On the Shallow Motion Approximations

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ABSTRACT

The approximate equations for shallow motions are derived mainly by following the approach of Spiegel and Veronis and the subsequent development of Dutton and Fichtl. Other derivations are also briefly noted. While each derivation assumes shallow flow, the conditions on the time scale and auxiliary assumptions vary between derivations. In the present study, the shallow motion approximations are found to be valid for a wider range of conditions than included in earlier derivations.

The more restrictive Boussinesq or "shallow convection" approximations form a subclass of shallow motions. Existing derivations of the full Boussinesq approximations do not apply to near-neutral conditions even though they are often applied to such conditions. The conditions required for the validity of the Boussinesq approximations are reformulated into criteria that are easier to evaluate.

Finally, the use of the shallow motion approximations in concert with Reynolds averaging is examined in some detail. Additional necessary conditions resulting from Reynolds averaging appear to be violated only in rather special situations, at least for atmospheric flows.

1. Introduction

The Boussinesq (1903) approximation is usually summarized in two parts: (i) the variation of perturbation density is neglected in the mass continuity equation and in the equation of motion except through its influence as a buoyancy term; and (ii) the influence of perturbation pressure on buoyancy can be neglected. Analogous approximations were used in the early study of Oberbeck (1888).

Considerable work has been devoted to establishing sufficient conditions for the validity of the Boussinesq approximations. The construction of these conditions has assumed a variety of forms and is not as straightforward as many would assume. Spiegel and Veronis (1960) and others have shown that the depth of the perturbation motion must be small compared to the scale height of the basic state flow and that the perturbations of the thermodynamic variables must be small compared to basic state values.

These conditions are necessary but not sufficient. (See Table 1, section 2.) The validity of the Boussinesq approximations requires additional conditions including restrictions on the time scale of the perturbation flow. The development of Spiegel and Veronis implicitly assumes that the Eulerian time scale is the same order of magnitude as the advective time scale. Ogura and Phillips (1962) derive equations analogous to those of Boussinesq by assuming that the time scale of the flow is comparable to that of the Brunt–Väisälä time scale of the perturbation flow. Dutton and Fichtl (1969) (see also Dutton, 1976) impose conditions on the Eulerian and advective changes of density separately resulting in a series of conditions which appear to be satisfied for a wide class of shallow atmospheric motions. Still other versions assume small Mach number, which together with a variety of more restrictive auxiliary conditions, such as constant entropy, leads to the Boussinesq approximation. A recent derivation in this class is presented by Businger (1982).

The assumptions of small amplitude of the thermodynamic perturbations, small depth of the flow, and conditions on the time scale allow application of the first part of the Boussinesq approximation. This allows use of the incompressible mass continuity equation and linearization of the ideal gas law. We will refer to this part of the Boussinesq approximation as the "shallow motion" approximations. The second part of the Boussinesq approximations requires additional conditions. In particular, the existing derivations of these approximations require the importance of the buoyancy term in the vertical equation of motion. Following Dutton and Fichtl, we will refer to this subclass of shallow motions, where the full Boussinesq approximations apply, as "shallow convection."

The above developments have led to substantial advances in atmospheric dynamics, which would otherwise be impossible or more complicated. In some studies, failure to recognize the organization afforded by the above developments has led to assumptions that are inconsistent or incorrect interpretations of the original heuristic arguments of Boussinesq. Sometimes advection of momentum or the perturbation pressure gradient in the momentum equation are incorrectly excluded with misuse of the term "Boussinesq." This led to neglect of the perturbation pressure gradient without discussion in most studies of cold air drainage, especially prior to Manins and Sawford (1979).

The present study reexamines the derivations of the approximate equations for shallow motions developed in Spiegel and Veronis and extended in Dutton and Fichtl. These developments appear to be less specialized than in other studies. It will be found that the requirements for application of the shallow motion approximations can be partially relaxed. At the same time existing derivations of the full Boussinesq approximations do not apply to some situations where they are commonly applied.

Certain Reynolds terms will be considered that have been neglected in previous studies. The Boussinesq approximations have been frequently used with Reynolds averaging. The validity of this use needs to be evaluated. It will be found that several new restrictions are required, although it appears that such restrictions are easily met for most atmospheric applications.

2. Mass continuity

Conventionally, the total flow ϕ_T is decomposed into a basic state part ϕ_0 and the perturbation motion ϕ such that

$$\phi_T = \phi_0 + \phi. \tag{1}$$

The basic state is assumed to be motionless and dependent only on height as in the study of Oberbeck (1888).

