

AN ABSTRACT OF THE THESIS OF

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OF VISCOELASTIC FLUIDS BETWEEN ROTATING SPHERES

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By various experiments, it has been found that the response of real materials to external forces is, in general, nonlinear in character. In classical continuum mechanics, the use of ordinary measures of strain have forced the constitutive equations to take complex forms and since the orders of these measures are not fixed, many unknown response coefficients have to be introduced into the constitutive equations. In general, there is no basis of choosing these coefficients. Seth attempted to resolve this difficulty by introducing generalized measures in continuum mechanics and Narasimhan and Sra extended these measures in such a way as to adequately explain some rheological behavior of materials. The constitutive equation of Narasimhan and Sra essentially contains two terms and four rheological constants and, unlike some previous theories, it does not

contain any unknown functions of the invariants of kinematic matrices while at the same time explains many viscoelastic phenomena.

In the present investigation, a theorem has been proved establishing certain criteria for fixing the orders of generalized measures suitably so as to predict different types of viscoelastic phenomena, such as dilatancy. We have found during the course of this investigation that the constitutive equation of Narasimhan and Sra does not adequately explain such physical phenomena as pseudoplasticity. However, in order to construct a constitutive equation so that it does explain such phenomena, we have found it necessary to construct combinations of sets of generalized measures. The resulting constitutive equation is found to be quite general and is able to explain a vast range of physical behavior of fluids.

To illustrate the use of this constitutive equation based upon combined generalized measures, we have investigated the important problem of secondary flows for fluids in the presence of moving boundaries. This problem is very important, since the investigation of secondary flows allows us to obtain a clearer picture of the actual motion of the fluid. For the problem of flow of a fluid in the annulus of two rotating spheres, we have obtained the solu-

tion for the velocity and pressure fields. In order to investigate the secondary flow pattern more thoroughly, we have obtained the streamline function of the flow. The streamlines in meridian planes containing the axis of rotation are found to be closed loops and the nature of these closed loops is found to be strongly dependent upon the viscoelastic parameter S . For S less than critical value, the flow is found to be very much like that of a Newtonian fluid, with the fluid advancing toward the inner sphere along the pole and outward along the equator. At the critical value the flow region is found to split into two subregions, each containing closed loops of streamlines. As S is increased further, another critical value is reached whereby the streamlines again become one set of closed loops, with the sense of rotation reversed from that of Newtonian fluid.

Generalized Deformation-Rates in Secondary Flows
of Viscoelastic Fluids Between Rotating Spheres

by

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Notation

We will be using the notation of tensor analysis and hence shall assume some knowledge of tensors. We shall explain the meaning of the various symbols used as and when they arise. However, we shall list the more frequently used symbols which may be referred to whenever needed:

<u>Symbol</u>	<u>Meaning</u>
a_i	i^{th} component of acceleration.
a, b, c	physical components of acceleration in spherical-polar coordinates along the r^- , θ^- , and ϕ^- directions respectively.
A, B, C	non-dimensional components of acceleration in spherical-polar coordinates along the r^- , θ^- , and ϕ^- directions respectively.
α_1, α_2	dimension correcting constants.
b_{ij}	second deformation-rate tensor.

<u>Symbol</u>	<u>Meaning</u>
b^*_{ij}	generalized second deformation-rate tensor.
\tilde{B}	second deformation-rate matrix.
\tilde{B}^*	generalized second deformation-rate matrix
\tilde{B}^{**}	combined generalized second deformation-rate matrix
β	radius of inner sphere
d_{ij}	first deformation-rate tensor
d^*_{ij}	generalized first deformation-rate tensor
\tilde{D}	first deformation-rate matrix
\tilde{D}^*	generalized first deformation-rate matrix
\tilde{D}^{**}	combined generalized first deformation-rate matrix
δ_{ij}	kronecker delta
Δ	Laplacian in spherical-polar coordinates
Δ^*	non-dimensional Laplacian in spherical-polar coordinates
$\frac{D}{Dt}$	material derivative
$\frac{\delta}{\delta t}$	intrinsic derivative
\tilde{e}	strain tensor
e_{ij}	strain tensor

<u>Symbol</u>	<u>Meaning</u>
e^*_{ij}	generalized strain tensor
\tilde{E}	strain matrix
\tilde{E}^*	generalized strain matrix
η	dimension correcting constant
γ	dimension correcting constant
h	non-dimensional rheological parameter
H	coefficient determinant
\tilde{I}	unit matrix
I_B, II_B, III_B	first, second, and third invariants of the second deformation-rate matrix
I_D, II_D, III_D	first, second, and third invariants of the first deformation-rate matrix
I_e, II_e, III_e	first, second, and third invariants of the strain tensor
J, J_0, J_e	set of positive integers, positive odd integers, positive even integers respectively
k, k_1, k_2, k'	dimension correcting constants
K	non-dimensional rheological parameter
l, l_0	deformed and undeformed lengths respectively
L	length
λ	ratio of outer sphere to inner sphere
$m, m_1, m_2, m', \underline{m}$	dimension correcting constant
M	mass
μ	dimension correcting constant

<u>Symbol</u>	<u>Meaning</u>
n, n_1, n_2, n'	measure indices
ν_a	apparent viscosity
Ω_1, Ω_2	angular velocities of the inner and out sphere respectively
p	hydrostatic pressure
P	non-dimensional hydrostatic pressure
q, q_1, q_2, q'	irreversibility indices
r, θ, ϕ	spherical-polar coordinate system
ρ	density of fluid
S	non-dimensional rheological parameter
t	time
t_{ij}	stress tensor
t'_{ij}	deviatoric part of stress tensor
$t_{(xy)}, t_{(yz)}, t_{(xz)},$ etc.	physical components of the stress tensor in the rectangular coordinate system
$t_{(rr)}, t_{(r\theta)}, t_{(r\phi)}$ etc.	physical components of the stress tensor in the spherical-polar coordinate system
\tilde{T}	stress matrix
T	time
u, v, w	physical components of velocity in the spherical-polar coordinate system
U, V, W	non-dimensional components of velocity in the spherical-polar coordinate system

SymbolMeaning

v_i	i^{th} component of velocity
$w_1(x)$	velocity component in the rectangular coordinate system in z-direction
x^i	spatial coordinates
x, y, z	cartesian coordinates

GENERALIZED DEFORMATION-RATES IN SECONDARY FLOWS
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CHAPTER I

INTRODUCTION

1.1 Classical Strain Measures

The fundamental conservation laws of continuum mechanics are applicable to all types of media. But different materials with the same mass and geometry when subjected to the same external forces in an identical manner are found to respond differently. This diversity in behavior is owing to the differences in their internal constitution. Therefore, to explain the response of a material when a force is applied, we need to establish a relationship between the loading to which the material is subjected and its deformation or motion. For a viscous medium we must find a relationship between the stresses and the rate of deformation, while for an elastic medium we must find a relationship between the stresses and the resulting deformation. This leads to the formulation of the constitutive equation of the material, which is a relation between the stress tensor and the deformation or the rate of deformation tensor. It should be observed that the measure of stress is a definite quantity, that is, the force per unit area; however, the measure of strain or deformation is quite flexible, as it should be, to allow for different

types of deformations and flows.

In elasticity theory, various measures of strain have been used. In an uniaxial strain of a loaded wire, if ℓ_0 and ℓ represent the undeformed and the deformed lengths of the wire respectively, then the various strain measures already in use in elasticity are (Seth, 1966b)

$$\text{Cauchy measure: } e^C = \frac{\ell - \ell_0}{\ell_0}, \quad (1.1)$$

$$\text{Swainger measure: } e^S = \frac{\ell - \ell_0}{\ell}, \quad (1.2)$$

$$\text{Hencky measure: } e^H = \log\left(\frac{\ell}{\ell_0}\right), \quad (1.3)$$

$$\text{Almansi measure: } e^A = \frac{1}{2}\left[1 - \left(\frac{\ell_0}{\ell}\right)^2\right], \quad (1.4)$$

$$\text{Green measure: } e^G = \frac{1}{2}\left[\left(\frac{\ell}{\ell_0}\right)^2 - 1\right]. \quad (1.5)$$

In the classical theory of elasticity, the Cauchy measure is used when the strain is small and only the linear terms of displacement gradients referred to the unstrained state are retained; the Swainger measure uses linear displacement gradients, but referred to the strained state; the Hencky measure is useful in plasticity; and the Almansi and Green measures are used in the theory of finite deformations.

In fluid dynamics, the rate of strain must be used and the classical measure of the deformation-rate is

$$d_{ij} = \frac{1}{2}(v_{i;j} + v_{j;i}), \quad (1.6)$$

where

$$v_i \equiv i^{\text{th}} \text{ component of velocity,}$$

$_{;i}$ and $_{;j}$ denote covariant partial differentiation with respect to x^i and x^j , respectively.

1.2 Present Constitutive Theory of Materials and Their Limitations

We now briefly mention some of the commonly used constitutive equations in current literature and point out their limitations.

Newtonian Fluid

$$\tilde{T} = (-p + \lambda_1 \theta) \tilde{I} + 2\mu \tilde{D} \quad (1.7)$$

where

$$\begin{aligned} \tilde{T} &\equiv \text{stress matrix,} \\ \mu, \lambda_1 &\equiv \text{coefficients of viscosity,} \\ &\quad \text{constant at given temperature,} \\ \tilde{D} &\equiv \text{deformation-rate matrix,} \\ \theta &\equiv \text{dilatation,} \\ P &\equiv \text{hydrostatic pressure.} \end{aligned}$$

Even though equation (1.7) describes very well the behavior of some fluids, such as certain gases, water, alcohol, and so on, many materials such as high polymer solutions, pastes, paints, colloidal solutions, paper pulp, etc. exhibit certain phenomena which cannot be described by the classical Newtonian stress-strain velocity relation (1.7). Examples of such phenomena are: the swelling of a fluid at the exit of a tube, known as the Merrington effect (Merrington, 1943), and the climbing of a fluid along the inner cylinder of a viscometer under rotation, known as the Weissenberg effect (Weissenberg, 1947), both very common in technological and biophysical experiments; variable viscosity, visco-elasticity, plasticity, creep, fatigue, time dependency, pseudo-plasticity, dilatancy, relaxation, etc. which abound in everyday experiences, as well as in industry. Secondary flows are worthy of special mention (for more details on secondary flows, see Chapter IV). In order to describe these phenomena, numerous investigators have posed various constitutive equations, of which the main ones are:

Reiner (1945) and Rivlin (1948) Fluids

$$\underline{T} = -p\underline{I} + \alpha_1 \underline{D} + \alpha_2 \underline{D}^2, \quad (1.8)$$

where

$\tilde{T} \equiv$ stress matrix,
 $\alpha_1 \equiv$ coefficient of viscosity,
 $\tilde{D} \equiv$ deformation-rate matrix,
 $\alpha_2 \equiv$ coefficient of cross-viscosity,
 $p \equiv$ hydrostatic pressure.

This equation appears mathematically simpler than others, but the rheological coefficients α_1 and α_2 are unknown functions of the invariants of the deformation-rate tensor and the theory offers no means of specifying them explicitly. This theory always predicts the existence of two equal normal stresses in certain steady viscometric flows, which is contrary to experiments performed by Padder and Dewitt (1954).

Oldroyd Fluids (1950)

$$\begin{aligned}
 (1 + \lambda_1 \frac{\partial}{\partial t}) t'_{ij} - 2k_1 (d_{im} t'_{jm} + d_{jm} t'_{im}) \\
 = 2\mu (1 + \lambda_2 \frac{\partial}{\partial t}) d_{ij} - 8\mu k_2 d_{im} d_{jm}^m, \quad (1.9)
 \end{aligned}$$

where

$\lambda_1 \equiv$ relaxation-time constant,

$\frac{\partial}{\partial t} \equiv$ intrinsic derivative,

$$\frac{\partial A_{ij}}{\partial t} \equiv \frac{\partial A_{ij}}{\partial t} + A_{ij;m} v^m + A_{mj;i} v^m + A_{im;j} v^m,$$

A_{ij} being any given tensor,
 $t'_{ij} \equiv$ deviatoric part of the stress tensor,
 k_1 and $k_2 \equiv$ arbitrary scalar constants,
 $d_{ij} \equiv$ deformation-rate tensor,
 $\mu \equiv$ coefficient of viscosity,
 $\lambda_2 \equiv$ retardation-time constant.

Oldroyd's theory involves a number of material constants and the non-linearity has been introduced in a very arbitrary manner.

Rivlin and Ericksen Fluids (1955)

$$\underline{\underline{T}} = a_0 \underline{\underline{I}} + \sum_{k=1}^N a_k (\pi_k + \pi_k^*), \quad (1.10)$$

where

a_k ($k = 0, 1, \dots, N$) are unknown functions of the invariants of the kinematic matrices $\underline{\underline{D}}^{(1)}, \underline{\underline{D}}^{(2)}, \dots, \underline{\underline{D}}^{(n)}$;

π_k are certain matrix products formed from the kinematic matrices $\underline{\underline{D}}^{(1)}, \underline{\underline{D}}^{(2)}, \dots, \underline{\underline{D}}^{(n)}$;

π_k^* is the transpose of π_k ;

the kinematic matrix $\underline{\underline{D}}^{(r)} \equiv \|\| d_{ij}^{(r)} \|\|$ is defined

by $d_{ij}^{(1)} \equiv \frac{1}{2}(v_{i;j} + v_{j;i})$ and

$$d_{ij}^{(r)} \equiv \frac{\partial (d_{ij}^{(r-1)})}{\partial t} + v^m d_{ij;m}^{(r-1)} + d_{im}^{(r-1)} v^m_{;j} + d_{jm}^{(r-1)} v^m_{;i}$$

$$(r \geq 2).$$

Rivlin and Ericksen's theory is successful in obtaining normal stresses which need not be equal, but the resulting constitutive equation has been made very complicated by the introduction of higher order kinematic matrices and unknown functions of their invariants.

Green and Rivlin Fluids (1957)

$$t_{ij} = \sum_{N=0}^R \int_{-\infty}^t \cdots \int_{-\infty}^t \phi_{ijp_1q_1 \cdots p_Nq_N}(t, \tau_1, \tau_2, \dots, \tau_N)$$

$$\times g_{p_1q_1}(\tau_1) \cdots g_{p_Nq_N}(\tau_N) d\tau_1 \cdots d\tau_N, \quad (1.11)$$

where

the kernels $\phi_{ij \cdots p_N q_N}$ are continuous functions of the indicated arguments and also of $D_{pq}^{(0)}$, $D_{pq}^{(1)}$, \dots , $D_{pq}^{(v)}$, defined in Rivlin and Ericksen's theory,

$$g_{pq}(\tau) \equiv \chi_{k,p}(\tau) \chi_{k,q}(\tau),$$

$$t_{ij} \equiv \text{stress tensor.}$$

Green and Rivlin's theory is a further generalization of

Rivlin and Ericksen's theory, hence the remarks on the limitations of Rivlin and Ericksen's theory apply here as well. Equations (1.11) become quite unwieldy except possibly in a few simple cases.

Noll Fluids (1958)

$$\underline{T}' = \int_{s=0}^{\infty} (G(s)), \quad (1.12)$$

where

\mathcal{J} is the constitutive functional and
 $G(s)$ is the history of the relative deformation gradient.

Noll's theory, somewhat similar in idea to the theory of Green and Rivlin, depends upon the experimental determination of the viscosity function and two normal stress functions.

The source of all the complications in the constitutive equations mentioned above is the use of ordinary measures. Often the linear measures of strain or strain-rate are used even though we know from experiments that the strains and strain-rates are nonlinear in character. Since the order of these measures is not fixed, there is no basis of choosing the response coefficients in these constitutive equations.

In order to overcome this difficulty, Seth (1966a, 1966b) introduced generalized measures of strain into elasticity. He also suggested the use of generalized measures of deformation-rate in viscoelasticity. But Narasimhan and Sra (1968) showed that the mere use of generalized measures of deformation-rate always predicts two of the normal stresses to be equal in certain viscometric flows. Since the viscoelastic behavior depends not only on the deformation-rate involving velocity gradients but also on a second deformation-rate involving acceleration gradients, they suggested the use of generalized measures of a second deformation-rate along with the generalized measure of the first deformation-rate to overcome the above difficulty. They illustrated this point of view by constructing a constitutive equation for dilatant fluids.

1.3 Purpose and Need for the Present Investigation

In the present investigation, we find that the mere use of just one set of generalized measures of deformation-rates is inadequate to describe phenomena such as pseudoplasticity, creep, relaxation, and so on. It will be shown in the text of this thesis that it is necessary to combine generalized measures of different orders to explain a variety of viscoelastic or viscoplastic phenomena. In order to illustrate the need for combining such generalized mea-

asures, suitable constitutive equations based on the measures are constructed for pseudoplastic fluids, which are characterized by the property that their apparent coefficient of viscosity decreases with the increase in rate of shear. The problem of secondary flows of such fluids between two rotating concentric spheres is formulated and solved to illustrate the pseudoplastic behavior.

1.4 Plan of the Present Investigation

The work is divided into six chapters. In Chapter I, we give the present constitutive theory of materials and their limitations. Chapter II, deals with the present state of the measure concept in continuum mechanics. Chapter III is devoted to classifying viscoelastic fluid based on orders of measures and developing a constitutive equation based on combined generalized measures. In Chapter IV, we discuss primary and secondary flows in the presence of moving boundaries. In Chapter V, we use the constitutive equation based on combined generalized measures to illustrate the secondary flows in the annulus of two spheres rotating about a common axis. Chapter VI contains the summary, discussion and scope of further work.

