Subalgebras of the Split Octonions

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1 Introduction

It is well known that the only proper subalgebras of the octonions are the reals, complexes, and quaternions. This statement is false for the split octonions. We will provide here a complete list of all possible signatures of the subalgebras of the split octonions.

1.1 Complex numbers

The complex numbers, $\mathbb{C}$, can be constructed by taking the real numbers and adding a square root of -1; call it i. This creates $\mathbb{C}$ in the form $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$. Hence, a complex number z can be written as a pair of real numbers (x, y) where $z = x + iy$. The complex numbers are attributed to Rafael Bombelli, who in 1572 formalized the rules with complex arithmetic. However, the complex numbers were first named imaginary by René Descartes in 1637.

The complex numbers both associate and commute, that is, for $x, y, z \in \mathbb{C}$,

$$(xy)z = x(yz), \tag{1}$$

$$xy = yx. \tag{2}$$

The complex conjugate $\overline{z}$ of a complex number (x, y) is defined as $\overline{z} = (x, -y)$, changing the sign of the imaginary part of z. The norm $|z|^2$ of a complex number is defined as $|z|^2 = z\overline{z}$. The only complex number with norm zero is zero and the norms of the complex numbers satisfy,

$$|y|^2 \, |z|^2 = |yz|^2 \,. \tag{3}$$

1.2 Quaternions

Now add another independent square root of -1, call it j. We must then check what is ij. In 1843, Sir William Rowan Hamilton suggested $k = ij$, where k is another square root of -1, as the three independent generators $1, i$, and j will not close without this fourth independent generator [1]. Furthermore, Hamilton proposed the multiplication table should be cyclic, that is,

$$ij = k = -ji \tag{4}$$

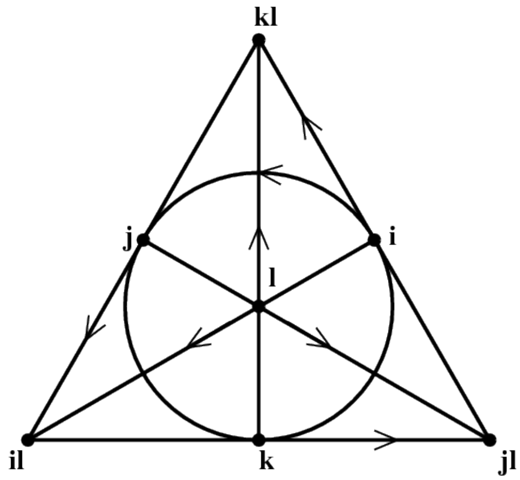

Figure 1: Fano plane depicting multiplication of octonions

where i, j, and k are considered imaginary quaternionic units. Observe that these units anticommute. The quaterions are denoted by $\mathbb{H}$.

Any quaternion can be written as four real elements or two complex elements, i.e. $q = q_1 + q_2 i + q_3 j + q_4 k = z_1 + z_2 j$ or $q = (z_1, z_2)$. Thus, we write $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$. The quaternions are associative, as $q(rs) = (qr)s$, but not commutative. The norm of a quaternion is

$$|q|^2 = q\overline{q} = q_1^2 + q_2^2 + q_3^2 + q_4^2.$$

Quaternionic conjugation satisfies $\overline{pq} = \overline{q}\,\overline{p}$ where $\overline{q} = (\overline{z_1}, -z_2)$ and it follows that $|pq|^2 = |p|^2\,|q|^2$.

1.3 Octonions

Again, we add another independent square root of -1, call it ℓ. Once we add the corresponding products $i\ell, j\ell, k\ell$, any element $x \in \mathbb{O}$ can be expressed as a pair of quaternions, q_1, q_2 as $x = (q_1, q_2)$. Conjugation can be written as $\overline{(q_1, q_2)} = (\overline{q_1}, -q_2)$. Thus, we see $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$. The multiplication table can be expressed using the Fano plane, as seen in Figure 1. Both John T. Graves and Arthur Cayley are independently given credit for the discovery of the octonions [1].

Octonions neither associate nor commute. However, we do have the property of the norm that $|pq|^2 = |p|^2\,|q|^2$.

1.4 The Cayley–Dickson Process

Observe that by doubling smaller algebras, we have built new algebras. Consider the following:

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i, \tag{5}$$
$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j, \tag{6}$$
$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell. \tag{7}$$

All of these are examples of Cayley–Dickson algebras where

$$\begin{aligned} \overline{(a,b)} &= (\overline{a}, -b) \\ (a,b)(c,d) &= (ac - \epsilon\overline{d}b, da + b\overline{c}) \\ (ab)\overline{(a,b)} &= (|a|^2 + \epsilon\,|b|^2, 0), \end{aligned}$$

for $\epsilon = \pm 1$. These definitions of conjugation, multiplication, and norm, respectively, extend those operations in a given algebra $\mathbb{K}$ to a new algebra $\mathbb{K} \oplus \mathbb{K}$. With these examples as motivation, new algebras can be generated by smaller composition algebras with successive choices of ϵ at each level. Here we note an alegbra is a vector space that closes under multiplication.

These algebras are also composition algebras, as a composition algebra possesses a norm, namely a nondegenerate quadratic form satisfying the identity

$$|pq|^2 = |p|^2\,|q|^2\,. \tag{8}$$

Posthumously in 1923, the Hurwitz Theorem was published, which states that the reals, complexes, quaternions, and octonions are the only real composition algebras with positive-definite norm.