The starting point is the mass continuity equation, which we express in terms of α, the specific volume:

$$\frac{\partial \alpha}{\partial t} + \mathbf{v}_H \cdot \nabla_H \alpha + w \frac{\partial \alpha}{\partial z} + w \frac{\partial \alpha_0}{\partial z} = \alpha_0 \nabla_H \cdot \mathbf{v}_H + \alpha_0 \frac{\partial w}{\partial z}, \tag{2}$$

where $\mathbf{v}_H$ is the horizontal velocity vector and ∇_H the horizontal gradient operator; other symbols are defined in the usual sense for Cartesian coordinates and we have assumed that $\alpha \ll \alpha_0$ as recorded in Table 1. Note that the inequality $\nabla\alpha \ll \nabla\alpha_0$ does not follow. Here we distinguish between Eulerian and advective terms, which will allow easier generalization to cases of multiple time and velocity scales.

In terms of usual scaling variables, the mass continuity equation can be written as

$$\frac{\alpha^*}{\tau} + V\frac{\alpha^*}{L} + W\frac{\alpha^*}{D} + W\frac{\alpha_0}{H} = \alpha_0\frac{V}{L} + \alpha_0\frac{W}{D}, \tag{3}$$

where α^* is the perturbation scale for specific volume, τ the Eulerian time scale, V the horizontal velocity scale, W the vertical velocity scale, L the horizontal length scale, D the vertical depth scale and H the basic state depth scale

$$H \equiv \alpha_0 \Big/ \frac{\partial \alpha_0}{\partial z}, \tag{4}$$

which for an isothermal basic state atmosphere is about 8 km.

a. Spiegel and Veronis

Spiegel and Veronis assume that $\nabla \cdot \mathbf{V}$ scales the same as d/dt in their mass continuity equation (their Eq. 18). In other words the advective and Eulerian time scales are the same order of magnitude

$$\tau \sim \frac{L}{V} \sim \frac{D}{W}. \tag{5}$$

TABLE 1. Assumptions for validity of shallow motion approximations using present notation, and the resulting relationships. Here $\mathbf{v}$ is the three-dimensional velocity vector, $\boldsymbol{\Omega}$ the earth's rotation vector and the overbar and prime notation refer to the decomposition applied in section 4.

Dutton & Fichtl	Spiegel & Veronis	Present
Shallow motion relationships		
$\phi^* \ll \phi_0$	$\phi^* \ll \phi_0$	$\phi^* \ll \phi_0$
$D \ll H$	$D \ll H$	$D \ll H$
$\tau \sim W\Big/\left(g\dfrac{\alpha^*}{\alpha_0}\right)$	$\tau \sim \dfrac{L}{V}$	$\tau \gg \dfrac{\alpha^*}{\alpha_0}\min\left[\dfrac{L}{V}, \dfrac{D}{W}\right]$
$\dfrac{V}{W} \ll \dfrac{L}{D}$	—	—
mass continuity	$\nabla \cdot \mathbf{v} \approx 0$	
ideal gas law	$\dfrac{\alpha}{\alpha_0} \approx \dfrac{T}{T_0} - \dfrac{p}{p_0}$	
momentum equation	$\dfrac{d\mathbf{v}}{dt} = -\alpha_0\nabla p - 2\Omega x\mathbf{v} - (g\alpha/\alpha_0)\mathbf{k}$	
Shallow convection subclass		
$H \sim H_i$	$H = H_i$ [1]	$\dfrac{R}{g}\dfrac{\partial T_0}{\partial z} \ll 1$
$\alpha_0\dfrac{P^*}{D} \ll g\dfrac{\alpha^*}{\alpha_0}$	$\alpha_0\dfrac{P^*}{D} \sim g\dfrac{\alpha^*}{\alpha_0}$	$F^2 \ll 1$
mass continuity	$\nabla \cdot \mathbf{v} \approx 0$	
ideal gas law	$\dfrac{\alpha}{\alpha_0} \approx \dfrac{T}{T_0}$	
momentum equation	$\dfrac{d\mathbf{v}}{dt} = -\alpha_0\nabla p - 2\Omega x\mathbf{v} - (gT/T_0)\mathbf{k}$	
Auxillary assumption		*Reynolds term restrictions*
$V\dfrac{\alpha^*}{\alpha_0} \ll V_{\mathrm{div}}$		$(\overline{\bar{\mathbf{v}}\phi'}, \overline{\mathbf{v}'\bar{\phi}}) \ll \overline{\mathbf{v}'\phi'}$
		$\nabla \cdot \overline{\rho'\mathbf{v}'} \ll \rho_0 \nabla \cdot \bar{\mathbf{v}}$ or (33, 34)
		$\overline{\alpha'\dfrac{\partial p'}{\partial z}} \ll g\dfrac{\bar{\alpha}}{\alpha_0}$ or (39)
		$\overline{\alpha'\dfrac{\partial p'}{\partial x_i}} \ll \bar{u}_j\dfrac{\partial \bar{u}_i}{\partial x_j}$ or (40)

[1] Here H_i is the depth of a fluid of constant density; this relationship is actually a result of the derivation rather than an assumption.

