0.1. Namely, we show that

$$2\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 1}\frac{(-1)^{n+1}q^{n^2+kn}A_\ell(q^{kn})}{(1+q^{kn})(1-q^{kn})^\ell}$$

is the generating series for the ℓth positive moments of the ranks of a family of B-partitions.

Definition 5.2.1 (Morrill [17]). A buffered Frobenius representation of the first kind, or a B_1-representation for short, is a B-partition

$$\nu \in \begin{pmatrix} A_1 & A_2 & \dots & A_k \\ B_1 & B_2 & \dots & B_k \end{pmatrix},$$

in which

(1) A_1 is the set of non-empty partitions α_1 into distinct parts.

(2) A_2 is the set of non-empty partitions α_2 with $\#(\alpha_2) \leq \sigma(\alpha_1)$.

(3) For all $i \geq 3$, the set A_i is the set of non-empty partitions α_i with $\#(\alpha_i)$ less than or equal to the number of occurrences of the largest part of α_{i-1}.

(4) B_1 is the set of overpartitions β_1 into $\#(\alpha_1)$ nonnegative parts with $\overline{\sigma}(\beta_1) \leq \#(\lambda_2)$.

(5) For all $2 \leq i < k$, the set B_i is the set of partitions into $\#(\alpha_i)$ nonnegative parts with at least $\#(\lambda_{i+1})$ occurrences of its largest part.

(6) B_k is the set of partitions into $\#(\alpha_k)$ nonnegative parts.

We also define the empty array to be a B_1-representation with $k = 0$.

For a nontrivial example, take ν as in (5.1.3),

$$\nu = \begin{pmatrix} (4,\widehat{3,2,}1) & (\widehat{1,1,1}) & (1,1) \\ (\overline{3},\overline{2},2,2) & (\widehat{1,1,1}) & (3,3) \end{pmatrix}.$$

On the top row, α_1 is a partition into distinct parts, which satisfies condition (1). Next, α_2 is a partition into three parts. Because $\sigma(\alpha_1) = 4$, this satisfies condition (2). Finally, λ_3 is a partition into two parts. Since α_2 has three occurrences of its largest part, we have satisfied condition (3).

On the bottom row, β_1 is an overpartition into three parts with $\overline{\sigma}(\beta_1) = 3$, which satisfies condition (4). Next, β_2 is a partition into three parts. Since $\overline{\sigma}(\beta_1) = 3$, we have satisfied condition (5). Finally, β_3 is a partition into two parts, which satisfies condition (6).

Additionally, α_1, α_2, and β_2 have been marked with hats, and that α_3 and β_3 have not been marked. Therefore ν is a B_1-representation.

Let $\mathcal{B}_1$ denote the set of B_1-representations. We have already seen an example where the jigsaw map sends ν to the first Frobenius representation $j(\nu)$ of an overpartition. This is true for any $\nu \in \mathcal{B}_1$.

Proposition 5.2.1 (Morrill)**.** *Let $\mathcal{F}_1$ denote the set of first Frobenius representations of overpartitions. Then $j : \mathcal{B}_1 \twoheadrightarrow \mathcal{F}_1$.*

Proof. Given

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix},$$

we have seen that

$$j(\nu) = \begin{pmatrix} \alpha' \\ \\ \beta' \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_{k'} \\ \\ b_1 & b_2 & \dots & b_{k'} \end{pmatrix}$$

is an F-partition. It suffices to show that $j(\nu)$ is the first Frobenius representation of an overpartition, as defined in Chapter 3. That is, we must show that α' is a partition into distinct parts and that β' is an overpartition into nonnegative parts. Because α_1 is a partition into distinct positive parts, so is α'. Because β_1 is an overpartition into nonnegative parts, so is β'.

To see that j is surjective, let

$$\nu' = \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ \\ b_1 & b_2 & \dots & b_k \end{pmatrix}$$

be the first Frobenius representation of an overpartition. Let $\alpha_1 = (a_1, a_2, \dots, a_k)$ and $\beta_1 = (b_1, b_2, \dots, b_k)$. Then α_1 is a partition into distinct parts and β_1 is an overpartition into nonnegative parts. Let

$$\nu = \begin{pmatrix} \alpha_1 \\ \\ \beta_1 \end{pmatrix} \in \mathcal{B}_1.$$

Then we see that $j(\nu) = \nu'$. □

Note that the jigsaw map is not injective, since

$$j\left(\begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix}\right) = j\left(\begin{pmatrix} \widehat{\alpha_1} & \widehat{\alpha_2} & \dots & \widehat{\alpha_{k-1}} & \alpha_k \\ \\ \widehat{\beta_1} & \widehat{\beta_2} & \dots & \widehat{\beta_{k-1}} & \beta_k \end{pmatrix}\right).$$

By applying the bijection from Proposition 3.4.1, we see that every B_1-representation ν is a non-unique representation of an overpartition λ.

5.21 Ranks

Recall that $\mathcal{B}_1$ is the set of B_1-representations. Let

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix} \in \mathcal{B}_1.$$

We seek to define rank functions $\rho_1^i : \mathcal{B}_1 \to \mathbb{Z}$ which detect whether each component α_i and β_i is marked with a hat. For $i \geq 1$, we define the indicator functions $\chi_i : \mathcal{B}_1 \to \{-1, 0, 1\}$ to be

$$\chi_i(\nu) := \begin{cases} 1, & \text{if } \alpha_i \text{ is marked with a hat, and } \beta_i \text{ is not marked with a hat} \\ -1, & \text{if } \beta_i \text{ is marked with a hat, and } \alpha_i \text{ is not marked with a hat} \\ 0, & \text{ otherwise.} \end{cases}$$

Note that this definition implies $\chi_i(\nu) = 0$ if $k < i$. Recall the definitions of partition rank and CL-rank from Sections 2.5 and 3.21 We also define $h(\nu)$ to be the number of entries of ν which are marked with a hat. Note that the $\chi_i(\nu)$ and $h(\nu)$ are defined for any B-partition ν.

Definition 5.2.2 (Morrill [17])**.** Let

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix} \in \mathcal{B}_1$$

The *first rank* of ν is

$$\rho_1^1(\nu) := r(\alpha_1) - (\overline{r}_{CL}(\beta_1) + 1) + \chi_1(\nu), \tag{5.2.1}$$

that is, the partition rank of α_1 plus one, minus the CL-rank of β_1 plus χ_1. We also define $\rho_1^1(\emptyset) := 0$.

An example of the first rank of

$$\nu = \begin{pmatrix} (4, \widehat{3, 2,} 1) & (\widehat{1, 1, 1}) & (1, 1) \\ (\overline{3}, \overline{2}, 2, 2) & (\widehat{1, 1, 1}) & (3, 3) \end{pmatrix}.$$

is shown in Table 5.1. Recall that the definition of B_1-representations puts restrictions on $\sigma(\alpha_1)$ and $\overline{\sigma}(\beta_1)$, but not $\sigma(\alpha_i)$ or $\sigma(\beta_i)$ for $i > 1$. Because of this difference, our definition of the ith rank takes a different form for $i > 1$.

Definition 5.2.3 (Morrill [17]). Let

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix} \in \mathcal{B}_1.$$

For $i > 1$, the *ith rank* of ν is defined to be

$$\rho_1^i(\nu) := \begin{cases} (\ell(\lambda_i) - 1) - \ell(\mu_i) + \chi_i(\nu), & \text{if } i \leq k \\ 0, & \text{if } i > k \end{cases} \tag{5.2.2}$$

An example of the ith ranks of

$$\nu = \begin{pmatrix} (4,\widehat{3,2},1) & (\widehat{1,1,1}) & (1,1) \\ (\overline{3},\overline{2},2,2) & (\widehat{1,1,1}) & (3,3) \end{pmatrix}. \tag{5.2.3}$$

is shown in Table 5.1. Now equipped with the ith rank functions, we may define the *full rank* of a B_1 representation ν.

TABLE 5.1: The ranks of the B_1-representation ν given in (5.2.3).

	$r(\alpha_1)$	$(\overline{r}_{CL}(\beta_1) + 1$	$\chi_1(\nu)$	$\rho_1^1(\nu)$
1	0	2	1	-1
i	$\ell(\alpha_i) - 1$	$\ell(\beta_i)$	$\chi_i(\nu)$	$\rho_1^i(\nu)$
2	0	1	0	-1
3	0	3	0	-3
4	–	–	0	0

Definition 5.2.4 (Morrill [17]). Let

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix} \in \mathcal{B}_1$$

The full rank of ν is

$$\rho_1(\nu) = \sum_{i\geq 1} \rho_1^i(\nu). \tag{5.2.4}$$

Note that the sum in (5.2.4) converges, since the summands $\rho_1^i(\nu)$ vanish for $i > k$. For example, the full rank of ν in (5.2.3) is given by

$$\rho_1(\nu) = (-1) + (-1) + (-3) + 0 + 0 + \cdots = -5.$$

We define

$$\mathcal{B}_1^k := \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_j \\ \beta_1 & \beta_2 & \dots & \beta_j \end{pmatrix} \in \mathcal{B}_1 \;\middle|\; j \leq k \right\},$$

that is, $\mathcal{B}_1^k$ is the set of B_1-representations with at most k columns.