Using the Cayley–Dickson process, we can construct three new algebras, with non-positive definite norms, namely the split versions of the algebras we have. For the split complexes, $\mathbb{C}'$, we apply the Cayley–Dickson process with $\epsilon = -1$ and the reals. Thus $\mathbb{C}' = \mathbb{R} \oplus \mathbb{R}L$, where $L^2 = 1$ and $|L|^2 = -1$. Then for an element $z = (a,b)$, $|z|^2 = (a^2 - b^2, 0)$. We also observe that $\mathbb{C}'$ contains null elements such as $(1 + L)$, since

$$\begin{aligned} |1+L|^2 &= (1+L)(1-L) \\ &= 1 - L^2 \\ &= 0. \end{aligned} \tag{9}$$

$\times$	1	I	J	K	L	IL	JL	KL
1	1	I	J	K	L	IL	JL	KL
I	I	-1	K	$-J$	IL	$-L$	$-KL$	JL
J	J	$-K$	-1	I	JL	KL	$-L$	$-IL$
K	K	J	$-I$	-1	KL	$-JL$	IL	$-L$
L	L	$-IL$	$-JL$	$-KL$	1	$-I$	$-J$	$-K$
IL	IL	L	$-KL$	JL	I	1	K	$-J$
JL	JL	KL	L	$-IL$	J	$-K$	1	I
KL	KL	$-JL$	IL	L	K	J	$-I$	1

Table 1: This is the multiplication table of the generators of the split octonions.

Here $(1 + L)$ is both a null element and a zero divisor.

Similarly, the Cayley–Dickson process can be used to construct the split quaternions, $\mathbb{H}' = \mathbb{C} \oplus \mathbb{C}L$. To better differentiate the split algebras from their non-split counterparts, we will use capital letters to denote the imaginary generators. Thus, $\mathbb{H}'$ has one possible basis of $\{1, K, L, KL\}$ where $K^2 = -1$ and $(KL)^2 = 1$.

The basis $\langle 1, K, L, KL \rangle$ is orthonormal, meaning that each basis element has norm one, and has dot product zero with all other basis elements, where norm is calculated by multiplying the vector with its conjugate. This algebra has signature (2,2,0), where signature is found by calculating the norm of each basis element and counting the number of positive, negative and null normed elements. In this subalgebra, two basis elements have norm greater than zero, two have norm less than zero, and none having norm zero.

Last but not least, the split octonions can be constructed as $\mathbb{O}' = \mathbb{H} \oplus \mathbb{H}L$, generated by $\{1, I, J, K, L, IL, JL, KL\}$. The multiplication table can be seen in Table 1. The split octonions are not commutative or associative. It is easy to check that $\mathbb{O}'$ has signature (4,4,0). This is the algebra whose subalgebras we will be investigating.

1.5 Signature

To obtain a signature, the generating basis must be orthogonal, otherwise the signature will be inconsistent. Consider the subalgebras $\langle 1, I, L, IL \rangle$ with apparent signature (2,2,0) and $\langle 1, L, IL, I + IL \rangle$ with apparent signature (1,2,1). But the basis, $\{1, L, IL, I + IL\}$ is not orthogonal. Its associated orthogonal basis is $\{1, I, L, IL\}$. As such, the true signature is (2,2,0).

From the squared norms of a set of independent orthogonal generators, we find the signature of a space. We observe for $a \in \mathbb{H}$, $|a|^2 > 0$. For $b \in \mathbb{H}L$, $|b|^2 < 0$. For example, the signature of $\mathbb{O}'$ is (4,4,0) as $1, I, J, K$ all have squared norms of $+1$; L, IL, JL, KL all have squared norms of -1, and there are no basis elements of norm 0.

Also, in the split octonions we find idempotent elements, namely elements that square to themselves. One such element is $\frac{1}{2}(1 - L)$.

A degenerate subspace of a subalgebra is a set of elements orthogonal to every other element in the space.

Proposition 1.1. *If a degenerate subspace of a subalgebra of $\mathbb{O}'$ closes, then it is an ideal.*

Proof. Without loss of generality, assume the algebra as signature (P, M, N) with $P > 0$. Thus 1 is in the nondegenerate part of the algebra. Hence we can assume without loss of generality that the degenerate element has the form $J+JL$. Neither J nor JL can be in the nondegenerate part of the subalgebra, otherwise, $J + JL$ would not be in the degenerate subspace. Products of 1 or L with $J + JL$ are in the degenerate subspace. Any other element in the nondegenerate subspace can be assumed without loss of generality to be either I or IL; in both cases the product is $K - KL$ up to a sign. However, if $K - KL$ is in the nondegenerate subspace, then so must be $K + KL$. But the product of $K + KL$ with I or IL now forces $J - JL$ to be in the subalgebra which is a contradiction. □

2 1-D Subalgebras

There are three possible signatures of a 1-D subalgebra, namely (1,0,0), (0,1,0), and (0,0,1). A subalgebra with signature (1,0,0) is the real line, $\mathbb{R}$, which we know and love. This is the only subalgebra of $\mathbb{O}'$ with signature (1,0,0), as I, J, K all square to -1 and any linear combinations of these will square to a real number, thus necessitating a real element in their span. The corresponding 1-D subset does not close.

Next, signature $(0, 1, 0)$ is similarly impossible, as any timelike element or elements of $\mathbb{H}L$ will square to a real number. Thus, the subalgebra will not close, as the element multiplied with itself generates something not in its span.

For signature $(0, 0, 1)$, we find two possible case: idempotent and null nilpotent.