This condition also appeared as a scaling argument in the derivation of the simplified mass continuity equation presented in Businger (1982). Condition (5) along with the assumption of shallow motion ($D \ll H$; $\alpha^* \ll \alpha_0$) allows application of incompressible mass continuity.

If the local time scale τ becomes small compared to the advective time scales L/V and D/W, then (5) is not valid. That is, condition (5) is invalid when density changes occur primarily due to effects other than mass transport. Such violations include propagation of energy by pressure fluctuations and density changes due to radiation flux divergence or phase change. Condition (5) also does not include steady state conditions where τ becomes large compared to the advective time scale.

b. Modified Spiegel and Veronis

Condition (5), implicitly used by Spiegel and Veronis and others, can be partially relaxed with no loss of approximation by comparing the Eulerian derivative of density directly with the largest of the divergence terms in (3). The Eulerian density change can be neglected if

$$\tau \gg \frac{\alpha^*}{\alpha_0}\min[L/V, D/W], \qquad (6)$$

which is a weaker assumption than (5). Condition (6) allows density changes due to compression and diabatic heating to exceed density changes due to mass transport. Since α^*/α_0 is generally $O(10^{-2})$ or less, (6) defines a minimum time scale that is much smaller than the advective or divergence time scale. At the same time (6) allows the Eulerian time scale to be arbitrarily large so that steady state conditions are allowed.

If condition (6) is valid, then only the additional condition $\alpha^*/\alpha_0 \ll 1$ is required in order that the first three terms in (3) be small compared to the right-hand side, in which case the scale version of the mass continuity equation becomes

$$W\frac{\alpha_0}{H} = \alpha_0\frac{V}{L} + \alpha_0\frac{W}{D}. \qquad (7)$$

Note that the horizontal advection of specific volume has been neglected by comparing with horizontal divergence and so forth. Vertical advection due to the basic state gradient can be neglected compared to the vertical divergence on the right-hand side for shallow motion ($D/H \ll 1$) in which case the mass continuity is approximately incompressible so that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad (8)$$

In deep motions where D/H is $O(1)$, the vertical advection term must be retained. Note that α_0 can be approximated as a constant coefficient for shallow motions while its height dependence must be included for deep motions.

c. Dutton and Fichtl

Dutton and Fichtl consider the case where buoyancy controls Eulerian accelerations (their Eq. 3.14) to the extent that (in present notation)

$$\tau \sim W/(g\alpha^*/\alpha_0), \qquad (9)$$

where we are assuming that momentum changes are characterized by the same time scale as that for density changes. Restriction (9) eliminates near-neutral flows where the buoyancy influence is weak. Then the Eulerian derivative of density or specific volume can be neglected with the additional trivial condition

$$\tau \gg D/g. \qquad (10)$$

However, advection of density is eliminated only after the auxiliary condition

$$\frac{V}{L}\frac{D}{W} \le O(1). \qquad (11)$$

Since one intent of the development is to prove the applicability of incompressible mass continuity for shallow convection, this condition seems partially circular, although the practical limitations of this assumption appear to be minimal.

In summary, use of condition (6) to derive the incompressible mass continuity approximation extends the application of this approximation to include steady state flows and flows where density advection is unimportant (excluded by 5) and to include near neutral flows (excluded by 9).

d. Nondivergent flow

The above developments are based on the usual assumption that the horizontal flow can be described by a single velocity scale. This assumption is not possible if a significant fraction of the flow is horizontally nondivergent, in which case the horizontal flow must be decomposed as

$$\mathbf{V}_H = \mathbf{V}_{\rm non} + \mathbf{V}_{\rm div},$$

$$\nabla_H \cdot \mathbf{V}_H \sim \mathbf{V}_{\rm div}/L.$$

The nondivergent part of the horizontal flow can be associated with translation or rotation and could be defined to absorb a basic state flow. Note that the time scales for advection and divergence are no longer necessarily the same order of magnitude.

The analysis leading to (6) can be restored by replacing V with $V_{\rm div}$ on the right-hand side of (6). Then the horizontal advection term in (3) can be neglected compared to the divergence term only if

$$V\frac{\alpha^*}{\alpha_0} \ll V_{\rm div}, \qquad (12)$$

where V is now the scale velocity for the entire horizontal flow. The ratio α^*/α_0 is generally $O(10^{-2})$ or smaller, in which case (12) will be satisfied if the divergent part of the flow is at least 10% of the total flow. An example of a possible exception is "two-dimensional" turbulence or certain vortex flows where horizontal divergence is small by definition. Then vertical motion forced by horizontal advection of density by the nondivergent flow may become significant. In such cases, the incompressibility approximation no longer applies. While the problem of two-dimensional turbulence is receiving increasing attention, it is difficult to isolate such phenomena with observations and violation of (12) remains an unverified possibility.