CHAPTER II

MEASURE CONCEPT IN CONTINUUM MECHANICS

2.1 Introduction

In recent years, many research workers have devoted themselves to the construction of suitable constitutive equations in order to explain the rheological behavior of materials. These constitutive equations involve many non-linear terms as well as unknown response coefficients. There are two types of non-linearities which are commonly introduced. One may be termed as parametric non-linearity involving a variable rheological parameter in the constitutive equation and the other is the well known tensorial non-linearity involving a non-linear tensorial relationship between stress and strain or strain-rate. Both of these types of non-linearities have led to the introduction of many unknown functions and response coefficients in the constitutive equations. The source of all this trouble is the use of ordinary measures of strain or strain-rate in the constitutive equations. Any such restriction placed on the measure of strain will naturally result in complicating the constitutive equation since the order of the measure used is not fixed in relation to the actual physical phenomena. In order to overcome this difficulty, Seth (1966a, 1966b) introduced generalized measures of strain

and explained a number of irreversible phenomena in continuum mechanics.

2.2 Generalized Measures of Deformation in Elasticity

In section 1.1, the various classical strain measures used in elasticity theory were listed. Seth (1966b) served that these strain measures (1.1)-(1.5) could be given Riemann integral representations with suitable weight functions. These are

$$e^C = \int_{\ell_0}^{\ell} \frac{d\ell}{\ell_0}, \quad (2.1)$$

$$e^S = \int_{\ell_0}^{\ell} \left(\frac{\ell_0}{\ell}\right)^2 \frac{d\ell}{\ell_0}, \quad (2.2)$$

$$e^H = \int_{\ell_0}^{\ell} \left(\frac{\ell_0}{\ell}\right) \frac{d\ell}{\ell_0}, \quad (2.3)$$

$$e^A = \int_{\ell_0}^{\ell} \left(\frac{\ell_0}{\ell}\right)^3 \frac{d\ell}{\ell_0}, \quad (2.4)$$

$$e^G = \int_{\ell_0}^{\ell} \left(\frac{\ell_0}{\ell}\right)^{-1} \frac{d\ell}{\ell_0}. \quad (2.5)$$

Upon examination of these integral representations, Seth noted the following generalizations: If e^* represents a general strain measure, then

$$e^* = \frac{1}{n} \left[1 - \left(\frac{\ell_0}{\ell}\right)^n \right] = \int_{\ell_0}^{\ell} \left(\frac{\ell_0}{\ell}\right)^{n+1} \frac{d\ell}{\ell_0}, \quad (2.6)$$

with

$$\left(\frac{\ell_0}{\ell}\right)^{n+1}, \text{ as the weight function.} \quad (2.7)$$

Note that, for $n = -1, 1, 0, 2, -2$ in (2.6) we obtain (2.1)-(2.5) respectively. If e_{ii}^C , e_{ii}^S , e_{ii}^A , and e_{ii}^G (no sum on i) are the principal Cauchy, Swainger, Almansi, and Green measures respectively, then the generalized principal measures e_{ii}^* (no sum on i) in Cartesian coordinates may be written as

$$e_{ii}^* = \int_0^{e_{ii}^C} (1 + e_{ii}^C)^{-n-1} de_{ii}^C \quad (2.8)$$

$$= -\frac{1}{n}[(e_{ii}^C + 1)^{-n-1}],$$

$$e_{ii}^* = \int_0^{e_{ii}^S} (1 - e_{ii}^S)^{n-1} de_{ii}^S \quad (2.9)$$

$$= \frac{1}{n}[1 - (1 - e_{ii}^S)^n],$$

$$e_{ii}^* = \int_0^{e_{ii}^A} (1 - 2e_{ii}^A)^{\frac{n}{2} - 1} de_{ii}^A \quad (2.10)$$

$$= \frac{1}{n}[1 - (1 - 2e_{ii}^A)^{\frac{n}{2}}],$$

$$\begin{aligned}
 e_{ii}^* &= \int_0^{e_{ii}^G} (1 + 2e_{ii}^G)^{\frac{-n}{2}} - 1 de_{ii}^G \quad (2.11) \\
 &= -\frac{1}{n} [(1 + 2e_{ii}^G)^{\frac{-n}{2}} - 1],
 \end{aligned}$$

where

n may be called the measure index.

A further generalization is (Seth, 1966b)

$$e_{ii}^* = \frac{k}{(mn)^q} \left[1 - (1 - 2me_{ii})^{\frac{n}{2}} \right]^q, \quad (2.12)$$

where k and m are dimension correcting constants and n and q are the measure and irreversibility indices respectively.

If n is a positive even integer and q a positive integer, then by the use of the Cayley-Hamilton theorem all powers of e_{ii} higher than the second can be expressed in terms of e_{ii} and e_{ii}^2 , obtaining

$$e_{ii}^* = F_0 + F_1 e_{ii} + F_2 e_{ii}^2, \quad (2.13)$$

where F_i ($i = 0, 1, 2$) contain only a finite number of terms in the invariants I_e , II_e , III_e of the strain tensor \underline{e} . By a rigid rotation of the frame of

reference, (2.13) becomes

$$e_{ij}^* = F_0 \delta_{ij} + F_1 e_{ij} + F_2 e_{im} e_{mj}, \quad (2.14)$$

where F_i ($i = 0, 1, 2$) remain unchanged since I_e , II_e , III_e are invariant under rigid body rotations. For general values of n and q , F_i ($i = 0, 1, 2$) are infinite series.

Likewise, the generalized measure (2.12) can also be written as

$$e_{ij}^* = \frac{k}{m^n q^n} \left[\delta_{ij} - (\delta_{ij} - 2m e_{ij})^{\frac{n}{2}} \right]^q \quad (2.15)$$

or in matrix form

$$\tilde{E}^* = \frac{k}{m^n q^n} \left[\tilde{I} - (\tilde{I} - 2m \tilde{E})^{\frac{n}{2}} \right]^q, \quad (2.16)$$

where

$$\tilde{I} = \|\delta_{ij}\|$$

and

$$\tilde{E} = \|e_{ij}\|.$$

The constitutive equation based on the generalized strain measure \tilde{E}^* can be constructed from the classical Hookean constitutive equation

$$\underline{\underline{T}} = \lambda' \theta \underline{\underline{I}} + 2\eta \underline{\underline{E}}, \quad (2.17)$$

where

$$\underline{\underline{T}} \equiv \text{stress matrix}$$

and

$$\lambda', \eta \equiv \text{Lame's constants,}$$

by replacing $\underline{\underline{E}}$ by $\underline{\underline{E}}^*$ to obtain

$$\underline{\underline{T}} = \lambda \theta \underline{\underline{I}} + 2\eta \underline{\underline{E}}^*, \quad (2.18)$$

or from (2.14) we obtain

$$\underline{\underline{T}}' = F'_0 \underline{\underline{I}} + F'_1 \underline{\underline{E}} + F'_2 \underline{\underline{E}}^2 \quad (2.19)$$

where

$$F'_i (i = 0, 1, 2)$$

are known functions of the invariants of $\underline{\underline{E}}$. This constitutive relation (2.19) has been successfully used by Seth in explaining a number of phenomena in non-linear elasticity.

2.3 Generalized Measures of Deformation-rate in Fluid Dynamics

Using the concept of section 2.2, Seth (1966b) generalized the measure of deformation-rate appropriate to fluid

dynamics to obtain

$$\begin{aligned} d_{ii}^* &= \int_0^{d_{ii}} (1-2md_{ii})^{\frac{n}{2}-1} d(d_{ii}), \\ &= \frac{1}{mn} [1 - (1-2md_{ii})^{\frac{n}{2}}], \end{aligned} \quad (2.20)$$

where m is a dimension correcting constant and

$d_{ij} = \frac{1}{2}[v_{i;j} + v_{j;i}]$. A further generalization (Seth, 1966b) would be

$$d_{ii}^* = \frac{k}{m^n n^q} \left[1 - (1-2md_{ii})^{\frac{n}{2}} \right]^q \quad (2.21)$$

where k is again a dimension correcting constant and n and q are the measure and irreversibility indices, respectively. Following section 2.2, by use of the Cayley-Hamilton theorem, (2.21) can be expressed as

$$d_{ii}^* = G_0 + G_1 d_{ii} + G_2 d_{ii}^2, \quad (2.22)$$

where $G_i (i = 0, 1, 2)$ contain only a finite number of terms in the invariants I_D , II_D , III_D of the deformation-rate tensor \underline{D} if n is an even positive integer and q a positive integer. By a rigid rotation of the frame of reference, (2.22) becomes

$$d_{ij}^* = G_0 \delta_{ij} + G_1 d_{ij} + G_2 d_{im} d_{mj}, \quad (2.23)$$

where G_i ($i = 0, 1, 2$) remain unchanged since I_D , II_D , and III_D are invariant under rigid body rotations.

Likewise, the generalized measures can also be written as (Narasimhan and Sra, 1968)

$$\tilde{D}^* = \frac{k}{m \tilde{q}_n \tilde{q}} \left[\tilde{I} - (\tilde{I} - 2m\tilde{D})^{\frac{n}{2}} \right] \tilde{q} \quad (2.24)$$

where

$$\tilde{D} = \|d_{ij}\|,$$

$$\tilde{I} = \|\delta_{ij}\|.$$

The constitutive equation based on the generalized deformation-rate measure \tilde{D}^* can be constructed from the classical Newtonian constitutive equation by replacing \tilde{D} by \tilde{D}^* to obtain for incompressible fluids (Narasimhan and Sra, 1968)

$$\tilde{T} = -P\tilde{I} + 2\mu\tilde{D}^* \quad (2.25)$$

or from (2.23)

$$\tilde{T} = (-P + G'_0)\tilde{I} + G'_1\tilde{D} + G'_2\tilde{D}^2 \quad (2.26)$$

where

$$\tilde{T} \equiv \text{stress tensor,}$$

G_i ($i = 0, 1, 2$) are known functions of the invariants of \tilde{D} .

Hence the need for assuming unknown response functions in constitutive equations is eliminated and the non-linearity is condensed into just two terms.

2.4 Generalized Measures of Deformation-Rates in Viscoelasticity

In certain viscometric flows of viscoelastic fluids, equations (2.25) and (2.26) always predicted two normal stresses, contrary to experiments (Narasimhan and Sra, 1968). Since the viscoelastic behavior depends not only on the deformation-rate involving velocity gradients but also on a second deformation-rate involving acceleration gradients, Narasimhan and Sra (1968) suggested the use of generalized measures of a second deformation-rate along with that of the first deformation-rate to remedy the above difficulty. It should be noted here that the deformation-rate matrix \underline{D} of (1.6) shall henceforth be called the first deformation-rate matrix and \underline{B} shall be called the second deformation-rate matrix. The generalization of the measure of the second deformation-rate, following the same procedure as for the first deformation-rate, led to the equations

$$\underline{B}^* = \frac{k'}{m'q' n'q'} \left[\underline{I} - (\underline{I} - 2m'\underline{B}) \frac{n'}{2} \right] q', \quad (2.27)$$

where

$\tilde{B}^* \equiv ||b_{ij}^*|| \equiv$ generalized second deformation-rate matrix,

$$\tilde{B}^* \equiv ||b_{ij}||, \text{ with } b_{ij} = \frac{1}{2}(a_{i;j} + a_{j;i} + v_{m;i} v_{;j}^m) \quad (2.28)$$

and

$$a_i = \frac{Dv_i}{Dt}, \text{ } i^{\text{th}} \text{ component of acceleration,}$$

k' and $m' \equiv$ dimension correcting constants,

n' and $q' \equiv$ measure and irreversibility indices, respectively,

and

$$\tilde{B}^* = H_0 \tilde{I} + H_1 \tilde{B} + H_2 \tilde{B}^2, \quad (2.29)$$

where $H_i (i = 0, 1, 2)$ are known functions of the invariants I_B, II_B, III_B of the second-deformation-rate tensor \tilde{B} .

The constitutive equation based on the generalized deformation-rate tensors \tilde{D}^* and \tilde{B}^* can be constructed from the classical Newtonian constitutive equation for incompressible fluids $\tilde{T} = -p\tilde{I} + 2\mu\tilde{D}$ by replacing \tilde{D} by the generalized deformation-rate tensors \tilde{D}^* and \tilde{B}^* to obtain (Narasimhan and Sra, 1968)

$$\tilde{T} = -p\tilde{I} + 2\mu\tilde{D}^* + 4\eta\tilde{B}^* \quad (2.30)$$

or

$$\tilde{T} = (-p + G_0 + H_0)\tilde{I} + G_1\tilde{D} + G_2\tilde{D}^2 + H_1\tilde{B} + H_2\tilde{B}^2 \quad (2.31)$$

where G_i ($i = 0,1,2$) are known functions of the invariants I_D , II_D , III_D , and H_i ($i = 0,1,2$) are known functions of the invariants I_B , II_B , III_B .

Narasimhan and Sra (1968) observed that the use of generalized measures of deformation-rates in viscoelastic fluids helps to simplify the constitutive equations by prescribing a priori the order of the measures of deformation-rates, to eliminate the need for assuming additional response coefficients, and to explain adequately the rheological phenomena usually observed in experiments.

CHAPTER III

EXTENSION OF GENERALIZED MEASURE CONCEPT

3.1 Introduction

The constitutive equation (2.25), using only the generalized measure of the first deformation-rate tensor, always predicts two of the normal stresses to be equal in certain viscometric flows, contrary to experiments. Narasimhan and Sra observed that viscoelastic behavior depends not only on the deformation-rate involving velocity gradients but also upon deformation-rate involving acceleration gradients. Hence, to eliminate the deficiency of equation (2.25), they suggested the use of generalized measures of the second deformation-rate along with those of the first deformation-rate, resulting in equation (2.30). By using the generalized measures of deformation-rates in viscoelasticity, a constitutive equation is constructed which contains known response coefficients and is adequate in explaining most rheological phenomena. However, we observe that the mere use of the generalized measure mentioned above is inadequate to describe such well known phenomena as pseudoplasticity, creep, stress-relaxation, and so on. In order to explain the above mentioned phenomena, it is found necessary to construct combinations of sets of generalized measures of different orders instead

of just using one set of generalized measures. In this chapter, we extend the generalized measure concept by combining two sets of generalized measures of different orders and construct a constitutive equation which will adequately describe pseudoplasticity.

3.2 A Theorem on Classification of Viscoelastic Fluids Based on Orders of Measures

Using the constitutive equation (2.30) by Narasimhan and Sra (1968), we shall classify viscoelastic fluids by specifying the measure and irreversibility indices n, n' and q, q' respectively in equations (2.24) and (2.27). But first, we must define precisely what is meant by a fluid exhibiting pseudoplastic, Newtonian, or dilatant behavior.

Definition 3.1. The apparent coefficient of viscosity ν_a of a fluid is defined to be the ratio of the shear stress to the rate of shear.

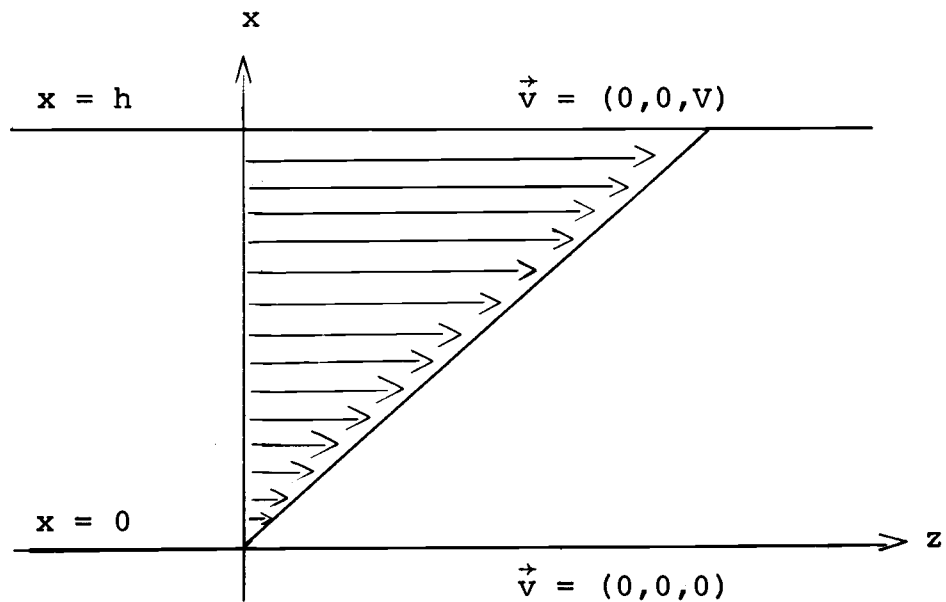
The apparent coefficient of viscosity is defined thus in order to compare it with the classical viscosity coefficient and determine the behavior of a viscoelastic fluid in relation to the classical viscous fluid. For a real fluid, the apparent coefficient of viscosity should be positive.

Definition 3.2. Fluids for which the apparent coefficient of viscosity decreases with increase in rate of shear are defined to be pseudoplastic fluids, while those for which it increases with increase in rate of shear are defined to be dilatant fluids.

Fluids which satisfy equation (1.7) are classified as Newtonian fluids and it should be noted that the coefficients of viscosity for these fluids are constant for given temperature. It is interesting to note that, theoretically at least, it should be possible by choosing a suitable combination of orders of measures to obtain a fluid for which the apparent coefficient of viscosity turns out to be a constant, quite analogous to the situation in the classical viscous case but the fluid is not described by the Newtonian constitutive equation (1.7).

