AN ABSTRACT OF THE THESIS OF

Annette Klute for the degree of Master of Arts in Mathematics presented on March 28, 1991.

Title: Real Algebraic Geometry and the Pierce-Birkhoff Conjecture.

Redacted for Privacy

Abstract approved: ______________________

Robby Robson

The Pierce-Birkhoff Conjecture has recently become of more and more interest in the field of real algebraic geometry. Originally, the question arose in lattice theory and the conjecture was first stated by G. Birkhoff and R.S. Pierce in 1956. Today's version says the following: Given a piecewise polynomial function $h : \mathbb{R}^n \to \mathbb{R}$, there are finitely many polynomials $f_{ij} \in \mathbb{R}[x_1, \ldots, x_n]$ such that $h = \sup_j \inf_i \{f_{ij}\}$, i.e. h can be defined globally in terms of finitely many polynomials on $\mathbb{R}^n$.

The conjecture has been proved in dimensions one and two. This paper will present the proof of the conjecture as given by C.N. Delzell in 1989 and also illustrate a new approach undertaken by J. Madden in order to translate the problem into a more general statement that might be proved by applying results of abstract real algebraic geometry.

The first chapters will introduce the basic concepts of real algebraic geometry and provide the background and tools needed for the proof. Chapter VIII contains the actual proof of the conjecture; in chapters IX and X the problem will be analysed from a more abstract point of view.

Real Algebraic Geometry
and the
Pierce-Birkhoff Conjecture

by

Annette Klute

A THESIS

submitted to

Oregon State University

in partial fulfillment of

the requirements for the

degree of

Master of Arts

Completed March 28, 1991

Commencement June 1991

APPROVED:

Redacted for Privacy

Associate Professor of Mathematics in charge of major

Redacted for Privacy

Head of Department of mathematics

Redacted for Privacy

Dean of Graduate School

Date thesis is presented: March 28, 1991

Typed by the researcher for Annette Klute

To my parents.

I went this far with him: "Sir, allow me to ask you one question. If the Church should say to you, 'two and three make ten', what would you do? "Sir," said he, "I should believe it, and I should count like this: one, two, three, four, ten." I was now fully satisfied.

Boswell's Journal for 31st May, 1764

ACKNOWLEDGEMENTS

I would like to thank the Department of Mathematics for providing financial aid in form of a teaching assistantship, without which I would not have been able to pursue graduate studies at Oregon State University. I would also like to mention the generous grant received from the Seminar on Real Algebraic Geometry and Quadratic Forms held at Berkeley and thank the Mathematical Sciences Research Institute for providing research facilities and a computer account. Most of all, I want to express my deepest gratitude to Prof. Robby Robson and his wife Susan Andrzejczak for their tremendous effort to give me a home away from home, to help me overcome cultural and language barriers and to introduce me — very gently — to the American way of life. I thank Robby Robson for his helpful comments on my manuscript, his availability and willingness to straighten out my math problems and — his sense of humor.

TABLE OF CONTENTS

REAL ALGEBRAIC GEOMETRY AND THE PIERCE–BIRKHOFF CONJECTURE

INTRODUCTION

What is real algebraic geometry? It is not easy to give a precise definition of this relatively young field, so we will rather provide some historical background. In the nineteenth century, a number of algebraists were interested in locating the real zeros of polynomials in one variable. Important results were obtained by J.C.F Sturm (1803 – 1855), Ch. Hermite (1822 – 1901), C.G.J Jacobi (1804 – 1851), C.W Borchardt (1817 – 1880), J.J. Sylvester (1814 – 1897) and A. Cayley (1821 – 1895), to name just a few.

Today, one is rather interested in sets defined by finitely many polynomial inequalities $f(x_1, \ldots, x_n) > 0$ (or $f(x_1, \ldots, x_n) \geq 0$), the coefficients coming from some ordered field k, typically the real numbers. Questions such as "how many inequalities are needed in order to describe a given semialgebraic set" have been answered. Semialgebraic geometry depends very much on how the underlying field is ordered, and a theory of ordered fields was developed by Artin and Schreier in the 1920's. Later on, their ideas were generalized to commutative rings, but is was not until the late seventies, when things were put into perspective by M. Coste and M.F. Roy, who introduced the real spectrum of a ring A as the set of all orderings on A, together with a topology that reveals its 'spectral' features. The real spectrum proves to be very useful for deriving results on semialgebraic sets, when A is taken to be the ring of polynomials in n variables over an ordered field. One of the central theorems of real algebraic geometry is the Ultrafilter Theorem by L. Bröcker that establishes an intimate relationship between the abstract positivity of a polynomial $f \in k[x_1, \ldots, x_n]$ and the semialgebraic set $\{f \in k^n : \ f(x) > 0\}$ defined by f.

Another area that plays an important role in this context is the theory of valuations and real places, whose main concepts are due to W. Krull (see Journal Reine Angewandte Mathematik, 1932). A highlight of this theory is the so-called Artin Lang Homomorphism Theorem that appears in S. Lang's paper "The Theory of Real Places" in the Annals of Mathematics in 1953.

An interesting problem where some techniques of real algebraic geomtry can be applied, is the Pierce-Birkhoff-Conjecture. First stated by G. Birkhoff and R.S. Pierce in 1956, the conjecture claims that a piecewise polynomial function

$$h : \mathbb{R}^n \to \mathbb{R}$$

can be reexpressed in terms of suprema and infima of finitely many polynomials, i.e.

$$h = \sup_j \inf_i \{f_{ij}\}$$

for some $f_{ij} \in \mathbb{R}[x_1, \ldots, x_n]$.

The outline of this paper is as follows: Chapters I and II present the Artin Schreier theory for fields and commutative rings, in chapter III the real spectrum will be introduced and further examined in chapter IV, which also contains the Ultrafilter Theorem. Chapter V provides some examples illustrating the Ultrafilter Theorem; and in chapter VI the basic concepts of valuation theory will be developed. In chapter VII, we will focus on semialgebraic subsets of $\mathbb{R}^n$ and describe the cylindrical algebraic decomposition of semialgebraic sets (due to P.J. Cohen), thus providing an important tool for the proof of the Pierce-Birkhoff Conjecture in chapter VIII. In chapter X we will take a different point of view and 'attack' the Pierce-Birkhoff Conjecture with methods of abstract real algebraic geometry as suggested by J. Madden in 1989. Some facts about semialgebraic functions that will be needed for Maddens's approach are presented in chapter IX.

CHAPTER I

ORDERED FIELDS

Before we can develop semialgebraic geometry, we need to settle the notion of positivity and 'order', i.e. we have to define what it means for a polynomial to be positive on a subset of its domain. So in this chapter, and in a more general context in chapters II and III, the basic concepts of orderings on fields will be introduced.

Definition 1.1: An ordering P of a field K is a subset $P \subset K$ satisfying the following conditions:

$$(P1)\ P+P \subset P \quad (P2)\ P \cdot P \subset P \quad (P3)\ P \cap -P = \{0\} \quad (P4)\ P \cup -P = K$$

Remarks:

1. Conditions (P2) and (P4) imply that $K^2 \subset P$: Since either $a \in P$ or $-a \in P$, $a^2 = a \cdot a = (-a)(-a) \in P$.
2. Condition (P3) is equivalent to (P3'): $-1 \notin P$. Since if $-1 \in P$ then $1 = 1^2 \in -P \cap P = \{0\}$ and conversely, if $0 \neq a \in P \cap -P$ then $-a^2 \in P$, so $-1 \in P$.
3. $P \subset Q \Rightarrow P = Q$: Suppose $q \in Q \backslash P$, then $q \in -P \subset -Q$, so $q \in Q \cap -Q = \{0\}$.

Given an ordering P on K, one can define a total ordering on K by

$$a \leq b \ \Leftrightarrow b - a \ \in P.$$

Then for $a, b, c \in K$:

(i) $a \leq b \Rightarrow a + c \leq b + c$

(ii) $a \leq b,\ c \geq 0 \Rightarrow ac \leq bc.$

Definition 1.2: A preordering T on K is a subset of K with the following properties:

$$(T1)\ T+T \in T \quad (T2)\ T \cdot T \in T \quad (T3)\ T \cap -T = \{0\}\ (T4)\ K^2 \in T$$

Since (P4) implies (T4), every ordering is a preordering.

Lemma 1.3. *If T is a preordering on K and $a \notin T$ then $T - aT$ is a preordering on K.*

Proof. (1), (2) and (4) follow from the fact that T is a preordering. Suppose $-1 = b - ac$, where $b, c \in T$. Then $c \neq 0$ and $a = \frac{1}{c}(b+1) = \frac{1}{c^2}(cb+c) \in T$, a contradiction.

Definition 1.4: A field K is said to be *formally real* iff -1 is not a sum of squares in K.

Theorem 1.5. *K has an ordering if and only if K is formally real.*

Proof. The 'only if'-part follows from the axioms. Suppose K is formally real. This means that $\sum K^2$ is a preordering on K. Let $M := \{T\colon\ T$ preordering on K and $T \supset \sum K^2\}$. M is nonempty and partially ordered by inclusion. If $\{T_\alpha\}$ is a chain in M then $T := \cup_\alpha T_\alpha$ is an upper bound; so by Zorn's Lemma, M contains a maximal element P. Claim: P is an ordering. Suppose $a \notin P$. Since $P = P - aP$, $0 - a \cdot 1 \in P$, hence $a \in -P$.

Example: Let $(K, \leq)$ be a totally ordered field, $K(t)$ the rational function field in one variable. For any $a \in K$ the sets

$$P_{a,+} \ := \ \{0\} \cup \{(t-a)^r f(t) \ : f(a) \neq \infty,\ f(a) > 0,\ r \in \mathbb{Z}\}$$

$$P_{a,-} \ := \ \{0\} \cup \{(a-t)^r f(t) \ : f(a) \neq \infty,\ f(a) > 0,\ r \in \mathbb{Z}\}$$

are orderings on $K(t)$ that extend the ordering on K. In $P_{a,+}$ we have $a < t < b$ for any $b \in K$ with $b > a$ and in $P_{a,-}$ $b < t < a$ for any $b < a$.

Definition 1.6: Let $K|L$ be a field extension, P an ordering on K. An ordering Q on L is an *extension* of P if $P = Q \cap K$.

Definition 1.7: A field K is said to be *real closed* if it is formally real and has no proper formally real extension that is algebraic over K.

Theorem 1.8. *Let K be a field. The following are equivalent:*

(1) *K is real closed*

(2) *$P = K^2$ is the unique ordering on K and every polynomial of odd degree has a root in K.*

In order to prove Theorem 1.8 we need the following result, which is due to T.A. Springer:

Lemma 1.9. *Let $L|K$ be a finite algebraic extension of K where $[L : K]$ is an odd number, $q =< a_1, \dots, a_m >$ an anisotropic quadratic form over K. Then q is anisotropic over L.*

Proof. We may assume that $L = K(\alpha)$, since if $L = K(\alpha)(\beta)$, then $[L : K] = [L : K(\alpha)] \cdot [K(\alpha) : K]$ and both extensions are of odd degree, so we can iterate the argument. Thus assume $L = K(\alpha)$ and let $f \in K[x]$ be the minimal polynomial of α over K. We will use induction on $n = \deg(f)$, n odd. If $n = 1$, there is nothing to show, so assume $n > 1$. Let $q =< a_1, \dots, a_m >$ be anisotropic over K. Suppose q is isotropic over L. Then we find $g_1, \dots, g_m, h \in K[x]$, not all $g_i = 0$ and $\deg(g_i) < n$ such that

$$(*) \qquad a_1 g_1(x)^2 + \dots + a_m g_m(x)^2 = f(x)h(x)$$

holds in $K[x]$. We may also assume that $\gcd(g_1, \dots, g_m) = 1$. Let $d := \max\{\deg(g_1), \dots, \deg(g_m)\}$. Then $d < n$ and since $2d = n + \deg(h)$ it follows that $\deg(h) < n$ and odd. In particular, h has an irreducible factor h_1 of odd degree. Now, the algebraic extension $E := K[x]/(h_1)$ of K satisfies $[E : K] < n$ and odd, therefore, by induction, q must be anisotropic over E. On the other hand, $(*)$ implies that q is isotropic over E, a contradiction.

Proof of Theorem 1.8. (1)$\Rightarrow$(2): Let $a \in P$. If a is not a square in K then $K(\sqrt{a}) = K[x]/(x^2 - a)$ is an algebraic extension of K which is not formally real. So we find an equation

$$-1 = \sum_{i=1}^{n} (x_i + y_i \sqrt{a})^2$$

and therefore

$$-1 = \sum_{i=1}^{n}(x_i{}^2 + y_i{}^2 a)$$

Since K is formally real, $\sum_{i=1}^{n} y_i{}^2 \neq 0$, so

$$-a = (\sum_{i=1}^{n} y_i{}^2)^{-1}(1+\sum_{i=1}^{n} x_i{}^2) = \frac{\sum y_i{}^2}{(\sum y_i{}^2)^2}(1+\sum x_i{}^2) \in P,$$

a contradiction. Therefore every positive element in K is a square in K.
Now suppose $f \in K[x]$ is a polynomial of degree n (n odd) that has no root in K. Without loss of generality assume f irreducible (otherwise at least one of its irreducible factors has odd degree). The field $L := K[x]/(f)$ is an algebraic extension of K with $[L : K] = n$. If we can show that P can be extended to an ordering Q on L we shall be done, for this contradicts the assumption that K is real closed. Consider the set

$$T := \{p_1 b_1{}^2 + \cdots + p_n b_n{}^2 : \ p_i \in P, \ b_i \in L, \ n \in \mathbb{N}\}$$

T satisfies $P \subset T$, $T + T \subset T$, $T \cdot T \subset T$ and $L^2 \subset T$.
Claim: T is a preordering on L. It remains to show that $-1 \notin T$. Suppose

$$-1 = p_1 b_1{}^2 + \cdots + p_m b_m{}^2 \text{ for some } p_i \in P, \ b_i \in L.$$

This means that the quadratic form $< 1, p_1 \ldots, p_m >$ is isotropic over L, contradicting Lemma 1.9. Therefore T is a preordering which can be enlarged to an ordering Q on L. Q extends P since $P \in T$.

(2) $\Rightarrow$ (1): Let $L|K$ be a proper algebraic extension of K. Then L contains a finite extension $L' \supset K$ and by assumption, $[L' : K] = 2^m$, $m \in \mathbb{N}$. This implies that there exist a field F, $K \subset F \subset L$, such that $[F : K] = 2$. Therefore $F = K(\sqrt{a})$ for some $a \in K$. Since $a \notin K^2$, there exists $b \in K$ with $a = -b^2$. Hence $-1 = \frac{a}{b^2} = (\frac{\sqrt{a}}{b})^2$ is a square in F, so F is not formally real. ♣♣

Definition 1.10: Let P be an ordering on a field K. A *real closure* $\hat{K}$ of K is an algebraic extension of K that is real closed and whose ordering contains P.

From Theorem 1.8 follows that $\hat{K}^2$ is the unique ordering on $\hat{K}$.

Theorem 1.11. *Every ordered field has a real closure and any two real closures of a field K are isomorphic.*

The proof uses a version of

Sturm's Theorem: (without proof) *Let R be a real closed field, $f \in R[x]$ without multiple roots, $f_0, \dots, f_k$ a sequence of polynomials constructed in the following way:*

$$f_0 = f, \; f_1 = f', \dots, f_{i-2} = f_{i-1} q_i - f_i$$

with $\deg(f_i) < \deg(f_{i-1})$ *for $i = 2, \dots, k$ and $f_k \in R \backslash \{0\}$. (In particular $f_k = gcd(f, f')$)*

If $v_f(+\infty)$ $(v_f(-\infty))$ denotes the number of sign changes of the leading coefficients of the polynomials $f_0(x), \dots, f_k(x)$ $(f_0(-x), \dots, f_k(x))$ then the number of roots of f in R is equal to $v_f(-\infty) \; - \; v_f(+\infty)$.

Proof of Theorem 1.11.

(a) Existence of a real closure by Zorn's Lemma: Let K be al field with ordering P and $\overline{K}$ an algebraic closure of K. Let

$$\mathcal{E} := \{(F, P_F :) \quad K \subset F \subset \overline{K} \text{ and } P \subset P_F\}.$$

The family $\mathcal{E}$ is ordered by

$$(F, P_F) \prec (F', P_{F'}) :\Leftrightarrow \; F \subset F' \text{ and } P_F \subset P_{F'}.$$

By Zorn's Lemma, $\mathcal{E}$ contains a maximal element $(\hat{K}, P_{\hat{K}})$. Claim: $\hat{K}$ is real closed. We have to show that every positive element is a square in $\hat{K}$. Suppose $a \in P_{\hat{K}}$ but a is not a square, then the semiring T generated by $P_{\hat{K}}$ and elements of the form $(c + d\sqrt{a})^2$, $c, d \in \hat{K}$, is a preordering in the field $\hat{K}(\sqrt{a})$: If

$$-1 = \sum_{i=1}^{n} b_i (c_i + d_i \sqrt{a})^2, \quad c_i, d_i \in \hat{K}, \; b_i \in P_{\hat{K}}$$

then

$$-1 = \sum_{i=1}^{n} b_i({c_i}^2 + {d_i}^2 a) \ \in P_{\hat{K}}$$

which is impossible.

Therefore T is contained in an ordering on $\hat{K}(\sqrt{a})$ that extends the ordering on $\hat{K}$, contradicting the maximality of $\hat{K}$.

(b) Now let R and R' be two real closures of K. Our task is to show that they are isomorphic. Consider the family

$$\mathcal{F} \ := \ \{\phi : \ F \to R' : \quad K \subset F \subset R, \ \phi \text{ is order preserving}\}$$

$\mathcal{F}$ is partially ordered in the following way: $\phi_1 \prec \phi_2$ if one has a commutative diagramm

$$\begin{tikzcd} F_1 \arrow[r, hook] \arrow[dr, "\phi_1"'] & F_2 \arrow[d, "\phi_2"] \\ & R' \end{tikzcd}$$

Again, by Zorn's Lemma, there is a maximal $\Phi : \ L \ \to R'$. It remains to show that $L = R$. If not, there exists $a \in R \backslash L$ with minimal polynomial $f \in L[x]$. As an ordered field L has characteristic zero and is therefore separable, so f has no multiple roots. Let $a_1 < \cdots < a_n$ be the roots of f in R, with $a = a_j$. Applying Sturm's Theorem, one gets a sequence of polynomials $f_0,\ldots,f_k$ in $L[x]$. Since Φ is order preserving, the polynomial $\Phi(f) = \sum_{i=1}^{n} \Phi(a_i)x^i$ has the same number of roots in R' as f in R:

$$v_f(-\infty) \ - \ v_f(+\infty) \ = \ v_{\Phi(f)}(-\infty) \ - \ v_{\Phi(f)}(+\infty)$$

If we denote the roots of $\Phi(f)$ with $b_1 < \cdots < b_n$, we obtain a homomorphism $\Psi : L(a) \to R$ by $\Psi(a) = b_j$.

Theorem 1.12. *(Fundamental Theorem of Algebra) If R is a real closed field, then $R(\sqrt{-1})$ is algebraically closed.*

Proof. Suppose $C = R(\sqrt{-1}) = R(i)$ is not algebraically closed, then, since by Theorem 1.8 any finite algebraic extension of R is of degree 2^m, $m \geq 1$, there exists an extension L with $[L : C] = 2$. So we can find $\alpha = a + bi$, $a, b \in R$ such that α is not a square in L. Since R is real closed, $b \neq 0$, and $a^2 + b^2 = c^2$ for some $c > 0$, $c \in R$. This implies that both $c - a$ and $c + a$ are positive (since $c^2 \geq a^2$). So we find $x, y \in R$ with $x^2 = \frac{c+a}{2}$, $y^2 = \frac{c-a}{2}$ and we choose $x > 0$, $\text{sign}(y) = \text{sign}(b)$. Then $(x + iy)^2 = x^2 + 2xyi - y^2 = a + 2xyi$. Since $(2xy)^2 = c^2 - a^2 = b^2$ and $x > 0$, it follows that $2xy = b$, hence $\alpha = (x + iy)^2$, contradicting our assumption that α is not a square in C.

Corollary 1.13. *Let R be a real closed field and $K \subset R$ a subfield. K is real closed iff K is algebraically closed in R.*

Proof. Suppose K is not algebraically closed in R. Then we have the field extensions $K \subset L \subset R$, $L|K$ algebraic. But R induces an ordering on L (by restriction), which contradicts the fact that K is real closed.
Conversely, assume K is algebraically closed in R. Then $K(\sqrt{-1})$ is algebraically closed in $R(\sqrt{-1})$ for, if $\alpha = a + bi$ $(a, b \in R)$ is algebraic over $K(i)$, it is algebraic over K, and so are $\bar{\alpha}$, $\frac{\alpha+\bar{\alpha}}{2}$, $\frac{\alpha-\bar{\alpha}}{2i}$, hence $a, b \in K$ and $\alpha = a + bi \in K(i)$. As R is real closed, Theorem 1.12 applies and it follows that its algebraic closure is $R(i)$. Therefore the algebraic closure of K is $K(i)$. But this means that K is real closed because there are no algebraic extensions of K properly contained in $K(i)$ and -1 is a square in $K(i)$.

CHAPTER II

ORDERED RINGS

The concept of reality and orderings can be generalized to rings. It turns out that we need to relax condition (P4) and require that $P \cap -P$ be a prime ideal in the ring being considered. As a consequence, not every ordering P on a ring A defines a total order anymore and it is possible to have chains of orderings $P \subset Q \subset \ldots$.

Definition 2.1: Let A be a commutative ring with unity. A subset P of A is called an ordering on A if it satisfies the following properties:

$$(1)\ P + P \subset P \quad (2)\ P \cdot P \subset P \quad (3)\ P \cup -P = A$$

$$(4)\ p = P \cap -P \text{ is a (proper) prime ideal of } A$$

p is called the support of P, also denoted by $\mathrm{supp}(P)$.

Remarks:

(1) As in the field case, $A^2 \subset P$ and $-1 \notin P$, for otherwise $1 = 1^2 \in P \cap -P$, so $p = A$.

(2) If $P \underset{\neq}{\subset} Q$, then $\mathrm{supp}(P) \underset{\neq}{\subset} \mathrm{supp}(Q)$: Suppose $\mathrm{supp}(P) = \mathrm{supp}(Q)$ and $a \in Q \backslash P$.Then $a \in -P \subset -Q$, so $a \in \mathrm{supp}(Q) = \mathrm{supp}(P) \subset P$, a contradiction.

Given an ordering P one introduces the following notation:

$$a >_P 0 :\Leftrightarrow a \in P \backslash -P$$

$$a \geq_P 0 :\Leftrightarrow a \in P$$

$$a =_P 0 :\Leftrightarrow a \in \mathrm{supp}(P)$$

Note that there are rings that cannot be ordered: let $A = \mathbb{Z}/n\mathbb{Z}$; then $-1 = \sum_{i=1}^{n-1} 1$, contradicting that $-1 \notin P$.

If P is an ordering on a field F, then $p = \mathrm{supp}(P)$ is the zero ideal.

Starting with an ordering P on A one can pass to the domain $\overline{A} = A/supp(P)$ and one obtains the ordering $\overline{P}$ on A. Since $\overline{P}$ has support zero, it extends uniquely to an ordering on $k(p) :=$ quot (A/p): $\frac{\bar{a}}{\bar{b}} \in \overline{P}$ iff $\bar{a} \cdot \bar{b} \in \overline{P}$. Conversely, given a prime ideal p of A and an ordering $\overline{P}$ on $k(p)$, the set $P = \{a : \bar{a} \in \overline{P} \cap A/p\}$ is an ordering on A with support p.

In contrast to the field case one distinguishes two notions of reality:

Definition 2.2: Let A be a commutative ring with unity.

(1) A is said to be *semireal* if -1 is not a sum of squares in A

(2) A is said to be *real* if for $a_1, \ldots, a_n \in A$

$$\sum_{i=1}^{n} {a_i}^2 = 0 \quad \Rightarrow \quad a_i = 0, \; i = 1, \ldots, n.$$

If A is real, then it is semireal: if $-1 = \sum_{i=1}^{n} {a_i}^2$ then $0 = 1^2 + \sum_{i=1}^{n} {a_i}^2$.
If K is a field, these two concepts coincide: suppose ${a_1}^2 + \cdots + {a_n}^2 = 0$ but $a_1 \neq 0$ (say), then $-1 = (\frac{a_2}{a_1})^2 + \cdots + (\frac{a_n}{a_1})^2$.

Also Note that there are rings which are semireal but not real:
$A := \mathbb{R}[x_1, \ldots, x_n]/({x_1}^2 + \cdots + {x_n}^2)$ is not real because $\sum_{i=1}^{n} {x_i}^2 = 0$ but $x_i \neq 0$ for all i. A is semireal since it admits a homomorphism into the field of real numbers via the map

$$\phi : A \rightarrow \mathbb{R}, \; \phi(f + ({x_1}^2 + \cdots + {x_n}^2)) = f(0).$$

In analogy to the field case one defines a preordering T on A as a subset satisfying the following conditions:

$$(T1)\; T + T \subset T \quad (T2)\; T \cdot T \subset T \quad (T3)\; A^2 \subset T \quad (T4)\; -1 \notin T$$

Lemma 2.3. *If T is a preordering on A and $xy \in -T$ for $x, y \in A$, then either $T + xT$ or $T + yT$ is a preordering on A.*

Proof. If none of them is a preordering one has equations $-1 = t_1 + xt_2$, $-1 = t_3 + yt_4$ for suitable $t_i \in T$ (conditions (T1)–(T3) hold for $T + xt$ and $T + yT$). Combining these equations one gets

$$xyt_2t_4 = (-1-t_1)(-1-t_3) = 1+t_1+t_3+t_1t_3 \;\Rightarrow\; -1 = -xyt_2t_4+t_1+t_3+t_1t_3 \in T,$$

a contradiction.

Lemma 2.4. *Any preordering T is contained in an ordering.*

Proof. Using Zorn's Lemma (applied to the set of preorderings containing T, partially ordered by inclusion) one can enlarge T to a maximal preordering M. In order to prove that M is an ordering one has to show that (1) $M \cup -M = A$ and (2) $M \cap -M$ is a prime ideal of A.
As to (1), suppose $x \notin M$. Since $M - xM$ is a preordering and M maximal, $M = M - xM$, hence $-x \in M$. As to (2), show first that $M \cap -M$ is an ideal in A: $m_1, m_2 \in M \cap -M$ implies $m_1 - m_2 \in M \cap -M$. If $m \in M \cap -M$ and $a \in A$, then either $M + aM = M$ or $M - aM = M$ and since both imply that $\pm am \in M$, one concludes that $M \cap -M$ is an ideal in A. Now suppose $ab \in M \cap -M$, but $a \notin M \cap -M$ and assume $a \notin M$. Then $M + aM \not\subset M$. Since $\pm ab \in -M$, Lemma 2.3 implies that $M \pm bM = M$, so that $b \in M \cap -M$.

Theorem 2.5. *A ring A can be ordered if and only if A is semireal.*

Proof. Let P be an ordering on A. If A is not semireal then $-1 \in \sum A^2 \subset P$, a contradiction. Conversely, assume A is semireal. Then $-1 \notin \sum A^2$, so $\sum A^2$ is a preordering which is contained in an ordering on A.

CHAPTER III

THE REAL SPECTRUM OF A COMMUTATIVE RING

In this Chapter we will describe the set of orderings as a topological space. This was first done by M. Coste and M.F. Roy in 1979. For a very detailed presentation of the material consult [3]. We will see that this space has some curious properties.

Definition 3.1: The *real spectrum*, Sper A, of the ring A is defined as the set of all pairs $\alpha = (p, \overline{P})$ such that p is a prime ideal of A and $\overline{P}$ is an ordering on $k(p)$.

Since every pair (p, P) corresponds to an ordering P on A with support p, one can interpret Sper A as the set of all orderings on A. If $k(\alpha)$ denotes a real closure of $k(p)$, there is the canonical homomorphism

$$\phi_\alpha \,:\, A \longrightarrow k(p) \longrightarrow k(\alpha),$$

where the first arrow stands for projection and the second for inclusion. For given $f \in A$ we will write $f(\alpha)$ rather than $\phi_\alpha(f)$.