An example of an idempotent subalgebra is the subalgebra generated by the element $\frac{1}{2}(1 + L)$. This element is projective, since it squares to itself, i.e.

$$\left(\frac{1}{2}(1+L)\right)^2 = \frac{1}{4}(2+2L) = \frac{1}{2}(1+L), \tag{10}$$

and $\frac{1}{2}(1 + L)$ is null, because its norm is zero, i.e.

$$\left|\frac{1}{2}(1+L)\right|^2 = \frac{1}{4}(1+L)(1-L) = 0. \tag{11}$$

Other idempotent elements include elements of the form $\frac{1}{2}(1 \pm aL)$ for $a \in \mathbb{H}$ where $\mid a \mid= 1$.

On the other hand, $\frac{a}{2}(1 + L)$ for $a \in \mathbb{H}$ with $|a|^2 = 1$ and $Re(a) = 0$ will produce nilpotent elements. Subalgebras of signature (0,0,1) that are nilpotent have generating vectors that square to zero. Another example is $a + bL$ for $a \in \{I, J, K\}$ and $b \in \mathbb{H}$ with $(bL)^2 = 1$. Consider

$$\begin{aligned}(a+bL)^2 &= a^2 + a(bL) + (bL)a + (bL)(bL) \\ &= -1 + a(bL) - a(bL) + 1 \\ &= 0.\end{aligned} \tag{12}$$

Thus, all elements of the form $a \pm bL$ for $a \in \mathbb{H} - \mathbb{R}$ and $b \in \mathbb{H}$ where $a^2 = -1$ and $(bL)^2 = 1$ are nilpotent elements that generate completely nilpotent 1-D subalgebras.

In general, elements with both $\mathbb{H}$ and $\mathbb{H}L$ type parts can be made idempotent with appropriate coefficients for each term. Consider $q = (a + b + cL)$ where $a \in \mathbb{R}$, $b \in Im\mathbb{H}$, $c \in \mathbb{H}$. Then the norm is

$$\begin{aligned}|q^2| &= |a+b+cL|^2 \\ &= (a+b+cL)(a-b-cL) \\ &= a^2 - |ab| - a(cL) + |ab| - |b|^2 - b(cL) + a(cL) - (bc) - (cL)b - (cL)^2) \\ &= a^2 + |b|^2 - |c|^2.\end{aligned} \tag{13}$$

Furthermore, the square of q is

$$\begin{aligned}q^2 &= (a+b+cL)^2 \\ &= a^2 + 2ab + |2ac|L - b^2 + |c|^2 \\ &= (a^2 - |b|^2 + |c|^2) + (2ab) + |2ac|L.\end{aligned} \tag{14}$$

Signature	
(1,0,0)	$\mathbb{R} = \langle 1 \rangle$
(0,1,0)	Not possible
(0,0,1)	Two kinds: $\langle 1 \pm L \rangle$(idempotent) and $\langle I \pm IL \rangle$ (nilpotent)

Table 2: All possible signatures of 1-D subalgebras and an example of a corresponding subalgebra, if it exists.

Thus, q can produce different kinds of elements, supposing that the element, q is null or $a^2+b^2-c^2=0$. Idempotent elements square to themselves, so

$$a^2 - b^2 + c^2 = a, \tag{15}$$

$$2ab = b, \tag{16}$$

$$2ac = c. \tag{17}$$

Thus we see,

$$c^2 - b^2 = \frac{1}{4} \tag{18}$$

and

$$a = \frac{1}{2}. \tag{19}$$

Equation 14 can extend to nilpotent elements as well. Nilpotent elements square to zero, i.e. $a^2 - b^2 + c^2 = 2ab = 2ac = 0$, implying $a = 0$ and $b^2 = c^2$. These results are summarized in Table 2.

$\times$	$1+L$	$I+IL$
$1+L$	$2(1+L)$	0
$I+IL$	$2(I+IL)$	0

Table 3: Products of $1+L$ and $I+IL$.

$\times$	$I+IL$	$J-JL$
$I+IL$	0	0
$J-JL$	0	0

Table 4: Products of $I+IL$ and $J-JL$

3 2-D Subalgebras

In the case of 2-D subalgebras, we need the signature to add up to 2. As such, there are three possible "2-of-a-kind" signatures and three combinations of two 1's. Thus, the possible signatures are (2,0,0), (0,2,0), (0,0,2), (1,0,1), (1,1,0), (0,1,1).

We will start with the most familiar, which is signature $(2,0,0)$. This signature is generated from the Cayley–Dickson process. As per the Hurwitz theorem, the only real composition algebras with positive-definite norm are $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$, with dimensions 1, 2, 4 and 8, respectively. Thus, with the Cayley–Disckson process we exhaust subalgebras of signature (2,0,0) with examples such as $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}I$ as well as any other $\mathbb{R} \oplus \mathbb{R}a$ for $a \in \mathbb{H}$, $a \notin \mathbb{R}$.

On the other hand, subalgebras of signature (0,2,0) are not possible. Let's start by considering two elements $aL, bL \in \mathbb{H}L$ where $a, b \in \mathbb{H}$. We see the following, using the Moufang identity,

$$(aL)(bL) = (L\bar{a})(bL) = L(\bar{a}b)L = \bar{b}a \in \mathbb{H}. \tag{20}$$

In this way, the subspace does not close for any two elements in $\mathbb{H}L$.