The behavior of apparent coefficient of viscosity can be studied by considering a shear model in which a flow is maintained between two parallel plates, one of which ($x = h$) is moving relative to the other ($x = 0$) with velocity V parallel to the z -axis (see figure 3.1). Since the channel is taken to be of infinite extent in the y -direction, all quantities will be independent of y . Assume the pressure gradient to be zero. Then from the equation of continuity, it follows that if $\vec{v} = \vec{v}(u, v, w)$,

$$\frac{\partial w_1}{\partial z} = 0 \quad \text{or} \quad w_1 = f(x). \quad (3.1)$$



(Figure 3.1) Simple Shear Model

In this case the first and second deformation-rate tensors as given by equations (1.6) and (2.28) are

$$\underline{D} = \left\| d_{ij} \right\| = \begin{vmatrix} 0 & 0 & \frac{1}{2}w_1'(x) \\ 0 & 0 & 0 \\ \frac{1}{2}w_1'(x) & 0 & 0 \end{vmatrix} \quad \text{and}$$

$$\underline{B} = \left\| b_{ij} \right\| = \begin{vmatrix} [w_1'(x)]^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

where the prime indicates differentiation with respect to x .

Then it follows if p' is a positive integer that

$$\tilde{D}^{2p'-1} = \begin{vmatrix} 0 & 0 & [\frac{1}{2}w'_1(x)]^{2p'-1} \\ 0 & 0 & 0 \\ [\frac{1}{2}w'_1(x)]^{2p'-1} & 0 & 0 \end{vmatrix}, \quad (3.2)$$

$$\tilde{D}^{2p'} = \begin{vmatrix} [\frac{1}{2}w'_1(x)]^{2p'} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & [\frac{1}{2}w'_1(x)]^{2p'} \end{vmatrix}, \quad (3.3)$$

and

$$\tilde{B}^{p'} = \begin{vmatrix} [w'_1(x)]^{2p'} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}. \quad (3.4)$$

Definition 3.3 Define $J = \{x: x \text{ is a positive integer}\}$,

$J_0 = \{x: x \text{ is a positive odd integer}\}$,

and

$J_E = \{x: x \text{ is a positive even integer}\}$.

Theorem Suppose the measure indices $n, n' \in J_E$ and the irreversibility indices $q, q' \in J$ in the equations (2.24) and (2.27) and further

- (i) if $n = 2$ or 4 and $q = 1$, then the fluid described by (2.30) is, in general, non-Newtonian

in nature but exhibits a constant apparent coefficient of viscosity.

- (ii) If $n = 2$, $q > 2$ with $q \in J_0$, or if $n > 4$, $q = 1$, or if $n > 2$, $q > 1$ with $q \in J$, then the fluid described by (2.30) either exhibits dilatant behavior or does not exist because the apparent coefficient of viscosity is non-positive.
- (iii) If $n = 2$, $q \in J_E$, then the fluid described by (2.30) does not exist because the apparent coefficient of viscosity is non-positive.

Proof Since we have required n and $n' \in J_E$, let $n = 2r$ and $n' = 2s$ with r and $s \in J$. Then the constitutive equation (2.30) becomes

$$\underline{\underline{T}} = -p\underline{\underline{I}} + 2\mu\underline{\underline{D}}^* + 4\eta\underline{\underline{B}}^*, \quad (3.5)$$

where

$$\underline{\underline{D}}^* = \frac{k}{(2rm)^q} \left[\underline{\underline{I}} - (\underline{\underline{I}} - 2m\underline{\underline{D}})^r \right]^q \quad (3.6)$$

and

$$\underline{\underline{B}}^* = \frac{k'}{(2sm')^{q'}} \left[\underline{\underline{I}} - (\underline{\underline{I}} - 2m'\underline{\underline{B}})^s \right]^{q'}. \quad (3.7)$$

By use of the binomial expansion, (3.6) and (3.7) become

$$\underline{\underline{D}}^* = \frac{k}{(-2rm)^q} \left[\sum_{\ell=1}^r \binom{r}{\ell} (-2m\underline{\underline{D}})^\ell \right]^q \quad (3.8)$$

and

$$\tilde{B}^* = \frac{k'}{(-2sm')^q} \left[\sum_{\ell=1}^s \binom{s}{\ell} (-2m'B)^\ell \right]^q. \quad (3.9)$$

Since \tilde{B}^ℓ contributes only to the normal stress effects in the simple shear model (see 3.4), then B^* contributes nothing to the shear stress and hence nothing to the apparent coefficient of viscosity. Thus the only term in the constitutive equation (3.5) which contributes to the shear stress is \tilde{D}^* and hence \tilde{D}^* is the only term we shall examine. Since m in equation (3.6) is an arbitrary rheological parameter which can take on either positive, negative, or zero values, then for convenience, we replace m by $-\underline{m}$ in (3.8) to obtain

$$\tilde{D}^* = \frac{k}{(2r\underline{m})^q} \left[\sum_{\ell=1}^r \binom{r}{\ell} (2\underline{m}D)^\ell \right]^q. \quad (3.10)$$

Case (i): If $n = 2$ ($r = 1$) and $q = 1$, then (3.10) becomes

$$\tilde{D}^* = k \tilde{D} \quad (3.11)$$

and equation (3.5) becomes

$$\tilde{T} = -p\tilde{I} + 2\alpha\tilde{D} + 4\eta\tilde{B}^* \quad (3.12)$$

where

$$\alpha \equiv \mu k.$$

For $m' = 0$ or $k' = 0$ in B^* , equation (3.12) reduces to the classical Newtonian equation $\underline{T} = -p\underline{I} + 2\alpha\underline{D}$ for incompressible fluids. For $m' \neq 0$ and $k' \neq 0$ in B^* , the shear stress components are

$$t_{(xy)} = 0 = t_{(yz)} \quad \text{and} \quad t_{(xz)} = \alpha\omega'(x) \quad (3.15)$$

and the apparent coefficient of viscosity becomes

$$v_a = \frac{\text{shear stress}}{\text{shear rate}} = \frac{t_{xz}}{\omega'(x)} = \alpha, \text{ a constant.} \quad (3.14)$$

Again, if $n = 4(r=2)$ and $q = 1$, then (3.10) becomes

$$\underline{D}^* = k\underline{D} + k\underline{m}\underline{D}^2 \quad (3.15)$$

and equation (3.5) becomes

$$\underline{T} = -p\underline{I} + 2\alpha\underline{D} + 2\alpha\underline{m}\underline{D}^2 + 4\eta\underline{B}^*. \quad (3.16)$$

The shear stress components are

$$t_{(xy)} = 0 = t_{(yz)} \quad \text{and} \quad t_{(xz)} = \alpha\omega'(x) \quad (3.17)$$

and the apparent coefficient of viscosity becomes

$$v_a = \alpha, \text{ a constant,} \quad (3.18)$$

which proves case (i).

Case (ii): If $n = 2(r=1)$ and $q > 2$ with $q \in J_0$, then (3.10) becomes

$$\underline{D}^* = k \underline{D}^{2\ell+1}, \quad (3.19)$$

where $q = 2\ell + 1$ and $\ell \in J$. Equation (3.5) becomes

$$\underline{T} = -p\underline{I} + 2\alpha\underline{D}^{2\ell+1} + 4\eta\underline{B}^*, \quad (3.20)$$

the shear stress components are

$$t_{(xy)} = 0 = t_{(yz)} \quad \text{and} \quad t_{(xz)} = \frac{\alpha}{2^{2\ell}} \left[w'(x) \right]^{2\ell+1} \quad (3.21)$$

and the apparent coefficient of viscosity is

$$v_a = \alpha \left[\frac{w'(x)}{2} \right]^{2\ell}, \quad (3.22)$$

an increasing function of the rate of shear $w'(x)$ since $\alpha > 0$ for a real fluid in view of (3.22).

If $n > 4(r > 2)$ and $q = 1$, then (3.10) becomes

$$\underline{D}^* = \frac{k}{2\underline{m}r} \sum_{\ell=1}^r \binom{r}{\ell} (2\underline{m}\underline{D})^\ell \quad (3.23)$$

and (3.5) becomes

$$\underline{T} = -p\underline{I} + \frac{\alpha}{\underline{m}r} \sum_{\ell=1}^r \binom{r}{\ell} (2\underline{m}\underline{D})^\ell + 4\eta\underline{B}^*. \quad (3.24)$$

Note that only the odd powers of \underline{D} contribute to the

shear stress, with the even powers contributing to the normal stress. The shear components are

$$t_{(xy)} = 0 = t_{(yz)} \quad \text{and} \quad t_{(xz)} = \frac{\alpha}{r} \sum_{i=0}^R \binom{r}{2i+1} \underline{m}^{2i} [w'(x)]^{2i+1}, \quad (3.25)$$

where

$$R = \begin{cases} (r-1)/2 & \text{if } r \in J_0, \\ (r-2)/2 & \text{if } r \in J_E, \end{cases}$$

and the apparent coefficient of viscosity is

$$v_a = \frac{\alpha}{r} \sum_{i=0}^R \binom{r}{2i+1} [\underline{m} w'(x)]^{2i}, \quad (3.26)$$

a monotonically increasing function of shear rate since $\alpha > 0$, for otherwise if $\alpha < 0$, $v_a < 0$ which is impossible for a real fluid.

If $n > 2$ ($r > 1$) and $q > 1$ with $q \in J$, then using (3.10), equation (3.5) becomes

$$\underline{T} = -p\underline{I} + \frac{2\alpha}{(2mr)^q} \left[\sum_{\ell=1}^r \binom{r}{\ell} (2m\underline{D})^\ell \right]^q + 4\eta\underline{B}^*. \quad (3.27)$$

Since only the odd powers of \underline{D} contribute to the shear stress, the shear stress components become

$$t_{xy} = 0 = t_{yz} \quad \text{and} \quad t_{xz} = \frac{2\alpha}{(2r)^q} \sum_{i=Q}^R b_{2i+1} m^{2i+1-q} [w'(x)]^{2i+1} \quad (3.28)$$

where b_{2i+1} is the appropriate coefficient and is positive for each i since it consists only of combination of $\binom{n}{\ell}$,

$$Q \equiv \begin{cases} (q-1)/2 & \text{if } q \in J_0, \\ q/2 & \text{if } q \in J_E, \end{cases}$$

and

$$R = \begin{cases} (rq-1)/2 & \text{if } rq \in J_0, \\ (rq-2)/2 & \text{if } rq \in J_E. \end{cases}$$

The apparent coefficient of viscosity becomes

$$v_a = \frac{2\alpha}{(2r)^q} \sum_{i=Q}^R b_{2i+1} m^{2i+1-q} [w'(x)]^{2i}. \quad (3.29)$$

We must now examine the coefficient of each $[w'(x)]^{2i}$ to see its nature, whether negative or positive, since it determines the nature of the apparent coefficient of viscosity. That is, we must examine $\frac{2\alpha m^{2i+1-q}}{(2r)^q}$ since $2b_{2i+1}/(2r)^q > 0$ for all i .

(a)

For $\alpha > 0$, $\underline{m} > 0$, and $q \in J$ with $q > 1$, $\alpha \underline{m}^{2i+1-q} \sim (+)(+) > 0$;

for $\alpha < 0$, $\underline{m} < 0$, and $q \in J_E$, $\alpha \underline{m}^{2i+1-q} \sim (-)(-) > 0$;

for $\alpha > 0$, $\underline{m} < 0$, and $q \in J_0$ with $q > 1$, $\alpha \underline{m}^{2i+1-q} \sim (+)(+) > 0$;

then v_a is a monotonically increasing function of shear-rate.

(b)

For $\alpha < 0$, $\underline{m} < 0$, and $q \in J_0$, $\alpha \underline{m}^{2i+1-q} \sim (-)(+) < 0$;

for $\alpha > 0$, $\underline{m} < 0$, and $q \in J_E$, $\alpha \underline{m}^{2i+1-q} \sim (+)(-) < 0$;

for $\alpha < 0$, $\underline{m} > 0$, and $q \in J$, $\alpha \underline{m}^{2i+1-q} \sim (-)(+) < 0$;

then v_a is negative.

Here the case (b) is impossible for a real fluid. Thus case (ii) is proved.

Case (iii): If $n = 2(r=1)$ and $q \in J_E$, then (3.10) becomes

$$\underline{D}^* = k \underline{D}^{2\ell} \quad (3.30)$$

where $q = 2\ell$ and $\ell \in J$. Then equation (3.5) becomes

$$\underline{T} = -p\underline{I} + 2\alpha \underline{D}^{2\ell} + 4\eta \underline{B}^* \quad (3.31)$$

and the shear stress components are

$$t_{(xy)} = t_{(yz)} = t_{(xz)} = 0 \quad (3.32)$$

which implies that the apparent coefficient of viscosity is zero. Hence the proof of the theorem.

3.3 Combined Generalized Measures in Viscoelasticity

As we have seen in the theorem proved in the last article, the study of the behavior of the apparent coefficient of viscosity provides useful information regarding non-Newtonian behavior of materials. We also find that the mere use of generalized measures of the deformation-rates in the constitutive equation is not adequate to explain non-Newtonian phenomena such as pseudoplasticity. In order to describe such phenomena, it is found necessary to combine generalized deformation-rate measures of different orders. In this article, a suitable constitutive equation is constructed based on the combined generalized measures in order to explain pseudoplastic behavior of fluids.

Since the generalized measures of the rates of deformation are involved, one can combine these generalized measures of different orders to obtain a suitable constitutive equation based on the combined measures. In an actual physical situation involving creeping motion or pseudoplastic flow, these orders of measures will be determined by experiments. By choosing the orders of the generalized measures $(m_r \times n_r \times k_r \times q_r)$ for the first deformation-rate \dot{D} and $(m'_r \times n'_r \times k'_r \times q'_r)$ for the second deforma-

tion-rate \underline{B} , $r = 1$ and 2 , and combining these resulting sets of generalized measures, we obtain

$$\underline{D}^{**} = \underline{D}_1^* + \underline{D}_2^* \quad (3.33)$$

where

$$\underline{D}_1^* = \frac{k_1}{q_1 n_1} \left[\underline{I} - (\underline{I} - 2m_1 \underline{D})^{\frac{n_1}{2}} \right]^{q_1} \quad (3.34)$$

and

$$\underline{D}_2^* = \frac{k_2}{q_2 n_2} \left[\underline{I} - (\underline{I} - 2m_2 \underline{D})^{\frac{n_2}{2}} \right]^{q_2} \quad (3.35)$$

and

$$\underline{B}^{**} = \underline{B}_1^* + \underline{B}_2^* \quad (3.36)$$

where

$$\underline{B}_1^* = \frac{k'_1}{m'_1 q_1 n'_1} \left[\underline{I} - (\underline{I} - 2m'_1 \underline{B})^{\frac{n'_1}{2}} \right]^{q'_1} \quad (3.37)$$

and

$$\underline{B}_2^* = \frac{k'_2}{m'_2 q'_2 n'_2} \left[\underline{I} - (\underline{I} - 2m'_2 \underline{B})^{\frac{n'_2}{2}} \right]^{q'_2} . \quad (3.38)$$

In the next article we obtain constitutive equations based on the combined generalized measures.

3.4 Constitutive Equation Based on Generalized Measures

The constitutive equation based on the combined generalized measures \underline{D}^{**} and \underline{B}^{**} can be constructed from the constitutive equation (2.30) (Narasimhan and Sra, 1968) by replacing \underline{D}^* by \underline{D}^{**} and \underline{B}^* by \underline{B}^{**} to obtain for incompressible fluids

$$\underline{T} = -p\underline{I} + 2\mu\underline{D}^{**} + 4\eta\underline{B}^{**} \quad (3.39)$$

where \underline{D}^{**} and \underline{B}^{**} are given by (3.33) and (3.36) respectively.

This constitutive equation (3.39) is very general and includes many of the previously constructed constitutive equations as special cases. By choosing the orders of the measures $(m_r \times n_r \times k_r, q_r)$ for \underline{D}^{**} and $(m'_r \times n'_r \times k'_r \times q'_r)$ for \underline{B}^{**} , with $r = 1, 2$, we obtain the following constitutive equations:

If $(m_1 \times 2 \times k_1 \times 1)$, $(m_2 \times n_2 \times 0 \times q_2)$, $(m'_r \times n'_r \times 0 \times q'_r)$ with $r = 1$ and 2 , then equation (3.39) becomes

$$\underline{T} = -p\underline{I} + 2\alpha\underline{D},$$

where $\alpha = \mu k_1$, the classical Newtonian equation for incompressible fluids.

If $(m_1 \times 4 \times k_1 \times 1)$, $(m_2 \times n_2 \times 0 \times \sigma_2)$, and $(m'_r \times n'_r \times 0 \times \sigma'_r)$ with $r = 1$ and 2 , then equation (3.39) becomes

$$\tilde{T} = -p\tilde{I} + 2\alpha_1\tilde{D} + 2\alpha_2\tilde{D}^2,$$

where $\alpha_1 = \mu k_1$ and $\alpha_2 = \mu m_1 k_1$, a special case of the Reiner and Rivlin equation (1.8) for incompressible fluids.

If $(m_1 \times n_1 \times k_1 \times \sigma_1)$, $(m_2 \times n_2 \times 0 \times q_2)$ and $(m'_1 \times n'_1 \times k'_1 \times \sigma'_1)$, $(m'_2 \times n'_2 \times 0 \times \sigma'_2)$, then equation (3.39) becomes equation (2.30), due to Marasimhan and Sra (1968).