As seen in chapter I, the real closure of an ordered field is unique up to isomorphism, one may introduce the following equivalence relation: given ring homomorphisms $\phi: A \to K$, $\phi': A \to K'$ into real closed fields K, K', denote by k (k') the real closures of $A/\ker\phi$ ($A/\ker\phi'$). One says that ϕ and ϕ' are equivalent (in symbols $\phi \sim \phi'$) if there is an isomorphism $\psi : k \to k'$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\phi'"'] & k \arrow[dl, "\psi"] \\ k' & \end{tikzcd}$$

is commutative. This allows us to characterize Sper A as the set of all equivalence classes of homomorphisms $\phi: A \to K$, where K is a real closed field and K^2 its unique ordering. The corresponding point α_ϕ in Sper A is the pair

$$\alpha_\phi \;=\; (ker\phi,\; \overline{P}),$$

where $\overline{P} = K^2 \cap A/ker\phi$ is the ordering on $A/ker\phi$ induced by the ordering K^2 on K.

We are now going to introduce the promised topology on Sper A, which in the literature is commonly denoted as the Harrison topology on Sper A. A subbasis for this topology is given by the sets

$$H_A(f) := \{P \in \text{Sper } A : \quad f >_P 0\};$$

accordingly, a basis for the Harrison topology is given by the sets

$$H_A(f_1, \ldots, f_n) := \{P \in \text{Sper } A : \quad f_1 >_P 0, \ldots, f_n >_P 0\}$$

For convenience, let's also define the sets

$$\overline{H}(f_1, \ldots, f_n) := \{P \in \text{Sper } A : \quad f_1 \geq_P 0, \ldots, f_n \geq_P 0\}$$
$$Z(f_1, \ldots, f_n) := \{P \in \text{Sper } A : \quad f_1 =_P 0, \ldots, f_n =_P 0\}.$$

Here we observe that $Z(f_1, \ldots, f_n) = Z(f)$, where $f = \sum_{i=1}^n f_i^2$, since

$$f_1 =_P 0, \ldots, f_n =_P 0 \iff \phi_P(f_1) = 0, \ldots, \phi_P(f_n) = 0 \iff \phi_P(\sum f_i^2) = 0,$$

as $k(P)$ is formally real.

Theorem 3.2. *Let P and Q be orderings on A. Then*

$$P \in \overline{\{Q\}} \Leftrightarrow Q \subset P$$

Corollary 3.3. *Maximal orderings are closed points in Sper A.*

Proof of Theorem 3.2. $P \in \overline{\{Q\}} \Leftrightarrow (P \in H_A(f) \Rightarrow Q \in H_A(f)) \Leftrightarrow P\backslash -P \subset Q\backslash -Q \Leftrightarrow Q \subset P$. As to the last '$\Leftrightarrow$'- sign, suppose $a \in Q\backslash P$, then $a \in -P\backslash P \subset -Q\backslash Q$ and if $a \in P\backslash -P$ but $a \notin Q\backslash -Q$, then $a \in -Q \subset -P$.

Remark: Theorem 3.2 implies that the closure of a point $\{P\}$ in Sper A is totally ordered by inclusion: suppose $T \supset P, T' \supset P$ and $a \in T \backslash T'$, $b \in T' \backslash T \Rightarrow T \in H(a-b)$, $T' \in H(b-a)$ and $H(a-b) \cap H(b-a) = \emptyset$, contradicting $T, T' \in \overline{\{P\}}$. So closures of points look like 'spears', which explains why one chose the abbreviation 'Sper A' for the real spectrum.

Definition 3.4 Given two points $P, Q \in$ Sper A, $P \subset Q$. Q is called a *specialization* of P and P is called a *generalization* of Q.

Now let R be a real closed field and A an affine R-algebra, i.e.

$$A \simeq R[x_1, \dots, x_n]/\wp,$$

where $\wp$ is an ideal in $R[x_1, \dots, x_n]$. Let $V(R) = \{x \in R^n : f(x) = 0 \text{ for all } f \in \wp\}$. For any $a = (a_1, \dots, a_n) \in V(R)$ consider the ordering $P_a = \{f \in A : \ f(a) \geq 0\}$, where '$\geq$' refers to the unique ordering R^2 on R. The support of P_a is the maximal ideal $m_a = \{f \in A : \ f(a) = 0\}$. m_a is maximal since $A/m_a \simeq R$ via the map $\phi(f + m_a) = f(a)$.

Hence the map $a \mapsto P_a$ defines an embedding of $V(R)$ into Sper A, or more precisely, into the subspace of maximal points of Sper A. In order to see that $a \mapsto P_a$ is in fact injective, suppose $P_a = P_b$ but $a \neq b$, say $a_1 < b_1$. Since $P_a = P_b$, $f(a) \geq 0 \Leftrightarrow f(b) \geq 0$, so $f(a) \cdot f(b) \geq 0$ for any $f \in A$. But for $g := x_1 - \frac{1}{2}(a_1 + b_1)$ $g(a) \cdot g(b) < 0$.

It turns out that the topology induced on $V(R)$ by the Harrison topology of Sper A is just the strong topology, i.e. the topology coming from the interval topology on R:

Theorem 3.5. *The relative topology on $V(R)$ as a subset of Sper A is precisely the strong topology on $V(R)$.*

Proof. Consider a basic open set $H(f_1, \dots, f_n)$ of Sper A. Then

$$H(f_1, \dots, f_n) \cap V(R) = \{P_a : \ f_i \in P_a \backslash -P_a\} = \{a : \ f_i(a) > 0\}$$

which is open in the strong topology. Conversely, if $U := \{a \in V : \, b_i < a_i < c_i, \; i = 1, \ldots, n\}$ is an open set in $V(R)$, then

$$U = H(x_1 - b_1, \ldots, x_n - b_n, c_1 - x_1, \ldots, c_n - x_n) \cap V(R),$$

which is open in the induced Harrison topology on $V(R)$. Even more is true:

Theorem 3.6. *$V(R)$ is a dense subspace of Sper A.*

This is a consequence of the **Artin-Lang-Homomorphism-Theorem** (proved in a later section):

Let A be an R-affine algebra, f_i, g_j $(i = 1, \ldots, n, j = 1, \ldots, n)$ elements of A. If there exists an ordering T on A such that $f_i >_T 0$ and $g_j \geq_T 0$ for all i, j then there exists a R-algebra homomorphism $\phi : \; A \to R$ such that $\phi(f_i) > 0$ and $\phi(g_j) \geq 0$ for all i, j.

Proof of Theorem 3.6. It suffices to show that every basic open set of Sper A meets $V(R)$. Let $f_1, \ldots, f_r \in A$ such that $H = H(f_1, \ldots, f_r) \neq 0$, say $P \in H$. Then $f_i >_P 0$ and there exists $\phi : \; A \to R$ such that $\phi(f_i) > 0, \; i = 1, \ldots, r$. Let $a_l := \phi(\bar{x}_l), \; l = 1, \ldots, n$. The point $a = (a_1, \ldots, a_n)$ is in $V(R)$, since for any $g \in \wp$, $g(a_1, \ldots, a_n) = g(\phi(\bar{x}_1), \ldots, \phi(\bar{x}_n)) = \phi(g) = 0$, and since $\phi(f_i) = f_i(a) > 0$ for all i, the ordering P_a belongs to $H \cap V(R)$.

CHAPTER IV

THE REAL SPECTRUM IN THE LIGHT OF LATTICE THEORY

Definition 4.1: A partially ordered set $(\mathcal{L}, \leq)$ is called a *lattice* (with 0 and 1) if every finite subset $M \in \mathcal{L}$ has a greatest lower bound $(\inf M)$ and a least upper bound $(\sup M)$ in $\mathcal{L}$. In particular, $0 := \inf \mathcal{L} := \sup \emptyset$ and $1 := \sup \mathcal{L} := \inf \emptyset$.

Some notation:

$$x \wedge y := \inf\{x, y\} \qquad x \vee y := \sup\{x, y\} \text{ for } x, y \in \mathcal{L}$$

$\mathcal{L}$ is called a *Boolean* lattice if also the following conditions hold:

(a)
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \text{(distributivity)}$$

(b) for $x \in \mathcal{L}$ there exists $x' \in \mathcal{L}$ such that

$$x \vee x' = 1 \text{ and } x \wedge x' = 0 \qquad \text{(complement)}$$

Definition 4.2:
(a) $\mathcal{C}(A)$ denotes the Boolean sublattice of $2^{\text{Sper } A}$ (the power set of Sper A) generated by the Harrison subbasis $\{H(f),\ f \in A\}$. The elements of $\mathcal{C}(A)$ are called *constructible sets.*
(b) The constructible topology on Sper A is generated by the constructible sets, which form a basis for this topology.

$\mathcal{C}(A)$ is the smallest subset of $2^{\text{Sper } A}$ that contains all Harrison sets and is closed under finite unions and complements. More precisely, $\mathcal{C}(A)$ consists of finite unions of sets of the form

$$H(f_1, \dots, f_m) \cap \overline{H}(g_1, \dots, g_n) \qquad \text{or} \qquad H(f_1, \dots, f_m) \cap Z(g)$$

for $f_i, g_i, g \in A$. Note that

$$H(g_1,\dots,g_l)^c = \cup_{i=1}^{l}\overline{H}(-g_i) \qquad \text{and} \qquad Z(g) = H(g^2)^c.$$

The constructible topology is usually finer than the Harrison topology. However, they agree in case that $A = K$ for some field K, since in this case $H(f)^c = H(-f)$ for $0 \neq f \in K$.

In order to avoid confusion, the following terminology will be introduced: A set $H \in$ Sper A will be called $\mathcal{C}$-closed ($\mathcal{C}$-open, $\mathcal{C}$-compact, etc.) if it is closed (open, compact...) in the constructible topology. Attributes without this prefix and formations like $A \mapsto \overline{A}$ refer to the Harrison topology.

Theorem 4.3. *In the constructible topology, Sper A is a totally disconnected compact space. A subset $C \in A$ is constructible if and only if it is $\mathcal{C}$-open and $\mathcal{C}$-closed.*

Proof. Let $Z = \Pi_{f\in A}\{0,1\}$, where $\{0,1\}$ has the discrete topology and Z the product topology. Z is totally disconnected:

Suppose there exists a connected subset $M \in Z$ containing at least two points z_1, z_2. Then one can find an $f \in A$ such that $z_1(f) \neq z_2(f)$, say $z_1(f) = 1, z_2(f) = 0$. The open sets $U_1 := \{z \in Z :\ z(f) = 1\}$ and $U_2 := \{z \in Z :\ z(f) = 0\}$ are disjoint and cover Z. Therefore $M = (M \cap U_1) \cup (M \cap U_2)$, $M \cap U_i \neq \emptyset$, a contradiction. By Tychonov's Theorem, Z is also compact.

The idea of the proof is to embed Sper A into Z and to show that it is closed in Z, so that the theorem follows. For this purpose one identifies an ordering P on A with the characteristic function on $P\backslash -P$, that is

$$P \leftrightarrow 1_{P\backslash -P} \qquad \text{where} \qquad 1_{P\backslash -P}(f) = \begin{cases} 0 & f \in -P \\ 1 & f \in P\backslash -P \end{cases}$$

That this is in fact an embedding, follows from the fact that $P = Q \Leftrightarrow P\backslash -P = Q\backslash -Q$ as seen earlier in the proof of Theorem 3.2.
Given the function $1_{P\backslash -P} \in Z$, we recover the ordering P by

$$P = (-1_{P\backslash -P}^{-1})(\{0\}) = \{a \in A : \ 1_{P\backslash -P}(-a) = 0\}.$$

Now, assuming $z \in Z\backslash \text{Sper } A$, one has to show that z is contained in an open set U that does not meet Sper A. To the function z attach the set $S := -z^{-1}(\{0\})$. Since S is not an ordering, it violates at least one of the following axioms:

(1) $S + S \subset S$	(2) $S \cdot S \subset S$	(3) $-1 \notin S$
(4) $S \cup -S = A$	(5) $S \cap -S$ is a prime ideal	

However, this implies that the particular axiom is violated in an entire neigh borhood of z: suppose, for instance, S does not satisfy (1), i.e. there exist $a, b \in S$ such that $a + b \notin S$. Then the set

$$U_1 := \ \{z \in Z : \ z(-a) = 0, \ z(-b) = 0, \ z(-a-b) = 1\}$$

is a neighborhood contained in (Sper $A)^c$, since any $w \in U_1$ violates (1), hence cannot come from an ordering. Likewise, if S violates (2), the set

$$U_2 := \{z \in Z : \ z(-a) = 0, \ z(-b) = 0, \ z(ab) = 1\}$$

is the desired neighborhood for S; the other cases are handled similarly.

Let's turn to the second statement of the theorem: by definition of the constructible topology a constructible set is both $\mathcal{C}$-closed and $\mathcal{C}$-open. Conversely, let C be a $\mathcal{C}$-clopen set. Since C is $\mathcal{C}$-open, it is a union of basic $\mathcal{C}$-open, i.e. constructible sets. C is also $\mathcal{C}$-closed, therefore it is $\mathcal{C}$-compact (as a subset of a $\mathcal{C}$-compact space). Consequently C is a union of *finitely many* constructible sets, which renders C constructible.

Corollary 4.4. *Sper A is quasicompact with respect to the Harrison topology.*

Proof. The Harrison topology is coarser than the constructible topology, therefore any open cover of Sper A is a $\mathcal{C}$-open cover of Sper A.

Definition 4.5: A set $Y \in$ Sper A is called ***proconstructible*** if it is an (arbitrary) intersection of constructible sets of Sper A.

Note: $Y \in$ Sper A is proconstructible iff Y is $\mathcal{C}$-closed.

Proof. As an intersection of constructible (therefore $\mathcal{C}$-closed) sets Y must be $\mathcal{C}$-closed, and if Y is $\mathcal{C}$-closed, then $Y^c = \cup_\alpha C_\alpha$, C_α constructible, so $Y = \cap_\alpha C_\alpha^c$ is proconstructible.

Theorem 4.6. *Let $Y \in$ Sper A be proconstructible. Then*

$$\overline{Y} = \bigcup_{y \in Y} \overline{\{y\}}.$$

Proof. "$\supset$" is obvious. As to the other inclusion suppose $z \in Y$. then for all open constructible sets U with $z \in U$, the intersection $U \cap Y$ is nonempty. Let

$$\mathcal{U} := \{U \text{ open constructible } : z \in U\}$$

and consider $V := \cap_{U \in \mathcal{U}}(U \cap Y) = (\cap_{U \in \mathcal{U}} U) \cap Y$. Since $U \cap Y \neq \emptyset$ for each U, $(\cap_{U \in \mathcal{U}} U) \cap Y \neq \emptyset$ (otherwise, by $\mathcal{C}$-compactness of Y, there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that $U_1 \cap, \ldots, \cap U_n \cap Y = \emptyset$, but $U_1 \cap, \ldots, \cap U_n$ is a neighborhood of z and therefore meets Y). For $y \in (\cap_{U \in \mathcal{U}}) \cap Y$ it follows that $z \in \overline{\{y\}}$ because any neighborhood of z contains an open constructible set $U \ni z$.

Corollary 4.7. *A proconstructible set $Y \in$ Sper A is closed iff it is closed under specialization; a constructible set $C \in$ Sper A is open iff it is closed under generalization.*

Proof. Since $\overline{Y} = \cup_{y \in Y} \overline{\{y\}}$, the first part of the Corollary follows. Now let $C \in \mathcal{C}(A)$ be open, $Q \in C$ and $P \in$ Sper A such that $P \subset Q$. We need to show that $P \in C$.

If not, then $P \in C^c = \cup_{T \in C^c} \overline{\{T\}}$, so there exists $T \in C^c$ such that $T \subseteq P$. But then $T \subseteq P \subseteq Q$, so $Q \in C^c$ by the first part of the theorem, a contradiction. Conversely, let U be constructible and closed under generalization. In order to show that U is open, it suffices to show that U^c is closed under specialization. To this end let $T \in U^c$, $P \in \text{Sper } A$ with $P \supseteq T$. If $P \in U$, then $T \in U$, again a contradiction.

Note that the second statement in Corollary 4.7 becomes false if we only require that C be proconstructible: Consider the ordering $P_{\infty_+} \in \text{Sper } \mathbb{R}[t]$ as described in detail in the next chapter. P_{∞_+} is proconstructible, for

$$P_{\infty_+} = \bigcap_{n \in \mathbb{N}} H(t-n).$$

Also, P_{∞_+} is closed under generalization, since *supp* $P_{\infty_+} = (0)$. But

$$\overline{\text{Sper } \mathbb{R}[t] \backslash P_{\infty_+}} = \text{Sper } \mathbb{R}[t]$$

by Theorem 4.9, so P_{∞_+} cannot be open.

Corollary 4.8. *Let Y be a closed subset of Sper A. Y is irreducible (i.e not a union of two proper closed subsets of Y) iff $Y = \overline{\{y\}}$ for some $y \in$ Sper A. Moreover, this y is unique.*

Proof. Assume Y is irreducible and let

$$\mathcal{U} := \{U \subset \text{Sper } A : \ U \text{ open constructible and } U \cap Y \neq \emptyset\}.$$

Define $Z := \cap_{U \in \mathcal{U}} U$. Since for any finite collection $U_1, \ldots, U_n$ the intersection $Y \cap \cap_{i=1}^n U_i$ is nonempty (if $Y \cap U_i \neq \emptyset$, $i = 1, 2$ but $Y \cap (U_1 \cap U_2) = \emptyset$, then $Y = (Y \cap U_1^c) \cup (Y \cap U_2^c)$, so Y reducible), $\mathcal{C}$-compactness of Y implies that $Y \cap Z \neq \emptyset$. For $y \in Y \cap Z$ one has $Y = \overline{\{y\}}$, since if $x \in Y \backslash \overline{\{y\}}$, there exists an open constructible set V that meets Y but not $\{y\}$, so $y \notin Z$.

For the converse, let $Y = \overline{\{y\}}$ and assume $Y = C_1 \cup C_2$, C_i closed in Y and $C_1 \cap C_2 = \emptyset$. Then $y \in C_1$ or $y \in C_2$ and since C_i closed, $\overline{\{y\}} \subset C_1$ or $\overline{\{y\}} \subset C_2$.

To see that y is unique, suppose $Y = \overline{\{y\}} = \overline{\{y'\}}$. Theorem 3.2 implies that $y \subset y'$ and $y' \subset y$, hence $y = y'$.

Theorem 4.9. *If $C \subset$ Sper A is constructible, then $C \cap V(R)$ is $\mathcal{C}$-dense in C and consequently dense in C.*

Proof. It suffices to show that every basic open set $U \subset C$ meets $C \cap V(R)$, i.e. $U \cap C \cap V(R) \neq \emptyset$. To this end, since $U \cap C$ is constructible, it is enough to prove the following:

$$(*) \qquad D \subset \text{Sper } A \text{ constructible, } D \neq \emptyset \ \Rightarrow \ D \cap V(R) \neq \emptyset.$$

Without loss of generality one may assume $D = Z(f) \cap H(f_1, \ldots, f_m)$, $D \neq \emptyset$ $(f, f_i \in A)$. Let $B := A/(f)$ and $W(R) := V_B(R)$. It will be shown that $W(R) \cap H(\bar{f_1}, \ldots, \bar{f_m}) \neq \emptyset$. As a subset of Sper $A/(f)$, $H(\bar{f_1}, \ldots, \bar{f_m}) \neq \emptyset$, so by Theorem 3.6 $H(\bar{f_1}, \ldots, \bar{f_m}) \cap W(R) \neq \emptyset$ and the preimage of $\overline{P} \in H(\bar{f_1}, \ldots, \bar{f_m}) \cap W(R)$ is an ordering in $V(R) \cap Z(f) \cap H(f_1, \ldots, f_m)$.

Corollary 4.10. *Let $P \in$ Sper A. $\{P\}$ is constructible if and only if $P \in V(R)$.*

Proof. $\{P\}$ constructible $\Rightarrow \{P\} \cap V(R) \neq \emptyset \Rightarrow P \in V(R)$ and conversely, if $P \in V(R) \Rightarrow P = P_a$ for some $a = (a_1, \ldots, a_n) \in V(R) \Rightarrow P = Z(f_1, \ldots, f_n)$, where $f_i = x_i - a_i$, $i = 1, \ldots, n$.

Definition 4.11 Let $\sigma(V(R)) := \{C \cap V(R) : C \in \mathcal{C}(A)\}$. Elements in $\sigma(V(R))$ are called *semialgebraic subsets* of $V(R)$.

If $C = H(f_1, \ldots, f_m) \cap Z(f)$, $f, f_i \in A$ then

$$\begin{aligned} C \cap V(R) &= \{a : \ f =_{P_a} 0, \ f_i >_{P_a} 0, \ i = 1, \ldots, m\} \\ &= \{a : \ f(a) = 0, \ f_i(a) > 0, \ i = 1, \ldots, m\}, \end{aligned}$$

so the semialgebraic sets in $V(R)$ can be described by finitely many polymial equations and inequalities.

The retraction map

$$\begin{aligned} r: \quad \mathcal{C}(\text{Sper } A) &\rightarrow \sigma(V(R)) \\ C &\mapsto C \cap V(R) \end{aligned}$$

is a one-to-one correspondence between the constructible sets in Sper A and the semialgebraic sets in $V(R)$. This can be seen as follows:

(a) Surjectivity: given a semialgebraic set $M = \{f = 0, g_i > 0, i = 1, \ldots, m\} \subset V(R)$, then $M = H(g_1, \ldots, g_m) \cap Z(f)$.

(b) The proof of injectivity will make use of $(*)$ in the proof of Theorem 4.9. Suppose $r(C_1) = r(C_2)$, i.e. $C_1 \cap V(R) = C_2 \cap V(R)$. Then $(C_1 \triangle C_2) \cap V(R) = C_1 \cap C_2^c \cap V(R) \cup C_2 \cap C_1^c \cap V(R) = \emptyset$, so by $(*)$ $C_1 \triangle C_2 = \emptyset$ which means $C_1 = C_2$.

The inverse map of r assigns to each semialgebraic set M the unique constructible set $\tilde{M}$ with the property $M = \tilde{M} \cap V(R)$. $\tilde{M}$ is the closure of M with respect to the constructible topology: $\tilde{M}$ is $\mathcal{C}$-closed as a constructible set and it is the $\mathcal{C}$-closure of M because $\tilde{M} \cap V(R)$ is $\mathcal{C}$-dense in $\tilde{M}$ by Theorem 4.9.

Definition 4.12: Let $\mathcal{L}$ be a boolean lattice of $2^{Sper\ A}$. A filter $\mathcal{F} \subset \mathcal{L}$ is a nonempty subset of $\mathcal{L}$ satisfying

(F1) $\emptyset \notin \mathcal{F}$

(F2) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

(F3) $A \in \mathcal{F},\ B \supset A \Rightarrow B \in \mathcal{F}$

Maximal filters are called ultrafilters; these always exist by Zorn's Lemma (applied to the set of filters partially ordered by inclusion; an upper bound of a chain is the union of the filters in the chain). Ultrafilters can be characterized in the following way:

Proposition 4.13. *A filter $\mathcal{F}$ of a boolean lattice $\mathcal{L}$ is maximal iff for any $A \in \mathcal{L}$, either A or A^c belongs to $\mathcal{F}$.*

Proof. Of course not both A and A^c are in $\mathcal{F}$, since $A \cap A^c = \emptyset \notin \mathcal{F}$. Moreover, either A or A^c intersects all elements in $\mathcal{F}$: if not, there exist $F, F' \in \mathcal{F},\ F \subset A,\ F' \subset A^c$

and $F \cap F' = \emptyset$. Without loss of generality assume $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. The collection

$$\mathcal{F}' := \{B \in \mathcal{L} : B \supset (F \cap A) \text{ for some } F \in \mathcal{F}\}$$

is a filter containing $\mathcal{F}$; hence by maximality of $\mathcal{F}$, $\mathcal{F} = \mathcal{F}'$, so $A \in \mathcal{F}$.
Conversely, let $\mathcal{F}$ be a filter such that A of A^c belongs to $\mathcal{F}$ for any $A \in \mathcal{L}$. If $\mathcal{F}$ is not maximal, then $\mathcal{F} \subset \mathcal{F}'$ for some maximal filter $\mathcal{F}'$ and one finds a set $B \in \mathcal{F}' \backslash \mathcal{F}$. But the $B^c \in \mathcal{F} \subset \mathcal{F}'$, so $B \cap B^c = \emptyset \in \mathcal{F}$, a contradiction.

Lemma 4.14. *There is a one-to-one correspondence between the set of filters on $\mathcal{C}(A)$ and the set of proconstructible sets ($\neq \emptyset$) of Sper A, given by the map*

$$\mathcal{F} \xrightarrow{\Phi} \bigcap_{F \in \mathcal{F}} F =: Y_{\mathcal{F}}$$

$Y_{\mathcal{F}} \neq \emptyset$ since Sper A is $\mathcal{C}$-compact. The inverse map, defined for Y proconstructible and non-empty, is

$$Y \xrightarrow{\Phi^{-1}} \mathcal{F}_Y := \{B \in \mathcal{L} : B \supseteq Y\}.$$

Proof. Φ is onto since F_Y (which clearly is a filter) maps to Y. To prove injectivity it is enough to show that $\Phi^{-1} \circ \Phi(\mathcal{F}) = \mathcal{F}$ for any filter $\mathcal{F}$. So for $\mathcal{F}$ and $Y := \bigcap_{A \in \mathcal{F}} A$ one has to show that $\mathcal{F} = \mathcal{F}_Y$. $\mathcal{F} \subset \mathcal{F}_Y$ since any $A \in \mathcal{F}$ contains Y. If B is any constructible set containing Y, then $B^c \subset \bigcup_{A \in \mathcal{F}} A^c$. B^c is $\mathcal{C}$-compact, so $B^c \subset A_1^c \cup \cdots \cup A_n^c$ for finitely many A_i and $B \supset A_1 \cap \cdots \cap A_n$. But $A_1 \cap \cdots \cap A_n \in \mathcal{F}$, so $B \in \mathcal{F}$.

Corollary 4.15.
(a) Ultrafilter Theorem (L. Bröcker, 1981):
There is a bijective correspondence between Sper A and the set of ultrafilters on $\sigma(V(R))$ via the map

$$\phi : \quad Sper\ A \longrightarrow Ultra\ \sigma(V(R)), \qquad \alpha \mapsto \mathcal{F}_\alpha = \{M : \alpha \in \tilde{M}\}$$

with inverse

$$\phi^{-1}: \quad Ultra\ \sigma(V(R)) \longrightarrow Sper\ A, \qquad \mathcal{F} \mapsto \bigcap_{M\in\mathcal{F}} \tilde{M}.$$

(b) The set of filters on $\sigma(V(R))$ corresponds bijectively to the set of nonempty proconstructible sets of Sper A via $\mathcal{F} \mapsto \bigcup_{M\in\mathcal{F}} \tilde{M}$.

In this respect, given an ordering $P \in \text{Sper } A$, an element $f \in A$ is positive with respect to P iff the semialgebraic set $\{x \in V(R) : f(x) > 0\}$ is contained in the filter $\mathcal{F}_P$. Thus, the 'abstract' positivity of f with respect to P can be described in terms of the unique ordering on $k(\alpha)$, where $\alpha = (p, \overline{P})$.

Proof of Corollary 4.15. (a) For any $\alpha \in \text{Sper } A$, $\mathcal{F}_\alpha$ is maximal since for $A \in$ Sper A either $\alpha \in A$ or $\alpha \in A^c$ and therefore A or A^c belongs to $\mathcal{F}_\alpha$. This shows $\phi(\text{Sper } A) \subset Ultra\ \sigma(V(R))$. Now, any maximal filter of $\sigma(V(R))$ is of the form $\mathcal{F}_\alpha$ for some $\alpha \in \text{Sper } A$, since Sper A is hausdorff in the constructible topology: suppose $\alpha, \beta \in \bigcap_{U\in\mathcal{U}} U$,where $\mathcal{U}$ is an ultrafilter. (This intersection cannot be empty because Sper A is $\mathcal{C}$-compact.) There are neighborhoods V of α, W of β with $V \cap W = \emptyset$, so that only one of them is in $\mathcal{U}$. ϕ is injective since it can be interpreted as the restriction of the map Φ^{-1} introduced in Lemma 4.14.
(b) follows directly from Lemma 4.14 because of the correspondence

$$\sigma(V(R)) \leftrightarrow \mathcal{C}(A).$$

CHAPTER V

SOME EXAMPLES ARE IN ORDER

Example 1. $A = \mathbb{R}[t]$, where $\mathbb{R}$ is the field of real numbers. $\mathbb{R}$ is real closed, for every positive number is a square in $\mathbb{R}$ and every polynomial in t of odd degree has a root in $\mathbb{R}$ by the Intermediate Value Theorem. What are the possible orderings on $\mathbb{R}[t]$? According to the previous chapter, each ordering corresponds to an ultrafilter consisting of semialgebraic subsets of $V(\mathbb{R}) = \mathbb{R}$, so let's try to determine the ultrafilters first.