3. Elimination of perturbation pressure

Use of incompressible mass continuity is one of the two main advantages of the Boussinesq approximation. The other main advantage is the expression of buoyancy in terms of temperature instead of specific volume, which is more difficult to measure.

Consider the ideal gas law $P\alpha = RT$ which is linearized by using the assumption that deviations of variables of state are small compared to basic-state values. Then,

$$\frac{P}{P_0} + \frac{\alpha}{\alpha_0} \approx \frac{T}{T_0}. \tag{13}$$

Depassier and Spiegel (1982) have studied errors in this relationship associated with significant relative variations of the expansion coefficients for density (see Spiegel and Veronis, Eq. 11). These errors become important in some laboratory flows containing motions with relatively large horizontal scales. However, in the atmosphere, significant errors in (13) due to horizontal variations occur only on global scales where the condition of shallow motion ($D/H \ll 1$) is normally not appropriate. One geophysical exception to (13) results from enormous horizontal temperature gradients developing at the edge of the Martian polar ice cap or in response to shading by rather extreme Martian topography.

With the shallow convection conditions, it can be shown that the influence of pressure fluctuations in the ideal gas law (13) can be omitted, leading to

$$\frac{\alpha}{\alpha_0} = \frac{T}{T_0}. \tag{14}$$

Now buoyancy can be exclusively related to temperature variations. To justify the neglect of pressure perturbations, Dutton and Fichtl as well as Spiegel and Veronis begin with the vertical equation of motion for the perturbation flow,

$$\frac{dw}{dt} = g\frac{\alpha}{\alpha_0} - \alpha_0\frac{\partial p}{\partial z}, \tag{15}$$

where the perturbation variables of state are again assumed to be small compared to the basic state values, and, viscous terms are neglected. For shallow motions, the vertical variation of basic state variables can be neglected.

To study the role of pressure fluctuations in the vertical equation of motion, the linearized ideal gas law (13) is substituted into (15) in which case one obtains

$$\frac{dw}{dt} = g\frac{T}{T_0} - g\frac{p}{p_0} - \alpha_0\frac{\partial p}{\partial z}. \tag{16}$$

a. *Spiegel and Veronis*

Spiegel and Veronis compare the pressure contribution to the buoyancy with the vertical pressure gradient which in terms of present scale values yields the ratio

$$\frac{gp^*/p_0}{\alpha_0 p^*/D} \sim \frac{D}{H_i}, \tag{17}$$

where p^* is the scale value for pressure fluctuations and the scale depth for the perturbation pressure is assumed to be the same as that for density. This relationship implies that the pressure contribution to the buoyancy is small compared to the pressure gradient term in the vertical equation of motion if the motion is shallow. No additional conditions are needed.

Spiegel and Veronis noted that this relationship (17) can also be used to eliminate the pressure influence on the buoyancy. Although not explicitly stated, this inference requires that the buoyancy term must be the same order of magnitude as the pressure gradient term.

b. *Dutton and Fichtl*

The requirement of important buoyancy was explicitly expressed in Dutton and Fichtl. They assume that the motion is controlled by buoyancy (thus the name "shallow convection") to the extent that the scale value of the pressure gradient is either of comparable magnitude or less than the magnitude of the buoyancy term (their Eq. 3.13) which in present notation becomes

$$\alpha_0\frac{p^*}{D} \leqslant g\frac{\alpha^*}{\alpha_0}. \tag{18}$$

This condition is related to relationship (9). Then using the basic state ideal gas law $p_0\alpha_0 = RT_0$, one obtains

$$\frac{p^*}{p_0} \leqslant \frac{D}{H_i}\frac{\alpha^*}{\alpha_0}, \tag{19}$$

$$H_i \equiv RT_0/g, \tag{20}$$

where H_i is the depth of an isothermal atmosphere and is generally the same order of magnitude as H, as discussed below.

Then for shallow motion ($D/H_i \ll 1$), the ideal gas law for the perturbation flow

$$\frac{p}{p_0} + \frac{\alpha}{\alpha_0} = \frac{T}{T_0} \tag{21}$$

becomes

$$\frac{p}{p_0} \ll \frac{\alpha}{\alpha_0} \approx \frac{T}{T_0} \tag{22}$$

and the potential temperature of the perturbation flow becomes approximately

$$\frac{\theta}{\theta_0} \approx \frac{T}{T_0}.$$

This development shows that the buoyancy can be estimated in terms of temperature or potential temperature if the buoyancy term is at least as large as the pressure gradient term. In other terms, the vertical pressure gradient is constrained by the vertical equation of motion (16) and the assumed importance of buoyancy (18). This restriction on the vertical gradient of pressure and the condition of the shallow motion guarantee that the total pressure perturbation is sufficiently small to insure result (22). If the buoyancy term is much smaller than the pressure gradient term, then all three terms in the linearized ideal gas law (21) might be required.

c. Pressure influence

The development leading to (22) does not apply to flows where the influence of stratification or buoyancy is weak, such as turbulent motions with small temperature fluctuations. In these flows, (18) breaks down because vertical accelerations may be more controlled by fluctuating pressure gradients than buoyancy effects.