Definition 5.2.5 (Morrill [17])**.** Fix $k \geq 1$. We define

$$\overline{N[k]}(m,n) := \sum_{\substack{\nu\in\mathcal{B}_1^k \\ \rho_1(\nu)=km}} (-1)^{h(\nu)} \left(\prod_{i=1}^{k} \zeta_k^{(i-1)\rho_1^i(\nu)} \right)$$

That is, $\overline{N[k]}(m,n)$ is equal to the weighted count of the full ranks of B_1^k-representations $\nu \in \mathcal{B}_1$ such that $\rho_1(\nu) = km$, where the count is weighted by

$$(-1)^{h(\nu)} \left(\prod_{i=1}^{k} \zeta_k^{(i-1)\rho_1^i(\nu)} \right).$$

5.22 Generating Series of $\overline{N[k]}(m,n)$

We construct a generating series for the ranks of B_1-representations $\nu \in \mathcal{B}_1^k$ using Lemma 2.7.1 and Lemma 3.5.1. As in Chapter 4, given nonnegative integers $n_1, n_2, \dots, n_k$, we write $N_0 = 0$ and $N_i = n_1 + n_2 + \cdots + n_i$ for $1 \leq i \leq k$.

Theorem 5.2.2 (Morrill [17]). *Fix $k > 0$. The coefficient of $x_1^{m_1} x_2^{m_2} \dots x_k^{m_k} q^n$ in*

$$\sum_{\substack{n_1 \geq 0 \\ \vdots \\ n_k \geq 0}} (-1;q)_{N_k} q^{\frac{N_k^2 - N_k}{2}} \prod_{i=1}^{k} \frac{(1 - x_{k-i+1})(1 - x_{k-i+1}^{-1}) q^{N_i}}{(x_{k-i+1} q^{N_{i-1}}, x_{k-i+1}^{-1} q^{N_{i-1}}; q)_{n_i+1}} \tag{5.2.5}$$

is equal to the weighted count of B_1-representations

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_j \\ \beta_1 & \beta_2 & \dots & \beta_j \end{pmatrix} \in \mathcal{B}_1^k$$

such that $|\nu| = n$ and $\rho_1^i(\nu) = m_i$, where the count is weighted by $(-1)^{h(\nu)}$.

Proof. We begin by analyzing an arbitrary summand

$$(-1;q)_{N_k} q^{\frac{N_k^2 - N_k}{2}} \prod_{i=1}^{k} \frac{(1 - x_{k-i+1})(1 - x_{k-i+1}^{-1}) q^{N_i}}{(x_{k-i+1} q^{N_{i-1}}, x_{k-i+1}^{-1} q^{N_{i-1}}; q)_{n_i+1}}, \tag{5.2.6}$$

where $n_1, n_2, \dots, n_k$ are nonnegative integers. If $n_1 = \dots = n_k = 0$, then (5.2.6) reduces to 1. This corresponds to the empty array, which is defined to have weight and rank equal to 0.

Otherwise, $n_i > 0$ for some i. Let j be the smallest index so that $n_j > 0$. Then for all $i < j$, we have $n_i = 0$, $N_i = 0$, and $N_{i-1} = 0$. We can reduce the multiplicand

$$\frac{(1 - x_{k-i+1})(1 - x_{k-i+1}^{-1}) q^{N_i}}{(x_{k-i+1} q^{N_{i-1}}, x_{k-i+1}^{-1} q^{N_{i-1}}; q)_{n_i+1}} = \frac{(1 - x_{k-i+1})(1 - x_{k-i+1}^{-1})}{(1 - x_{k-i+1})(1 - x_{k-i+1}^{-1})} = 1.$$

We see that the summand in (5.2.6) reduces to

$$(-1;q)_{N_k} q^{\frac{N_k^2 - N_k}{2}} \prod_{i=j}^{k} \frac{(1 - x_{k-i+1})(1 - x_{k-i+1}^{-1}) q^{N_i}}{(x_{k-i+1} q^{N_{i-1}}, x_{k-i+1}^{-1} q^{N_{i-1}}; q)_{n_i+1}}. \tag{5.2.7}$$

We also reindex the summation to obtain

$$(-1;q)_{N_k} q^{\frac{N_k^2 - N_k}{2}} \prod_{i=1}^{k-j+1} \frac{(1 - x_i)(1 - x_i^{-1}) q^{N_{k-i+1}}}{(x_i q^{N_{k-i}}, x_i^{-1} q^{N_{k-i}}; q)_{n_{k-i+1}+1}}. \tag{5.2.8}$$

We claim that the coefficient of $z^{m_i}q^n$ in the ith factor of (5.2.8) is equal to the number of possible columns $(\alpha_i, \beta_i)^T$ in a B_1-representation

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{k-j+1} \\ \beta_1 & \beta_2 & \dots & \beta_{k-j+1} \end{pmatrix},$$

where $n = |\alpha_i| + |\beta_i|$, $N_{k-i+1} = \#(\alpha_i)$ and $m_i = \rho_1^i(\nu)$.

When $i = 1$, we use Lemma 2.7.1 and Lemma 3.5.1, we see that the coefficient of $x_1^{m_1}q^n$ in the term

$$\frac{(-1;q)_{N_k} q^{\frac{N_k^2+N_k}{2}}}{(x_1 q^{N_{k-1}}, x_1^{-1} q^{N_{k-1}}; q)_{n_k+1}} \tag{5.2.9}$$

equal to the number of pairs (α_1, β_1) as follows. First, α_1 is a partition with $\#(\alpha_1) = N_k$ and $\sigma(\alpha_1) \geq N_{k-1}$. Second, β_1 is an overpartition into nonnegative parts with $\#(\beta_1) = N_k$ and $\overline{\sigma}(\beta_1) \geq N_{k-1}$. These bounds correspond to criteria (1), (2), (4), and (5) in the definition of B_1-representations. Finally, we see that $m_1 = r(\alpha_1) - (\overline{r}(\beta_1) + 1)$.

Given an arbitrary (α_1, β_1), the coefficient of $x_1^{m_1}$ in $(1 - x_1)(1 - x_1^{-1})$ is equal to the weighted count of ways of marking the entries α_1 and β_1 of ν with hats, where $m_1 = \chi_1(\nu)$ and the count is weighted by $(-1)^{\chi_1(\nu)}$. Neither of the configurations (α_1, β_1) and $(\widehat{\alpha_1}, \widehat{\beta_1})$ change $\rho_1^1(\nu)$, nor the weight of the count, $(-1)^{h(\nu)}$. The configuration $(\widehat{\alpha_1}, \beta_1)$ increases $\rho_1^1(\nu)$ by 1, and the configuration $(\alpha_1, \widehat{\beta_1})$ decreases $\rho_1^1(\nu)$ by 1. Each of the latter two configurations weights the count by an additional factor of -1. Therefore, the coefficient of $x_i^{m_i}q^n$ in the term

$$(-1;q)_{N_k} q^{\frac{N_k^2+N_k}{2}} \frac{(1 - x_1)(1 - x_1^{-1})}{(x_1 q^{N_{k-1}}, x_1^{-1} q^{N_{k-1}}; q)_{n_k+1}} \tag{5.2.10}$$

is equal to the weighted count of possible columns $(\alpha_1, \beta_1)^T$ of a B_1-representation ν such that $\#(\alpha_1) = N_k$ and $\rho_1^1(\nu) = m_1$, where the count is weighted by $(-1)^{\chi_1(\nu)}$.

We now consider $2 \leq i \leq k - j$. Note that $k - i \geq j$, which implies that $N_{k-i} \neq 0$. As in the proof of Lemma 2.7.1, the coefficient of $x_i^{m_i}q^n$ in the term

$$\frac{q^{N_{k-i+1}}}{(x_iq^{N_{k-i}}, x_i^{-1}q^{N_{k-i}}; q)_{n_{k-i+1}+1}}.$$

is equal to the number of pairs (α_i, β_i) as follows. First, α_i is a partition into N_{k-i+1} parts with at least N_{k-i} occurrences of its largest part. Next, β_i is a partition into N_{k-i+1} nonnegative parts with at least N_{k-i} occurrences of its largest part. These bounds correspond to criteria (2) and (5) in the definition of B_1-representations. Finally, we see that $m_i = (\ell(\alpha_1) - 1) - \ell(\beta_1)$.