Since *Sper* A induces the usual interval topology on $\mathbb{R}$, consider the boolean lattice $\mathcal{L}$ generated by the open intervals $\{]a, \infty[: \ a \in \mathbb{R}\}$ and look for possible ultrafilters $\mathcal{U}$ in $\mathcal{L}$.

We certainly have filters of type

(A) $$\mathcal{U}_a = \{U \in \mathcal{L} : \ a \in U\} \text{ for fixed } a \in \mathbb{R}$$

Ultrafilters of another kind are

(B) $$\mathcal{U}_{a_+} = \{U \in \mathcal{L} :]a - \epsilon, a[\subset U \text{ for some } \epsilon > 0\}$$

$$\mathcal{U}_{a_-} = \{U \in \mathcal{L} :]a, a + \epsilon[\subset U \text{ for some } \epsilon > 0\}$$

for fixed $a \in \mathbb{R}$.

Proof. (for $\mathcal{U}_{a_+}$ only) Suppose $B \in \mathcal{L}$, $B = \cup_{j=1}^{n} I_j$, for some intervals $I_j \subset \mathbb{R}$ and $B \notin \mathcal{U}_{a_+}$. Then for all $\epsilon > 0$ $]a, a + \epsilon[\not\subset B$, so a cannot lie in the interior of B. Also, a cannot be a left endpoint of a nondegenerate interval I_k of B, so it is either a right endpoint of some I_k or an interior point of B^c, and in either case $B^c \in \mathcal{U}_{a_+}$.

A third kind of ultrafilters are those of type

(C) $$\mathcal{U}_{\infty_+} = \{U \in \mathcal{L} :]a, +\infty[\subset U \text{ for some } a \in \mathbb{R}\}$$

$$\mathcal{U}_{\infty_-} = \{U \in \mathcal{L} :] - \infty, a[\subset U \text{ for some } a \in \mathbb{R}\}$$

Given any interval Iin $\mathbb{R}$, either I or I^c is unbounded, so that these filters are in fact maximal.

Lemma. *Any ultrafilter of $\mathcal{L}$ is either of type (A), (B) or (C).*

Proof. Let $\mathcal{U}$ be an ultrafilter and assume $\mathcal{U}$ is not of type (A).
Case 1: There exist $c < d$ in $\mathbb{R}$ such that $]c, d[\, \in \mathcal{U}$. Then the intersection

$$V := \bigcap_{U \in \mathcal{U}} \overline{U}$$

is not empty since it is contained in the compact interval $[c, d]$ and any finite intersection $U_1 \cap \cdots \cap U_n$, $U_i \in \mathcal{U}$ is nonempty by the filter axioms. V cannot contain two distinct elements $a \neq b$ because if $a, b \in V$, $a \neq b$, then for $\epsilon := \frac{1}{2}|a - b|$ either $]a - \epsilon, a + \epsilon[\, \in \mathcal{U}$ or $]b - \epsilon, b + \epsilon[\, \in \mathcal{U}$. So for $\{a\} = V$ it follows $a \in \overline{U}$ for all $U \in \mathcal{U}$, and depending on whether $]a, a + \epsilon[\, \in \mathcal{U}$ or $]a - \epsilon, a[\, \in \mathcal{U}$ (ϵ arbitrary), $\mathcal{U} = \mathcal{U}_{a_+}$ or $\mathcal{U} = \mathcal{U}_{a_-}$.
Case 2: $\mathcal{U}$ contains no finite interval. Then $]-\infty, a[\cup]b, \infty[\in \mathcal{U}$ for any pair $a, b \in \mathbb{R}$ and consequently either $]-\infty, a[\, \in \mathcal{U}$ for all $a \in \mathbb{R}$ or $]b, \infty[\, \in \mathcal{U}$ for all $b \in \mathbb{R}$ and $\mathcal{U}$ is of type (C).

Having thus determined the possible ultrafilters on $\mathbb{R}$, what are the corresponding orderings on $\mathbb{R}[t]$?

Type (A) : Denote the ordering corresponding to $\mathcal{U}_a$ with P_a. Since both $]a - \epsilon, a]$ and $[a, a + \epsilon[$ belong to $\mathcal{U}_a$ (ϵ arbitrary), both functions $t - a$ and $a - t$ are in P_a, so *supp* $(P_a) = (t - a)$ and $P_a = \{f \in \mathbb{R}[t] : \ f(a) \geq 0\}$.

Type (B) : Let $\mathcal{U} = \mathcal{U}_{a_-}$ for some $a \in \mathbb{R}[t]$. Since $]a - \epsilon, a[\in \mathcal{U}$ and $]a, a + \epsilon[\notin \mathcal{U}$ for all $\epsilon > 0$, one concludes that $a - t \in P_{a_-}$ but $t - a \notin P_{a_-}$. Write $f \in \mathbb{R}[t]$ in the form $(a - t)^r g(t)$ with $g(a) \neq 0$ to see that $f > 0$ on an interval $]a - \epsilon, a[$ iff $g(a) > 0$. Thus

$$P_{a_-} = \{0\} \cup \{(a - t)^r g(t) : \ g(a) > 0\}, \text{ and likewise}$$
$$P_{a_+} = \{0\} \cup \{(t - a)^r g(t) : \ g(a) > 0\}.$$

Note that P_{a_-} and P_{a_+} both generalize the ordering P_a and

$$supp\ (P_{a_-}) = supp\ (P_{a_+}) = (0) \subsetneq (t-a) = supp\ (P_a),$$

which is a maximal ideal.

Type (C) : Let $\mathcal{U} = \mathcal{U}_{\infty_-}$. In this case $]-\infty, a[\ \in \mathcal{U}$ for all $a \in \mathbb{R}$, so $f > 0$ with respect to P_{∞_-} iff $f > 0$ on an interval $]-\infty, a[$,which means that $\lim_{t \to -\infty} f(t) = +\infty$ Therefore

$$\begin{aligned} P_{\infty_-} &= \{0\} \cup \{a_0 + a_1 t + \cdots + a_n t^n : \ (-1)^n a_n > 0\}, \text{ and likewise} \\ P_{\infty_+} &= \{0\} \cup \{a_0 + a_1 t + \cdots + a_n t^n : \ a_n > 0\}. \end{aligned}$$

Note that P_{∞_-} and P_{∞_+} are maximal orderings although their support is zero. To see this, suppose $P_{\infty_+} \subsetneq P$ for some $P \in Sper\ \mathbb{R}[t]$. Then P must have nonzero support and $supp\ (P) = (f)$ for some irreducible $f \in \mathbb{R}[t]$. Thus f is either of the form $t - a$ for some $a \in \mathbb{R}$ or $f = t^2 + bt + c$ with $b^2 - 4c < 0$ $(b, c \in \mathbb{R})$. But the ring $B := \mathbb{R}[t]/(t^2 + bt + c)$ is not semireal, for

$$-1 \ = \ \frac{4(t + \frac{b}{2})^2}{4c - b^2} \ \in B^2.$$

So $f = t - a$ and $P = P_a$ for some $a \in \mathbb{R}$, but $h = t - (a+1) \in P_{\infty_+} \backslash P_a$

Example 2. $A = \mathbb{R}[x, y]$, $V(\mathbb{R}) = \mathbb{R}^2$. Consider the following filter $\mathcal{F}$:

$$F \in \mathcal{F} :\Leftrightarrow \begin{cases} \exists \epsilon > 0\ \exists g \in \mathbb{R}[x] \text{ with } g(x) > 0 \text{ for } x \in]0, \epsilon[\\ \text{and the set } M_g^\epsilon := \{(a, b) \in \mathbb{R}^2 : \ 0 < a < \epsilon, \ 0 < b < g(a)\} \subset F \end{cases}$$

$\mathcal{F}$ is a filter: suppose $F, F' \in \mathcal{F}$, then there exist $M_g^\epsilon \subset F, M_{g'}^{\epsilon'} \subset F'$ and for

$$\epsilon_0 := min \ \left\{\{c \in \mathbb{R}^+ \backslash \{0\} : \ (g - g')(c) = 0\}, \epsilon, \epsilon'\right\}$$

either $M_g^{\epsilon_0}$ or $M_{g'}^{\epsilon_0}$ is contained in $F \cap F'$.

$\mathcal{F}$ is maximal: suppose $B \in \sigma\ (\mathbb{R}^2)$ and $B \notin \mathcal{F}$. Then for any $g \in \mathbb{R}[x]$ such that $g(x) > 0$ on $]0,\epsilon[$, the set $M_g^\epsilon \not\subset B$. One only needs to consider the case that $(0,0) \in \overline{B}$ and $B \subset \{(a,b):\ a > 0, b > 0\}$. Without loss of generality assume

$$B = \{(x,y) \in \mathbb{R}^2 :\ f(x,y) = 0, g_1(x,y) > 0, \ldots, g_n(x,y) > 0\}.$$

One can further reduce to the case $B = \{f = 0\}$ or $B = \{g > 0\}$, f, g irreducible: first, a set $\{fg > 0\}$ can be decomposed into $(\{f > 0\} \cap \{g > 0\}) \cup (\{f < 0\} \cap \{g < 0\})$ and secondly, if $F = F_1 \cap \cdots \cap F_n \notin \mathcal{F}$ then there exists an $i \in \{1, \ldots, n\}$ such that $U_i \notin \mathcal{F}$, thus $U_i^c \in \mathcal{F}$ and consequently $U^c \in \mathcal{F}$. So we might as well assume
(a) $B = \{f = 0\}$ for some $f \in \mathbb{R}[x,y]$. $\{f = 0\}$ is a closed set with empty interior, so B^c contains a set of the required form.
(b) If $B = \{f > 0\}$ with $f = \sum_{i=1}^n a_i(x) y^i$, $a_i(x) = \sum_{j=0}^{n_i} b_{ij} x^j$, $a_0(x) \not\equiv 0$, let $m := ord\,(a_0(x))$ and $D := \sum_{i,j} |b_{ij}|$, $i = 1, \ldots, n, j = 0, \ldots, n_i$. Then for $\epsilon := \frac{|b_{0m}|}{2D}$ the sign of f is determined by b_{0m} whenever $0 < x < \epsilon$ and $0 < y < x^{m+1}$, since

$$\begin{aligned} |f(x,y) - b_{0m}x^m| &= |b_{0m+1}x^{m+1} + \cdots + b_{0n_0}x^{n_0} + y\bar{f}(x,y)| \text{ for suitable } \bar{f} \\ &\leq x^{m+1}(|b_{0m+1}| + \cdots + |\bar{f}(x,y)|) \\ &\leq x^m \cdot \epsilon D \\ &< |b_{0m}| x^m. \end{aligned}$$

By assumption, B does not contain the set

$$M = M_{x^{m+1}}^\epsilon = \{(x,y):\ 0 < x < \epsilon,\ 0 < y < x^{m+1}\}.$$

Also, f does not change sign on M, thus $f \leq 0$ on M and $B^c \in \mathcal{F}$.

The ordering P corresponding to $\mathcal{F}$ can be described as follows:
Functions of the form $a - x^m$, $m \geq 0$, $a \in \mathbb{R}^+$ and $x^m - y$ are in P because the sets $\{a - x^m > 0\}$, $\{x^m - y > 0\}$ are in $\mathcal{F}$. From this one can see that x is 'infinitesimal' with respect to $\mathbb{R}^+ \backslash \{0\}$ and y is 'infinitesimal' with respect to x, in symbols $0 \ll_P y \ll_P x \ll_P \mathbb{R}^+ \backslash \{0\}$.

What are possible specializitions of P? First note that $supp\ (P) = (0)$. Considering the prime ideal chain

$$(0) \underset{\neq}{\subset} (y) \underset{\neq}{\subset} (x,y) \quad \text{in } \mathbb{R}[x,y]$$

one finds $P \subset P' \subset P''$, where

$$P' = \{f \in \mathbb{R}[x,y] : \ f(x,0) \geq 0 \text{ on }]0,\epsilon[\text{ for some } \epsilon > 0\} \text{ and}$$
$$P'' = \{f \in \mathbb{R}[x,y] : \ f(0,0) \geq 0\}.$$

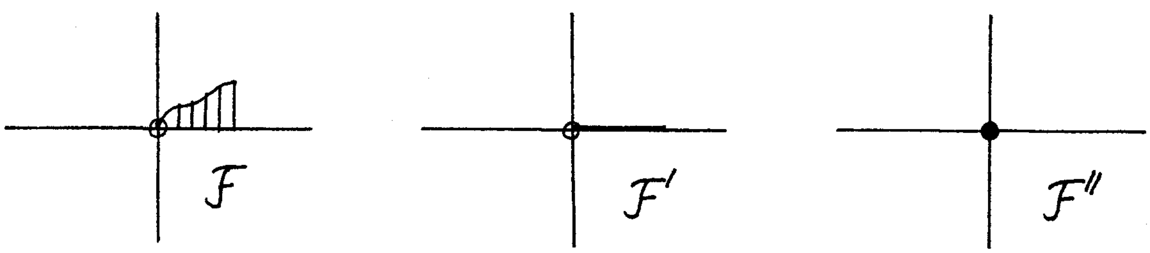
Here P' corresponds to the ultrafilter

$$\mathcal{F}' = \{F \in \sigma(\mathbb{R}^2) : \ \exists \epsilon > 0 \text{ such that }]0,\epsilon[\times \{0\} \subseteq F\}$$

and P'' corresponds to

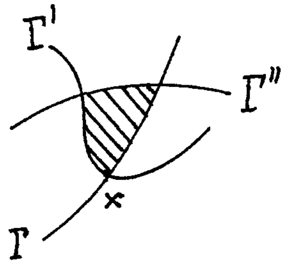
$$\mathcal{F}'' = \{F \in \sigma(\mathbb{R}^2) : \ (0,0) \in F\}.$$

So in this case one has a nice geometric interpretation of the chain of orderings $P \subset P' \subset P''$ via the ultrafilters



In general, one can construct an ordering P on $\mathbb{R}[x,y]$ by taking an algebraic curve Γ, fixing a point on it and setting

$$f \geq_P 0 :\Leftrightarrow \text{the set } \{(x,y) : \ f(x,y) \geq 0\} \text{contains a segment of the form}$$



where Γ' and Γ'' are also algebraic curves and Γ' passes through x.

CHAPTER VI

VALUATIONS, PLACES AND THE LANG HOMOMORPHISM THEOREM

VALUATIONS

Definition 6.1: Let A be an integral domain with quotient field K. A is called a *valuation domain* (or *valuation ring)* if for every $a \in K\backslash\{0\}$ either $a \in A$ or $a^{-1} \in A$.

Definition 6.2: A ring A is called a *local ring* if it has only one maximal ideal (denoted by $\mathfrak{m}_A$ or simply $\mathfrak{m}$, if the reference is clear). The field $k(A) := A/\mathfrak{m}_A$ is the residue field of A.

Lemma 6.3. *If A is a valuation domain, then A is local.*

Proof. It suffices to show that the set $\mathfrak{m}$ = set of non-units forms an ideal in A. Given $a \in A$, $m \in \mathfrak{m}$, am cannot be a unit, for if $amb = 1$ for some $b \in A$, then $m = (ab)^{-1}$, but m was assumed to be a non-unit. If $a, b \in \mathfrak{m}$, then either $ab^{-1} \in A$ or $a^{-1}b \in A$ and assuming $ab^{-1} \in A$ one gets $a + b = b(ab^{-1} + 1) \in \mathfrak{m}$.

Definition 6.4: Let A be a subring of a field K, P an ordering on K. A is said to be *convex* in K (with respect to P) if for $a, b \in K$ the following holds:

$$0 < b < a,\ a \in A\ \Rightarrow\ b \in A$$

Accordingly, one defines the *convex hull* $ch_P(A)$ of A in K with respect to P as the set

$$ch_P(A) :=\ \{k \in K :\ \exists a \in A \text{ such that } |k|_P \leq a\},$$

where $|a|_P = a$ if $a \in P$, $|a|_P = -a$ otherwise.

Lemma 6.5. *Let A a subring of K, P an ordering of K.*

(1) *$ch_P(A)$ is a convex subring of K*

(2) *A is convex in K iff $[0,1] \subset A$. In particular, any ring B that contains a convex ring A is convex.*

Proof. (1): For $x, y \in ch_P(A)$ there exist $a, b \in A$ such that $a \pm x, b \pm y \in P$. Hence $(a+b) \pm (x+y) \in P$ and $|x+y|_P \leq_P a+b$. Moreover,

$$ab + xy = \frac{1}{2}[(a+x)(b+y) + (a-x)(b-y)] \geq_P 0$$
$$ab - xy = \frac{1}{2}[(a+x)(b-y) + (a-x)(b+y)] \geq_P 0$$

hence $|xy|_P \geq_P 0$ and thus $x+y, xy \in ch_P(A)$. As to convexity of $ch_P(A)$, note that if $0 < k < b$, $k \in K, b \in A$, then there exists $a \in A$ such that $0 < k < b < a$, thus $k \in ch_P(A)$.

(2): If A is convex, then $[0,1] \subseteq A$, since $0, 1 \in A$. Conversely, if $[0,1] \subseteq A$ and $a \in A, b \in K$ with $0 < b < a$, then $0 < a^{-1}b < 1$ and $b = a(a^{-1}b) \in A$.

Note: $ch_P(A)$ is a valuation ring: let $a \in K$, then either a or $a^{-1} \in [-1,1]$.

Theorem 6.6. *Let A be a valuation ring of K.*

(1) *If $\mathfrak{p}$ is a prime ideal of A then $\mathfrak{p} = \mathfrak{p}A_{\mathfrak{p}}$.*

(2) *If B is a subring of K, then $B \supset A$ implies $\mathfrak{m}_B \subset A$, hence $\mathfrak{p} := \mathfrak{m}_B$ is a prime ideal of A. Furthermore, $B = A_{\mathfrak{p}}$. Consequently, any overring of A is the localization of A at some prime ideal $\mathfrak{p} \lhd A$.*

(3) *The set of overrings of A is totally ordered by inclusion and the same holds for the set of prime ideals of A.*

Proof. (1): Let $p \in \mathfrak{p}$, $a \in A \backslash \mathfrak{p}$. It is to show that $pa^{-1} \in \mathfrak{p}$. First observe that $pa^{-1} \in A$ for otherwise $p^{-1}a \in A$ and $a = p(p^{-1}a) \in \mathfrak{p}$. Now, $p = (pa^{-1})a \in \mathfrak{p}$ and $a \notin \mathfrak{p}$, therefore $pa^{-1} \in \mathfrak{p}$.

(2): If $\mathfrak{m}_B \not\subset A$, one finds an element $m \in \mathfrak{m}_B \backslash A$, so that $m^{-1} \in A \subset B$, a contradiction. $A_\mathfrak{p}$ is contained in B because $\mathfrak{p}A_\mathfrak{p} = \mathfrak{m}_B$ and $(A_\mathfrak{p})^* \subset B^*$. For the other inclusion, suppose $b \in B$. If $b \notin A$ then $b^{-1} \in A$, so $b \in B^* = B\backslash\mathfrak{m}_B$ and $b = 1/b^{-1} \in A_\mathfrak{p}$.

(3): Suppose $B \supset A$, $B' \supset A$ and $b \in B\backslash B'$, $b' \in B'\backslash B$. If $b(b')^{-1} \in A \subset B'$ then $b = b(b')^{-1}b' \in B'$ and if $b^{-1}b' \in A \subset B$ then $b' = bb^{-1}b' \in B$, so neither $b(b')^{-1}$ nor $b^{-1}b' \in A$, a contradiction.

Remark: The smallest convex subring of an ordered field K is $ch_P(\mathbb{Z})$, since $\mathbb{Z} \subset A \subset K$ for any subring $A \subset K$.

Definition 6.7: Let K be a field an Γ a totally ordered abelian group. A *valuation* of K is a map $v: K \to \Gamma \cup \{\infty\}$ such that for $a, b \in K$

(v1) $\quad v(a) = \infty \iff a = 0$

(v2) $\quad v(ab) = v(a) + v(b)$

(v3) $v(a+b) \geq \min\{v(a), v(b)\}$

$v(K^*) = \Gamma_v$ is called the *value group* of K. (it is tacitly assumed that $\gamma < \infty$ for all $\gamma \in \Gamma$, $\gamma + \infty = \infty + \gamma = \infty$, $\infty + \infty = \infty$.)

Remarks:

(1) If a is a root of unity, then $v(a) = 0$, since (a) $v(1) = v(1^2) = v(1) + v(1) \Rightarrow v(1) = 0$ and (b) $0 = v(a^n) = nv(a) \Rightarrow v(a) = 0$ (Γ is totally ordered, thus torsion free).

(2) $v(a^{-1}) = -v(a)$ for $a \in K$ since $0 = v(1) = v(aa^{-1}) = v(a) + v(a^{-1})$

(3) $v(-a) = v(-1) + v(a) = v(a)$

(4) If $v(a) \neq v(b)$ then $v(a+b) = \min\{v(a), v(b)\}$:
Suppose $v(a+b) > \min\{v(a), v(b)\}$ and, say, $v(a) < v(b)$. Then $v(a+b) > v(a)$ implies $v(a) = v((a+b)-b) \geq \min\{v(a+b), v(b)\} > v(a)$.

Theorem 6.8. *Let $v: K \to \Gamma \cup \{\infty\}$ be a valuation of K. Then $A_V := \{a \in K: v(a) \geq 0\}$ is a valuation ring of K with maximal ideal $\mathfrak{m}_A := \{a \in K: v(a) > 0\}$.*

Proof. The fact that A_V is a ring follows from Γ being totally ordered. Suppose $a \in K \backslash A_V$, i.e. $v(a) < 0$, then $v(a^{-1}) > 0$ and $a^{-1} \in A_V$, thus A_V is a valuation domain. If $a \in \mathfrak{m}_A$ then $v(a^{-1}) < 0$, so $a^{-1} \notin A$ and a is a non-unit of A.

Example: 1 Let $F = K(t)$, K a field, and $v: K^* \to \mathbb{Z}$ be the order of a function $f = \frac{g}{h}$ in t, i.e. if $f = t^r \frac{g'}{h'}$, $g'(0), h'(0) \neq 0$, then $v(f) = \text{ord}(f) = r$.
To prove axiom (v3), suppose $f = t^r \sum_{i=0}^n a_i t^i$, $g = t \sum_{i=0}^m b_i t^i$, $a_0, b_0 \neq 0$, $r \leq s$.

$$v(f+g) \quad \begin{cases} > s & \text{if } r = s \text{ and } a_s = -b_s \\ = s & \text{if } r = s \text{ and } a_s \neq b_s \\ r & \text{otherwise,} \end{cases}$$

so in all cases $v(f+g) \geq \min\{v(f), v(g)\}$. The corresponding valuation ring is

$$A_V = \{f \in K(t): v(f) \geq 0\} = K(t)_{(t)}$$

Given a valuation domain A of a field K, let us construct a valuation $v: K \to \Gamma \cup \{\infty\}$ such that $A_V = A$. For this purpose let $\Gamma := K^*/A^*$ and define

$$aA^* \leq bA^* :\Longleftrightarrow a^{-1}b \in A$$

This defines a total ordering on Γ since for any $k \in K$, either k or $k^{-1} \in A$. Let v be the the projection of K^* onto K^*/A^*, i.e. $v(a) = aA^*$ for $a \in K^*$ and $v(0) = \infty$. The axioms (v1)–(v3) are satisfied:
(v1): holds by definition

(v2): $v(ab) = abA^* = aA^* \cdot bA^*$ by definition of the multiplication in Γ

(v3): for $a, b \neq 0$ and $v(a) \leq v(b)$ ($\Rightarrow a^{-1}b \in A$)

$$v(a+b) = (a+b)A^* = a(1+a^{-1}b)A^* \geq aA^* = v(a)$$

$A_V = A$ since $a \in A_V \iff v(a) \geq 0 \iff aA^* \geq 0 \iff a \in A$.

$v = v_A$ is often called the *canonical valuation* associated to A.

Definition 6.9: Let (K, P) be an ordered field, $v : K \to \Gamma \cup \{\infty\}$ a valuation on K. v is said to be *compatible* with P if A_V is convex in (K, P).

Lemma 6.10. *The following statements are equivalent:*

(1) *v is compatible with P*

(2) $1 + \mathfrak{m}_A \subset P$

(3) $0 <_P a <_P b \Rightarrow v(a) \geq v(b) \quad (a, b \in K)$

Proof. (1) $\Rightarrow$ (2): obvious for $m \in \mathfrak{m}_A$, $m \geq_P 0$. So suppose $m \in \mathfrak{m}_A$ and $m <_P 0$. $m^{-1} \notin A \Rightarrow -m^{-1} >_P 1 \Rightarrow 1 >_P -m \Rightarrow 1 + m >_P 0$.

(2) $\Rightarrow$ (3): suppose $0 <_P a <_P b$ but $v(a) < v(b)$, then $ba^{-1} \in \mathfrak{m}$ and $0 <_P a \cdot a^{-1} <_P ba^{-1}$, so $1 - ba^{-1} <_P 0$ contradicting $1 + \mathfrak{m} \subset P$.

(3) $\Rightarrow$ (1): suppose $0 <_P a <_P b$, $b \in A$, $a \in K$, then $v(a) \geq v(b) \geq 0$, hence $a \in A$.

Example 2: Consider the situation described in the previous example (after Theorem 6.8) and assume K is ordered by$P$. We are now going to construct an ordering Q on $K(t)$ with $Q \supset P$ and Q compatible with v.

For $f \in K(t)$ let $L(f)$ be the coefficient of f corresponding to its lowest order term, i.e. if $f = \sum_{i=-s}^{n} a_i t^i$, $a_{-s} \neq 0$, then $L(f) = a_{-s} = a_{v(f)}$. Define

$$Q := \{0\} \cup \{f \in K(t) : L(f) > 0\}$$

Q is an ordering since $L(fg) = L(f) \cdot L(g)$ and

$$L(f+g) = \begin{cases} L(f) + L(g) & \text{if } v(f) = v(g) \text{ and } L(f) \neq -L(g) \\ L(f) & \text{if } v(f) < v(g) \end{cases}$$

The case $L(f) = -L(g)$ does not occur for $f, g \in Q$. Now suppose $0 < f < g$ with respect to Q but $v(f) < v(g)$. This implies $v(f-g) = v(f)$ and $L(f-g) = L(f) > 0$, thus $f - g \in Q$ and $f \geq_Q g$, a contradiction. So the pair (v, Q) meets condition (3) in Lemma 6.10, which means that v is compatible with Q.

For $K = \mathbb{R}$ the ordering Q on $\mathbb{R}(t)$ is exactly the extension of P_{0_+} on $\mathbb{R}[t]$ to its quotient field (see chapter V).

Example 3: Let α be the ordering described in the second example in chapter V. For $f \in \mathbb{R}[x_1, \ldots, x_n]$, $f = y^m \sum_{i=0}^k a_i(x) y^i$, where $a_0(x) = \sum_{j=l}^s a_{0j} x^j$, define $v(f) := (\operatorname{ord}_y f, \operatorname{ord}_x a_0(x))$ and extend v to $\mathbb{R}(x,y)$ via $v(\frac{f}{g}) = v(f) - v(g)$. (Set $v(0) := \infty$, as usual.) v is a valuation on $\mathbb{R}(x_1, \ldots, x_n)$:

(v1): holds by definition.