If the buoyancy term is not of primary importance in the vertical equation of motion, existing justification for neglect of the influence of pressure effects on the buoyancy does not apply. This case is of considerable interest since small buoyancy effects in the vertical equation of motion may still lead to important buoyancy effects in the kinetic energy equation, which is derived from the vertical equation of motion. The buoyancy flux term in the kinetic energy equation represents conversion between kinetic and potential energy and is of the form

$$\frac{g}{T_0} wT - \frac{g}{p_0} wp.$$

The second term is almost always neglected in atmospheric studies, and should not be confused with the often-retained pressure transport term originating from the pressure gradient term in the vertical equation of motion. With neglect of the pressure contribution to the buoyancy, the buoyancy flux term becomes proportional to the heat flux.

As an example of potential violation of this simplification, consider the case where the shallow motions are shear-driven turbulent eddies, in which the terms in the kinetic energy equation are usually averaged over some longer time scale. Even a small buoyancy flux term in the kinetic energy equation for such motions can completely alter their nature and associated transfer coefficients. It is important that the existing justification for the neglect of the influence of pressure fluctuations on the buoyancy flux is not valid for the case of small buoyancy flux. Direct determination of the pressure fluctuations is needed to resolve this issue.

Therefore, it is of interest to establish conditions when (18) is satisfied and the buoyancy flux can be safely related to the heat flux. Since pressure variations are usually difficult to measure it is useful to pose (18) in a form where the pressure is related to variables which are easier to measure.

Since the hydrostatic part of the perturbation pressure automatically satisfies (18), the validity of (18) requires restrictions on the magnitude of the nonhydrostatic part of the pressure induced by the motion itself. It is then useful to pose assumption (18) in terms of the Froude number. This is done by noting that the total derivative in the horizontal equation of motion can be scaled in terms of the parcel time scale L/V for the case of a single horizontal velocity scale and negligible Coriolis terms. Then

$$\frac{V^2}{L} \approx \alpha_0 \frac{p^*}{L}. \tag{23}$$

Substituting (23) into the inequality (18), we obtain

$$\frac{V^2}{D} \leqslant g \frac{\alpha^*}{\alpha_0} \tag{24}$$

or

$$F^2 \equiv V^2/Dg\left(\frac{\alpha^*}{\alpha_0}\right) \leqslant 1. \tag{25}$$

This limitation on permitted values of the Froude number expresses assumption (18) in terms of restriction on the flow speed for a given magnitude of buoyancy. Even with significant stratification, this inequality may break down on sufficiently small scales.

In summary, the influence of pressure fluctuations on the buoyancy term in the vertical equation of motion can be neglected compared to the pressure gradient term for shallow motions with no additional assumptions, as shown by Spiegel and Veronis. However, the influence of pressure fluctuations on the buoyancy and in the ideal gas law cannot be categorically neglected unless the buoyancy term is important in the vertical equation of motion as explicitly assumed by Dutton and Fichtl. The latter condition requires the Froude number to be on the order of one or smaller. These conditions are summarized in Table 1.

d. The two basic state depth scales

Note that incompressible mass continuity is a useful assumption if the depth of the perturbation motion is small compared to the scale height of the basic state specific volume, H, defined by (4). On the other hand, neglect of pressure perturbations in the buoyancy term requires that the motion depth scale be small compared to the depth of an isothermal atmosphere H_i, defined by (20). We now examine atmospheric situations where H is significantly smaller than H_i, in which case the incompressible mass continuity incurs some error even though the motion is shallow.

The relationship between H and H_i can be shown by differentiating the logarithmic form of the basic state ideal gas law:

$$\frac{1}{\alpha_0}\frac{\partial \alpha_0}{\partial z} + \frac{1}{p_0}\frac{\partial p_0}{\partial z} = \frac{1}{T_0}\frac{\partial T_0}{\partial z}. \tag{26}$$

Using the hydrostatic condition for the basic state fluid and definitions of H and H_i, we obtain

$$H^{-1} - H_i^{-1} = \frac{1}{T_0}\frac{\partial T_0}{\partial z}.$$

Rearranging,

$$H = H_i \Big/ \left(1 + \frac{H_i}{T_0}\frac{\partial T_0}{\partial z}\right);$$

Applying this relationship to an isothermal atmosphere, we recover the result

$$H = H_i.$$

For the usual case where the basic state temperature decreases with height

$$H > H_i$$

and the assumption of incompressible mass continuity for the perturbation motion is valid for greater motion depths compared to the usual example of isothermal basic state. However, the value of H remains close to that of H_i (Dutton and Fichtl) since the temperature decrease with height is limited by the dry adiabatic lapse rate.