As above, the coefficient of $x_i^{m_i}$ in $(1-x_i)(1-x_i^{-1})$ is equal to the weighted count of ways of marking an arbitrary α_i or β_i with hats, where $m_i = \chi_i(\nu)$ and the count is weighted by $(-1)^{\chi_i(\nu)}$. Therefore, the coefficient of $x_i^{m_i}q^n$ in

$$\frac{(1-x_i)(1-x_i^{-1})q^{N_{k-i+1}}}{(x_iq^{N_{k-i}}, x_i^{-1}q^{N_{k-i}}; q)_{n_{k-i+1}+1}}$$

is equal to the weighted count of possible columns $(\alpha_i, \beta_i)^T$ of a B_1-representation ν such that $\#(\alpha_i) = N_{k-i+1}$ and $\rho_1^i(\nu) = m_i$, where the count is weighted by $(-1)^{\chi_i(\nu)}$.

For $i = k - j + 1$, we can reduce the term

$$\frac{(1-x_{k-j+1})(1-x_{k-j+1}^{-1})q^{N_j}}{(x_{k-j+1}q^{N_{j-1}}, x_{k-j+1}^{-1}q^{N_{j-1}}; q)_{n_j+1}} = \frac{q^{N_j}}{(x_{k-j+1}q, x_{k-j+1}^{-1}q; q)_{n_j}}, \tag{5.2.11}$$

since $N_{j-1} = 0$. As above, we see that the coefficient of $x_{k-j+1}^{m_{k-j+1}}q^n$ in (5.2.11) is equal to the number of pairs $(\alpha_{k-j+1}, \beta_{k-j+1})$ such that α_{k-j+1} is a partition into N_j parts, β_{k-j+1} is a partition into N_j nonnegative parts, and $m_{k-j+1} = (\ell(\alpha_{k-j+1})-1)-\ell(\beta_{k-j+1})$. These bounds correspond to criteria (3) and (6) in the definition of B_1-representations.

Recall that entries in the last column of ν, cannot be marked with a hat. Therefore, the coefficient of $x_j^{m_j}q^n$ in (5.2.11) is equal to the number of possible columns $(\alpha_{k-j+1}, \beta_{k-j+1})^T$ of B_1-representation ν such that $\#(\alpha_{k-j+1}) = N_1$ and $\rho_1^{k-j+1}(\nu) = m_{k-j+1}$.

Given $\alpha_1, \alpha_2, \ldots, \alpha_j$ and $\beta_1, \beta_2, \ldots, \beta_j$ as above,

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_j \\ \beta_1 & \beta_2 & \ldots & \beta_j \end{pmatrix}$$

is a B_1-representation with $\rho_1^i(\nu) = m_i$ and $|\nu| = n$, and $\#(\alpha_i) = N_{k-i+1}$. By the definition of $h(\nu)$ and $\chi_i(\nu)$, we have that

$$(-1)^{\chi_1(\nu)}(-1)^{\chi_2(\nu)} \ldots (-1)^{\chi_{k-j+1}(\nu)} = (-1)^{h(\nu)}$$

By multiplying the coefficients of the factors (5.2.8), we find that the coefficient of $x_1^{m_1} x_2^{m_2} \ldots x_{k-j+1}^{m_{k-j+1}} q^n$ in (5.2.8) is equal to the weighted count of B_1-representations

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{k-j+1} \\ \beta_1 & \beta_2 & \ldots & \beta_{k-j+1} \end{pmatrix},$$

where $n = |\nu|$, and for all $1 \leq i \leq k-j+1$ we have $N_{k-i+1} = \#(\alpha_i)$ and $m_i = \rho_1^i(\nu)$, where count is weighted by $(-1)^{h(\nu)}$.

We now return to the full sum

$$\sum_{\substack{n_1 \geq 0 \\ \vdots \\ n_k \geq 0}} (-1;q)_{N_k} q^{\frac{N_k^2 - N_k}{2}} \prod_{i=1}^{k} \frac{(1 - x_{k-i+1})(1 - x_{k-i+1}^{-1}) q^{N_i}}{(x_{k-i+1} q^{N_{i-1}}, x_{k-i+1}^{-1} q^{N_{i-1}}; q)_{n_i+1}} \tag{5.2.12}$$

By summing over all possible values of $n_1, n_2, \ldots, n_k$, we see that the coefficient of $x_1^{m_1} x_2^{m_2} \ldots x_k^{m_k} q^n$ in (5.2.5) is equal to the weighted count of B_1-representations with $|\nu| = n$ and $\rho_1^i(\nu) = m_i$, where the count is weighted by $(-1)^{h(\nu)}$, as desired. □

Corollary 5.2.1 (Morrill [17])**.** *Fix $k \geq 1$. Then*

$$\begin{aligned} \overline{R[k]}(z,q) &:= \frac{(-q;q)_\infty}{(q;q)_\infty} \left(1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+kn}}{(1 - zq^{kn})(1 - z^{-1} q^{kn})} \right) \\ &= \sum_{m \in \mathbb{Z}} \sum_{n \geq 0} \overline{N[k]}(m,n) z^m q^n \end{aligned}$$

That is, $\overline{R[k]}(z,q)$ is the generating series for $\overline{N[k]}(m,n)$.

Proof. By Theorem 4.0.2, we see that

$$\sum_{\substack{n_1\geq 0\\ \vdots \\ n_k\geq 0}} (-1;q)_{N_k} q^{\frac{N_k^2-N_k}{2}} \prod_{i=1}^{k} \frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})q^{N_i}}{(x_{k-i+1}q^{N_{i-1}}, x_{k-i+1}^{-1}q^{N_{i-1}};q)_{n_i+1}}$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2+kn}\prod_{i=1}^{k}\frac{(1-x_i)(1-x_i^{-1})}{(1-x_iq^n)(1-x_i^{-1}q^n)}\right). \quad (5.2.13)$$

By Theorem 5.2.2, the coefficient of $x_1^{m_1}x_2^{m_2}\dots x_k^{m_k}q^n$ in (5.2.13) is equal to the weighted count of B_1-representations $\nu\in\mathcal{B}_1^k$ with $|\nu|=n$ and $\rho_1^i(\nu)=m_i$, where the count is weighted by $(-1)^{h(\nu)}$.

Let ζ_k be a primitive kth root of unity, and substitute $x_i\mapsto\zeta_k^{i-1}z^{\frac{1}{k}}$ in (5.2.13). This maps

$$x_1^{m_1}x_2^{m_2}\dots x_k^{m_k}q^n \to \left(\prod_{i=1}^{k}(\zeta_k^{i-1}z^{\frac{1}{k}})^{m_i}\right)q^n = \zeta_k^{m_2+2m_3+\cdots+(k-1)m_k}z^{\frac{m_1+m_2+\cdots+m_k}{k}}q^n.$$

Naively, this substitution produces

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2+kn}\prod_{i=1}^{k}\frac{(1-\zeta_k^{i-1}z^{\frac{1}{k}})(1-(\zeta_k^{i-1}z^{\frac{1}{k}})^{-1})}{(1-\zeta_k^{i-1}z^{\frac{1}{k}}q^n)(1-(\zeta_k^{i-1}z^{\frac{1}{k}})^{-1}q^n)}\right) \quad (5.2.14)$$

as an element of $\mathbb{Z}[\zeta_k]((z^{\frac{1}{k}}))[[q]]$, the ring of formal power series in the variable q whose the coefficients are in the ring $\mathbb{Z}[\zeta_k]((z^{\frac{1}{k}}))$. Recall that $\mathbb{Z}[\zeta_k]((z^{\frac{1}{k}}))$ is the ring formal Laurent series in the variable $z^{\frac{1}{k}}$ with coefficients in $\mathbb{Z}[\zeta_k]$.

We have already defined ρ_1 to be the sum of the column ranks,

$$\rho_1(\nu)=\rho_1^1(\nu)+\rho_1^2(\nu)+\cdots+\rho_1^k(\nu).$$

In this light, the coefficient of $(z^{\frac{1}{k}})^m q^n$ in (5.3.16) is equal to weighted count the number of B_1-representations $\nu\in\mathcal{B}_1^k$, with $\rho_1(\nu)=m$ and $|\nu|=m$, where the count is weighted by

$$(-1)^{h(\nu)}\zeta_k^{\rho_1^2(\nu)+2\rho_1^3(\nu)+\cdots+(k-1)\rho_1^k(\nu)}, \quad (5.2.15)$$

which is the weight of the count in the definition of $\overline{N[k]}(m,n)$.