For signature (0,0,2), we can find two possible types of subalgebras, both extensions of 1-D subalgebras. One type of subalgebra is generated by an idempotent element and a nilpotent element; the other type of subalgebra is generated by two nilpotent elements. In the former case, an example is $\langle 1+L, I+IL \rangle$. Consider the multiplication table as shown in Table 3. Thus $\langle 1+L, I+IL \rangle$ is not completely null. The composition of two nilpotent elements, will produce a completely null subalgebra. Consider $\langle I+IL, J-JL \rangle$, whose multiplication table as shown in Table 4. In both cases, the two

$\times$	1	$\frac{1}{2}(1+L)$
1	1	$\frac{1}{2}(1+L)$
$\frac{1}{2}(1+L)$	$\frac{1}{2}(1+L)$	$\frac{1}{2}(1+L)$

Table 5: Products of 1 and idempotent element $\frac{1}{2}(1+L)$.

$\times$	1	$I+IL$
1	1	$I+IL$
$I+IL$	$I+IL$	0

Table 6: Products of 1 and nilpotent element $I+IL$.

generators have square norm zero. Thus by $|qp|^2 = |p|^2|q|^2$, the product of the two elements will have norm zero.

Continuing with our signature exploration, we come to a possible signature of (1,0,1), with one element of $\mathbb{H}$ and another element with norm zero. A simple example with this signature is generated by 1 and a null element such as $\frac{1}{2}(1+L)$ or $I+IL$. The multiplication tables for both cases are listed in Table 5 and Table 6.

The sub algebra generated by 1 and an idempotent element such as $\frac{1}{2}(1+L)$ is contained in the subalgebra generated by 1 and L with signature (1,1,0). Thus the sub algebra generated by 1 and an idempotent element such as $\frac{1}{2}(1+L)$ is not signature (1,0,1).

There exist subalgebras with signature (1,1,0), namely $\langle 1, bL\rangle$ for $b \in \mathbb{H}$. These subalgebras are the split complex numbers, $\mathbb{C}'$, where, instead of the Cayley–Dickson process using $\epsilon = -1$ to create the complex numbers, we use $\epsilon = +1$. It suffices to show $\langle 1, L\rangle$ is closed. Observe the multiplication table in Table 7.

The last signature to consider for 2-D subalgebras is signature (0,1,1). However, any element q such that $|q|^2 = -a$ for $a > 0$ will have $|qq|^2 = |q|^2|q|^2 = (-a)(-a) = a^2$, with $a^2 > 0$. Thus, q^2 is not in our subspace as its norm is not possible in our signature. Thus, subspaces of signature (0,1,1)

$\times$	1	L
1	1	L
L	L	1

Table 7: Products of 1 and L.

Signature	
(2,0,0)	$\mathbb{C}$
(0,2,0)	Not possible
(0,0,2)	Two kinds $\langle 1+L, I+IL\rangle$ and $\langle I+IL, J-JL\rangle$
(1,0,1)	Nilpotent $\langle 1, I+IL\rangle$
(1,1,0)	Split Complex:$\langle 1, L\rangle$
(0, 1, 1)	Not possible

Table 8: All possible signatures of 2-D and an example of a corresponding subalgebra, if it exists.

do not form subalgebras.

To summarize, we see in Table 8 there are only four possible signatures of 2-D subalgebras that exists.

	$I+IL$	$J+JL$	$K-KL$
$I+IL$	0	$2(K-KL)$	0
$J+JL$	$-2(K-KL)$	0	0
$K-KL$	0	0	0

Table 9: This is the multiplication table of the algebra $\langle I+IL, J+JL, K-KL\rangle$.

4 3-D Subalgebras

As the signature must add up to the dimension of the space, we know that the signature of 3-D subalgebras adds up to 3. Keeping this in mind, there are 3 possible signatures with three of the same kind of element, 6 possible signatures with two of one kind and one of another, and 1 possible signature of one of each kind of element. This gives us seven possible signatures.

Several of the examples here are found in Dray and Manogue's "The Geometry of the Octonions" (2015)[1]. This section expands on their contributions and lists all possible signatures.

4.1 Three-of-a-kind signatures.

We observe signature (3,0,0) is not possible, as noted by Hamilton as well as apparent by the Cayley–Dickson process.

Signature (0,3,0) is also not possible, as any unit needs 1 to close. Another way to see this is (0,2,0) will not close because any two elements with negative square norm, will have product with a positive square norm. In this way, the multiplication table will contain elements with positive norm, contradicting the assumed signature. Thus, signature (0,3,0) is not possible.

On the other hand, it is possible to have a signature of (0,0,3). Consider the algebra $\langle I+IL, J+JL, K-KL\rangle$. The multiplication table for this subalgebra is given in Table 9. As $|K-KL|^2=0$, this is a totally nilpotent subalgebra.

4.2 Two-of-a-kind signatures

Just as there are no subalgebras with signature (0,3,0), subspaces with signature (0,2,1) and (0,1,2) do not close, as they require at least one element

$\times$	1	$I+IL$	$J-JL$
1	1	$I+IL$	$J-JL$
$I+IL$	$I+IL$	0	0
$J-JL$	$J-JL$	0	0

Table 10: These are the possible products of $1, I+IL$, and $J-JL$.

with positive norm. For any a such that $|a|^2 < 0$, we see

$$\begin{aligned}|a^2|^2 &= |a|^2|a|^2 \\ &= (-b)^2 \\ &> 0. \end{aligned} \tag{21}$$

Thus a^2 contradicts the assumed signature and so these subspaces will not close.

Furthermore, any subspace of signature (2,1,0) or (2,0,1) does not close. In the case of signature (2,1,0), the generators have the form $\langle 1, a, b\rangle$ where $a \in \{I, J, K\}$ and $b \in \{L, IL, JL, KL\}$. 1 has to be included as $a^2 = -1$ and $b^2 = 1$. Now ab will be in $\mathbb{H}L$ where $ab \neq b$, thus this subspace does not close.