On the other hand, with strong inversion conditions for the basic state fluid where temperature increases with height

$$H < H_i$$

and the conditions for application of incompressible mass continuity become more restrictive. In particular, the restriction

$$\frac{H_i}{T_0}\frac{\partial T_0}{\partial z} \approx \frac{R}{g}\frac{\partial T_0}{\partial z} \approx 29\ \mathrm{K}^{-1}\ \mathrm{m}\frac{\partial T_0}{\partial z} \ll 1, \tag{27}$$

must be satisfied.

As a numerical example of strong inversion conditions, we chose $\partial T_0/\partial z = 3$ K/100 m. Then $H = \frac{1}{2}H_i$, which leads to the restriction $D \ll \frac{1}{2}H_i$ or $D \ll 4$ km. The basic state atmosphere is most strongly stratified over relatively thin layers so that motions confined to such layers are also automatically thin. As an academic example, stratification in the lower polar troposphere sometimes exceeds an average value 3 K/100 m over a depth of 1 km. The application of incompressibility to motions occupying such a layer would lead to errors in excess of 25%. However, gravity waves may be the most likely possibility since the vertical extent of other motions would be seriously constrained by the stratification. Since D is the scale value for vertical derivatives, the appropriate value of D for gravity waves would be one-half the vertical wavelength.

We conclude that the scale height for the basic state specific volume is approximately equal to the depth of an isothermal atmosphere; exceptions with shallow atmospheric motions are rather special.

4. Transport between scales

In actuality, shallow atmospheric motions occur simultaneously on a variety of horizontal scales with exchange of properties between scales. Then for meaningful analysis, it is necessary to divide the perturbation flow into an averaged part $\bar{\phi}$ and a deviation part ϕ' such that

$$\phi_{\text{total}} = \phi_0 + \bar{\phi} + \phi'. \tag{28}$$

The part ϕ' generally includes the unresolved portion of the flow. Since the basic-state flow is time-independent, an equation for $\partial\bar{\phi}/\partial t$ can be constructed by substituting the above decomposition into the equation for ϕ and applying the averaging operator (an overbar) to the entire equation. We will loosely refer to ϕ' as turbulent fluctuations, although such terminology is only mathematical. The component ϕ' may not satisfy the physical requirements for turbulence and may include, for example, gravity waves. Such a decomposition may involve the use of spectral filtering of actual data, or definition of resolvable and unresolved motions in a numerical model implying grid volume averaging. Transport of ϕ by the turbulence leads to a mean contribution to the change of $\bar{\phi}$. Such terms are often referred to as Reynolds terms. In atmospheric studies, this exchange is not usually considered in the mass continuity equation but is often modeled as a diffusive process in the momentum and thermodynamic equations where energy is transferred from the larger to smaller scales. Exceptions include turbulence in stratified flows where turbulence kinetic energy may be transformed to larger scale gravity waves and two-dimensional motions (e.g., Lilly, 1983).

If the time or space scale of the averaged shallow motion is sufficiently separated from that of the turbulence, then it is useful to define the averaging operator in terms of simple (unweighted) averaging. In this case, the usual Reynolds' averaging rules hold and only the "turbulent" transport $\nabla\cdot\overline{\mathbf{v}'\phi'}$ results. Except for vertical motion close to the ground, a separation of scales does not occur systematically for atmospheric variables.

Without a spectral gap or separation of scales, physical interpretation of actual data requires detrending and/or high-pass filtering in order to partially remove sampling errors. With such operations included in the averaging operator, additional terms result:

$$\overline{\bar{\mathbf{v}}\phi'} \quad \text{and} \quad \overline{\mathbf{v}'\bar{\phi}}. \tag{29}$$

The extra terms (29) are normally neglected without comment, although their potential importance has been noted by Thompson (1954), Charnock (1957), Lester (1972) and others. For measurements of turbulence in stratified flows, we have found the terms (29) to be important when the record is not at least an order of magnitude longer than the scale defining the overbar operator. However, no general statements can be made and this subject needs more attention.

a. Mass continuity

The mass continuity equation for the above decomposition (28) is most conveniently applied in flux form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0. \tag{30}$$

We substitute the above decomposition (28) into (30), then average the entire equation, and neglect the extra Reynolds terms (29). Noting that the basic state flow is motionless and dependent only on height, we obtain

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\mathbf{v}}_H \cdot \nabla \bar{\rho} + \bar{w}\left(\frac{\partial \bar{\rho}}{\partial z} + \frac{\partial \rho_0}{\partial z}\right) = -(\rho_0 + \bar{\rho})\nabla \cdot \bar{\mathbf{v}} - \nabla \cdot (\overline{\rho' \mathbf{v}'}). \tag{31}$$

The analogy with (1)–(2) is not complete since the mean variable $\bar{\phi}$ is now formally defined in terms of an averaging operator in contrast to previous sections where such a commitment was not necessary.