Since

$$\prod_{i=0}^{k-1}(1-\zeta_k^i X) = 1 - X^k,$$

the right hand side of (5.2.13) reduces to $\overline{R[k]}(z,q)$. Moreover, since $\overline{R[k]}(z,q)$ is defined in terms of the variable z, not $z^{\frac{1}{k}}$, we see that the weighted count vanishes for B_1-representations ν with $\rho_1(\nu) \notin k\mathbb{Z}$. Therefore, the coefficient of $z^m q^n$ in $\overline{R[k]}(z,q)$ is equal to $\overline{N[k]}(m,n)$. □

Note that Corollary 5.2.1 implies that $\overline{N[k]}(m,n) \in \mathbb{Z}$ for all k, m, and n.

5.23 The ℓth Positive Moments of $\overline{N[k]}(m,n)$

We now work towards the proof of Theorem 1.0.1. The first half of Theorem 1.0.1 involves the ℓth positive moments of $\overline{N[k]}(m,n)$. We require a proposition of Larsen, Rust and Swisher. Recall that $A_\ell(q)$ denotes the ℓth Eulerian polynomial.

Proposition 5.2.3 (Larsen, Rust, Swisher [14])**.** *Let $\ell \geq 1$. Then,*

$$\left(z\frac{\partial}{\partial z}\right)^{\ell-1} \frac{z}{(1-zq^n)^2} = \frac{zA_\ell(zq^n)}{(1-zq^n)^{\ell+1}}$$

We may now prove the first half of Theorem 1.0.1.

Theorem 5.2.4 (Morrill)**.** *For all $k \geq 1$, the ℓth positive moment generating series for $\overline{N[k]}(m,n)$ is given by*

$$\sum_{n\geq 0}\left(\sum_{m\geq 1} m^\ell \overline{N[k]}(m,n)\right) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 1} \frac{(-1)^{n+1} q^{n^2+kn} A_\ell(q^{kn})}{(1+q^{kn})(1-q^{kn})^\ell}.$$

Proof. By Corollary 5.2.1,

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} \overline{N[k]}(m,n) z^m q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \frac{(1-z)(1-z^{-1})q^{kn}}{(1-zq^{kn})(1-z^{-1}q^{kn})}\right).$$

We can verify that

$$\frac{(1-z)(1-z^{-1})q^{kn}}{(1-zq^{kn})(1-z^{-1}q^{kn})} = 1 - \frac{1-q^n}{1+q^n}\sum_{m=0}^{\infty} z^m q^{kn} - \frac{1-q^n}{1+q^n}\sum_{m=1}^{\infty} z^{-m} q^{kn}$$
$$= 1 - \frac{1-q^n}{1+q^n}\frac{1}{1-zq^n} - \frac{1-q^n}{1+q^n}\sum_{m=1}^{\infty} z^{-m} q^{kn}$$

By Proposition 3.8.1 and Proposition 5.2.3

$$\sum_{n\geq 0}\left(\sum_{m\geq 1} m^\ell \overline{N[k]}(m,n)\right) q^n = \left[\left(z\frac{\partial}{\partial z}\right)^{\ell-1} \frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\frac{z(1-q^n)}{(1-zq^{kn})}\right)\right]_{z=1}$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\left[\left(z\frac{\partial}{\partial z}\right)^{\ell-1}\frac{z(1-q^n)}{(1-zq^{kn})}\right]_{z=1}\right)$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 1}\frac{(-1)^{n+1}q^{n^2+kn}A_\ell(q^{kn})}{(1+q^{kn})(1-q^{kn})^\ell},$$

as desired. □

5.3 Buffered Frobenius Representations of the Second Kind

Next, we work towards the proof of the second half of Theorem 1.0.1. Namely, we show that

$$2\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 1}\frac{(-1)^{n+1}q^{n^2+2kn}A_\ell(q^{2kn})}{(1+q^{2kn})(1-q^{2kn})^\ell}$$

is the generating series for the ℓth positive moments of the ranks of a second family of B-partitions.

Definition 5.3.1 (Morrill [17]). A buffered Frobenius representation of the second kind, or a B_2-representation, is a B-partition

$$\nu \in \begin{pmatrix} A_1 & A_2 & \dots & A_k \\ B_1 & B_2 & \dots & B_k \end{pmatrix}$$

where

(1) A_1 is the set of nonempty overpartitions α_1 into odd parts[9].

(2) A_2 is the set of nonempty partitions α_2 into even parts, with $\#(\lambda_2) \leq \overline{\sigma}_2(\lambda_1)$.

(3) For all $3 < i \leq k$, A_i is the set of nonempty partitions α_i into even parts with $\#(\lambda_i)$ less than or equal to the number of occurrences of the largest part of λ_{i-1}

(4) B_1 is the set of partitions β_1 into $\#(\lambda_1)$ nonnegative parts where odd parts may not repeat, with $\sigma_2(\mu_1) \leq \#(\lambda_2)$.

(5) For all $2 \leq i < k$, B_i is the set of partitions β_i into $\#(\lambda_i)$ nonnegative even parts and at most $\#(\lambda_{i+1})$ occurrences of their largest part.

(6) B_k is the set of partitions β_i into $\#(\lambda_i)$ nonnegative even parts.

We also define the empty array to be a B_1-representation with $k = 0$.

For a nontrivial example, consider the array

$$\nu = \begin{pmatrix} \widehat{(3, \overline{1})} & (2,2) & (4) \\ (6,5) & (2,0) & (2) \end{pmatrix}. \tag{5.3.1}$$

On the top row, from left to right, α_1 is an overpartition into odd parts with $\overline{\sigma}_2(\alpha_1) = 2$, which satisfies condition (1). Next, α_2 is a partition into two even parts, with two occurrences of its largest part, which satisfies condition (2). Last, α_3 is an partition into a single even part, which satisfies (3).

On the bottom row, β_1 is a partition into two parts and has no repeating odd parts, with $\sigma_2(\mu_1) = 2$, which satisfies condition (4). We see that β_2 is a partition into two nonnegative even parts with a single occurrence of its largest part. Because there are only

[9]Note that the overpartition row has been switched from Section 5.2. This is done in order to retain notation for the second Frobenius representation of an overpartition as seen in [16].

three columns, we only need to verify that β_3 has a single part to satisfy condition (5). Therefore, ν is a buffered Frobenius representation of the second kind with three columns.

As was the case for B_1-representations, we see that B_2-representations lie over the second Frobenius representations of overpartitions.

Proposition 5.3.1 (Morrill)**.** *Let $\mathcal{B}_2$ denote the set of buffered Frobenius representations of the second kind, and let $\mathcal{F}_2$ denote the set of second Frobenius representations of overpartitions. Then $j : \mathcal{B}_2 \twoheadrightarrow \mathcal{F}_1$.*

Proof. The proof is nearly identical to that of Proposition 5.2.1. If

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix}$$

we only need to show that

$$j(\nu) = \begin{pmatrix} a_1 & a_2 & \dots & a_{\#(\alpha_1)} \\ b_1 & b_2 & \dots & b_{\#(\beta_1)} \end{pmatrix}$$

is a second Frobenius representation of an overpartition as in Chapter 3. That is, we must show that α' is an overpartition where odd parts may not repeat, and that β' is a partition whose odd parts may not repeat. The former follows from the fact that α_1 is an overpartition into odd parts and that the other α_i each have even parts. Similarly, the latter is proven since β_1 is an partition into nonnegative parts whose odd parts may not repeat, and all other β_i each have even parts. Thus, combining the parts of the partitions under the jigsaw map cannot change the parity of the restricted parts.

To see that j is surjective, let

$$\nu' = \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ b_1 & b_2 & \dots & b_k \end{pmatrix}$$

be the second Frobenius representation of an overpartition. Let $\alpha_1 = (a_1, a_2, \ldots, a_k)$ and $\beta_1 = (b_1, b_2, \ldots, b_k)$. Then α_1 is an overpartition and β_1 is an partition into nonnegative parts whose odd parts are distinct. Let

$$\nu = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \in \mathcal{B}_1.$$

Then we see that $j(\nu) = \nu'$.

This is equivalent to the second Frobenius representation of an overpartition as defined in Chapter 3. □

In the proof, we see that buffered Frobenius representations of the second kind subsume the second Frobenius representation of an overpartition in the terminology of [16]. As in Section 5.2, we have generalized the second Frobenius representation of an overpartition.

Note that the jigsaw map is not injective, since

$$j\left(\begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_k \\ \beta_1 & \beta_2 & \ldots & \beta_k \end{pmatrix}\right) = j\left(\begin{pmatrix} \widehat{\alpha_1} & \widehat{\alpha_2} & \ldots & \widehat{\alpha_{k-1}} & \alpha_k \\ \widehat{\beta_1} & \widehat{\beta_2} & \ldots & \widehat{\beta_{k-1}} & \beta_k \end{pmatrix}\right).$$

By applying the bijection from Proposition 3.4.2, we see that every B_2-representation ν is a non-unique representation of an overpartition.