(v2): Consider polynomials $f, g \in \mathbb{R}[x, y]$ first. Let

$$f = y^m \sum_{i=0}^{k_1} a_i(x) y^i, \text{ where } a_0(x) = \sum_{j=l_1}^{r_1} a_{0j} x^j$$

and

$$g = y^s \sum_{i=0}^{k_2} b_i(x) y^i, \text{ where } b_0(x) = \sum_{j=l_2}^{r_2} b_{0j} x^j.$$

Then

$$\begin{aligned} v(fg) &= v(y^{m+s}(\sum_0^{k_1} a_i(x) y^i)(\sum_0^{k_2} b_j(x) y^j)) = (m+s, \operatorname{ord}(a_0(x) + b_0(x)) \\ &= (m+s, l_1 + l_2) = v(f) + v(g). \end{aligned}$$

Now let $h_1 = \frac{f_1}{g_1}$, $h_2 = \frac{f_2}{g_2} \in \mathbb{R}(x, y)$. In this case

$$\begin{aligned} v(h_1 h_2) &= v(\frac{f_1 f_2}{g_1 g_2}) = v(f_1 f_2) - v(g_1 g_2) \\ &= v(f_1) + v(f_2) - v(g_1) - v(g_2) = v(h_1) + v(h_2). \end{aligned}$$

(v3): Again, we will consider polynomials $f, g \in \mathbb{R}[x, y]$ first. Using the notation set up in (v2) we may assume that $m \leq s$ to obtain

$$v(f+g) = \begin{cases} (m, l), \text{ where } l \geq \min\{l_1, l_2\} & \text{if } m < s \text{ or } (m = s \text{ and} \\ & \quad a_0(x) + b_0(x) \neq 0) \text{ (see Ex. 2)} \\ (i, t), \text{ where } i > m & \text{if } m = s \text{ and } a_0(x) + b_0(x) = 0. \end{cases}$$

So in any case, $v(f+g) \geq \min\{v(f), v(g)\}$. If $h_1 = \frac{f_1}{g_1}$, $h_2 = \frac{f_2}{g_2}$, then

$$\begin{aligned} v(h_1 + h_2) &= v(\frac{f_1}{g_1} + \frac{f_2}{g_2}) \\ &= v(\frac{f_1 g_2 + f_2 g_1}{g_1 g_2}) \\ &= v(f_1 g_2 + f_2 g_1) - v(g_1 g_2) \\ &\geq \min\{v(f_1) + v(g_2), v(f_2) + v(g_1)\} - v(g_1) - v(g_2) \\ &= \min\{v(f_1) + v(g_2) - v(g_1) - v(g_2), v(f_2) + v(g_1) - v(g_1) - v(g_2)\} \\ &= \min\{v(h_1), v(h_2)\}. \end{aligned}$$

Claim: v is compatible with α. We are going to show:

$$v(a) < v(b) \Rightarrow b(\alpha) \leq a(\alpha)$$

for $a, b \in \mathbb{R}(x,y)$, $a(\alpha), b(\alpha) \geq 0$. First assume that $a, b \in \mathbb{R}[x,y]$ and let $v(a) = (m_1, l_1)$, $v(b) = (m_2,, l_2)$. Then $a = y^{m_1}(a_{m_1}(x) + \cdots)$, $b = y^{m_2}(b_{m_2}(x) + \cdots)$. Since by assumption $(m_1, l_1) < (m_2, l_2)$, we either have $m_1 < m_2$ or $m_1 = m_2$ and $l_1 < l_2$. According to the estimations we made in chapter V, example 2, we conclude that the set $\{(x,y) : a(x,y) > 0\}$ contains a set of the form $\{(x,y) : 0 < x < \epsilon, 0 < y < x^{l_1+1}\}$ for some $\epsilon > 0$, and so does $\{(x,y) : (a-b)(x,y) > 0\}$, hence $(a-b)(\alpha) > 0$.

Now let $h_1 = \frac{f_1}{g_1}$, $h_2 = \frac{f_2}{g_2}$, where $h_1(\alpha) > 0$, $h_2(\alpha) > 0$ and $v(h_1) < v(h_2)$. Then $v(f_1) - v(g_1) < v(f_2) - v(g_2)$, so $v(f_1 g_2) < v(f_2 g_1)$. The previous calculations imply that $(f_1 g_2)(\alpha) \geq (f_2 g_1)(\alpha)$, which gives the desired result.

Theorem 6.11. *Let A be a valuation ring in a field K, $v : K \to \Gamma \cup \{\infty\}$ a surjective valuation, $\Gamma_+ := \{\gamma \in \Gamma : \gamma \geq 0\}$.*

(1) *The map*

$$\Phi : \mathfrak{p} \mapsto \Delta_{\mathfrak{p}} := v(A \backslash \mathfrak{p}) \cup -v(A \backslash \mathfrak{p}) = v(A_{\mathfrak{p}}^*)$$

is a one-to-one correspondence between the prime ideals of A and the convex subgroups of Γ.

(2) *$\wp \triangleleft A$ is prime if and only if $\wp$ is a radical ideal.*

Proof. (1): If $\gamma,\ \delta \in \Delta_{\mathfrak{p}}$ there exist $r, s \in A_{\mathfrak{p}}^*$, such that $v(r) = \gamma,\ v(s) = \delta$. Then $\gamma - \delta = v(r) - v(s) = v(rs^{-1})$. If $rs^{-1} \in \mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}$, then $s^{-1} \in \mathfrak{p}$, but $s \in A_{\mathfrak{p}}^*$. This shows that $\Delta_{\mathfrak{p}}$ is a subgroup of Γ. Now suppose $0 < \delta' < \delta$ with $\delta \in \Delta_{\mathfrak{p}}$. We will find $r, s \in A_{\mathfrak{p}}^*$ such that $v(r) = \delta,\ v(s) = \delta - \delta'$, and thus $\delta' = v(r) - v(s) = v(rs^{-1})$. If $rs^{-1} \in \mathfrak{p}$ then $s^{-1} \in \mathfrak{p}$, but on the other hand $v(s^{-1}) = \delta' - \delta < 0$. Therefore $rs^{-1} \notin \mathfrak{p}$ and $\delta' \in \Delta_{\mathfrak{p}}$, so $\Delta_{\mathfrak{p}}$ is convex.

If $\mathfrak{p} \underset{\neq}{\subset} \mathfrak{q}$ are two prime ideals of A, then $A_{\mathfrak{q}} \underset{\neq}{\subset} A_{\mathfrak{p}}$ and $v(A_{\mathfrak{q}}^*) \underset{\neq}{\subset} v(A_{\mathfrak{p}}^*)$, since for $a \in A_{\mathfrak{p}}^* \backslash A_{\mathfrak{q}}^*$ it follows $|v(a)| > |v(b)|$ (otherwise, by convexity of $\Delta_{\mathfrak{q}}$, $v(a) \in \Delta_{\mathfrak{q}} \Rightarrow v(a) = v(b)$ for some $b \in A_{\mathfrak{q}}^* \Rightarrow ab^{-1} \in A^* \subset A_{\mathfrak{q}}^* \Rightarrow a = abb^{-1} \subset A_{\mathfrak{q}}^*$.) This shows injectivity of Φ and it remains to show that Φ is onto. Consider the set

$$\mathfrak{p}_{\Delta} := \ \{0\} \cup v^{-1}(\Gamma_+ \backslash \Delta),$$

where Δ is a convex subgroup of Γ.

(a) $\mathfrak{p}_{\Delta}$ is an ideal in A: let $a, b \in \mathfrak{p}_{\Delta}$ and suppose $a + b \in v^{-1}(\Delta)$. Since $v(a+b) \geq \min\{v(a), v(b)\}$, this implies that at least one of $v(a), v(b)$ lies in Δ, so either $a \in v^{-1}$ or $b \in v^{-1}$.

(b) $\mathfrak{p}_{\Delta}$ is a prime ideal: let $a, b \notin \mathfrak{p} \Rightarrow v(a), v(b) \in \Delta \Rightarrow v(a) + v(b) = v(ab) \in \Delta \Rightarrow ab \notin \mathfrak{p}$.

(c) $\Phi(\mathfrak{p}_{\Delta}) = \Delta:\ \Phi(\mathfrak{p}_{\Delta}) = v(A \cap \mathfrak{p}_{\Delta}{}^c) \cup -v(A \cap \mathfrak{p}_{\Delta}{}^c) = (\Gamma_+ \cap \Delta) \cup (\Gamma_- \cap \Delta) = \Delta$.

(2): Let $\wp \triangleleft A$ such that $\sqrt{\wp} = \wp$. Suppose $ab \in \wp$ and assume $ab^{-1} \in A$ Then $abab^{-1} = a^2 \in \wp$, so $a \in \wp$.

Definition 6.12:

(1) Let Γ be a totally ordered abelian group. The *rank* of Γ is the number of convex subgroups of Γ properly contained in Γ.

(2) Let A be a valuation domain. The rank of A is defined as the rank of its canonical value group $\Gamma = K^*/A^*$ where $K = \text{quot}\ (A)$.

According to the previous theorem, rank A is the number of prime ideals of A (other than the zero ideal) which in this case agrees with the Krull dimension of A, since the set of prime ideals is totally ordered by inclusion. Rank A also coincides with the number of overrings of A in K $(\neq K)$ by Theorem 6.6.

REAL PLACES

Let K be a field and $\tilde{K} := K \cup \{\infty\}$ together with the following relations:

$a + \infty = \infty + a = \infty, \quad a \cdot \infty = \infty \cdot a = \infty \ (a \in K)$

$-\infty = \infty, \quad 0^{-1} = \infty, \quad \infty^{-1} = 0, \quad \infty \cdot \infty = \infty$

The relations $\infty + \infty$, $0 \cdot \infty$ and $\infty \cdot 0$ remain undefined.

Definition 6.13: Let K, L be fields. A *place* of K with values in L is a map $\lambda : \tilde{K} \to \tilde{L}$ satisfying

(1) $\lambda(a) + \lambda(b)$ is defined in $\tilde{L} \Rightarrow a + b$ is defined in $\tilde{K}$ and $\lambda(a+b) = \lambda(a) + \lambda(b)$

(2) $\lambda(a)\lambda(b)$ is defined $\Rightarrow ab$ is defined and $\lambda(ab) = \lambda(a)\lambda(b)$

(3) $\lambda(1) = 1$.

If K and L are extensions of a common ground field k and $\lambda|_k = id_k$, then λ is called a *k-place* of K.

Remark. Places have the following properties:

(a) $\lambda(\infty) = \infty, \ \lambda(0) = 0$

(b) $\lambda(-a) = -\lambda(a), \ \lambda(a^{-1}) = [\lambda(a)]^{-1}$ for $a \in \tilde{K}$

(c) If $\lambda' : \tilde{L} \to \tilde{M}$ is a place, then $\lambda' \circ \lambda : \tilde{K} \to \tilde{M}$ is a place.

Proof. (a): If $\lambda(\infty) \neq \infty$, then $\lambda(\infty) + \lambda(\infty)$ is defined and so is $\infty + \infty$, but in fact, it's not.

(b) holds for $a \in \{0, \infty\}$ because of (a). If $\lambda(a) = \lambda(-a) = \infty$ then $\lambda(a) = -\lambda(a)$ and $\lambda(a^{-1}) = 0 = [\lambda(a)]^{-1}$. Otherwise $\lambda(a) + \lambda(-a), \lambda(a) \cdot \lambda(a^{-1})$ are defined and $\lambda(a) + \lambda(-a) = \lambda(0) = 0, \lambda(a) \cdot \lambda(a^{-1}) = \lambda(1) = 1$, which proves (b).

(c) holds because a place acts like a homomorphism whenever it takes values in L and is the identity on $\{0, \infty\}$.

Theorem 6.14. *If* $\lambda : K \cup \{\infty\} \to L \cup \{\infty\}$ *is a place, then*

$$A_\lambda := \{a \in K : \lambda(a) \neq \infty\}$$

is a valuation ring with maximal ideal $m_\lambda := \{a \in K : \lambda(a) = 0\}$.

Proof. It is clear that A_λ is a ring. For any $a \in K$, at least one of $\lambda(a), \lambda(-a) \neq \infty$, so A is a valuation ring. $a \in A$ is a nonunit of $A \iff \lambda(a^{-1}) = \infty \iff \lambda(a) = 0 \iff a \in m_\lambda$, hence m_λ is the maximal ideal of A.

Given a valuation ring $A \subset K = \operatorname{quot} A$, the *associated canonical place* is

$$\lambda: \quad K \cup \{\infty\} \to k(A) \cup \infty = A/\mathfrak{m}_A \cup \infty$$

$$\lambda(a) = \begin{cases} a + \mathfrak{m}_A & \text{if } a \in A \\ \infty & \text{otherwise} \end{cases}$$

Then $A = \lambda^{-1}(k(A))$ and $a \in \mathfrak{m}_A \iff \lambda(a) = 0$.

Definition 6.15: A place $\lambda: K \cup \{\infty\} \to L \cup \{\infty\}$ is called *real* if L is formally real. If P, Q are orderings on K, L respectively, then $\lambda: K \cup \{\infty\} \to L \cup \{\infty\}$ is said to be *compatible* with P, Q, if it is order preserving, i.e. $\lambda(P) \subset Q \cup \{\infty\}$.

Theorem 6.16. *Let* $\lambda: K \to L \cup \{\infty\}$ *be a place,* P *an order on* K, Q *an order on* L.
λ *is compatible with* P *and* $Q \iff A_\lambda$ *is convex with respect to* P *and the induced homorphism* $\bar{\lambda}: A \to A/\mathfrak{m}_A \to L$, $\bar{\lambda}(a + \mathfrak{m}_A) = \lambda(a)$ *satisfies* $\bar{\lambda}(\overline{P}) \subset Q$.

Proof. $\Rightarrow$: Suppose A is not convex, then there is $m \in \mathfrak{m}_A$ such that $-1 - m \in P$ and $\lambda(-1 - m) = -1 \notin Q$, so λ does not preserve order. Since $\bar{\lambda}(\bar{a}) = \lambda(a)$ for $a \in A_\lambda$, $\bar{\lambda}$ preserves order if λ does.
$\Leftarrow$: $\lambda(a) \in Q \cup \{\infty\} \iff \lambda(a^{-1}) \in Q \cup \{\infty\}$, so consider $a \in A_\lambda \cap P$. Then $\bar{a} = a + \mathfrak{m}_A \in \overline{P}$ and $\lambda(a) = \bar{\lambda}(\bar{a}) \in Q$.

THE ARTIN LANG HOMOMORPHISM

In this section, the Lang-Homomorphism-Theorem, as stated in chapter III , will be proved. However, we will not present Lang's original proof but a slightly different version as presented in [12], chapter 5.

Theorem 6.17. *Let $A = k[x_1, \ldots, x_n]/\mathfrak{a}$, where k is a real closed field and $K =$ quot A. Given $a_1, \ldots, a_k \in K$, there exists a k-place $\phi : K \to \cup\{\infty\}$ such that $\phi(a_i)$ is finite for all i.*

Corollary 6.18. *(Lang's Homomorphism Theorem) Let A be a real affine domain over k. Then there exists a k-homomorphism $\phi : A \to k$.*

Remark: This is another way of saying that A has an ordering, given by $P = \phi^{-1}(k^2)$. Define $a_i := \phi(\bar{x}_i), i = 1, \ldots, n$, then $a := (a_1, \ldots, a_n) \in V_A(k)$ and

$$\phi^{-1}(k^2) = \{f \in A : \phi(f) \geq 0\} = \{f \in A \; f(a_1, \ldots, a_n) \geq 0\} = P_a,$$

which is the maximal ordering corresponding to the real point a.

Proof of Corollary 6.18. Apply Theorem 6.17: if $\phi : K \to k \cup \{\infty\}$ is a k-place such that $\phi(x_i + \mathfrak{a}) < \infty$ for all i, then $A \in a_\phi = \phi^{-1}(k)$ and $\phi|_A$ is the desired homomorphism.

Corollary 6.19. *Let A be an affine k-algebra, k real closed, and f_i, g_j, $i = 1, \ldots, r, j = 1, \ldots, s$ be given elements in A. If $P \in$ Sper A is an ordering such that $p \in H(f_1, \ldots, f_r) \cap \overline{H}(g_1, \ldots, g_s)$ then there exists a k-algebra homomorphism $\phi \; A \to k$ such that $\phi(f_i) > 0, \phi(g_j) \geq 0$ for all i, j.*

Proof. It is enough to consider the case supp$(P) = (0)$, since if supp$(P) = \wp \neq (0)$, then apply the result to $\overline{A} := A/\wp$ to get $\bar{\phi} : A/\wp \to k$ such that $\bar{\phi}(\bar{f}_i) > 0$, $\bar{\phi}(\bar{g}_j) \geq 0$ for all i, j, and $\phi : A \to k$, $\phi(f) := \bar{\phi}(\bar{f})$ is the desired map. So suppose

$\operatorname{supp}(P) = (0)$, so that P extends uniquely to an ordering $\tilde{P}$ on $K = \text{quot } A$ with $f_i \in \tilde{P}\backslash\{0\}$, $g_j \in \tilde{P}$. Since K has an ordering, K is real and so is A. Now apply Corollary 6.18 to the algebra $A' = A[1/\prod_{i=1}^{r} f_i, \sqrt{f_1}, \ldots, \sqrt{f_r}, \sqrt{g_1}, \ldots, \sqrt{g_s}]$, in which all f_i and g_j are squares, which implies that $\phi(f_i) \geq 0, \phi(g_j) \geq 0$ for all i, j. In particular, the f_i are units in A', so that $\phi(f_i) = 0$ is impossible. It remains to show that the algebra A' is in fact real.

To this end it suffices to show that $K' = \text{quot } A' = K(\sqrt{f_1}, \ldots, \sqrt{f_r}, \sqrt{g_1}, \ldots, \sqrt{g_s})$ is real. Without loss of generality assume that $K' = K(\sqrt{f})$ for some f, since one can iterate the argument for each square root adjoined. Suppose

$$
\begin{aligned}
-1 &= \sum_{i=1}^{m} a_i^2 , \qquad \text{where } a_i = \alpha_i + \beta_i\sqrt{f}, \alpha_i, \beta_i \in K \\
\Rightarrow\ -1 &= \sum_{i+1}^{m} (\alpha_i^2 + \beta_i^2 f) \\
\Rightarrow\ -f\sum_{i+1}^{m} \beta_i^2 &= 1 + \sum_{i=1}^{m} \alpha_i^2,
\end{aligned}
$$

where the right-hand side is in $P\backslash\{0\}$, but the left-hand side is not, since f was assumed to be in P.

The proof of Theorem 6.17 uses the following result:

($*$) *Let $K|F$ be a finite algebraic extension of formally real fields, π : Sper $K \to$ Sper F defined by $\pi(P) = P \cap F$. Then π(Sper K) is open in Sper F.*

Based on ($*$) we will prove the following

Lemma 6.20. *Let K be real function field over k and $\text{trdeg}_k(K) = 1$. For any transcendental $x \in K$ there exists $\delta \in k$ and $P \in$ Sper K such that $(x - \delta)^{-1} \notin ch_P(k)$ (the convex hull of k in K with respect to P).*

Proof. Fix $P_0 \in$ Sper K and consider its restriction Q_0 to $F = k(x)$. By ($*$), there exists an open neighborhood $H = H(f_1, \ldots, f_r)$ of Q_0 that is contained in ϕ(Sper K). The goal is to find $P \in H$ such that for some $\delta \in k$, $(x - \delta)$ is infinitesimal (i.e. $|\delta|_P < k^+$) and positive (with respect to P), so that $(x-\delta)^{-1}$ is not bounded by any element in k. One may assume that $f_i \in k[x]\backslash\{0\}$, since if $f = \frac{g}{h}, h \neq 0$, then $f \in P \iff g \cdot h \in P$ and $g \cdot h \in k[x]$. Moreover, since any polynomial of odd degree has a root in k, f is a product of linear factors and quadratic irreducible polynomials. But if g is an irreducible polynomial of degree 2, it must be a sum of two squares, and dropping this factor from f does not affect its sign modulo P. Therefore one may assume that each f_i is a product of distinct linear factors. Consider the sets

$$\begin{aligned} A_i(Q) := & \ \{a \in k : \ f_i(a) = 0, \ a < x\} \\ B_i(Q) := & \ \{a \in k : \ f_i(a) = 0, \ a > x\} \end{aligned}$$

for any $Q \in \operatorname{Sper} k(x)$. Then if b_i denotes the cardinality of $B_i(Q)$, $sign_Q(f_i) = (-1)^{b_i}$. So for any $Q \in \operatorname{Sper} k(x)$ that satisfies $B_i(Q) = B_i(Q_0)$ for all i, it follows that $Q \in H$ and Q is the restriction of an ordering on K. Our task now is to find Q such that $B_i(Q) = B_i(Q_0)$, $i = 1, \ldots, r$ and $(x-\delta)$ is infinitesimal with respect to Q for some $\delta \in k$. Let

$$a := \max\,(\bigcup_{i=1}^{r} A_i(Q_0)), \quad b := \min\,(\bigcup_{i=1}^{r} B_i(Q_0)),$$

i.e. a is the largest root of the f_i which is smaller than x, and b is the smallest root bigger than x. In case that one of the sets $\bigcup_{i=1}^{r} A_i(Q_0))$, $\bigcup_{i=1}^{r} B_i(Q_0))$ is empty, define $a = b$. We have $a <_{Q_0} x <_{Q_0} b$. Now consider $Q := P_{\delta_+}$ for $\delta := \frac{1}{2}(a+b)$. In this ordering, $0 < x - \delta < \kappa$ for any $\kappa \in k^+\backslash\{0\}$ and $B_i(Q) = B_i(Q_0)$ for all i. Therefore $Q \in H$ and since $(x-\delta)^{-1}$ is unbounded with respect to k, $\pi^{-1}(Q)$ is the desired ordering.

Remark If $P := \Phi^{-1}(Q)$, Q as above, then $x \in A := ch_P(k)$ and $A/\mathfrak{m}_A = k$.

Proof. Since $0 < |x-\delta| < |\kappa|$ for any $0 \neq \kappa \in k$, $x - \delta \in \mathfrak{m}_A$ and $x = (x-\delta)+\delta \in A$. $A/\mathfrak{m}_A$ is ordered by $\overline{P}_0$, where $\overline{P}_0 = P \cap A$, so if one can show that $A/\mathfrak{m}_A$ is algebraic over k, it follows that $A/\mathfrak{m}_A = k$, because k is real closed. Suppose one can find

$y \in A$ such that $\bar{y}$ is transcendental over k. Then $\bar{x}, \bar{y}$ are algebraically independent, for otherwise $f(\bar{x}, \bar{y}) = 0 \Rightarrow f(\delta, \bar{y}) = 0$ and thus $\bar{y}$ is algebraic over k. Therefore $\mathrm{trdeg}_k K \geq 2$, contradicting the assumption that $\mathrm{trdeg}_k K = 1$.

Lemma 6.21. *Let A be a residually real valuation ring and $a_1, \ldots, a_n \in K = \mathrm{quot}\, A$. If $\sum_{i=1}^n a_i^2 \in A$, then $a_i \in A$, $i = 1, \ldots, n$.*

Proof. Since for any i, j either $a_i/a_j \in A$ or $a_j/a_i \in A$, one may assume that $a_i/a_1 \in A$ for all i. Then $a_1^2(1 + \sum_{i=2}^n (\frac{a_i}{a_1})^2) \in A$ and $1 + \sum_{i=2}^n (\frac{a_i}{a_1})^2$ is a unit in A, for otherwise $-1 = \sum_{i=2}^n (\frac{\bar{a}_i}{\bar{a}_1})^2$ in $A/\mathfrak{m}_A$, but $A/\mathfrak{m}_A$ is real. Thus $a_1 \in A$, which implies $a_i \in A$ for all i.

We now come to the

Proof of Theorem 6.17. (by induction on $d = \mathrm{trdeg}_k K$)
If $d = 0$, then K is algebraic over k, hence $K = k$ and there is nothing to show. So Let $d = 1$ and $x := \sum_{i=1}^m a_i^2 \in K$.
Case 1: x is transcendental over k. Let P be the ordering on K as constructed in Lemma 6.20. Then $x \in A := ch_P(k)$, so by Lemma 6.21 all $a \in A$. The canonical place λ_A associated to the valuation ring A has the desired properties.
Case 2: x is algebraic over k. Then $x \in k$, since $k(x)$ is real and k is real closed. So take any transcendental $y \in K$ and use the construction described in Case 1 to get a k-place $\phi : K \to k \cup \{\infty\}$. Since $x \in k$, $\phi(x) < \infty$, so $x \in A_\phi = \phi^{-1}(k)$ and again all $a_i \in A_\phi$.
Now assume $d > 1$. Let $\tilde{K}$ be a real closure of K with respect to some ordering P. Choose a subfield $L \subset K$ with $\mathrm{trdeg}_L K = 1$ and denote its relative algebraic closure in $\tilde{K}$ by $\overline{L}$. Then, by Corollary 1.13, $\overline{L}$ is real closed and the field $\overline{L}(K)$ is a function field over $\overline{L}$ of transcendence degree 1. By induction, there exists an $\overline{L}$-place $\phi : \overline{L} \to \overline{L} \cup \{\infty\}$ with $\phi(a_i) < \infty$ for all i. The restriction $\phi|_K$ is a place from K to $\overline{L} \cup \{\infty\}$ and the ring $A := \phi^{-1}|_K(\overline{L})$ is a valuation ring in K that contains all

a_i. The associated canonical place $\lambda_A : K \to A/\mathfrak{m}_A = \phi^{-1}(\overline{L})/\phi^{-1}(0)$ is finite on all a_i. $A/\mathfrak{m}_A$ is real as it is isomorphic to a subfield of $\overline{L}$, and $\mathrm{trdeg}_k A/\mathfrak{m}_A \leq d-1$. By induction we find a place $\lambda' : A/\mathfrak{m}_A \to k \cup \{\infty\}$ with $\lambda'(a_i) < \infty$ for all i. Finally, the composition $\lambda := \lambda_A \circ \lambda'$ is the place we were looking for. $\heartsuit\heartsuit$

CHAPTER VII

SEMIALGEBRAIC FUNCTIONS ON $\mathbb{R}^n$

Definition 7.1: Let $S \subset \mathbb{R}^n$ be a semialgebraic set. A function $f : S \to \mathbb{R}$ is called *semialgebraic* if for all semialgebraic sets $T \subset \mathbb{R}^{p+1}$ the set

$$\{(x,t) \in \mathbb{R}^{n+p} : \; x \in S \text{ and } (t, f(x)) \in T\}$$

is semialgebraic.

Remarks.

(1) Polynomials are semialgebraic:
If $S \subset \mathbb{R}^n$ is described by polynomials $p_1 \geq 0, \ldots, p_r \geq 0$, $p_i \in \mathbb{R}[x_1, \ldots, x_n]$, and $T \subset \mathbb{R}^{p+1}$ by $q_1, \ldots, q_s$, $q_i \in \mathbb{R}[x_1, \ldots, x_{p+1}]$, then for any $Q \in \mathbb{R}[x_1, \ldots, x_n]$ the set

$$\{(x,t) \in \mathbb{R}^{n+p} : \; x \in S \;\wedge\; (t, Q(x)) \in T\}$$

is given by

$$\{(x,t) \in \mathbb{R}^{n+p} : \; p_1(x) \geq 0, \ldots, p_r(x) \geq 0, q_1(t, Q(x)) \geq 0, \ldots, q_s(t, Q(x)) \geq 0\},$$

which is of course semialgebraic.

(2) The graph $\Gamma(f) = \{(x, f(x)) : \; x \in S\}$ of a semialgebraic function $f : \; S \to \mathbb{R}$ is semialgebraic in $\mathbb{R}^{n+1}$: Let $T \subset \mathbb{R}^2$, $T = \; \{(x,y) \in \mathbb{R}^2 : \; x - y = 0\}$. Then

$$\begin{aligned} &\{(x,t) \in \mathbb{R}^{n+1} : \; x \in S \;\wedge\; (t, f(x)) \in T\} \\ =&\{(x,t) : \; x \in S \;\wedge\; t = f(x)\} = \{(x, f(x)) : \; x \in S\} \end{aligned}$$

It will turn out that a function $f : \; S \to \mathbb{R}$ is semialgebraic iff its graph $\Gamma(f)$ is semialgebraic as a set in $S \times \mathbb{R}$, which makes it much easier to check whether a given function is semialgebraic of not. Using this result, one can show that, for instance, the function

$$f : \; \mathbb{R} \to \mathbb{R}, \quad f(x) = +\sqrt{|x|}$$

is semialgebraic, since

$$\begin{aligned}\Gamma(f) &= \{(x,y):\ y = +\sqrt{|x|}\}\\ &= \{(x,y):\ x \geq 0, y \geq 0 \wedge y^2 - x = 0\} \cup \{(x,y):\ x < 0, y < 0 \wedge y^2 + x = 0\}\end{aligned}$$

More generally, functions describing the roots of polynomials with respect to a specified variable are semialgebraic where they are defined.

Semialgebraic functions need not be continuous: the function $f:\ \mathbb{R} \to \mathbb{R},\ f(x) = \operatorname{sign}(x)$ is a counterexample.

(3) Definition 7.1 ensures that sets defined by finitely many sign conditions on semialgebraic functions are semialgebraic: if $\mathbb{R} \supset T =]0, \infty[$, $p = 0$ and $S = \mathbb{R}^n$, then

$$\{x \in \mathbb{R}^n : f(x) \in T\} = \{x \in \mathbb{R}^n : f(x) > 0\}$$

is semialgebraic for a semialgebraic function f.