The Reynolds term in (31) is generally neglected without formal justification although molecular transport of density is included in some treatments of laboratory flows. Noting that the correlation coefficient is bounded by unity, the magnitude of this term can be estimated in terms of the basic length scales. Then

$$\nabla \cdot \overline{\rho' \mathbf{v}'} \leqslant O\left(\frac{\sigma_\rho \sigma_w}{D} + \frac{\sigma_\rho \sigma_{\mathbf{v}H}}{L}\right), \tag{32}$$

where σ represents the standard deviation.

We now assume that both σ_w and $\sigma_{\mathbf{v}H}$ can be scaled by the turbulent velocity scale q so that

$$\sigma_w \approx \sigma_{\mathbf{v}H} \sim q.$$

Since this is an order of magnitude restriction, it is much weaker than assuming isotropic turbulence. Classically, q is related to the most important mean velocity scale such that

$$q = C \max[V, W].$$

For mean flows that are thin and principally horizontal, turbulence is often generated by vertical shear of the horizontal flow in which case q is related to V. Next, consider narrow convective thermals or plumes where the turbulence is generated more by horizontal shear of the updraft than small scale buoyancy. Then, q is related to W. In the intermediate case, $V \approx W$. In practice C depends on relative position, the role of stratification or small scale buoyancy generation of turbulence, stage of development and so forth. Of importance here is that C is almost always small compared to unity.

In the case of narrow convective flow, the turbulent transport is dominated by the second term on the right-hand side of (32) and horizontal velocity fluctuations scale as $\sigma_{\mathbf{v}H} \sim CW$. Then the turbulent transport term scaled by the mean divergence term is of the order

$$\sigma_\rho \frac{CW}{L} \bigg/ \rho_0(W/D) \sim C \frac{\sigma_\rho}{\rho_0} \frac{D}{L}. \tag{33}$$

Since the usual value of C and the ratio of densities are both small, the turbulence transport term becomes important only for extremely narrow convection. However, turbulence may prevent the convection from becoming narrower than a critical value. As an example, consider typical boundary layer thermals where $C \sim 10^{-1}$, $\sigma_\rho/\rho_0 \sim 10^{-2}$ and $D/L \sim 10$. Then (33) indicates that the transport of density by the smaller scale turbulence is estimated to be two orders of magnitude smaller than the mean flow divergence.

In the case of thin horizontal flow, the ratio of the turbulent transport to the mean flow divergence becomes

$$C \frac{\sigma_\rho}{\rho_0} L/D, \tag{34}$$

where again turbulent transport of density becomes important only in the case of extreme aspect ratio.

We conclude that for conditions where the turbulence can be described by a single velocity scale, which is in turn proportional to the dominant mean velocity scale, the Reynolds term in the mass continuity equation can be neglected.

b. Equation of motion

To derive the appropriate form of the equations of motion, we first substitute the decomposition (28) into the equations of motion, average, neglect all extra Reynolds terms (e.g., 29) and assume the turbulent flow is approximated by incompressible mass continuity. Then we use the assumptions of basic state hydrostatic flow and $\alpha \ll \alpha_0$. Neglecting viscous terms, the resulting equation of motion in tensor form is

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\alpha_0 \frac{\partial \bar{p}}{\partial x_i} - \overline{\alpha' \frac{\partial p'}{\partial x_i}} + \frac{\bar{\rho}}{\rho_0} g \delta_{i3} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}, \tag{35}$$

where repeated indices are summed. The extra term associated with correlation between density and pressure gradient is usually neglected. This term is not automatically small, since the assumption

$$\nabla p' \ll \nabla \bar{p} \tag{36}$$

does not follow from $p' \ll \bar{p}$.

The magnitude of the pressure fluctuations is difficult to estimate but some information can be obtained by decomposing p' into hydrostatic and nonhydrostatic parts

$$p' = p'_{\text{non}} + p'_{\text{hyd}}. \tag{37}$$

The hydrostatic part of the pressure fluctuation leads to zero vertical acceleration by definition.

The nonhydrostatic part of the pressure fluctuation can be most simply estimated from the usual scale analysis (23). Applying this scaling to the horizontal equation of motion for the fluctuating part of the flow yields

$$p'_{\text{non}} \approx \rho_0 q^2$$

where q is again the turbulent velocity scale. Then the pressure correlation term in (35) becomes

$$\overline{\alpha' \frac{\partial p'}{\partial x_i}} \sim \frac{(\alpha'/\alpha_0)q^2}{l} \tag{38}$$

where l is the turbulent length scale. Using (38), numerous ratios can be constructed from (35) to indicate the relative importance of the pressure correlation terms.