5.31 Ranks

Recall that $\mathcal{B}_2$ is the set of B_2-representations. Let

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_k \\ \beta_1 & \beta_2 & \ldots & \beta_k \end{pmatrix} \in \mathcal{B}_2.$$

We seek to define rank functions $\rho_2^i : \mathcal{B}_2 \to \mathbb{Z}$ which detect whether each component α_i and β_i is marked with a hat. Recall the definition of $\chi_i(\nu)$ and $h(\nu)$ from Section 5.21. We also require the second partition rank and the second overpartition rank from Sections 2.8 and 3.22.

Definition 5.3.2 (Morrill [17])**.** Let

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix} \in \mathcal{B}_2$$

The *first rank* of ν is

$$\rho_2^1(\nu) := \overline{r}_2(\alpha_1) - r_2(\beta_1) + \chi_1(\nu), \tag{5.3.2}$$

that is, the second rank of α_1, minus the second partition rank of β_1, plus χ_1. We also define $\rho_2^1(\emptyset) := 0$.

An example of the second rank of

$$\nu = \begin{pmatrix} \widehat{(3,\overline{1})} & (2,2) & (4) \\ (6,5) & (2,0) & (2) \end{pmatrix} \tag{5.3.3}$$

is shown in Table 5.2.

Recall that the definition of B_2-representations puts restrictions on $\overline{\sigma}_2(\alpha_1)$ and $\sigma_2(\beta_1)$, but not $\sigma(\alpha_i)$ or $\sigma(\beta_i)$ for $i > 1$. Because of this difference, our definition of the ith rank takes a different form for $i > 1$.

Definition 5.3.3 (Morrill [17])**.** Let

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix} \in \mathcal{B}_2.$$

For $i > 1$, the *ith rank* of ν is defined to be

$$\rho_1^i(\nu) := \begin{cases} \frac{\ell(\alpha_i)}{2} - 1 - \frac{\ell(\beta_i)}{2} + \chi_i(\nu) & \\ 0, & \text{if } i > k \end{cases} \tag{5.3.4}$$

Note that (5.3.4) defines an integer for all $\nu \in \mathcal{B}_2$, since λ_i and μ_i consist of even parts when $i > 1$. An example of the ith ranks of

$$\nu = \begin{pmatrix} \widehat{(3,\overline{1})} & (2,2) & (4) \\ (6,5) & (2,0) & (2) \end{pmatrix} \tag{5.3.5}$$

is shown in Table 5.2. Now equipped with the ith rank functions, we may define the *full rank* of a B_2 representation ν.

TABLE 5.2: The ranks of the B_2-representation ν given in (5.3.5).

	$r(\alpha_1)$	$(\overline{r}_{CL}(\beta_1)+1$	$\chi_1(\nu)$	$\rho_1^1(\nu)$
1	0	2	1	-1
i	$\ell(\alpha_i)-1$	$\ell(\beta_i)$	$\chi_i(\nu)$	$\rho_1^i(\nu)$
2	0	1	0	-1
3	0	3	0	-3
4	–	–	0	0

For example, take ν as in (5.3.1),

$$\nu = \begin{pmatrix} \widehat{(3,\overline{1})} & (2,2) & (4) \\ (6,5) & (2,0) & (2) \end{pmatrix}. \tag{5.3.6}$$

Then $\chi_1(\nu) = 1$, and $\chi_2(\nu) = \chi_3(\nu) = 0$. The first rank of ν is equal to $0 - 2 + 1 = -1$, the second rank of ν is equal to $1 - 1 - 1 + 0 = -2$, and the third rank of ν is equal to $2 - 1 - 1 + 0 = 0$. For $i > 3$, we see that $\rho_i(\nu) = 0$ by definition.

Definition 5.3.4 (Morrill [17])**.** Let

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix} \in \mathcal{B}_2$$

The full rank of ν is

$$\rho_2(\nu) = \sum_{i\geq 1} \rho_2^i(\nu). \tag{5.3.7}$$

Note that the sum in (5.3.7) converges, since the summands $\rho_2^i(\nu)$ vanish for $i > k$. For example, the full rank of ν in (5.2.3) is given by

$$\rho_1(\nu) = (-1) + (-1) + (-3) + 0 + 0 + \cdots = -5.$$

We define

$$\mathcal{B}_2^k := \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_j \\ \beta_1 & \beta_2 & \dots & \beta_j \end{pmatrix} \in \mathcal{B}_2 \;\middle|\; j \leq k \right\},$$

that is, $\mathcal{B}_2^k$ is the set of B_1-representations with at most k columns.

Definition 5.3.5 (Morrill [17]). Fix $k \geq 1$. We define

$$\overline{N2[k]}(m,n) := \sum_{\substack{\nu \in \mathcal{B}_2^k \\ \rho_1(\nu)=km}} (-1)^{h(\nu)} \left(\prod_{i=1}^{k} \zeta_k^{(i-1)\rho_2^i(\nu)} \right)$$

That is, $\overline{N2[k]}(m,n)$ is equal to the weighted count of the full ranks of B_2^k-representations $\nu \in \mathcal{B}_2$ such that $\rho_2(\nu) = km$, where the count is weighted by

$$(-1)^{h(\nu)} \left(\prod_{i=1}^{k} \zeta_k^{(i-1)\rho_2^i(\nu)} \right).$$

5.32 Generating Series of $\overline{N2[k]}(m,n)$

We now construct a generating series for the ranks of B_2-representations $\nu \in \mathcal{B}_2^k$ using Lemma 2.9.1 and Lemma 3.6.1. As in Section 5.22, given nonnegative integers $n_1, n_2, \dots, n_k$, we write $N_0 = 0$ and $N_i = n_1 + n_2 + \cdots + n_i$ for $1 \leq i \leq k$.

We see the generating function for the ith ranks of a buffered Frobenius representation with exactly k columns given in the theorem below.

Theorem 5.3.2 (Morrill [17]). *Fix $k > 0$. The coefficient of $x_1^{m_1} x_2^{m_2} \dots x_k^{m_k} q^n$ in*

$$\sum_{\substack{n_1 \geq 0 \\ \vdots \\ n_k \geq 0}} \frac{(-1;q)_{2N_k}}{q^{N_k}} \prod_{i=1}^{k} \frac{(1 - x_{k-i+1})(1 - x_{k-i+1}^{-1}) q^{2N_i}}{(x_{k-i+1} q^{2N_{i-1}}, x_{k-i+1}^{-1} q^{2N_{i-1}}; q^2)_{n_i+1}} \tag{5.3.8}$$

is equal to the weighted count of B_2-representations

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_j \\ \beta_1 & \beta_2 & \dots & \beta_j \end{pmatrix} \in \mathcal{B}_2^k$$

such that $|\nu| = n$ and $\rho_2^i(\nu) = m_i$, where the count is weighted by $(-1)^{h(\nu)}$.

Proof. We begin by analyzing an arbitrary summand

$$\frac{(-1;q)_{2N_k}}{q^{N_k}} \prod_{i=1}^{k} \frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})q^{2N_i}}{(x_{k-i+1}q^{2N_{i-1}}, x_{k-i+1}^{-1}q^{2N_{i-1}}; q^2)_{n_i+1}} \tag{5.3.9}$$

where $n_1, n_2, \ldots, n_k$ are nonnegative integers. If $n_1 = \cdots = n_k = 0$, then (5.3.9) reduces to 1. This corresponds to the empty array, which is defined to have weight and rank equal to 0.

Otherwise, $n_i > 0$ for some i. Let j be the smallest index so that $n_j > 0$. Then for all $i < j$, we have $n_i = 0$, $N_i = 0$, and $N_{i-1} = 0$. We can reduce the multiplicand

$$\frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})q^{2N_i}}{(x_{k-i+1}q^{2N_{i-1}}, x_{k-i+1}^{-1}q^{2N_{i-1}}; q^2)_{n_i+1}} = \frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})}{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})} = 1.$$

We see that the summand in (5.3.9) reduces to

$$\frac{(-1;q)_{2N_k}}{q^{N_k}} \prod_{i=j}^{k} \frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})q^{2N_i}}{(x_{k-i+1}q^{2N_{i-1}}, x_{k-i+1}^{-1}q^{2N_{i-1}}; q^2)_{n_i+1}} \tag{5.3.10}$$

We also reindex the summation to obtain

$$\frac{(-1;q)_{2N_k}}{q^{N_k}} \prod_{i=1}^{k-j+1} \frac{(1-x_i)(1-x_i^{-1})q^{2N_{k-i+1}}}{(x_i q^{2N_{k-i}}, x_i^{-1}q^{2N_{k-i}}; q^2)_{n_{k-i+1}+1}}. \tag{5.3.11}$$

We claim that the coefficient of $x_i^{m_i}q^n$ in the ith factor of (5.3.11) is equal to the number of possible columns $(\alpha_i, \beta_i)^T$ in a B_2-representation

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{k-j+1} \\ \beta_1 & \beta_2 & \ldots & \beta_{k-j+1} \end{pmatrix},$$

where $n = |\alpha_i| + |\beta_i|$, $N_{k-i+1} = \#(\alpha_i)$ and $m_i = \rho_2^i(\nu)$.