If $(m_1 \times 2 \times k_1 \times 1)$, $(m_2 \times 2 \times k_2 \times 3)$ and $(m'_1 \times 2 \times k'_1 \times 1)$, $(m'_2 \times n'_2 \times 0 \times \sigma'_2)$, then equation (3.39) becomes

$$\tilde{T} = -p\tilde{I} + 2\alpha_1\tilde{D} - 2\alpha_2\tilde{D}^3 + \gamma\tilde{B}, \quad (3.40)$$

where

$$\alpha_1 = \mu k_1,$$

$$\alpha_2 = -\mu k_2$$

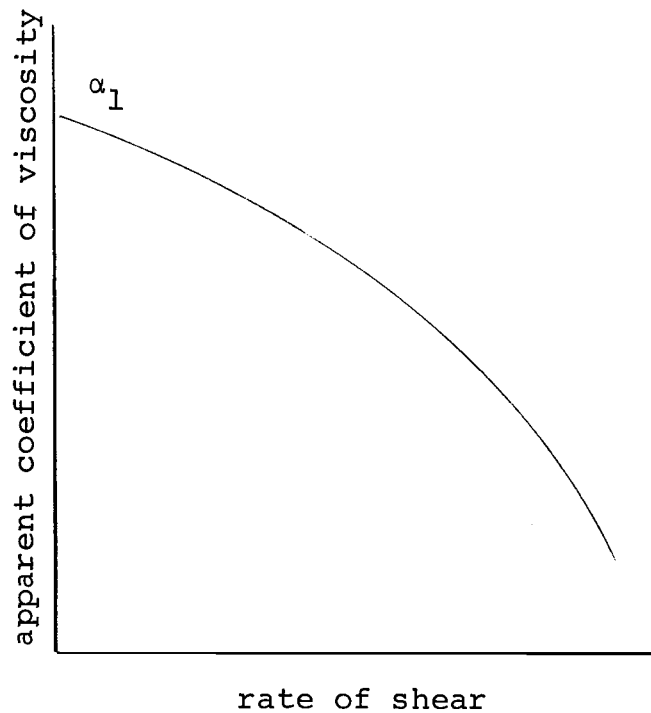
and

$$\frac{\nu}{\gamma} = 4\eta k'.$$

Using the simple shear model of section 3.2, the apparent coefficient of viscosity ν_a is

$$\nu_a = \alpha_1 - \frac{\alpha_2}{4}[w'(x)]^2.$$

A graph of the apparent coefficient of viscosity vs. rate of shear is shown in figure (3.2). Since the apparent viscosity is a decreasing function of the shear rate, the fluid is called pseudoplastic.



(Figure 3.2) Graph of apparent coefficient of viscosity for a pseudoplastic fluid.

CHAPTER IV
PRIMARY AND SECONDARY FLOWS IN THE PRESENCE
OF MOVING BOUNDARIES

4.1 Introduction

In the study of non-Newtonian fluids, it is convenient to break the flow into two categories, primary flow and secondary flow.

Definition 4.1 The primary flow is defined to be the motion of the fluid which results by considering only the flow contributed by the inclusion of the Newtonian coefficient of viscosity, and neglecting the contribution from the inertial and non-Newtonian terms.

Definition 4.2 The secondary flow is defined to be the motion due to the contribution from the inclusion of the inertial and non-Newtonian terms which is super-imposed upon the primary motion.

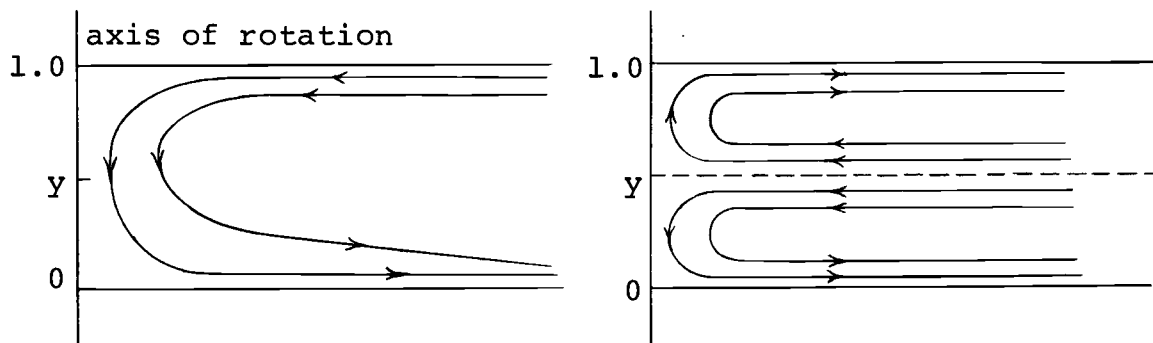
Hence, the inertial effects and non-Newtonian effects, regarded small in comparison to the effect of the viscosity term, are considered only as secondary effects. By neglecting the secondary flow in fluid flow problems, especially in problems dealing with moving boundaries, a valid picture of the actual flow of the fluid is not obtained. Thus, the study of secondary flow allows a more thorough

understanding of the behavior of non-Newtonian fluids. Also, by investigating the effects due to the non-Newtonian terms, that is, secondary flows, one obtains a means of evaluating the rheological parameters in the constitutive equation by relating the experimental results obtained by using some non-Newtonian fluid with the corresponding theoretical results.

4.2 Secondary Flows in Classical Fluids (Theoretical Results)

Rosenblat, S. (1960) investigated the flow of a classical viscous fluid between two parallel infinite plates which oscillate torsionally about a common axis. Two cases were of specific interest: (i) only one disk is in motion, and (ii) both disks oscillate with the same frequency and amplitude, but 180° out of phase. Solutions for the equations were obtained for both small and large Reynolds numbers. For one disk oscillating, the fluid is drawn inwards along the axis towards the moving disk and thrown radially outward near the oscillating disk (see figure 4.1). For both disks oscillating 180° out of phase with the same amplitude and frequency, the flow field consists of two regions symmetric about the midplane, with no flow across it. This plane acts as an interface separating the fluid into two self-contained portions, in each of which

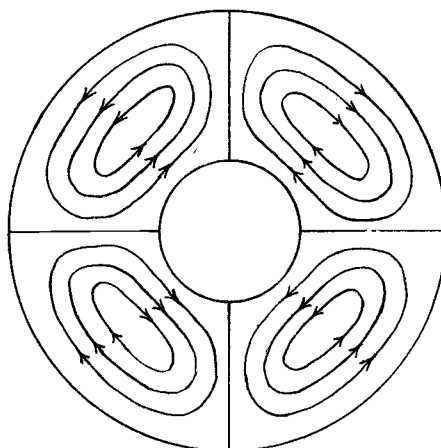
there is axial flow towards the moving disk (see figure 4.2).



(Figure 4.1) One disk oscillating ($y=0$): typical streamlines of steady radial-axial flow.

(Figure 4.2) Two disks oscillating: typical streamlines of steady radial-axial flow.

Haberman, W.L. (1962) considered the problem of secondary flow of a classical Newtonian fluid for a sphere rotating inside a coaxially rotating spherical container. The equations of motion were made linear and the results showed that the secondary flow does not appear to contribute to the turning moment required to maintain rotation of each sphere. The flow field consists of four regions each of which contains streamlines that are closed loops (see figure 4.3).



(Figure 4.3)

Streamlines of secondary flow
about a sphere rotating in a
fixed spherical container.

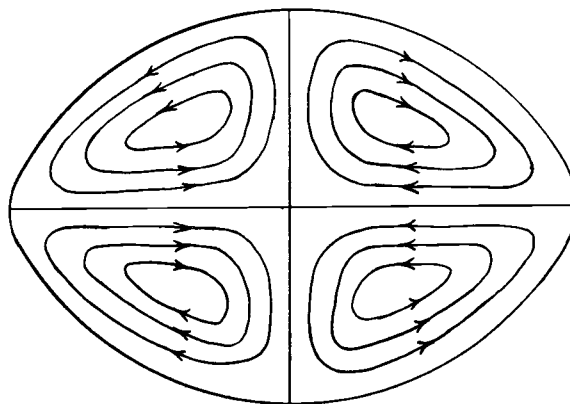
4.3 Secondary Flows in Non-Newtonian Fluids (Theoretical Results)

Green, A.E. and R.S. Rivlin (1956) investigated the flow of the fluid described by the constitutive equation

$$\underline{\underline{T}} = -p\underline{\underline{I}} + \alpha_1(\text{II}_{\underline{\underline{D}}}, \text{III}_{\underline{\underline{D}}})\underline{\underline{D}} + \alpha_2(\text{II}_{\underline{\underline{D}}}, \text{III}_{\underline{\underline{D}}})\underline{\underline{D}}^2$$

in a non-circular tube. They discovered that it is not possible to have a rectilinear flow in a non-circular tube without an appropriate body force distribution in addition to a uniform pressure gradient along the tube. They observed that for a tube with an elliptical cross-section, without the application of body forces, a steady-state flow

was produced consisting of a primary flow (rectilinear flow) plus a flow (secondary flow) super-imposed in planes perpendicular to the axis of the tube (see figure 4.4).



(Figure 4.4)

Secondary flow in a tube
with elliptical cross-section

Other investigators of the problem of flow through a non-circular tube are Ericksen (1956), Stone (1957), and Criminale, Ericksen, and Filbey (1958).

Jones, J.R. (1960) examined the flow of a fluid described by the constitutive equation

$$\underline{\underline{T}} = -p\underline{\underline{I}} + \alpha_1\underline{\underline{D}} + \alpha_2\underline{\underline{D}}^2$$

in a curved pipe with circular cross-section. In the analysis, it was assumed that the curvature of the pipe was small, that is, the radius of the circle in which the line of the pipe is curved is large in comparison with the

radius of the cross-section of the pipe. The method of solution was by successive approximations, taking the zeroth order approximation to be the Newtonian flow. It was found that to the order of the approximation considered, the relation between the axial pressure gradient and the rate of out-flow is the same as if the pipe were straight. However, a secondary flow was present which affected the disposition of the streamlines.

Bhatnagar, P.L. and G.K. Rajeswari (1962) investigated the flow of the non-Newtonian fluid described by the constitutive equation

$$\underline{T} = -p\underline{I} + 2\alpha_1\underline{D} + 4\alpha_2\underline{D}^2 + 2\alpha_3\underline{B}$$

between two parallel infinite plates which oscillate torsionally about a common axis. Like Rosenblat (1960), they investigated two specific cases: (i) only one disk in motion, and (ii) both disks oscillate with the same frequency and amplitude, but 180° out of phase. Also, solutions for the equations were obtained for both small and large Reynolds numbers. For small values of the non-Newtonian parameters, the results showed that the fluid behaves like Newtonian fluid and possesses radial-axial flows similar to the one predicted by Rosenblat (1960). However, it was found that there exists critical values of the non-Newtonian parameters, depending upon the Reynolds number above

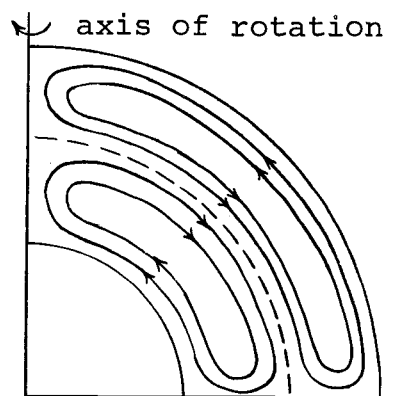
which the flow corresponding to the steady part reverses in sense. Also, for fixed values of the non-Newtonian parameters, the authors observed that a value of the Reynolds can be found for which the flow field divides into subregions having distinct flow characteristics.

The same authors in a later paper (Bhatnagar, P.L. and G.K Rajeswari, 1963) considered the problem of secondary flow in the annulus between two concentric spheres, rotating about a common axis. Again, they used the non-Newtonian fluid described by the constitutive equation

$$\underline{\underline{T}} = -p\underline{\underline{I}} + 2\alpha_1\underline{\underline{D}} + 4\alpha_2\underline{\underline{D}}^2 + 2\alpha_3\underline{\underline{B}}.$$

The primary motion, obtained by just considering the viscosity term, consists of the streamlines which are concentric circles with centers on the axis of rotation. In the case of the viscosity term with inclusion of inertia terms, the secondary flow was observed to be the same as described by Haberman (1962). With the inclusion of the non-Newtonian parameters, it was observed that there exist critical values, depending upon the Reynolds number, in which the flow field changes its nature. For values of these parameters less than the critical values, the secondary flow resembles the secondary flow in the case of Newtonian fluid with the inclusion of inertia effects. However, when the

values of these parameters are chosen to be greater than the critical values, it was observed that the quadrants are further subdivided into sub-regions by a streamline running between vertical and horizontal radii. In each of these sub-regions the streamlines form closed loops and the sense of the flow in these sub-regions is opposite (see figure 4.5). Also, it was found, after the critical values have been reached, the interface between the two regions moves towards the outer sphere as the non-Newtonian parameters are increased and for a definite value of the parameters, the outer part vanishes and each quadrant is filled up with closed loops as prior to the breaking, but of opposite sense.



(Figure 4.5)

The secondary flow pattern in one quadrant of an annulus of two spheres for fluid whose parameters have passed the critical values.

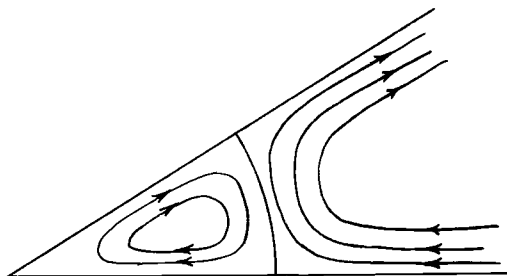
Other investigators working on the problem of flow in the annulus of two rotating spheres are Langlois, W.E. (1962) and Waters, N.D. (1964).

Bhatnagar, P.L. and S.L. Rathna (1963) examined the secondary flow between two rotating coaxial cones using the non-Newtonian fluid described by the constitutive equation

$$\underline{\underline{T}} = -p\underline{\underline{I}} + 2\alpha_1\underline{\underline{D}} + 4\alpha_2\underline{\underline{D}}^2 + 2\alpha_3\underline{\underline{B}}.$$

Starting with the rectilinear motion of a Newtonian fluid in which the streamlines are circles with centers on the common axis of the cones, the equations were built up for studying the modification of the flow by taking into account the inertia terms according to Oseen's scheme and regarding the effects of the non-Newtonian terms as small. With the inclusion of the inertia terms in Oseen's approximation, they observed that a secondary flow occurs in which the fluid particles move away from the vertex almost parallel to the surfaces of the inner cone, which is rotating faster than the outer one and the particles near the outer surface move parallel to it towards the vertex. When the non-Newtonian terms were included, it was found that the flow field divided into two domains. In the domain nearer the vertex, the streamlines of the secondary flow formed closed loops, and outside this domain, the fluid particles moved nearly parallel to the generators of the cones to

which they were closest, moving toward the vertex near one cone and away near the other (see figure 4.6)

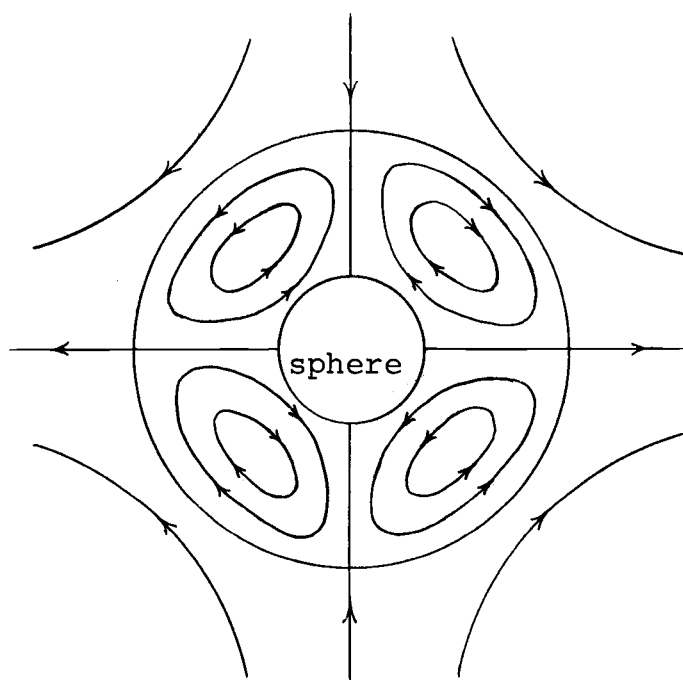


(Figure 4.6)

The flow pattern of a viscoelastic fluid between two rotating cones.

Thomas, R.H. and K. Walters (1964) considered the flow of an elastico-viscous liquid designated as liquid B', due to a sphere immersed in it and rotating about a vertical diameter. The method of solution is by successive approximations of a certain parameter. The primary flow is the zeroth order approximation and is the same as for a Newtonian fluid. The results show that to first order approximation, the motion of fluid B' due to the rotating sphere is strongly dependent on a parameter, say m , described in the problem. It was found that there exist three different cases depending upon the values of m , which are: (i) $0 \leq m \leq \frac{1}{12}$, (ii) $\frac{1}{12} < m < \frac{1}{4}$, and (iii) $m \geq \frac{1}{4}$. For case (i), the flow pattern for fluid B' is similar to that for a purely viscous liquid ($m=0$); the liquid re-

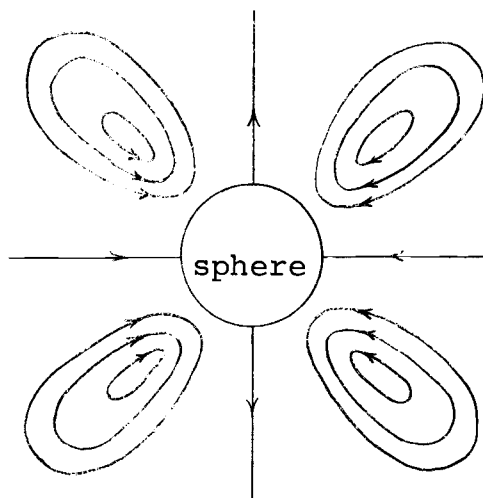
cedes from the sphere at the equator and approaches it again at the poles. For case (ii), the flow field is divided by a streamline in the form of a sphere. Inside the sphere, the streamlines are closed curves and the liquid recedes from the sphere at the poles and approaches it again at the equator. Outside this sphere, the curves are open and the liquid recedes from the sphere at the equator and approaches it again at the poles (see figure 4.7).