Similarly for signature (2,0,1) we would have $\langle 1, a, b\rangle$ where $a \in \mathbb{H} - \mathbb{R}$ and $|b|^2 = 0$. Then

$$\begin{aligned}|ab|^2 &= |a|^2|b|^2 \\ &= |a|^2(0) \\ &= 0 \end{aligned} \tag{22}$$

However $a \notin \mathbb{R}$ so we know $ab \neq \langle b\rangle$, thus ab has norm contradicting the signature. Thus, subspaces of signature (2,0,1) do not close.

Now let us consider signature (1,2,0). Here we have $\langle 1, a, b\rangle$ where $|a|^2 < 0$ and $|b|^2 < 0$. Again 1 must be included, as any element of $\mathbb{H}L$ will square to an element of $\mathbb{R}$. For this algebra to close, ab must be a linear combination of elements in our basis. If $a, b \in \mathbb{H}L$, it suffices to consider $ab = (cL)(dL) = L(cd)L = d\bar{c} \in \mathbb{H}$ where $d\bar{c} \notin \mathbb{R}$ and a, b are independent. Hence, this subspace does not close.

On the other hand, we can find subalgebras of signature (1,0,2). Consider $\langle 1, I+IL, J-JL\rangle$, whose multiplication table is seen in Table 10. This subalgebra is the sum of $\mathbb{R}$ and a nilpotent 2-D algebra, namely the subalgebra of signature (0,0,2) generated by $\langle I + IL, J - JL\rangle$. The subalgebra $\langle I + IL, J - JL\rangle$ is an ideal within the subalgebra $\langle 1, I + IL, J - JL\rangle$.

As in the 2-D case, the subalgebra, generated by $\langle 1, 1 + L, I + IL\rangle$ does not have signature (1,0,2) as the subalgebra generated by $\langle 1, 1 + L\rangle$ does not have signature (1,0,1). Thus, the only possible type of algebra for signature (1,0,2) is the extension of the nilpotent 2-D subalgebra with $\mathbb{R}$.

4.3 One-of-each-kind signature

Last but not least, subalgebras with signature (1,1,1) can be formed. Consider the subalgebra generated by $\langle 1, L, I + IL\rangle$. This subalgebra's multiplication table is seen in Table 11.

$\times$	1	L	$I + IL$
1	1	L	$I + IL$
L	L	1	$-(I + IL)$
$I + IL$	$I + IL$	$I + IL$	0

Table 11: These are the possible products of $1, L$, and $I + IL$.

In this way, composition subalgebras of $\langle 1, L\rangle$ and $\langle a + bL\rangle$ for $a = b \in Im(\mathbb{H})$ close and have signature (1,1,1). If $a \neq b$, we find $L(a+bL) = -aL-b$ and $-(b+aL)$ is not in our subalgebra. If we choose a different $\mathbb{H}L$ element to compose with, we need to have all three non-real elements of $\mathbb{H}$ represented, i.e. we need $\langle 1, aL, b + cL\rangle$ where $a \neq b \neq c$ for $a, b, c \in \mathbb{H} - \{1\}$. This can be seen in the subalgebra $\langle 1, IL, J \pm KL\rangle$.

4.4 Summary

The possible signatures and examples of their subalgebras are listed in Table 12.

Signature	
(3,0,0)	Not possible
(0,3,0)	Not possible
(0,0,3)	Nilpotent: $\langle I+IL, J+JL, K-KL\rangle$
(2,1,0)	Not possible
(2,0,1)	Not possible
(1,2,0)	Not possible
(1,0,2)	Mixed $\mathbb{R}$+nilpotent: $\langle 1, I+IL, J-JL\rangle$
(0,2,1)	Not possible
(0,1,2)	Not possible
(1,1,1)	Mixed $\mathbb{C}$+nilpotent : $\langle 1, L, I+IL\rangle$

Table 12: Possible signatures of 3-D subalgebras and their validity.

5 4-D Subalgebras

As the signature must add up to four, we find several possible 4-D subalgebras.

5.1 Four-of-a-kind

As discussed previously, we find $\mathbb{H}$, via the Cayley–Dickson process, has signature (4,0,0). As we ascend in dimension, we will find this is the last time an X-D, X-of-a-kind subalgebra closes.

Signature (0,4,0) is not possible. This follows previous arguments in lower dimensions, as any element $a \in \mathbb{H}L$, squares to an element that contradicts the signature of a space with no real elements.

To have a signature of (0,0,4), we need four independent, orthogonal elements of $\mathbb{O}'$ whose norms are each zero. Consider the subalgebra generated by $\langle 1+L, I+IL, J+JL, K-KL\rangle$. The multiplication table is displayed in Table 13

From Table 13 we observe the 3-D subalgebra generated by $\langle I+IL, J+JL, K-KL\rangle$ is an ideal of our signature (0,0,4) subalgebra since the subalgebra $\langle I+IL, J+JL, K-KL\rangle$ is a completely nilpotent subalgebra. As the 4-D algebra $\langle 1+L, I+IL, J+JL, K-KL\rangle$ contains an ideal, it is considered non-simple.

	$1+L$	$I+IL$	$J+JL$	$K-KL$
$1+L$	$2(1+L)$	0	0	$2(K-KL)$
$I+IL$	$2(I+IL)$	0	$2(K-KL)$	0
$J+JL$	$2(J+JL)$	$-2(K-KL)$	0	0
$K-KL$	0	0	0	0

Table 13: Multiplication table of subalgebra $\langle 1+L, I+IL, J+JL, K-KL\rangle$.