When $i = 1$, we use Lemma 2.9.1 and Lemma 3.6.1 to see that the coefficient of $x_1^{m_1}q^n$ in the term

$$\frac{(-1;q)_{2N_k}q^{2N_k}}{(x_1 q^{N_{k-1}}, x_1^{-1}q^{2N_{k-1}}; q^2)_{n_k+1}q^{N_k}} = \frac{(-1,-q;q^2)_{N_k}q^{N_k}}{(x_1 q^{N_{k-1}}, x_1^{-1}q^{2N_{k-1}}; q^2)_{n_k+1}}$$

is equal to the number of pairs (α_1, β_1) as follows. First, α_1 is an overpartition with $\#(\alpha_1) = N_k$ and $\overline{\sigma}_2(\alpha_1) \geq N_{k-1}$. Second, β_1 is an partition into nonnegative parts with $\#(\beta_1) = N_k$ and $\sigma_2(\beta_1) \geq N_{k-1}$. These bounds correspond to criteria (1), (2), (4), and (5) in the definition of B_1-representations. Finally, we see that $m_1 = \overline{r}_2(\alpha_1) - r_2(\beta_1)$.

Given an arbitrary (α_1, β_1), the coefficient of $x_1^{m_1}$ in $(1 - x_1)(1 - x_1^{-1})$ is equal to the weighted count of ways of marking the entries α_1 and β_1 of ν with hats, where $m_1 = \chi_1(\nu)$ and the count is weighted by $(-1)^{\chi_1(\nu)}$. Neither of the configurations (α_1, β_1) and $(\widehat{\alpha_1}, \widehat{\beta_1})$ change $\rho_2^1(\nu)$, nor the weight of the count, $(-1)^{h(\nu)}$. The configuration $(\widehat{\alpha_1}, \beta_1)$ increases $\rho_2^1(\nu)$ by 1, and the configuration $(\alpha_1, \widehat{\beta_1})$ decreases $\rho_2^1(\nu)$ by 1. Each of the latter two configurations weights the count by an additional factor of -1. Therefore, the coefficient of $x_i^{m_i} q^n$ in the term

$$\frac{(1 - x_i)(1 - x_i^{-1})q^{2N_{k-i+1}}}{(x_i q^{2N_{k-i}}, x_k^{-1} q^{2N_{k-i}}; q^2)_{n_{k-i+1}+1}} \tag{5.3.12}$$

is equal to the weighted count of possible columns $(\alpha_1, \beta_1)^T$ of a B_2-representation ν such that $\#(\alpha_1) = N_k$ and $\rho_2^1(\nu) = m_1$, where the count is weighted by $(-1)^{\chi_1(\nu)}$.

We now consider $2 \leq i \leq k - j$. Note that $k - i \geq j$, which implies that $N_{k-i+1} \neq 0$. As in the proof of Lemma 3.6.1, the coefficient of $x_i^{m_i} q^n$ in the term

$$\frac{q^{2N_{k-i+1}}}{(x_i q^{2N_{k-i}}, x_k^{-1} q^{2N_{k-i}}; q^2)_{n_{k-i+1}+1}}$$

is equal to the number of pairs (α_i, β_i) as follows. First, α_i is a partition into N_{k-i+1} even parts with at least N_{k-i} occurrences of its largest part. Next, β_i is a partition into N_{k-i+1} nonnegative even parts with at least N_{k-i} occurrences of its largest part. These bounds correspond to criteria (2) and (5) in the definition of B_1-representations. Finally, we see that

$$m_i = \frac{\ell(\alpha_i)}{2} - 1 - \frac{\ell(\beta_i)}{2},$$

which is $\rho_2^i(\nu)$.

As above, the coefficient of $x_i^{m_i}$ in $(1-x_i)(1-x_i^{-1})$ is equal to the weighted count of ways of marking an arbitrary α_i or β_i with hats, where $m_i = \chi_i(\nu)$ and the count is weighted by $(-1)^{\chi_i(\nu)}$. Therefore, the coefficient of $x_i^{m_i} q^n$ in

$$\frac{(1-x_i)(1-x_i^{-1})q^{2N_{k-i+1}}}{(x_i q^{2N_{k-i}}, x_k^{-1} q^{2N_{k-i}}; q^2)_{n_{k-i+1}+1}}$$

is equal to the weighted count of possible columns $(\alpha_i, \beta_i)^T$ of a B_2-representation ν such that $\#(\alpha_i) = N_{k-i+1}$ and $\rho_2^i(\nu) = m_i$, where the count is weighted by $(-1)^{\chi_i(\nu)}$.

For $i = k-j+1$, we can reduce the term

$$\frac{(1-x_{k-j+1})(1-x_j^{-1})q^{2N_j}}{(x_{k-j+1} q^{2N_{j-1}}, x_{k-j+1}^{-1} q^{2N_{j-1}}; q)_{n_j+1}} = \frac{q^{2N_j}}{(x_k q^2, x_k^{-1} q^2; q^2)_{n_j}}, \tag{5.3.13}$$

since $N_{j-1} = 0$. As above, we see that the coefficient of $x_{k-j+1}^{m_{k-j+1}} q^n$ in (5.2.11) is equal to the number of pairs $(\alpha_{k-j+1}, \beta_{k-j+1})$ such that α_{k-j+1} is a partition into N_j parts, β_{k-j+1} is a partition into N_j nonnegative parts, and $m_{k-j+1} = (\ell(\alpha_{k-j+1})-1) - \ell(\beta_{k-j+1})$. These bounds correspond to criteria (3) and (6) in the definition of B_1-representations.

Recall that entries in the last column of ν, cannot be marked with a hat. Therefore, the coefficient of $x_j^{m_j} q^n$ in (5.3.13) is equal to the number of possible columns $(\alpha_{k-j+1}, \beta_{k-j+1})^T$ of B_2-representation ν such that $\#(\alpha_{k-j+1}) = N_1$ and $\rho_1^{k-j+1}(\nu) = m_{k-j+1}$.

Given $\alpha_1, \alpha_2, \ldots, \alpha_j$ and $\beta_1, \beta_2, \ldots, \beta_j$ as above,

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_j \\ \beta_1 & \beta_2 & \ldots & \beta_j \end{pmatrix}$$

is a B_2-representation with $\rho_1^i(\nu) = m_i$ and $|\nu| = n$, and $\#(\alpha_i) = N_{k-i+1}$. By the definition of $h(\nu)$ and $\chi_i(\nu)$, we have that

$$(-1)^{\chi_1(\nu)}(-1)^{\chi_2(\nu)} \ldots (-1)^{\chi_{k-j+1}(\nu)} = (-1)^{h(\nu)}$$

By multiplying the coefficients of the factors (5.3.11), we find that the coefficient of $x_1^{m_1}x_2^{m_2}\dots x_{k-j+1}^{m_{k-j+1}}q^n$ in (5.3.11) is equal to the weighted count of B_2-representations

$$\nu = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{k-j+1} \\ \beta_1 & \beta_2 & \dots & \beta_{k-j+1} \end{pmatrix},$$

where $n = |\nu|$, and for all $1 \le i \le k-j+1$ we have $N_{k-i+1} = \#(\alpha_i)$ and $m_i = \rho_1^i(\nu)$, where count is weighted by $(-1)^{h(\nu)}$.