(Figure 4.7)

The projections of the particle paths on any plane containing the axis of rotation when $m = 1/6$.

For case (iii), the streamlines are closed curves and their directions are such that the liquid approaches the sphere at the equator and recedes from it at the poles (see figure 4.8).



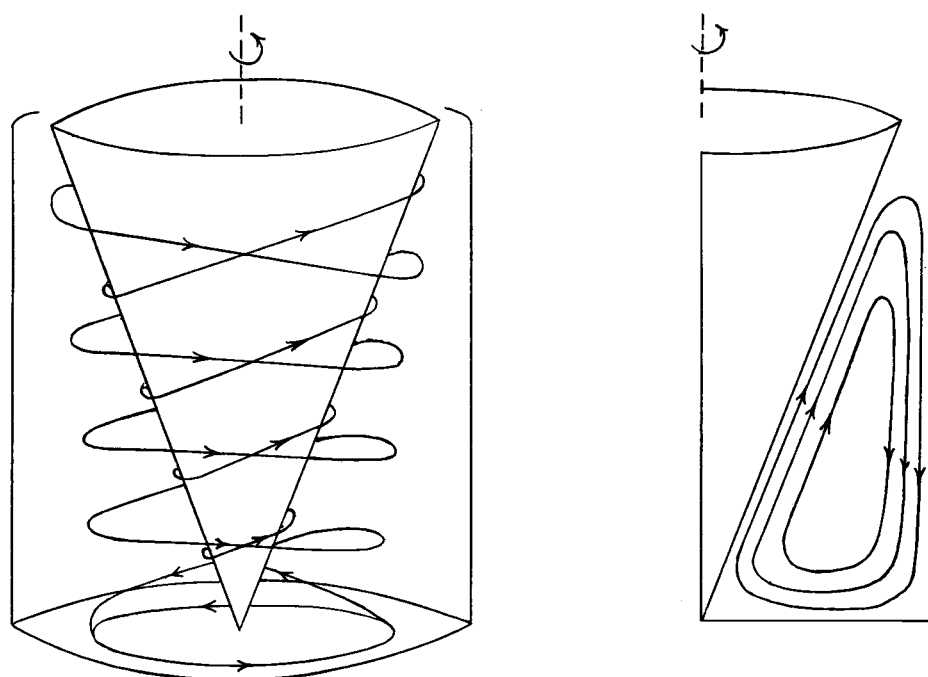
(Figure 4.8)

The projections of the particle path on any plane containing the axis of rotation when $m = 1/4$.

4.4 Experimental Results on Secondary Flow

Hoppman, W.H. and C.H. Miller (1963) used the technique of injecting a dye into the moving fluid by a hypodermic needle to observe the secondary flow of the Newtonian fluid U.S.P. castor oil in a cone-plate viscometer. The experiment was conducted four times, each time using a cone of different apex half angle, with the idea of seeing how different shaped cones affected the fluid flow.

The results of the experiment showed that the fluid particles do not simply rotate in concentric circular paths about the axis of rotation of the cone, as expected, but spiral around the central curve of a vortex as they move around the cone axis (see figure 4.9).



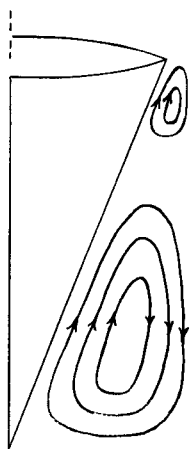
(Figure 4.9)

Drawing showing flow characteristics.

- (left) Schematic diagram of a streamline.
- (right) Schematic diagram of cross-section of toroidal-like surface of flow.

Hoppman, W.H. and Baronet (1964), using the same geometric configuration as above, investigated the secondary

flow using three Newtonian fluids, which were castor oil, water, and decalin, and two non-Newtonian fluids, which were sweetened condensed milk and solutions of polyisobutylene in cetane. It was observed that the flow phenomena for non-Newtonian fluids are much more complicated than for Newtonian fluids. On examination of the median cross-sections, vortices were found and the velocity field of the flow depended sharply upon the concentration of the fluid. For example, for 10% solution of polyisobutylene in cetane, only one vortex center was observed, but the direction of the flow was opposite from that of a Newtonian fluid. For 7.5% solution of the fluid, two vortex centers were observed (see figure 4.10), and for 5% solution, the fluid behaved much like Newtonian fluid, with one vortex center and flow in the same direction as that of the Newtonian fluid.



(Figure 4.10)

Vortices generated by a rotating cone immersed in
7.5% solution of Polyisobutylene in cetane.

In the preceding work, (see figure 4.10), the secondary flow was obtained for fluids which were described by using classical deformation-rate measures and not generalized measures. The use of classical measures leads to unknown response coefficients and in order to use such equations, the response coefficients are arbitrarily assigned. In chapter 5, we illustrate the use of generalized measures in describing the secondary flow of a fluid. By the use of generalized measures, the response coefficients are known exactly.

CHAPTER V

APPLICATION OF GENERALIZED MEASURES OF DEFORMATION-RATES TO SECONDARY FLOWS OF VISCOELASTIC FLUIDS BETWEEN ROTATING SPHERES

5.1 Formulation of the ProblemThe Fluid

In this chapter we shall illustrate the secondary flows using the constitutive equation (3.39) based on combined generalized measures by fixing the orders of the measures as

$$(m_1 \times 2 \times k_1 \times 1), (m_2 \times 2 \times k_2 \times 3)$$

and

(5.1)

$$(m'_1 \times 2 \times k'_1 \times 1), (m'_2 \times n'_2 \times 0 \times q'_1)$$

and obtaining the constitutive equation

$$\underline{\underline{T}} = -p\underline{\underline{I}} + 2\alpha_1\underline{\underline{D}} - 2\alpha_2\underline{\underline{D}}^3 + \gamma\underline{\underline{B}}, \quad (5.2)$$

or by the Cayley-Hamilton theorem

$$\underline{\underline{T}} = -p\underline{\underline{I}} + 2\alpha_1\underline{\underline{D}} - 2\alpha_2(\text{III}_{\underline{\underline{D}}}\underline{\underline{I}} - \text{II}_{\underline{\underline{D}}}\underline{\underline{D}}) + \gamma\underline{\underline{B}}, \quad (5.3)$$

where

$$\underline{\underline{T}} \equiv \|t^i_j\| \equiv \text{stress matrix,}$$

$$p \equiv \text{hydrostatic pressure,}$$

$$\underline{\underline{I}} \equiv \|\delta^k_j\| \equiv \text{unit matrix,}$$

$\alpha_1 \equiv \mu k_1$, a rheological parameter,

$\underline{D} \equiv \|d^i_j\| \equiv$ first deformation-rate matrix,

$\alpha_2 \equiv -\mu k_2$, a rheological parameter,

$II_{\underline{D}}$ and $III_{\underline{D}} \equiv$ second and third invariants of \underline{D} , respectively,

$\gamma \equiv 4\eta k'_1$, a rheological parameter,

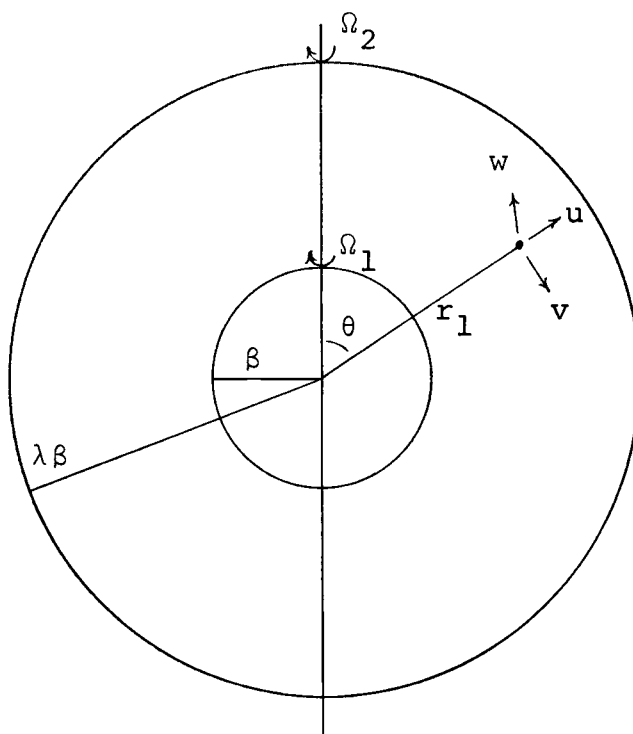
and

$\underline{B} \equiv \|b^i_j\| \equiv$ secondary deformation-rate matrix.

For more discussion on the fluid, see (3.40).

The Geometric Configuration

The fluid behavior is investigated in the annulus of two concentric spheres which rotate slowly with respect to each other about a common axis. The spherical-polar coordinate system (r_1, θ, ϕ) is used with the origin at the common center of the spheres and the polar angle θ and the azimuthal angle ϕ being measured from the common axis of rotation and some convenient meridian plane, respectively. Let the spheres be of radii β and $\lambda\beta$ ($\lambda > 1$), and rotating with angular velocities Ω_1 and Ω_2 , respectively (see figure 5.1).



(Figure 5.1) Notation

The Equations of Motion

Since we are assuming an incompressible fluid in steady-state motion, the density ρ of the fluid is constant and all quantities are independent of time. By symmetry with respect to the axis of rotation, all quantities are independent of ϕ . If u, v, w are the physical components of velocity; a, b, c are the physical components of acceleration; and $t_{(rr)}, t_{(r\theta)}, t_{(r\phi)}$, etc. are the physical components of the stress tensor, then the physical components of stress are:

$$t_{(rr)} = -p + 2\alpha_1 u_{r_1} - 2\alpha_2 [III_D - II_D u_{r_1}] + \gamma [a_{r_1} + u_{r_1}^2 + v_{r_1}^2 + w_{r_1}^2], \quad (5.4a)$$

$$t_{(\theta\theta)} = -p + \frac{2\alpha_1}{r_1}(u+v_\theta) - 2\alpha_2 [III_D - \frac{II_D}{r_1}(u+v_\theta)] \\ + \frac{\gamma}{r_1^2} [r_1 b_\theta + r_1 a + (u_\theta - v)^2 + (u+v_\theta)^2 + w_\theta^2], \quad (5.4b)$$

$$t_{(\phi\phi)} = -p + \frac{2\alpha_1}{r_1}(u+v\text{ctn}\theta) - 2\alpha_2 [III_D - \frac{II_D}{r_1}(u+v\text{ctn}\theta)] \\ + \frac{\gamma}{r_1^2} [r_1 a + r_1 b\text{ctn}\theta + (u+v\text{ctn}\theta)^2 + w^2 \text{csc}^2\theta], \quad (5.4c)$$

$$t_{(r\theta)} = \frac{\alpha_1}{r_1}(u_\theta + r_1 v_{r_1} - v) + \frac{\alpha_2 II_D}{r_1} [u_\theta + r_1 v_{r_1} - v] \\ + \frac{\gamma}{r_1} [\frac{1}{2}(a_\theta + r_1 b_{r_1} - b) + u_{r_1}(u_\theta - v) + v_{r_1}(u+v_\theta) + w_{r_1} w_\theta], \quad (5.4d)$$

$$t_{(r\phi)} = \frac{\alpha_1}{r_1}(r_1 w_{r_1} - w) + \frac{\alpha_2 II_D}{r_1} [r_1 w_{r_1} - w] \\ + \frac{\gamma}{r_1} [\frac{1}{2}(r_1 c_{r_1} - c) - w u_{r_1} - w v_{r_1} \text{ctn}\theta + w_{r_1}(u+v\text{ctn}\theta)], \quad (5.4e)$$

$$t_{\theta\phi} = \frac{\alpha_1}{r_1}(w_\theta - w\text{ctn}\theta) + \frac{\alpha_2 II_D}{r_1} [w_\theta - w\text{ctn}\theta] \\ + \frac{\gamma}{r_1} [\frac{1}{2}(r_1 c_\theta - r_1 c \text{ctn}\theta - w(u_\theta - v) - \text{ctn}\theta w(u+v_\theta) + w_\theta(u+v\text{ctn}\theta)]; \quad (5.4f)$$

the equation of continuity is

$$u_{r_1} + \frac{2u}{r_1} + \frac{v_{\theta}}{r_1} + \frac{v \operatorname{ctn} \theta}{r_1} = 0; \quad (5.5)$$

and the momentum equations are:

$$\begin{aligned} \rho a = -p_{r_1} + \alpha_1 \left[\Delta u - \frac{2u}{r_1^2} - \frac{2v_{\theta}}{r_1^2} - \frac{2 \operatorname{ctn} \theta}{r_1^2} v \right] \\ + \gamma S_1(r_1, \theta) - \alpha_2 K_1(r_1, \theta), \end{aligned} \quad (5.6a)$$

$$\begin{aligned} \rho b = \frac{-p_{\theta}}{r_1} + \alpha_1 \left[\Delta v + \frac{2u_{\theta}}{r_1^2} - \frac{v}{r_1^2 \sin^2 \theta} \right] \\ + \gamma S_2(r_1, \theta) - \alpha_2 K_2(r_1, \theta), \end{aligned} \quad (5.6b)$$

$$\rho c = \alpha_1 \left[\Delta w - \frac{w}{r_1^2 \sin^2 \theta} \right] + \gamma S_3(r_1, \theta) - \alpha_2 K_3(r_1, \theta), \quad (5.6c)$$

where

$$\Delta \equiv \frac{\partial^2}{\partial r_1^2} + \frac{2}{r_1} \frac{\partial}{\partial r_1} + \frac{1}{r_1^2} \frac{\partial^2}{\partial \theta^2} + \frac{\operatorname{ctn} \theta}{r_1^2} \frac{\partial}{\partial \theta}, \quad (5.7)$$

$$a \equiv uu_{r_1} + \frac{v}{r_1} u_{\theta} - \frac{1}{r_1} (v^2 + w^2), \quad (5.8a)$$

$$b \equiv uv_{r_1} + \frac{v}{r_1} v_{\theta} + \frac{uv}{r_1} - \frac{w^2 \operatorname{ctn} \theta}{r_1}, \quad (5.8b)$$

$$c \equiv uw_{r_1} + \frac{vw_{\theta}}{r_1} + \frac{uw}{r_1} + \frac{vw \operatorname{ctn} \theta}{r_1}, \quad (5.8c)$$

$$\begin{aligned} S_1(r_1, \theta) &\equiv a_{r_1 r_1} + \frac{1}{2r_1^2} a_{\theta\theta} + \frac{1}{2r_1} b_{r_1\theta} + \frac{2a_{r_1}}{r_1} + \frac{\operatorname{ctn} \theta}{2r_1^2} a_{\theta} \\ &- \frac{2a}{r_1^2} + \frac{\operatorname{ctn} \theta}{2r_1} b_{r_1} - \frac{3b_{\theta}}{2r_1^2} - \frac{3\operatorname{ctn} \theta b}{2r_1^2} + \left[u_{r_1}^2 + v_{r_1}^2 + w_{r_1}^2 \right]_{r_1} \\ &+ \frac{1}{r_1^2} \left[u_{r_1} (u_{\theta} - v) + v_{r_1} (u + v_{\theta}) + w_{r_1} w_{\theta} \right]_{\theta} + \frac{2}{r_1} \left[u_{r_1}^2 + v_{r_1}^2 + w_{r_1}^2 \right] \\ &- \frac{1}{r_1^3} \left[(u_{\theta} - v)^2 + (v_{\theta} + u)^2 + w_{\theta}^2 \right] + \frac{\operatorname{ctn} \theta}{r_1^2} \left[u_{r_1} (u_{\theta} - v) + v_{r_1} (u + v_{\theta}) + w_{r_1} w_{\theta} \right] \\ &- \frac{1}{r_1^3} \left[u + v \operatorname{ctn} \theta \right]^2 - \frac{w^2}{r_1^3 \sin^2 \theta}, \quad (5.9a) \end{aligned}$$

$$\begin{aligned} S_2(r_1, \theta) &\equiv \frac{1}{2r_1} a_{r_1\theta} + \frac{1}{2} b_{r_1 r_1} + \frac{1}{r_1^2} b_{\theta\theta} + \frac{2a_{\theta}}{r_1} + \frac{1}{r_1} b_{r_1} \\ &+ \frac{\operatorname{ctn} \theta}{r_1^2} b_{\theta} - \frac{b}{r_1^2 \sin^2 \theta} + \frac{1}{r_1} \left[u_{r_1} (u_{\theta} - v) + v_{r_1} (u + v_{\theta}) + w_{r_1} w_{\theta} \right]_{r_1} \\ &+ \frac{1}{r_1^3} \left[(u_{\theta} - v)^2 + (u + v_{\theta})^2 + w_{\theta}^2 \right]_{\theta} + \frac{2}{r_1^2} \left[u_{r_1} (u_{\theta} - v) + v_{r_1} (u + v_{\theta}) + w_{r_1} w_{\theta} \right] \\ &+ \frac{\operatorname{ctn} \theta}{r_1^3} \left[(u_{\theta} - v)^2 + (v_{\theta} + u)^2 + w_{\theta}^2 \right] - \frac{\operatorname{ctn} \theta}{r_1^3} \left[w^2 \operatorname{csc}^2 \theta + (u + v \operatorname{ctn} \theta)^2 \right], \quad (5.9b) \end{aligned}$$