Lemma 7.2. *Let $S \subset \mathbb{R}^n$ be semialgebraic. If f is a polynomial in m variables and $g_1, \ldots, g_m :\ S \to \mathbb{R}$ are semialgebraic, then $f(g_1, \ldots, g_m) :\ S \to \mathbb{R}$ is semialgebraic.*

Corollary 7.3. *The semialgebraic functions $f :\ S \to \mathbb{R}$ form a ring under addition and multiplication.*

Proof. Let $f, g :\ S \to \mathbb{R}$ be semialgebraic functions, define $\alpha, \beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\alpha(x,y) = x + y$, $\beta(x,y) = x \cdot y$, then by the previous Lemma, $\alpha(f,g) = f + g$ and $\beta(f,g) = f \cdot g$ are semialgebraic.

Proof of Lemma 7.2. We need to show that for given $T \subset \mathbb{R}^{p+1}$, T semialgebraic, the set

$$\mathcal{S} :=\ \{(x,t) \in \mathbb{R}^{n+p} :\ (t, f(g_1(x), \ldots, g_m(x)) \in T\}$$

is semialgebraic. Without loss of generality, assume T is given by a single inequality, i.e. $T = \{(t,y) \in \mathbb{R}^p \times \mathbb{R} :\ p(t,y) \geq 0\}$. Now, for any m-tuple $(y_1, \ldots, y_m) \in \mathbb{R}^m$,

$p(t, f(y_1, \ldots, y_m)) = P(y_1, \ldots, y_m)$, where the coefficients of P are polynomials in t that depend on p and f only. Rewrite P in the form

$$P(y_1, \ldots, y_m) = \sum_{i=0}^{s} Q_i(y_1, \ldots, y_{m-1}) y_m^i, \quad Q_i = Q_i(t).$$

Since g_m is semialgebraic, the set

$$\{(x, t, y_1, \ldots, y_{m-1}) : \ P(y_1, \ldots, y_{m-1}, g_m(x)) = \sum_{i=1}^{s} Q_i(t)(y_1, \ldots, y_{m-1}) g_m^i(x) \geq 0\}$$

is semialgebraic, hence is the union of sets of the form

$$\{(x, t, y_1, \ldots, y_{m-1}) : \ P_i'(x, t, y_1, \ldots, y_{m-1}) \geq 0, \ i = 1, \ldots, k\}.$$

By the same argument, the set

$$\{(x, t, y_1, \ldots, y_{m-2}) : \ P_i'(x, t, y_1, \ldots, y_{m-2}, g_{m-1}(x)) \geq 0\}$$

is semialgebraic, which implies that

$$\{(x, t, y_1, \ldots, y_{m-2}) : \ P_i(y_1, \ldots, y_{m-2}, g_{m-1}(x), g_m(x)) \geq 0\}$$

is semialgebraic. Repeating this argument, one eventually concludes that $\mathcal{S}$ is semialgebraic and the proof is done.

We will now present an algorithm that allows us to decompose semialgebraic subsets into cylinders. This method will find its application in chapter VIII, where the Pierce-Birkhoff Conjecture will be proved. Cylindrical Algebraic Decomposition is due to P.J. Cohen and its original version can be found in [5]. The proof presented here is taken from [6].

Theorem 7.4. *(Cylindrical Algebraic Decomposition)*
Let $P(x, y) = P(x_1, \ldots, x_n, y) \in \mathbb{R}[x_1, \ldots, x_n][y]$ be given. There exists a partition of $\mathbb{R}^n$ into semialgebraic sets $A_1, \ldots, A_m$ such that for all i exactly one of the following cases hold:

(1) *sign $P(x, y)$ $(\in \{+, -, 0\})$ is constant for all $x \in A_i$ and all $y \in \mathbb{R}$*

(2) *The zeros of P that lie in $A_i \times \mathbb{R}$ are given by continuous semialgebraic functions $Z_1(x), \ldots, Z_{t_i}(x)$ such that $Z_1(x) < \ldots, < Z_{t_i}(x)$ for all $(x, y) \in A_i \times \mathbb{R}$ and the sign of $P(x, y)$ only depends on $y - Z_j(x)$, $j = 1, \ldots, t_i$. ($t_i \leq \deg_y P$).*

Proof. By induction on $n = \deg_y(P)$. If $n = 0$, let

$$A_1 = \{x \in \mathbb{R}^n : \ P(x) = 0\}$$

$$A_2 = \{x \in \mathbb{R}^n : \ P(x) > 0\}$$

$$A_3 = \{x \in \mathbb{R}^n : \ P(x) < 0\}$$

and P will not change sign on $A_i \times \mathbb{R}$.

Now let $n \geq 1$ and suppose the theorem is proved for all Q such that $\deg_y(Q) < n$. If $\deg_y(P) = n$, then $\deg_y(\dfrac{\partial P}{\partial y}) = n - 1$ and by induction, $\mathbb{R}$ can be decomposed into sets $A_1, \ldots, A_m$ such that the zeros of $\dfrac{\partial P}{\partial y}$ are given by $\tilde{Z}_1 < \cdots < \tilde{Z}_{k_i}$ on A_i and the $\tilde{Z}_j$ are continuous and semialgebraic. Thus one can decompose the A_i into smaller semialgebraic pieces $B_1, \ldots, B_s$ on which the signs of $P(\tilde{Z}_j)$, $i = 1, \ldots, m, j = 1, \ldots, k_i$ are constant. For fixed $x_i \in B_i$ and $y \in]\tilde{Z}_j(x), \tilde{Z}_{j+1}(x)[$ the function $P(x, y)$ is monotone and has therefore at most one zero in this interval. This implies that the zeros of P on B_i are given by functions $Z_1 < \cdots < Z_{t_i}$ such that for all $x \in B_i$ and each Z_l exactly one of the following relations holds:

(1) $Z_1(x) < \tilde{Z}_1(x)$

(2) $Z_{t_i}(x) > \tilde{Z}_{k_i}(x)$

(3) $\tilde{Z}_j(x) < Z_l(x) < \tilde{Z}_{j+1}(x)$ for suitable j

(4) $Z_l(x) = \tilde{Z}_j(x)$ for some j

On each of the pieces

$$C = \{(x, y) \in B_i \times \mathbb{R} : \ Z_j(x) < y < Z_{j+1}(x)\}, \ j = 1, \ldots, t_i$$

P does not change its sign, for if it did, we could find $(x_0, y_0) \in C$ with $P(x_0, y_0) = 0$. Hence there would be points (x_1, y_1), $(x_2, y_2) \in C$ with the properties

(a) $Z_j(x_0) < y_1 \leq y_0 \leq y_2 < Z_{j+1}(x_0)$
(b) $\dfrac{\partial P}{\partial y}(x_0, y_1) = \dfrac{\partial P}{\partial y}(x_0, y_2) = 0$
(c) P is monotone on the fiber $\{x_0\} \times]y_1, y_2[$.
Since by construction of the B_i, sign $P(x_0, y_1) =$ sign $P(x_0, y_2)$, one concludes that $P(x_0, y_1) = P(x_0, y_2) = 0$, however $Z_j(x_0) \underset{\neq}{<} y_1, y_2 \underset{\neq}{<} Z_{j+1}(x_0)$.

For the same reason, sign(P) is constant on $\{(x, y) : \ x \in B_i \ \wedge \ -\infty < y < Z_1(x)\}$ and on $\{(x, y) : \ x \in B_i \ \wedge \ Z_{t_i}(x) < y < \infty\}$.

As to continuity of the Z_l, note first that in case $Z_l = \tilde{Z}_j$ for some j continuity follows from the induction hypothesis. So suppose $\tilde{Z}_j < Z_l < \tilde{Z}_{j+1}$ on B_i for suitable j (the cases $-\infty < Z_l < \tilde{Z}_1$ and $\tilde{Z}_{t_i} < Z_j < \infty$ are handled similarly). Fix $x_0 \in B_i$ and $a, b \in \mathbb{R}$ such that

$$\tilde{Z}_j(x_0) < a < Z_l(x_0) < b < \tilde{Z}_{j+1}(x_0).$$

Without loss of generality assume $P(x_0, a) < 0, \ P(x_0, b) > 0$. Since $\tilde{Z}_j, \ \tilde{Z}_{j+1}$ and P are continuous, one can find a neighborhood U of x_0 in B_i such that $\tilde{Z}_j(x) < a, \ \tilde{Z}_{j+1}(x) > b, \ P(x, a) < 0, \ P(x, b) > 0$ on U. By the intermediate value theorem $a < Z_j(x) < b$ for all $x \in U$.
It remains to show that the functions Z_j are semialgebraic. Given a set of the form

$$A := \ \{(x, t) \in \mathbb{R}^n \times \mathbb{R}^p : \ x \in B_i \ \wedge \ Q(t, Z_j(x)) > 0\},$$

where $Q \in \mathbb{R}[x_1, \ldots, x_{n+p}]$, we have to show that A is semialgebraic.
If $P_n(x)$ denotes the leading coefficient of P in y, one can assume that $P_n(x)$ is of constant sign on B_i, for otherwise partition B_i into smaller sets where it remains constant. If $P_n(x)$ happens to be zero on B_i, semialgebraicity of Z_j follows from the induction hypothesis since in this case P is of lower degree on B_i. So let's assume $P_n(x) > 0$ on B_i, the other case is treated the same.
Upon dividing $\frac{P(x,y)}{P_n(x,y}$ by $Q(t, y)$ in the ring $\mathbb{R}(t, x)[y]$ one obtains

$$P_n^k(x)Q(t, y) = P(x, y) \cdot S(t, x, y) + R(t, x, y),$$

where $k \geq 0$ and $\deg_y(R) < \deg_y(P)$. Substituting $Z_j(x)$ for y yields

$$P_n^k(x)Q(t, Z_j(x)) = R(t, x, Z_j(x)),$$

so that

$$A = \{(x,t) \in \mathbb{R}^{n+p} : \; x \in B_i \;\wedge\; R(t,x,Z_j(x)) > 0\}.$$

Using the induction hypothesis on R, one can decompose $\mathbb{R}^{n+p}$ into semialgebraic sets $C_1, \ldots, C_k$ with corresponding semialgebraic zero-functions $W_1, \ldots, W_{s_j}$, $j = 1, \ldots, k$ and in refining this partition where necessary, one may as well assume that the functions $\tilde{Z}_l(x) - W_j(t,x)$ and $P(x, W_j(t,x))$ (all l, j) have constant signs on C_i (here we used the fact that the W_j are semialgebraic).

<u>Claim:</u> the sign of $R(t,x,Z_j(x))$ on each C_i is determined by the signs of $\tilde{Z}_l - W_j$ and $P(x, W_j(t,x))$, and therefore remains constant on C_i.

Once the claim is established, the theorem follows, since the set A is the union of those C_i, where $R(t,x,Z_j(x))$ is positive, which renders A semialgebraic.

<u>Proof of the Claim:</u> Fix $(t_0, x_0) \in C_i$. Since $x_0 \in B_j$ for some j, it is possible to locate $Z(x_0)$ in one of the intervals

$$]-\infty, \tilde{Z}_1(x_0)[, \ldots,]\tilde{Z}_{t_j}(x_0), \infty[$$

(each of these intervals contains at most one of the Z_j). Since $\tilde{Z}_l(x) - W_j(t,x)$ does not change sign on C_i, one can find out, which of the $W_j(t_0, x_0)$ lie in the same interval as $Z(x_0)$. Evaluation of P at $(x_0, W_j(t_0, x_0))$ for those W_j in question, tells the position of $Z(x_0)$ between, say, W_k and W_{k+1} (i.e. P has different signs on W_k and W_{k+1}). But on

$$\{(t,x,y) : \; (t,x) \in C_i \;\wedge\; W_k(t,x) < y < W_{k+1}(t,x)\}$$

the sign of R remains constant and the claim is proved. ♠♠

The result can be extended to a finite number of polynomials $P_i \in \mathbb{R}[x_1, \ldots, X_n,]$:

Corollary 7.5. *For given polynomials $P_1(x,y),\dots,P_s(x,y)$ there exists a partition of $\mathbb{R}^n$ into semialgebraic sets $A_1,\dots,A_m$ such that the zeros of all polynomials $P_1,\dots,P_s$ on $A_i\times\mathbb{R}$ are given by semialgebraic functions $Z_1<\dots<Z_{t_i}$ for all i and sign$P_j(x,y)$ depends only on $y-Z_k(x)$, $k=1,\dots,t_i$.*

Proof. By induction on s = number of polynomials. Case $s=1$ is covered by Theorem 7.4, so let $s+1$ polynomials be given. For the first s ones one has a decomposition of $\mathbb{R}^n$ into $B_1,\dots,B_p$ with zeros $Z_1,\dots,Z_{k_i}$ on B_i. For P_{s+1} one has another decomposition into $C_1,\dots,C_l$ with functions $W_1,\dots,W_k$ describing the zeros of P_{s+1}. Thus form all possible intersections $B_i\cap C_j$ and partition these into semialgebraic subsets where the functions Z_i-W_j (all i,j) have constant sign.

Corollary 7.6. *If $S\subset\mathbb{R}^{n+1}$ is semialgebraic, then its projection $\pi(S)\subset\mathbb{R}^n$ is semialgebraic.*

Proof. Let S be given by $\{p_1\geq 0,\dots,p_s\geq 0\}$. Construct a cylindrical decomposition of $\mathbb{R}^n$ into $A_1,\dots,A_m$ according to Corollary 7.5. Then $\pi(S)$ is the union of those A_i such that there exists $(x_i,y_i)\in A_i\times\mathbb{R}$ and $p_1(x_i,y_i),\dots,p_s(x_i,y_i)\geq 0$.

Corollary 7.7. *(Tarski-Seidenberg-Principle) Suppose $S\subset\mathbb{R}^n$ is given by a formula Ψ consisting of a finite number of quantifiers (among '$\exists,\forall$') and sign conditions on finitely many polynomials. Then it is possible to describe S in terms of polynomial relations only, i.e. all quantifiers can be eliminated.*

Proof. First, it is enough to consider the symbol '$\exists$', since

$$\{x\in\mathbb{R}^n:\ \forall y\ \Psi(x,y)\}=\{x\in\mathbb{R}^n:\ \exists y \text{ such that } (\neg\Psi)(x,y)\}^c.$$

Secondly, using induction, it suffices to prove the statement for formulas involving one '$\exists$' only, since

$$\{x\in\mathbb{R}^n:\ \exists y\exists z\dots\Psi(x,y,z\dots)\}=\pi\{(x,y)\in\mathbb{R}^{n+1}:\ \exists z\dots\Psi(x,y,z\dots)\}$$

So, given a set $S = \{x \in \mathbb{R}^n : \exists y \ni: \Psi(x,y)\}$, where Ψ consists of the sign data $P_1 \geq 0, \dots, P_m \geq 0$, then

$$S = \pi\{(x,y) \in \mathbb{R}^{n+1} : P_1(x,y) \geq 0, \dots, P_m(x,y) \geq 0\},$$

hence S is semialgebraic by Corollary 7.6 and can therefore be written in terms of sign conditions of certain polynomials in n variables.

Corollary 7.8. *A function $f : S \to \mathbb{R}$ is semialgebraic if and only if its graph $\Gamma(f) \subset \mathbb{R}^{n+1}$ is semialgebraic.*

Proof. The 'only if'-part was done earlier. Suppose $\Gamma(f) = \{(x, f(x)) : x \in S\}$ is described by a formula Φ and $T \subset \mathbb{R}^{p+1}$ is given by Ψ. Then

$$\{(x,t) \in \mathbb{R}^p : s \in S \wedge \Psi(t, f(x))\} = \{(x,t) \in \mathbb{R}^p : \exists y \text{ such that } \Psi(t,y) \wedge \Phi(x,y)\}$$

is semialgebraic by the previous Corollary.

Given a finite number of polynomials $P_1(x,y), \dots, P_t(x,y)$ in $\mathbb{R}[x_1, \dots, x_n, y]$, one can look at all intersections

$$(*) \qquad A_{\epsilon_1,\dots,\epsilon_t}(P_1, \dots, P_t) = \bigcap_{i=1}^{t} \{(x,y) \in \mathbb{R}^{n+1} : P_i(x,y)\epsilon_i 0\},$$

where $\epsilon_i \in \{>, <, 0\}$. Via cylindrical algebraic decomposition we see that each $A_{\epsilon_1,\dots,\epsilon_t}(P_1, \dots, P_t)$ is the union of sets of the form

$$\{(x,y) : x \in B \text{ and } \zeta(x) < y < \eta(x)\} \text{ or } \{(x,y) : x \in B \text{ and } \zeta(x) = y\}$$

for some semialgebraic set $B \subset \mathbb{R}^n$ and semialgebraic functions $\zeta, \eta : B \to \mathbb{R}$.
In general, $A_{\epsilon_1,\dots,\epsilon_t}(P_1, \dots, P_t)$ need not be connected. However, by adding some polynomials to the list $P_1, \dots, P_t$, one can achieve that the sets of the form $(*)$ be connected. How this is done in the one-dimensional case, shows the following

Lemma 7.9. *(Thom's Lemma) Let* $\{P_1, \ldots, P_m\} \subset \mathbb{R}[t]$ *be closed under derivation (i.e.* $P \in \{P_1, \ldots, P_m\} \Rightarrow P' \in \{P_1, \ldots, P_m\}$*) and let* $A = A_{\epsilon_1,\ldots,\epsilon_m}(P_1, \ldots, P_m)$ *for given* $\epsilon_i \in \{<, >, 0\}$*. Then* A *is either empty or connected (i.e a point or an interval) in* $\mathbb{R}$*.*

Proof. By induction on m = number of polynomials. For $m = 0$ there is nothing to show. So consider a collection of $m + 1$ polynomials and without loss of generality assume P_{m+1} is of maximal degree among all of them. Since $\{P_1, \ldots, P_m\}$ is closed under derivation, the set

$$A' := \bigcap_{i=1}^{m} \{x \in \mathbb{R} : \; P_i(x)\epsilon_i 0\}$$

is connected. Suppose it is not empty and let

$$A := A' \cap \{x \in \mathbb{R} : \; P_{m+1}(x)\epsilon_{m+1} 0\}.$$

If A' is a point, then A is empty or a point; so suppose A' is an interval. P'_{m+1} is one of the polynomials $P_1, \ldots, P_m$, thus P_{m+1} is strictly monotone or constant on A'. If P_{m+1} is constant, then, depending on ϵ_{m+1}, A is either empty of $A = A'$. Otherwise P_{m+1} is injective on A', so if $\epsilon_{m+1} \in \{=\}$, A is a point; if $\epsilon_{m+1} \in \{<, >\}$, then A is an interval contained in A'.

This idea can be generalized to n variables:

Theorem 7.10. *Let* $\{P_1, \ldots, P_r\} \subset \mathbb{R}[x_1, \ldots, x_n]$*. Then we can find polynomials* $P_{r+1}, \ldots, P_{r+s} \in \mathbb{R}[x_1, \ldots, x_n]$ *such that for any* $\epsilon = (\epsilon_1, \ldots, \epsilon_{r+s}) \in \{<, >, =\}^{r+s}$ *the sets*

$$A(\epsilon) = \bigcap_{i=1}^{r+s} \{x \in \mathbb{R}^n : \; P_i(x)\epsilon_i 0\}$$

are either empty or connected.

Proof. The proof is based on induction on n = number of variables and uses Thom's Lemma in dimension 1, applied to a suitable cylindrical decomposition.

If $n = 1$, we have to add the derivatives in all orders of all the given P_i in order to get the result.

Now assume $\{P_1, \ldots, P_r\} \subset \mathbb{R}[x_1, \ldots, x_n, x_{n+1}]$. First we add to this list the partial derivatives with respect to x_{n+1} of all ordersand of all P_i to obtain a list $\{P_1, \ldots, P_{r+s}\}$. Then we decompose $\mathbb{R}^n$ into semialgebraic subsets $B_1, \ldots, B_p$, where on each B_i the zeros of $\{P_1, \ldots, P_{r+s}\}$ are given by $Z_1, \ldots, Z_{l_i}$. Using the induction hypothesis, one may assume that the B_i are connected and given by sign conditions on polynomials $P_{r+s+1}, \ldots, P_{r+s+k}$.

Claim: For $\epsilon \in \{<, >, =\}^{r+s+k}$ the set $A_\epsilon(P_1, \ldots, P_{r+s+k})$ is connected. Fix $x_0 \in A(\epsilon)$. Then $\pi(x_0) =: x_0' \in \mathbb{R}^n$ is in one of the B_i and by Thom's Lemma, $A(\epsilon) \cap \pi^{-1}(x_0')$ consists either only of the point x_0 (and therefore $x_0 = (x_0', Z_j(x_0'))$ for some $j \in \{1, \ldots, l_i\}$) or

$$A(\epsilon) \cap \pi^{-1}(x_0') = \{(x_0', y) : \ Z_j(x_0') < y < Z_{j+1}(x_0')\}$$

for some j. Since on B_i, the $\{P_1, \ldots, P_{r+s}\}$ do not change their signs, one concludes that in the first case,

$$A(\epsilon) = \{(x, y) \in B_i \times \mathbb{R} : \ y = Z_j(x)\}$$

and in the second case

$$A(\epsilon) = \{(x, y) \in B_i \times \mathbb{R} : \ Z_j(x) < y < Z_{j+1}(x)\}.$$

If $A(\epsilon)$ is of the first type, it is connected since it is the graph of a continuous function on a connected domain. Otherwise, assume $A(\epsilon) = A \cup B$, A and B relatively open in $A(\epsilon)$ and disjoint. Then $\pi(A \cup B) = \pi(A) \cup \pi(B) = B_i$; and since B_i is connected and $\pi(A), \pi(B)$ are nonempty open sets (projections are open mappings), we find $x_0 \in \pi(A) \cap \pi(B)$. But then, for some $y_1, y_2 \in \mathbb{R}$, $(x_0, y_1) \in A$, $(x_0, y_2) \in B$ and the fiber $F = \{x_0\} \times]Z_j(x_0), Z_{j+1}(x_0)[$ can be written as $A_1 \cup B_1$, where $A_1 := F \cap A$, $B_1 := F \cap B$, A_1, B_1 nonempty and open in F. This, however, contradicts the connectedness of F.

CHAPTER VIII

THE PIERCE–BIRKHOFF CONJECTURE

Definition 8.1: A continuous function $h: \mathbb{R}^n \to \mathbb{R}$ is called *piecewise polynomial* (pwp) if there exist semialgebraic sets $A_1, \ldots, A_m$ and polynomials $g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_n]$ such that $\mathbb{R}^n = \bigcup_{i=1}^{m} A_i$ and $h = g_i$ on A_i.

Note that the set of piecewise polynomial functions on $\mathbb{R}^n$ is a ring under addition and multiplication: if $h = g_i$ on A_i and $h' = g'_j$ on A'_j, then $h \dotplus h' = g_i \dotplus g'_j$ on $A_i \cap A'_j$.

The so-called *Pierce-Birkhoff-Conjecture,* first stated by G. Birkhoff and R.S. Pierce in 1956, says the following:

Conjecture 8.2. *If $h: \mathbb{R}^n \to \mathbb{R}$ is pwp, then there exist polynomials $f_{ij} \in \mathbb{R}[x_1, \ldots, x_n]$, $i = 1, \ldots, k, j = 1, \ldots, l$ such that*

$$h(x) = \sup_j \inf_i \{f_{ij}(x)\} \text{ for all } x \in \mathbb{R}^n.$$

The conjecture has been proved in 1984 by L. Mahé for $n = 2$. The proof that will be presented here uses Mahé's ideas and is based on a paper by C.N. Delzell, published in the Rocky Mountain Journal of Mathematics in 1989.

It is rather easy to see that the converse of Conjecture 7.2 holds:
Since for two continuous functions f, g, $\sup\{f, g\}$ and $\inf\{f, g\}$ are given by

$$\sup\{f, g\} = \frac{1}{2}(f-g) + \frac{1}{2}|f-g|, \qquad \inf\{f, g\} = \frac{1}{2}(f-g) - \frac{1}{2}|f-g|,$$

one concludes that h given as in $(*)$ is continuous. To see that it is pwp, reindex the f_{ij} to obtain a list of polynomials $f_1, \ldots, f_m$ and look at the differences $f_i - f_j$, $1 \leq j < i \leq m$. If the latter polynomials are denoted by $g_1, \ldots, g_s$, form the sets

$$A(\epsilon_1, \ldots, \epsilon_s) := \{x \in \mathbb{R}^n : g_i(x)\epsilon_i 0,\ i = 1 \ldots, s\},\ \epsilon_i \in \{>, <, =\}.$$

On each of such $A(\epsilon)$, we have

$$f_{\sigma(1)} \leq \cdots \leq f_{\sigma(m)}$$

for some permutation $\sigma \in \mathcal{S}(m)$ and therefore $\sup_j \inf_i\{f_{ij}\} = f_{i_0 j_0}$ throughout $A(\epsilon)$ for some i_0, j_0, which says that h is piecewise polynomial.

The proof of the conjecture for $n \leq 2$ proceeds in several steps. Starting out with the given A_i and corresponding polynomials f_i, we construct a list $g_1, \ldots, g_s$ by forming the differences $f_i - f_j$ and close this family up under derivation and another, not yet defined operation. Using cylindrical decomposition, we obtain a partition of $\mathbb{R}^n$ into connected algebraic subsets $E_1, \ldots, E_t$ such that $h = f_{\nu(E_i)}$ on E_i. Finally, we will construct sup-inf-polynomially definable functions h_{ij} such that

$$h_{ij} \leq f_{\nu(E_i)} \text{ on } E_i$$

$$h_{ij} \geq f_{\nu(E_j)} \text{ on } E_j$$

and $H = \sup_j \inf_i\{h_{ij}, f_j\}$ agrees with h on $\mathbb{R}^2$.

Before we turn to the proof of Conjecture 8.2, some

Preliminaries:

Definition 8.3: Denote by SIPD($\mathbb{R}^n$) the lattice generated by $\mathbb{R}[x_1, \ldots, x_n]$ and the operations sup and inf, i.e. if $f, g \in$ SIPD($\mathbb{R}^n$) then $\sup\{f,g\}, \inf\{f,g\} \in$ SIPD($\mathbb{R}^n$).

Lemma 8.4. *SIPD($\mathbb{R}^n$) is a ring under addition and multiplication.*

Proof. Given $f, g, h \in$ SIPD($\mathbb{R}^n$), then

$$-\sup\{f,g\} = \inf\{-f,-g\}$$

$$h + \sup\{f,g\} = \sup\{h+f, h+g\}$$

$$h + \inf\{f,g\} = \inf\{h+f, h+g\},$$

and since for $f_1, \ldots, f_n$

$$\sup\{f_1, \ldots, f_n\} = \sup\{f_1, \sup\{f_2, \sup\{\ldots \sup\{f_{n-1}, f_n \ldots\} \ldots\},$$

the sum and difference of two functions in SIPD($\mathbb{R}^n$) can be computed successively from these building blocks. Before we go on to prove that SIPD($\mathbb{R}^n$) is closed under multiplication, we make a

Definition: Given $f \in$ SIPD($\mathbb{R}^n$), we will define the *height* of f in the following way: any function $f \in$ SIPD($\mathbb{R}^n$) can be obtained in finitely many steps from taking suprema or infima of no more than two functions in SIPD($\mathbb{R}^n$) at a time, i.e. the definition of f resembles a tree structure.

Example: If

$$f = \sup\{f_1, \inf\{f_2, f_3\}, \sup\{f_4, f_5, f_6\}\}$$

then f can be reexpressed as

$$f = \sup\{f_1, \sup\{\inf\{f_2, f_3\}, \sup\{f_4, \sup\{f_5, f_6\}\}\}\}.$$

Let the height of such an f be the length of this so-obtained tree, i.e. the maximal number of suprema and infima taken successively. The height of the function f in our example would be 4.