5.2 Three-of-a-kind 3-D without nilpotent elements

Firstly, any 3-D signature without nilpotent generators that is not possible will not be made possible by adjoining a nilpotent generator to the generating set. Consider for a, b such that $|a|^2 \neq 0$, and $|b|^2 = 0$. Then we see the following as in Lemma 1:

$$\begin{aligned} |ab|^2 &= |a|^2 \, |b|^2 \\ &= 0 \end{aligned} \tag{23}$$

Since the 3-D subalgebra set has at least two elements that are not 1, without loss of generality, $ab \neq b$, thus the 4-D subspace does not close. This argument eliminates possible signatures of (3,0,1), (0,3,1), (2,1,1), and (1,2,1).

Furthermore, adding a "time-like" or $\mathbb{H}$-unit to the generating set of a three-of-a-kind subalgebra, resulting in a signature of (3,1,0), (1,3,0), and (0,1,3), is still not possible. Consider a subalgebra with signature (3,1,0). As seen in 3-D, three quaternionic elements will not close without their fourth, which is another 1 norm element. Similarly for signature (1,3,0), three independent elements of $\mathbb{H}L$ will not close without all of $\mathbb{H}$. We know that three quaterions do not close, thus you can generate your subalgebra with one, one and an imaginary quaterion, or all four. Since for any $a, b \in \mathbb{H}L$, $ab \in \mathbb{H}$, with three independent elements of $\mathbb{H}L$ you will produce three distinct elements of $\mathbb{H}$. This subspace of signature (1,3,0) will not close. Finally, signature (0,1,3) will not have an associated subalgebra as a single element of norm -1 will not close without a real component.

Of three-of-a-kind signatures, (1,0,3) remains to be considered. An example of a possible subalgebra is generated by $\langle 1, I+IL, J+JL, K-KL\rangle$. The multiplication table of this subalgebra is listed in Table 14 We observe within this subalgebra, a 3-D ideal generated by $\langle I+IL, J+JL, K-KL\rangle$. Thus the subalgebra is generated by $\langle 1, I+IL, J+JL, K-KL\rangle$ is non-simple.

	1	$I+IL$	$J+JL$	$K-KL$
1	1	$I+IL$	$J+JL$	$K-KL$
$I+IL$	$I+IL$	0	$2(K-KL)$	0
$J+JL$	$J+JL$	$-2(K-KL)$	0	0
$K-KL$	$K-KL$	0	0	0

Table 14: Here is the multiplication table of a sample subalgebra of signature (1,0,3).

$\times$	1	K	$I+IL$	$J-JL$
1	1	K	$I+IL$	$J-JL$
K	K	-1	$J-JL$	$-I-IL$
$I+IL$	$I+IL$	$-J+JL$	0	0
$J-JL$	$J-JL$	$I+IL$	0	0

Table 15: These are the possible products of $1, K, I+IL$, and $J-JL$.

This leaves us to consider two-of-a-kind signatures.

5.3 Extensions of Two-of-a-kind signatures

Above, signatures (2,1,1) and (1,2,1) were shown not to be possible. Consider signature (2,2,0). This is the signature of the split quaterions as generated using the Cayley–Dickson process. An example of a such a subalgebra is generated by $\langle 1, I, L, IL\rangle$.

Signature (0,2,2) does not have an algebra, as any negative norm element will square to a positive norm element.

Now consider signature (2,0,2). A possible subalgebra might be generated by $\langle 1, K, I+IL, J-JL\rangle$. Consider the multiplication table as seen in Table 15.

Consider signature (1,1,2). A possible subalgebra of such a signature is generated by $\langle 1, L, I+IL, J-JL\rangle$. Observe the multiplication table given in Table 16. We observe that there exists an ideal in these subalgebras, where the ideal is generated by $\langle I+IL, J-JL\rangle$, a completely nilpotent 2-D subalgebra. This means the subalgebras, $\langle 1, L, I+IL, J-JL\rangle$ and $\langle 1, K, I+IL, J-JL\rangle$ are non-simple subalgebras.

$\times$	1	L	$I+IL$	$J-JL$
1	1	L	$I+IL$	$J-JL$
L	L	1	$-(I+IL)$	$J-JL$
$I+IL$	$I+IL$	$I+IL$	0	0
$J-JL$	$-J+JL$	$-(J-JL)$	0	0

Table 16: These are the possible products of $1, L, I+IL$, and $J-JL$.

Signature	
(4,0,0)	$\mathbb{H}$
(0,0,4)	Completely Nilpotent: $\langle 1+L, I+IL, J+JL, K-KL\rangle$
(2,2,0)	$\mathbb{H}'$
(2,0,2)	$\mathbb{C}$+2-D Nilpotent: $\langle 1, K, I+IL, J-JL\rangle$
(1,0,3)	$\mathbb{R}$+3-D Nilpotent: $\langle 1, I+IL, J+JL, K-KL\rangle$
(1,1,2)	$\mathbb{C}'$+2-D Nilpotent: $\langle 1, L, I+IL, J-JL\rangle$

Table 17: Signatures of 4-D subalgebras and their validity.

5.4 Summary

The possible signatures and examples of their subalgebras are listed in Table 17.

6 5-D, 6-D, and 7-D Subalgebras

The 5-D, 6-D,and 7-D subalgebras will be considered together, as many of the arguments against one kind of signature, are the same in these cases.