$$\begin{aligned}
S_3(r_1, \theta) \equiv & \frac{1}{2}c_{r_1 r_1} + \frac{1}{2r_1^2}c_{\theta\theta} + \frac{1}{r_1}c_{r_1} + \frac{\text{ctn}\theta}{2r_1^2}c_{\theta} - \frac{c}{2r_1^2 \sin^2\theta} \\
& - \frac{1}{r_1^3} \left[r_1^2 (w_{r_1} u + w_{r_1} v \text{ctn}\theta - w_{r_1} (u + v \text{ctn}\theta)) \right]_{r_1} \\
& - \frac{1}{r_1^3 \sin^2\theta} \left[\sin^2\theta (w(u_{\theta} - v) + w \text{ctn}\theta (u + v_{\theta}) - w_{\theta} (u + v \text{ctn}\theta)) \right]_{\theta}, \quad (5.9c)
\end{aligned}$$

$$\begin{aligned}
K_1(r_1, \theta) \equiv & 2 \left[\text{III}_D - \text{II}_D u_{r_1} \right]_{r_1} - \frac{1}{r_1^2} \left[\text{II}_D (u_{\theta} + r_1 v_{r_1} - v) \right]_{\theta} \\
& + \frac{\text{II}_D}{r_1^2} \left[4u - 4r_1 u_{r_1} - u_{\theta} \text{ctn}\theta + 3v \text{ctn}\theta - r_1 v_{r_1} \text{ctn}\theta + 2v_{\theta} \right], \quad (5.10a)
\end{aligned}$$

$$\begin{aligned}
K_2(r_1, \theta) \equiv & -\frac{1}{r_1} \left[\text{II}_D (u_{\theta} + r_1 v_{r_1} - v) \right]_{r_1} + \frac{2}{r_1} \left[\text{III}_D - \frac{\text{II}_D}{r_1} (u + v_{\theta}) \right]_{\theta} \\
& + \frac{2\text{II}_D}{r_1^2} \left[v \text{csc}^2\theta - v_{\theta} \text{ctn}\theta - r_1 v_{r_1} - u_{\theta} \right], \quad (5.10b)
\end{aligned}$$

and

$$\begin{aligned}
K_3(r_1, \theta) \equiv & -\frac{1}{r_1} \left[\text{II}_D (r_1 w_{r_1} - w) \right]_{r_1} - \frac{1}{r_1^2} \left[\text{II}_D (w_{\theta} - w \text{ctn}\theta) \right]_{\theta} \\
& - \frac{2\text{II}_D}{r_1^2} \left[(r_1 w_{r_1} - w) + \text{ctn}\theta (w_{\theta} - w \text{ctn}\theta) \right], \quad (5.10c)
\end{aligned}$$

with

$$\begin{aligned} \text{II}_D &= \frac{u_{r_1}}{r_1}(u+v_\theta) + \frac{1}{r_1^2}(u+v_\theta)(u+v\text{ctn}\theta) + \frac{u_{r_1}}{r_1}(u+v\text{ctn}\theta) \\ &- \frac{1}{4r_1^2} \left[(u_\theta + r_1 v_{r_1} - v)^2 + (r_1 w_{r_1} - w)^2 + (w_\theta - w\text{ctn}\theta)^2 \right], \end{aligned} \quad (5.11a)$$

$$\begin{aligned} \text{III}_D &= \frac{u_{r_1}}{r_1} \left[(u+v_\theta)(u+v\text{ctn}\theta) - \frac{1}{4}(w_\theta - w\text{ctn}\theta)^2 \right] \\ &+ \frac{1}{4r_1^3} \left[(u_\theta + r_1 v_{r_1} - v)(r_1 w_{r_1} - w)(w_\theta - w\text{ctn}\theta) \right] \\ &- \frac{1}{4r_1^3} \left[(u_\theta + r_1 v_{r_1} - v)^2 (u+v\text{ctn}\theta) + (r_1 w_{r_1} - w)^2 (u+v_\theta) \right], \end{aligned} \quad (5.11b)$$

and w_r and w_θ denote partial differentiation of w with respect to r and θ respectively.

The above equations are subjected to the following boundary conditions:

$$\begin{aligned} u = 0 = v, \quad w &= \beta\Omega_1 \sin\theta \quad \text{at } r_1 = \beta, \\ u = 0 = v, \quad w &= \lambda\beta\Omega_2 \sin\theta \quad \text{at } r_1 = \lambda\beta. \end{aligned} \quad (5.12)$$

Let A, B, C be the non-dimensional components of acceleration and U, V, W be the non-dimensional components of velocity. If the following substitutions:

$$a = \frac{v^2}{\beta^3} A, \quad b = \frac{v^2}{\beta^3} B, \quad \text{and} \quad c = \frac{v\Omega}{\beta} C; \quad (5.13)$$

$$u = \frac{v}{\beta} U, \quad v = \frac{v}{\beta} V, \quad \text{and} \quad w = \beta \Omega W; \quad (5.14)$$

$$p = \frac{\rho v^2}{\beta^2} P, \quad (5.15)$$

and

$$r_1 = \beta r, \quad (5.16)$$

where

$$v = \frac{\alpha_1}{\rho} \quad \text{and} \quad \Omega = \Omega_1 - \Omega_2 ,$$

are made in equations (5.6), then we obtain the

Non-Dimensional Equations of Motion:

equation of continuity

$$U_r + 2\frac{U}{r} + \frac{V_\theta}{r} + \frac{V \text{ctn} \theta}{r} = 0 \quad (5.17)$$

and momentum equations

$$A = -P_r + \left[\Delta^* U - \frac{2U}{r^2} - \frac{2V_\theta}{r^2} - \frac{2 \text{ctn} \theta V}{r^2} \right] + SS_1^*(r, \theta) + KK_1^*(r, \theta), \quad (5.18a)$$

$$B = -\frac{P_\theta}{r} + \left[\Delta^* V + \frac{2U_\theta}{r^2} - \frac{V}{r^2 \sin^2 \theta} \right] + SS_2^*(r, \theta) + KK_2^*(r, \theta), \quad (5.18b)$$

$$C = \left[\Delta^* W - \frac{W}{r^2 \sin^2 \theta} \right] + SS_3^*(r, \theta) + KK_3^*(r, \theta), \quad (5.18c)$$

where

$$\Delta^* = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\text{ctn}\theta}{r^2} \frac{\partial}{\partial \theta} , \quad (5.19)$$

$$A = UU_r + \frac{V}{r} U_\theta - \frac{V^2}{r} - \frac{hW^2}{r} , \quad (5.20a)$$

$$B = UV_r + \frac{V}{r} V_\theta + \frac{UV}{r} - \frac{hW^2 \text{ctn}\theta}{r} , \quad (5.20b)$$

$$C = UV_r + \frac{V}{r} W_\theta + \frac{UW}{r} + \frac{VW}{r} \text{ctn}\theta , \quad (5.20c)$$

$$\begin{aligned} S_1^*(r, \theta) &= A_{rr} + \frac{1}{2r^2} A_{\theta\theta} + \frac{1}{2r} B_{r\theta} + \frac{2}{r} A_r + \frac{\text{ctn}\theta}{2r^2} A_\theta \\ &- \frac{2A}{r^2} + \frac{\text{ctn}\theta}{2r} B_r - \frac{3}{2r^2} B_\theta - \frac{3\text{ctn}\theta}{2r^2} B + \left[U_r^2 + V_r^2 + hW_r^2 \right]_r \\ &+ \frac{1}{r^2} \left[U_r (U_\theta - V) + V_r (U + V_\theta) + hW_r W_\theta \right]_\theta + \frac{2}{r} \left[U_r^2 + V_r^2 + hW_r^2 \right] \\ &- \frac{1}{r^3} \left[(U_\theta - V)^2 + (V_\theta + U)^2 + hW_\theta^2 \right] + \frac{\text{ctn}\theta}{r^2} \left[U_r (U_\theta - V) + V_r (U + V_\theta) + hW_r W_\theta \right] \\ &- \frac{1}{r^3} \left[U + V \text{ctn}\theta \right] - \frac{hW^2}{r^3 \sin^2 \theta} , \end{aligned} \quad (5.21a)$$

$$\begin{aligned} S_2^*(r, \theta) &= \frac{1}{2r} A_{r\theta} + \frac{1}{2} B_{rr} + \frac{1}{r^2} B_{\theta\theta} + \frac{2}{r^2} A_\theta + \frac{B_r}{r} \\ &+ \frac{\text{ctn}\theta}{r^2} B - \frac{B}{r^2 \sin^2 \theta} + \frac{1}{r} \left[U_r (U_\theta - V) + V_r (U + V_\theta) + hW_r W_\theta \right]_r \\ &+ \frac{1}{r^3} \left[(U_\theta - V)^2 + (U + V_\theta)^2 + hW_\theta^2 \right] + \frac{2}{r^2} \left[U_r (U_\theta - V) + V_r (U + V_\theta) + hW_r W_\theta \right] \end{aligned}$$

$$+ \frac{\text{ctn}\theta}{r^3} \left[(U_\theta - V)^2 + (V_\theta + U)^2 + hW_\theta^2 \right] - \frac{\text{ctn}\theta}{r^3} \left[hW^2 \csc^2\theta + (U + V \text{ctn}\theta)^2 \right], \quad (5.21b)$$

$$S_3^*(r, \theta) = \frac{1}{2} C_{rr} + \frac{1}{2r^2} C_{\theta\theta} + \frac{1}{r} C_r + \frac{\text{ctn}\theta}{2r^2} C_\theta - \frac{C}{2r^2 \sin^2\theta} \\ - \frac{1}{r^3} \left[r^2 (WU_r + WV_r \text{ctn}\theta - W_r (U + V \text{ctn}\theta)) \right]_r \\ - \frac{1}{r^3 \sin^2\theta} \left[\sin^2\theta (W(U_\theta - V) + W \text{ctn}\theta (U + V_\theta) - W_\theta (U + V \text{ctn}\theta)) \right]_\theta, \quad (5.21c)$$

$$K_1^*(r, \theta) = 2 \left[III_D^* - II_D^* U_r \right]_r - \frac{1}{r^2} \left[II_D^* (U_\theta + rV_r - V) \right]_\theta \\ + \frac{II_D^*}{r^2} \left[4U - 4rU_r - U_\theta \text{ctn}\theta + 3V \text{ctn}\theta - rV_r \text{ctn}\theta + 2V_\theta \right], \quad (5.22a)$$

$$K_2^*(r, \theta) = -\frac{1}{r} \left[II_D^* (U_\theta + rV_r - V) \right]_r + \frac{2}{r} \left[III_D^* - \frac{II_D^*}{r} (U + V_\theta) \right]_\theta \\ + \frac{2II_D^*}{r^2} \left[V \csc^2\theta - V_\theta \text{ctn}\theta - rV_r - U_\theta \right], \quad (5.22b)$$

$$K_3^*(r, \theta) = -\frac{1}{r} \left[II_D^* (rW_r - W) \right]_r - \frac{1}{r^2} \left[II_D^* (W_\theta - W \text{ctn}\theta) \right]_\theta \\ - \frac{2II_D^*}{r^2} \left[(rW_r - W) + \text{ctn}\theta (W_\theta - W \text{ctn}\theta) \right], \quad (5.22c)$$

with

$$II_D^* = \frac{U_r}{r} (U + V_\theta) + \frac{1}{r^2} (U + V_\theta) (U + V \text{ctn}\theta) + \frac{U_r}{r} (U + V \text{ctn}\theta) \\ - \frac{1}{4r^2} \left[(U_\theta + rV_r - V)^2 + h(rW_r - W) + h(W_\theta - W \text{ctn}\theta)^2 \right], \quad (5.23a)$$

$$\begin{aligned}
III_D^* &= \frac{U_r}{r^2} \left[(U+V_\theta) (U+V \text{ctn}\theta) - \frac{h}{4} (W_\theta - W \text{ctn}\theta)^2 \right] \\
&+ \frac{h}{4r^3} \left[(U_\theta + rV_r - V) (rW_r - W) (W_\theta - W \text{ctn}\theta) \right] \\
&- \frac{1}{4r^3} \left[(U_\theta + rV_r - V)^2 (U+V \text{ctn}\theta) + h (rV_r - W)^2 (U+V_\theta) \right], \quad (5.23b)
\end{aligned}$$

$$S \equiv \gamma/\rho\beta^2 \quad \text{with} \quad [S] = ML^{-1}/ML^{-3}L^2 = M^0L^0T^0, \quad (5.24a)$$

$$K \equiv \frac{-\alpha_2 v}{\rho\beta^4} \quad \text{with} \quad [K] = \frac{ML^{-1}T \cdot T^{-1}L^2}{ML^{-3} \cdot L^4} = M^0L^0T^0, \quad (5.24b)$$

and

$$h \equiv \left(\frac{\Omega\beta^2}{v} \right)^2 \quad \text{with} \quad [h] = \frac{T^{-2} \cdot L^4}{T^{-2} L^4} = M^0L^0T^0, \quad (5.24c)$$

where [] denotes the dimensions of the quantity in question in terms of mass M, length L, and time T.

The boundary conditions in non dimensional form are

$$\begin{aligned}
U = 0 = V, \quad W &= \frac{\Omega_1}{\Omega} \sin\theta \quad \text{at} \quad r = 1 \\
U = 0 = V, \quad W &= \frac{\Omega_2}{\Omega} \lambda \sin\theta \quad \text{at} \quad r = \lambda
\end{aligned} \quad (5.25)$$

5.2 Method of Solution of the Equations of Motion

If the spheres are rotating in the same direction with nearly the same velocities, that is, if $\Omega = \Omega_1 - \Omega_2$ is "small", if v is "large" meaning α_1 is large, then

$$h = \left(\frac{\Omega \beta^2}{\nu} \right)^2 \quad (5.26)$$

is very small so that $h^2 \ll h$. The method of solution will be to expand the expression for pressure and the velocity components in ascending powers of the parameter h . That is, assume

$$P = P_0 + hP_1 + h^2P_2 + \dots \quad (5.27)$$

$$U = U_0 + hU_1 + h^2U_2 + \dots \quad (5.28a)$$

$$V = V_0 + hV_1 + h^2V_2 + \dots \quad (5.28b)$$

$$W = W_0 + hW_1 + h^2W_2 + \dots \quad (5.28c)$$

5.3 Primary Flows in Pseudoplastic Fluids

We shall call the rectilinear motion with the neglect of the inertia terms and the non-Newtonian S - and K -terms in the equations of motion the primary motion. We shall specify the primary motion by the zero subscript. The equations of motion for the primary flow are:

continuity equation

$$U_{0r} + \frac{2U_0}{r} + \frac{V_{0\theta}}{r} + \frac{V_0 \text{ctn}\theta}{r} = 0; \quad (5.29)$$

and

momentum equations

$$-P_{0r} + \left[\Delta^*U_0 - \frac{2U_0}{r} - \frac{2V_{0\theta}}{r^2} - \frac{2\text{ctn}\theta}{r^2} V_0 \right] = 0, \quad (5.30a)$$

$$- \frac{P_{0\theta}}{r} + \left[\Delta^*V_0 + \frac{2U_{0\theta}}{r^2} - \frac{V_0}{r^2 \sin^2\theta} \right] = 0, \quad (5.30b)$$

$$\Delta^*W_0 - \frac{W_0}{r^2 \sin^2\theta} = 0. \quad (5.30c)$$

The boundary conditions for the primary flow are

$$\begin{aligned} U_0 = 0 = V_0, \quad W_0 = \frac{\Omega_1}{\Omega} \sin\theta \quad \text{at } r = 1, \\ U_0 = 0 = V_0, \quad W_0 = \frac{\Omega_2}{\Omega} \lambda \sin\theta \quad \text{at } r = \lambda. \end{aligned} \quad (5.31)$$

In the primary motion, the streamlines in any plane perpendicular to the common axis of rotation are concentric circles with centers on the axis. Thus

$$U_0 = 0 = V_0 \quad (5.32)$$

and

$$P_{0r} = 0 = P_{0\theta} \quad \text{which implies } P_0 = \text{constant} \quad (5.33)$$

Let $W_0 = f(r)\sin\theta$, then (5.28c) becomes

$$\frac{d^2f}{dr^2} + \frac{2}{r} \frac{df}{dr} - \frac{2f}{r^2} = 0 \quad (5.34)$$

and

$$f(r) = \left(\frac{d}{r^2} + er \right), \quad (5.35)$$

where

$$d = \frac{\lambda^3}{\lambda^3 - 1} \quad \text{and} \quad e = \frac{\Omega_2 \lambda^3 - \Omega_1}{\Omega(\lambda^3 - 1)}. \quad (5.36)$$

$$\text{Thus} \quad w_0 = \left(\frac{d}{r^2} + er \right) \sin\theta \quad (5.37)$$