In general, the height of $f \in$ SIPD($\mathbb{R}^n$) might depend on the way f is defined, but this will not have an effect on the proof. Notice that

$$\sup\{f, g\} = g + \sup\{f - g, 0\}$$
$$\inf\{f, g\} = g + \inf\{f - g, 0\}$$
$$\inf\{f - g, 0\} = -\sup\{-(f - g), 0\},$$

so that

$$\begin{aligned} h \cdot \sup\{f, g\} &= h \cdot (g + \sup\{f - g, 0\}) \\ h \cdot \inf\{f, g\} &= h \cdot (g + \inf\{f - g, 0\}) \\ &= h \cdot (g - \sup\{-(f - g), 0\}) \end{aligned}$$

and it is enough to show that

$$h \cdot \sup\{f, 0\} \in \mathrm{SIPD}(\mathbb{R}^n)$$

for given $h, f \in \mathrm{SIPD}(\mathbb{R}^n)$, using induction on the height of functions in $\mathrm{SIPD}(\mathbb{R}^n)$. We will use the following identity:

$$(*) \qquad h \cdot \sup\{f, 0\} = \sup\{\inf\{hf, h^2 f + f\}, \inf\{0, -h^2 f - f\}\}$$

Proof of the identity: Since $\inf\{0, -h^2 f - f\} = (-1 - h^2)\sup\{f, 0\}$, the right hand side is equal to

$$(**) \qquad \sup\{\inf\{hf, f(1 + h^2)\}, (-1 - h^2)\sup\{0, f\}\}.$$

Case A: $f > 0$, so $f(1 + h^2) > 0$.
If $h > 0$, $(**) \ = \ \inf\{hf, f(1 + h^2\}$ and
if $h \leq 0$, $(**) \ = \ \sup\{hf, -f(1 + h^2\}$

Case B: $f \leq 0$, so $f(1 + h^2) \leq 0$
If $h > 0$, $(**) = 0 = h \sup\{f, 0\}$ and
if $h \leq 0$, $(**) = 0 = h \sup\{f, 0\}$

In Case A, note that $|hf| \leq |f(1+h^2)|$ (since $f^2(1+h^2)^2 - h^2 f^2 = f^2(1+h^2+h^4) \geq 0$), so that $(**) = hf = h \sup\{f, 0\}$ as well. Now let

$$A := \{h \in \mathrm{SIPD}(\mathbb{R}^n) : h \cdot \sup\{f, 0\} \in \mathrm{SIPD}(\mathbb{R}^n) \text{ for all } f \in \mathrm{SIPD}(\mathbb{R}^n)\}.$$

Note that if $h \in A$ and $f \in \mathrm{SIPD}(\mathbb{R}^n)$, then $hf \in \mathrm{SIPD}(\mathbb{R}^n)$, since $f = \sup\{f, 0\} - \sup\{-f, 0\}$. Our task is to show that $A = \mathrm{SIPD}(\mathbb{R}^n)$. We will proceed in several steps.

Step 1: We will show that $\mathbb{R}[x_1, \ldots, x_n] \subset A$. For this purpose, let $h \in \mathbb{R}[x_1, \ldots, x_n]$, $f \in \mathrm{SIPD}(\mathbb{R}^n)$ and, for the time being, assume that also that $f \in \mathbb{R}[x_1, \ldots, x_n]$. Then, by identity $(*)$,

$$h \sup\{f, 0\} = \sup\{\inf\{hf, h^f + f\}, \inf\{0, -hf - h^2 f\}\} \in \mathrm{SIPD}(\mathbb{R}^n).$$

Now suppose $f \in \mathrm{SIPD}(\mathbb{R}^n)$, $f = \sup\{f_1, f_2\}$, where $f_1, f_2 \in \mathrm{SIPD}(\mathbb{R}^n)$. By induction on the height of f, we may assume that $hf_i, h^2 f_i \in \mathrm{SIPD}(\mathbb{R}^n)$. We will apply $(*)$ twice in order to get the result:

$$\begin{aligned} h\sup\{f,0\} &= \sup\{\inf\{hf, h^2 f + f\}, \inf\{\dots\}\} \\ &= \sup\{\inf\{h\sup\{f_1,f_2\}, h^2\sup\{f_1,f_2\} + \sup\{f_1,f_2\}\}, \inf\{\dots\}\} \end{aligned}$$

and we have to show that $h\sup\{f_1,f_2\}, h^2\sup\{f_1,f_2\} \in \mathrm{SIPD}(\mathbb{R}^n)$. But this can be done using $(*)$ and the identity

$$h\sup\{f_1,f_2\} = f_2 + \sup\{f_1 - f_2, 0\}.$$

Step 2: A is closed under the operation $h \to \sup\{h,0\}$. Let $h \in A$, $f \in \mathrm{SIPD}(\mathbb{R}^n)$.

$$\begin{aligned} \sup\{h,0\}\cdot f &= \sup\{h,0\}(\sup\{f,0\} - \sup\{-f,0\}) \\ &= \sup\{h,0\}\cdot\sup\{f,0\} - \sup\{h,0\}\cdot\sup\{-f,0\} \\ &= \sup\{h\sup\{f,0\},0\} - \sup\{h\sup\{-f,0\},0\} \\ &\in \mathrm{SIPD}(\mathbb{R}^n) \end{aligned}$$

by $(*)$ and the fact that $h \in A$.

Step 3: A is closed under suprema, since $\sup\{h_1,h_2\} = h_2 + \sup\{h_1 - h_2, 0\}$ and A is closed under addition. By step 1, A contains all polynomials, so $A = \mathrm{SIPD}(\mathbb{R}^n)$.

Definitions 8.5: Let $\mathcal{P} = \{P_1, \dots, P_t\}$ be a collection of finitely many polynomials $P_1, \dots, P_t \in \mathbb{R}^n$. As before, for $\epsilon = (\epsilon_1, \dots, \epsilon_t) \in \{>,<,=\}^t$ we write

$$A_{\mathcal{P}}(\epsilon) = \cap_{i=1}^t \{x \in \mathbb{R}^n : \ P_i(x)\epsilon_i 0\}.$$

Let $\mathcal{A}(\mathcal{P}) = \{A_{\mathcal{P}}(\epsilon) : \epsilon \in \{>,<,=\}^n$ and $A_{\mathcal{P}}(\epsilon)$ is open in $\mathbb{R}^n\}$, i.e. $\mathcal{A}(\mathcal{P})$ is the collection of the open semialgebraic sets of $\mathbb{R}^n$ defined by $\mathcal{P}$. Here $\epsilon_i \in \{=\}$ occurs only in the case that the corresponding polynomial P_i is the zero polynomial.

Claim: $\mathcal{U} := \bigcup_{A \in \mathcal{A}(\mathcal{P})} A$ is dense in $\mathbb{R}^n$.

Proof: First, if A, B are closed and $\overset{\circ}{A} = \overset{\circ}{B} = \emptyset$, then $(A \cup B)^\circ = \emptyset$: Suppose $U \subset A \cup B$, U open in $\mathbb{R}^n$. Then $U \cup A^c \neq \emptyset$ and $U \cap B^c \neq \emptyset$. Also, $(U \cap A^c) \cap B^c \neq \emptyset$, for otherwise $U \cap A^c \subset B$, but B does not contain open sets. Therefore $U \cap A^c \cap B^c \neq \emptyset$, contradicting the assumption. Secondly, $\mathcal{U}^c$ is the finite union of closed sets of the form $B := \cap_{i=1}^t \{P_i \epsilon_i 0\}$, where $\epsilon_i \in \{\leq, \geq, =\}$ and there exists at least one index j such that $P_j \not\equiv 0$ and $\epsilon_j \in \{=\}$. But if B contains an open set V, then $P_j = 0$ on V, hence $P_j \equiv 0$. Therefore $\overset{\circ}{B} = \emptyset$, which implies $\mathcal{U}^c$ is nowhere dense.

Suppose, we have a cylindrical algebraic decomposition of $\mathbb{R}^{m-1} \times \mathbb{R}^{n-m}$ into $B_1, \ldots, B_l$ with respect to the m-th variable x_m, i.e. if ζ describes the zeros of P_i, then $\zeta = \zeta(x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n)$. A set $C := B_i \times \mathbb{R}$ will be called an m-cylinder of $\mathcal{P}$ and the set of open m-cylinders of $\mathcal{P}$ will be denoted by $\mathcal{C}_m(\mathcal{P})$ (i.e. its base, being one of the B_i, is open. Again, the set

$$V = \bigcup_{C \in \mathcal{C}_m(\mathcal{P})} C$$

is dense in $\mathbb{R}^n$.

For given $\mathcal{P} \subset \mathbb{R}[x_1, \ldots, x_n] \backslash \{0\}$ let $\Gamma_m(\mathcal{P})$ be the smallest set containing $\mathcal{P}$, such that $\Gamma_m(\mathcal{P})$ is closed under the following two operations:

(1) $P \mapsto \dfrac{\partial P}{\partial x_m}$ (2) $P \mapsto R = P - x_m \dfrac{\partial P}{\partial x_m}$

Correspondingly, we have the sets $\mathcal{A}(\Gamma_m(\mathcal{P}))$ and $\mathcal{C}_k(\Gamma_m(\mathcal{P}))$ (we need only the case $k = m$).

Finally, let $x := (x_1, \ldots, x_n)$ and $\hat{x}_m := (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n)$.

Lemma 8.6. *Let $\mathcal{P} \subset \mathbb{R}[x_1, \ldots, x_n]$, $1 \leq m \leq n$ and $0 \neq P \in \Gamma_m(\mathcal{P})$. Fix a cylinder $C \in \mathcal{C}_m(\Gamma_m(\mathcal{P}))$ and $P \in \mathcal{P}$. Let*

$$\zeta_1 < \cdots < \zeta_t$$

be the zeros of P in C ($t \leq \deg_{x_m} P$). Then, for each $i \in \{0, \ldots, t+1\}$, there exists a function $c_{P,i} \in SIPD$ such that

$$c_{P,i}(x) = \begin{cases} P(x) & \text{if } x_m > \zeta_i(\hat{x}_m) \\ 0 & \text{otherwise} \end{cases}$$

In other words, on a given cylinder $\mathcal{C} \subset \mathbb{R}^n$, $c_{P,i}(x)$ truncates the polynomial P such that $c_{P,i}(x) = P$ above the i-th zero-function and $c_{P,i}(x) = 0$ below.

Proof. By induction on $d = \deg_{x_m}(P)$. If $d = 0$, let $c_{P,0}(x) = P$ and $c_{P,t+1}(x) = 0$. Now let $d > 0$. By induction on d one may assume that for all $Q \in \Gamma_m(\mathcal{P})$ with $\deg_{x_m} Q < d$, all $c_{P,i}(x)$ have been constructed as well as for P, all $c_{P,j}(x)$, for $1 \le j \le i-1$ by induction on i. We are now going to construct $c_{P,i}(x)$ for P in the following way:
If $\zeta'_1 < \cdots < \zeta'_{t'}(\ \zeta^r_1 < \cdots < \zeta^r_{t_r})$ denote the zeros of $\dfrac{\partial P}{\partial x_m}$ $\quad (R = P - x_m \dfrac{\partial P}{\partial x_m})$, let j be the smallest index such that $\zeta_i \le \zeta'_j$ and k the smallest index such that $\zeta'_j \le \zeta^r_k$. Define

$$e(x) := \frac{x_m}{d} c_{P,j}(x) + c_{R,k}(x),$$

then

$$e(x) = \begin{cases} 0 & \text{if } x_m \le \zeta'_j \\ P(x) - R(x) & \text{if } \zeta'_j \le x_m \le \zeta^r_k \\ P(x) & \text{if } \zeta^r_k \le x_m. \end{cases}$$

If $P(\zeta'_j) = 0$, then $R(\zeta'_j) = P(\zeta'_j) - \zeta'_j \dfrac{\partial P}{\partial x_m}(\zeta'_j) = 0$ and therefore $\zeta_i = \zeta'_j = \zeta^r_k$ and one can take

$$c_{P,i}(x) = e(x).$$

Otherwise, assume $P(\zeta'_j) > 0$ ($P(\zeta'_j) < 0$ is similar). In this case, since $R(\zeta'_j) = P(\zeta'_j) > 0$,

(1) $\qquad R(x) > 0$ for $\zeta'_j \le x_m < \zeta^r_k$ $\quad$ and

(2) $\qquad P(x) < 0$ for $\zeta_{i-1} < x_m < \zeta_i$.

(2) holds since by assumption, ζ_i is no multiple root and therefore P must change sign at ζ_i. By (1),

$$\sup\{P, e\} = \begin{cases} P & \text{if } x_m \ge \zeta_i \\ \sup\{P, 0\} & \text{if } x_m \le \zeta_i \end{cases}$$

Now define

$$c_{P,i} := \inf\{\sup\{c_{P,i-1}, 0\}, \sup\{P, e\}\}.$$

It remains to show that $c_{P,i}$ has the desired properties.

If $P(\zeta_j) > 0$ then $c_{P,i-1} < 0$ for $\zeta_{i-1} < x < \zeta_i$ and $\sup\{c_{P,i-1}, 0\} = 0$ for $x \le \zeta_i$. Since $\sup\{P, e\} \ge 0$ for $x \le \zeta_i$, we conclude that $c_{P,i}(x) = 0$ for $x \le \zeta_i$. For $x \ge \zeta_i$, $\sup\{P, e\} = P = c_{P,i-1}$, and the proof is done.

For the rest of this chapter, let h be a piecewise polynomial function on $\mathbb{R}^n$, $h = g_i$ on A_i and let

$$\mathcal{P} := \{g_i - g_j : 1 \le i < j \le s\}.$$

Here it is assumed that the g_i are distinct, hence $\mathring{A}_i \cap \mathring{A}_j = \emptyset$. Also, we may assume that the A_i be closed, since h is continuous, and $h = g_i$ on A_i forces $h = g_i$ on $\overline{A}_i$. We have the following

Proposition 8.7. *Let $C \in \mathcal{C}_m(\Gamma_m(\mathcal{P}))$ $(1 \le m \le n)$. For each connected component of C there exists a function $q \in$SIPD($\mathbb{R}^n$) which coincides with h on A.*

Proof. Fix a cylinder $C \in \mathcal{C}_m(\Gamma_m(\mathcal{P}))$ for some m and without loss of generality assume that C is connected. Let $\zeta_1 < \cdots < \zeta_t$ be the zeros of all functions in $\Gamma_m(\mathcal{P})$ over C. The sets

$$D_i := \{x \in \mathbb{R}^n : \zeta_i(\hat{x}_m) < x_m < \zeta_{i+1}(x_m)\}, \; i = 0, \ldots, t+1$$

are disjoint, open and connected and their union is dense in C. (here we set $\zeta_0 := -\infty$ and $\zeta_{t+1} := +\infty$). For each $k = 1, \ldots, t$ there exists a unique $\mu(k)$ such that $D_k \in A_{\mu(k)}$, since h is continuous and D_k is connected. (If D_k was not connected, say $D_k \subset A_i \cup A_j$, $i \ne j$, we would find points $x_i, x_j \in D_k$ such that $h(x_i) = g_i(x_i)$, $h(x_j) = g_j(x_j)$, so on a path connecting x_i and x_j there would be a point x with $g_i(x) = g_j(x)$, but all functions $g_i - g_j$ are nonzero on D_k).
If $t = 1$ (i.e. $D_0 = C$) then define $q := g_{\mu(D_0)}$. Otherwise, let

$$d_k := g_{\mu(D_k)} - g_{\mu(D_{k-1})} \quad \text{for } k = 1, \ldots, t+1.$$

Note that $d_k = 0$ on $\overline{D}_k \cap \overline{D}_{k-1}$ because h is continuous, so that the D_k are separated by the graphs of the zero-functions of the $g_i - g_j$. If $d_k \neq 0$ on D_k, there is exactly one function among $\zeta_1, \ldots, \zeta_t$ whose graph separates D_k from D_{k-1}. Hence, by Lemma 8.6, there exists $c_{d_k,i(k)}$ ∈SIPD with

$$c_{d_k,i(k)} = \begin{cases} d_k(x) & \text{if } x_m > \zeta_{i(k)}(\hat{x}_m) \\ 0 & \text{otherwise.} \end{cases}$$

Now define

$$q = g_{\mu(D_0)} + \sum_{k=1}^{t+1} c_{d_k,i(k)} \quad \in \mathrm{SIPD}(\mathbb{R}^n).$$

For any $x \in U := \cup_{i=0}^{t+1} D_i$, $x \in D_i$ for a unique i and

$$\begin{aligned} q(x) &= g_{\mu(D_0)}(x) + \sum_{k=1}^{t+1} c_{d_k,i(k)}(x) \\ &= g_{\mu(D_0)}(x) + \sum_{k=1}^{i} [g_{\mu(D_0)}(x) - g_{\mu(D_{k-1})}(x)] \\ &= g_{\mu(D_i)}(x) = h(x). \end{aligned}$$

If $x \in \overline{D}_k \cap \overline{D}_{k-1}$ for some $k \geq 1$, then $x_m = \zeta_{i(k)}(\hat{x}_m)$, so $c_{d_k,i(k)}(x) = 0$ and $q(x) = g_{\mu(D_{k-1})}(x) = g_{\mu(D_k)}(x) = h(x)$.

Remark: For $n = 1$, Proposition 8.7 implies the Pierce-Birkhoff-Conjecture, since the zeros are just points in $\mathbb{R}$ and the unique cylinder in $\mathcal{C}_1(\Gamma_1(\mathcal{P}))$ is all of $\mathbb{R}^1$.

Lemma 8.8. *Suppose $\delta \leq \zeta \in \mathbb{R}$ and $b: \mathbb{R} \to \mathbb{R}$ is pwp. If $\delta = \zeta$ assume further that $b(\zeta) = 0$. Then it is possible to construct a function u ∈SIPD($\mathbb{R}^n$) such that $u(t) \geq b(t)$ for $t \geq \zeta$ and $u(t) = 0$ for $t \leq \delta$.*

Proof. In case $\delta < \zeta$ define $u(t)$ by

$$u(t) = \begin{cases} b(t) & \text{if } \zeta \leq t \\ \dfrac{b(\zeta)}{\zeta - \delta}(t - \delta) & \text{if } \delta < t \leq \zeta \\ 0 & \text{otherwise} \end{cases}$$

and if $\delta = \zeta$, let $u(t) = b(t)$ for $t \geq \zeta$, $u(t) = 0$ otherwise.

In either case, u is piecewise polynomial, hence in SIPD($\mathbb{R}^n$) by the preceding remark.

Lemma 8.9. *Let $Q \subset \mathbb{R}[x_1, \dots, x_n]$, Q finite. There exists a finite subset Q' of $\mathbb{R}[x_1, \dots, x_n]$ containing Q such that each $A \in \mathcal{A}(Q')$ is connected and one obtains a function $\nu : \mathcal{A}(Q') \mapsto \{1, \dots, s\}$ such that for all $A \in \mathcal{A}(Q')$ $h = g_{\nu(A)}$ on A.*

Proof. The existence of Q' follows from Theorem 7.10. On each $A \in \mathcal{A}(Q)$, the functions $g_i - g_j$ have constant sign (either $<$ or $>$), and therefore $A \cap (A_i \cap A_j) \neq \emptyset$ for each pair i, j since $g_i - g_j = 0$ on $A_i \cap A_j$. Suppose $A \subset A_k \cup A_l$, $A \not\subset A_k, A \not\subset A_l$. Since $A \cap (A_k \cap A_l) = \emptyset$, $A \subset A_k \triangle A_l$ (symmetric difference). A is open, hence $A \subset (A_k \cap A_l)^\circ \subset A_k^\circ \cup A_l^\circ$. But $A_k^\circ \cap A_l^\circ = \emptyset$ and so $A \subset A_k$ (say) and define $\nu(A) = k$.

Proof of Conjecture 8.2. for $n = 2$.
For $\mathcal{P} := \{g_i - g_j : 1 \le i < j \le n\} \subset \mathbb{R}[x_1, x_2]$ let $\mathcal{P}' := \Gamma_1(\mathcal{P}) \cup \Gamma_2(\mathcal{P})$ and let $\mathcal{C}_1(\mathcal{P}'), \mathcal{C}_2(\mathcal{P}')$ be cylindrical algebraic decompositions of $\mathcal{P}'$ such that all cylinders in $\mathcal{C}_1(\mathcal{P}')$ and $\mathcal{C}_2(\mathcal{P}')$ are connected. If $\mathcal{C}_1$ ($\mathcal{C}_2$) is the set of polynomials in $\mathbb{R}[x_2]$ ($\mathbb{R}[x_1]$) describing the cylinders in $\mathcal{C}_1(\mathcal{P}')$ ($\mathcal{C}_2(\mathcal{P}')$), let $Q := \mathcal{P}' \cup \mathcal{C}_1 \cup \mathcal{C}_2$. Then each $A \in \mathcal{A}(Q)$ is connected. Denote the sets in $\mathcal{A}(Q)$ by $E_1, \dots, E_t$. We will construct functions h_{ij} $\in$SIPD($\mathbb{R}^n$), $i, j = 1, \dots, t$ such that

(1) $$h_{ij} \le g_{\nu(E_i)} \text{ on } E_i$$

(2) $$h_{ij} \ge g_{\nu(E_j)} \text{ on } E_j$$

Then for $H := \sup_j \inf_i \{h_{ij}, g_{\nu(E_j)}\}$ we have $H = h$ on $E := \cup_{i=1}^t E_i$ and since E is dense in $\mathbb{R}^2$, it extends uniquely to a continous function $\hat{H}$ on $\mathbb{R}^2$ which must be h. To see that $H = h$ on E, suppose $x \in E_k$ for some k. Then for each j the function $h_j := \inf_i \{h_{ij}, g_{\nu(E_j)}\}$ satisfies

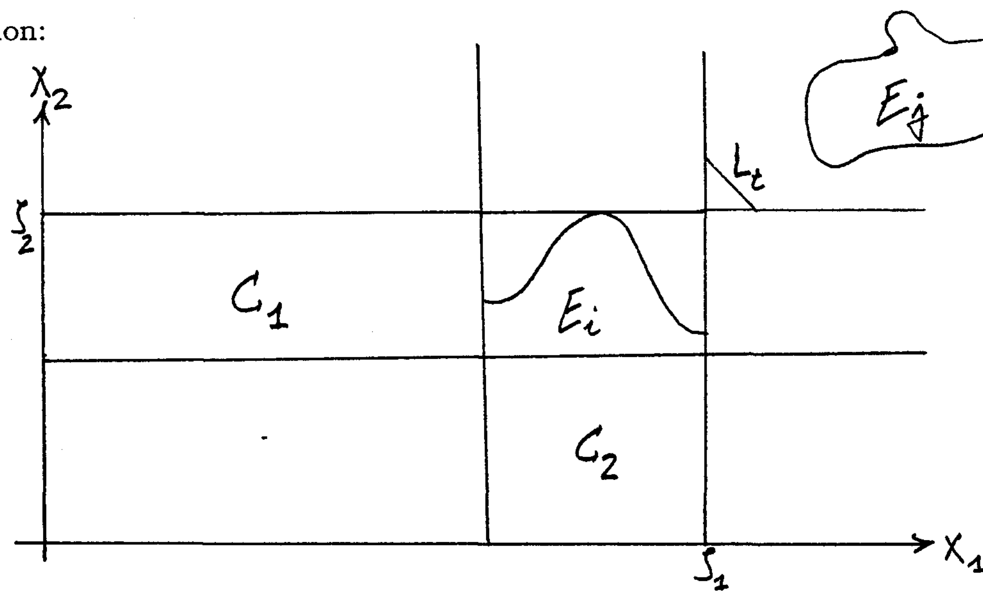
(3) $$h_j(x) \le g_{\nu(E_i)}(x) \text{ for } x \in E_i \text{ by (1) and}$$

(4) $$h_j(x) = g_{\nu(E_j)}(x) \text{ for } x \in E_j \text{ by (2),}$$

so for $x \in E_k$, one has $h_k(x) = g_{\nu(E_k)}(x)$ by (4) and $h_j(x) \le g_{\nu(E_k)}(x)$ for $j \neq k$, hence $\sup_j h_j(x) = g_{\nu(E_k)}(x)$.

If E_i, E_j (i, j are fixed from now on) lie in a common cylinder C of $\mathcal{C}_1(\mathcal{P}')$ or $\mathcal{C}_2(\mathcal{P}')$, there exists a function q that agrees with h on C by Proposition 7.7 and one may take q for h_{ij}.

So assume E_i and E_j do not lie in a common cylinder and we have the following situation:



E_i lies in a unique cylinder C_1 in x_1-direction and in another cylinder C_2 in x_2-direction. Let (ζ_1, ζ_2) be the upper right vertex of the rectangle $C_1 \cap C_2$. Here it is assumed that E_j is somewhat above and to the right of E_i, the other cases can be treated in a similar way. Let

$$L_t := \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 + x_2 = t \text{ and } (x_1 - \zeta_1)(x_2 - \zeta_2) \geq 0\}$$

and

$$I(t) := \{1 \leq k \leq s : \ A_k \cap L_t \neq \emptyset\}.$$

Since $\mathbb{R}^2 = \cup_{i=1}^{s} A_k$, each $I(t) \neq \emptyset$. Define

$$p(t) := \max_{(x_1,x_2)\in L_t} (h - g_{\nu(E_i)})(x_1, x_2).$$

$p(t)$ is well defined because L_t is compact for each t.

$$p(t) \leq \max_{\substack{k\in I(t) \\ (x_1,x_2)\in L_t}} (g_k - g_{\nu(E_i)})(x_1, x_2)$$

since for each $(x_1, x_2) \in L_t$, $h(x_1, x_2) = g_k(x_1, x_2)$ for some $k \in I(t)$.

Now translate the point (ζ_1, ζ_2) to the origin and rotate the axes by $\frac{\pi}{4}$ radians, applying the following coordinate change:

$$Y_1 := (X_1 - \zeta_1) + (X_2 - \zeta_2), \qquad Y_2 = (X_1 - \zeta_1) - (X_2 - \zeta_2)$$

For each k, expand $g_k - g_{\nu(E_i)}$ in powers of Y_1 and Y_2:

$$(g_k - g_{\nu(E_i)}) = a_{k00} + a_{k10}Y_1 + a_{k01}Y_2 + a_{k20}Y_1^2 + a_{k11}Y_1Y_2 + a_{k02}Y_2^2 + \ldots$$

for finitely many $a_{klj} \in \mathbb{R}$.

Claim:

$$(*) \qquad |(x_1 - \zeta_1) \pm (x_2 - \zeta_2)| \leq |t - \zeta_1 - \zeta_2| \qquad \text{for } (x_1, x_2) \in L_t$$

Proof: Since $x_1 + x_2 = t$, $|x_1 - \zeta_1 + x_2 - \zeta_2| = |t - \zeta_1 - \zeta_2|$. To prove $(*)$ for the '$-$'-sign, distinguish two cases:

(1) $x_1 \geq \zeta_1$ and $x_2 \geq \zeta_2$. Then

$$|(x_1-\zeta_1)-(x_2-\zeta_2)| \leq \max\{(x_1-\zeta_1),(x_2-\zeta_2)\} \leq (x_1-\zeta_1)+(x_2-\zeta_2) = |t-\zeta_1-\zeta_2|$$

(2) $x_1 \leq \zeta_1$ and $x_2 \leq \zeta_2$. Then

$$|(x_1 - \zeta_1) - (x_2 - \zeta_2)| \leq \max\{(x_1 - \zeta_1), (x_2 - \zeta_2)\} \leq \zeta_1 + \zeta_1 + x_1 - x_2 = |t - \zeta_1 - \zeta_2|$$

So by $(*)$, $|Y_1| \leq |t - \zeta_1 - \zeta_2|$ and $|Y_2| \leq |t - \zeta_1 - \zeta_2|$ and we obtain

$$\begin{aligned} p(t) \leq \max_{k \in I(t)} \big((|a_{k00}| + (|a_{k10}| + |a_{k01}|)|t - \zeta_1 - \zeta_2| \\ + (|a_{k20}| + |a_{k10}| + |a_{k01}|)|t - \zeta_1 - \zeta_2|^2 + \ldots \big). \end{aligned}$$

Denote the righthand side by $b(t)$. Let $\delta \in \mathbb{R}$ be the smallest number such that

$$\delta < t < \zeta_1 + \zeta_2 \Rightarrow L_t \cap E_i = \emptyset.$$

Of course $\delta \leq \zeta_1 \leq \zeta_2$. If $\delta = \zeta_1 + \zeta_2$ then (1) $b(\delta) = 0$ and (2) $\lim_{t \to 0} b(t) = 0$.

Proof of (1): If $\delta = \zeta_1 + \zeta_2$ then $I(\delta) = \{k : \ (\zeta_1, \zeta_2) \in A_k\}$. If $(\zeta_1, \zeta_2) \in \overset{\circ}{A}_l$ for some l, then $I(l) = \{l\}$ since the $\overset{\circ}{A}_k$ are mutually disjoint. But then $l = \nu(E_i)$, for if $l \neq \nu(E_i)$, one can find a neighborhood U of (ζ_1, ζ_2) with

(a) $L_t \subset U$ for $\zeta_1 + \zeta_2 - \epsilon < t < \zeta_1 + \zeta_2$ and some $\epsilon > 0$

(b) $U \cap E_i = \emptyset$.