6.1 Signatures (P,0,0), (0,M,0), (0,0,N) for $P, M, N \geq 5$

The number of possible subalgebras decreases from this dimension on up. There are only four independent units with norm 1, namely $1, I, J, K$, and four distinct units with norm -1, namely L, IL, JL, KL, in $\mathbb{O}'$. As we need independent generators to classify signatures of subspaces, signatures (5,0,0) and (0,5,0) are impossible and similarly for (6,0,0), (0,6,0), (7,0,0), and (0,7,0).

Similarly, as we progress to higher dimensions such as 6-D and 7-D, we find that subalgebras with signatures (P,M,N) will not exists for $P \geq 5$ or $M \geq 5$. Therefore signatures (5,1,0), (1,5,0), (5,0,1), (0,5,1), (5,2,0), (5,1,1), (5,0,2), (1,5,1), (2,5,0), (0,5,2), (6,1,0), (6,0,1), (0,6,1), and (1,6,0) do not have subalgebras associated to them. We do not need to go any higher as this is an 8 dimensional space.

Furthermore, consider $N \geq 5$. For this signature, we need at least five independent, orthogonal null elements and we have eight generators in $\mathbb{O}'$. Since none of the generators of $\mathbb{O}'$ have norm 0, to create null elements we need pairs from $\mathbb{O}'$, each with a quaternionic and a $\mathbb{H}L$ part. Hence any orthogonal basis has at most four independent null generators and in this case all the octonions are represented among the set of generators. Thus, the set of four generators is independent and orthogonal, but any larger, say five or more null generators, would no longer be orthogonal. Thus signatures (0,0,5), (0,0,6), (0,0,7), (0,1,5), (1,0,5), (0,2,5), (2,0,5), (1,1,5), (1,0,6), and (0,1,6) do not have subalgebras associated with them as each has $P \geq 5$ or $M \geq 5$. We do not need to go any higher as this is an 8 dimensional space with signature (4,4,0).

6.2 Signatures $(0, M, N)$

Any subset with signature $(0, M, N)$ will not close, as shown previously, since any element of $\mathbb{H}L$ squares to a real number. This argument eliminates signatures (0,4,1), (0,1,4), (0,2,4), (0,4,2), (0,3,4), (0,3,3), (0,3,2), (0,2,3), and (0,4,3).

6.3 Signatures (P, M, N): $P \neq M$ & $P, M \neq 0$

There are two cases to consider $P > M$ and $P < M$ both such that $P \neq 0$ and $M \neq 0$. For $P > M$ we see $M < P \leq 4$. Choose a,b in the subalgebra with $a \in \mathbb{H}$ and $b \in \mathbb{H}L$; then ab in $\mathbb{H}L$. Since there are P independent choices of a, but only $M < P$ independent possibilities for ab, the algebra can not close. One such example of this is $\langle 1, I, J, K, IL, JL, KL \rangle$ as $IL(I) = L \notin \langle 1, I, J, K, IL, JL, KL \rangle$.

For $P < M$, Choose b as above, but now a also in $\mathbb{H}L$. Then ab is in $\mathbb{H}$. Since there are M independent choices for a, but only $P < M$ independent possibilities for ab, the subspace can not close. Thus the subspace does not close. One example of this is $\langle 1, I, L, IL, JL, KL \rangle$ as $IL(JL) = K \notin \langle 1, I, L, IL, JL, KL \rangle$.

The remaining signatures are then (1,0,4), (1,1,3), (1,1,4), (2,0,3), (2,0,4), (2,2,1), (2,2,2), (2,2,3), (3,0,2), (3,0,3), (3,0,4), (3,3,0), (3,3,1), (4,0,1), (4,0,2), and (4,0,3).

6.4 Four-of-a-kind

Our possible four of a kind subalgebras have signatures (1,0,4), (1,1,4), (2,0,4), (3,0,4), (4,0,1), (4,0,2), and (4,0,3).

Now we consider signatures (4,0,1) and (1,0,4). To deduce the signature, we need independent generators. It suffices to consider units. Then a subalgebra of signature (4,0,1) can be generated by $\langle 1, I, J, K, a \rangle$ where $|a|^2 = 0$. However, in order for a to be null, it must be a linear combination of elements of $\mathbb{H}$ and $\mathbb{H}L$. This means the true signature of this space will be (4,1,0), with generators $\langle 1, I, J, K, b \rangle$ such that $b = a - m$ with $m \in \mathbb{H}$. This same argument can be applied to signatures (4,0,2) and (4,0,3).

For signatures (1,0,4), (2,0,4), (3,0,4), and (1,1,4), we need four independent and orthogonal null generators whose multiplication table closes and at least the generator, 1 as any other choice of $\mathbb{H} - \mathbb{R}$ on its own will square to -1 and not close. As we need four independent null elements and we have eight generators in $\mathbb{O}'$, and since none of the generators of $\mathbb{O}'$ have norm 0, to create null elements we need pairs from $\mathbb{O}'$, each with a quaternionic and a $\mathbb{H}L$ part. This means any basis has at most four independent null generators, all the octonions are represented among the set of generators. Thus extending this basis by adding a generator of $\langle 1 \rangle$ make the new generating set dependent. This means the true signature is not (1,0,4), but (1,1,3).

	1	L	$I+IL$	$J+JL$	$K-KL$
1	1	L	$I+IL$	$J+JL$	$K-KL$
L	L	1	$-(I+IL)$	$-(J+JL)$	$K-KL$
$I+IL$	$I+IL$	$I+IL$	0	$2(K-KL)$	0
$J+JL$	$J+JL$	$J+JL$	$-2(K-KL)$	0	0
$K-KL$	$K-KL$	$-(K-KL)$	0	0	0

Table 18: Here is the multiplication table of a sample subalgebra of signature (1,1,3).