5.4 Secondary Flows in Pseudoplastic Fluids

In this investigation, we shall consider only the contribution due to the first order terms of h . By including the first order terms of h , we include the effects due to the inertia terms and the non-Newtonian S - and K - terms. The equations of motion for the first-order approximation are

the equation of continuity

$$U_{1r} + \frac{2U_1}{r} + \frac{V_{1\theta}}{r} + \frac{V_1 \text{ctn}\theta}{r} = 0, \quad (5.38)$$

and the momentum equations

$$-\frac{W_0^2}{r} = -P_{1r} + \left[\Delta^* U_1 - \frac{2U_1}{r^2} - \frac{2V_{1\theta}}{r^2} - \frac{2\text{ctn}\theta V_1}{r^2} \right] + SS_1^{**}(r, \theta) \\ + KK_1^{**}(r, \theta), \quad (5.39a)$$

$$-\frac{W_0^2}{r} \operatorname{ctn}\theta = -\frac{P_{1\theta}}{r} + \left[\Delta^* V_1 + \frac{2U_{1\theta}}{r^2} - \frac{V_1}{r^2 \sin^2\theta} \right] + SS_2^{**}(r, \theta) + KK_2^{**}(r, \theta), \quad (5.39b)$$

$$C^*(r, \theta) = \left[\Delta^* W_1 - \frac{W_1}{r^2 \sin^2\theta} \right] + SS_3^{**}(r, \theta) + KK_3^{**}(r, \theta), \quad (5.39c)$$

where

$$S_1^{**}(r, \theta) = \frac{-36d^2}{r^7} \sin^2\theta, \quad (5.40a)$$

$$S_2^{**}(r, \theta) = 0, \quad (5.40b)$$

$$S_3^{**}(r, \theta) = \frac{1}{2} C_{rr}^* + \frac{1}{2r^2} C_{\theta\theta}^* + \frac{1}{r} C_{r\theta}^* + \frac{\operatorname{ctn}\theta}{2r^2} C_{\theta\theta}^* - \frac{C^*}{2r^2 \sin^2\theta} - \frac{1}{r^3} \left[r^2 \{ W_0 U_{1r} + W_0 V_{1r} \operatorname{ctn}\theta - W_{0r} (U_1 + V_1 \operatorname{ctn}\theta) \} \right]_r - \frac{1}{r^3 \sin^2\theta} \left[\sin^2\theta \{ W_0 (U_{1\theta} - V_1) + W_0 \operatorname{ctn}\theta (U_1 + V_1 \operatorname{ctn}\theta) - W_{0\theta} (U_1 + V_1 \operatorname{ctn}\theta) \} \right]_{\theta} \quad (5.40c)$$

$$K_1^{**}(r, \theta) = 0 \quad (5.41a)$$

$$K_2^{**}(r, \theta) = 0 \quad (5.41b)$$

$$K_3^{**}(r, \theta) = \frac{81d^3}{2r^{10}} \sin^3\theta \quad (5.41c)$$

$$C^*(r, \theta) = U_1 W_{0r} + \frac{V_1 W_{0\theta}}{r} + \frac{U_1 W_0}{r} + \frac{V_1 W_0}{r} \operatorname{ctn}\theta, \quad (5.42)$$

$$W_0 = \left(\frac{d}{r^2} + er \right) \sin\theta, \quad \text{as given in (5.37).}$$

Boundary Conditions

By the no-slip condition of a viscous fluid on the boundary, the boundary conditions for the first approximation of the secondary flow are

$$\begin{aligned} U_1 = V_1 = W_1 = 0 \quad \text{at } r = 1 \\ U_1 = V_1 = W_1 = 0 \quad \text{at } r = \lambda \end{aligned} \quad (5.43)$$

Solution for the Velocity Field

Solutions of the coupled equations (5.39a-b) are obtained by eliminating the pressure term P in the equations and introducing the streamline function $\psi_1(r, \theta)$ where

$$U_1 = \frac{1}{r^2 \sin \theta} \frac{\partial \psi_1}{\partial \theta} \quad \text{and} \quad V_1 = \frac{-1}{r \sin \theta} \frac{\partial \psi_1}{\partial r}, \quad (5.44)$$

which satisfy the continuity equation

$$(r^2 \sin \theta U_1)_r + (r \sin \theta V_1)_\theta = 0 \quad (5.45)$$

exactly. Thus, equations (5.39) become

$$D^4 \psi_1 = -6 \left(\frac{d^2}{r^5} + \frac{de}{r^2} \right) \sin^2 \theta \cos \theta + \frac{72 S d^2}{r^7} \sin^2 \theta \cos \theta \quad (5.46)$$

where

$$D^4 \psi_1 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\text{ctn}\theta}{r^2} \frac{\partial}{\partial \theta} \right)^2 \psi_1 \quad (5.47)$$

and $d, e,$ and S are given by (5.36) and (5.24a) respectively.

From (5.43), the streamlines $\psi_1(r, \theta)$ must satisfy the boundary conditions

$$\begin{aligned} \psi_{1\theta}(r, \theta) = 0 = \psi_{1r}(r, \theta) \quad \text{at } r = 1, \\ \psi_{1\theta}(r, \theta) = 0 = \psi_{1r}(r, \theta) \quad \text{at } r = \lambda. \end{aligned} \quad (5.48)$$

In order to solve equation (5.46) for the streamline $\psi_1(r, \theta)$ assume a solution of the form

$$\psi_1(r, \theta) = g(r) \sin^2 \theta \cos \theta \quad (5.49)$$

and note that

$$D^2 \psi_1 = \left[\frac{d^2 g(r)}{dr^2} - \frac{6g(r)}{r^2} \right] \sin^2 \theta \cos \theta \quad (5.50)$$

Then equation (5.44) becomes

$$\frac{d^2 z(r)}{dr^2} - \frac{6z(r)}{r^2} = -6 \left(\frac{d^2}{r^2} + \frac{de}{r^2} \right) + \frac{72Sd^2}{r^7} \quad (5.51)$$

where

$$z(r) = \frac{d^2 g(r)}{dr^2} - \frac{6g(r)}{r^2} \quad (5.52)$$

Solving (5.51) for $z(r)$, then (5.52) for $g(r)$, we obtain

$$\psi_1(r, \theta) = \left(\sum_j \sigma_j r^{j-4} \right) \sin^2 \theta \cos \theta, \quad (5.53)$$

and by (5.44)

$$U_1(r, \theta) = \left(\sum_j \sigma_j r^{j-6} \right) (2-3\sin^2 \theta), \quad (5.54a)$$

$$V_1(r, \theta) = \left(\sum_j \sigma_j (j-4) r^{j-6} \right) \sin \theta \cos \theta, \quad (5.54b)$$

where

$$\sigma_1 = sd^2/2,$$

$$\sigma_2 = - \left[(2\lambda^7 - 5\lambda^4 + 3\lambda^2)C_2 + 15(\lambda^4 - \lambda^2)(C_1 - C_3) + (3\lambda^5 - 5\lambda^3 + 2)C_4 \right] / H,$$

$$\sigma_3 = d^2/4,$$

$$\sigma_4 = \left[(4\lambda^7 + 21\lambda^2 - 25)C_1 + (2\lambda^7 - 7\lambda^2 + 5)C_2 + (-25\lambda^4 + 21\lambda^2 + 4/\lambda^3)C_3 \right. \\ \left. + (5\lambda^5 - 7\lambda^3 + 2/\lambda^2)C_4 \right] / H \quad (5.55)$$

$$\sigma_5 = 0,$$

$$\sigma_6 = -de/4,$$

$$\sigma_7 = \left[(-5\lambda^4 + 7\lambda^2 - 2/\lambda^3)C_2 + 10(1/\lambda^3 - \lambda^4)(C_1 - C_3) \right. \\ \left. + (-2\lambda^5 + 7 - 5/\lambda^2)C_4 \right] / H$$

$$\sigma_8 = 0,$$

$$\sigma_9 = \left[(3\lambda^2 - 5 + 2/\lambda^3)C_2 + 6(\lambda^2 - 1/\lambda^3)(C_1 - C_3) + (2\lambda^3 - 5 + 3/\lambda^2)C_4 \right] / H$$

and

$$\sigma_j = 0 \quad \text{for all } j < 1 \quad \text{and } j > 9;$$

$$C_1 = (de - d^2 - 2sd^2)/4,$$

$$C_2 = (2de + d^2 + 6sd^2)/4,$$

$$C_3 = (de\lambda^2 - d^2/\lambda - 2sd^2/\lambda^3)/4,$$

$$C_4 = (2de\lambda + d^2/\lambda^2 + 6sd^2/\lambda^4)/4,$$

with

$$H = (4\lambda^{10} - 25\lambda^7 + 42\lambda^5 - 25\lambda^3 + 4)/\lambda^3 \quad (5.56a)$$

or

$$H = \frac{(\lambda-1)^4}{\lambda^3} (4\lambda^6 + 16\lambda^5 + 40\lambda^4 + 55\lambda^3 + 40\lambda^2 + 16\lambda + 4) \quad (5.56b)$$

Note that $H > 0$ for $\lambda > 1$ and < 0 for $0 < \lambda < 1$.

Substituting U_1 (5.54a) and V_1 (5.54b) into equation (5.39c), we obtain

$$\Delta^* W_1 - \frac{W_1}{r^2 \sin^2 \theta} = \left(\sum_j \omega_j r^{j-11} \right) \sin \theta + \left(\sum_j \pi_j r^{j-11} \right) \sin^3 \theta \quad (5.57)$$

where

$$\omega_j = - \left[2(j-5)\sigma_{j-2} + s(3j^2 - 27j + 78)\sigma_j \right] d - 2(j-11)\sigma_{j-5} e \quad (5.58a)$$

$$\pi_j = \left[(2j-9)\sigma_{j-2} + s(3j^2 - 15j + 48)\sigma_j / 2 \right] d + 2(j-12)\sigma_{j-5} e - \frac{81}{2} \kappa d^2 \delta_j^1, \quad (5.58b)$$

with

$$\delta_j^1 = 1 \quad \text{if } j = 1 \quad \text{and } 0 \quad \text{if } j \neq 1,$$

and σ_j and d, e given by (5.55) and (5.36) respectively.

By assuming a solution of the form

$$W_1(r, \theta) = N_1(r) \sin \theta + N_2(r) \sin^3 \theta, \quad (5.59)$$

equation (5.57) can be written as the system of equations

$$\frac{d^2 N_1(r)}{dr^2} + \frac{2}{r} \frac{dN_1(r)}{dr} - \frac{2N_1(r)}{r^2} + \frac{8N_2(r)}{r^2} = \sum_j \omega_j r^{j-11}, \quad (5.60a)$$

$$\frac{d^2 N_2(r)}{dr^2} + \frac{2}{r} \frac{dN_2(r)}{dr} - \frac{12N_2(r)}{r^2} = \sum_j \pi_j r^{j-11}, \quad (5.60b)$$

with

$$N_1(1) = N_2(1) = 0$$

$$N_1(\lambda) = N_2(\lambda) = 0.$$

Solving the system of equations (5.60), we have

$$N_1(r) = \delta_5 r^{-2} + \delta_6 r + \sum_{j \neq 5, 7, 10, 12} \frac{\omega_j (j-5)(j-12) - 8\pi_j}{(j-5)(j-7)(j-10)(j-12)} r^{j-9} \\ - \frac{1}{3} \left(\omega_7 + \frac{4\pi_7}{5} \right) r^{-2} \ln r + \frac{1}{3} \left(\omega_{10} - \frac{4\pi_{10}}{5} \right) r \ln r$$

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$$\begin{aligned}
& + \frac{4}{35}\pi_5 r^{-4} \left(\frac{7}{10} + \ln r \right) + \frac{4}{35} \pi_{12} r^3 \left(\frac{7}{10} - \ln r \right) + \frac{(\omega_{12} - 8\delta_8) r^3}{16} \\
& + \frac{(\omega_5 - 8\delta_7)}{10} r^{-4}, \tag{5.61a}
\end{aligned}$$

$$\begin{aligned}
N_2(r) & = \delta_7 r^{-4} + \delta_8 r^3 + \sum_{j \neq 5, 12} \frac{\pi_j}{(j-5)(j-12)} r^{j-9} \\
& - \frac{\pi_5}{7} r^{-4} \ln r + \frac{\pi_{12}}{7} r^3 \ln r, \tag{5.61b}
\end{aligned}$$

where

$$\begin{aligned}
\delta_5 & = \frac{\lambda^2}{\lambda^3 - 1} \left\{ -\lambda \left[\sum_{j \neq 5, 7, 10, 12} \frac{\omega_j (j-5)(j-12) - 8\pi_j}{(j-5)(j-7)(j-10)(j-12)} + \frac{2}{25} (\pi_5 + \pi_{12}) \right. \right. \\
& + \frac{1}{16} (\omega_{12} - 8\delta_8) + \frac{1}{10} (\omega_5 - 8\delta_7) \left. \right] + \left[\sum_{j \neq 5, 7, 10, 12} \frac{\omega_j (j-5)(j-12) - 8\pi_j}{(j-5)(j-7)(j-10)(j-12)} \lambda^{j-9} \right. \\
& - \frac{1}{3} (\omega_7 + \frac{4\pi_7}{5}) \lambda^{-2} \ln \lambda + \frac{1}{3} (\omega_{10} - \frac{4}{5}\pi_{10}) \lambda \ln \lambda \\
& + \frac{4}{35} \pi_5 \lambda^{-4} \left(\frac{7}{10} + \ln \lambda \right) + \frac{4}{35} \pi_{12} \lambda^3 \left(\frac{7}{10} - \ln \lambda \right) + (\omega_{12} - 8\delta_8) \frac{\lambda^3}{16} \\
& \left. + \frac{(\omega_5 - 8\delta_7)}{10} \lambda^{-4} \right] \left. \right\}, \tag{5.62a}
\end{aligned}$$

$$\begin{aligned}
\delta_6 & = \frac{1}{1-\lambda^3} \left\{ -\left[\sum_{j \neq 5, 7, 10, 12} \frac{\omega_j (j-5)(j-12) - 8\pi_j}{(j-5)(j-7)(j-10)(j-12)} + \frac{2}{25} (\pi_5 + \pi_{12}) \right. \right. \\
& + \frac{1}{16} (\omega_{12} - 8\delta_8) + \frac{1}{10} (\omega_5 - 8\delta_7) \left. \right] + \lambda^2 \left[\sum_{j \neq 5, 7, 10, 12} \frac{\omega_j (j-5)(j-12) - 8\pi_j}{(j-5)(j-7)(j-10)(j-12)} \lambda^{j-9} \right. \\
& - \frac{1}{3} (\omega_7 + \frac{4\pi_7}{5}) \lambda^{-2} \ln \lambda + \frac{1}{3} (\omega_{10} - \frac{4}{5}\pi_{10}) \lambda \ln \lambda
\end{aligned}$$

$$\begin{aligned}
& + \frac{4}{35}\pi_5\lambda^{-4} \left(\frac{7}{10} + \ell n\lambda \right) + \frac{4}{35}\pi_{12}\lambda^3 \left(\frac{7}{10} - \ell n\lambda \right) + (\omega_{12} - 8\delta_8) \frac{\lambda^3}{16} \\
& \quad + \frac{(\omega_5 - 8\delta_7)}{10} \lambda^{-4} \left. \right\} , \tag{5.62b}
\end{aligned}$$

$$\begin{aligned}
\delta_7 = \frac{\lambda^4}{\lambda^7-1} \left\{ -\lambda^3 \sum_{j \neq 5,12} \frac{\pi_j}{(j-5)(j-12)} + \sum_{j \neq 5,12} \frac{\pi_j \lambda^{j-9}}{(j-5)(j-12)} \right. \\
\left. + \frac{\ell n\lambda}{7} (\pi_{12}\lambda^3 - \pi_5\lambda^{-4}) \right\} \tag{5.62c}
\end{aligned}$$

$$\begin{aligned}
\delta_8 = \frac{1}{\lambda^7-1} \left\{ -\lambda^4 \left(\sum_{j \neq 5,12} \frac{\pi_j \lambda^{j-9}}{(j-5)(j-12)} + \frac{\ell n\lambda}{7} [\pi_{12}\lambda^3 - \pi_5\lambda^{-4}] \right) \right. \\
\left. + \sum_{j \neq 5,12} \frac{\pi_j}{(j-5)(j-12)} \right\} , \tag{5.62d}
\end{aligned}$$

with ω_j and π_j given by (5.58).

Solution for the Pressure Field

The equations for the pressure field are obtained by substituting (5.37) and (5.54) into equations (5.39) and (5.39b) as:

$$\begin{aligned}
P_{1r} = \left[\left(\frac{d^2}{r^5} + \frac{2de}{r^2} + e^2 r \right) - 3 \sum_j \sigma_j (j-2)(j-7) r^{j-8} - \frac{36d^2 S}{r^7} \right] \sin^2 \theta \\
+ 2 \sum_j \sigma_j (j-2)(j-7) r^{j-8} , \tag{5.63a}
\end{aligned}$$

$$P_{1\theta} = \left[\left(\frac{d^2}{r^4} + \frac{2de}{r} + e^2 r^2 \right) - \sum_j (j-2)(j-6)(j-7)\sigma_j r^{j-7} \right] \sin\theta \cos\theta, \quad (5.63b)$$

where

$$P_{1r} = \frac{\partial P_1}{\partial r} \quad \text{and} \quad P_{1\theta} = \frac{\partial P_1}{\partial \theta} .$$

Solving these systems of equations, we have

$$P_1 = \left[\left(\frac{d^2}{r^4} + \frac{2de}{r} + e^2 r^2 \right) - \sum_j (j-2)(j-6)(j-7)\sigma_j r^{j-7} \right] \frac{\sin^2 \theta}{2} + \sum_{j \neq 7} 2\sigma_j (j-2)r^{j-7} + P_{10} \quad (5.64)$$

where the constant of integration P_{10} can be determined as the pressure corresponding to any given r and θ , say $r = r_0$ with $1 \leq r_0 \leq \lambda$ and $\theta = 0$; d and e are given by (5.36); σ_j given by (5.55).

Summary of Solutions

Primary Flow (zeroth approximation):

$$U_0 = 0 = V_0 ,$$

$$P_0 = \text{constant},$$

$$W_0 = \left(\frac{d}{r^2} + er \right) \sin\theta ,$$

where d and e are given by (5.36).