However, (a) and (b) imply $\delta < \zeta_1 + \zeta_2$, a contradiction. Therefore $l = \nu(E_i)$ and

$$b(\delta) = \max_{k \in I(\delta)} (g_k - g_{\nu(E_i)})(\zeta_1, \zeta_2) = (g_{\nu(E_i)} - g_{\nu(E_i)})(\zeta_1, \zeta_2) = 0.$$

If $(\zeta_1, \zeta_2) \in \partial A_l$ for some l, then $I(\delta) = \{k : \ (\zeta_1, \zeta_2) \in \partial A_k\}$ and for $j, l \in I(\delta)$, $g_l(\zeta_1, \zeta_2) = 0$. Moreover $\nu(E_i) \in I(\delta)$ since there exists $\epsilon > 0$ such that

$$\delta - \epsilon < t < \delta \Rightarrow L_t \cap E_i \neq \emptyset,$$

so that $(\zeta_1, \zeta_2) \in \overline{E}_i \subset A_{\nu(E_i)}$. (The A_i were assumed to be closed). Again, $b(t) = \max_{k \in I(\delta)} (g_k - g_{\nu(E_i)})(\zeta_1, \zeta_2) = 0$.

Proof of (2): $b(\delta) = 0$ implies $a_{k00} = 0$ for all $k \in I(\delta)$, hence $\lim_{t \to \delta} b(t) = 0$.

By Lemma 8.8 we can construct a function u $\in$SIPD($\mathbb{R}$) with $u(t) \geq b(t) \geq p(t)$ for $t \geq \zeta_1 + \zeta_2$ and $u(t) = 0$ for $t \leq \delta$. It follows that

(a) $h_{ij}(x_1, x_2) \geq g_{\nu(E_j)}(x_1, x_2)$ on E_j since for any $(x_1, x_2) \in E_j$ $x_1 \geq \zeta_1, x_2 \geq \zeta_2$ and thus

$$u(x_1 + x_2) \geq p(x_1 + x_2) = \max_{(x_1, x_2) \in L_{x_1 + x_2}} (h - g_{\nu(E_i)}) \leq g_{\nu(E_j)} - g_{\nu(E_i)}.$$

(b) $h_{ij}(x_1, x_2) = g_{\nu(E_i)}(x_1, x_2)$ on E_i since here $x_1 + x_2 \leq \delta$ and $u(x_1 + x_2) = 0$ $\quad \diamondsuit\diamondsuit$.

CHAPTER IX

ABSTRACT SEMIALGEBRAIC FUNCTIONS

Abstract semialgebraic functions are defined on constructible subsets of the real spectrum of a given commutative ring A. In the situation where A is a polynomial ring over a real closed field R, the constructible sets of Sper A correspond to semialgebraic subsets of R^n, and the idea is to extend "ordinary" semialgebraic functions and make them 'work' over the real spectrum. It turns out that the ring of abstract semialgebraic functions over a constructible set C in Sper A is isomorphic to the ring of semialgebraic functions on the semialgebraic subset in R^n that corresponds to C (this will not be proved here, also it is by no means obvious that the set of abstract semialgebraic functions forms a ring, as shown by N. Schwartz). In this chapter, we will set up the notion of an abstract semialgebraic function and prove a certain 'continuity property', referring to its values on points $x, y \in$ Sper A, where y specializes x. First, we will introduce the concept of *sections,* which of course can be done in a much more general context than needed here.

Definition 9.1: Let A be a ring, $A[t]$ the ring of polynomials in one indeterminate over A. We have the projection map

$$\begin{aligned} \pi : Sper\ A[T] &\rightarrow \text{Sper } A \\ \beta &\mapsto \beta \cap A. \end{aligned}$$

Given $\alpha \in$ Sper A, the fiber $\pi^{-1}(\alpha) = \{\beta \in \text{Sper } A[T] : \pi(\beta) = \alpha\}$ is homeomorpic to Sper $k(\alpha)[T]$ with respect to the Harrison topology. Here, $k(\alpha)$ denotes the real closure of *quot* $A/\text{supp}(\alpha)$. (For a proof of this, consult [2]).

Note that $\pi^{-1}(\alpha)$ is $\mathcal{C}$-clopen as it is the homeomorphic image of the $\mathcal{C}$-clopen set Sper $k(\alpha)[T]$, thus $\pi^{-1}(\alpha)$ is constructible in Sper $A[T]$ by Theorem 4.3.

Definition 9.2: Let $X \in$ Sper A be constructible. A *section* $s : X \rightarrow$ Sper $A[T]$ is a set theoretic map satisfying $\pi \circ s = id_X$. s is called a *constructible section* if the image $s(X)$ is constructible in Sper $A[T]$.

In other words, a section s assigns each $x \in X$ an element $s(x)$ of the fiber $\pi^{-1}(x)$. In the same vein one may define sections $s: S \to R^{n+1}$ where $S \in R^n$ is a semialgebraic set (R a real closed field). In this situation, any function $f: S \to R$ yields a section $s_f : S \to \Gamma(f) \subset S \times R$ by $s_f(x) = (x, f(x))$, $x \in S$. It is exactly this model that will be used to construct sections on Sper $A[T]$.

If $s: X \to$ Sper $A[T]$ is a constructible section, the point $s(x) = \pi^{-1}(x) \cap s(X)$ is constructible in Sper $k(x)[T]$ for any $x \in X$. However, this implies that $s(x) \in k(x)$ according to Corollary 4.10, and therefore $k(s(x)) = k(x)$ (here $k(s(x))$ denotes the real closure of $F = quot\ A[T]/\mathrm{supp}(s(x))$). In this sense we may identify the ordering $s(x)$ with the image $T(s(x))$ in $k(s(x)) = k(x)$ and we obtain a 'function' $f_s: X \to \bigcup_{x \in X} k(x)$, $f_s(x) = T(s(x))$, i.e. $f_s(x) \in \prod_{x \in X} k(x)$. The section $s(X)$ is completely determined by f_s since for any $x \in X$ the ordering $s(x)$ corresponds to the '$k(x)$-rational' point $T(s(x))$.

We have the injective maps

$$A/\mathrm{supp}(x) \xrightarrow{p} A[T]/\mathrm{supp}(s(x)) \xrightarrow{i} k(x),$$

where i is just the inclusion and p defined by $p(a + \mathrm{supp}(x)) = a + \mathrm{supp}(s(x))$. p is injective because for $a \in A$

$$a \in \mathrm{supp}(s(x)) \iff a \in A \cap s(x) \cap -s(x) \iff a \in \mathrm{supp}(x).$$

Hence $A/\mathrm{supp}(x) \subset A[T]/\mathrm{supp}(s(x)) \subset k(x)$.

How do we obtain extensions of orderings on A to $A[T]$? Given $\alpha \in$ Sper A, $a \in A$, we have the projection map

$$\pi_\alpha(a) := a(\alpha) = a + \mathrm{supp}(\alpha) \in A/\mathrm{supp}(\alpha) \subset k(\alpha).$$

Define $\bar{s}_a(\alpha) := \bar{\beta} = \{f \in K(\alpha)[T] : f(a(\alpha)) \geq 0\}$. $\bar{\beta}$ is the ordering on $k(\alpha)[T]$ corresponding to the point $a(\alpha) \in k(\alpha)$ with $k(\beta) = k(\alpha)[T]/(T - a(\alpha)) = k(\alpha)$. The desired ordering β on $A[T]$ then is

$$s_a(\alpha) := \beta = \pi_\alpha^{-1}(\beta) \cap A/\mathrm{supp}(\alpha).$$

Doing this for each $\alpha \in$ Sper A, we obtain the section $s_a :$ Sper $A \to$ Sper $A[T]$ and since for any α, the ordering $s_a(\alpha)$ is associated to the point $a(\alpha) \in k(\alpha)$, we may identify s_a with $\tilde{a} := (a(\alpha))_{\alpha \in \text{Sper } A} \in \prod_{\alpha \in \text{Sper } A} k(\alpha) =: \tilde{A}$.

Let's go back to the setting where $A = R[x_1, \ldots, x_n]/\mathfrak{a}$, $S \subset V_A(R)$ semialgebraic set and $f : S \to R$ a semialgebraic function. G. Brumfiel has shown that f is continiuous iff $\Gamma(f)$ is a closed subset of $S \times R$ and f is locally bounded, i.e. for $0 < \epsilon \in R$ and $x \in S$ there exists an open neighborhood U of x such that $|f(y)| < |f(x)| + \epsilon$ for all $y \in U$. This gives rise to the following definitions:

Definition 9.3: Let s be a constructible section of X. s is *locally bounded* if for all $x \in X$ there exists an open neighborhood $U(x)$ such that for all $y \in U(x)$

$$f_s^2(y) \leq \frac{a(y)}{b(y)} \qquad \text{for some } a, b \in A \text{ with } b(y) \neq 0.$$

Definition 9.4: An *abstract semialgebraic function* on X is a constructible section s of X satisfying

(1) s is locally bounded

(2) $s(X)$ is closed in $\pi^{-1}(X)$.

Functions so defined have an interesting property:

Theorem 9.5. *Let f be an abstract semialgebraic function on X. For any $x, y \in X$ such that $x \subset y$ there exists a specialization z of $f(x)$ with $\pi(z) = y$. (Recall, $\pi(z)$ denotes the projection of z onto Sper A).*

The proof of Theorem 9.5 will make use of valuation theory. Let us set up some notation first:

Given points $x, y \in X$ with $x \subset y$, define $\mathcal{M}(x, y)$ to be the set of all convex valuation rings $C \supset A/\text{supp}(x)$ of $k(x)$ that satisfy the following condition:

$$(*) \qquad \mathbf{m}_C \cap A/\text{supp}(x) = \text{supp}(y)/\text{supp}(x),$$

where $\mathbf{m}_c$ is the maximal ideal of C. By Theorem 5.6, $\mathcal{M}(x,y)$ is totally ordered by inclusion, its smallest element being the convex hull $ch_{k^2(x)}(\overline{A}_{\bar{\mathbf{p}}})$, where $\overline{A} = A/\mathrm{supp}(x)$ and $\bar{\mathbf{p}} = \mathrm{supp}(y)/\mathrm{supp}(x)$.

Proof. Denote the smallest element of $M(x,y)$ by $D(x,y)$. $D(x,y)$ is obviously convex. Also, any $C \in \mathcal{M}(x,y)$ contains $\overline{A}_{\bar{\mathbf{p}}}$, since if $\frac{\bar{a}}{\bar{s}} \in C$ for some $\bar{a} \in \overline{A}$, $\bar{s} \notin \bar{\mathbf{p}}$, then $\frac{\bar{s}}{\bar{a}} \in \mathbf{m}_C$, hence $\bar{s} \in \mathbf{m}_C \cap \overline{A} = \bar{\mathbf{p}}$, a contradiction. By convexity of C then $D(x,y) \subset C$. Note that $\mathbf{p}\overline{A}_{\mathbf{p}}$ is convex in $\overline{A}_{\mathbf{p}}$ with respect to the ordering $k^2(x)$ on $k(x)$: Suppose $0 < \frac{\bar{a}}{\bar{s}} \leq \frac{\bar{b}}{\bar{t}}$, $\bar{b} \in \mathbf{p}$, $\bar{s}, \bar{t} \notin \mathbf{p}$. Then $0 \leq_y at \leq_y bs$ since y specializes x. By assumption $b \in \mathrm{supp}(y)$, hence $a \in \mathrm{supp}(y)$, so $\frac{\bar{a}}{\bar{s}} \in \mathbf{p}$.
It remains to show that $D(x,y)$ satisfies $(*)$. This will be guaranteed by the following

Lemma 9.6. *Let (K,P) be an ordered field, A a local subring of K whose maximal ideal is convex in A with respect to P. Then*

$$\mathbf{m}_{ch_P(A)} = \mathbf{m}_A,$$

where $ch_P(A)$ is the convex hull of A with respect to P.

Proof. $\mathbf{m}_{ch_P(A)} \subset \mathbf{m}_A$ is clear since every unit in A is a unit in $ch_P(A)$. Suppose $m \in \mathbf{m}_A \backslash \mathbf{m}_{ch_P(A)}$, $m > 0$. Then $0 < m^{-1} < a$ for some $a \in A$, so $m > a^{-1} > 0$ which implies $a^{-1} \in \mathbf{m}_A$, a contradiction.

The maximal element of $\mathcal{M}(x,y)$ is given by

$$C(x,y) := \bigcup_{C \in \mathcal{M}(x,y)} C.$$

$C(x,y)$ is a ring since $\mathcal{M}(x,y)$ forms a chain under inclusion; it is a valuation domain as an overring of $D(x,y)$, and it is convex because $0 < a < b$, $b \in C(x,y)$ implies $a \in C \subset C(x,y)$ for some $C \in \mathcal{M}(x,y)$. We now turn to the

Proof of Theorem 9.5. Let $x \subset y$ be given points in X, $\mathcal{M}(x,y)$ as above. As f is locally bounded, there exists $U \ni y$, U open such that for all $x' \in U$: $f_s(x')^2 \leq$

$\frac{a(x')}{b(x')}$, $b \neq 0$ on U. In particular, $x \in U$ since $y \in \overline{\{x\}}$, and $f_s(x)^2 \leq \frac{a(x)}{b(x)}$. $b(y) \neq 0$ implies that $\frac{a(x)}{b(x)} \notin \bar{\mathbf{p}}$, hence $\frac{a(x)}{b(x)} \in D(x,y)$ which forces $f_s(x)^2 \in D(x,y)$ and thus $f_s(x) \in D(x,y)$. Fix any $C \in \mathcal{M}(x,y)$. Let us now construct the specialization z of $s(x)$ that projects down to $y \in \operatorname{Sper} A$. As seen earlier, the ring $A[T]/\operatorname{supp}(s(x))$ is a subring of $k(x)$. It is also contained in C, for $T(s(x)) = f_s(x) \in C$. $\mathbf{m}_C$ is convex in C and so is the prime ideal $\mathbf{m}_C \cap A[T]/\operatorname{supp}(s(x)) =: q/\operatorname{supp}(s(x))$ in the ring $A[T]/\operatorname{supp}(s(x))$. Define

$$z := s(x) \cup q.$$

Claim: z is an ordering on $A[T]$.We will check the axioms (P1)–(P4).

(P1): $z + z \subset z$: Let $z_1, z_2 \in z$. If both are either in $s(x)$ or q, their sum will be as well. So suppose $z_1 \in s(x) \backslash q$, $z_2 \in q$. If $|z_1| > |z_2|$ with respect to the ordering $s(x)$, then $z_1 + z_2 \in s(x)$, otherwise $z_1 \in q$ by convexity of q. In either case, $z_1 + z_2 \in z$.

(P2): $z \cdot z \subset z$ and

(P3): $z \cup -z \subset z$ are clear.

(P4): $z \cap -z = (s(x) \cup q) \cap (-s(x) \cup q) = q$.

Thus z is an ordering on $A[T]$. We have $s(x) \subset z$; and $\pi(z) = y$ since

$$z \cap A = (s(x) \cap A) \cup (q \cap A) = x \cup \mathbf{m}_C \cap A = x \cup \operatorname{supp}(y) = y.$$

Example: Let $A = R[x_1, \ldots, x_n]/\mathfrak{a}$, $S \subset V_A(R)$, R a real closed field. Consider a continuous semialgebraic function $f : \to R$. f induces a section $s_f : \ S \to S \times R$ via the map $s_f(x) = (x, f(x))$ for $x \in S$. Let $\tilde{S}$ be the $\mathcal{C}$-closure of S in $\operatorname{Sper} A$. s_f extends to a constructible section $\tilde{s}_f : \ \tilde{S} \to \operatorname{Sper} A[T]$ in the following way:

If $x \in \tilde{S} \cap V_A(R) = S$, then $\tilde{s}_f(x) = T(x, f(x)) = f(x)$, hence we get our original function back and $\tilde{s}_f(x)$ is the ordering associated to the point $(x, f(x)) \in \Gamma(f)$, i.e.

$$\tilde{s}_f(x) = \{P \in A[T] : \ P(x, f(x)) \geq 0\}.$$

For $x \in \tilde{S}$ we are going to describe $\tilde{s}_f(x)$ via the ultrafilter $\mathcal{F}_x$ attached to x: According to Corollary 4.15,

$$\mathcal{F}_x = \{M \in \sigma(V_A(R)) : \ x \in \tilde{M}\}.$$

First, restrict $\mathcal{F}_x$ to semialgebraic sets contained in S, i.e.

$$\mathcal{F}_{|S} := \{M \cap S : \; M \in \mathcal{F}_x\}.$$

For convenience, let's denote $\mathcal{F}_{|S}$ again by $\mathcal{F}_x$. The section $s_f(x)$ maps each $M \in \mathcal{F}_x$ to the semialgebraic set $M \times f(M) \subset \Gamma(f)$. Now consider the set

$$\mathcal{F}_{f(x)} = \{N \subset \sigma(V(R) \times R) : \; \exists M \in \mathcal{F}_x \text{ such that } M \times f(M) \subset N\}.$$

Claim: $\mathcal{F}_{f(x)}$ is a filter. We need to check the following axioms:
(F1) $\emptyset \notin \mathcal{F}_{f(x)}$: clear, since $\emptyset \notin \mathcal{F}_x$.
(F2) $A, B \in \mathcal{F}_{f(x)} \Rightarrow A \cap B \in \mathcal{F}_{f(x)}$: A contains a set $M_1 \times f(M_1)$ and B a set $M_2 \times f(M_2)$, hence $A \cap B$ contains the set $M_1 \cap M_2 \times f(M_1 \cap M_2)$, where $M_1 \cap M_2 \in \mathcal{F}_x$.
(F3) $A \subset B,\; A \in \mathcal{F}_{f(x)} \Rightarrow B \in \mathcal{F}_{f(x)}$: by construction.
$\mathcal{F}_{f(x)}$ is maximal for, if $N \notin \mathcal{F}_{f(x)}$ for some $N \in \sigma(V(R) \times R)$, then

$$(*) \qquad \forall\, M \in \mathcal{F}_x : \; M \times f(M) \not\subset N.$$

Fix $M \in \mathcal{F}_x$. $\mathcal{F}_x$ being maximal, either $A := \pi(N \cap (M \times f(M))$ or $M \backslash A$ belongs to $\mathcal{F}_x$. If $A \in \mathcal{F}_x$, then $A \times f(A) \subset N$, which contradicts $(*)$, hence $N^c \in \mathcal{F}_{f(x)}$.
$\mathcal{F}_{f(x)}$ corresponds to an ordering $y \in \text{Sper } A[T]$ and we define $\tilde{s}_f(x) = y$. We have $\pi(y) = x$ since

$$a \in y \cap A \iff \{a \geq 0\} \in \mathcal{F}_{f(x)} \iff \pi(\{a \geq 0\}) \in \mathcal{F}_x \iff a \in x.$$

In order to prove that $\tilde{s}_f$ is constructible, we need to show that the set $\tilde{s}_f(\tilde{S})$ is a constructible subset of Sper A. Recall that $f: \; S \to R$ is a semialgebraic function in the ordinary sense, so its graph $\Gamma(f)$ can be written in the form

$$\Gamma(f) = \{x \in S \times R : \; \Psi(x)\}$$

for a formula Ψ. We claim that

$$\tilde{s}_f(\tilde{S}) = \{y \in \text{Sper } A[T] : \; \Psi(y)\},$$

i.e. the 'abstract graph' of $\tilde{s}_f$ is just $\widetilde{\Gamma(f)}$.

Proof. "$\subset$": Let $y \in \tilde{s}_f(\tilde{S})$. There exists $x \in \tilde{S}$ with $y = \tilde{s}_f(x)$ and $\mathcal{F}_y = \mathcal{F}_{f(x)}$. $\mathcal{F}_y$ contains the set $S \times f(S) = \Gamma(f)$, thus $\Psi(y)$ is true and we conclude that $y \in \Sigma$.
"$\supset$": Let $y \in \operatorname{Sper} A[T]$ such that $\Psi(y)$ holds. We need to find $x \in \tilde{S}$ such that $y = \tilde{s}_f(x)$. Again, by assumption, $\Gamma(f) \in \mathcal{F}_y$. Thus for any $F \in \mathcal{F}_y$, $F \cap \Gamma(f) \in \mathcal{F}_y$ and $F \cap \Gamma(f)$ is of the form $M \times f(M)$ for some semialgebraic set $M \subset S$. Let

$$\mathcal{G} := \{M \in \sigma(V(R)) : \exists N \in \mathcal{F}_y \ni : \pi(N) \subset M\}.$$

$\mathcal{G}$ is a filter (in $\sigma(V(R))$) since if $\pi(N) \subset M$, $\pi(N') \subset M'$, then $\pi(N \cap N') \subset \pi(N) \cap \pi(N') \subset M \cap M'$. If $M \notin \mathcal{G}$ for some $M \in \sigma(V(R))$, then M does not contain any semialgebraic subset S' of S such that $S' \times f(S') \in \mathcal{F}_y$. In particular, $(M \cap S') \times f(M' \cap S') \notin \mathcal{F}_y$ for such S'. Maximality of $\mathcal{F}_y$ now implies that $(S' \backslash M) \times f(S' \backslash M) \in \mathcal{F}_y$, hence $M^c \in \mathcal{G}$. This shows that $\mathcal{G}$ is an ultrafilter and therefore defines a point $x$in Sper A. It remains to show that $x \in \tilde{S}$: this follows from S being in $\mathcal{G}$.

We are now going to prove that $\tilde{s}_f$ is indeed an abstract semialgebraic function on $\tilde{S}$.

For this purpose we need to show (a) $\tilde{s}_f$ is locally bounded and (b) $\tilde{s}_f(\tilde{S})$ is closed in $\pi^{-1}(\tilde{S})$.

As seen earlier, $\tilde{s}_f(\tilde{S}) = \widetilde{\Gamma_f(S)}$, so that (b) follows. As to (a), note first that $\Gamma(f)$ (and therefore $\widetilde{\Gamma(f)}$) has only finitely many connected components (cf. Theorem 6.10). For each component C_i we can find polynomials p_i that vanish on C_i (C_i has empty interior). Writing p_i in the form $p_i = \sum_{j=0}^{n_i} a_{ij}(x)y^j$, $a_{in_i} \not\equiv 0$, we see that $\sum_{j=0}^{n_i} a_{ij} f(x)^i = 0$, $x \in C_i$. For fixed $x_0 \in C_i$, the set of zeros of $p_i(x_0, T)$ is contained in the set

$$\{(x_0, y) : \ |y| < 1 + \sum_{j=0}^{n_i} |\frac{a_{ij}(x_0)}{a_{n_i}(x_0)}|\}.$$

To see this, let $a_{in_i}y^{n_i} + \cdots + a_0 = 0$. Then

$$y^{n_i} = -\frac{a_{n_i-1}}{a_{n_i}}y^{n-1} - \cdots - \frac{a_0}{a_{n_i}},$$

and if $|y| \geq 1$ then

$$|y| \leq |\frac{a_{n-1}}{a_{n_i}}||\frac{1}{y}| + \cdots + |\frac{a_0}{a_{n_i}}||\frac{1}{y^{n_0-1}}| \leq \sum |\frac{a_i}{a_{n_i}}|,$$

so in any case $|y| \leq 1 + \sum |\frac{a_i}{a_{n_i}}|$.
U_i is semialgebraic and open in $V(R)$, so

$$\tilde{U}_i := \{x \in \tilde{S} : \ |\tilde{s}_f(x)| < 1 + \frac{1}{|a_{i0}(x)|} \sum_{i=1} n_i |a_{ij}(x)|, \ a_{i0}(x) \neq 0\}$$

is constructible and open in Sper A (this is not covered by any theorem proved in this paper, but follows from the fact that $U \subset V(R)$ is open $\iff$ $\tilde{U} \subset$ Sper A is open, cf. [2]).In particular, $\tilde{S} \subset \cup_i U_i$ and since $\tilde{s}_f$ is bounded on each U_i, we conclude that $\tilde{s}_f$ is locally bounded.□

A whole lot more can be said about abstract semialgebraic functions. Here are some further results we will need in the next chapter:

Definition 9.7: Given a ring A, we denote by $SA(A)$ the set of abstract semialgebraic functions on Sper A.

$SA(A)$ is a ring under addition and multiplication when defined componentwise, i.e. if $f, g \in SA(A)$, then $f(x), g(x) \in k(x)$, so set $(f \dot{+} g)(x) = f(x) \dot{+} g(x)$ in $k(x)$. This guarantees that $\pi(f(x) \dot{+} g(x)) = x$. Moreover, $SA(A)$ is a lattice, i.e. it is closed under suprema and infima (taken componentwise). In the special case that $S \subset \mathbb{R}^n$ is a semialgebraic set and A the ring of continuous semialgebraic functions, it can be shown that $SA(A)$ is isomorphic to $\tilde{A} = \prod_{\alpha \in \text{Sper } A} a(\alpha)$. (See N. Schwartz: Real closed spaces, Habilitationsschrift, Műnchen 1984)

Another important feature that $SA(A)$ shares with its 'ordinary' counterpart is that subsets of Sper A defined by finitely many sign conditions on functions of $SA(A)$ are constructible.

Now let's see what abstract functions can do for us.

CHAPTER X

THE PIERCE–BIRKHOFF CONJECTURE: AN ABSTRACT APPROACH

In the recent past, the Pierce-Birkhoff Conjecture has challenged quite a few mathematicians. In 1989, J. Madden was able to restate the problem and place it in a much more general setting. Before discussing his results, let's recall briefly what the Pierce-Birkhoff Conjecture says: It claims that a continuous piecewise polynomial function h on $\mathbb{R}^n$ can be expressed as

$$h = \sup_j \inf_k \{f_{ik}\}$$

for finitely many polynomials f_{ik}. If we want to transfer this to the abstract setting, we first of all need a

Definition 10.1: For any ring A, let PWP(Sper A) denote the set of functions $f \in SA(A)$ such that for all $\alpha \in$ Sper A there exists $a \in A$ such that $f(\alpha) = a(\alpha)$. Let SIPD(Sper A) be the sublattice of $SA(A)$ generated by $\tilde{A}$, i.e. SIPD(Sper A) is closed under taking suprema and infima. As in the concrete case, SIPD(Sper A) and PWP(Sper A) are rings and SIPD(Sper A) $\subset$ PWP(Sper A).

Consider a function $f \in$ PWP(Sper A) and fix $\alpha \in$ Sper A. There is $a \in A$ such that $f(\alpha) = a(\alpha)$. As f is semialgebraic, the set $\{\alpha \in \text{Sper } A : \ f(\alpha) = a(\alpha)\}$ is constructible, so that f agrees with $\tilde{a}$ on a $\mathcal{C}$-open set U. By $\mathcal{C}$-compactness, there are finitely many $U_k \subset$ Sper A and $a_k \in A$ such that Sper $A = \cup_{k=1}^n U_k$ and $f = \tilde{a}_k$ on U_k. In the case that $A = \mathbb{R}[x_1, \ldots, x_n]$ this says that f is 'piecewise polynomial' on Sper A and we have derived an abstract analogue to the concept of piecewise polynomial functions on $\mathbb{R}^n$. Consequently, given a pwp function h on $\mathbb{R}^n$, $h = f_i$ on semialgebraic sets A_i, then the map $h \mapsto \tilde{h}$, where $\tilde{h} = \tilde{f}_i$ on A_i, establishes an isomorphism between PWP($\mathbb{R}^n$) and PWP(Sper $\mathbb{R}^n$). The next definition relates SIPD(Sper A) and PWP(Sper A):

Definition 10.2: We call A a Pierce-Birkhoff ring if

$$\text{SIPD}(\text{Sper } A) = \text{PWP}(\text{Sper } A).$$

Then the Pierce-Birkhoff Conjecture translates into the statement that $\mathbb{R}[x_1, \ldots, x_n]$ is a Pierce-Birkhoff ring.