We now return to the full sum

$$\sum_{\substack{n_1\ge 0\\ \vdots \\ n_k\ge 0}} \frac{(-1;q)_{2N_k}}{q^{N_k}} \prod_{i=1}^{k} \frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})q^{2N_i}}{(x_{k-i+1}q^{2N_{i-1}}, x_{k-i+1}^{-1}q^{2N_{i-1}};q^2)_{n_i+1}} \tag{5.3.14}$$

By summing over all possible values of $n_1, n_2, \dots, n_k$, we see that the coefficient of $x_1^{m_1}x_2^{m_2}\dots x_k^{m_k}q^n$ in (5.3.8) is equal to the weighted count of B_2-representations with $|\nu| = n$ and $\rho_2^i(\nu) = m_i$, where the count is weighted by $(-1)^{h(\nu)}$, as desired. □

Corollary 5.3.1 (Morrill [17]). *Fix $k \ge 0$. Then*

$$\begin{aligned}\overline{R[2k]}(z,q) &:= \frac{(-q;q)_\infty}{(q;q)_\infty}\left(1 + 2\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+2kn}}{(1-zq^{2kn})(1-z^{-1}q^{2kn})}\right) \\ &= \sum_{m\in\mathbb{Z}}\sum_{n\ge 0} \overline{N2[k]}(m,n) z^m q^n\end{aligned}$$

That is, $\overline{R[2k]}(z,q)$ is the generating series for $\overline{N2[k]}(m,n)$.

Proof. By Theorem 4.0.3, we see that

$$\begin{aligned}&\sum_{\substack{n_1\ge 0\\ \vdots \\ n_k\ge 0}} \frac{(-1;q)_{2N_k}}{q^{N_k}} \prod_{i=1}^{k} \frac{(1-x_{k-i+1})(1-x_{k-i+1}^{-1})q^{2N_i}}{(x_{k-i+1}q^{2N_{i-1}}, x_{k-i+1}^{-1}q^{2N_{i-1}};q^2)_{n_i+1}} \\ &\qquad = \frac{(-q;q)_\infty}{(q;q)_\infty}\left(1 + 2\sum_{n=1}^{\infty} q^{n^2+2kn} \prod_{i=1}^{k} \frac{(1-x_i)(1-x_i^{-1})}{(1-x_iq^{2n})(1-x_i^{-1}q^{2n})}\right). \end{aligned} \tag{5.3.15}$$

By Theorem 5.3.2, the coefficient of $x_1^{m_1}x_2^{m_2}\dots x_k^{m_k}q^n$ in (5.3.15) is equal to the weighted count of B_2-representations with $|\nu| = n$ and $\rho_2^i(\nu) = m_i$, where the count is weighted by $(-1)^{h(\nu)}$.

Let ζ_k be a primitive kth root of unity, and substitute $x_i \mapsto \zeta_k^{i-1}z^{\frac{1}{k}}$ in (5.2.13).

This maps

$$x_1^{m_1}x_2^{m_2}\dots x_k^{m_k}q^n \to \left(\prod_{i=1}^{k}(\zeta_k^{i-1}z^{\frac{1}{k}})^{m_i}\right)q^n = \zeta_k^{m_2+2m_3+\cdots+(k-1)m_k}z^{\frac{m_1+m_2+\cdots+m_k}{k}}q^n.$$

Naively, this substitution produces

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2+2kn}\prod_{i=1}^{k}\frac{(1-\zeta_k^{i-1}z^{\frac{1}{k}})(1-(\zeta_k^{i-1}z^{\frac{1}{k}})^{-1})}{(1-\zeta_k^{i-1}z^{\frac{1}{k}}q^{2n})(1-(\zeta_k^{i-1}z^{\frac{1}{k}})^{-1}q^{2n})}\right) \tag{5.3.16}$$

as an element of $\mathbb{Z}[\zeta_k]((z^{\frac{1}{k}}))[[q]]$, the ring of formal power series in the variable q whose the coefficients are in the ring $\mathbb{Z}[\zeta_k]((z^{\frac{1}{k}}))$. Again, $\mathbb{Z}[\zeta_k]((z^{\frac{1}{k}}))$ is the ring formal Laurent series in the variable $z^{\frac{1}{k}}$ with coefficients in $\mathbb{Z}[\zeta_k]$.

Since we defined ρ_2 to be the sum of the column ranks

$$\rho_2(\nu) = \rho_2^1(\nu) + \rho_2^2(\nu) + \cdots + \rho_2^k(\nu).$$

In this light, the coefficient of $(z^{\frac{1}{k}})^m q^n$ in (5.3.16) is equal to weighted count the number of B_2-representations $\nu \in \mathcal{B}_2^k$, with $\rho_2(\nu) = m$ and $|\nu| = m$, where the count is weighted by

$$(-1)^{h(\nu)}\zeta_k^{\rho_2^2(\nu)+2\rho_2^3(\nu)+\cdots+(k-1)\rho_2^k(\nu)}, \tag{5.3.17}$$

which is the weight of the count in the definition of $\overline{N2[k]}(m,n)$.

Since

$$\prod_{i=0}^{k-1}(1-\zeta_k^i X) = 1 - X^k,$$

the right hand side of (5.2.13) reduces to $\overline{R[2k]}(z,q)$. Moreover, since $\overline{R[k]}(z,q)$ is defined in terms of the variable z, not $z^{\frac{1}{k}}$, we see that the weighted count vanishes for B_2-representations ν with $\rho_2(\nu) \notin k\mathbb{Z}$. Therefore, the coefficient of $z^m q^n$ in $\overline{R[2k]}(z,q)$ is equal to $\overline{N2[k]}(m,n)$. □

5.33 The ℓth Positive Moments of $\overline{N[2k]}(m,n)$

Curiously, Corollary 5.2.1 and Corollary 5.3.1 imply that $\overline{N2[k]}(m,n) = \overline{N[2k]}(m,n)$ for all k, m, and n. We then obtain the second half of Theorem 1.0.1 as a corollary of Theorem 5.2.4.

Corollary 5.3.2. *For all $k \geq 1$, the ℓth positive moment generating series for $\overline{N2[k]}(m,n)$ is given by*

$$\sum_{n\geq 0}\left(\sum_{m\geq 1} m^{\ell}\overline{N2[k]}(m,n)\right) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 1}\frac{(-1)^{n+1}q^{n^2+2kn}A_\ell(q^{2kn})}{(1+q^{2kn})(1-q^{2kn})^{\ell}},$$

where $A_\ell(q)$ denotes the ℓth Eulerian polynomial in the variable q.

This concludes our presentation of results.

6 CONCLUSION

Our goal for this dissertation was to develop counting functions $\overline{N[k]}(m,n)$ and $\overline{M[k]}(m,n)$ of overpartition ranks and cranks whose ℓth positive moments satisfy

$$\sum_{n\geq 0}\left(\sum_{m\geq 1} m^{\ell}\overline{N[k]}(m,n)\right)q^n = 2\left(\prod_{i\geq 1}\frac{1+q^i}{1-q^i}\right)\sum_{n\geq 1}(-1)^{n+1}\frac{q^{n^2+kn}A_{\ell}(q^{kn})}{(1+q^{kn})(1-q^{kn})^{\ell}}$$
$$\sum_{n\geq 0}\left(\sum_{m\geq 1} m^{\ell}\overline{M[d]}(m,n)\right)q^n = \left(\prod_{i\geq 1}\frac{1+q^i}{1-q^i}\right)\sum_{n\geq 1}(-1)^{n+1}\frac{q^{k\frac{n^2+n}{2}}A_{\ell}(q^{kn})}{(1-q^{kn})^{\ell}},$$

where $A_{\ell}(q)$ is the ℓth Eulerian polynomial in the variable q. To begin with, $\overline{N[1]}(m,n)$ and $\overline{N[2]}(m,n)$ count the Dyson ranks and M_2-ranks of overpartitions, respectively, and $\overline{M[1]}(m,n)$ and $\overline{M[2]}(m,n)$ count the first residual cranks and second residual cranks of overpartitions, respectively.

Generalizing the crank function turned out to be the easier task. In Section 3.7, we defined the kth residual crank of an overpartition λ to be $cr_k(\lambda) := cr(\lambda_k)$, where $cr(\lambda_k)$ denotes the crank of the partition λ_k and λ_k is an ordinary partition whose parts are $\frac{1}{k}$th the nonoverlined parts of λ that are divisible by k. We then defined the counting function $\overline{M[k]}(m,n)$ to be equal to the number of overpartitions λ of n with $cr_k(\lambda) = m$, and found that

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0}\overline{M[k]}(m,n)z^m q^n = \frac{(-q;q)_{\infty}(q^k;q^k)_{\infty}^2}{(q;q)_{\infty}(zq^k,z^{-1}q^k;q^k)_{\infty}} \tag{6.0.1}$$

This allowed us to calculate the generating series for the ℓth positive moments of $\overline{M[k]}(m,n)$

$$\sum_{n\geq 0}\left(\sum_{m\geq 1} m^{\ell}\overline{M[k]}(m,n)\right)q^n = \left(\prod_{i\geq 1}\frac{1+q^i}{1-q^i}\right)\sum_{n\geq 1}(-1)^{n+1}\frac{q^{k\frac{n^2+n}{2}}A_{\ell}(q^{kn})}{(1-q^{kn})^{\ell}},$$

as desired.