In this way, for 5-D, 6-D, and 7-D, none of the four-of-a-kind signatures have corresponding subalgebras.

6.5 Three-of-a-kind

As three elements of $\mathbb{H}$ cannot close, signatures (3,3,0), (3,3,1), (3,0,2), and (3,0,3) do not have associated subalgebras.

This leaves us with possible signatures of (1,1,3), (2,0,3), and (2,2,3). For signature (2,0,3), there are four possible $\mathbb{H}$ units to choose from for the two norm 1 generators. In choosing three independent null generators, at least three elements of $\mathbb{H}$ are represented. It is not possible to choose 2 more elements of $\mathbb{H}$ and have the generating set be independent. In this way, signature (2,0,3) is not the true signature and thus signature (2,0,3) has no associated subalgebra. Similarly for signature (2,2,3).

This leaves signature (1,1,3). Consider the subalgebra generated by $\langle 1, L, I+IL, J+JL, K-KL\rangle$. The corresponding multiplication table is displayed in Table 18. As seen in Table 18, the subalgebra generated by $\langle 1, L, I+IL, J+JL, K-KL\rangle$ closes. Thus signature (1,1,3) is possible. Notice that this subalgebra is non simple as the subalgebra $\langle I+IL, J+JL, K-KL\rangle$ forms an ideal.

6.6 Two-of-a-kind

The remaining signatures to consider are (2,2,1) and (2,2,2). Suppose the generators of the subalgebra with signature (2,2,1) are independent. Then this algebra has subalgebras of $\mathbb{H}'$ and a nilpotent 1-D subalgebra. The nilpotent 1-D subalgebra will form an ideal, by our usual norm argument.

$\times$	1	K	L	KL	$I+IL$	$J-JL$
1	1	K	L	KL	$I+IL$	$J-JL$
K	K	-1	KL	$-L$	$J-JL$	$-(I+IL)$
L	L	$-KL$	1	$-K$	$-(I+IL)$	$J-JL$
KL	KL	L	K	1	$J-JL$	$I+IL$
$I+IL$	$I+IL$	$-(J-JL)$	$I+IL$	$-(J-JL)$	0	0
$J-JL$	$J-JL$	$I+IL$	$-(J-JL)$	$-(I+IL)$	0	0

Table 19: Here is the multiplication table of a sample subalgebra of signature (2,2,2).

Signature	
(1,1,3)	$\mathbb{C}'$+3-D Nilpotent: $\langle 1, L, I+IL, J+JL, K-KL\rangle$
(2,2,2)	$\mathbb{H}'$+2-D Nilpotent: $\langle 1, K, L, KL, I+IL, J-JL\rangle$

Table 20: Possible signatures of 5-D subalgebras and their validity.

However for some $a \in \mathbb{H} - \mathbb{R}$ and nilpotent element b, $ab \neq b$. Thus this subspace will not close.

Finally for signature (2,2,2) consider the subalgebra $\langle 1, K, L, KL, I+IL, J-JL\rangle$. These are independent and orthogonal. The multiplication table is shown in Table 19. Thus signature (2,2,2) is manifest in subalgebra $\langle 1, K, L, KL, I+IL, J-JL\rangle$.

6.7 5-D, 6-D, 7-D summary

Table 20 lists the possible signatures of subalgebras of dimension 5, 6, or 7 subalgebras.

7 Summary

The possible signatures of subalgebras of the split octonions are listed in Table 21. All of these algebras are either completely null, known Cayley–Dickson algebras, or sums of null and Cayley–Dickson algebras.

Signature	
(1,0,0)	$\mathbb{R} = \langle 1 \rangle$
(0,0,1)	Two kinds: $\langle 1 \pm L \rangle$(idempotent) and $\langle I \pm IL \rangle$ (nilpotent)
(2,0,0)	$\mathbb{C}$
(0,0,2)	Two kinds $\langle 1 + L, I + IL \rangle$ and $\langle I + IL, J - JL \rangle$
(1,0,1)	Nilpotent $\langle 1, I + IL \rangle$
(1,1,0)	Split Complex:$\langle 1, L \rangle$
(0,0,3)	Two kinds $\langle 1 + L, J + JL, K - KL \rangle$ and Nilpotent: $\langle I + IL, J + JL, K - KL \rangle$
(1,0,2)	Mixed $\mathbb{R}$+nilpotent: $\langle 1, I + IL, J - JL \rangle$
(1,1,1)	Mixed $\mathbb{C}$+nilpotent : $\langle 1, L, I + IL \rangle$
(4,0,0)	$\mathbb{H}$
(0,0,4)	Completely Nilpotent: $\langle 1 + L, I + IL, J + JL, K - KL \rangle$
(2,2,0)	$\mathbb{H}'$
(2,0,2)	$\mathbb{C}$+2-D Nilpotent: $\langle 1, K, I + IL, J - JL \rangle$
(1,0,3)	$\mathbb{R}$+3-D Nilpotent: $\langle 1, I + IL, J + JL, K - KL \rangle$
(1,1,2)	$\mathbb{C}'$+2-D Nilpotent: $\langle 1, L, I + IL, J - JL \rangle$
(1,1,3)	$\mathbb{C}'$+3-D Nilpotnet: $\langle 1, L, I + IL, J + JL, K - KL \rangle$
(2,2,2)	$\mathbb{H}'$+2-D Nilpotent: $\langle 1, K, L, KL, I + IL, J - JL \rangle$

Table 21: All possible signatures of subalgebras of $\mathbb{O}'$.

References

[1] Dray, T. and Manogue, C. A. (2015). The Geometry of the Octonions, (World Scientific Publishing).