If we go back and look at the proof of the conjecture in Chapter 8, we see that the main idea was to construct functions h_{ij} such that

$$h_{ij} \leq g_{\nu(E_i)} \text{ on } E_i \text{ and } h_{ij} \geq g_{\nu(E_j)} \text{ on } E_j,$$

where $\mathbb{R}^2 = \cup_j E_j$ and $h = g_{\nu(E_j)}$ on E_j. This idea is reflected in the following

Lemma 10.3. *A is a Pierce-Birkhoff ring if and only if for each $s \in PWP(Sper\ A)$ and any two points $\alpha, \beta \in Sper\ A$ there exists $h \in A$ such that $h(\alpha) \geq s(\alpha)$ and $h(\beta) \leq s(\beta)$.*

Proof. $\Rightarrow$: (contrapositive). Suppose we can find $s \in \text{PWP}(\text{Sper } A)$ such that for all $h \in A$, $h(\alpha) \geq s(\alpha)$ implies $h(\beta) > s(\beta)$. Pick finitely many $h_1, \ldots, h_n$ with this property. Since $\inf_j h_j(\alpha) \geq s(\alpha)$, this implies $\inf_j h_j(\beta) > s(\beta)$.— Now suppose that $\sup_i \inf_j h_{ij} = s(\alpha)$ for some $h_{ij} \in A$. If for all i, $\inf_j h_{ij}(\beta) < s(\beta)$, then $\sup_i \inf_j h_{ij}(\beta) < s(\beta)$, thus we can find i_0 such that $\inf_j h_{i_0 j} \geq s(\beta)$. But then, by the above consideration, $\inf_j h_{i_0 j} > s(\beta)$, hence $\sup_i \inf_j h_{ij}(\beta) > s(\beta)$. This means that $s \notin \text{SIPD}(\text{Sper } A)$.
$\Leftarrow$: For any two points $\alpha, \beta \in \text{Sper } A$ pick $h_{\alpha\beta} \in A$ such that

$$h_{\alpha\beta} \geq s(\alpha)$$
$$h_{\alpha\beta} \leq s(\beta).$$

In particular, $h_{\alpha\alpha}(\alpha) = s(\alpha)$. We can find $U(\alpha, \beta)$ and $V(\alpha, \beta)$ such that

$$h_{\alpha\beta}(\gamma) \geq s(\gamma) \qquad \text{for } \gamma \in U(\alpha, \beta)$$
$$h_{\alpha\beta}(\delta) \geq s(\delta) \qquad \text{for } \delta \in V(\alpha, \beta).$$

Fix α. Since Sper $A = \bigcup_{\beta \in Sper A} V(\alpha, \beta)$, by compactness there are $\alpha = \beta_0, \beta_1, \ldots, \beta_n$ such that Sper $A = \cup_{j=0}^{n} V(\alpha, \beta_j)$. Because $h_{\alpha\beta_j}(\delta) \leq s(\delta)$ on $V(\alpha, \beta_j)$, $h_\alpha := \inf_j\{h_{\alpha\beta_j}\} \leq s$ on all of Sper A. Let $U(\alpha) := \cap_{j=0}^{n} U(\alpha, \beta_j)$. Then $h = s$ on $U(\alpha)$ since $U(\alpha) \subset U(\alpha, \alpha)$, $h_{\alpha\alpha} = s$ on $U(\alpha, \alpha)$ and $h_{\alpha\beta_j} \geq s(\alpha)$ on $U(\alpha)$. Again, by compactness, Sper $A = \cup_{i=1}^{m} U(\alpha_i)$, where $h_{\alpha_i} = s$ on $U(\alpha_i)$ and $h_{\alpha_i} \leq s$ elsewhere. Hence, the function $h := \sup_i h_{\alpha_i} \in$ SIPD(Sper A) represents s.

We may interpret a function $h_{\alpha\beta}$ as described in Lemma 10.3 as a "separator of α and β along s", which lead J. Madden to the following

Definition 10.4: For $\alpha, \beta \in$ Sper A let $\langle \alpha, \beta \rangle$ be the ideal in A generated be all elements $a \in A$ such that $a(\alpha) \geq 0$ and $a(\beta) \leq 0$. $\langle \alpha, \beta \rangle$ will be called the *separating ideal* of α and β.

Note that *supp* α + *supp* $\beta \subset \langle \alpha, \beta \rangle$. We have a nice characterization of elements in $\langle \alpha, \beta \rangle$:

Lemma 10.5. *Let $a \in A$, $a(\alpha) \geq 0$. Then $a \in \langle \alpha, \beta \rangle$ if and only if there exists $b \in A$ such that $b(\alpha) \geq a(\alpha)$ and $b(\beta) \leq 0$.*

Thus elements in $\langle \alpha, \beta \rangle$ are 'dominated' by those which actually change sign.

Proof. If $a(\beta) \leq 0$, then $a \in \langle \alpha, \beta \rangle$ by definition and we may take $b = a$. So assume $a(\beta) \geq 0$. If $a \in \langle \alpha, \beta \rangle$ then $a = \sum_{i=1}^{m} f_i h_i$, where $h_i(\alpha) \geq 0$ and $h_i(\beta) \leq 0$. We may assume that $f_i(\alpha) \geq 0$ for all i, for if $f_j(\alpha) \leq 0$ for some j, then $(a - f_j h_j)(\alpha) \geq a(\alpha) \geq 0$and if we find h such that $h(\alpha) \geq (a - f_j h_j)(\alpha)$ and $h(\beta) \leq 0$, h works for a as well. So assume all f_i are positive with respect to α. Define

$$g_i = \begin{cases} f_i \text{ if } f_i(\beta) \geq 0 \\ 1 \ \text{ if } f_i(\beta) < 0 \text{ and } 0 \leq f_i(\alpha) \leq 1 \\ f_i^2 \text{ if } f_i(\beta) < 0 \text{ and } f_i(\alpha) > 1 \end{cases}$$

Then $g_i(\alpha) \geq f_i(\alpha) \geq 0$ and $g_i(\beta) \geq 0$, so that $g_i(\alpha)h_i(\alpha) \geq f_i(\alpha)$ and $g_i(\beta)h_i(\beta) \leq 0$. Hence $h := \sum_{i=1}^m g_i h_i$ has the required properties.
Conversely, assume $h(\alpha) \geq a(\alpha) \geq 0$ and $h(\beta) \leq 0$ for some $h \in A$. Then $h - a, h \in \langle \alpha, \beta \rangle$, thus $a = -(h-a) + h \in \langle \alpha, \beta \rangle$.

Corollary 10.6. $\langle \alpha, \beta \rangle = \alpha \cap (-\beta) + (-\alpha) \cap (\beta)$, *i.e. every element in the separating ideal is the sum of two elements which change sign.*

Proof. "$\supset$": by definition of $\langle \alpha, \beta \rangle$.
"$\subset$": trivial for $a \in \langle \alpha, \beta \rangle$, $a(\alpha) \geq 0, a(\beta) \leq 0$. So assume a is positive on α and β, then we find $h \in \langle \alpha, \beta \rangle$ such that $h(\alpha) \geq a(\alpha) \geq 0$ and $h(\beta) \leq 0$. Thus $a = (a-h) + h$, where $a - h \in -\alpha \cap \beta$, $h \in \alpha \cap (-\beta)$.

Note: If α and β are contained in a common specialization γ, then necessarily $\langle \alpha, \beta \rangle \subset supp\ \gamma$.

Proposition 10.7.

(1) *In $A/supp\ \alpha$ (resp. $A/supp\ \beta$), $\langle \alpha, \beta \rangle / supp\ \alpha$ ($\langle \alpha, \beta \rangle / supp\ \beta$) is convex.*

(2) *Both α and β induce the same total order on $A/\langle \alpha, \beta \rangle$ and $\langle \alpha, \beta \rangle$ is the smallest ideal with this property.*

(3) *If $\langle \alpha, \beta \rangle$ is proper, then $\wp : \sqrt{\langle \alpha, \beta \rangle}$ is prime and α, β induce the same total order on $A/\wp$. Moreover, $A/\wp$ with this order is the least common specialization of α and β in Sper A.*

Proof. (1) For $A/supp\ \alpha$ only: Let $0 < \bar{a} < \bar{b}$, $\bar{b} \in \langle \alpha, \beta \rangle / supp\ \alpha$. There is $\bar{h} \in A/supp\ \alpha$ with $0 < \bar{a} < \bar{b} < \bar{h}$ and $h(\beta) \leq 0$, so by Lemma 10.5, $\bar{a} \in \langle \alpha, \beta \rangle / supp\ \alpha$.
(2) The orderings induced by α and β are total orders because $supp\ \alpha$, $supp\ \beta \subset \langle \alpha, \beta \rangle$; they are the same since everything that changes sign is in $\langle \alpha, \beta \rangle$. Suppose $\bar{\alpha} = \bar{\beta}$ on A/I, where $I \supset supp\ \alpha + supp\ \beta$. Then $\bar{a}(\bar{\alpha}) \geq 0$, $\bar{a}(\bar{\beta}) \leq 0$ implies $\bar{a} = 0$,

hence $\langle\alpha,\beta\rangle \subset I$.

(3) As $\sqrt{\langle\alpha,\beta\rangle}$ is the nilradical of the ring $A/\langle\alpha,\beta\rangle$, it suffices to show that the nilradical of a totally ordered ring is convex and prime. Assume $0 < a < b$ and $b^n = 0$ for some n. Then $0 \le a^2 \le ab$, thus $0 \le a^{2n} \le (ab)^n = 0$, which shows that $\sqrt{0}$ is convex. The same argument shows that $\sqrt{0}$ is prime: suppose $ab \in \sqrt{0}$ and $0 < a \le b$. (This assumption is valid, since we may switch signs to achieve this.) As before, $a^{2n} \in \sqrt{0}$ for some n, hence $a \in \sqrt{0}$. So if we lift the ordering $\bar{\alpha} = \bar{\beta}$ on $A/\sqrt{\langle\alpha,\beta\rangle}$ to an ordering γ on A, we get $\gamma \supset \alpha$, $\gamma \supset \beta$ and *supp* $\gamma = \sqrt{\langle\alpha,\beta\rangle}$. But $\sqrt{\langle\alpha,\beta\rangle}$ is the smallest prime ideal containing $\langle\alpha,\beta\rangle$ and by (2) it is necessary that $\langle\alpha,\beta\rangle$ be contained in *supp* γ, hence γ is the least common specialization of α and β.

Note: (3) implies that $\sqrt{\langle\alpha,\beta\rangle} = A$ if α and β have no common specialization.

Now let $s \in \text{PWP}(\text{Sper } A)$, $\alpha,\beta \in \text{Sper } A$. Denote by s_α any element $a \in A$ with the property $s(\alpha) = a(\alpha)$. Then we can state the following

Theorem 10.8. *(J. Madden, 1989) A is a Pierce-Birkhoff ring if and only if for all $s \in PWP(Sper\ A)$ and all pairs $\alpha,\beta \in Sper\ A$, $s_\alpha - s_\beta \in \langle\alpha,\beta\rangle$.*

Proof. "$\Rightarrow$:" Let $s \in \text{PWP}(\text{Sper } A), \alpha,\beta \in \text{Sper } A$ and $h \in A$ such that $h(\alpha) \ge s(\alpha) = s_\alpha$, $h(\beta) \le s(\beta) = s_\beta$. We may assume that $(s_\alpha - s_\beta)(\alpha) \ge 0$. A little calculation yields $h(\alpha) - s_\beta(\alpha) \ge s_\alpha(\alpha) - s_\beta(\alpha) \ge 0$ and $h(\beta) - s_\beta(\beta) \le 0 \Rightarrow$ $h(\alpha) + s_\beta(\alpha) \ge s_\alpha$ and $h(\beta) + s_\beta(\beta) \le s_\beta(\beta)$.

Comment: Although Theorem 10.8 is a very clear result, it does not seem to make the proof of the Conjecture easier. It basically comes down to the following: given a pwp function h on $\mathbb{R}^n$ such that $h = g_i$ on A_i, $h = g_j$ on A_j, we know that $g_i - g_j$ vanishes outside $(\mathring{A}_i)^c \cap (\mathring{A}_j)^c$, since h is continous. So suppose a polynomial g has zeros outside two open and disjoint semialgebraic sets A and B, one has to show that for any pair (α,β) such that $\alpha \in \tilde{A}$, $\beta \in \tilde{B}$, $g \in \langle\alpha,\beta\rangle$.

Separating ideals themselves are an interesting object to study and in some situations it is possible to interpret them geometrically. To see this, let us start with a

Definition 10.9: Let $A = R[x_1, \dots, x_n]/\wp$, R a real closed field. Fix $\alpha \in$ Sper A, $f, g \in$ Sper A. We call f infinitesimal with respect to g (in symbols $f(\alpha) \ll g(\alpha)$) if for all $r \in R$, $sign\ g(\alpha) + rf(\alpha) = sign\ g(\alpha)$.

Note:

(1) $f \in supp\ \alpha \Rightarrow f(\alpha) \ll g(\alpha)$ (trivial)

(2) $f(\alpha) \ll g(\alpha)$ and $|h(\alpha)| < r$ for some $r \in R \Rightarrow (hf)(\alpha) \ll g(\alpha)$: We have $sign\ g(\alpha) + rsf(\alpha) = sign\ g(\alpha)$ for all $s \in R$ $\Rightarrow sign\ g(\alpha) + sh(\alpha)f(\alpha) = sign\ g(\alpha)$ for all s

(3) $f(\alpha) \ll g(\alpha)$ and $h(\alpha) \ll g(\alpha) \Rightarrow (f+h)(\alpha) \ll g(\alpha)$.

Consider two points α, β in Sper A such that the coordinate functions x_i are bounded in both orderings, i.e. there exist $r_i, s_i \in R$ with $|x_i|(\alpha) \leq r_i$ and $|x_i(\beta)| \leq s_i$ for all i. As A is Noetherian, $\langle \alpha, \beta \rangle =< f_1, \dots, f_m >$ for $f_i \in A$, $f_i(\alpha) \geq 0$ and $f_i(\beta) \leq 0$. Then

$$\langle \alpha, \beta \rangle =< f_1, f_1 + f_2, f_1 + f_2 + f_3, \dots, f_1 + \dots + f_m >:=< g_1, \dots, g_m >$$

and

$$0 \leq g_1(\alpha) \leq \dots \leq g_m(\alpha) \quad 0 \leq -g_1(\beta) \leq \dots \leq -g_m(\beta).$$

If $h \in \langle \alpha, \beta \rangle$, then $h = \sum_{i=1}^m h_i g_i$ for some $h_i \in A$, and since all h_i are bounded, we conclude that

$$(*) \qquad g_m(\alpha) \not\ll h(\alpha) \quad \text{and} \quad g_m(\beta) \not\ll h(\beta),$$

i.e. in both orderings g_m is *not* infinitesimal with respect to h.

Conversely, assume $h \in A$ has property $(*)$. We may also assume that $h(\alpha) \geq 0$. We find $r, s \in R$ such that

(1) $sign\ (h + rg_m)(\alpha) \neq sign\ h(\alpha)$, so $(h + rg_m)(\alpha) \leq 0$ and $r < 0$

(2) $sign\ (h + sg_m)(\beta) \neq sign\ h(\beta)$.

If $h(\beta) \geq 0$, then necessarily $s < 0$, so $h + sg_m \in \langle\alpha, \beta\rangle$ and therefore $h \in \langle\alpha, \beta\rangle$. If $h(\beta) \leq 0$ then $h \in \langle\alpha, \beta\rangle$ by definition of $\langle\alpha, \beta\rangle$. Thus we have proved

Lemma 10.10. *Let $A = R[x_1, \ldots, x_n]/\mathfrak{a}$ and $\alpha, \beta \in$ Sper A be such that all coordinate functions are bounded with respect to α, β. Then there exists $f \in A$ such that*

$$a \in \langle\alpha, \beta\rangle \iff a(\alpha) \not\gg f(\alpha) \text{ and } a(\beta) \not\gg f(\beta).$$

In this respect, f represents the "highest level of positivity" an element in the separating ideal can assume.

Definition 10.11: We call f as in Lemma 10.10 an *indicator* of the separating ideal $\langle\alpha, \beta\rangle$.

Corollary 10.12. *Let A as in Lemma 10.10 and fix $\alpha \in$ Sper A such that α specializes to a point $a \in V_A(R)$ The set*

$$\{\langle\alpha, \beta\rangle : \beta \in \text{Sper } A\}$$

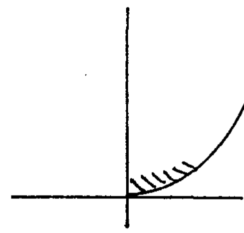
forms a chain under inclusion.

Proof. First note that all coordinate functions are bounded with respect to α, because $f(a) < \infty$ for all $f \in A$. Given $\beta, \gamma \in$ Sper A, we can assume that both specialize to a for otherwise at least one of the ideals $< \alpha, \beta >$, $< \alpha, \gamma >$ is the whole ring and we get the desired inclusion. So we find indicators $f_{\langle\alpha,\beta\rangle}, f_{<\alpha,\gamma>} \in A$ for the respective separating ideals. Without loss of generality assume $f_{\langle\alpha,\beta\rangle}(\alpha) \not\ll f_{<\alpha,\gamma>}(\alpha)$. Then $f_{<\alpha,\gamma>} \in \langle\alpha, \beta\rangle$ and therefore $< \alpha, \gamma > \subset \langle\alpha, \beta\rangle$.

Example: Consider the ordering α as given in chapter V, example 2. Let

$$\beta := \{f \in \mathbb{R}[x, y] : \text{ the set } \{f \geq 0\} \text{ contains a segment of the form}$$
$$\{0 < x < \epsilon, x^m \leq y < g(x)\} \text{ for some } \epsilon > 0, \text{ where } g(x) > 0 \text{ on } (0, \epsilon)\}.$$

β is determined by the curve $y = x^m$:



Claim: $\langle \alpha, \beta \rangle = \langle y, x^m \rangle$. First notice that the functions $y - cx^m$, $0 < c < 1$ change sign, so that $\langle y, x^m \rangle \subset \langle \alpha, \beta \rangle$. For the other inclusion , suppose $f = a_0(x) + y\bar{f}(x,y) \in \langle \alpha, \beta \rangle$, where $\mathrm{ord}_x a_0(x) < m$. We may also assume that f changes sign, say $f(\alpha) > 0$, $f(\beta) < 0$. However, for $s := \mathrm{ord}_x a_0(x)$ $(s > 0)$ we can find $\epsilon_s > 0$ such that the sign of f is constant on the set $\{(x,y) : 0 < x < \epsilon, 0 < y < 2x^{s+1}\}$. Since $f(\alpha) > 0$ and $s + 1 \leq m$, f must be positive on this set (the curve $y = 2x^{s+1}$ lies above the curve $y = x^m$), so $f(\beta) > 0$ and f does not change sign, contradicting our assumption.

Our result implies that

$$\bigcap_{\beta \in \mathrm{Sper}\, \mathbb{R}[x,y]} \langle \alpha, \beta \rangle \subset (y).$$

We are now going to prove that, in fact,

$$\bigcap_{\beta \in \mathrm{Sper}\, \mathbb{R}[x,y]} \langle \alpha, \beta \rangle = (y).$$

We only need to consider orderings β such that α and β have a common specialization, for otherwise $\langle \alpha, \beta \rangle$ will be the whole ring (c.f. the remark after Proposition 10.7).

If the least common specialization of α and β is the ordering associated to the point $(0,0)$, then $\sqrt{\langle \alpha, \beta \rangle} = (x,y)$ and $\langle \alpha, \beta \rangle$ contains a power of its radical. Therefore $x^m \in \langle \alpha, \beta \rangle$ for some m, and since $y(\alpha) \ll x^m(\alpha)$ for all m, $y \in \langle \alpha, \beta \rangle$.

If α and β specialize to the ordering

$$\gamma = \{f \in \mathbb{R}[x,y] : f(x,0) > 0 \text{ for } x > 0 \text{ sufficiently small }\}$$

then β is just the ordering that corresponds to the other side of the x-axis, and y changes sign between α and β. So again $y \in \langle\alpha, \beta\rangle$ and the statement is established.

If we consider the valuation v_α associated to α (cf. chapter 6, example 3), we see that the infinitesimality of a function with respect to another can be expressed in terms of their values:

$$f(\alpha) \ll g(\alpha) \iff v_\alpha(f) > v_\alpha(g).$$

So the separating ideal contains all elements whose value is greater than a prescribed value $\gamma \in \mathbb{Z} \times \mathbb{Z}$. In this respect, $\langle\alpha, \beta\rangle$ can be interpreted as a 'valuation ideal' inside the polynomial ring $\mathbb{R}[x, y]$. This also explains why $y \in \langle\alpha, \beta\rangle$ for all β: $v_\alpha(y) = (1, 0) > v_\alpha(f)$ for any f that has a monomial in x. Given any β such that the least common specialization of α and β is the point $(0, 0)$, the separating ideal $\langle\alpha, \beta\rangle$ will contain such an f, and consequently $y \in \langle\alpha, \beta\rangle$.

So, in general, the intersection

$$\bigcap_{\beta \in \text{Sper } \mathbb{R}[x_1, \ldots, x_n]} \langle\alpha, \beta\rangle,$$

where the intersection is taken over those β which neither generalize nor specialize α, seems to depend on the value group Γ_α associated to v_α, and it would be interesting to know under which conditions

$$\bigcap_{\substack{\beta \in \text{Sper } \mathbb{R}[x_1, \ldots, x_n] \\ \beta \not\subset \alpha \wedge \alpha \not\subset \beta}} \langle\alpha, \beta\rangle = \text{supp } \alpha.$$

In different terms: What conditions do we need to impose on α such that for all f that do not belong to the support of α we can find an ordering β with $f \notin \langle\alpha, \beta\rangle$, i.e. when can we find an ordering β that is closer to α than f, so to speak.

Another way of interpreting separating ideals is the following: suppose $\alpha \in$ Sper $\mathbb{R}[x_1, \ldots, x_n]$ is given by an arc of an algebraic curve $\Gamma_\alpha \subset \mathbb{R}^n$ passing through the origin. α can be described in the following way: $f(\alpha) \geq 0$ iff there is $\epsilon > 0$

such that $f \geq 0$ on $B(\mathbf{0}, \epsilon) \cap \Gamma_\alpha$, where $B(\mathbf{0}, \epsilon)$ is the ball with radius ϵ centered at $\mathbf{0}$. Given $f \in \mathbb{R}[x_1, \dots, x_n]$, we can define the rate of growth of f along Γ_α: for any $r > 0$ we intersect the sphere $S^{n-1}(r)$ of radius r with Γ_α and evaluate f at this point. In this way we obtain a function $f_\alpha : \mathbb{R} \to \mathbb{R}$, $f_\alpha(r) = f(S^{n-1}(r) \cap \Gamma_\alpha)$. Observe that f_α is semialgebraic:

$$\begin{aligned} Graph(f) &= \{(x, y) \in \mathbb{R}^2 : y = f(x), \exists x_1, \dots, x_n : \ \Psi_\alpha(x_1, \dots, x_n) \\ &\qquad \wedge \ x^2 = x_1^2 + \cdots + x_n^2\}. \end{aligned}$$

Here Ψ_α is the formula that determines Γ_α. There exists a neighborhood of $\mathbf{0}$ such that f_α is represented by a convergent Puiseux series $\zeta_f^\alpha = \sum_{\delta \geq 1} a_\delta r^\delta$ and we define the rate of growth $\phi_\alpha(f)$ of f along Γ_α to be the smallest exponent occurring in ζ_f^α, i.e. $\phi_\alpha(f) = ord\ \zeta_f^\alpha$. We see that $f(\alpha) \ll g(\alpha) \iff \phi_\alpha(f) < \phi_\alpha(g)$ and $f(\alpha) > 0 \iff a_{\delta_0} > 0$, $\delta_0 = \phi_\alpha(f)$. Given two algebraic arcs Γ_α, Γ_β that intersect in the origin, we can look at the separating ideal of the attached orders α and β. Let $\langle \alpha, \beta \rangle$ be generated by $f_1, \dots, f_m$ where $f_i(\alpha) \geq 0$ and $f_i(\beta) \leq 0$, $i = 1, \dots, m$. Define $f_{\alpha\beta} := f_1 + \cdots + f_m$ and $c_\alpha = \min\{\phi_\alpha(f_1), \dots, \phi_\alpha(f_m)\}$. Then

$$\phi_\alpha(f_{\alpha\beta}) = c_\alpha = \min\{\phi_\alpha(f) : \ f \in \langle \alpha, \beta \rangle\}.$$

To see this, let $f_{i_1}, \dots, f_{i_r}$ be the generators with $\phi_\alpha(f_{i_j}) = c_\alpha$, $i = 1, \dots, r$, i.e.

$$\zeta_{f_{i_j}}^\alpha = a_{i_j} r^{c_\alpha} + \langle \text{terms of higher orders} \rangle.$$

Since all f_{i_j} are positive with respect to α, all $a_{i_j} > 0$, so $\sum_{j=1}^r a_{i_j} > 0$ and ord $\zeta_f^\alpha = \text{ord} \sum \zeta_{f_{i_j}}^\alpha = c_\alpha$. The second equality holds since for $f = \sum g_i f_i$,

$$\text{ord } \zeta_f^\alpha = \text{ord} \sum \zeta_{g_i}^\alpha \zeta_{f_i}^\alpha \geq \min\{\text{ord } \zeta_{f_i}^\alpha, \ i = 1, \dots, m\} = c_\alpha.$$

It is understood that the addition and multiplication of Puiseux-series is valid only on the intersection of their respective neighborhoods of convergence, i.e. if $f_\alpha(r) = \zeta_f^\alpha$ on U_f, $g_\alpha(r) = \zeta_g^\alpha$ on U_g where U_f, U_g are neighborhoods of $\mathbf{0}$, then $(f_\alpha \dot{+} g_\alpha)(r) = \zeta_f^\alpha \dot{+} \zeta_g^\alpha$ on $U_f \cap U_g$.

Also, $\phi_\alpha(f) \geq c_\alpha$ implies that $f \in \langle \alpha, \beta \rangle$, for in this case we will find $\lambda > 0$ such that $(\lambda f_{\alpha\beta} - f)(\alpha) \geq 0$, and since $\lambda f_{\alpha\beta}(\beta) \leq 0$, it follows from Lemma 10.5 that $f \in \langle \alpha, \beta \rangle$.

The same observations hold for $c_\beta := \min\{\phi_\beta(f_1), \ldots, \phi_\beta(f_m)\}$ and we conclude that

$$f \in \langle \alpha, \beta \rangle \iff \phi_\alpha(f) \geq c_\alpha \quad \text{and} \quad \phi_\beta(f) \geq c_\beta.$$

Thus in this situation infinitesimality can be described in terms of the rate of growth along the curves Γ_α and Γ_β and a function f is in the separating ideal $\langle \alpha, \beta \rangle$ if and only if its rate of growth along these two curves does not exceed a certjain value (note that, since α and β are centered at $\mathbf{0}$, things go the other way: a lower order in the Puiseux expansion corresponds to a larger rate of growth).

References

[1] M.F. Atiyah, I.G. MacDonald, *Introduction to Commutative Algebra*, Addison Wesley Publishing Company, 1969.

[2] E. Becker, *On the real spectrum of a ring and its applications to semialgebraic geometry*, Bull. Am. Math. Soc. **19** (1986), 19–60.

[3] J. Bochnak, M. Coste, M.-F. Roy, *Géométrie algébrique réelle*, Ergebnisse der Mathematik und ihre Grenzgebiete, 3.Folge, Band 12, Springer Verlag 1987.

[4] G.W. Brumfiel, *Partially Ordered Rings and Semialgebraic Geometry*, Cambridge University Press, 1979.

[5] P.J. Cohen, *Decomposition Procedures for Real and P-Adic Fields*, Commun. Pure and Applied Math **22** (1969), 131 – 151.

[6] M. Coste, *Ensembles semi-algébriques*, Lecture Notes in Mathematics **959** (1981), 109–136.

[7] H. Delfs, *The homotopy axiom in semialgebraic cohomology*, J. reine und angew. Math. **355** (1985), 108–128.

[8] C. Delzell, *On the Pierce-Birkhoff conjecture over ordered fields*, Rocky Mountain J. Math. **19** (1989), 651–668.

[7] M. Henriksen and J. Isbell, *Lattice-ordered rings and function rings*, Pacific J. Math. **11** (1962), 533–566.

[9] K. Jänich, *Topologie*, 2. Auflage, Springer Verlag, 1987.

[11] M. Knebusch, C. Scheiderer, *Einführung in die reelle Algebra*, Vieweg Verlag, Braunschweig, 1989.

[12] T.Y. Lam, *An introduction to real algebra*, Rocky Mountain J. Math. **14** (1984), 767–814.

[13] T.Y. Lam, *Orderings, valuations and quadratic forms*, CBMS Regional Conf. Ser. in Math., Am. Math. Soc. **52** (1983).

[14] J. Madden, *Pierce-Birkhoff rings*, Arch. Math. **52** (1989), 565–570.

[15] K. Meyberg, *Algebra Teil 2*, Carl Hanser Verlag, München, 1976.

[16] N. Schwartz, *The basic theory of real closed spaces*, Memoirs of the Am. Math. Soc. **397** (1989).