It would be interesting to see if the kth residual cranks give combinatorial proof of any overpartition congruences. We expect classical results, such as the automorphic properties of (6.0.1), to follow from established results on the generating series for the cranks of partitions,

$$\frac{(q;q)_\infty}{(zq,z^{-1}q;q)_\infty}.$$

In Chapter 5, we introduced the notion of B-partitions. This extends the notion of F-partitions, which played a role in Lovejoy's development [15] [16] of the Dyson rank and the M_2-rank of overpartitions.

In Section 5.2, we defined $\mathcal{B}_1^k$ to be the set B_1-representations with at most k columns. This gave rise to the weighted counting function

$$\overline{N[k]}(m,n) := \sum_{\substack{\nu\in\mathcal{B}_1^k \\ \rho_1(\nu)=km}} (-1)^{h(\nu)} \left(\prod_{i=1}^{k} \zeta_k^{(i-1)\rho_1^i(\nu)}\right),$$

where $\rho_1^i(\nu)$ denotes the ith rank of ν and $\rho_1(\nu) = \rho_1^1(\nu) + \rho_1^2(\nu) + \cdots + \rho_1^k(\nu)$. We then found that

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} \overline{N[k]}(m,n) z^m q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}\left(1 + 2\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+kn}}{(1-zq^{kn})(1-z^{-1}q^{kn})}\right).$$

This allowed us to calculate the generating series for the ℓth positive moments of $\overline{N[k]}(m,n)$

$$\sum_{n\geq 0}\left(\sum_{m\geq 1} m^\ell \overline{N[k]}(m,n)\right) q^n = 2\left(\prod_{i\geq 1}\frac{1+q^i}{1-q^i}\right)\sum_{n\geq 1}(-1)^{n+1}\frac{q^{n^2+kn}A_\ell(q^{kn})}{(1+q^{kn})(1-q^{kn})^\ell},$$

as desired.

In Section 5.3, we defined $\mathcal{B}_2^k$ to be the set of B_2-representations with at most k columns. This gave rise to the counting function

$$\overline{N2[k]}(m,n) := \sum_{\substack{\nu\in\mathcal{B}_2^k \\ \rho_1(\nu)=km}} (-1)^{h(\nu)} \left(\prod_{i=1}^{k} \zeta_k^{(i-1)\rho_2^i(\nu)}\right),$$

where $\rho_2^i(\nu)$ denotes the ith rank of ν and $\rho_2(\nu) = \rho_2^1(\nu) + \rho_2^2(\nu) + \cdots + \rho_2^k(\nu)$. We then found that

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0}\overline{N2[k]}(m,n)z^m q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}\left(1 + 2\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+2kn}}{(1-zq^{2kn})(1-z^{-1}q^{2kn})}\right).$$

Curiously, this implies that

$$\overline{N2[k]}(m,n) = \overline{N[2k]}(m,n). \tag{6.0.2}$$

It follows that

$$\sum_{n\geq 0}\left(\sum_{m\geq 1} m^\ell \overline{N2[k]}(m,n)\right) q^n = 2\left(\prod_{i\geq 1}\frac{1+q^i}{1-q^i}\right)\sum_{n\geq 1}(-1)^{n+1}\frac{q^{n^2+2kn}A_\ell(q^{2kn})}{(1+q^{2kn})(1-q^{2kn})^\ell}.$$

Unlike the first and second Frobenius representations of overpartitions from Section 3.4, B_1-representations and B_2-representations are in many-to-one correspondence with overpartitions. Thus, we are presently unable to find a rank function $\overline{r}_k : \overline{P} \to \mathbb{Z}$ such that $\overline{N[k]}(m,n)$ is equal to the number of overpartitions λ with $|\lambda| = n$ and $\overline{r}_k(\lambda) = m$. It does seem conspicuous that

$$\sum_{m\in\mathbb{Z}}\overline{N[k]}(m,n) = \sum_{m\in\mathbb{Z}}\overline{N2[k]}(m,n) = \overline{p}(n).$$

As a special case, we found that the sets $\mathcal{B}_1^1$ and $\mathcal{B}_2^1$ were equivalent to the first Frobenius representations and second Frobenius representations of overpartitions, respectively. This fact suggests that there may be a kth Frobenius representation of overpartitions for $k \geq 3$, or at least a canonical choice of B_1-representation or B_2-representation. We would then be able to pull back the rank function ρ_1 or ρ_2 in order to define $\overline{r}_k$. It may be fruitful to pursue a combinatoric argument to explain $\overline{N2[1]}(m,n) = \overline{N[2]}(m,n)$ in terms of mapping B_1-representations with two columns to the second Frobenius representations of overpartitions.

Searching for a canonical choice of B_1-representation or B_2-representation for overpartitions will be more difficult. The weighted count in $\overline{N[k]}(m,n)$ vanishes for B_1-representations whose full rank is not a multiple of k. The same overpartition may be represented by B_1-representations and B_2-representations with different numbers of entries. Even with a fixed number of entries, columns of the Young tableaux may also be exchanged between adjacent entries on the same row without disturbing the underlying overpartition.

We see potential in establishing another Frobenius representation of overpartitions by way of F-partitions rather than B-partitions. Consider the limiting case of Jackson's transformation,

$$ {}_6\Phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d \end{matrix} ; \, q, \frac{aq}{bcd} \right] = \frac{(aq, aq/cd, aq/bd, aq/bc; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}, $$

which is given as (3.3.1.3) in Slater's book on hypergeometric series [19]. Next, substitute

$$ a \mapsto -a, \quad b \mapsto -1/b, \quad c \mapsto -q/c, \quad d \mapsto -q^2/d, \quad q \mapsto q^4. $$

This gives us

$$ 1 + \sum_{n=1}^{\infty} \frac{(1 + aq^{8n})(-aq^4; q^4)_{n-1}(-1/b, -q/c, -q^2/d; q^4)_n}{(q^4, abq^4, acq^3, adq^2; q^4)_n} (abcdq)^n $$
$$ = \frac{(-acdq, -abdq^2, -abcq^3, -aq^4; q^4)_\infty}{(abcdq, adq^2, acq^3, abq^4; q^4)_\infty}. \quad (6.0.3) $$

On the right hand side of (6.0.3), the coefficient of $a^i b^j c^k d^\ell q^n$ is equal to the number of overpartitions λ with $|\lambda| = n$, $\#(\lambda) = i$, where

- λ has j parts which are congruent to 2 or 3 modulo 4 and overlined, or are congruent to 0 or 1 modulo 4 and not overlined.
- λ has k parts which are congruent to 1 or 3 modulo 4.
- λ has ℓ parts which are congruent to 1 or 2 modulo 4.

The left hand side of (6.0.3) bears a resemblance to the generating series of the first Frobenius representations of overpartitions and the second Frobenius representations of overpartitions. Because the definition of first Frobenius representations of overpartitions does not involve residues, and the definition of second Frobenius representations of overpartitions involves residues modulo 2, 6.0.3 suggests a fourth Frobenius representation of overpartitions. At this time we cannot offer a candidate for a third Frobenius representation of overpartitions.

When we set $a = b = c = d = 1$ in (6.0.3), there are two factors of $(q^4; q^4)_n^{-1}$. This suggests a rank-like generating function

$$1 + \sum_{n=1}^{\infty} \left[\frac{(-1, -q, -q^2; q^4)_n}{(zq^4; q^4)_n} \right] \left[\frac{(1+q^{8n})(-q^4; q^4)_{n-1}}{(q^2, q^3, z^{-1}q^4; q^4)_n} \right] q^n =: \sum_{n \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}} F[4](m,n) z^m q^n, \quad (6.0.4)$$

which in turn suggests the existence of an overpartition rank function counted by F[4](m,n). The natural comparison to be drawn is to $\overline{N[4]}(m,n)$. However, computation shows that the series (6.0.4) is not equal to the generating series for $\overline{N[4]}(m,n)$,

$$\frac{(-q;q)_\infty}{(q;q)_\infty} \left(1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+4n}}{(1-zq^{4n})(1-z^{-1}q^{4n})} \right). \quad (6.0.5)$$

In particular, the series agree for overpartitions $|\lambda| \leq 7$, after which $F[4](0,8) > \overline{N[4]}(0,8)$.

We began this investigation by looking at positive moment generating series for the ranks and cranks of overpartitions and the inequality

$$\overline{N2}_\ell^+(n) < \overline{M2}_\ell^+(n) < \overline{N}_\ell^+(n).$$

Now that we have established combinatorial objects whose positive moments generalize $\overline{M}_\ell^+(n)$ and $\overline{N}_\ell^+(n)$ in a natural way, we ask if the moment inequalities continue to hold between the kth residual cranks and the full ranks of B_1-representations and B_2-representations.

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