Secondary Flow (first order approximation):

$$\text{Streamlines -- } \psi_1(r, \theta) = \left(\sum_j \sigma_j r^{j-4} \right) \sin^2 \theta \cos \theta,$$

$$\text{Velocity Components -- } U_1(r, \theta) = \left(\sum_j \sigma_j r^{j-4} \right) (2 - 3 \sin^2 \theta),$$

$$V_1(r, \theta) = - \left(\sum_j \sigma_j (j-4) r^{j-6} \right) \sin \theta \cos \theta,$$

$$W_1(r, \theta) = N_1(r) \sin \theta + N_2(r) \sin^3 \theta,$$

where σ_j , N_1 and N_2 are given by (5.55), (5.62a) and (5.62b) respectively.

Pressure Field -- P_1 is given by (5.64).

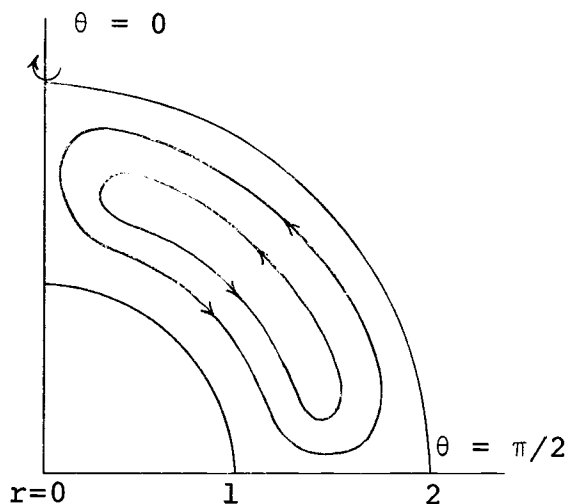
5.5 Discussion of the Results

In order to make a complete investigation of the flow pattern, we first obtain the streamline function ψ_1 from equation (5.53) by using the expressions for the velocity components given by (5.54a and b). We find that the streamline function ψ_1 vanishes at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2},$ and 2π for all values of r and is zero at $r = 1$ and $r = \lambda$ for all values of θ . Thus the secondary flow in a meridian plane is divided into four quadrants by the axis of rotation and the equator ($\theta = \frac{\pi}{2}$). For a Newtonian fluid ($S = 0, K = 0$), the secondary flow consists of closed loops,

with the direction of flow being toward the common center of the spheres at the poles and away from this common center at the equator (see figure 5.2). The secondary flow pattern for a non-Newtonian fluid is, to a first approximation in h , strongly dependent upon the rheological parameter S characterizing viscoelasticity. There exist parametric values, say S_0 and S_1 , which are dependent upon the rate of rotations of the spheres, for which the flow pattern changes. For $S < S_0$, the secondary flow pattern is very similar to that of a Newtonian fluid with the inclusion of inertial terms. At the critical value S_0 , the secondary flow in each quadrant is further broken into two subregions, each containing closed loops (see figure 5.3). As the parameter S is increased from S_0 to S_1 , the interface between the two regions moves towards the outer sphere, and at S_1 , the outer subregion vanishes and once again each quadrant is filled up with the same type of loops. The direction of these loops is, however, in the opposite direction from that of Newtonian flow (see figure 5.4). This is due to the viscoelastic nature of the fluid.

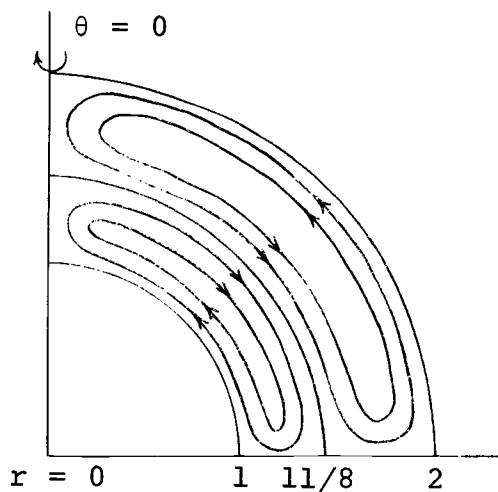
To illustrate the above and give precise breaking values S_0 and S_1 of the streamlines, we let the inner sphere rotate with angular velocity $\Omega_1 = 0.2$ (non-dimensional form) and the outer sphere rotates with $\Omega_2 = 0.1$

so that h (5.26) $\sim .01$ and $h^2 \ll h$. Letting $\lambda = 2$, we obtain, by use of the computer program VELØCITY found in the appendix, $S_0 = 0.38$ and $S_1 = 0.56$.



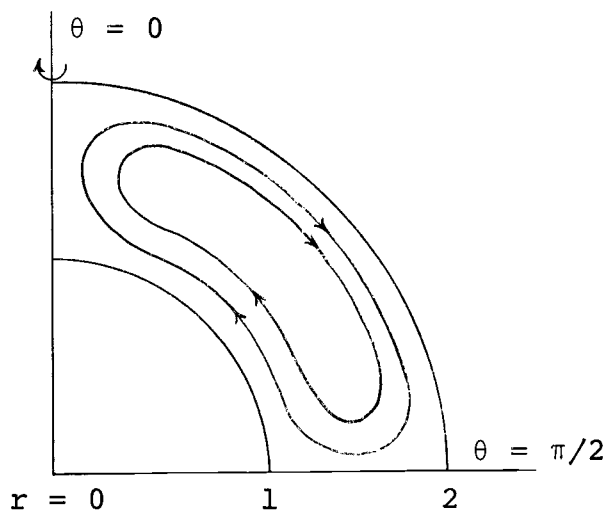
(Figure 5.2)

Typical streamlines of the secondary flow for the case $S' = 0$ [$\Omega_1 = 0.2$, $\Omega_2 = 0.1$, and $\lambda = 2$].



(Figure 5.3)

Typical streamlines of the secondary flow for the case $S = 4.5$. [$\Omega_1 = 0.2$, $\Omega_2 = 0.1$, and $\lambda = 2$].



(Figure 5.4)

Typical streamlines of the secondary flow for the case $S' = 5.5[\Omega_1 = 0.2, \Omega_2 = 0.1, \text{ and } \lambda = 2]$.

CHAPTER VI

SUMMARY, DISCUSSION, AND SCOPE OF FURTHER WORK

6.1 Summary and Discussion

The response of real materials to external loading is, in general, nonlinear in character. The classical theories which were designed to explain the behavior of materials subjected to deforming forces use classical measures of deformation and assume complicated constitutive equations involving many unknown response coefficients. Hence a wide spread interest in the search for more general theories has arisen. In our present work, we have given a brief outline of the various theories governing the behavior of real materials. The pioneering work done by Reiner, Rivlin, Ericksen, Green, Oldroyd, Noll, etc. have used ordinary measures of deformation or deformation-rate. The use of linear or classical measures of deformation have strained the constitutive equations into complex forms involving many unknown response coefficients and powers and products of ordinary measures. The main source of the difficulty is the use of classical measures instead of generalized measures. Another pioneer, Seth, however, attempted to resolve this difficulty by the introduction of generalized measures of deformation or deformation-rate in continuum mechanics. Using this idea, Narasimhan and Sra extended these general-

ized measures of Seth in such a way as to explain adequately some rheological behavior of materials. The constitutive equations they set up, based on generalized measures, contain essentially two terms and at most four rheological constants. Unlike some previous theories, these constitutive equations do not contain any unknown functions of the invariants of kinematic matrices, etc.

In the present investigation, we have proved an important theorem concerning the generalized measures of deformation rates which enables one to predict precisely, by fixing the orders of these measures, a variety of viscoelastic phenomena such as dilatancy, pseudoplasticity, and so on. We also found that the constitutive equation of Narasimhan and Sra does not adequately describe the important physical phenomenon pseudoplasticity, which abounds in every day life. To remedy this, we found it necessary to construct combinations of generalized measures. We found that the constitutive equation using the combined measures still does not contain any unknown functions of the invariants of kinematic matrices, etc.; but is much more general and hence is able to describe a wider range of physical phenomena.

To illustrate the use of combined generalized measures in constitutive equations, we have considered the very interesting and important problem of secondary flows in

fluids in the presence of moving boundaries. For this purpose we arbitrarily fixed certain orders of the measures of the rates of deformation D^{**} and B^{**} . In actual practice however, the choice of the orders of measures will have to be determined by experiments. In examining the fluid flow in the presence of moving boundaries, we found it necessary to investigate secondary flows in order to obtain a true picture of the actual flow. For the problem of flow of a fluid in the annulus of two rotating spheres, we have, for the first time, obtained a satisfactory scientific basis for explaining the secondary flow phenomena based on the combined generalized measure concept. Expressions for the velocity and pressure fields have been obtained by solving the basic field equations. In order to study the secondary flow pattern, the streamline function has been obtained. The streamlines of the flow in meridian planes containing the axis of rotation are found to be closed loops and the nature of these loops is found to be strongly dependent upon the viscoelastic parameter S . There exist parametric values, say S_0 and S_1 , which are dependent upon the rate of rotations of the spheres, for which the flow pattern changes. For S less than the critical value S_0 , the flow is very much like

that of a Newtonian fluid, with the fluid advancing toward the inner sphere along the pole and outward along the equator. At S_0 , the flow region splits into two subregions, each containing closed loops of streamlines. As S increases toward S_1 , the interface between the two subregions advances toward the outer sphere, and at S_1 , the streamlines again become one set of closed loops, with the sense reversed from that of Newtonian fluid. With the aid of the CDC 3300 computer, we were able to find precise values of S_0 and S_1 . With $\Omega_1 = 0.2$, $\Omega_1 = 0.1$, and $\lambda = 2$, we found that $S_0 = .38$ and $S_1 = .56$.

6.2 Scope of Further Work

It appears at this time that nonlinear theory based on generalized measures has a clear advantage over the nonlinear theories based on ordinary measures. Therefore, there is a natural desire to exploit the idea of generalized measures in the investigation of nonlinear phenomena. We saw by combining sets of generalized measures, we were able to adequately describe pseudoplastic behavior. Another extension of the generalized measure concept would be to describe materials which have memory, a very important phenomenon in mechanics of deformable bodies.

BIBLIOGRAPHY

- Bhatnagar, P.L. and G.K. Rajeswari. 1962. The secondary flows induced in a non-Newtonian fluid between two parallel infinite oscillating planes. *Journal of the Indian Institute of Science* 44:219-238.
- _____ 1963. Secondary flow of non-Newtonian fluid between two concentric spheres rotating about an axis. *Indian Journal of Mathematics* 15:93-112.
- Bhatnagar, P.L. and S.L. Rathna. 1963. Flow of a fluid between two rotating coaxial cones having the same vertex. *Quarterly Journal of Mechanics and Applied Mathematics* 16:329-346.
- Criminale, W.O., J.L. Ericksen and G.L. Filbey, Jr. 1958. Steady shear flow of non-Newtonian fluids. *Archive for Rational Mechanics and Analysis* 1:410-417.
- Ericksen, J.L. 1956. Overdetermination of the speed in rectilinear motion of non-Newtonian fluids. *Quarterly of Applied Mathematics* 14:318-321.
- Eringen, A.C. 1962. *Nonlinear theory of continuous media*. New York, McGraw-Hill. 477 p.
- Green, A.E. and R.S. Rivlin. 1956. Steady flow of non-Newtonian fluids through tubes. *Quarterly of Applied Mathematics* 14:299-308.
- _____ 1957. The mechanics of non-linear materials with memory. Part I. *Archive for Rational Mechanics and Analysis* 1:1-21.
- Haberman, W.L. 1962. Secondary flow about a sphere rotating in a viscous liquid inside a coaxially rotating spherical container. *The Physics of Fluids* 5:625-626.
- Hoppmann, W.H., II and C.N. Baronet. 1964. Flow generated by a cone rotating in a liquid. *Nature* 201 (Part 2): 1205-1207.
- Hoppmann, W.H., II and C.H. Miller. 1963. Rotational fluid flow generator for studies in rheology.

- Jones, J.R. 1960. Flow of a non-Newtonian liquid in a curved pipe. Quarterly Journal of Mechanics and Applied Mathematics 13:428-443.
- Langlois, W.E. 1963. Steady flow of a slightly viscoelastic fluid between rotating spheres. Quarterly of Applied Mathematics 21:61-71.
- Merrington, A.C. 1943. Flow of viscoelastic materials in capillaries. Nature 159:310-311.
- Narasimhan, M.N.L. and K.S. Sra. 1968. Generalized measures of deformation-rates in viscoelasticity and their applications to rectilinear and simple shearing flows. International Journal of Nonlinear Mechanics. (In press).
- Noll, W. 1958. A mathematical theory of the mechanical behavior of continuous media. Archive for Rational Mechanics and Analysis 2:197-226.
- Oldroyd, J.G. 1951. The motion of elastico-viscous liquid contained between coaxial cylinders, I. Quarterly Journal of Mechanics and Applied Mathematics 4:271-282.
- Reiner, M. 1945. A mathematical theory of dilatancy. American Journal of Mathematics 67:350-362.
- Rivlin, R.S. 1948. The hydrodynamics of non-Newtonian fluids, I. Proceedings of the Royal Society (London), Ser. A, 193:260-281.
- Rivlin, R.S. and J.L. Ericksen. 1955. Stress-deformation relations for isotropic materials. Journal of Rational Mechanics and Analysis 4:323-425.
- Rosenblat, S. 1960. Flow between torsionally oscillating disks. Journal of Fluid Mechanics 8:388-399.
- Seth, B.R. 1966a. Generalized strain and transition concept for elastic-plastic deformation-creep and relaxation. In: Proceedings of the Eleventh International Congress of Applied Mechanics, Munich, 1964. Berlin, Springer, p. 383-389.
- _____ 1966b. Measure-concept in mechanics. International Journal of Non-linear Mechanics. 1:35-40.

- Stone, D.E. 1957. On non-existence of rectilinear motion in plastic solids and non-Newtonian fluids. Quarterly of Applied Mathematics 15:257-262.
- Thomas, R.H. and K. Walter. 1964. The motion of an elastico-viscous liquid due to a sphere rotating about a diameter. Quarterly Journal of Mechanics and Applied Mathematics. 17:39-53.
- Waters, N.D. 1964. A note on the secondary flow about a sphere rotating in an elastico-viscous liquid inside a rotating concentric spherical container. British Journal of Applied Physics 15:600-602.
- Weissenberg, K. 1947. A continuum theory of rheological phenomena. Nature 159:310-311.

Appendix

To aid us in finding numerically the velocity components U_1 and V_1 ((5.54a) and (5.54b) respectively), for each r and θ , we used the following Fortran program. First, however, we shall explain the symbols used in the program.

<u>Notation:</u>	<u>Problem:</u>
A	λ
AD	increment change in θ
B1, B2	end points of θ range
C1, C2, C3, C4	C_1, C_2, C_3, C_4 respectively
D	d
E	e
H	H
OM1, OM2	Ω_1, Ω_2 respectively
OM	Ω
RR	radius of inner sphere
S	S, rheological parameter
SI(J)	σ_j
U(,), V(,)	U_1 and V_1 , respectively
W	increment change in θ

Fortran Program

```

PROGRAM VELOCITY
DIMENSION SI(9), U( , ), V( , )
READ (60,1) A,OM1,OM2,S,RR,W,B2,B1,AD
1 FORMAT ( )

```

```

D=A**3/(A**3-1)
OM=OM1-OM2
E=(OM2*A**3-OM1)/(OM*(A**3-1.0))
C1=0.25*(D*E-D*D-2*S*D*D)
C2=0.25*(2*D*E+D*D+6*S*D*D)
C3=0.25*(D*E*A**2-D*D/A-2*S*D*D/A**3)
C4=0.25*(2*D*E*A+D*D/A**2+6*S*D*D/A**4)
H=(4*A**10-25*A**7+42*A**5-25*A**3+4)/A**3
SI(1)=0.5*S*D*D
SI(2)=-((2*A**7-5*A**4+3*A**2)*C2+15*(A**4-A**2)*
(C1-C3)+(3*A**5-5*A**3+2)*C4)/H
SI(3)=0.25*D*D
SI(4)=((4*A**7+21*A**2-25)*C1+(2*A**7-7*A**2+5)*C2+
(-25*A**4+21*A**2+4/A**3)*C3+(5*A**5-7*A**3+2/A**2)*
C4)/H
SI(5)=0.0
SI(6)=-0.25*D*E
SI(7)=((-5*A**4+7*A**2-2/A**3)*C2+10*(1/A**3-A**4)*
(C1-C3)+(-2*A**5+7-5/A**2)*C4)/H
SI(8)=0.0
SI(9)=((3*A**2-5+2/A**3)*C2+6*(A**2-1/A**3)*(C1-C3
+(2*A**3-5+3/A**2)*C4)/H
MAX R=(A-1.0)*RR/W+1.0
MAX B=(B2-B1)/AD+1.0
DØ 3 K=1, MAX R
DØ 2 L=1, MAX B
2 U(K,L)=0.0
3 V(K,L)=0.0
DØ 7 K=1, MAX R
R=RR+(K-1)*W
DØ 6 L=1, MAX B
B=B1+(L-1)*AD
DØ 4 J=1, 9

```



```
4 U(K,L)=U(K,L)+SI(J)*R**(J-6)
  U(K,L)=U(K,L)*(2-3*sin(B)*sin(B))
  DØ 5   J=1, 9
5 V(K,L)=V(K,L)+SI(J)*(J-4)*R**(J-6)
6 V(K,L)=-V(K,L)*sin(B)*cos(B)
7 CØNTINUE
  PRINT 8
8 FØRMAT (   )
  STØP
  END
```