A sufficient condition for neglect of the pressure correlation term in the vertical equation of motion can be derived by scaling with respect to the buoyancy term. Then the pressure correlation term in the vertical equation of motion can be neglected if

$$\overline{\alpha' \frac{\partial p'}{\partial z}} \Big/ (g\bar{\alpha}/\alpha_0) \sim \frac{\alpha'}{\bar{\alpha}} q^2/gl \ll 1. \tag{39}$$

Since q^2 will normally be much smaller than gl, (39) indicates that the pressure correlation term is at least several orders of magnitude smaller than other terms in the vertical equation of motion.

The importance of the pressure fluctuation term in the horizontal momentum equation can be estimated by scaling (38) with the advection terms. Then this pressure fluctuation term can be neglected if the ratio

$$\overline{\alpha' \nabla_H p'} \Big/ \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} \sim \frac{\alpha'}{\alpha_0} \frac{q^2}{l} \Big/ \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} \tag{40}$$

is small. As one example, consider the case where the largest advection term scales as V^2/L in which case the above ratio (40) becomes

$$\frac{\alpha'}{\alpha_0} \frac{q^2}{V^2} \frac{L}{l}.$$

Since α'/α_0 is of $O(10^{-2})$ and the ratio of velocity scales is normally small compared to one, the pressure fluctuation term appears to be generally unimportant. Very small turbulent-length scale l usually implies very small turbulent-velocity scale q so that it is difficult to construct geophysical examples where the above ratio is not small.

Comparison of the pressure fluctuation term with the usual Reynolds stress term also indicates that the pressure fluctuation term is not likely to be important for geophysical examples. However, the careful investigator may wish to evaluate (39)–(40) on an individual flow basis.

The pressure fluctuation term can be avoided entirely by posing the equation of motion in flux form

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} - \rho g \delta_{i3}.$$

After using decomposition (28) and Reynolds averaging, covariance terms involving the buoyancy flux result in lieu of the pressure fluctuation term. The buoyancy flux terms appear to be small although it is difficult to construct general conditions which can be readily evaluated for atmospheric situations.

5. Conclusions

The derivation of conditions for the validity of the Boussinesq approximations is not as straightforward as many would assume. In the literature, a variety of sets of conditions have been assumed which, if satisfied, allow application of the Boussinesq approximations. The Boussinesq approximation can be divided into two parts. The first group of assumptions allows use of incompressible mass continuity and linearization of the ideal gas law, which are referred to as the shallow motion approximations. Additional restrictions allow neglect of the pressure influence on buoyancy. This more restrictive subclass of shallow motions is equivalent to the full Boussinesq approximations, also referred to as the shallow convection approximations.

The different derivations of the shallow motion approximations share the following conditions (Table 1):

(a) the perturbations of variables of state must be small compared to basic state averaged values;

(b) the motion must be shallow compared to the scale depth of the basic state flow; and

(c) restrictions on the time scale are required.

The condition on the time scale developed in the present study (6) is less restrictive than in previous developments. Condition (6) still rules out a class of flows where the density fluctuations are due primarily to processes other than mass transport. Such violations include compression waves and the special case of strong diabatic heating with little motion. For most shallow motions of atmospheric interest, condition (6) on the time scale is easily satisfied, in which case incompressible mass continuity is a good approximation.

Following the approach of Spiegel and Veronis, no additional assumptions are needed in order to neglect the influence of pressure fluctuations on the buoyancy term in the vertical equation of motion. However, in order to neglect the influence of pressure changes in the ideal gas law and the expression for buoyancy, existing derivations require that the buoyancy term must be important in the vertical equation of motion. This requirement for the full Boussinesq approximations was implicitly assumed in Spiegel and Veronis and explicitly used in Dutton and Fichtl. This requirement implies that present derivations of the full Boussinesq approximations relevant to atmospheric flows do not apply to near-neutral flows. Therefore, existing derivations of the turbulence kinetic energy equation where the buoyancy flux is linearly related to the heat flux are not necessarily valid for the case where the buoyancy flux is small but still exerts an important influence on turbulent motions.

In special circumstances, it may be necessary to test additional criteria which have not been considered in previous studies. Advection of density by the nondivergent part of the flow could become important if condition (12) is not satisfied. Notable possibilities include two-dimensional turbulence and vortex flows. Relationship (27) shows that with strong basic state stratification, incompressible mass continuity is valid over a thinner depth compared to the depth over which one can neglect the influence of perturbation pressure on the buoyancy. Certain Reynolds terms can be neglected with certainty only if (29), (33)–(34) and (39)–(40) are small. These extra terms, as well as the influence represented by (27), seem negligible for most shallow atmospheric motions.

With some complications, the considerations presented above can be extended to include deep motions, important Coriolis effects, large vorticity and more complicated basic state flow.

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